PERTURBATIONS AND BIFURCATIONS IN THE THREE DIMENSIONAL KOLMOGOROV MODEL

by

Muhammad Abdus Sattar

B.Sc. (Honours), University of Rajshahi, 1965

M.Sc., University of Dacca, Bangladesh, 1966

M.S., University of Windsor, Canada, 1982

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics and Statistics

SIMON FRASER UNIVERSITY

August 1986

APPROVAL

Name: Muhammad Abdus Sattar

Degree: Doctor of Philosophy

Title of Thesis: Perturbations and Bifurcations in the Three Dimensional Kolmogorov Model.

Examining Committee:

Chairman: Dr. A.R. Freedman

Dr. G.N. Bojadziev
Senior Supervisor

Dr. M. Singh

Dr. D.L. Sharma

Dr. G.A.C. Graham

Dr. H.I. Freedman
External Examiner
Professor
Department of Mathematics
University of Alberta
Edmonton, Alberta

Date Approved: August 26, 1986

PARTIAL COPYRIGHT LICENSE

Title of Thesis/Project/Extended Essay

PERTURBATIONS AND BIFURCATIONS IN THE THREE DIMENSIONAL KOLMOGOROV MODEL

Author: ____________
(signature)

Muhammad Abdus Sattar
(name)

Sept. 17, 1986
(date)

ABSTRACT

In this thesis we investigate the stabilizing and destabilizing influence and branching effect of small perturbation on the equilibria of a dynamical system of three nonlinear autonomous ordinary differential equations of the Kolmogorov-type with small perturbation. It is assumed that the unperturbed system has at least one simple or multiple equilibrium in the first octant and that the equilibria of the perturbed system originating from the multiple equilibrium of the unperturbed system are simple.

By using the qualitative theory and bifurcation theory of differential equations, the nature and stability of the simple equilibria of the unperturbed as well as the perturbed systems are examined in the three dimensional phase space. In order to illustrate the theory, the qualitative behaviors of the equilibria of some three dimensional perturbed population models are compared with those of the corresponding unperturbed models.

We have shown that, under the influence of small perturbation, although the nature of the hyperbolic equilibria of the Kolmogorov model may or may not change, the stability of the equilibria remains the same, and both the nature and stability of the simple nonhyperbolic equilibria change. We have also proved that, if the Jacobian of the unperturbed terms of the dynamical system is zero and that up to the second degree perturbed terms is different from zero, then depending on the rank of the Jacobian matrix of the unperturbed terms, the multiple unperturbed equilibrium bifurcates into at least two or at most eight branches of simple perturbed equilibria.

DEDICATION

To

my parents Miraj Ali and Hurjan Nesa

my wife Lailun Nahar Sattar (Lily)

my daughters Shanta and Setu

ACKNOWLEDGEMENTS

I wish to express my sincere admiration, appreciation and gratitude to my supervisor, Dr. G.N. Bojadziev for his advice, encouragement, guidance and financial support provided through the course of this work.

I would like to take this opportunity to thank everyone who contributed in one way or another to the completion of my thesis. Specifically, I am thankful to Dr. G.A.C. Graham, Chairman of the Department of Mathematics and Statistics for providing me with Teaching Assistantships, and to the Departmental Graduate Studies Committee and the Senate Committee on Scholarships, Awards and Bursaries for awarding me an SFU Open Scholarship and a President's Research Grant Stipend. Thanks also to Kathy Hammes, Departmental Assistant; Cindy Lister, Diane Pogue, and Jane Fabricius, Office Staff for their indirect help. Special thanks to Sylvia Holmes, Departmental Graduate Secretary for her efficient handling of the typing.

The greatest debt is of course to my family, Lily, Shanta and Setu, to whom my gratitude can hardly be expressed. It would be an understatement to say that this thesis could not have been completed without their support - moral and emotional. I am really grateful to you, young ladies.

I also want to thank Professor H.I. Freedman for his suggestions and Dr. Steve Kloster for his help in the computation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

The mathematical investigation of a great number of physical and biological problems leads naturally to the solutions of differential equations - ordinary or partial, linear or nonlinear, autonomous or non-autonomous. In this work we shall discuss problems that can be described by a dynamical system of three ordinary nonlinear autonomous differential equations of the Kolmogorov-type. Dynamical systems corresponding to physical and biological problems usually contain a certain number of parameters. Our work will involve only one small positive parameter representing perturbation in the dynamical system.

The purpose of this thesis is the study of the effect of small perturbations on a nonlinear system of three species interactions represented by the three dimensional perturbed Kolmogorov model. The object of the study is to investigate the behavior of solutions involving changes and no changes in the nature and stability of the equilibrium of the Kolmogorov model in both the cases when the parameter does not vary and is varied. This study is of mathematical significance and important for various applications. Three dimensional dynamical systems of the Kolmogorov-type are frequently used as examples in bifurcation theory, chaos, invariant manifold theory, population dynamics, and some other branches of modern applied mathematics.

In order to achieve our objective, we shall derive the criteria for the existence of simple equilibria of the three dimensional perturbed Kolmogorov model corresponding to the simple and multiple equilibria of the unperturbed model; study bifurcation of multiple unperturbed equilibrium into simple perturbed equilibria; provide parameter conditions for all possible types of simple equilibria; examine the nature and stability of the unperturbed as well as the perturbed simple equilibria in the three dimensional phase space; and compare the qualitative behaviors of the equilibria of the perturbed system with those of the unperturbed system.

CHAPTER 1

THE SURVEY AND THE PROPOSAL

In this chapter I have done a survey of research works dealing with the qualitative analyses of two and three dimensional Kolmogorov-type models with and without perturbation. The important results of these works have been summarized in brief. The last section of this chapter contains the proposal for the dissertation.

1.1 TWO DIMENSIONAL KOLMOGOROV MODEL

In 1936, Kolmogorov [37] proposed a system of two equations

$$N_i' = N_i F_i(N_1, N_2), \qquad i = 1,2, \tag{1.1}$$

$' \equiv d/dt$, as a model of predator-prey problems, where N_1 and N_2 are the number of prey and predator populations, and F_1 and F_2 are two given functions of N_1 and N_2. Kolmogorov provided a set of sufficient conditions, global in nature, for the existence of either a stable equilibrium point or a stable limit cycle.

Utz and Waltman [59], and Waltman [60] examined the possibilities of periodic solutions in certain less general Kolmogorov-type models. Utz and Waltman [59] considered

$$N_1' = N_1 F_1(N_2), \qquad N_2' = N_2 F_2(N_1) \ , \tag{1.2}$$

to describe competition between two species, and by using the separability of the phase plane equation, derived sufficient conditions for the existence of a limit cycle. The question of boundedness was considered and the conditions that populations are bounded away from zero were given. Using a bifurcation theorem of K.O. Friedrichs [29], Waltman [60] derived sufficient conditions for the existence of periodic solutions of a system of differential equations

$$N_1' = \alpha N_1 F_1(N_1, N_2), \qquad N_2' = N_2 F_2(N_1, N_2), \tag{1.3}$$

used to describe competition between two species, where α is a parameter.

In 1967, Rescigno and Richardson [56] re-investigated the system (1.1). Besides Kolmogorov's conditions, they also provided global conditions which in one case simulated competition and in another case symbiosis, and analysed the behaviors of the solutions for both these cases. In 1972, Brauer [15], and May [43] re-examined (1.1). Brauer used the system to describe a predator-prey relationship, and considered the behavior of solutions near an equilibrium point and the approximate location of the nonlinear equilibrium point. He concluded that it might not be possible to describe the qualitative nature of the equilibrium points in more than two dimensions, but it should be possible to find the approximate locations of the equilibrium points of the nonlinear system (1.1) and to determine whether they are asymptotically stable. On the basis of the results that essentially all models that have been proposed for predator-prey systems are shown to possess either a stable equilibrium point or a stable limit cycle, May [44] noted that such a limit cycle provides a satisfactory explanation for those species communities in which populations are observed to oscillate in a rather reproducible periodic manner. In 1974, Albrecht et al [1] investigated the qualitative behavior of the solutions of (1.1) and proved a theorem of Kolmogorov concerning predator-prey interactions under slightly different hypotheses.

The principle of competitive exclusion states that two species competing for the same resources cannot co-exist stably in the same habitat. But in 1972, from an experimental report, Ayala [9] showed that two species of fruit fly do co-exist stably in unqualified competition. On the basis of this result, Brauer [16], in 1974, concluded that the mathematical model with linear growth rate cannot fully describe a biological system. He then proposed a Kolmogorov-type competition model

$$N_1' = N_1[F_1(N_1) - F_2(N_2)], \quad N_2' = N_2[-K_1(N_1) + K_2(N_2)], \qquad (1.4)$$

with nonlinear growth rates, and provided conditions for the existence of an asymptotically stable equilibrium solution.

Following a suggestion of Samuelson [58], Freedman [24], in 1975, examined a two dimensional Kolmogorov model with perturbation

$$N_i' = N_i F_i(N_1, N_2, \varepsilon), \qquad i = 1,2, \qquad (1.5)$$

for the existence of a perturbed equilibrium point for small positive ε. He investigated the nature and stability of this equilibrium point both for the non-critical and the critical cases. He also derived sufficient conditions for the existence of periodic solutions. In the same year

Freedman and Waltman [26,27] considered a particular form of (1.5) describing a predator-prey interaction with perturbation, and provided local conditions on the nonlinear functions guaranteeing the existence of periodic solutions both in the cases of a perturbed equilibrium point [26] and an unperturbed equilibrium point [27]. G. Bojadziev and M. Bojadziev [12] have investigated a particular case of the model (1.5) from the point of view of control and structural stability.

In 1976, Bulmer [18] considered (1.1) and derived criteria to formulate a general predator-prey model, and discussed the conditions for the occurrence of limit cycles. He also investigated the effects of random environmental fluctuations on a stable equilibrium and on a limit cycle. In 1977, Rescigno [54] discussed the general properties of a Kolmogorov-type model

$$N_1' = N_1F_1(N_1,N_2), \qquad N_2' = F_2(N_1,N_2), \tag{1.6}$$

describing a single species N_1 living in a limited environment in the presence of its own pollutant N_2. In 1978, Hastings [32], re-studied the unperturbed two dimensional Kolmogorov model (1.1) and provided sufficient conditions for the global stability of the equilibrium. In 1981, Butler and Freedman [19] considered a Kolmogorov-type predator-prey system with periodic coefficients, i.e.,

$$N_i' = N_iF_i(t,N_1,N_2), \qquad i = 1,2, \tag{1.7}$$

with $F_i(t+\omega,N_1,N_2) = F_i(t,N_1,N_2)$, $i = 1,2$, and provided conditions under which periodic solutions exist. Then, they applied the results to a predator-prey system with periodic carrying capacity.

1.2 THREE DIMENSIONAL KOLMOGOROV MODEL

In 1968, Rescigno [52] extended Kolmogorov's model to three dimensions. He proposed a set of three equations

$$N_i' = N_i F_i (N_1, N_2, N_3), \qquad i = 1,2,3, \tag{1.8}$$

describing three species living in competition in the same environment. He analysed some of the properties of the system (1.8), and in particular, he found that, under certain conditions, the size of the populations can oscillate.

In 1972, Rescigno and Jones [55] used the same model (1.8) and discussed the hypotheses and properties of a three species predator-prey chain. They also gave geometrical interpretations of the model (1.8). They showed that only the populations of the first and second species in the chain must necessarily oscillate around the point of equilibrium if they do not come to the equilibrium. The other species may or may not oscillate.

In 1977, Rescigno [53] re-examined his own model (1.8) and analysed the properties of the system describing two predators competing for the same prey. In particular, he found that, under certain conditions, both predators can survive, with or without oscillations in the prey populations.

In 1984, Freedman and Waltman [28] studied the unperturbed three dimensional Kolmogorov model (1.8) describing a three level food web, two competing predators feeding on a single prey, or a single predator feeding on two competing prey; and provided conditions under which all three populations persist. Bojadziev [11] has considered perturbed models in R^2 and R^3 describing the growth of a single population. Hausrath [33] examined a particular case of (1.8) representing a perturbed three dimensional food chain and showed that the qualitative behavior of solutions of an asymptotically stable system remains the same under the influence of small perturbation.

1.3 THE PROPOSAL

Extending the work of H.I. Freedman [24], we propose a perturbed system of three nonlinear autonomous ordinary differential equations of the Kolmogorov-type

$$N_i' = N_i F_i(N_1, N_2, N_3, \varepsilon), \qquad i = 1,2,3, \tag{1.9}$$

where ε is a small positive parameter. For $\varepsilon = 0$, the perturbed system (1.9) reduces to the unperturbed system

$$N_i' = N_i F_i(N_1, N_2, N_3, 0), \qquad i = 1,2,3. \tag{1.10}$$

By using the qualitative methods of ordinary differential equations we analyse the nature and stability of the simple equilibria of the unperturbed system (1.10) as well as the perturbed system (1.9) in the three dimensional phase space.

The model (1.9) and the basic assumptions are discussed in Chapter 2. Some definitions and explanations of certain concepts are also provided in this chapter.

Chapter 3 contains a short review of the qualitative theory and the bifurcation theory of ordinary differential equations.

In Chapter 4, the nature and stability of a simple equilibrium of the nonlinear unperturbed model (1.10) are investigated.

In Chapter 5, the simple equilibrium of the perturbed system (1.9) originating from a simple equilibrium of the unperturbed system (1.10) is analysed qualitatively in the phase space. The qualitative behaviors of some particular perturbed population models are compared with those of the unperturbed models.

Chapter 6 contains an analysis of bifurcation of a multiple unperturbed equilibrium into simple perturbed equilibria. The nature and stability of the simple perturbed equilibria are examined. Two perturbed food chain models involving bifurcations are discussed.

CHAPTER 2

THE MODEL AND THE PRELIMINARIES

In this chapter we discuss the model (1.9) and state some basic assumptions and properties concerning the functions F_i and their arugments N_i , $i = 1,2,3$, and ε . Also, we give some definitions and explain certain concepts which are used in this thesis.

2.1 THE MODEL AND THE BASIC ASSUMPTIONS

We study qualitatively a system of three autonomous nonlinear ordinary differential equations of the form

$$N_i' = N_i F_i(N_1,N_2,N_3,\varepsilon), \qquad i = 1,2,3, \tag{2.1}$$

where $' \equiv d/dt$, and ε , which represents perturbation in the system (2.1), is a small positive parameter. The set of evolution equations (2.1) is called the three dimensional perturbed Kolmogorov model or the *perturbed model*. The unknown functions $N_i(t)$ represent the size of the ith species, and the given nonlinear functions of three real variables $F_i(N_1,N_2,N_3,\varepsilon)$ represent the specific growth rate of populations $N_i(t)$, $i = 1,2,3$. We assume that the rate of increase or decrease of the populations does not depend on time and that the populations are so large as to be measurable with real numbers. We also assume that the functions F_i are defined and continuously differentiable for all nonnegative values of N_i , $i = 1,2,3$, such that solutions for initial-value problems of (2.1) with $\varepsilon = 0$ exist. Further, it is supposed that there exists at least one solution $N_0(t)$ of (2.1) for $\varepsilon = 0$ in the first octant. We consider the problem of what solutions of (2.1) exist for small positive values of the parameter ε . In doing so, we shall consider the solutions of (2.1) for $\varepsilon \neq 0$ which have initial conditions close to those of $N_0(t)$.

For $\varepsilon = 0$, the system (2.1) becomes

$$N_i' = N_i F_i(N_1,N_2,N_3,0), \qquad i = 1,2,3, \tag{2.2}$$

which is called the three dimensional unperturbed Kolmogorov model or the *unperturbed model*. The properties and assumptions which are valid for N_i and F_i , $i = 1,2,3$, of the perturbed model (2.1) are also true for N_i and F_i , $i = 1,2,3$, of the unperturbed model (2.2).

2.2 SOME CONCEPTS AND DEFINITIONS

In order to explain certain concepts and state some definitions, we consider the following system of three autonomous nonlinear ordinary differential equations of the Kolmogorov-type

$$Z_i' = Z_i f_i(Z_1,Z_2,Z_3), \qquad i = 1,2,3, \tag{2.3}$$

where f_i , $i = 1,2,3$, are analytic in a domain G of R^3 . A system of autonomous ordinary differential equations is called a *dynamical system*.

The system (2.3) is called the three dimensional dynamical system of the Kolmogorov-type.

2.2.1 Phase Space and Phase Portrait

The solutions of (2.3) can be represented by surfaces in the (Z_1, Z_2, Z_3)-space. This three dimensional Euclidean space is called the *phase space* or the three dimensional *state space*. The solution surface passing through a certain initial point in G is known as the integral surface of (2.3). The phase space diagram represented by the family of integral surfaces is called the *phase portrait* or topological structure of the dynamical system (2.3).

2.2.2 Equilibrium Point

In the theory of autonomous ordinary differential equations, an important part is played by equilibrium points. The points for which all the right hand sides of the autonomous system of differential equations equal zero are called *equilibrium points* or *equilibria*. The equilibrium points are treated by the qualitative theory of differential equations. They enable us to assess qualitatively, under certain conditions, the shape of the integral surfaces in the neighborhood of the equilibrium point.

If $Z^0(Z_1^0, Z_2^0, Z_3^0) \in G$ is such that

$$f_i(Z_1^0, Z_2^0, Z_3^0) = 0, \qquad i = 1,2,3, \tag{2.4}$$

then Z^0 is called an *equilibrium point* or *equilibrium* of the system (2.3). The multiplicity of the equilibrium point is defined as the multiplicity of the intersection point Z^0 of the three surfaces in (2.4). An equilibrium point of multiplicity one is called a *simple equilibrium* and an equilibrium point of multiplicity greater than one is called a *multiple equilibrium*.

The matrix

$$j_0 = \left[\frac{\partial f_i}{\partial Z_j}(Z_1^0, Z_2^0, Z_3^0)\right], \qquad i,j = 1,2,3, \tag{2.5}$$

is called the *Jacobian matrix* of (2.4). If $\det j_0 \neq 0$, the system (2.4) has simple solutions and therefore the equilibrium Z^0 is called an *isolated equilibrium* or simple equilibrium of (2.3). This means that there exists a neighborhood of Z^0 such that the only equilibrium point of (2.3) in that neighborhood is Z^0. On the other hand, if $\det j_0 = 0$, the system (2.4) has multiple solutions, and then Z^0 is called a *degenerate equilibrium* or multiple equilibrium of (2.3). This means that there exists more than one solution of (2.4) at Z^0.

If all the eigenvalues of the matrix (2.5) have nonzero real parts, then the equilibrium Z^0 is called a *hyperbolic equilibrium*, while if at least one of the eigenvalues of (2.5) has a zero real part, then Z^0 is called a *nonhyperbolic equilibrium*.

By the *nature of an equilibrium* we mean the local phase portrait or the local topological structure represented by the equilibrium in the phase space, and by the *character of an equilibrium* we mean both the nature and the stability or instability property of the concerned equilibrium in the phase space.

The equilibrium points of the perturbed system (2.1), given by the solutions of

$$F_i(N_1,N_2,N_3,\varepsilon) = 0, \qquad i = 1,2,3, \tag{2.6}$$

are called perturbed equilibrium points or *perturbed equilibria.* The equilibria of the unperturbed system (2.2), obtained by solving the system of equations

$$F_i(N_1,N_2,N_3,0) = 0, \qquad i = 1,2,3, \tag{2.7}$$

are known as unperturbed equilibrium points or *unperturbed equilibria.*

Equilibrium points representing the rest positions of the species of a physical system are an important class of solutions of the associated system. These are also referred to as *equilibrium solutions* of autonomous systems. The equilibrium points of a linearized system are also called trivial points and hence the equilibrium solution of a linearized system is known as the *trivial solution. Bifurcating solutions* are equilibrium solutions which form intersecting branches in a suitable state space.

2.2.3 Linearization

Linearization is an invariant operation, i.e., an operation which is independent of the coordinate system. Therefore, studying a neighborhood of an equilibrium point means studying how the process evolves when its initial conditions deviate slightly from their equilibrium values.

To investigate a dynamical system in a neighborhood of an equilibrium Z^0, it is natural to make a Taylor series expansion of the system in the given neighborhood. The first term of the Taylor series is linear, and the process of dropping the remaining terms is called *linearization*. The linearized system can be regarded as an example of a system with an equilibrium Z^0. On the other hand, it might be expected that the behavior of the nonlinear system is close to that of the linear system, since small quantities of higher order are dropped in making the linearization. Of course, the problem of relation between the solutions of the original system and those of the linearized system requires special investigation. The linearizations commonly practiced are approximating devices that are good enough for most purposes. There are, however, also certain cases in which linear treatments may not be applicable at all.

2.2.4 Variational Matrix

The matrix

$$A = \left[Z_i \frac{\partial f_i}{\partial Z_j} (Z_1^0, Z_2^0, Z_3^0) \right], \qquad i,j = 1,2,3, \tag{2.8}$$

evaluated at the equilibrium Z^0 is called the *variational matrix* for the linearized part of (2.3). The variational matrix for higher degree terms of (2.3) can similarly be constructed by evaluating higher order partials at Z^0 . The variational matrix of a dynamical system, its eigenvalues, and the corresponding eigenvectors play an important part in the investigation of the topological structure and stability of an equilibrium of the associated system. The *noncritical case* of the system (2.3) corresponds to the condition $\det A \neq 0$ and ensures the existence of a simple equilibrium for (2.3). On the other hand, the *critical case* of the system (2.3), corresponding to the condition $\det A = 0$, implies the existence of a multiple equilibrium for the system (2.3).

2.2.5 Jordan Canonical Form

The Jordan canonical form of a square matrix has a major role in the qualitative analysis of an equilibrium of a dynamical system. The dimension of the solution space of a system is determined by the number of linearly independent eigenvectors of the variational matrix for the linearized part of the system. The direction of the solution surfaces and the nature of the trajectory of a dynamical system depend respectively on the signs of the eigenvalues and the types of eigenvalues with the corresponding number of linearly independent eigenvectors of the variational matrix of the dynamical system. From the Jordan canonical form of a matrix, one can easily determine the types and signs of eigenvalues and the corresponding number of linearly independent eigenvectors of the same matrix. Therefore, with the aid of Jordan canonical form of a variational matrix of an unperturbed system, it is possible to determine, relatively quickly and easily, the nature and stability of the equilibrium of the unperturbed system. On the otherhand, since the eigenvalues of a variational matrix of a perturbed system are, in general, assumed to be distinct, the nature and stability of an equilibrium of a perturbed system can be determined by the characteristic equation of the variational matrix of the perturbed system. Here we shall recall, without proof, how the Jordan canonical form of a matrix can be constructed.

The linearized part of the system (2.3) can be expressed in the matrix form

$$Z' = A Z , \tag{2.9}$$

where $Z = (Z_1,Z_2,Z_3)^T$ is a 3×1 matrix, T represents the transpose of a matrix, and the 3×3 matrix A is given by (2.8). It is known from linear algebra that for a 3×3 constant (real or complex) matrix A, there always exists a coordinate transformation

$$W = B Z , \tag{2.10}$$

where $W = (W_1,W_2,W_3)^T$ is a 3×1 matrix, and B is a certain 3×3 nonsingular matrix whose columns are the eigenvectors of the matrix A. The transformation (2.10) reduces the system (2.9) to the form

$$W' = BZ' = BAZ = BAB^{-1}W = j_1 W, \tag{2.11}$$

where the matrix j_1, defined by

$$j_1 = BAB^{-1} , \tag{2.12}$$

is called the *Jordan canonical form* of the matrix A . The matrix j_1 consists of elementary block matrices whose main diagonal consists of one and the same eigenvalue, while all of the elements of the right adjacent diagonal are unity, and the rest of the elements of the block are zero. The number of blocks of j_1 depends on the number of linearly independent eigenvectors of A corresponding to its eigenvalues. Hence, the main diagonal elements of j_1 are the eigenvalues of A , the elements of the right adjacent diagonal, depending on the number of elementary blocks, may either be zero, or unity, or a combination of zero and unity, and all the remaining elements of j_1 are zero.

CHAPTER 3

TECHNIQUES OF QUALITATIVE ANALYSIS

The methods of differential equations that have been used in this thesis to study the qualitative behavior of three dimensional dynamical systems of the Kolmogorov-type are presented in this chapter. More specifically, this chapter contains a short review of the qualitative theory and bifurcation theory of ordinary differential equations.

3.1 QUALITATIVE THEORY

Basically, most of the physical problems are nonlinear from the outset. Although, we have some known methods for solutions of most linear and some nonlinear systems of ordinary differential equations, there are very few methods for solutions for more extensive classes of nonlinear differential equations. The solution of nonlinear differential equations, in general, involves the solution of nonlinear algebraic - and sometimes nonalgebraic equations, which we are often unable to solve with accuracy. This, of course, is very disappointing. However, it is not necessary, in most applications, to find the solutions of nonlinear problems explicitly. Rather, we are interested in the qualitative properties of the nonlinear system concerning the following questions: (i) Do there exist equilibrium solutions? (ii) Are the solutions stable? (iii) Is there a periodic solution? Remarkably, we can often give satisfactory answers to these questions, even though we cannot solve the nonlinear system explicitly.

Hence, we will be concerned with the qualitative theory of differential equations.

Qualitative theory of differential equations originates in the giant developments due to Pioncaré [48-50], Birkhoff [10], and Liapunov [39-40]. The modern methods of qualitative analysis of differential equations have also their origins in the works of Andronov and co-workers [2-6], Nemytskii and Stepanov [45], and Coddington and Levinson [21]. Some recent works on qualitative theory includes Arnold [7-8], Hirsch and Smale [35], Lefschetz [38], Cronin [22], Iooss and Joseph [36], Chow and Hale [20], and Guckenheimer and Holmes [30].

The qualitative method is based on the study of the representation of the solutions of differential equations in the state space, their stabilities and the existence of periodic solutions. The qualitative properties of solutions: topological structure, stability property, and periodicity yield a coherent and esthetically pleasing theory which has important applications in physical and life sciences.

3.1.1 Topological Method

The topological method of phase portrait analysis is due to Poincaré. By this method the solutions of differential equations are sought not as explicit functions of time, but as integral curves/surfaces in a state space. It is one of the important means of investigating the various phenomena of nonlinear oscillations. Considerable insight into the qualitative aspects of the solution, and some quantitative informations as well, can be obtained through a study of the integral curves/surfaces.

The topological method is used to examine the nature of the phase portrait of the equilibria of a nonlinear dynamical system

$$Z' = AZ + h(Z), \tag{3.1}$$

by making use of the linearized system

$$Z' = AZ , \tag{3.2}$$

where $Z = (Z_1, Z_2, Z_3)^T$ is a 3×1 matrix, and A is a 3×3 constant real valued matrix. Here the 3×1 matrix $h = (h_1, h_2, h_3)^T$ represents the nonlinear terms of (3.1), and there exist numbers $\beta > 1$ and $\alpha \geq 0$, such that

$$|h(Z)| \geq \alpha |Z|^{\beta} , \tag{3.3}$$

holds in a neighborhood of the equilibrium Z^0 of (3.2).

The substitution $Z = Ke^{\lambda t}$, where $K = (K_1, K_2, K_3)^T$ is a 3×1 constant matrix, into the system (3.2) results in the characteristic equation

$$\det(A - \lambda I) = 0 , \tag{3.4}$$

where I is a 3×3 identity matrix. From (3.4) one may calculate the eigenvalues and the corresponding eigenvectors of A.

According to the principal axes theorem of linear algebra, the dimension of the solution space and thus the equilibrium point of (3.2) depends on the number of linearly independent eigenvectors corresponding to the eigenvalues of A . Hence, the classification of equilibrium points of (3.1), under the condition (3.3), is governed by the eigenvalues and the corresponding eigenvectors of A . From Reyn [57], we obtain the following types of simple equilibrium points for the dynamical system (3.2).

(i) If A has three distinct nonzero real eigenvalues having same signs, then the equilibrium Z^0 is called a three branched node.

(ii) If A has three distinct nonzero real eigenvalues with at least two of them having different signs, then the equilibrium Z^0 is called a saddle node.

(iii) If A has three nonzero real with two equal eigenvalues having same signs, then the equilibrium Z^0 is called a star node or a two branched node when A has respectively three or two linearly independent eigenvectors corresponding to its eigenvalues.

(iv) If A has three nonzero real with two equal eigenvalues having at least two of them with different signs, then the equilibrium Z^0 is called a saddle star or a two branched node when A has respectively three or two linearly independent eigenvectors corresponding to its eigenvalues.

(v) If A has three nonzero repeated eigenvalues, then the equilibrium Z^0 is called a star, an antisymmetric node star, or a one branched node when A has respectively three, two, or one linearly independent eigenvector corresponding to its eigenvalues.

(vi) If A has one nonzero real and two complex (with nonzero real parts) eigenvalues, and the real eigenvalue and the real parts of the complex eigenvalues have the same signs, then the equilibrium Z^0 is called a node spiral.

(vii) If A has one nonzero real and two complex (with nonzero real parts) eigenvalues, and the real eigenvalue and the real parts of the complex eigenvalues have different signs, then the equilibrium Z^0 is called a saddle spiral.

(viii) If A has one nonzero real and two imaginary eigenvalues, then the equilibrium Z^0 is called a center focus.

In cases (i) - (vii) the nature of the equilibria is the same both for the linearized system (3.2) and the corresponding nonlinear system (3.1). Results in case (viii) are only valid for the linearized system (3.2). In case (viii) higher order terms of (3.1) may produce a node spiral or a saddle spiral.

3.1.2 Stability Methods

The stability theory was originated by Liapunov. The basic idea of stability study is to examine the question: under what conditions do equilibrium solutions of a dynamical system approach or stay close to a given equilibrium solution? Since the biological systems tend to be quite complicated, it is assumed that the disturbances of the system as described by the differential equations are constantly occuring. This suggests that only those solutions of the differential equations which have strong stability properties are biologically significant.

More recently, it has been clear that if a biological problem is formulated in terms of a system of ordinary differential equations, the stability theory must play an important role in the study of the system. In order to study the stability of an autonomous system, the '*Liapunov criterion*' for stability by the first approximation is examined. The '*method of first approximation*' is used to obtain results concerning the stability of the trivial solution of the nonlinear system (3.1) by making use of the linearized system (3.2) under the condition (3.3). Following Liapunov [40], the results concerning the stability and instability behaviors of (3.1) and (3.2) are characterized by the following:

(i) If all eigenvalues of A are negative or have negative real parts, then the trivial solutions of (3.2) as well as (3.1) are asymptotically stable.

(ii) If at least one eigenvalue of A is positive or has a positive real part, then the trivial solutions of (3.2) as well as (3.1) are unstable.

(iii) If A has one positive real and two imaginary eigenvalues, then the trivial solution of (3.2) is neutrally unstable and that of (3.1) is unstable.

(iv) If A has one negative real and two imaginary eigenvalues, then the trivial solution of (3.2) is neutrally stable. In this case, the higher order terms in (3.1) determines the stability or instability of the trivial solution of (3.1).

Hence, a necessary and sufficient condition for the asymptotic stability of (3.2) and thus (3.1) is given by the requirement that all the roots of (3.4) are negative or have negative real parts. Generally, it is not easy to find all the roots of (3.4). One thus makes use of the criteria which provide assertions about the signs of the real parts of the roots of the characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{3.5}$$

of (3.4) without having to resort to an actual solution of the above equation. The most important of these criteria is known as '*The Routh-Hurwitz criteria*' (see Cronin [22], Page 157). According to these criteria, a necessary and sufficient condition assuring that all roots of the cubic equation (3.5) have negative real parts is given by

$$a_1 > 0, \; a_3 > 0, \quad \text{and} \quad a_1 a_2 > a_3 \; . \tag{3.6}$$

3.2 BIFURCATION THEORY

Dynamical systems describing physical problems generally contain parameters. The word bifurcation means forked and is used in a broad sense for designating all possible qualitative reorganizations of various objects resulting from changing the parameters on which they depend. One of the most important classes of dynamical systems comprises of those systems whose topological structure in a given region does not change under small modifications of the parameters. Pexioto [46-47] called such systems as structurally stable systems. If small changes in the parameter lead to a change in the topological structure of the dynamical system, then the system is termed structurally unstable system. These changes in the parameter values are called bifurcation values.

A fundamental step towards modern bifurcation theory in differential equations occured with the definition of structural stability by Andronov and Pontryagin [5] in 1937 and the classification of structurally stable systems in the plane. With these concepts, Andronov and Leontovich [2] were able to make precise definitions of types of bifurcation points. These results were applied extensively to the theory of nonlinear oscillations by Andronov, Vitt and Khaikin [6], and Andronov, Leontovich, Gordon and Maier [3-4].

There are evident limitations as to how far one can proceed with a systematic bifurcation theory. In parameter regions consisting of structurally unstable systems, the detailed changes in the topological structure can be exceedingly complicated. We, therefore, shall focus upon the simplest bifurcation of individual equilibrium points. The analysis of such bifurcations is generally performed by examining the vector field near the degenerate equilibrium point.

The implicit function theorem is a basic mathematical tool used in bifurcation theory. Since we will require the theorem for our needs, and since its proof may be found in any book on bifurcation theory (see,e.g., [36]), a short review of the theorem will be given next.

3.2.1 Implicit Function Theorem

Consider the following system of equations:

$$f_i(Z_1,Z_2,Z_3,\varepsilon) = 0, \qquad i = 1,2,3, \tag{3.7}$$

where f_i , $i = 1,2,3$, are continuously differentiable in some open region of the (Z_1,Z_2,Z_3)-space. Assume that

$$f_i(Z_{10},Z_{20},Z_{30},\varepsilon_0) = 0, \qquad i = 1,2,3, \tag{3.8}$$

and that the Jacobian matrix

$$j_2 = \left[\frac{\partial f_i}{\partial Z_j}\right], \qquad i,j = 1,2,3, \tag{3.9}$$

computed at the point $(Z_{10},Z_{20},Z_{30},\varepsilon_0)$, has a nonzero determinant: $\det j_2 \neq 0$. Then there exist $\alpha > 0$, $\beta > 0$ such that the following assertions hold:

(i) There is a unique continuous set of functions Z_i , $i = 1,2,3$, defined for $\varepsilon_0 - \alpha < \varepsilon < \varepsilon_0 + \alpha$ satisfying $Z_{i0} - \beta < Z_i(\varepsilon) < Z_{i0} + \beta$, $i = 1,2,3$, and

$$f_i(Z_1(\varepsilon),Z_2(\varepsilon),Z_3(\varepsilon),\varepsilon) = 0, \qquad i = 1,2,3. \tag{3.10}$$

(ii) Moreover, Z_i, $i = 1,2,3$, are continuously differentiable for $\varepsilon_0 - \alpha < \varepsilon < \varepsilon_0 + \alpha$, and

$$\left[Z_i'(\varepsilon)\right] = - \frac{\left[\frac{\partial f_i}{\partial \varepsilon}(Z_1(\varepsilon),Z_2(\varepsilon),Z_3(\varepsilon),\varepsilon)\right]}{j_2(Z_1(\varepsilon),Z_2(\varepsilon),Z_3(\varepsilon),\varepsilon)}, \quad i = 1,2,3. \tag{3.11}$$

If f_i , $i = 1,2,3$, are analytic functions of all variables, then $Z_i(\varepsilon)$, $i = 1,2,3$, are analytic near $\varepsilon = \varepsilon_0$.

If $\det j_2 = 0$ for $\varepsilon_0 = 0$, we have to solve the nonlinear system (3.7) up to a certain power of ε in order to obtain the solution of the perturbed system (3.7). In our work, the perturbed nonlinear system is solved directly according to the rank of j_2 for $\varepsilon_0 = 0$. We note that, in such cases, the techniques of Freedman [23], and Loud [41] can also be used.

CHAPTER 4

A SIMPLE EQUILIBRIUM OF THE UNPERTURBED THREE DIMENSIONAL KOLMOGOROV MODEL

In this chapter we consider the noncritical case of the unperturbed three dimensional Kolmogorov model. Here we give the criteria for the existence of a simple equilibrium of the unperturbed three dimensional Kolmogorov model in the first octant and provide parameter conditions for all possible types of unperturbed simple equilibria. Further, we determine the Jordan canonical form for the variational matrix of the linearized part of this dynamical system and examine the nature and stability of the simple unperturbed equilibrium in the phase space.

4.1 EXISTENCE OF A SIMPLE UNPERTURBED EQUILIBRIUM

Since in the noncritical case, certain features of the trajectories of a differential system may be preserved under small perturbation, we study the unperturbed three dimensional Kolmogorov model

$$N_i' = N_i F_i(N_1, N_2, N_3, 0), \qquad i = 1,2,3. \tag{4.1}$$

We assume that the unperturbed system (4.1) has at least one equilibrium point $E^0\ (M_1^0, M_2^0, M_3^0)$, called the unperturbed equilibrium, in the interior of the first octant. This means that the system

$$F_i(N_1, N_2, N_3, 0) = 0, \qquad i = 1,2,3, \tag{4.2}$$

has at least one solution (M_1^0, M_2^0, M_3^0) such that

$$F_i(M_1^0, M_2^0, M_3^0, 0) = 0, \qquad M_i^0 > 0, \qquad i = 1,2,3. \tag{4.3}$$

Let J_0 be the matrix

$$J_0(M_1^0, M_2^0, M_3^0) = \left[F_{iN_j}^0 \right], \qquad i,j = 1,2,3, \tag{4.4}$$

where

$$F_{iN_j}^0 = \frac{\partial F_i}{\partial N_j}(M_1^0, M_2^0, M_3^0, 0), \qquad i,j = 1,2,3, \tag{4.5}$$

and assume that

$$|J_0| = \det J_0(M_1^0, M_2^0, M_3^0) \neq 0 . \tag{4.6}$$

The assumption (4.6) corresponds to the noncritical case of the implicit function theorem of the system (4.2) and ensures that E^0 is a simple equilibrium point of (4.1), i.e., E^0 is a point of intersection of the surfaces (4.2) such that the tangent planes to the surfaces at their common point exist and are distinct. Moreover, by the implicit function theorem, with (4.6), the equilibrium E^0 is isolated, i.e., there exists a neighborhood of E^0 containing no equilibrium states other than E^0 .

In order to linearize the nonlinear system (4.1), we set

$$N_i = M_i^0 + X_i, \qquad i = 1,2,3. \tag{4.7}$$

Substituting (4.7) into (4.1) and using the Taylor series expansion for $F_i(M_1^0+X_1, M_2^0+X_2, M_3^0+X_3, 0)$, $i = 1,2,3$, we obtain

$$X_i' = \left(\sum_{j=1}^{3} M_i^0 F_{iN_j}^0\right) X_i + \bar{E}_i^0, \qquad i = 1,2,3, \tag{4.8}$$

where $\bar{E}_i^0$ represents the nonlinear part of (4.1) and is given by

$$\bar{E}_i^0 = \bar{E}_i^0(X_1, X_2, X_3, 0) = (M_i^0+X_i) F_i(M_1^0+X_1, M_2^0+X_2, M_3^0+X_3, 0)$$

$$- \left(\sum_{j=1}^{3} M_i^0 F_{iN_j}^0\right) X_i, \qquad i = 1,2,3, \tag{4.9}$$

and $F_{iN_j}^0$ is given by (4.5).

The variational matrix for the linear part of (4.8) at the unperturbed equilibrium point E^0 is

$$\Delta_0(M_1^0, M_2^0, M_3^0) = \left[m_{ij}^0\right], \qquad i,j = 1,2,3, \tag{4.10}$$

where

$$m_{ij}^0 = M_i^0 \frac{\partial F_i}{\partial N_j}(M_1^0, M_2^0, M_3^0, 0), \qquad i,j = 1,2,3. \tag{4.11}$$

Further, because of (4.3) and (4.6), we must have

$$|\Delta_0| = \det \Delta_0(M_1^0, M_2^0, M_3^0) \neq 0 \; . \tag{4.12}$$

The characteristic equation for (4.10) is

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \; , \tag{4.13}$$

where

$$P_1 = - \sum_{i=1}^{3} m_{ii}^0 \; ,$$

$$P_2 = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (m_{ii}^0 m_{jj}^0 - m_{ij}^0 m_{ji}^0) , \qquad i \neq j, \tag{4.14}$$

$$P_3 = - \det \Delta_0 \; .$$

The assumption (4.12) ensures that the cubic equation (4.13) does not have any zero root. The eigenvalues of the matrix (4.10) are obtained by solving the characteristic equation (4.13) and the eigenvectors corresponding to these eigenvalues can also be determined. Moreover, the roots of the characteristic equation (4.13) of the linearized part of the unperturbed system (4.1) can be distinct or repeated. Hence, the variational matrix (4.10) for the linearized part of the unperturbed system (4.1) has distinct or repeated eigenvalues. While the variational matrix (4.10) has three linearly independent eigenvectors corresponding to three distinct eigenvalues, it may have one, two, or three linearly independent eigenvectors corresponding to the repeated eigenvalues. Therefore, the nature and stability of the equilibrium E^0 of the unperturbed system (4.1) can be determined from the Jordan canonical form of the variational matrix (4.10).

4.2 JORDAN CANONICAL FORM OF THE VARIATIONAL MATRIX

In order to find the Jordan canonical form of the variational matrix (4.10), we use the transformation

$$U = PX \tag{4.15}$$

where P is a certain 3×3 nonsingular matrix whose columns are the eigenvectors of the matrix Δ_0, and $U = (u,v,w)^T$ and $X = (X_1,X_2,X_3)^T$, T representing the transpose of a matrix, are 3×1 matrices. The substitution of (4.15) into (4.8) yields the transformed system

$$U' = (P\Delta_0 P^{-1})U + P\bar{E}^0(P^{-1}u, P^{-1}v, P^{-1}w), \tag{4.16}$$

where Δ_0 is given by (4.10) and $\bar{E}^0 = (\bar{E}_1^0, \bar{E}_2^0, \bar{E}_3^0)^T$ is a 3×1 matrix. The matrix $\Delta^0 = P\Delta_0 P^{-1}$ is the Jordan canonical form of the matrix Δ_0. The transformation (4.15) is equivalent to a rotation and stretch of axes and does not affect the character of the equilibrium points. Moreover, $P\Delta_0 P^{-1}$, which represents the Jacobian matrix for the linear part of the transformed system (4.16), is also the Jacobian matrix of the linear part of the original system (4.8). Further, since $|\Delta_0| \neq 0$, Δ^0 must be equivalent to one of the following forms:

$$
\Delta^0_{r3} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad
\Delta^0_{c3} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_0+i\omega_0 & 0 \\ 0 & 0 & \lambda_0-i\omega_0 \end{bmatrix},
$$

$$
\Delta^0_{i3} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & i\omega_0 & \\ 0 & 0 & -i\omega_0 \end{bmatrix}, \quad
\Delta^0_{23} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix},
$$

$$
\Delta^0_{22} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad
\Delta^0_{13} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \tag{4.17}
$$

$$
\Delta^0_{12} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad
\Delta^0_{11} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix},
$$

where λ_1, λ_2, and λ_3 are the real eigenvalues of the matrix (4.10) and λ_0 and ω_0 are the real and imaginary parts respectively of the complex eigenvalues of (4.10). Here Δ^0_{r3}, Δ^0_{c3}, Δ^0_{i3}, Δ^0_{23}, and Δ^0_{13} have three linearly independent eigenvectors corresponding to the three distinct nonzero real eigenvalues, one nonzero real and two complex eigenvalues with nonzero real points, one nonzero real and two imaginary eigenvalues, three nonzero real with two equal eigenvalues, and three repeated nonzero real eigenvalues respectively; Δ^0_{22} and Δ^0_{12} have two linearly independent eigenvectors corresponding to the three nonzero real with two equal eigenvalues and three repeated nonzero real eigenvalues respectively; and Δ_{11}^{0} has one linearly independent eigenvector corresponding to the three repeated nonzero real eigenvalues of the variational matrix (4.10).

4.3 NATURE AND STABILITY OF THE UNPERTURBED EQUILIBRIUM

The nature and stability of the equilibrium E^0 of (4.1) is determined by the types and signs of the eigenvalues of (4.10) and the number of linearly independent eigenvectors corresponding to these eigenvalues. The nature of the eigenvalues of (4.10) depends on the values

$$D_0 = \frac{1}{27}\left(P_2 - \frac{1}{3}P_1^{\,2}\right)^3 + \frac{1}{4}\left(\frac{2}{27}P_1^{\,3} - \frac{1}{3}P_1P_2 + P_3\right)^2 ,$$

$$H_0 = P_1P_2 - P_3 , \quad Q_0 = P_2 - \frac{1}{3}P_1^{\,2} , \quad R_0 = P_3 - \frac{1}{27}P_1^{\,3} , \tag{4.18}$$

where P_1, P_2, and P_3 are given by (4.14); and the number of linearly independent eigenvectors corresponding to the eigenvalues of (4.10) are given by the Jordan canonical forms (4.17).

The eigenvalues of (4.10) are the nonzero solutions of the characteristic equation (4.13). The cubic equation (4.13) has three distinct roots (with nonzero real parts) when $D_0 \neq 0$, at least two equal roots when $D_0 = 0$, and two imaginary roots when $D_0 > 0$ and $H_0 = 0$.

In [57], Reyn presented a detailed classification of the equilibrium points of a three dimensional linear differential system. We shall use some of his results in our work.

We now provide parameter conditions for all possible types of eigenvalues of (4.10) and combining each set of these eigenvalues with the corresponding Jordan canonical form we state the nature and stability of the equilibrium E^0 of the dynamical system (4.1).

CASE A.

If $D_0 \neq 0$, then the variational matrix (4.10) has three distinct eigenvalues with nonzero real parts.

Sub-Case A(i).

If $D_0 < 0$, then the variational matrix (4.10) has three distinct nonzero real eigenvalues λ_1, λ_2, and λ_3 and has the Jordan canonical form Δ^0_{r3}.

(1) If $P_1 > 0$, $P_3 > 0$, and $H_0 > 0$, then $\lambda_i < 0$, $i = 1,2,3$; and E^0 is an asymptotically stable three branched node (Fig. 1).

(2) If $P_3 < 0$ and $H_0 > 0$, then $\lambda_1 > 0$, $\lambda_2 < 0$, and $\lambda_3 < 0$; and E^0 is a three branched saddle node with stable two branched plane node (Fig. 2).

(3) If $P_3 > 0$ and $H_0 < 0$, then $\lambda_1 < 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$; and E^0 is a three branched saddle node with unstable two branched plane node.

(4) If $P_1 < 0$, $P_3 < 0$, and $H_0 < 0$, then $\lambda_i > 0$, $i = 1,2,3$; and E^0 is an unstable three branched node.

Sub-Case A(ii).

If $D_0 > 0$ and $H_0 \neq 0$, then the variational matrix (4.10) has one nonzero real and two complex (with nonzero real parts) eigenvalues λ_1 and $\lambda_0 \pm i\omega_0$ and has the canonical form Δ^0_{c3}.

(5) If $P_1 > 0$, $P_3 > 0$, and $H_0 > 0$, then $\lambda_1 < 0$ and $\lambda_0 < 0$; and E^0 is an asymptotically stable node spiral (Fig. 3). More specifically, E^0 is an asymptotically stable (a) blunt spiral when $\lambda_1 > \lambda_0$, (b) conical spiral when $\lambda_1 = \lambda_0$, and (c) pointed spiral when $\lambda_1 < \lambda_0$.

(6) If $P_3 < 0$ and $H_0 > 0$, then $\lambda_1 > 0$ and $\lambda_0 < 0$; and E^0 is a saddle spiral with stable plane focus (Fig. 4).

(7) If $P_3 > 0$ and $H_0 < 0$, then $\lambda_1 < 0$ and $\lambda_0 > 0$; and E^0 is a saddle spiral with unstable plane focus.

(8) If $P_1 < 0$, $P_3 < 0$, and $H_0 < 0$, then $\lambda_1 > 0$ and $\lambda_0 > 0$; and E^0 is an unstable node spiral. More specifically, E^0 is an unstable (a) blunt spiral when $\lambda_1 > \lambda_0$, (b) conical spiral when $\lambda_1 = \lambda_0$, and (c) pointed spiral when $\lambda_1 < \lambda_0$.

CASE B.

If $D_0 = 0$, then the variational matrix (4.10) has at least two equal eigenvalues.

Sub-Case B(i).

If $Q_0 \neq 0$, then the variational matrix (4.10) has three real with two equal eigenvalues λ_1, λ_2, and λ_2 and has either the Jordan canonical form Δ_{23}^0 or Δ_{22}^0.

(9) If $P_1 > 0$, $P_3 > 0$, and $H_0 > 0$, then $\lambda_i < 0$, $i = 1,2$; and E^0 is asymptotically stable. E^0 corresponding to Δ_{23}^0 represents either a pointed star node when $\lambda_1 > \lambda_2$ or a blunt star node when $\lambda_1 < \lambda_2$; and E^0 corresponding to

Δ_{22}^{0} represents either a wide two branched node when $\lambda_1 > \lambda_2$ or a slender two branched node when $\lambda_1 < \lambda_2$.

(10) If $P_3 < 0$ and $H_0 > 0$, then $\lambda_1 > 0$ and $\lambda_2 < 0$; and E^0 corresponding to Δ_{23}^{0} and Δ_{22}^{0} represents a saddle star with stable plane star and a two branched saddle node with stable one branched plane node respectively.

(11) If $P_3 > 0$ and $H_0 < 0$, then $\lambda_1 < 0$ and $\lambda_2 > 0$; and E^0 corresponding to Δ_{23}^{0} and Δ_{22}^{0} represents a saddle star with unstable plane star and a two branched saddle node with unstable one branched plane node respectively (Figs. 9-10).

(12) If $p_1 < 0$, $P_3 < 0$, and $H_0 < 0$, then $\lambda_i > 0$, $i = 1,2$; and E^0 is unstable . E^0 corresponding to Δ_{23}^{0} represents either a pointed star node when $\lambda_1 > \lambda_2$ or a blunt star node when $\lambda_1 < \lambda_2$; and E^0 corresponding to Δ_{22}^{0} represents either a wide two branched node when $\lambda_1 > \lambda_2$ or a slender two branched node when $\lambda_1 < \lambda_2$ (Figs. 5-8).

Sub-Case B(ii).

If $Q_0 = R_0 = 0$, then the variational matrix (4.10) has three repeated real eigenvalues λ_1, λ_1, and λ_1 and has one of the Jordan canonical forms Δ_{13}^{0}, Δ_{12}^{0}, or Δ_{11}^{0} .

(13) If $P_1 > 0$, $P_3 > 0$, and $H_0 > 0$, then $\lambda_1 < 0$; and E^0 is asymptotically stable. E^0 corresponding to Δ_{13}^{0}, Δ_{12}^{0}, and Δ_{11}^{0} represents a three dimensional star, an antisymmetric node star, and a one branched node respectively.

(14) If $P_1 < 0$, $P_3 < 0$, and $H_0 < 0$, then $\lambda_1 > 0$; and E^0 is unstable. E^0 corresponding to Δ^0_{13}, Δ^0_{12}, and Δ^0_{11} represents a three dimensional star, an antisymmetric node star, and a one branched node respectively (Fig. 11-13).

CASE C.

If $D_0 > 0$ and $H_0 = 0$, then the variational matrix (4.10) has one real and two imaginary eigenvalues λ_1 and $\pm i\omega_0$ and has the Jordan canonical form Δ^0_{i3} .

(15) If $P_1 > 0$ and $P_3 > 0$, then $\lambda_1 < 0$; and E^0 is a neutrally stable convergent center focus.

(16) If $P_1 < 0$ and $P_3 < 0$, then $\lambda_1 > 0$; and E^0 is a neutrally unstable divergent center focus (Fig. 14).

In order to give a more refined classification of the hyperbolic and simple nonhyperbolic equilibrium points of a dynamical system, we introduce the following definitions:

DEFINITION 1

The hyperbolic equilibria of a dynamical system corresponding to three distinct eigenvalues with nonzero real parts of the variational matrix for the linearized system are called *A-type equilibria.*

DEFINTION 2

The hyperbolic equilibria of a dynamical system corresponding to three nonzero real with at least two repeated eigenvalues of the variational matrix for the linearized system are called *B-type equilibria.*

DEFINITION 3

The simple nonhyperbolic equilibria of a dynamical system corresponding to one nonzero real and two imaginary eigenvalues of the variational matrix for the linearized system are called *C-type equilibria.*

We now define the following sets:

$$S_1 = \{(1),(5),(9),(13)\},$$

$$S_2 = \{(2),(3),(4),(6),(7),(8),(10),(11),(12),(14)\}, \qquad (4.19)$$

$$S_3 = \{(15)\}, \qquad S_4 = \{(16)\},$$

and assume that $F_i(N_1,N_2,N_3,0)$, $i = 1,2,3$, be such that the hypotheses of (h), $h = 1,2,\ldots,16$, hold.

Using the results in Cases A, B, and C, the Definitions 1, 2, and 3, and (4.19), we have established the following theorem.

THEOREM 1.

A-type and B-type equilibria of the unperturbed three dimensional Kolmogorov model satisfying the conditions stated in Cases A and B are always hyperbolic and C-type equilibria of the same model satisfying the conditions in Case C are always nonhyperbolic. The equilibrium E^0 *is asymptotically stable if* $(h) \in S_1$, *unstable if* $(h) \in S_2$, *neutrally stable if* $(h) \in S_3$, *and neutrally unstable if* $(h) \in S_4$.

Remarks: The nature and stability or instability property of the equilibria in Cases A and B are valid for both the linear and the corresponding nonlinear systems, while that in Case C are valid only for the linearized part of the nonlinear system. In order to examine the nature and stability of the equilibria of the nonlinear system in Case C, the effect of nonlinear terms must be taken into account. In Case C, the higher order terms of (4.8) may generate an asymptotically stable or unstable node spiral or a saddle spiral.

We record the following references for the figures quoted in this chapter. (i) Figures 1-2: Arnold [7], (ii) Figures 3,4, and 14: Reissig et al [51], and (iii) Figures 5-13: Reyn [57].

CHAPTER 5

PERTURBATIONS OF A SIMPLE EQUILIBRIUM OF THE THREE DIMENSIONAL KOLMOGOROV MODEL

In this chapter we consider the noncritical case of the perturbed three dimensional Kolmogorov model corresponding to the noncritical case of the unperturbed model and derive the characteristic equation of the variational matrix for the linearized system of the perturbed model. Further, we examine the nature and stability of the perturbed equilibrium emanating from a simple unperturbed equilibrium in the three dimensional phase space and compare the qualitative behaviors of some perturbed equilibria with those of the unperturbed equilibria.

5.1 EXISTENCE OF A SIMPLE PERTURBED EQUILIBRIUM

In order to establish a relationship between the topological structure and stability of an unperturbed and a perturbed simple equilibrium, we now study the noncritical case of the perturbed three dimensional Kolmogorov model

$$N_i' = N_i F_i(N_1, N_2, N_3, \varepsilon), \qquad i = 1,2,3, \tag{5.1}$$

corresponding to the noncritical case of the unperturbed model (4.1), where ε is a small positive parameter. For $\varepsilon = 0$, (5.1) reduces to the system (4.1) which has a simple equilibrium $E^0(M_1^0, M_2^0, M_3^0)$ in the first octant provided that the conditions (4.3) and (4.6) are satisfied. For $\varepsilon \neq 0$ the equilibria of the perturbed system (5.1) are obtained by solving the system of equations

$$F_i(N_1,N_2,N_3,\varepsilon) = 0 , \qquad i = 1,2,3, \tag{5.2}$$

subject to the conditions (4.3) and (4.6). By the implicit function theorem, with (4.6), the perturbed system (5.1) has a unique solution $M_i^*(\varepsilon)$ in the neighbourhood of the solution M_i^0 of the unperturbed system (4.1), such that $M_i^*(0) = M_i^0$, $i = 1,2,3,$ and

$$F_i(M_1^*,M_2^*,M_3^*,\varepsilon) = 0, \qquad i = 1,2,3. \tag{5.3}$$

Let J_0^* be the matrix

$$J_0^*(M_1^*,M_2^*,M_3^*) = \left[F_{iN_j}^{0*}(\varepsilon) \right], \qquad i,j = 1,2,3, \tag{5.4}$$

where

$$F_{iN_j}^{0*}(\varepsilon) = \frac{\partial F_i}{\partial N_j}(M_1^*,M_2^*,M_3^*,\varepsilon), \qquad i,j = 1,2,3, \tag{5.5}$$

such that $F_{iN_j}^{0*}(0) = F_{iN_j}^{0}$. The assumption (4.6) guarantees that

$$|J_0^*| = \det J_0^*(M_1^*,M_2^*,M_3^*) \neq 0, \tag{5.6}$$

which corresponds to the noncritical case for the perturbed system (5.3).

Hence, the perturbed model (5.1) has a simple equilibrium $E^*(M_1^*, M_2^*, M_3^*)$, called the simple perturbed equilibrium, which for $\varepsilon = 0$ moves to the simple unperturbed equilibrium E^0 of (4.1).

In order to find the solution of (5.3), we seek $M_i^*(\varepsilon)$ in terms of power series of ε in the neighbourhood of M_i^0 in the form

$$M_i^*(\varepsilon) = M_i^0 + \varepsilon m_i + \varepsilon^2 \ell_i + \dots , \qquad i = 1,2,3. \tag{5.7}$$

Since ε is small, in general, it is sufficient to evaluate m_i, $i = 1,2,3$. Substituting (5.7) into (5.3), expanding

$$F_i(M_1^0 + \varepsilon m_1,\ M_2^0 + \varepsilon m_2,\ M_3^0 + \varepsilon m_3,\ \varepsilon), \qquad i = 1,2,3,$$

in Taylor series, and equalizing the coefficient of ε to zero, we obtain the following linear system for m_i, $i = 1,2,3$:

$$\sum_{j=1}^{3} m_j F_{iN_j}^0 + F_{i\varepsilon} = 0 , \qquad i = 1,2,3, \tag{5.8}$$

where $F_{iN_j}^0$ is given by (4.5), and

$$F_{i\varepsilon} = \frac{\partial F_i}{\partial \varepsilon}(M_1^0, M_2^0, M_3^0, 0), \qquad i = 1,2,3. \tag{5.9}$$

The system of equations (5.8) is a set of three linear non-homogeneous equations whose Jacobian (5.6) is different from zero. Such a system, by Cramer's rule, has a unique solution given by

$$m_i = - \frac{1}{|\Delta_0|} \sum_{j=1}^{3} \Delta_{ji} F_{j\varepsilon} , \qquad i = 1,2,3, \qquad (5.10)$$

where Δ_{ji} is the cofactor of the element F_{jN_i} in the matrix $\left[F^0_{iN_j}\right]$, i,j = 1,2,3, and Δ_0 is given by (4.10).

For $\varepsilon = 0$, the equilibrium E^* of the perturbed system (5.1) turns to the equilibrium E^0 of the unperturbed system (4.1), and for $\varepsilon \neq 0$, the equilibrium E^0 moves to the equilibrium E^* . Thus, we say that the simple unperturbed equilibrium E^0 generates the simple perturbed equilibrium E^* .

Hence,we can state the following theorem concerning the existence of an equilibrium for the perturbed system.

THEOREM 2

In the neighbourhood of a simple equilibrium E^0 of the unperturbed system (4.1) there exists a unique simple equilibrium E^* of the perturbed system (5.1) for sufficiently small positive ε .

5.2 NATURE AND STABILITY OF THE PERTURBED EQUILIBRIUM

The nature and stability of the equilibrium E^* of the perturbed system (5.1) are determined by the characteristic equation of the variational matrix for the linearized part of (5.1). To find the variational matrix for the linear part of the perturbed model, first we linearize the nonlinear system (5.1). In order to linearize the perturbed system (5.1), we use the transformation

$$N_i = M_i^*(\varepsilon) + X_i^*(\varepsilon) , \qquad i = 1,2,3, \tag{5.11}$$

such that $M_i^*(0) = M_i^0$ and $X_i^*(0) = X_i$, $i = 1,2,3$. Substituting (5.11) into (5.1) and using the Taylor series expansion for

$$F_i(M_1^* + X_1^*, M_2^* + X_2^*, M_3^* + X_3^*, \varepsilon), \qquad i = 1,2,3,$$

we obtain

$$X_i^{*\prime}(\varepsilon) = [M_i^*(\varepsilon) \sum_{j=1}^{3} F_{iN_j}^{0*}(\varepsilon)]X_i^* + \bar{E}_i^*(\varepsilon), \qquad i = 1,2,3, \tag{5.12}$$

where $\bar{E}_i^*(\varepsilon)$ represents the nonlinear part of (5.1) and is given by

$$\bar{E}_i^*(\varepsilon) = \bar{E}_i^*(X_1^*,X_2^*,X_3^*,\varepsilon) = [M_i^*(\varepsilon) + X_i^*(\varepsilon)]F_i(M_1^* + X_1^*, M_2^* + X_2^*,$$

$$M_3^* + X_3^*, \varepsilon) - [M_i^*(\varepsilon) \sum_{j=1}^{3} F_{iN_j}^{0*}(\varepsilon)]X_i^*, \qquad i = 1,2,3, \tag{5.13}$$

and $F_{iN_j}^{0*}(\varepsilon)$ is given by (5.5). Further, $\bar{E}_i^*(\varepsilon)$ and $F_{iN_j}^{0*}(\varepsilon)$ are such that

$$\bar{E}_i^*(0) = \bar{E}_i^0 , \qquad i = 1,2,3,$$

and (5.14)

$$F_{iN_j}^{0*}(0) = F_{iN_j}^0 , \qquad i,j = 1,2,3,$$

where $\bar{E}_i^0$ and $F_{iN_j}^0$ are given by (4.9) and (4.5) respectively.

The variatonal matrix for the linear part of (5.12) at the perturbed equilibrium point is

$$\Delta_0^*(\varepsilon) = \Delta_0^*(M_1^*,M_2^*,M_3^*) = \left[m_{ij}^0 + \varepsilon m_{ij}^*\right], \qquad i,j = 1,2,3, \tag{5.15}$$

such that $\Delta_0^*(0) = \Delta_0$, where m_{ij}^0 is given by (4.11), and

$$m_{ij}^* = M_i^0\left[\sum_{k=1}^{3} m_k F_{iN_jN_k}^0 + F_{iN_j\varepsilon}^0\right] + m_i F_{iN_j}^0, \qquad i,j = 1,2,3, \tag{5.16}$$

with

$$F_{iN_jN_k}^0 = \frac{\partial^2 F_i}{\partial N_j \partial N_k}(M_1^0,M_2^0,M_3^0,0), \qquad i,j,k = 1,2,3,$$

and (5.17)

$$F_{iN_j\varepsilon}^0 = \frac{\partial^2 F_i}{\partial N_j \partial \varepsilon}(M_1^0,M_2^0,M_3^0,0), \qquad i,j = 1,2,3.$$

The assumption (4.12) ensures that

$$|\Delta_0^*| = \det \Delta_0^*(M_1^*,M_2^*,M_3^*) \neq 0 , \tag{5.18}$$

and (5.18) corresponds to the noncritical case of the perturbed model (5.1).

The characteristic equation of (5.15), up to the order of ε , is

$$\lambda^3 + (P_1 + \varepsilon q_1)\lambda^2 + (P_2 + \varepsilon q_2)\lambda + (P_3 + \varepsilon q_3) = 0 \ , \qquad (5.19)$$

where P_1, P_2, and P_3 are given by (4.14), and

$$q_1 = -\sum_{i=1}^{3} m^*_{ii} \ ,$$

$$q_2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} [\tfrac{1}{2} m_{ii} (m^*_{jj} + m^*_{kk}) - m_{ij} m^*_{ji}], \quad i \neq j \neq k, \qquad (5.20)$$

$$q_3 = -\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m^*_{31} & m^*_{32} & m^*_{33} \end{vmatrix} - \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m^*_{21} & m^*_{22} & m^*_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} - \begin{vmatrix} m^*_{11} & m^*_{12} & m^*_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} .$$

The condition (5.18) implies that the characteristic equation (5.19) does not have any zero root. Further, the roots of the characteristic equation for a perturbed system are, in general, assumed to be distinct. Hence,the variational matrix (5.15) of the perturbed model (5.1) has three distinct eigenvalues and thus three linearly independent eigenvectors corresponding to these eigenvalues. Therefore, the nature and stability of the perturbed equilibrium E^* can be determined by the types and signs of roots of the characteristic equation (5.19) of the perturbed system (5.1). Further, for $\varepsilon = 0$, the equation (5.19) reduces to the equation (4.13), and for $\varepsilon \neq 0$, the equation (4.13) produces (5.19). Thus, we say that the characteristic equation (4.13) of the unperturbed system (4.1) generates the characteristic equation

(5.19) of the perturbed system (5.1). Therefore, the roots of (4.13) generates the roots of (5.19).

Distinct roots of the characteristic equation (4.13) of the unperturbed system can generate only distinct roots of the characteristic equation (5.19) of the perturbed system. However, in general, repeated roots of (4.13) generate distinct roots of (5.19), and imaginary roots of (4.13) generate complex roots of (5.19). To facilitate the study of these cases we introduce the following notations.

$$
\begin{aligned}
D_0^* &= \frac{1}{9}\left(p_2 - \frac{1}{3}p_1^2\right)^2\left(q_2 - \frac{2}{3}p_1q_1\right) \\
&+ \frac{1}{18}\left(p_1p_2 - \frac{2}{9}p_1^3 - 3p_3\right)\left(p_1q_2 + p_2q_1 - \frac{2}{3}p_1^2q_1 - 3q_3\right), \\
H_0^* &= p_1q_2 + p_2q_1 - q_3, \\
Q_0^* &= q_2 - \frac{2}{3}p_1q_1, \quad R_0^* = q_3 - \frac{1}{9}p_1^2q_1,
\end{aligned}
\tag{5.21}
$$

where p_i and q_i, $i = 1,2,3$, are given by (4.14) and (5.20). Also, we use some small values $\mu_i(\varepsilon)$, $i = 1,2,3$, such that $\mu_i(0) = 0$. These small values may vary but $\mu_1 \neq \mu_2 \neq \mu_3$.

We now provide parameter conditions for all possible types of distinct roots of the characteristic equation (5.19) and examine the character (i.e., the nature of the phase portrait and the stability or the instability property) of the perturbed equilibrium E^* corresponding to these roots.

CASE A_1

Three distinct real roots λ_i , $i = 1,2,3$, of (4.13), satisfying $D_0 < 0$, can generate only three distinct real roots $\lambda_i + \mu_i$, $i = 1,2,3$, of (5.19). In this case the perturbed equilibrium E^* has exactly the same character as the unperturbed equilibrium E^0 in *Sub-Case A(i)*.

CASE A_2

One real and two complex roots λ_1 and $\lambda_0 \pm i\omega_0$, $\lambda_0 \neq 0$, of (4.13), satisfying $D_0 > 0$ and $H_0 \neq 0$, can generate only one real and two complex roots $\lambda_1 + \mu_1$ and $\lambda_0 + \mu_2 \pm i(\omega_0 + \mu_3)$ of (5.19). In this case the perturbed equilibrium E^* has exactly the same character as the unperturbed equilibrium E^0 in *Sub-Case A(ii)*.

Hence, from the results in Cases A, A_1, and A_2, and the Definition 1, we have the following theorem.

THEOREM 3

A-type equilibria of the unperturbed system (4.1) generate A-type equilibria of the perturbed system (5.1), i.e., nodes, saddle nodes, node spirals, and saddle spirals of (4.1) generate corresponding equilibria of (5.1) with the same nature and stability or instability property.

CASE B_1

Double roots λ_1, λ_2, and λ_2 of (4.13) satisfying $D_0 = 0$ and $Q_0 \neq 0$ can generate:

(a) Distinct roots $\lambda_1 + \mu_1$, $\lambda_2 + \mu_2$, and $\lambda_3 + \mu_3$ of (5.19) if $D_0^* < 0$. Then E^* has similar character as E^0 in *Sub-Case A(i)*.

(b) Complex roots $\lambda_1 + \mu_1$ and $\lambda_2 \pm i\mu_2$ of (5.19) if $D_0^* > 0$. Then E^* has similar character as E^0 in *Sub-Case A(ii)*.

Note. If $D_0^* = 0$, double roots of (4.13), depending on higher order terms of ε neglected in (5.19), may or may not generate double roots of (5.19) if $Q_0^* \neq 0$. We do not consider this case.

CASE B_2

If $Q_0 = R_0 = 0$, then $D_0^* = 0$. In this case the nature of the roots of (5.19) depends on Q_0^* and

$$D_0^{**} = \frac{1}{4}\left(R_0^* - \frac{1}{3} p_1 Q_0^*\right)^2 , \tag{5.22}$$

where Q_0^* and R_0^* are given by (5.21). Hence, under the conditions $Q_0 = R_0 = 0$, the triple roots λ_1, λ_1, and λ_1 of (4.13) can generate:

(a) Three distinct real roots $\lambda_1 + \mu_1$, $\lambda_1 + \mu_2$, and $\lambda_1 + \mu_3$ of (5.19) if $D_0^{**} = 0$, $Q_0^* < 0$. Then E^* has similar character as E^0 in 1 and 4 of *Sub-Case A(i)*.

(b) One real and two complex roots $\lambda_1 + \mu_1$ and $\lambda_1 \pm i\mu_2$ of (5.19) if $D_0^{**} \neq 0$ or $D_0^{**} = 0$, $Q_0^* > 0$. Then E^* has similar character as E^0 in 5(b) and 8(b) of *Sub-Case A(ii)*.

Note. Triple roots of (4.13), depending on higher order terms of ε neglected in (5.19), may or may not generate double roots of (5.19) if

$Q_0^* \neq 0$ or triple roots of (5.19) if $Q_0^* = R_0^* = 0$. We do not consider these cases.

Taking into consideration the Cases A, B, B_1, and B_2 and the Definitions 1 and 2, we establish the following theorem.

THEOREM 4

A B-type equilibrium of the unperturbed system (4.1) generates an A-type equilibrium of the perturbed system (5.1) if $D_0^* \neq 0$; $D_0^* = 0$, $D_0^{**} \neq 0$; *or* $D_0^* = D_0^{**} = 0$, $Q_0^* \neq 0$. *Three dimensional stars, node stars, antisymmetric node stars, two branched nodes, and one branched nodes of (4.1) generate either three branched nodes or node spirals of (5.1); and saddle stars and two branched saddle nodes of (4.1) generate saddle nodes or saddle spirals of (5.1). The stability or instability property of* E^0 *and* E^* *is the same.*

CASE C_1

Under the conditions $D_0 > 0$ and $H_0 = 0$, one real and two imaginary roots λ_1 and $\pm i\omega_0$ of (4.13) can generate one real and two complex roots $\lambda_1 + \mu_1$ and $\mu_2 \pm i(\omega_0 + \mu_3)$ of (5.19) if $H_0^* \neq 0$. The character of E^* is stated below:

(a) If $p_1 > 0$, $p_3 > 0$, and $H_0^* > 0$, then $\lambda_1 < 0$ and $\mu_2 < 0$; and E^* is an asymptotically stable blunt spiral.

(b) If $p_3 < 0$ and $H_0^* > 0$, then $\lambda_1 > 0$ and $\mu_2 < 0$; and E^* is a saddle spiral with stable plane focus.

(c) If $P_3 > 0$ and $H_0^* < 0$, then $\lambda_1 < 0$ and $\mu_2 > 0$; and E^* is a saddle spiral with unstable plane focus.

(d) If $p_1 < 0$, $p_3 < 0$, and $H_0^* < 0$, then $\lambda_1 > 0$ and $\mu_2 > 0$; and E^* is an unstable blunt spiral.

Hence, the nonhyperbolic equilibrium E^0 may change its stability property under small perturbation. This occurs when $\lambda_1 < 0$. Then a neutrally stable E^0 of the unperturbed system (4.1) generates an asymptotically stable E^* if $\mu_2 < 0$ and an unstable E^* if $\mu_2 > 0$ of the perturbed system (5.1).

Note. If $H_0^* = 0$, imaginary roots of (4.13), depending on higher order terms of ε neglected in (5.19), may or may not generate imaginary roots of (5.19). We do not consider this case.

On the basis of the results in Cases C and C_1, and the Definition 3, we derive the following theorem.

THEOREM 5

A C-type equilibrium of the unperturbed system (4.1) generates an A-type equilibrium of the perturbed system (5.1) if $H_0^* \neq 0$. *The stability or instability property of* E^0 *and* E^* *are not the same. A convergent center focus of (4.1) generates an asymptotically stable node spiral or a saddle spiral of (5.1) and a divergent center focus of (4.1) generates an unstable node spiral or a saddle spiral of (5.1).*

5.3 QUALITATIVE STUDIES OF THREE PERTURBED LOTKA-VOLTERRA MODELS

In order to illustrate some of the results obtained in Theorems 3, 4, and 5, we now discuss the following special cases of the general model (5.1).

Example 1. To illustrate a result of Theorem 3, the following three dimensional perturbed food chain model, studied by Hausrath [33], is considered.

$$N_1' = N_1[a_{10} - a_{12}N_2 + \varepsilon F_1(N_1,N_2,N_3)] ,$$

$$N_2' = N_2[-a_{20} + a_{21}N_1 - a_{23}N_3 + \varepsilon F_2(N_1,N_2,N_3)] , \qquad (5.23)$$

$$N_3' = N_3[a_{30} + a_{32}N_2 - a_{33}N_3 + \varepsilon F_3(N_1,N_2,N_3)] .$$

The system (5.23) models a real situation that occurred during this century in Isle Royale National Park (210 square mile island in Lake Superior). Here N_1 is the food supply for a moose population N_2, and N_3 is a wolf population which preys on the moose N_2 ; a_{ij} are positive coefficients and $\varepsilon N_i F_i$ are perturbations, $i = 1,2,3$, $j = 0,\ldots,4$. Hausrath shows that (5.23) for $\varepsilon = 0$ has the equilibrium point $E^0(M_1^0,M_2^0,M_3^0)$, where

$$M_1^0 = \frac{a_{10}}{a_{12}} , \quad M_2^0 = \frac{a_{30} + a_{32}M_1^0}{a_{32}}, \quad M_3^0 = \frac{a_{20} + a_{23}M_2^0}{a_{21}} .$$

The characteristic equation of the variational matrix of (5.23) for $\varepsilon = 0$ is (4.13) where

$$p_1 = a_{33}M_2^0,\ p_2 = a_{23}a_{32}M_1^0M_2^0 + a_{12}a_{21}M_1^0M_3^0,\ p_3 = a_{12}a_{21}a_{33}M_1^0M_2^0M_3^0 .$$

Use of the Routh-Hurwitz criterion

$$p_1 > 0,\ p_3 > 0,\quad H_0 = p_1p_2 - p_3 = \frac{a_{10}a_{23}a_{32}}{a_{12}a_{33}}\left(a_{30} + \frac{a_{10}a_{32}}{a_{12}}\right)^2 > 0 ,$$

gives that the characteristic equation has all roots with negative real parts which indicates that the equilibrium E^0 is asymptotically stable. Further, Hausrath proves several theorems concerning the relationship between the unperturbed and perturbed systems of (5.23). Part of these results, namely the persistance of the stable equilibrium E^0 can be obtained as a particular case of Theorem 2. Really, since E^0 is hyperbolic equilibrium which is asymptotically stable, the same is valid for the equilibrium E^* of the perturbed system (5.23). To find the nature of E^0 and hence E^*, one has to study the values D_0, Q_0, and R_0 given by (4.18) and make use of Theorems 3 and 4.

Example 2. To illustrate the interesting Case B_2(b) we consider the perturbed model

$$N_1' = N_1\{2 - 2N_1 - N_2 + N_3 + \varepsilon(\tfrac{1}{3} - \tfrac{\sqrt{2}}{6})N_3\} ,$$

$$N_2' = N_2\{5 - N_1 - N_2 - N_3 + \tfrac{4}{3}\varepsilon N_1\} , \tag{5.24}$$

$$N_3' = N_3\{-1 + N_1 + N_2 - N_3 + \varepsilon\frac{\sqrt{2}}{6}N_2\} .$$

In the unperturbed system of (5.24), the interaction between the species N_1 and N_2 is competitive, between N_1 and N_3 is mutualistic, and between N_2 and N_3 is predator-prey. The perturbational terms in (5.24), with factor $\varepsilon << 1$, change slightly the unperturbed system in the following way. The coefficients with factor ε indicate a weaker type of interaction between populations N_i and N_j which contributes little to the growth rate of N_i, $i,j = 1,2,3$, of the unperturbed system of (5.24).

The unperturbed equilibrium of (5.24) is $E^0(1,2,2)$ and the corresponding perturbed equilibrium, up to the order of ε, is $E(1 + 2\varepsilon/3,\ 2 - \sqrt{2}\ \varepsilon/6,\ 2 + (2/3 + \sqrt{2}/6)\varepsilon)$. The variational matrix (5.15) for the perturbed system (5.24) becomes

$$\begin{bmatrix} -2-\frac{4}{3}\varepsilon & -1-\frac{2}{3}\varepsilon & 1+(1-\frac{\sqrt{2}}{6})\varepsilon \\ -2+(\frac{8}{3}+\frac{\sqrt{2}}{6})\varepsilon & -2+\frac{\sqrt{2}}{6}\varepsilon & -2+\frac{\sqrt{2}}{6}\varepsilon \\ 2+(\frac{2}{3}+\frac{\sqrt{2}}{6})\varepsilon & 2+(\frac{2}{3}+\frac{1}{\sqrt{2}})\varepsilon & -2-(\frac{2}{3}+\frac{\sqrt{2}}{6})\varepsilon \end{bmatrix},$$

such that the condition (5.18) is satisfied.

The characteristic equation (5.19) of the variational matrix (5.15) for (5.24) is

$$\lambda^3 + (6 + 2\varepsilon)\lambda^2 + [12 + (8 + \sqrt{2})\varepsilon]\lambda + [8 + (8 + 2\sqrt{2})\varepsilon] = 0 \qquad (5.25)$$

comparing (5.25) with (5.19) we have

$$p_1 = 6, \quad p_2 = 12, \quad p_3 = 8, \quad q_1 = 2, \quad q_2 = 8 + \sqrt{2}, \quad q_3 = 8 + 2\sqrt{2}$$

and hence from (4.18), (5.21), and (5.22) we obtain

$$D_0 = D_0^* = D_0^{**} = Q_0 = R_0 = 0, \qquad Q_0^* = \sqrt{2}\ .$$

We note that the variational matrix (4.10) for the unperturbed system (5.24 with $\varepsilon = 0$) has one linearly independent eigenvector (-1,1,1) corresponding to the triple eigenvalue -2 and thus has the Jordan canonical form Δ_{11}^0 given by (4.17).

Therefore, the unperturbed equilibrium E^0 is an asymptotically stable one branched node [see (13) of *Sub-Case B(ii)*] which is a B-type equilibrium; and the perturbed equilibrium E^* is an asymptotically stable conical spiral [see (b) of Case B_2 along with 5(b) of *Sub-Case A(ii)*] which is an A-type equilibrium. This result is in conformity with Theorem 4.

Example 3. In order to show that a nonhyperbolic equilibrium of the unperturbed system generates a hyperbolic equilibrium of the perturbed system (Theorem 5), we propose the following perturbed model in R^3 .

$$N_1' = N_1(7 - 4N_1 - 3N_2 - \varepsilon N_2)\ ,$$

$$N_2' = N_2[-3 + 3N_1 + N_2 - \sqrt{3}N_3 + \varepsilon(2N_1 - \sqrt{3}N_3)], \tag{5.26}$$

$$N_3' = N_3(-5 + 3N_2 + 2\sqrt{3}N_3 + 3\varepsilon N_2)\ .$$

The unperturbed system of (5.26) describes a three-level food chain model, where N_1 is the lowest trophic level population or prey, N_2 is the middle trophic level population or first predator, and N_3 is the highest trophic level population or second predator. The small perturbational terms in (5.26) has the same meaning as those in (5.24).

The unperturbed and perturbed equilibrium points of (5.26) are $E^0(1,1,1/\sqrt{3})$ and $E^*(1 + 5\varepsilon,\ 1 - 7\varepsilon,\ 1/\sqrt{3} + 3\sqrt{3}\varepsilon)$ respectively. The variational matrix (5.15) for the system (5.26) is

$$\begin{bmatrix} -4 - 20\varepsilon & -3 - 16\varepsilon & 0 \\ 3 - 19\varepsilon & 1 - 7\varepsilon & -\sqrt{3} + 6\sqrt{3}\varepsilon \\ 0 & \sqrt{3} + 10\sqrt{3}\varepsilon & 2 + 18\varepsilon \end{bmatrix},$$

such that the condition (5.18) is satisfied. The characteristic equation for the above matrix, up to the order ε, is

$$\lambda^3 + (1 + 9\varepsilon)\lambda^2 + (2 - 97\varepsilon)\lambda + (2 + 20\varepsilon) = 0 . \qquad (5.27)$$

Comparing (5.27) with (5.19) we obtain

$$p_1 = 1,\quad p_2 = p_3 = 2,\quad q_1 = 9,\quad q_2 = -97,\quad q_3 = 20 .$$

Using these values in (4.18) and (5.21), we find that

$$D_0 = 0.53, \quad H_0 = 0, \quad \text{and} \quad H_0^* = -99 .$$

Thus, for $\varepsilon = 0$, we are in Case C, and for $\varepsilon \neq 0$ in Case C_1. The unperturbed equilibrium E^0 of (5.26) is a convergent centre focus [see (15) of Case C] which is a simple nonhyperbolic equilibrium and the perturbed equilibrium E^* of (5.26) is a saddle spiral with unstable plane focus [see (c) of Case C_1] which is an A-type equilibrium. This result is in agreement with Theorem 5.

CHAPTER 6

BIFURCATIONS OF A MULTIPLE EQUILIBRIUM OF THE THREE DIMENSIONAL KOLMOGOROV MODEL

In this chapter we consider the noncritical case of the perturbed three dimensional Kolmogorov model corresponding to the critical case of the unperturbed model and derive criteria for the existence of simple equilibria for the perturbed model. We study the bifurcation of a multiple unperturbed equilibrium into simple perturbed equilibria and examine the nature and stability of the perturbed equilibria in the three dimensional phase space. In the last section, bifurcations of the multiple unperturbed equilibria of two certain population models are investigated and the qualitative behaviors of the corresponding simple perturbed equilibria are examined.

6.1 EXISTENCE OF A MULTIPLE EQUILIBRIUM

In order to investigate the bifurcation of a multiple equilibrium of the unperturbed three dimensional Kolmogorov model into simple equilibria, we consider the noncritical case of the perturbed three dimensional Kolmogorov model

$$N_i' = N_i F_i(N_1, N_2, N_3, \varepsilon), \qquad i = 1,2,3, \tag{6.1}$$

where ε is a small positive parameter, corresponding to the critical case of the unperturbed model

$$N_i' = N_i F_i(N_1,N_2,N_3,0), \qquad i = 1,2,3. \tag{6.2}$$

we assume that the unperturbed system (6.2) has at least one equilibrium point $F^0(N_1^0,N_2^0,N_3^0)$, called the unperturbed equilibrium, in the interior of the first octant. This means that the system

$$F_i(N_1,N_2,N_3,0) = 0, \qquad i = 1,2,3, \tag{6.3}$$

has at least one solution (N_1^0,N_2^0,N_3^0), such that

$$F_i(N_1^0,N_2^0,N_3^0,0) = 0, \qquad N_i^0 > 0\ , \qquad i = 1,2,3. \tag{6.4}$$

Let J be the matrix

$$J(N_1^0,N_2^0,N_3^0) = \left[F_{iN_j}\right], \qquad i,j = 1,2,3, \tag{6.5}$$

where

$$F_{iN_j} = \frac{\partial F_i}{\partial N_j}(N_1^0,N_2^0,N_3^0,0), \qquad i = 1,2,3, \tag{6.6}$$

and assume that

$$|J| = \det J(N_1^0,N_2^0,N_3^0) = 0\ . \tag{6.7}$$

The assumption (6.7) corresponds to the critical case for the system (6.3) and ensures that F^0 is a multiple equilibrium of (6.2), i.e., F^0 is a point of intersection of the surfaces (6.3) such that the tangent planes to the surfaces at their common point exist and are coincident. To find the equilibrium of (6.1) for $\varepsilon \neq 0$, we have to solve the system of equations

$$F_i(N_1,N_2,N_3,\varepsilon) = 0, \qquad i = 1,2,3, \tag{6.8}$$

subject to the conditions (6.4) and (6.7). Under the condition (6.7), the system (6.8) may or may not have real solutions. For our problem we assume that the perturbed system (6.8) has a solution $N_i^*(\varepsilon)$ in the neighborhood of the solution N_i^0 of the unperturbed system (6.3), such that $N_i^*(0) = N_i^0$, and

$$F_i(N_1^*,N_2^*,N_3^*,\varepsilon) = 0, \qquad i = 1,2,3. \tag{6.9}$$

To find the solution of (6.9), we seek $N_i^*(\varepsilon)$ in terms of power series of ε in the neighborhood of N_i^0 in the form

$$N_i^*(\varepsilon) = N_i^0 + \varepsilon n_i + \varepsilon^2 t_i + \dots, \qquad i = 1,2,3, \tag{6.10}$$

where n_i and t_i, $i = 1,2,3$, are real number.

Let J^* be the matrix

$$J^*(N_1^0 + \varepsilon n_1, N_2^0 + \varepsilon n_2, N_3^0 + \varepsilon n_3, \varepsilon) = [F^*_{iN_j}], \quad i,j = 1,2,3, \tag{6.11}$$

where

$$F^*_{iN_j} = \frac{\partial F_i}{\partial N_j}(N^0_1,N^0_2,N^0_3,0) + \varepsilon\{\sum_{k=1}^{3} n_k \frac{\partial^2 F_i}{\partial N_j \partial N_k}(N^0_1,N^0_2,N^0_3,0)$$

$$+ \frac{\partial^2 F_i}{\partial N_j \partial \varepsilon}(N^0_1,N^0_2,N^0_3,0)\}, \qquad i,j = 1,2,3, \tag{6.12}$$

and assume that

$$|J^*| = \det J^* \neq 0 \ . \tag{6.13}$$

The assumption (6.13) corresponds to the noncritical case of the implicit function theorem for the system (6.9) and ensures that (6.8) has simple solutions. Hence, the perturbed model (6.1) has simple equilibria $F^*(N^*_1,N^*_2,N^*_3,\varepsilon)$, called the perturbed equilibria. For $\varepsilon = 0$, the equilibria F^* of the perturbed system (6.1) returns to the equilibrium F^0 of the unperturbed system (6.2). Thus, under the influence of small perturbation the multiple equilibrium F^0 of (6.2) satisfying (6.7) generates simple equilibria F^* of (6.1) provided the condition (6.13) is fulfilled.

6.2 BIFURCATION OF A MULTIPLE EQUILIBRIUM

In order to study bifurcations of a multiple equilibrium F^0 into simple equilibria F^*, we have to solve the system of equations (6.9) subject to the condition (6.7) and (6.13). Substituting (6.10) into (6.9), expanding

$$F_i(N_1+\varepsilon n_1+\varepsilon^2 t_1,\ N_2+\varepsilon n_2+\varepsilon^2 t_2,\ N_3+\varepsilon n_3+\varepsilon^2 t_3,\varepsilon),\quad i = 1,2,3,$$

in Taylor series, we obtain

$$\begin{aligned}
&F_{iN_1}n_1 + F_{iN_2}n_2 + F_{iN_3}n_3 + F_{i\varepsilon} + \varepsilon(\tfrac{1}{2}F_{iN_1N_1}n_1^2 + \\
&\tfrac{1}{2}F_{iN_2N_2}n_2^2 + \tfrac{1}{2}F_{iN_3N_3}n_3^2 + \tfrac{1}{2}F_{i\varepsilon\varepsilon} + F_{iN_1N_2}n_1n_2 + \\
&F_{iN_1N_3}n_1n_3 + F_{iN_2N_3}n_2n_3 + F_{iN_1\varepsilon}n_1 + F_{iN_2\varepsilon}n_2 + \\
&F_{iN_3\varepsilon}n_3 + F_{iN_1}t_1 + F_{iN_2}t_2 + F_{iN_3}t_3) = 0\ , \quad i = 1,2,3,
\end{aligned} \tag{6.14}$$

where higher order terms of the $0(\varepsilon^2)$ are neglected. Here F_{iN_j} are given by (6.6), and

$$F_{i\varepsilon} = \frac{\partial F_i}{\partial \varepsilon}(N_1^0,N_2^0,N_3^0,0), \qquad F_{iN_jN_k} = \frac{\partial^2 F_i}{\partial N_j \partial N_k}(N_1^0,N_2^0,N_3^0,0),$$

$$F_{iN_j\varepsilon} = \frac{\partial^2 F_i}{\partial N_j \partial \varepsilon}(N_1^0,N_2^0,N_3^0,0), \qquad F_{i\varepsilon\varepsilon} = \frac{\partial^2 F_i}{\partial \varepsilon^2}(N_1^0,N_2^0,N_3^0,0), \tag{6.15}$$

$$i,j,k = 1,2,3,$$

and it is assumed that at least one of the second partial derivatives

$$F_{iN_jN_k} \neq 0 , \qquad i,j,k = 1,2,3. \tag{6.16}$$

If all of $F_{iN_jN_k}$, $i,j,k = 1,2,3$, are zero, then we shall have to use the higher order terms neglected in (6.14) to resolve the problem. We do not consider this case. By the implicit function theorem with (6.12) the solutions of the system (6.14) are simple. Further, the system of equations (6.14) is a set of three quadratic equations in three unknowns, such a system may have one or more entire surfaces of solutions, or it may have two to eight real solutions; or no real solutions. If the system (6.14) does not have real solutions, no real values for n_i, $i = 1,2,3$, exist, and so a solution of the type sought in (6.13) does not exist. If the system (6.14) has a multiple root, higher order terms neglected in (6.14) are required to resolve the situation, so we do not handle the case. For simple equilibria of the type sought in (6.13), we are interested only in the simple real roots of (6.14).

The cases of branching will occur if the conditions (6.7), (6.12), and (6.16) are satisfied. The condition (6.7), i.e., det J = 0 requires that the rank of the Jacobian matrix J given by (6.5) be two, one, or zero. We will discuss each of these cases separately.

CASE I. *Jacobian Matrix J has Rank 2.*

In this case we assume that the rank of the matrix J given by (6.5) is two. This means that at least one of the second order minors of J is different from zero. We assume for definiteness that

$$\begin{vmatrix} F_{2N_2} & F_{2N_3} \\ F_{3N_2} & F_{3N_3} \end{vmatrix} \neq 0 . \tag{6.17}$$

Then from the second and third equations of (6.14) we find that n_2 and n_3 are the solutions of

$$F_{jN_2}n_2 + F_{jN_3}n_3 = - F_{j\varepsilon} - F_{jN_1}n_1 , \qquad j = 2,3 . \tag{6.18}$$

From the reduced system (6.18), the solutions for n_2 and n_3 as functions of n_1 are given by

$$n_j = \frac{1}{\Delta_{11}} (\Delta_{1j}n_1 + \Delta_{0j}), \qquad j = 2,3, \tag{6.19}$$

where Δ_{1i} are the cofactors of the elements F_{1N_i}, i = 1,2,3, in J and

$$\Delta_{02} = - \begin{vmatrix} F_{2\varepsilon} & F_{2N_3} \\ F_{3\varepsilon} & F_{3N_3} \end{vmatrix}, \qquad \Delta_{03} = \begin{vmatrix} F_{2\varepsilon} & F_{2N_2} \\ F_{3\varepsilon} & F_{3N_2} \end{vmatrix}. \tag{6.20}$$

Substituting (6.19) into the first equation of (6.14), the solution for n_1 is given by the following quadratic equation:

$$a_{10} + \varepsilon(a_{11}n_1^2 + a_{12}n_1 + a_{13}) = 0 \tag{6.21}$$

where

$$a_{10} = (\Delta_{02}F_{1N_2} + \Delta_{03}F_{1N_3} + \Delta_{11}F_{1\varepsilon})\Delta_{11}^2 ,$$

$$a_{11} = \frac{1}{2} R_{11}\Delta_{11}^2 + \frac{1}{2} R_{22}\Delta_{12}^2 + \frac{1}{2} R_{33}\Delta_{13}^2 + R_{12}\Delta_{11}\Delta_{12}$$

$$+ R_{13}\Delta_{11}\Delta_{13} + R_{23}\Delta_{12}\Delta_{23} ,$$

$$a_{12} = R_{22}\Delta_{02}\Delta_{12} + R_{33}\Delta_{03}\Delta_{13} + R_{12}\Delta_{02}\Delta_{11}$$

$$+ R_{13}\Delta_{03}\Delta_{11} + R_{23}(\Delta_{03}\Delta_{12} + \Delta_{02}\Delta_{13}) \tag{6.22}$$

$$+ R_{01}\Delta_{11}^2 + R_{02}\Delta_{11}\Delta_{12} + R_{03}\Delta_{11}\Delta_{13} ,$$

$$a_{13} = \frac{1}{2} R_{22}\Delta_{02}^2 + \frac{1}{2} R_{33}\Delta_{03}^2 + R_{23}\Delta_{02}\Delta_{03} +$$

$$R_{02}\Delta_{02}\Delta_{11} + R_{03}\Delta_{03}\Delta_{11} + R_{04}\Delta_{11}^2 , \quad \ldots$$

with

$$R_{ij} = \sum_{K=1}^{3} \Delta_{K1} F_{KN_iN_j}, \qquad i,j = 1,2,3,$$

$$R_{0i} = \sum_{K=1}^{3} \Delta_{K1} F_{KN_i\varepsilon}, \qquad i = 1,2,3, \tag{6.23}$$

$$R_{04} = \sum_{K=1}^{3} \Delta_{K1} F_{K\varepsilon\varepsilon},$$

and, because of (6.16), it is assumed that

$$a_{11} \neq 0 . \tag{6.24}$$

The condition $a_{11} = 0$ requires higher order terms neglected in (6.14) to resolve the case, and we do not treat it. Equation (6.21) is a quadratic equation in one variable and we need only simple real solutions of (6.21). The solution of (6.21) depends upon whether $a_{10} = 0$ or $a_{10} \neq 0$.

Sub-Case IA.

First we consider that

$$a_{10} = 0 . \tag{6.25}$$

Here the perturbed equilibria (6.13) have the forms

$$N_i^* = N_i + \alpha_{1i}\varepsilon + 0(\varepsilon^2), \qquad i = 1,2,3, \tag{6.26}$$

where α_{11} is a solution of the quadratic equation

$$a_{11}\alpha_{11}^2 + a_{12}\alpha_{11} + a_{13} = 0 , \tag{6.27}$$

and

$$\alpha_{1j} = \frac{1}{\Delta_{11}} (\Delta_{0j} + \Delta_{1j}\alpha_{11}), \qquad j = 2,3,$$

provided that

$$a_{12}^2 - 4a_{11}a_{13} > 0 . \tag{6.28}$$

Under the condition (6.28), the quadratic equation (6.27) has two simple nonzero real roots if $a_{13} \neq 0$ or one zero and one nonzero real roots if $a_{13} = 0$, $a_{12} \neq 0$. Thus, if the conditions (6.17), (6.24), (6.25), and (6.28) are satisfied, there will be two branches of simple perturbed equilibrium points of the form (6.26) if $a_{13} \neq 0$; or two branches of simple equilibrium points, one of them of the form (6.26) and the other coinciding with the unperturbed equilibrium (N_1^0,N_2^0,N_3^0) if $a_{13} = 0$, $a_{12} \neq 0$.

Sub-Case IB.

We now assume that

$$a_{10} \neq 0 . \tag{6.29}$$

In this subcase the equilibria of the perturbed system (6.1) have the forms

$$N_i^* = N_i + \beta_{1i}\varepsilon^{\frac{1}{2}} + O(\varepsilon), \qquad i = 1,2,3, \tag{6.30}$$

where

$$\beta_{1i} = \pm\left(\frac{-a_{10}}{a_{11}}\right)^{\frac{1}{2}} \cdot \frac{\Delta_{1i}}{\Delta_{11}}, \qquad i = 1,2,3,$$

such that

$$a_{10}a_{11} < 0 . \tag{6.31}$$

Thus, under the hypotheses of (6.17), (6.24), (6.29), and (6.31), the multiple unperturbed equilibrium F^0 bifurcates into two branches of simple perturbed equilibria of the form (6.30) for sufficiently small positive ε .

<u>CASE II</u>. *Jacobian matrix J has Rank 1.*

In this case we assume that the rank of the matrix J given by (6.5) is one. This means that at least one of the elements of J is nonzero. We suppose for definiteness that

$$F_{3N_3} \neq 0 . \tag{6.32}$$

Then from the third equation of (6.14) we can find n_3 as functions of n_1 and n_2 in the form

$$n_3 = -\frac{1}{F_{3N_3}}(F_{3N_1}n_1 + F_{3N_2}n_2 + F_{3\varepsilon}) . \tag{6.33}$$

Substituting (6.33) into the first and second equations of (6.14), we find that the solutions for n_1 and n_2 are given by

$$b_{i0} + \varepsilon(b_{i1}n_1^2 + b_{i2}n_2^2 + b_{i3}n_1n_2 + c_{i1}n_1$$

$$+ c_{i2}n_2 + c_{i3}) = 0 , \qquad i = 1,2, \tag{6.34}$$

where

$$b_{i0} = F_{3N_3}F_{i\varepsilon} - F_{iN_3}F_{3\varepsilon} ,$$

$$b_{ij} = (\tfrac{1}{2} F_{iN_jN_j}F_{3N_3} + \frac{1}{2}\frac{F_{iN_3N_3}F_{3N_j}^2}{F_{3N_3}} - F_{iN_jN_3}F_{3N_j})$$

$$- F_{iN_3}(\tfrac{1}{2} F_{3N_jN_j} + \frac{1}{2}\frac{F_{3N_3N_3}F_{3N_j}^2}{F_{3N_3}^2} - \frac{F_{3N_jN_3}F_{3N_j}}{F_{3N_3}}) ,$$

$$b_{i3} = (F_{iN_1N_2}F_{3N_3} - F_{iN_1N_3}F_{3N_2} - F_{iN_2N_3}F_{3N_1} + \frac{F_{iN_3N_3}F_{3N_1}F_{3N_2}}{F_{3N_3}})$$

$$- F_{iN_3}(F_{3N_1N_2} - \frac{F_{3N_1N_3}F_{3N_2}}{F_{3N_3}} - \frac{F_{3N_2N_3}F_{3N_1}}{F_{3N_3}} + \frac{F_{3N_3N_3}F_{3N_1}F_{3N_2}}{F_{3N_3}^2}) ,$$

$$c_{ij} = (F_{iN_j\varepsilon}F_{3N_3} - F_{iN_jN_3}F_{3\varepsilon} - F_{iN_3\varepsilon}F_{3N_j} + \frac{F_{iN_3N_3}F_{3N_j}F_{3\varepsilon}}{F_{3N_3}})$$

$$- F_{iN_3}(F_{3N_j\varepsilon} - \frac{F_{3N_jN_3}F_{3\varepsilon}}{F_{3N_3}} - \frac{F_{3N_3\varepsilon}F_{3N_j}}{F_{3N_3}} + \frac{F_{3N_3N_3}F_{3N_j}F_{3\varepsilon}}{F^2_{3N_3}}) ,$$

$$c_{i3} = (\frac{1}{2} F_{i\varepsilon\varepsilon}F_{3N_3} - F_{iN_3\varepsilon}F_{3\varepsilon} + \frac{1}{2}\frac{F_{iN_3N_3}F^2_{3\varepsilon}}{F_{3N_3}})$$

$$- F_{iN_3}(\frac{1}{2} F_{3\varepsilon\varepsilon} - \frac{F_{3N_3\varepsilon}F_{3\varepsilon}}{F_{3N_3}} + \frac{1}{2}\frac{F_{3N_3N_3}F^2_{3\varepsilon}}{F^2_{3N_3}}) ,$$

$$i,j = 1,2 . \qquad (6.35)$$

The condition (6.16) requires that at least one of

$$b_{ij} \neq 0 , \qquad i,j = 1,2 . \qquad (6.36)$$

If all of $b_{ij} = 0$, $i,j = 1,2$, higher order terms neglected in (6.14) are needed to analyse the situation, we do not treat it. The system (6.34) is a system of two quadratic equations in two unknowns. Such a system may have one or more entire curves of solutions; or it may have two to four real solutions; or no real solutions. It may be recalled that we need only simple real roots of (6.34). The solution of (6.34) depends on b_{i0} , $i = 1,2$. Thus, we have two possibilities: either $b_{i0} = 0$ or at least one of $b_{i0} \neq 0$, $i = 1,2$.

Sub-Case IIA.

First we consider that

$$b_{i0} = 0 \ , \qquad i = 1,2. \tag{6.37}$$

Then from (6.34), (6.33), and (6.13), the equilibria of the perturbed system (6.1) have the forms

$$N_i^* = N_i^0 + \alpha_{2i}\varepsilon + O(\varepsilon^2), \qquad i = 1,2,3, \tag{6.38}$$

where $(\alpha_{21},\alpha_{22})$ is a real root of

$$b_{j1}\alpha_{21}^2 + b_{j2}\alpha_{22}^2 + b_{j3}\alpha_{21}\alpha_{22} + c_{j1}\alpha_{21} +$$

$$c_{j2}\alpha_{22} + c_{j3} = 0 \ , \qquad j = 1,2, \tag{6.39}$$

and

$$\alpha_{23} = -\frac{1}{F_{3N_3}}(F_{3N_1}\alpha_{21} + F_{3N_2}\alpha_{22} + F_{3\varepsilon}) \ ,$$

such that the Jacobian matrix

$$\Delta_{21} = \begin{bmatrix} 2b_{11}\alpha_{21} + b_{13}\alpha_{22} + c_{11} & b_{13}\alpha_{21} + 2b_{12}\alpha_{22} + c_{12} \\ 2b_{21}\alpha_{21} + b_{23}\alpha_{23} + c_{21} & b_{23}\alpha_{21} + 2b_{22}\alpha_{22} + c_{22} \end{bmatrix}$$

of (6.39) is nonsingular, i.e.,

$$|\Delta_{21}| \neq 0 \ . \tag{6.40}$$

Under the condition (6.40), the system of equations (6.39) may have two to four simple nonzero real solutions or one zero and one to three nonzero real solutions. Thus, under the hypotheses (6.32), (6.36), (6.37), and (6.40), there will be two to four branches of simple perturbed equilibrium points, either all of them of the form (6.38) or one of them coinciding with the unperturbed equilibrium and the rest of the form (6.38).

Sub-Case IIB.

Here it is supposed that at least one of b_{10} and b_{20} is not equal to zero. Hence, this subcase has three possibilities: either $b_{10} \neq 0$, $b_{20} \neq 0$; or $b_{10} \neq 0$, $b_{20} = 0$; or $b_{10} = 0$, $b_{20} \neq 0$. Now we consider that

$$b_{i0} \neq 0 \ , \qquad i = 1,2. \tag{6.41}$$

Then from (6.34), (6.33), and (6.13), the equilibria of the perturbed system (6.1) have the forms

$$N_i^* = N_i^0 + \beta_{2i}\varepsilon^{\frac{1}{2}} + 0(\varepsilon) , \qquad i = 1,2,3, \tag{6.42}$$

where (β_{21},β_{22}) is a real solution of

$$b_{j1}\beta_{21}^2 + b_{j2}\beta_{22}^2 + b_{j3}\beta_{21}\beta_{22} + b_{j0} = 0, \quad j = 1,2, \tag{6.43}$$

and

$$\beta_{23} = -\frac{1}{F_{3N_3}}\left(F_{3N_1}\beta_{21} + F_{3N_2}\beta_{22}\right),$$

provided that the Jacobian matrix

$$\Delta_{22} = \begin{bmatrix} 2b_{11}\beta_{21} + b_{13}\beta_{22} & b_{13}\beta_{21} + 2b_{12}\beta_{22} \\ 2b_{21}\beta_{21} + b_{23}\beta_{22} & b_{23}\beta_{21} + 2b_{22}\beta_{22} \end{bmatrix}$$

of (6.43) is nonsingular, i.e.,

$$|\Delta_{22}| \neq 0 . \tag{6.44}$$

We observe that $(-\beta_{21},-\beta_{22})$ is also a solution of (6.43). Under the condition (6.44), the system of equations (6.43) has either two or four simple nonzero real solutions. Hence, if the hypotheses (6.32), (6.36), (6.41), and (6.44) are satisfied, there will be either two or four branches of perturbed equilibria of the form (6.42).

Exactly similar analyses as above are valid for the other two possibilities $b_{10} \neq 0$, $b_{20} = 0$; and $b_{10} = 0$, $b_{20} \neq 0$ of this sub-case.

CASE III. *Jacobian matrix J has Rank 0.*

In this case it is assumed that the rank of the matrix J given by (6.5) is zero, i.e.,

$$F_{iN_j} = 0\,, \qquad i,j = 1,2,3, \tag{6.45}$$

where F_{iN_j} are given by (6.6). Then from (6.14), we find that n_1, n_2, and n_3 are the solutions of

$$F_{i\varepsilon} + \varepsilon(\tfrac{1}{2} F_{iN_1N_1}n_1^2 + \tfrac{1}{2} F_{iN_2N_2}n_2^2 + \tfrac{1}{2} F_{iN_3N_3}n_3^2 +$$

$$F_{iN_1N_2}n_1n_2 + F_{iN_1N_3}n_1n_3 + F_{iN_2N_3}n_2n_3 + F_{iN_1\varepsilon}n_1 +$$

$$F_{iN_2\varepsilon}n_2 + F_{iN_3\varepsilon}n_3 + \tfrac{1}{2} F_{i\varepsilon\varepsilon}) = 0, \qquad i = 1,2,3. \tag{6.46}$$

The system (6.46) is a set of three quadratic equations in three unknowns. Such a system may have one or more entire surfaces of solutions; or it may have two to eight real solutions; or no real solutions. We recall that we are interested in the simple real solutions of (6.46). The solution of (6.46) depends on $F_{i\varepsilon}$, i = 1,2,3. There are two possible cases: either all of the numbers $F_{i\varepsilon}$, i = 1,2,3, vanish or at least one of them does not vanish.

Sub-Case IIIA.

Throughout this subcase we assume that

$$F_{i\varepsilon} = 0 \ , \qquad i = 1,2,3, \tag{6.47}$$

where $F_{i\varepsilon}$ is given by (6.15). Then the perturbed equilibria have the forms

$$N_i^* = N_i^0 + \alpha_{3i}\varepsilon + 0(\varepsilon^2), \qquad i = 1,2,3, \tag{6.48}$$

where $(\alpha_{31},\alpha_{32},\alpha_{33})$ is a real root of

$$\frac{1}{2} F_{iN_1N_1}\alpha_{31}^2 + \frac{1}{2} F_{iN_2N_2}\alpha_{32}^2 + \frac{1}{2} F_{iN_3N_3}\alpha_{33}^2 +$$

$$F_{iN_1N_2}\alpha_{31}\alpha_{32} + F_{iN_1N_3}\alpha_{31}\alpha_{33} + F_{iN_2N_3}\alpha_{32}\alpha_{33} +$$

$$F_{iN_1\varepsilon}\alpha_{31} + F_{iN_{2\varepsilon}}\alpha_{32} + F_{iN_{3\varepsilon}}\alpha_{33} + \frac{1}{2} F_{i\varepsilon\varepsilon} = 0, \qquad i = 1,2,3, \tag{6.49}$$

such that the Jacobian matrix

$$\Delta_{31} = \left[\sum_{k=1}^{3} F_{iN_jN_k}\alpha_{3k} + F_{iN_j\varepsilon}\right], \qquad i,j = 1,2,3,$$

of (6.49) is nonsingular, i.e.,

$$|\Delta_{31}| \neq 0 \ , \tag{6.50}$$

where $F_{iN_jN_k}$ and $F_{iN_j\varepsilon}$ are given by (6.15). Under the condition (6.50), the system of equations (6.49) has two to eight simple real solutions and either all the solutions are nonzero or one of them is a zero solution. Hence, in this sub-case there will be two to eight branches of equilibrium points originating from (N_1^0, N_2^0, N_3^0) for sufficiently small positive ε, and either none or one of them will coincide with (N_1^0, N_2^0, N_3^0).

Sub-Case IIIB.

We now assume that at least one of $F_{i\varepsilon}$, $i = 1,2,3$, is different from zero. Hence, we have three possibilities: (i) all three of $F_{i\varepsilon} \neq 0$, $i = 1,2,3$; (ii) two of $F_{i\varepsilon} \neq 0$ and the rest one of $F_{i\varepsilon} = 0$, $i = 1,2,3$; and (iii) one of $F_{i\varepsilon} \neq 0$ and the rest two of $F_{i\varepsilon} = 0$, $i = 1,2,3$.

First, let us consider that

$$F_{i\varepsilon} \neq 0 , \qquad i = 1,2,3. \tag{6.51}$$

In this case the perturbed equilibria have the forms

$$N_i^* = N_i^0 + \beta_{3i}\varepsilon^{\frac{1}{2}} + O(\varepsilon), \qquad i = 1,2,3, \tag{6.52}$$

where $(\beta_{31}, \beta_{32}, \beta_{33})$ is a real root of

$$\frac{1}{2} F_{iN_1N_1}\beta_{31}^2 + \frac{1}{2} F_{iN_2N_2}\beta_{32}^2 + \frac{1}{2} F_{iN_3N_3}\beta_{33}^2 +$$

$$F_{iN_1N_2}\beta_{31}\beta_{32} + F_{iN_1N_3}\beta_{31}\beta_{33} + F_{iN_2N_3}\beta_{32}\beta_{33}$$

$$+ F_{i\varepsilon} = 0 , \qquad i = 1,2,3, \tag{6.53}$$

provided that the Jacobian matrix

$$\Delta_{32} = \left[\sum_{k=1}^{3} F_{iN_jN_k} \beta_{3k} \right], \qquad i,j = 1,2,3,$$

is nonsingular, i.e.,

$$|\Delta_{32}| \neq 0 \ . \tag{6.54}$$

We note that $(-\beta_{31}, -\beta_{32}, -\beta_{33})$ is also a solution of (6.53). Under the condition (6.54), the system of equations (6.53) has two, four, six, or eight simple nonzero real solutions. Thus, in this sub-case there will be two, four, six, or eight branches of perturbed equilibrium points of of form (6.52).

Exactly similar analyses as above are valid for the other two possibilities of this subcase.

Remarks: We observe that the system (6.1) can be written in the form

$$N_i' = \varphi_i(N_1,N_2,N_3,0) + \varepsilon\psi_i(N_1,N_2,N_3,\varepsilon), \qquad i = 1,2,3, \tag{6.1a}$$

where the terms φ_i are independent of ε and ψ_i involve ε. It is assumed that

$$\det\left[\frac{\partial\varphi_i}{\partial N_j}(N_1^0,N_2^0,N_3^0,0)\right] = 0 \ , \ \text{and}$$

$$\det\left[\frac{\partial\varphi_i}{\partial N_j}(N_1^0,N_2^0,N_3^0,0) + \varepsilon\,\frac{\partial\psi_i}{\partial N_j}(N_1^0,N_2^0,N_3^0,\varepsilon)\right] \neq 0 \ , \quad i,j = 1,2,3, \tag{6.1b}$$

where $F^0(N_1^0,N_2^0,N_3^0)$ is an equilibrium of the unperturbed system of (6.1a) and thus (6.1). The conditions (6.1b) are in consistent with (6.7) and (6.12) and guarantee that the unperturbed equilibrium F^0 is a multiple equilibrium while the equilibria of the perturbed system (6.1a) originating from F^0 are simple. Furthermore, we note that if $\psi_i(N_1^0,N_2^0,N_3^0,\varepsilon) = 0$, $i = 1,2,3$, then F^0 is also an equilibrium of the perturbed system (6.1a) while if at least one of $\psi_i(N_1^0,N_2^0,N_3^0,\varepsilon) \neq 0$, $i = 1,2,3$, then F^0 is not an equilibrium of (6.1a). Thus, we have the following remarks.

Remark 1. If (6.1a) satisfies (6.1b) and all of $\psi_i(N_1^0,N_2^0,N_3^0,\varepsilon) = 0$, $i = 1,2,3$, then one of the perturbed equilibria of (6.1a) coincides with F^0 and the rest of the perturbed equilibria exist in the neighborhood of F^0 .

Remark 2. If (6.1a) fulfills (6.1b) and not all of $\psi_i(N_1^0,N_2^0,N_3^0,\varepsilon) \neq 0$, $i = 1,2,3$, then all the perturbed equilibria of (6.1a) exist in the neighborhood of F^0 .

Taking into consideration the results in Cases I, II, and III, and the Remarks 1 and 2, we have established the following theorem.

THEOREM 6

If the rank of the Jacobian matrix of the unperturbed three dimensional Kolmogorov model is $n = 2, 1,$ *or* 0, *and the conditions (6.12) and (6.16) are satisfied, then the multiple equilibrium of the unperturbed model bifurcates into at least* 2 *or at most* 2^m *simple equilibria of the corresponding perturbed model, where* $m = 1$ *for* $n = 2$, $m = 2$ *for* $n = 1$, *and* $m = 3$, *for* $n = 0$. *Further, either all the perturbed equilibria exist in the neighborhood of the unperturbed equilibrium or one of the perturbed equilibria coincides with the unperturbed equilibrium and the rest of the perturbed equilibria exist in the neighborhood of the unperturbed equilibrium.*

6.3 NATURE AND STABILITY OF THE PERTURBED EQUILIBRIA

The nature and stability of the equilibria F^* of the perturbed system (6.1) depends on the variational matrix of (6.1). To find the variational matrix of (6.1), we linearize the perturbed system (6.1) by using the transformations

$$N_i = N_i^*(\varepsilon) + Y_i^*(\varepsilon), \qquad i = 1,2,3, \tag{6.55}$$

such that $N_i^*(0) = N_i^0$, $i = 1,2,3$. Substituting (6.55) into (6.1) and using Taylor series expansion for

$$F_i(N_1^* + Y_1^*, N_2^* + Y_2^*, N_3^* + Y_3^*, \varepsilon), \qquad i = 1,2,3,$$

we obtain

$$Y_i^{*'}(\varepsilon) = [N_i^*(\varepsilon) \sum_{j=1}^{3} F^*_{iN_j}(\varepsilon)]Y_i^*(\varepsilon) + \bar{F}_i^*(\varepsilon), \quad i = 1,2,3, \qquad (6.56)$$

where $\bar{F}_i^*(\varepsilon)$ represents the higher order terms of ε involving the nonlinear terms of (6.1), and is given by

$$\bar{F}_i^*(\varepsilon) = \bar{F}_i^*(Y_1^*,Y_2^*,Y_3^*,\varepsilon) = [N_i^*(\varepsilon) + Y_i^*(\varepsilon)]F_i(N_1^*+Y_1^*,N_2^*+Y_2^*,N_3^*+Y_3^*,\varepsilon)$$

$$- [N_i^*(\varepsilon) \sum_{j=1}^{3} F^*_{iN_j}(\varepsilon)]Y_i^*(\varepsilon), \quad i = 1,2,3, \qquad (6.57)$$

and

$$F^*_{iN_j}(\varepsilon) = \frac{\partial F_i}{\partial N_j}(N_1^*,N_2^*,N_3^*,\varepsilon), \qquad i,j = 1,2,3,$$

with

$$F^*_{iN_j}(0) = F_{iN_j}, \qquad i,j = 1,2,3,$$

where F_{iN_j} is given by (6.6).

The variational matrix of (6.56) at the perturbed equilibrium point, up to the order of ε, is

$$\Delta^*(\varepsilon) = \left[n_{ij} + \varepsilon n^*_{ij}\right], \qquad i,j = 1,2,3, \qquad (6.58)$$

such that

$$\Delta^*(0) = \Delta = \left[n_{ij}\right], \qquad i,j = 1,2,3, \tag{6.59}$$

where

$$n_{ij} = N_i^0 \frac{\partial F_i}{\partial N_j}(N_1^0, N_2^0, N_3^0, 0), \qquad i,j = 1,2,3, \tag{6.60}$$

and

$$n_{ij}^* = N_i^0\left(\sum_{k=1}^{3} n_k F_{iN_jN_k} + F_{iN_j\varepsilon}\right) + n_i F_{iN_j}, \qquad i,j = 1,2,3. \tag{6.61}$$

The partial derivatives of F_i in (6.61) are given by (6.15). The assumptions (6.12) and (6.7) guarantee that

$$|\Delta^*(\varepsilon)| \neq 0 \quad \text{and} \quad |\Delta| = 0 , \tag{6.62}$$

which ensures that we are dealing with the noncritical case of the perturbed model (6.1) corresponding to the critical case of the unperturbed model (6.2).

The characteristic equation of (6.58), up to the order of ε, is

$$\lambda^3 + (r_1 + \varepsilon s_1)\lambda^2 + (r_2+\varepsilon s_2)\lambda + \varepsilon s_3 = 0 , \tag{6.63}$$

where

$$r_1 = -\sum_{i=1}^{3} n_{ii} ,$$

$$r_2 = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} (n_{ii}n_{jj} - n_{ij}n_{ji}), \qquad i \neq j,$$

and

$$s_1 = -\sum_{i=1}^{3} n_{ii}^* , \tag{6.64}$$

$$s_2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} [\tfrac{1}{2} n_{ii} (n_{jj}^* + n_{kk}^*) - n_{ij}n_{ji}^*], \quad i \neq j \neq k ,$$

$$s_3 = -\begin{vmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31}^* & n_{32}^* & n_{33}^* \end{vmatrix} - \begin{vmatrix} n_{11} & n_{12} & n_{13} \\ n_{21}^* & n_{22}^* & n_{23}^* \\ n_{31} & n_{32} & n_{33} \end{vmatrix} - \begin{vmatrix} n_{11}^* & n_{12}^* & n_{13}^* \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{vmatrix} .$$

It is assumed that $s_3 \neq 0$. If $s_3 = 0$, then higher order terms of ε neglected in (6.63) are required to resolve the situation. We do not consider this case. For $\varepsilon = 0$, the equation (6.63) reduces to

$$\lambda^3 + r_1\lambda^2 + r_2\lambda = 0 , \tag{6.65}$$

which is the characteristic equation of (6.59). Equation (6.65) has at least one zero root. The number of zero roots of (6.65) depends on the rank of the variational matrix Δ given by (6.59). The cubic equation (6.65) has one, two, or three zero roots provided the rank of Δ is two, one, or zero respectively. The condition (6.62) ensures that the characteristic equation (6.63) does not have any zero root.

Moreover, the roots of the characteristic equation of a perturbed system are, in general, assumed to be distinct. Hence, the variational matrix of the perturbed model (6.1) has three distinct eigenvalues and thus three linearly independent eigenvectors corresponding to these eigenvalues. Therefore, the nature and stability of the perturbed equilibria F^* can be determined by the types and signs of roots of the characteristic equation (6.63) of the perturbed system (6.1). Further, for $\varepsilon = 0$, the equation (6.63) reduces to (6.65), and we say that the characteristic equation (6.65) of the unperturbed system (6.2) generates the characteristic equation (6.63) of the perturbed system (6.1). Therefore, the roots of (6.65) generate the roots of (6.63).

Distinct roots of the chracteristic equation (6.65) of the unperturbed system (6.2) can generate only distinct roots of the characteristic equation (6.63) of the perturbed system (6.1). However, in general, repeated roots of (6.65) generate distinct roots of (6.63) and imaginary roots of (6.65) generate complex roots of (6.63). To facilitate the study of these cases we introduce the following notations:

$$D = \frac{1}{108} r_2^2(4r_2 - r_1^2), \quad H = r_1 r_2,$$

$$D^* = \frac{1}{9}(r_2 - \frac{1}{3} r_1^2)^2 (s_2 - \frac{2}{3} r_1 s_1)$$

$$+ \frac{1}{18}(r_1 r_2 - \frac{2}{9} r_1^3)(r_1 s_2 + r_2 s_1 - \frac{2}{3} r_1^2 s_1 - 3s_3), \tag{6.66}$$

$$H^* = r_1 s_2 + r_2 s_1 - s_3,$$

$$Q^* = s_2 - \frac{2}{3} r_1 s_1, \quad R^* = s_3 - \frac{1}{9} r_1^2 s_1,$$

where r_i and s_j, $i = 1,2$, $j = 1,2,3$, are given by (6.64). Also, we use some small values $\nu_i(\varepsilon)$, $i = 1,2,3$, such that $\nu_i(0) = 0$. These small values may vary but $\nu_1 \neq \nu_2 \neq \nu_3$.

We now provide parameter conditions for all possible types of distinct roots of the characteristic equation (6.63) and examine the character (i.e., the nature of the phase portrait and the stability or instability property) of the perturbed equilibria corresponding to these roots. We discuss all possible cases according to the rank of Δ given by (6.59).

CASE α. *Variational Matrix* Δ *has Rank 2*.

In this case equation (6.65) has one zero root. Thus, we have four possibilities: one zero and two distinct real roots, one zero and two complex (with nonzero real parts) roots, one zero and two repeated real roots, and one zero and two imaginary roots.

Sub-Case α_1: One Zero and Two Distinct Real Roots.

One zero and two distinct real roots 0 and ξ_i, $i = 2,3$, of (6.65) satisfying $D < 0$, can generate only three distinct real roots ν_1, $\xi_2 + \nu_2$, and $\xi_3 + \nu_3$ of (6.63). The perturbed equilibrium F^* has the following character.

(i) If $r_1 > 0$, $s_3 > 0$, and $H > 0$, then $\nu_1 < 0$ and $\xi_i < 0$, $i = 2,3$; and F^* is an asymptotically stable three branched node.

(ii) If $s_3 < 0$ and $H > 0$, then $\nu_1 > 0$ and $\xi_i < 0$, $i = 2,3$; and F^* is a three branched saddle node with stable two branched plane node.

(iii) If $s_3 < 0$ and $H < 0$, then $\nu_1 < 0$ and $\xi_i > 0$, $i = 2,3$; and F^* is a three branched saddle node with unstable two branched plane node.

(iv) If $r_1 < 0$, $s_3 < 0$, and $H < 0$, then $\nu_1 > 0$ and $\xi_i > 0$, $i = 2,3$; and F^* is an unstable three branched node.

Sub-Case α_2 : One Zero and Two Complex Roots.

One zero and two complex roots 0 and $\xi \pm i\eta$, $\xi \neq 0$ of (6.65) satisfying $D > 0$ and $r_1 \neq 0$, can generate only one real and two complex roots ν_1 and $\xi + \nu_2 \pm i(\eta + \nu_3)$ of (6.63). The perturbed equilibrium F^* has the following character.

(v) If $r_1 > 0$, $s_3 > 0$, and $H > 0$, then $\nu_1 < 0$ and $\xi < 0$; and F^* is an asymptotically stable pointed spiral.

(vi) If $s_3 < 0$ and $H > 0$, then $\nu_1 > 0$ and $\xi < 0$; and F^* is a saddle spiral with stable plane focus.

(vii) If $s_3 > 0$ and $H < 0$, then $\nu_1 < 0$ and $\xi > 0$; and F^* is a saddle spiral with unstable plane focus.

(viii) If $r_1 < 0$, $s_3 < 0$, and $H < 0$, then $\nu_1 > 0$ and $\xi > 0$; and F^* is an unstable pointed spiral.

Sub-Case α_3 : One Zero and Two Repeated Real Roots.

One zero and two repeated real roots 0, ξ_2, and ξ_2 of (6.65) satisfying $D = 0$, $r_1 \neq 0$, and $r_2 \neq 0$ can generate:

(a) Three distinct real roots ν_1, $\xi_2 + \nu_2$, and $\xi_2 + \nu_3$ of (6.63) if $D^* < 0$. Then F^* has similar character as F^* in *Sub-Case* α_1 .

(b) One real and two complex roots ν_1 and $\xi_2 \pm i\nu_2$ of (6.63) if $D^* > 0$. Then F^* has similar character as F^* in *Sub-Case* α_2 .

Note. Double nonzero roots of (6.65), depending on higher order terms of ε neglected in (6.63), may or may not generate double non-zero roots of (6.63) if $D^* = 0$. We do not treat it.

Sub-Case α_4 . <u>One Zero and Two Imaginary Roots.</u>

One zero and two imaginary roots 0 and $\pm i\eta$ of (6.65) satisfying $D > 0$ and $r_1 = 0$ generate one real and two complex roots ν_1 and $\nu_2 \pm i(\eta + \nu_3)$ of (6.63) if $H^* \neq 0$. The character of F^* is stated below:

(ix) If $s_1 > 0$, $s_3 > 0$, and $H^* > 0$, then $\nu_i < 0$, $i = 1,2$; and F^* is an asymptotically stable blunt spiral when $\nu_1 > \nu_2$ and pointed spiral when $\nu_1 < \nu_2$.

(x) If $s_3 < 0$ and $H^* > 0$, then $\nu_1 > 0$ and $\nu_2 < 0$; and F^* is a saddle spiral with stable plane focus.

(xi) If $s_3 > 0$ and $H^* < 0$, then $\nu_1 < 0$ and $\nu_2 > 0$; and F^* is a saddle spiral with unstable plane focus.

(xii) If $s_1 < 0$, $s_3 < 0$, and $H^* < 0$, then $\nu_i > 0$, $i = 1,2$; and F^* is an unstable blunt spiral when $\nu_1 > \nu_2$ and pointed spiral when $\nu_1 < \nu_2$.

Note. If $H^* = 0$, imaginary roots of (6.65), depending on higher order terms of ε neglected in (6.63), may or may not generate imaginary roots of (6.63). We do not consider this case.

CASE β. *Variational Matrix* Δ *has rank 1*.

In this case equation (6.65) has two zero roots. Hence, we have only one possibility: one nonzero real and two zero roots.

One nonzero real and two zero roots ξ_1, 0, and 0 of (6.65) satisfying $r_1 \neq 0$ and $r_2 = 0$ can generate:

(c) Three distinct nonzero real roots $\xi_1 + \nu_1$, ν_2, and ν_3 of (6.63) if $D^* < 0$. Then F^* has similar character as F^* in *Sub-Case* α_1 provided r_2 is replaced by s_2, H by H^*, ξ_2 by ν_2, and ξ_3 by ν_3.

(d) One nonzero real and two complex roots with nonzero real parts $\xi_1 + \nu_1$ and $\nu_2 \pm i\nu_3$ if $D^* > 0$ and $H^* \neq 0$. Then F^* has similar character as F^* in (vi) and (vii) of *Sub-Case* α_2 provided H is replaced by H^*, ν_1 by ξ_1, and ξ by ν_2. Also, F^* has the following character:

(xiii) If $r_1 > 0$, $s_3 > 0$, and $H^* > 0$, then $\xi_1 < 0$ and $\nu_2 < 0$; and F^* is an asymptotically stable blunt spiral.

(xiv) If $r_1 < 0$, $s_3 < 0$, and $H^* < 0$, then $\xi_1 > 0$ and $\nu_2 > 0$; and F^* is an unstable blunt spiral.

Note. Double zero roots of (6.65), depending on higher order terms of ε neglected in (6.63), may or may not generate double nonzero real roots of (6.63) provided $D^* = 0$ and $Q^* \neq 0$. We do not treat this possibility.

CASE γ . *Variational Matrix* Δ *has Rank 0* .

In this case equation (6.65) has three zero roots. This occurs when $r_1 = r_2 = 0$. We observe that $r_1 = r_2 = 0$ implies $D = D^* = 0$. Hence, three zero roots of (6.65) can generate only one nonzero real and two complex (with nonzero real parts) roots ν_1 and $\nu_2 \pm i\nu_3$ of (6.63). Here the equilibrium F^* has the following character:

(xv) If $s_3 < 0$, then $\nu_1 > 0$ and $\nu_2 < 0$; and F^* is a saddle spiral with stable plane focus.

(xvi) If $s_3 > 0$, then $\nu_1 < 0$ and $\nu_2 > 0$; and F^* is a saddle spiral with unstable plane focus.

Note. Triple zero roots of (6.65), depending on higher order terms of ε neglected in (6.63), may or may not generate nonzero double roots of (6.63) if $Q^* \neq 0$, or nonzero triple roots of (6.63) if $Q^* = R^* = 0$. We do not consider these cases. Further, we observe that triple zero roots can not generate three distinct nonzero real roots.

On the basis of the results obtained in Cases α, β, and γ , we establish the following theorem.

THEOREM 7

Multiple equilibria of the unperturbed system (6.2), for small positive ε *, generate A-type hyperbolic equilibria of the perturbed system (6.1) either automatically or provided* $D^* \neq 0$ *and/or* $H^* \neq 0$ *. Under the influence of small perturbation, the nature of the multiple* *unperturbed equilibrium always changes, while the stability property changes only if* $r_1 > 0$ *or* $r_1 = 0$, $s_1 > 0$; $r_2 > 0$ *or* $r_2 = 0$, $s_2 > 0$; *and* $s_3 > 0$.

CHAPTER 7

EXAMPLES AND NUMERICAL SOLUTIONS

In this chapter we present two examples and numerical solutions of one of the examples.

7.1 QUALITATIVE BEHAVIORS OF TWO PERTURBED FOOD-CHAIN MODELS

In this section we give two examples of population models involving bifurcations which are special cases of the general model (6.1).

Example 1. In order to illustrate a result of Theorem 6 in which one of the perturbed equilibria coincides with the unperturbed equilibrium, we consider the following perturbed three dimensional simple food chain.

$$N_1' = N_1[1 + N_1^2 - 2N_1N_2 + \varepsilon(N_1 - N_2)],$$

$$N_2' = N_2[-1 + 2N_1N_2 - \frac{1}{2}N_2^2N_3 + \varepsilon(2N_1 - N_3)], \qquad (7.1)$$

$$N_3' = N_3[-1 + N_2 + \varepsilon(2N_2 - N_3)].$$

In this three species food chain, N_3 eats N_2, and N_2 eats N_1; and hence N_1 is the prey, N_2 the first predator, and N_3 the second predator. The perturbation terms, i.e., the coefficients of ε, with $\varepsilon << 1$, indicate weaker types of interactions between the populations N_i and N_j, and contribute little to the growth rate of N_i, $i,j = 1,2,3$, of the unperturbed system of (7.1).

The unperturbed system of (7.1) has a double equilibrium $F^0(1,1,2)$. We observe that all the perturbational terms in (7.1) vanish at the unperturbed equilibrium point (1,1,2). The perturbed equilibria of (7.1) are $F^{*1}(1,1,2)$ and $F^{*2}(1-\varepsilon, 1, 1-4\varepsilon)$.

The variational matrices for the perturbed system (7.1) at the equilibrium points F^{*1} and F^{*2} are respectively

$$\Delta^{*1} = \begin{bmatrix} \varepsilon & -2-\varepsilon & 0 \\ 2+2\varepsilon & 0 & -\frac{1}{2}-\varepsilon \\ 0 & 2+4\varepsilon & -2\varepsilon \end{bmatrix} \quad \text{and } \Delta^{*2} = \begin{bmatrix} -\varepsilon & -2+3\varepsilon & 0 \\ 2+2\varepsilon & 2\varepsilon & -\frac{1}{2}-\varepsilon \\ 0 & 2 & -2\varepsilon \end{bmatrix} . \quad (7.2)$$

We observe that the conditions (6.7), (6.12), and (6.16) are fulfilled. The variational matrix for the unperturbed system of (7.1) has rank two thus we are in Case I. Further, the system (7.1) satisfies the conditions (6.24), (6.25), and (6.28), and the Remark 1 of Section 6.2. Hence, the result that the multiple unperturbed equilibrium F^0 bifurcates into two simple perturbed equilibria F^{*1} and F^{*2}, and one of the perturbed equilibria, here F^{*1}, coincides with the unperturbed equilibrium F^0 is in agreement with Theorem 6.

The characteristic equations for Δ^{*1} and Δ^{*2} of (7.2), up to the order of ε, are respectively

$$\lambda^3 + \varepsilon\lambda^2 + (5 + 10\varepsilon)\lambda + 7\varepsilon = 0 , \quad (7.3)$$

and

$$\lambda^3 + \varepsilon\lambda^2 + 5\lambda + 9\varepsilon = 0 \ . \tag{7.4}$$

Comparing (7.3) with (6.63) we find that

$$r_1 = 0, \quad r_2 = 5, \quad s_1 = 1, \; s_2 = 10, \; s_3 = 7 \ .$$

Using these values in (6.66) we obtain

$$D = 125/127, \quad H = 0, \quad H^* = -2 \ ,$$

and thus we are in (xi) of *Sub-Case* α_4 . Hence, the perturbed equilibrium F^{*1} is a saddle spiral with unstable plane focus. Comparing (7.4) with (6.63) we have

$$r_1 = 0, \quad r_2 = 5, \quad s_1 = 1, \quad s_2 = 0, \quad s_3 = 9 \ .$$

Then from (6.66) we obtain

$$D = 125/27, \quad H = 0, \quad H^* = -4 \ ,$$

and thus we are in (xi) of *Sub-Case* α_4 . Here the perturbed equilibrium F^{*2} is a saddle spiral with unstable plane focus. These results are in complete agreement with Theorem 7.

Example 2. To illustrate a case of Theorem 6 where all the perturbed equilibria exist in the neighborhood of the multiple unperturbed equilibrium, we study the following perturbed population model

$$N_1' = N_1[1 + N_1^2 - 2N_1N_2 + \varepsilon(4N_1 - 6N_2),$$

$$N_2' = N_2[-1 + 2N_1N_2 - \tfrac{1}{2}N_2^2N_3 + \varepsilon(N_1 - \tfrac{1}{2}N_3)], \qquad (7.5)$$

$$N_3' = N_3[-1 + N_2 + \varepsilon N_2].$$

The model (7.5) has the same interpretation as the model (7.1). Further, the unperturbed part of (7.5) is the same as that of (7.1), and thus has the double equilibrium $F^0(1,1,2)$. We note that the perturbational terms associated with the first and the third equations do not vanish at the unperturbed equilibrium point (1,1,2). The perturbed equilibria of (7.5) are $F^{*1}(1-(3+\sqrt{3})\varepsilon,\ 1-\varepsilon,\ 2-(12+4\sqrt{3})\varepsilon)$ and $F^{*2}(1-(3-\sqrt{3})\varepsilon,\ 1-\varepsilon,\ 2-(12-4\sqrt{3})\varepsilon)$.

The variational matrices for the perturbed system (7.5) at the equilibrium points F^{*1} and F^{*2} are respectively

$$\Delta^{*1} = \begin{bmatrix} -2\sqrt{3}\varepsilon & -2+(6+4\sqrt{3})\varepsilon & 0 \\ 2-3\varepsilon & (8+2\sqrt{e})\varepsilon & -\frac{1}{2}+\varepsilon \\ 0 & 2-(10+4\sqrt{3})\varepsilon & 0 \end{bmatrix} \text{ and } \Delta^{*2} = \begin{bmatrix} 2\sqrt{3}\varepsilon & -2+(6-4\sqrt{3})\varepsilon & 0 \\ 2-3\varepsilon & (8-2\sqrt{3})\varepsilon & -\frac{1}{2}+\varepsilon \\ 0 & 2-(10-4\sqrt{3})\varepsilon & 0 \end{bmatrix}. \qquad (7.6)$$

We note that the conditions (6.7), (6.12), and (6.16) are fulfilled and we are in Case I. The system (7.5) also satisfies the conditions

(6.24), (6.25), and (6.28), and the Remark 2 of Section 6.2. Hence, the result that F^0 bifurcates into F^{*1} and F^{*2} ; and F^{*1} and F^{*2} exist in the neighborhood of F^0 is consistent with Theorem 6.

The characteristic equations for Δ^{*1} and Δ^{*2} of (7.6), up to the order of ε , are respectively

$$\lambda^3 - (8\varepsilon)\lambda^2 + \{5 - (25 + 10\sqrt{3})\varepsilon\}\lambda + 2\sqrt{3}\varepsilon = 0 , \tag{7.7}$$

and

$$\lambda^3 - (8\varepsilon)\lambda^2 + \{5 - (25 - 10\sqrt{3})\varepsilon\}\lambda - 2\sqrt{3}\varepsilon = 0 . \tag{7.8}$$

Comparing (7.7) with (6.63) we find that

$$r_1 = 0, \quad r_2 = 5, \quad s_1 = -8, \; s_2 = -25 - 10\sqrt{3}, \; s_3 = 2\sqrt{3} .$$

Using these values in (6.66) we have

$$D = 125/27 , \quad H = 0, \quad H^* = -40-2\sqrt{3},$$

and thus we are in (xi) of *Sub-Case* α_4 . Hence, the perturbed equilibrium F^{*1} is a saddle spiral with unstable plane focus.

Comparing (7.8) with (6.63) we get

$$r_1 = 0, \quad r_2 = 5, \quad s_1 = -8, \; s_2 = -25+10\sqrt{3}, \; s_3 = -2\sqrt{3}.$$

Substituting these values in (6.66) we obtain

$$D = 125/27, \quad H = 0, \quad H^* = -40 + 2\sqrt{3},$$

and thus we are in (xii) of *Sub-Case* α_4 . The cubic equation has one real and two complex roots, where the real part of the complex root is greater than the real root. Hence, the perturbed equilibrium F^{*2} is an unstable pointed spiral. The above results are in agreement with Theorem 7.

7.2 NUMERICAL SOLUTIONS OF A PERTURBED FOOD CHAIN

In this section we present various numerical solutions of the system of ordinary differential equations (7.5) for different values of ε . Each value of ε is treated in a separate case. In each case the system of equations (7.5) has been integrated for various times. These computer solutions are presented as a guide to the analysis and as some measure of verification of the results obtained in Example 2 of Section 7.1.

Many different runs using variety of initial values are made. We note that for arbitrary selected initial conditions in the vicinity of the double equilibrium $F^0(1,1,2)$, the behavior of solutions of (7.5) for different values of ε remains identical. By using a computer we find the solutions of (7.5) for $\varepsilon = 0, .05, .10$, and $.15$ with the initial values $N_1(0) = 1.0$, $N_2(0) = 1.1$, and $N_3(0) = 2.1$ within the time range 0 to 10 which is divided into 50 equal intervals; and then draw the diagrams for these solutions.

Table 1 (see Page 110) represents the solutions for the system of equations (7.5) when $\varepsilon = 0$. Here we find that all three populations oscillate in the neighborhood of the equilibrium F^0 . Figure 15 represents the diagram for Table 1. In this case the orbit is so near to a closed curve that on Figure 15 it looks like closed.

Tables 2, 3, and 4 (see Pages 111, and 112) represent the solutions for the system (7.5) when $\varepsilon = .05, .10$, and $.15$ respectively. From these tables we notice that as the value of the small parameter ε increases, the populations N_2 and N_3 exhibit increased oscillations while the population N_1 goes away from F^0 . Figures 16, 17, and 18 which portray the graphs for Tables 2, 3, and 4 respectively represent unstable pointed spirals. These figures exhibit that the solutions for N_2 and N_3 are almost periodic while N_1 spirals away from F^0 .

On comparison, it is found that the behavior of the solutions obtained numerically is consistent with the behavior of the solution obtained qualitatively close to the equilibrium F^{*2} (which is an unstable pointed spiral) of Example 2 in Section 7.1.

CONCLUSION

In this thesis we have considered a system of three autonomous nonlinear ordinary differential equations of the Kolmogorov-type involving small perturbation. Sufficient conditions for the existence of simple perturbed equilibria in the neighborhood of the simple and multiple unperturbed equilibria of the three dimensional Kolmogorov model have been derived. The nature and stability of the simple unperturbed as well as the perturbed equilibria have been investigated qualitatively in the three dimensional phase space. We have also examined the bifurcations of a multiple unperturbed equilibrium into simple perturbed equilibria. In order to illustrate the theory, the qualitative behaviors of the equilibria of some unperturbed three dimensional population models have been compared to those of the perturbed models. It has been shown that small perturbation has an stabilizing or a destabilizing influence in the noncritical case and a branching effect in the critical case.

A model in population dynamics, like many physical and engineering models, represents an idealization and simplification of a real situation. Besides, it involves parameters which cannot be measured exactly. All this gives greater credibility to models with certain qualitative properties that do not change under the influence of small perturbations. One such important property (perhaps the most important) is the hyperbolic nature of an equilibrium point.

According to Theorems 2-4 it is enough to establish that the unperturbed system (4.1) has a hyperbolic equilibrium E^0. It will persist under the influence of small perturbations. Hence, we do not need to study the nature of the equilibrium E^* of the perturbed system (5.1). Example 1 in Section 5.3 (Hausrath [33]) is an excellent illustration in support of the usefullness of Theorem 3. Further, in Theorem 5, conditions have been presented under which a simple non-hyperbolic equilibrium E^0 of the unperturbed model generates a hyperbolic equilibrium E^* of the perturbed model. E^0 and E^* may or may not have the same stability or instability property. This theorem has not only mathematical interest, its physical and biological interpretation is that a system having a nonhyperbolic equilibrium is too fragile and its relevance as a suitable model of a real situation may be questioned.

Theorem 6 gives the criteria for the existence of simple equilibria F^* of the perturbed system (6.1) in the neighborhood of a multiple equilibrium F^0 of the unperturbed system (6.2). The multiple equilibrium F^0 is always unstable and does not persist under small perturbation. Hence, we need to examine the nature and stability of the perturbed equilibria F^*. In Theorem it has been shown that the number of simple perturbed equilibria F^* depends on the number of multiplicity of the unperturbed equilibrium F^0 and either all the perturbed equilibria exist in the neighborhood of the unperturbed equilibrium or one of the perturbed equilibria coincides with the unperturbed equilibrium and the rest lie in the neighborhood of the unperturbed equilibrium. Sufficient conditions under which a multiple equilibrium of the unperturbed system generates A-type simple equilibria of the perturbed system and under which the stability property of the multiple equilibrium changes are presented in Theorem 8.

A nonlinear study of the perturbed three dimensional Kolmogorov model reveals the existence of asymptotically stable equilibrium solutions along with unstable equilibrium solutions and bifurcation solutions. Biologically, the result is interesting as a description of the complexities that nonlinearities can introduce even into the simplest equations of population dynamics. Mathematically, the model illustrates some arbitrary dynamical behaviors for three dimensional nonlinear systems.

The present work allows further investigations. One immediate extension of the work is the study of the existence and bifurcation of periodic solutions in the three dimensional Kolmogorov model with small perturbation. Further, we note that we have investigated the time-independent three dimensional Kolmogorov model with small perturbation. Hence, a parallel problem dealing with the qualitative analysis of the time-dependent three dimensional Kolmogorov model with or without perturbation can be explored. The solution of the problem is expected to be more complicated than that presented here.

Some sections of the work presented in this thesis have been summarized in journal articles [13-14].

FIGURES

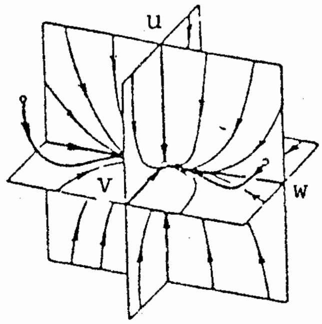

Figure 1

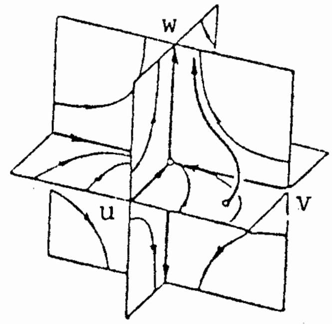

Figure 2

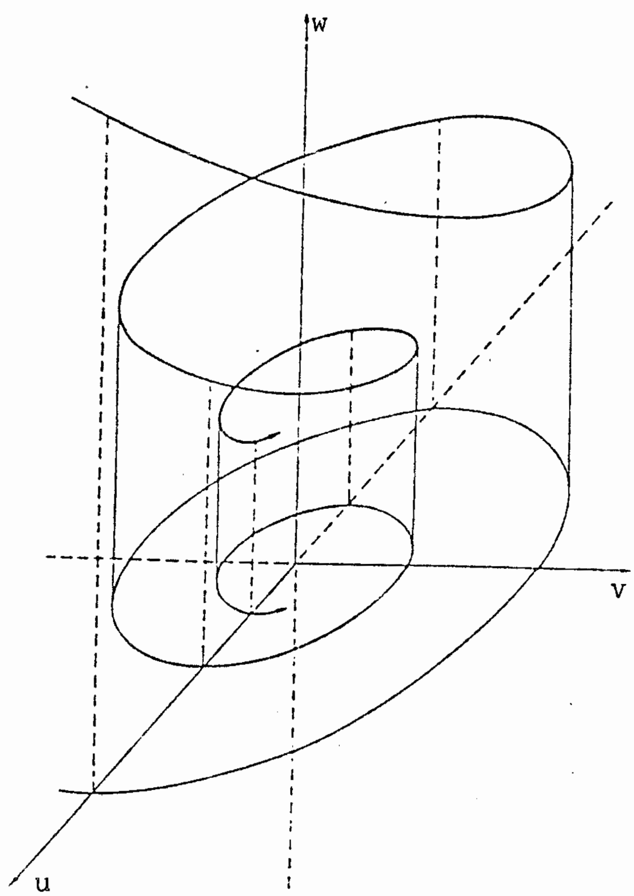

Figure 3

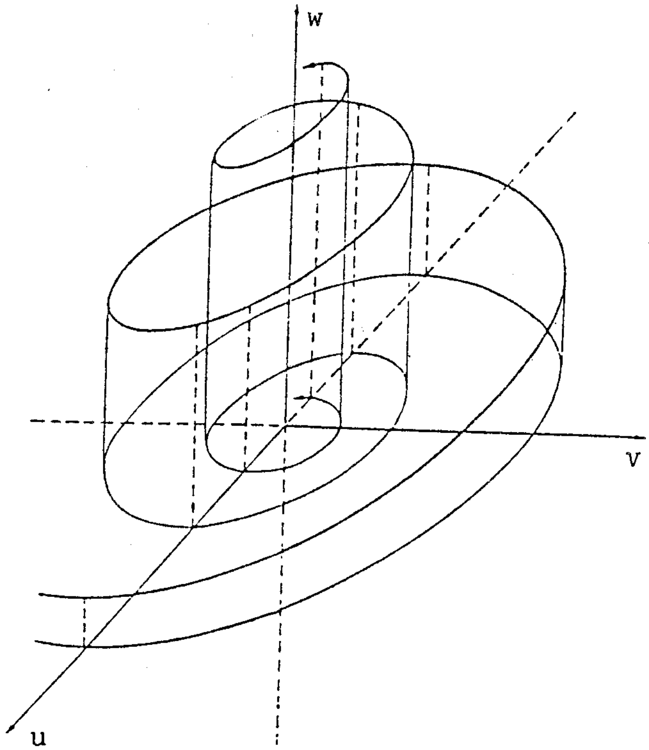

Figure 4

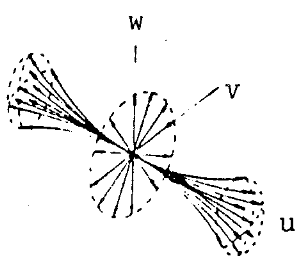

Figure 5

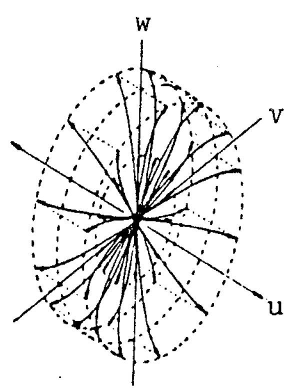

Figure 6

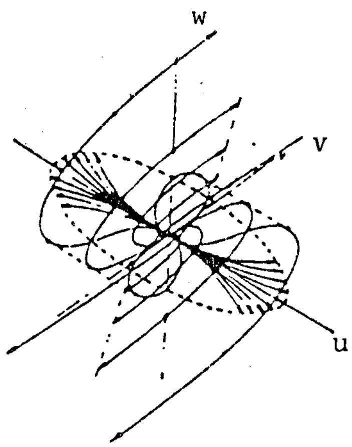

Figure 7

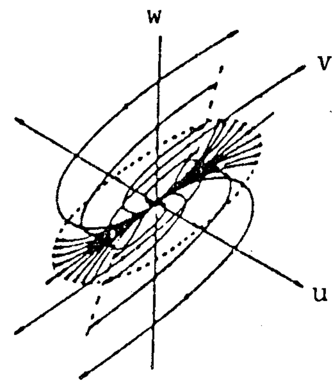

Figure 8

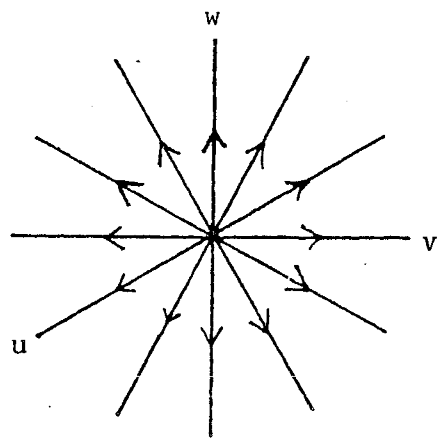

Figure 11

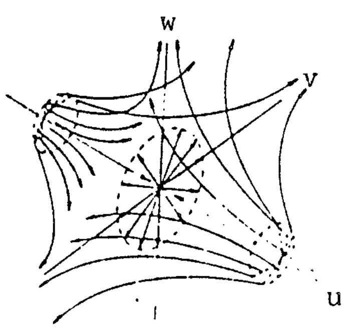

Figure 9

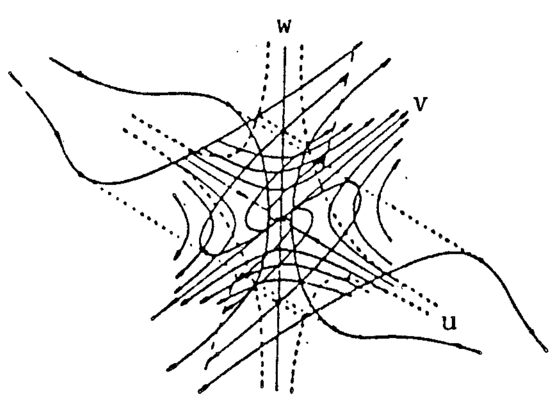

Figure 10

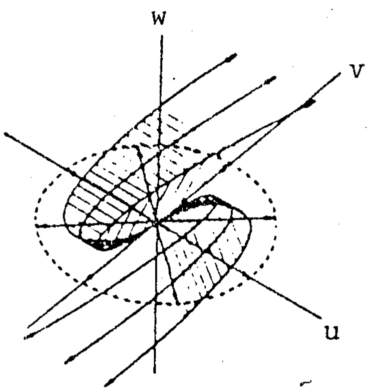

Figure 12

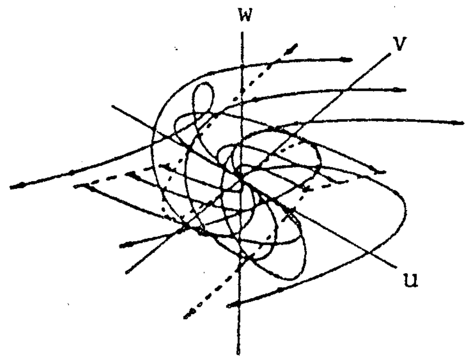

Figure 13

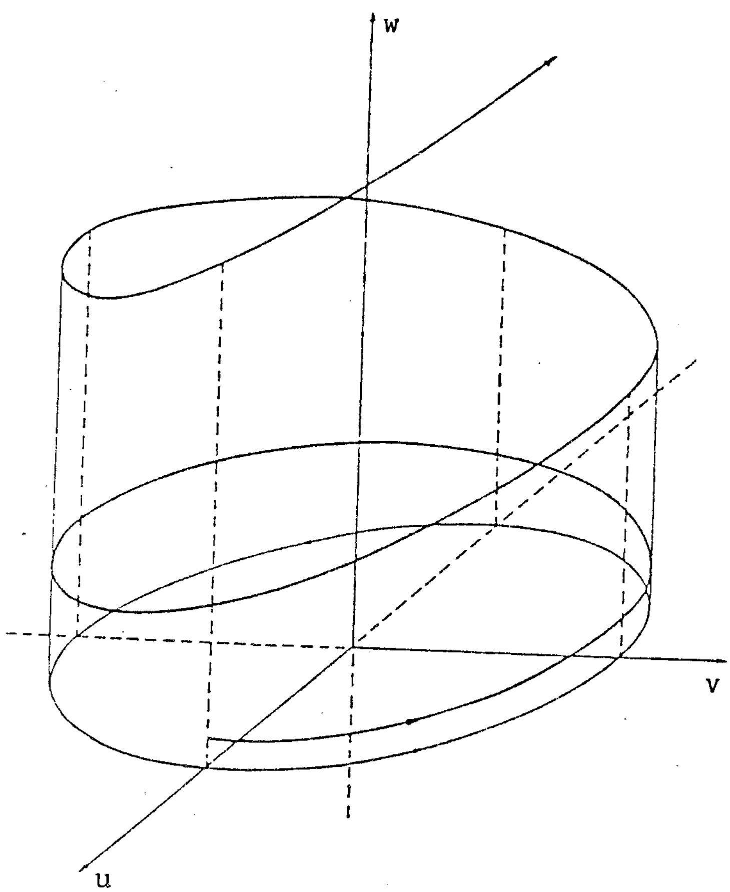

Figure 14

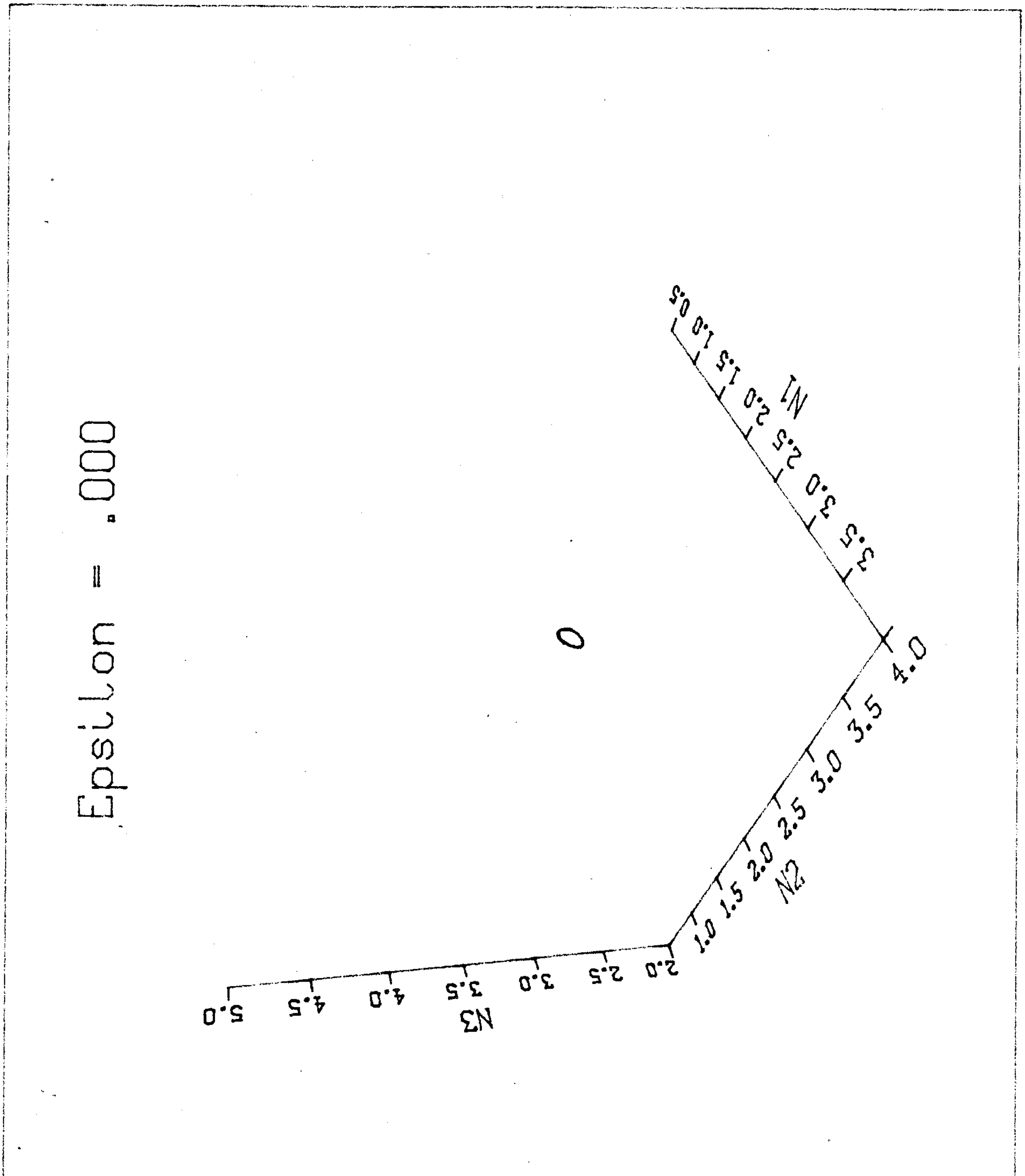
Epsilon = .000
0
N1
0.5
1.0
1.5
2.0
2.5
3.0
3.5
N2
1.0
1.5
2.0
2.5
3.0
3.5
4.0
N3
2.0
2.5
3.0
3.5
4.0
4.5
5.0

Figure 15

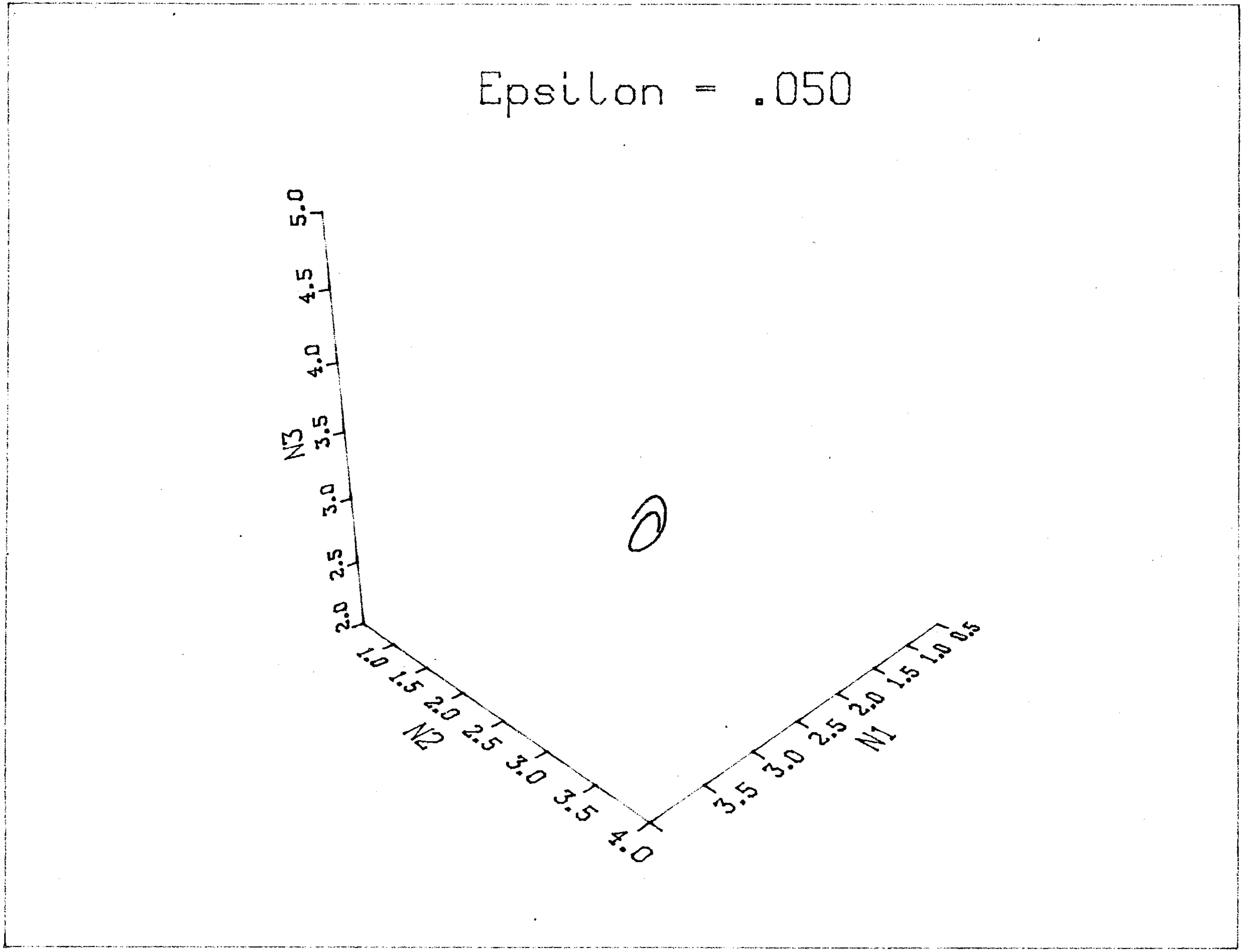
Epsilon = .050
N1
0.5 1.0 1.5 2.0 2.5 3.0 3.5
N2
1.0 1.5 2.0 2.5 3.0 3.5 4.0
N3
2.0 2.5 3.0 3.5 4.0 4.5 5.0

Figure 16

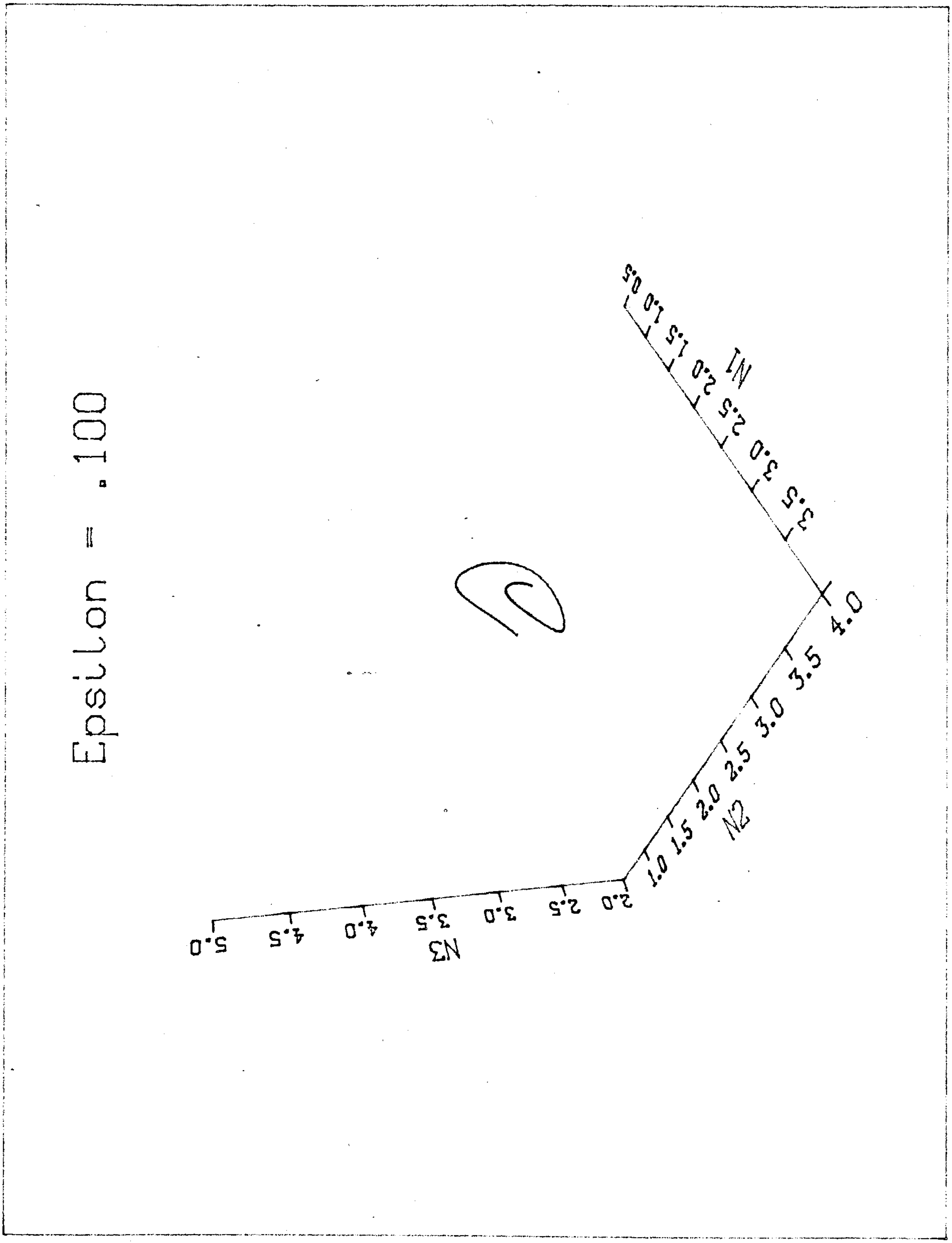
Epsilon = .100
N1
0.5 1.0 1.5 2.0 2.5 3.0 3.5
N2
1.0 1.5 2.0 2.5 3.0 3.5 4.0
N3
2.0 2.5 3.0 3.5 4.0 4.5 5.0

Figure 17

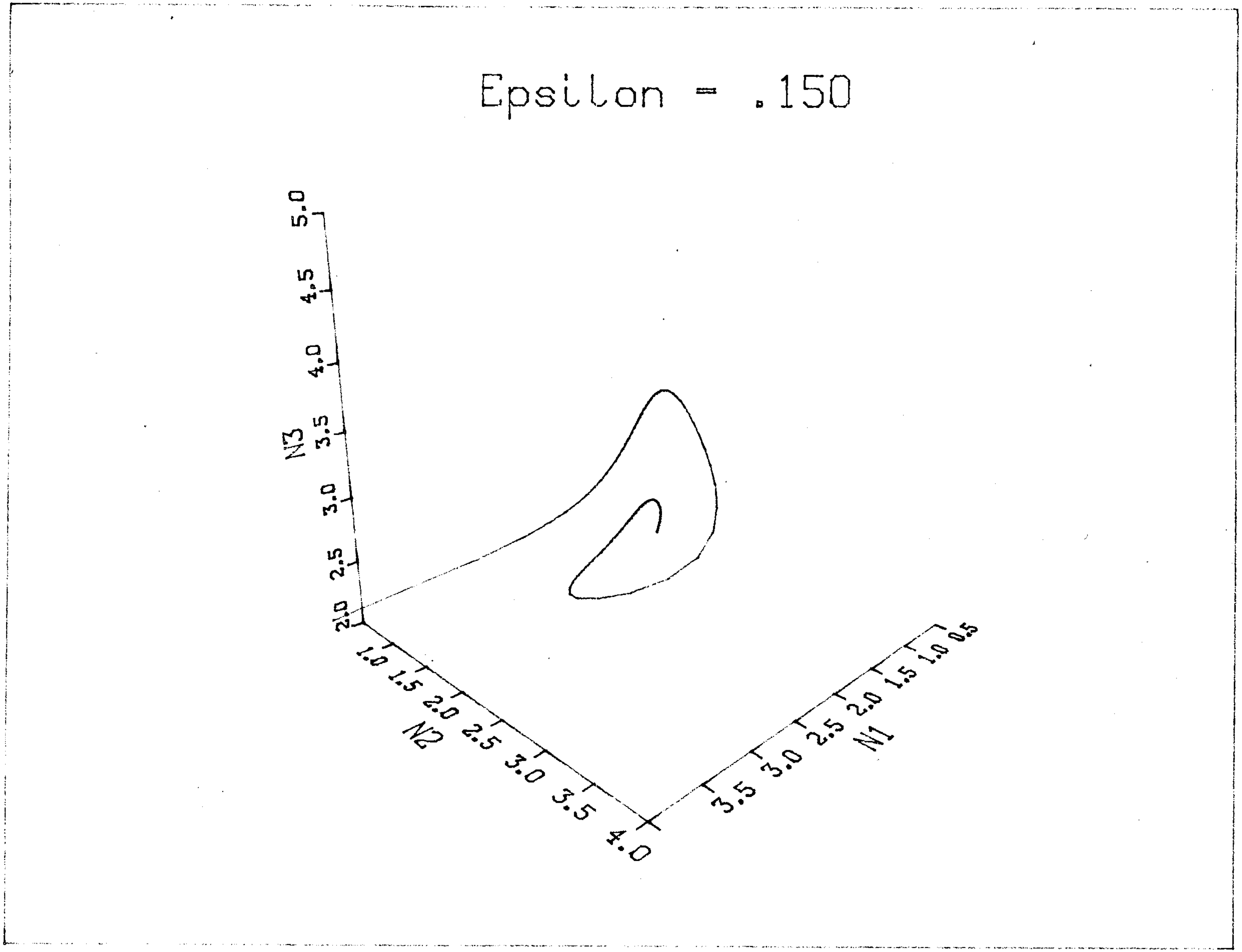
Epsilon = .150
N3
5.0
4.5
4.0
3.5
3.0
2.5
2.0
N2
1.0
1.5
2.0
2.5
3.0
3.5
4.0
N1
0.5
1.0
1.5
2.0
2.5
3.0
3.5

Figure 18

t	N_1	N_2	N_3
EPS= 0.0			
0.0	1.00000	1.10000	2.10000
0.20	0.96581	1.07432	2.13759
0.40	0.94589	1.03571	2.16151
0.60	0.94119	0.99449	2.16798
0.80	0.95049	0.95805	2.15744
1.00	0.97165	0.93091	2.13323
1.20	1.00196	0.91586	2.10033
1.40	1.03779	0.91476	2.06454
1.60	1.07390	0.92892	2.03199
1.80	1.10291	0.95848	2.00874
2.00	1.11610	1.00073	2.00023
2.20	1.10694	1.04779	2.01000
2.40	1.07623	1.08641	2.03769
2.60	1.03369	1.10341	2.07765
2.80	0.99239	1.09403	2.11996
3.00	0.96184	1.06407	2.15430
3.20	0.94608	1.02419	2.17357
3.40	0.94530	0.98399	2.17522
3.60	0.95798	0.95000	2.16060
3.80	0.98183	0.92617	2.13360
4.00	1.01393	0.91495	2.09949
4.20	1.05025	0.91809	2.06421
4.40	1.08496	0.93662	2.03391
4.60	1.11015	0.97005	2.01456
4.80	1.11728	1.01437	2.01118
5.00	1.10151	1.05999	2.02637
5.20	1.06639	1.09298	2.05828
5.40	1.02331	1.10203	2.09974
5.60	0.98474	1.08577	2.14035
5.80	0.95852	1.05206	2.17049
6.00	0.94735	1.01161	2.18442
6.20	0.95078	0.97309	2.18089
6.40	0.96701	0.94214	2.16213
6.60	0.99357	0.92215	2.13256
6.80	1.02726	0.91534	2.09769
7.00	1.06354	0.92322	2.06357
7.20	1.09592	0.94648	2.03634
7.40	1.11605	0.98370	2.02176
7.60	1.11608	1.02925	2.02426
7.80	1.09350	1.07182	2.04517
8.00	1.05479	1.09754	2.08089
8.20	1.01237	1.09793	2.12289
8.40	0.97752	1.07509	2.16063
8.60	0.95627	1.03845	2.18561
8.80	0.95007	0.99826	2.19362
9.00	0.95791	0.96213	2.18470
9.20	0.97776	0.93485	2.16192
9.40	1.00692	0.91930	2.13017
9.60	1.04182	0.91744	2.09517
9.80	1.07729	0.93056	2.06304
10.00	1.10613	0.95875	2.03983
IFAIL=0			

Table 1

t	N_1	N_2	N_3
EPS= 0.05000			
0.0	1.00000	1.10000	2.10000
0.20	0.94223	1.06641	2.15971
0.40	0.90630	1.01045	2.19958
0.60	0.89407	0.94966	2.21232
0.80	0.90381	0.89536	2.19809
1.00	0.93341	0.85351	2.16185
1.20	0.98114	0.82757	2.11103
1.40	1.04507	0.82072	2.05418
1.60	1.12090	0.83741	2.00074
1.80	1.19728	0.88399	1.96144
2.00	1.24883	0.96625	1.94900
2.20	1.23780	1.07708	1.97703
2.40	1.14990	1.17454	2.05203
2.60	1.02928	1.19898	2.15898
2.80	0.93104	1.14331	2.26293
3.00	0.87399	1.05015	2.33319
3.20	0.85439	0.95606	2.35776
3.40	0.86528	0.87693	2.33930
3.60	0.90234	0.81741	2.28737
3.80	0.96431	0.77890	2.21363
4.00	1.05214	0.76358	2.13007
4.20	1.16706	0.77715	2.04898
4.40	1.30393	0.83237	1.98466
4.60	1.42861	0.95309	1.95719
4.80	1.43800	1.15839	1.99750
5.00	1.25067	1.35722	2.13418
5.20	1.02074	1.35689	2.33282
5.40	0.88083	1.20596	2.50259
5.60	0.82109	1.04174	2.59329
5.80	0.81287	0.91004	2.60444
6.00	0.84117	0.81326	2.55384
6.20	0.90129	0.74639	2.46176
6.40	0.99562	0.70638	2.34656
6.60	1.13349	0.69503	2.22453
6.80	1.33333	0.72343	2.11193
7.00	1.61952	0.82716	2.03094
7.20	1.92265	1.12135	2.02971
7.40	1.63740	1.71012	2.23485
7.60	1.06434	1.66688	2.64978
7.80	0.84763	1.28370	2.95379
8.00	0.78657	1.01774	3.07284
8.20	0.79281	0.84905	3.05664
8.40	0.84229	0.73946	2.95440
8.60	0.93257	0.66920	2.80279
8.80	1.07539	0.63077	2.62890
9.00	1.30542	0.62768	2.45461
9.20	1.72740	0.68766	2.30347
9.40	2.84600	0.99387	2.22923
9.60	1.87237	3.52938	3.00021
9.80	0.93259	1.49180	3.90261
10.00	0.83547	1.01073	4.12229
IFAIL=0			

Table 2

t	N_1	N_2	N_3
EPS= 0.10000			
0.0	1.00000	1.10000	2.10000
0.20	0.91872	1.05824	2.18188
0.40	0.86741	0.98526	2.23732
0.60	0.84826	0.90670	2.25539
0.80	0.85851	0.83725	2.23650
1.00	0.89629	0.78336	2.18773
1.20	0.96233	0.74841	2.11908
1.40	1.06016	0.73644	2.04181
1.60	1.19494	0.75586	1.96854
1.80	1.36527	0.82565	1.91565
2.00	1.52262	0.98545	1.91005
2.20	1.48142	1.26499	1.99980
2.40	1.16762	1.45288	2.21998
2.60	0.90544	1.32576	2.47785
2.80	0.78265	1.10472	2.64999
3.00	0.74315	0.92276	2.70907
3.20	0.75373	0.79045	2.67578
3.40	0.80337	0.69693	2.57857
3.60	0.89415	0.63378	2.44270
3.80	1.04180	0.59828	2.28898
4.00	1.29094	0.59662	2.13567
4.20	1.79444	0.66279	2.00437
4.40	3.87917	1.12347	1.95936
IFAIL=2			

Table 3

t	N_1	N_2	N_3
EPS= 0.15000			
0.0	1.00000	1.10000	2.10000
0.20	0.89531	1.04983	2.20410
0.40	0.82933	0.96022	2.27469
0.60	0.80386	0.86556	2.29718
0.80	0.81457	0.78328	2.27272
1.00	0.86002	0.71958	2.21095
1.20	0.94465	0.67725	2.12468
1.40	1.08209	0.66086	2.02775
1.60	1.30392	0.68367	1.93588
1.80	1.68097	0.79050	1.87307
2.00	2.25290	1.19208	1.90588
2.20	1.54241	2.16062	2.31409
2.40	0.82887	1.56869	2.94474
2.60	0.66492	1.08023	3.24951
2.80	0.63260	0.81688	3.29988
3.00	0.65856	0.65744	3.19651
3.20	0.73078	0.55308	3.00557
3.40	0.86129	0.48367	2.77077
3.60	1.09840	0.44244	2.52208
3.80	1.64820	0.43765	2.28262
IFAIL=2			

Table 4

BIBLIOGRAPHY

[1] Albrecht, F., Gatzke, H., Haddad, A., and Wax, N., The dynamics of two interacting populations, *J. Math. Anal. Appl.* 46: 658-670 (1974).

[2] Andronov, A.A. and Leontovich, F.A., Sur la theorie de la variation de la structure qualitative de la division du plan en trajectories, *Dokl. Akad. Nauk SSSR* 21: 427-430 (1938).

[3] Andronov, A.A., Leontovich, F.A., Gordon, I.I., and Maier, A.G., *Qualitative Theory of Second Order Dynamic Systems*, Wiley, New York, 1973.

[4] Andronov, A.A., Leontovich, F.A., Gordon, I.I., and Maier, A.G., *Theory of Bifurcations of Dynamical Systems on a Plane*, Wiley, New York, 1973.

[5] Andronov, A.A. and Pontryagin, L., Systemes grossiers, *Dokl. Akad. Nauk SSSR* 14: 247-251 (1937).

[6] Andronov, A.A., Vitt, E.A., and Khaikin, S.E., *Theory of Oscillations*, Pergamon Press, New York, 1966.

[7] Arnold, V. I., *Ordinary Differential Equations*, The MIT Press, Cambridge, Massachusetts, 1973.

[8] Arnold, V. I., *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1982.

[9] Ayala, F., Competition between species: frequency dependence, *Science* 171: 820-824 (1971).

[10] Birkhoff, G.D., *Dynamical systems*, American Mathematical Society Publications, Providence, 1927.

[11] Bojadziev, G.N., Perturbed higher order population models, in *Proceedings of the Fifth ICMM "Mathematical Modelling in Science and Technology"*, Berkeley, July 29-31, 1985.

[12] Bojadziev, G.N. and Bojadziev, M., Lotka-Volterra model controlled by small perturbations, in *Proceedings of the International AMSE Conference "Modelling and Simulations"*, Athens, June 27-29, 1984, Vol. 1-2, pp. 29-40.

[13] Bojadziev, G.N. and Sattar, M.A., Perturbations in the three dimensional Kolmogorov model, *Math. Biosci.* 78: 293-305 (1986).

[14] Bojadziev, G.N. and Sattar, M.A., Bifurcations in the three dimensional Kolmogorov model, *Math. Biosci.* (submitted).

[15] Brauer, F., A nonlinear predator-prey problem, in *Ordinary Differential Equations* (L. Weiss, Ed.), Academic Press, New York, 1972.

[16] Brauer, F., On the populations of competing species, *Math. Biosci.* 19: 299-306 (1974).

[17] Brauer, F. and Nohel, J.A., *The Qualitative Theory of Ordinary Differential Equations*, Benjamin INC., New York, 1969.

[18] Bulmer, M.G., The theory of prey-predator oscillations, *Theor. Pop. Biol.* 9: 137-150.

[19] Butler, G.J. and Freedman, H.I., Periodic solutions of a predator-prey system with periodic coefficients, *Math. Biosci* 55: 27-38 (1981).

[20] Chow, S.N. and Hale, J.K., *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.

[21] Coddington, E.A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[22] Cronin, J., *Differential Equations: Introduction and Qualitative Theory*, Marcel Dekker, New York, 1980.

[23] Freedman, H.I., The implicit function theorem in the scalar case, *Can. Math. Bull.* 12: 721-732 (1969).

[24] Freedman, H.I., A perturbed Kolmogorov-type model for the growth problem, *Math. Biosci.* 23: 127-149 (1975).

[25] Freedman, H.I., *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

[26] Freedman, H.I. and Waltman, P., Perturbation of two dimensional predator-prey equations, *SIAM J. Appl. Math.* 28: 1-10 (1975).

[27] Freedman, H.I. and Waltman, P., Perturbation of two dimensional predator-prey equations with an unperturbed critical point, *SIAM J. Appl. Math.* 29: 719-733 (1975).

[28] Freedman, H.I. and Waltman, P., Persistence in models of three interacting predator-prey populations, *Math. Biosci.* 68: 213-231 (1984).

[29] Friedrichs, K.O., *Advanced Ordinary Differential Equations*, New York University Lecture Notes, Sec. IV-6, 1949.

[30] Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[31] Hagedorn, P., *Nonlinear Oscillations*, Clarendon Press, Oxford, 1981.

[32] Hastings, A., Global stability in two species systems, *J. Math. Biol.* 5: 399-403 (1978).

[33] Hausrath, A.R., Stability properties of a class of differential equations modeling predator-prey relationships, *Math. Biosci.* 26: 267-281 (1975).

[34] Hayashi, C., *Nonlinear Oscillations in Physical Systems*, McGraw-Hill, New York, 1964.

[35] Hirsch, M.W. and Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[36] Iooss, G. and Joseph, D.D., *Elementary Stability and Bifurcation Theory*, Springer-Verlag, Berlin, 1981.

[37] Kolmogorov, A., Sulla teoria di Volterra della lotta per l'existenza, *Gior. Inst. Ital. Attuari* 7: 74-80 (1936).

[38] Lefschetz, S., *Ordinary Differential Equations: Geometric Theory*, Interscience, New York, 1977.

[39] Liapunov, A.M., *Sur les figures d'equilibre peu differentes des ellipsodies d'une masse liquide homogene donnee d'un movement de rotation*, Zap. Akad. Nauk. St. Petersburg, 1906.

[40] Liapunov, A.M., *Probleme general de la stabilite du movement*, Princeton University Press, Princeton, 1949.

[41] Loud, W.S., Some singular cases of the implicit function theorem, *Amer. Math. Monthly* 68: 965-977 (1961).

[42] Marsden, J.E., Qualitative methods in bifurcation theory, *Bull. Amer. Math. Soc.* 84: 1125-1148 (1979).

[43] May, R.M., Limit cycles in predator-prey communities, *Science* 177: 900-902 (1972).

[44] May, R.M., *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, 1973.

[45] Nemytskii, V.V. and Stepanov, V.V., *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, 1960.

[46] Peixoto, M., On structural stability, *Ann. of Math.* 69: 189-222 (1959).

[47] Peixoto, M., Structural stability on two dimensional manifolds, *Topology* 1: 101-120 (1962).

[48] Poincaré, H., Sur l'equilibre d'une masse fluids animes d'un mouvement de rotation, *Acta Math.* 7: 259-380 (1885).

[49] Poincaré, H., Sur les equations de la dynamique et le probleme de trois corps, *Acta Math.* 13: 1-270 (1890).

[50] Poincaré, H., *Les methodes nouvelles de la mecanique celeste,* Vol. 3, Gauthier-Villars, Paris, 1892.

[51] Reissig, R., Sansone, G., and Conti, R., *Nonlinear Differential Equations of Higher Order,* Noordhoff International Publishing, Leyden, The Netherlands, 1974.

[52] Rescigno, A., The struggle for life II: Three Competitors, *Bull. Math. Biophys.* 30: 291-297 (1968).

[53] Rescigno, A., The struggle for life IV: Two predators sharing a prey, *Bull. Math. Biol.* 39: 179-185 (1977).

[54] Rescigno, A., The struggle for life V: One species living in a limited environment, *Bull. Math. Biol.* 39: 479-485 (1977).

[55] Rescigno, A. and Jones, K.G., The struggle for life III: A predator-prey chain, *Bull. Math. Biophys.* 34: 521-532 (1972).

[56] Rescigno, A. and Richardson, I.W., The struggle for life I: Two species, *Bull. Math. Biophys.*29: 377-388 (1967).

[57] Reyn, J.W., Classification and description of singular points of a system of three linear differential equations, *Z. Agnew, Math. Phys.* 15: 540-557 (1964).

[58] Samuelson, P.A., Generalized predator-prey oscillations in ecological and economic equilibrium, *Proc. Nat. Acad. Sci. U.S.A.,* 68: 980-983 (1971).

[59] Utz, W.R. and Waltman, P.E., Periodicity and boundedness of solutions of generalized differential equations of growth, *Bull. Math. Biophys.* 25: 75-93 (1963).

[60] Waltman, P.E., The equations of growth, *Bull. Math. Biophys.* 26: 39-43 (1964).