LONGEST CYCLES IN GRAPHS

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ABSTRACT

There are four main parts in this thesis.

The first part contains a proof of the following result. If G is a k-connected claw-free ($K_{1,3}$-free) graph of order n such that

$$\sum_{v \in I} d(v) \geq n-k.$$

for any (k+1)-independent set I, then G contains a Hamilton cycle.

The second part deals with C. Thomassen's conjecture that any longest cycle of a 3-connected graph has a chord. We'll show that the conjecture is true for a planar graph if it is cubic or $\delta \geq 4$. We also show that if there is a minimum counterexample, then the subgraph outside of a chordless longest cycle is an independent set.

The third part is concerned with bridges of longest cycles in 3-connected non-hamiltonian graphs. Let G be such a graph and let

$$d(u)+d(v) \geq m$$

for each pair of non-adjacent vertices u and v. Let the length of its longest cycle C be r. Then the length of any bridge of C is at most r-m+2.

The final part presents a survey of results about longest directed cycles in digraphs. Since their proofs have been published elsewhere, they are omitted here.

To my grandmother

—— the first mathematics teacher

in my life

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PART A

INTRODUCTION

The subject of paths and cycles in graphs is fundamental to the study of graph theory. It is no surprise that there is a vast literature on the topic. Besides the thousands of papers dealing with the subject, there have been two conferences devoted to cycles in graphs, a book on cycles in graphs [2] has recently appeared and a book on Euler tours by H. Fleischner is forthcoming.

This thesis is concerned with cycles in graphs and directed cycles in digraphs. There are four sections but there is a unifying theme of looking at problems dealing with longest cycles in graphs. Of course, the longest cycle of a graph can be a Hamilton cycle, that is, a cycle containing every vertex of the graph. Some natural questions to ask are whether or not a given graph has a Hamilton cycle, whether or not the graphs in a family all have a Hamilton cycle, and what kinds of conditions must be imposed on a given family of graphs in order to guarantee that thay all have a Hamilton cycle.

If a graph does not have a Hamilton cycle, there are several directions that one can take in the investigation of such graphs. For example, what kinds of conditions can be imposed in order to guarantee that a graph does not have a Hamilton cycle? Another approach is to investigate properties of the longest cycles in non-hamiltonian graphs. This is the approach taken in this thesis.

One of the types of conditions that have been imposed on graphs to guarantee a Hamilton cycle have involved degrees of vertices. There are several classical results in this direction. A recent theorem of Mathews and Sumner [12] involves the minimum degree of the graph and the structural property that the graph has no induced $K_{1,3}$ subgraph. The latter condition simply says that no neighborhood of a vertex can have an independent set of size three. Their result is generalized in Part B of the thesis.

C. Thomassen has recently conjectured [2, p466] that every longest cycle in a 3-connected graph has a chord. In Part C, it is proved to be true for a planar graph G if either G is cubic or $\delta(G)\geq 4$. It is also shown that the subgraph outside of a chordless longest cycle in a minimum counterexample to the general conjecture must be an independent set. These results make the question more interesting for the class of 3-connected planar graphs.

Thomassen's conjecture in the previous part is saying that longest cycles in a 3-connected graph must have short bridges because a chord of a cycle is the shortest bridge a cycle can have. In Part D, we take the opposite viewpoint by asking how long might a bridge of a longest cycle in a 3-connected graph be. The theorem of Part D is given for non-hamiltonian graphs because the question is not interesting if the graph has a Hamilton cycle. The theorem establishes an upper bound on the length of a bridge.

The basic terminology and notation of this thesis may be found in [5].

PART B

HAMILTON CYCLES IN CLAW-FREE GRAPHS

A graph is called claw-free if G has no induced $K_{1,3}$ subgraph. Mathews and Sumner [12] showed that *if G is a claw-free 2-connected graph of order n with minimum degree δ such that* $3\delta\geq n-2$, *then G contains a Hamilton cycle.* In this chapter, we will give a result about Hamilton cycles in k-connected claw-free graph which generalizes the Mathews-Sumner Theorem.

THEOREM 1.1.

Let G be a k-connected claw-free graph of order n such that

$$\sum_{v\in I} d(v)\geq n-k,$$

for any (k+1)-independent set I. Then G contains a Hamilton cycle.

Bondy [6] conjectured that *if* G *is a* k*-connected graph of order* n *such that*

$$\sum_{v\in I} d(v)\geq n+k(k-1)$$

for any (k+1)*-independent set* I *of* G, *then the subgraph outside any longest cycle contains no path of length* k-1. The theorem in this chapter implies the conjecture in the case of claw-free graphs.

In this chapter, let

$$N_D(v)=\{u\in V(D)\ |(v,u)\in E(G)\},$$

where D is a subgraph of G. If V(D)=V(G), we simply write N(v) instead of $N_D(v)$. If $C=x_1\ldots x_r x_1$ is a cycle, $x_i C x_j$ denotes the interval $x_i x_{i+1}\ldots x_{j-1}x_j$ of C and $x_j\overline{C}x_i$ denotes the interval $x_j x_{j-1}\ldots x_{i+1}x_i$ of C.

PROOF OF THE THEOREM.

Let $G=(V,E)$ be a graph satisfying the conditions given in the theorem. Let $C=v_1\ldots v_r v_1$ be a longest cycle of G. Assume that C is not a Hamilton cycle. Let B be a component of $G\backslash V(C)$.

By k-connectivity, there are h edges joining B and C, $h\geq k$. Notice that $h\geq |N_C(x)|$ for any $x\epsilon V(B)$. Let these edges be $\{(x_i,v_{j_i})\ |i=1,\ldots,h\}$, where $x_i\epsilon V(B)$ and $v_{j_i}\epsilon V(C)$, for $i=1,\ldots,h$ and $1\leq j_1<j_2<\ldots<j_h\leq r$. Let x_iBx_j denote a path of B joining x_i and x_j.

<u>I.</u> Let's define some special sets on C by the following algorithm.

ALG(v_{j_μ}).

1. $w_1\ldots w_r \longleftarrow v_{j_\mu -1}\overline{C}v_{j_\mu}$,

   $S_\mu \longleftarrow \emptyset$,

   go to 2;

2. If there is an integer i such that $w_i,w_{i+1}\epsilon N(w_1)$, choose i as big as possible. The pair $\{w_i,w_{i+1}\}$ is called the <u>insertion</u> <u>pair</u> for w_1.

   $S_\mu \longleftarrow S_\mu \cup \{w_1\}$,

   $w_1\ldots w_r \longleftarrow w_2\ldots w_i w_1 w_{i+1}\ldots w_r$,

   go to 3;

3. Repeat 2 until either $w_{i\pm 1}\notin N(w_1)$ for all $w_i\epsilon N(w_1)$ or $w_1\epsilon S_\mu$ already;

   If $w_1\notin S_\mu$, then $S_\mu \longleftarrow S_\mu \cup \{w_1\}$;

   go to 4;

4. Stop.

Use this $Alg(v_{j_\mu})$ for each $\mu=1,\dots,h$ obtaining $S_1,\dots,S_h$.

Let
$$\bigcup_{v\in S_i} N_C(v)=N_i.$$

<u>II. Proposition 1.</u>

<u>We have</u>
$$S_1\subsetneq\{v_{j_h+1},\dots,v_{j_1-1}\},$$
$$S_t\subsetneq\{v_{j_{t-1}+1},\dots,v_{j_t-1}\} \qquad \underline{\text{for}}\ t=2,\dots,h$$
<u>and</u>
$$S_i\cap S_j=\emptyset, \qquad \underline{\text{if}}\ i\neq j.$$

<u>Proof</u>.

From $Alg(v_{j_\mu})$ it is obvious that S_μ is an interval on C for $\mu=1,\dots,h$. Now $v_{j_h}\notin S_1$, because otherwise, during the processing of $Alg(v_{j_1})$, we would have a path $w_1\dots w_r$ with $w_1=v_{j_h}$ and $w_r=v_{j_1}$. Then the cycle $w_1\dots w_r x_h B x_1 w_1$ would be longer than C. Hence, $S_1\subsetneq\{v_{j_h+1},\dots,v_{j_1-1}\}$. The other conclusions are similar.

<u>III.</u> Considering $S_1=\{v_{j_1-1},\dots,v_{j_1-|S_1|}\}$, there exists the least integer α_1 such that the insertion pair for $v_{j_1-\alpha_1}$ is not contained in $C\backslash[S_1\backslash\{v_{j_1-1},\dots,v_{j_1-\alpha_1}\}]$. When $Alg(v_{j_1})$ stops, either w_1 has no insertion pair or $w_1\in S_1$ already (that is, $w_1=v_{j_1-\alpha}$ has its insertion pair intersecting with $\{v_{j_1-\alpha-1},\dots,v_{j_1-|S_1|}\}$). Both cases guarantee the existence of $v_{j_1-\alpha_1}$. If $\alpha_1<|S_1|$, then $v_{j_1-\alpha_1}$ is the first vertex whose insertion pair intersects $\{v_{j_1-\alpha_1-1},\dots,v_{j_1-|S_1|}\}$ during the processing of $Alg(v_{j_1})$. Hence, <u>the insertion pair for</u> $v_{j_1-\alpha_1}$ <u>is contained in</u> $\{v_{j_1-\alpha_1-1},\dots,v_{j_1-|S_1|-1}\}$, <u>when</u> $\alpha_1<|S_1|$. <u>And</u> $v_{j_1-\alpha_1}$ <u>has no insertion pair, when</u> $\alpha_1=|S_1|$.

Similarly, considering $\mu=1,\ldots,h$, we will get $v_{j_\mu-\alpha_\mu}$.

The rest of the proof is going to show that

1. $I=\{x,v_{j_1-\alpha_1},\ldots,v_{j_h-\alpha_h}\}$ is an independent set, where $x\in B$; and
2. B, $N(v_{j_1-\alpha_1})$, ..., $N(v_{j_h-\alpha_h})$, $\{v_{j_1},\ldots,v_{j_h}\}$ and $\{v_{j_1-\alpha_1},\ldots,v_{j_h-\alpha_h}\}$ are disjoint sets.

If we can do so, I will be the independent set contradicting the hypotheses of the theorem.

IV. An operation defined in $\mathrm{Alg}(v_{j_\mu})$.

During the processing of $\mathrm{Alg}(v_{j_\mu})$, we produce an operation on some paths $P=w_1\ldots w_p$. Assume only one of $\{w_1,w_p\}$ is in S_μ (say, w_1) and let $\{w_i,w_{i+1}\}$ be the insertion pair for w_1. Define

$$Z_\mu(P)=w_2\ldots w_i w_1 w_{i+1}\ldots w_p.$$

(If $w_p\in S_\mu$, $Z_\mu(P)=w_1\ldots w_i w_p w_{i+1}\ldots w_{p-1}$.)

We can define the operation Z_μ on some paths $P=w_1\ldots w_p$ with respect to S_μ. Here, $Z_\mu(P)$ is well-defined only when $|\{w_1,w_p\}\cap S_\mu|=1$ and the insertion pair for the vertex in this intersection exists. When Z_μ is operating on P, the endvertex w_1 (or w_p) will be moved into its insertion pair. And $Z_\mu^\beta(P)$ denotes the compositions of the operation Z_μ on P (repeated β times).

V. We claim that $(v_{j_\mu},v_{j_\mu-\alpha_t})\notin E(G)$ and $\alpha_\mu>1$ for $\mu=1,\ldots,h$.

Without loss of generality, consider $\mu=1$. Let $w_1\ldots w_r=Z_1^{\alpha_1-1}(P)$ where $w_1=v_{j_1-\alpha_1}$, $w_r=v_{j_1}$. Assume that $(w_1,w_r)\in E(G)$. Then $w_1\ldots w_r w_1$ is also a longest cycle. Now

$\{x_1, w_1, w_{r-1}\} \subseteq N(w_r)$ and G being claw-free imply that $(w_1, w_{r-1}) \in E(G)$. So (w_{r-1}, w_r) is the insertion pair for w_1 which contradicts that the insertion pair for w_1 is not contained in $V(C)\backslash\{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|}\}$. Hence, we must have that $(w_1, w_r) = (v_{j_1}, v_{j_1-\alpha_1}) \notin E(G)$. Incidentally the same proof shows that $\alpha_1 > 1$ always holds.

VI. Let us consider $v_{j_1-\alpha_1}$ and $v_{j_\lambda-\alpha_\lambda}$ as an example. Let $P = v_{j_1-1}\bar{C}v_{j_1}$, $Q = v_{j_1-1}\bar{C}v_{j_\lambda}x_\lambda Bx_1 v_{j_1}Cv_{j_\lambda-1}$ (let $q = |Q|$).

Proposition 2.

Let $\alpha < |S_1|$, and $Z_1^\alpha(P) = w_1 \ldots w_r$.

1. $Z_1^\alpha(P)$ is well-defined;
2. $v_{j_\lambda-1} \notin N(w_1)$, and $v_{j_\lambda-1}$, v_{j_λ} are adjacent in $Z_1^\alpha(P)$;
3. $Z_1^\alpha(Q)$ is well-defined; and
4. let $z, z' \in N(w_1)$. Then z and z' are adjacent in $Z_1^\alpha(P)$ if and only if z and z' are adjacent in $Z_1^\alpha(Q)$.

Proof.

(1) is true for all α, $0 \le \alpha < |S_1|$.

(2) will be proved by induction on α. If $(v_{j_\lambda-1}, v_{j_1-1}) \in E(G)$, the cycle $v_{j_1-1}\bar{C}v_{j_\lambda}x_\lambda Bx_1 v_{j_1}Cv_{j_\lambda-1}v_{j_1-1}$ would be longer than C. So it is true for $\alpha=0$. Assume that it is true for $\alpha<\kappa$. Since $v_{j_\lambda-1}, v_{j_\lambda}$ are adjacent in $Z_1^{\kappa-1}(P)$, let $Z_1^{\kappa-1}(P) = u_1 \ldots u_r$ with $v_{j_\lambda} = u_i$, $v_{j_\lambda-1} = u_{i+1}$. Since $v_{j_\lambda-1} \notin N(u_1)$, the insertion pair for u_1 will not be $\{v_{j_\lambda-1}, v_{j_\lambda}\} = \{u_i, u_{i+1}\}$. Hence, $v_{j_\lambda-1}, v_{j_\lambda}$ are still adjacent in $Z_1^\kappa(P)$. If $v_{j_\lambda-1} \in N(w_1)$, then the cycle

$w_1 Z_1^{\kappa}(P) v_{j_\lambda} x_\lambda B x_1 w_r Z_1^{\kappa}(\bar{P}) v_{j_\lambda - 1} w_1$ would be longer than C.

By (2), the insertion pair for w_1 is always contained in either $\{w_1, \ldots, w_i\}$ or $\{w_{i+1}, \ldots, w_r\}$, where $v_{j_\lambda - 1} = w_{i+1}$ and $v_{j_\lambda} = w_i$. Hence,

$$Z_1^{\alpha}(Q) = w_1 Z_1^{\alpha}(P) w_i x_\lambda B x_1 w_r Z_1^{\alpha}(\bar{P}) w_{i+1}$$

and $Z_1^{\alpha}(Q)$ is well-defined. Let $z, z' \epsilon N(w_1)$. If z, z' are adjacent in $Z_1^{\alpha}(P)$, (respectively, in $Z_1^{\alpha}(Q)$), either $z, z' \epsilon \{w_1, \ldots, w_i\}$, or $z, z' \epsilon \{w_{i+2}, \ldots, w_r\}$. Hence, z, z' are adjacent in $Z_1^{\alpha}(Q)$ (respectively, in $Z_1^{\alpha}(P)$).

VII. Recall that $P = v_{j_1 - 1} \bar{C} v_{j_1}$ and $Q = v_{j_1 - 1} \bar{C} v_{j_\lambda} x_\lambda B x_1 v_{j_1} C v_{j_\lambda - 1}$ (let $q = |Q|$).

Proposition 3.

1. If $\alpha < |S_1|$ and $\beta < |S_\lambda|$, then $Z_1^{\alpha} Z_\lambda^{\beta}(Q)$ is well-defined.
2. Let $\{Z_{\mu_1}, \ldots, Z_{\mu_s}\}$ be a series of operations, where $\mu_1, \ldots, \mu_s \epsilon \{1, \lambda\}$. $Z_{\mu_s} Z_{\mu_{s-1}} \ldots Z_{\mu_1}(Q)$ is only dependent on the number of Z_1 and the number of Z_λ. In other words, any permutation of $\{\mu_1, \ldots, \mu_s\}$ would not make any difference in $Z_{\mu_s} \ldots Z_{\mu_1}(Q)$.
3. Let $Z_1^{\alpha}(P) = w_1 \ldots w_r$, $Z_1^{\alpha} Z_\lambda^{\beta}(Q) = u_1 \ldots u_q$ and $v, v' \epsilon N(w_1)$. Then v, v' are adjacent in $Z_1^{\alpha}(P)$ if and only if v, v' are adjacent in $Z_1^{\alpha} Z_\lambda^{\beta}(Q)$.

Proof.

We use induction on $\alpha + \beta$. When $\beta = 0$, (1) and (3) are true by Proposition 2, and (2) is true because $\beta = 0$. Symmetrically, the

Proposition is true when $\alpha=0$.

Assume that (1) and (2) are true for $\alpha+\beta<\kappa$, ($\kappa\geq 2$). Consider $\alpha+\beta=\kappa$, $\alpha<|S_1|$ and $\beta<|S_\lambda|$. We only need to show $Z_1^\alpha Z_\lambda^\beta(Q)$ is well-defined and

$$Z_1 Z_\lambda Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q)= Z_\lambda Z_1 Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q).$$

By the induction hypothesis, $Z_1^{\alpha-1}Z_\lambda^{\beta-1}(Q)$, $Z_\lambda Z_1^{\alpha-1}Z_\lambda^{\beta-1}(Q)$ and $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$ are well-defined. Let $Z_1^{\alpha-1}Z_\lambda^{\beta-1}(Q)=w_1\ldots w_q=Q^*$, $w_1\epsilon S_1$ and $w_q\epsilon S_\lambda$, let $\{w_a,w_{a+1}\}$ be the insertion pair for w_1 and let $\{w_b,w_{b+1}\}$ be the insertion pair for w_q , all of which exist because $Z_\lambda(Q^*)$ and $Z_1(Q^*)$ are well-defined. If $\{w_a,w_{a+1}\}=\{w_b,w_{b+1}\}$, we would get a cyle $w_1w_{a+1}Q^*w_qw_a\bar{Q}^*w_1$ longer than C. So $\{w_a,w_{a+1}\}\neq\{w_b,w_{b+1}\}$ and

$Z_1Z_\lambda(Q^*)=Z_\lambda Z_1(Q^*)=w_2\ldots w_aw_1w_{a+1}\ldots w_bw_qw_{b+1}\ldots w_{q-1}$ when $a<b$ or

$Z_1Z_\lambda(Q^*)=Z_\lambda Z_1(Q^*)=w_2\ldots w_bw_qw_{b+1}\ldots w_aw_1w_{a+1}\ldots w_{q-1}$ when $b<a$

and therefore (1) and (2) follow.

Assume that (3) is true for $\alpha+\beta<\kappa$ ($\kappa\geq 2$). Let us consider $\alpha+\beta=\kappa$. Let $v,v'\epsilon N(v_{j_1-\alpha-1})$. By the induction hypothesis, v,v' are adjacent in $Z_1^\alpha(P)$ if and only if v,v' are adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)=y_1\ldots y_q=Q^{**}$, (where $y_1=v_{j_1-\alpha-1}$). Suppose that v,v' are adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$. The insertion pair for y_q in Q^{**} is not $\{v,v'\}$. If so, let $\{v,v'\}=\{y_i,y_{i+1}\}$ and then the cycle $y_1Q^{**}y_iy_q\bar{Q}^{**}y_{i+1}y_1$ would be longer than C. Hence, v,v' are still adjacent in $Z_1^\alpha Z_\lambda^\beta(Q)$. Conversely, suppose that v,v' are adjacent in $Z_1^\alpha Z_\lambda^\beta(Q)$, but not in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)=Q^{**}$. Then $y_q\epsilon\{v,v'\}\subseteq N(y_1)$ which would give a cycle $y_1\ldots y_qy_1$ longer than C. So v,v' must be adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$.

VIII. We claim $B\cap N(v_{j_1-\alpha_1})=\emptyset$ and $(v_{j_1-\alpha_1},v_{j_\lambda-\alpha_\lambda})\notin E(G)$

If $x\in B\cap N(v_{j_1-\alpha_1})$, then the cycle $Z_1^{\alpha_1-1}(P)v_{j_1}x_1Bxv_{j_1-\alpha_1}$ would be longer than C. If $(v_{j_1-\alpha_1},v_{j_\lambda-\alpha_\lambda})\in E(G)$, then the cycle $Z_1^{\alpha_1-1}Z_\lambda^{\alpha_\lambda-1}(Q)v_{j_\lambda-\alpha_\lambda}v_{j_1-\alpha_1}$ would be longer than C. Hence, we have proved the first assertion suggested in III.

IX. We claim $N_1\cap S_\lambda=\emptyset$ and $N_\lambda\cap S_1=\emptyset$.

If not, let $v_{j_1-\alpha-1}\in N(v_{j_\lambda-\beta-1})\cap S_1\subseteq N_\lambda\cap S_1$, where $\alpha<|S_1|$ and $\beta<|S_\lambda|$. Then the cycle $Z_1^\alpha Z_\lambda^\beta(Q)v_{j_\lambda-\beta-1}v_{j_1-\alpha-1}$ would be longer than C. Hence, $v_{j_1-\alpha_1}\notin N_\lambda$ and $v_{j_\lambda-\alpha_\lambda}\notin N_1$, which is a part of the second assertion of III.

X. Let $Z_1^\alpha Z_\lambda^\beta(Q)=w_1\ldots w_q$, $(\alpha\leq\alpha_1-1,\ \beta\leq\alpha_\lambda-1)$. Then we claim $\{v_{j_1-\alpha-1},v_{j_1-\alpha-2},\ldots,v_{j_1-|S_1|-1}\}$ remains as an interval in $Z_1^\alpha Z_\lambda^\beta(Q)$ and

$$w_1=v_{j_1-\alpha-1},\ \ldots,\ w_{|S_1|-\alpha+1}=v_{j_1-|S_1|-1}.$$

By the choice of α_1, it is obviously true when $\beta=0$. We proceed by induction on β. Let $Q^*=Z_1^\alpha Z_\lambda^{\beta-1}(Q)=y_1\ldots y_q$. Since $N_\lambda\cap S_1=\emptyset$ (by IX), the insertion pair for y_q will not be contained in $\{y_1,\ldots,y_{|S_1|-\alpha+1}\}$. Hence,

$$\{y_1,\ldots,y_{|S_1|-\alpha+1}\}=\{w_1,\ldots,w_{|S_1|-\alpha+1}\}$$

remains as an interval in $Z_1^\alpha Z_\lambda^\beta(Q)$.

XI. We claim $N(v_{j_1-\alpha_1})\cap N(v_{j_\lambda-\alpha_\lambda})=\emptyset$ which is a part of the second assertion of III.

Let $Q^*=Z_1^{\alpha_1-1}Z_\lambda^{\alpha_\lambda-1}(Q)=w_1\ldots w_q$. By VIII, $N(v_{j_1-\alpha_1})\cap N(v_{j_\lambda-\alpha_\lambda})\cap V(B)=\emptyset$.

If $y\in N(v_{j_1-\alpha_1})\cap N(v_{j_\lambda-\alpha_\lambda})\subseteq[V(G)\setminus(C\cup B)]$, then $Z_1^{\alpha_1-1}Z_\lambda^{\alpha_\lambda-1}(Q)v_{j_\lambda-\alpha_\lambda}yv_{j_1-\alpha_1}$ would be a cycle longer than C.

Assume that $w_s\in N(v_{j_1-\alpha_1})\cap N(v_{j_\lambda-\alpha_\lambda})\cap V(C)$. If $(w_1,w_q)\in E(G)$, then the cycle $w_1\ldots w_qw_1$ would be longer than C. If $(w_q,w_{s-1})\in E(G)$ or $(w_1,w_{s+1})\in E(G)$, the cycle $w_1Q^*w_{s-1}w_q\bar{Q}^*w_sw_1$ or $w_1Q^*w_sw_q\bar{Q}^*w_{s+1}w_1$ would be longer than C. If $(w_{s-1},w_{s+1})\in E(G)$, the cycle $w_1Q^*w_{s-1}w_{s+1}Q^*w_qw_sw_1$ would be longer than C.

Since G is claw-free, the only remaining cases are that $(w_1,w_{s-1})\in E(G)$ when $s-1>1$ and $(w_q,w_{s+1})\in E(G)$ when $s+1<q$. Note that $s-1=1$ and $s+1=q$ cannot hold simultaneously as this would imply $3=q\geq|C|+1$. If $s-1>1$, by Proposition 3, then the adjacent pair $w_{s-1},w_s\in N(w_1)$ implies the existence of the insertion pair for w_1 in $Z_1^{\alpha_1-1}(P)$. By the choice of α_1,

$$\{w_{s-1},w_s\}\subseteq\{v_{j_1-\alpha_1-1},\ldots,v_{j_1-|S_1|-1}\}.$$

Similarly, if $s+1<q$, then

$$\{w_{s+1},w_s\}\subseteq\{v_{j_\lambda-\alpha_\lambda-1},\ldots,v_{j_\lambda-|S_\lambda|-1}\}.$$

If $s-1>1$ and $s+1<q$, then

$$w_s\in\{v_{j_1-\alpha_1-1},\ldots,v_{j_1-|S_1|-1}\}\cap\{v_{j_\lambda-\alpha_\lambda-1},\ldots,v_{j_1-|S_1|-1}\}$$

which contradicts Proposition 1.

So without loss of generality, let $s-1>1$ and $s+1=q$. By X, $\{w_1,\ldots,w_s\}\subseteq\{v_{j_1-\alpha_1},\ldots,v_{j_1-|S_1|-1}\}$ implies that $2\geq|Q\setminus\{v_{j_1-\alpha_1},\ldots,v_{j_1-|S_1|}\}|\geq|Q\setminus S_1|$. It contradicts that $|Q\setminus S_1|\geq|S_\lambda|+|B|+|\{v_{j_\lambda}\}|\geq3$.

XII. We now wish to show that $(v_{j_\lambda}, v_{j_1-\alpha_1}) \notin E(G)$ and $\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\} \cap N(v_{j_1-\alpha_1}\} = \emptyset$

Since $\{v_{j_\lambda-1}, v_{j_\lambda+1}, x_\lambda\} \subseteq N(v_{j_\lambda})$, G is claw-free and C is a longest cycle, note that $(v_{j_\lambda-1}, v_{j_\lambda+1}) \in E(G)$. Suppose $(v_{j_\lambda}, v_{j_1-\alpha_1}) \in E(G)$, let α be the least integer such that $(v_{j_\lambda}, v_{j_1-\alpha}) \in E(G)$. Then the insertion pair for $v_{j_1-\gamma}$ is not $\{v_{j_\lambda}, v_{j_\lambda+1}\}$ for $\gamma=1,\ldots,\alpha-1$. Hence, v_{j_λ} and $v_{j_\lambda+1}$ are adjacent in $Z_1^{\alpha-1}(P)=P^*=w_1\ldots w_r$ where $w_1=v_{j_1-\alpha}$ and $w_r=v_{j_1}$. Also, v_{j_λ} and $v_{j_\lambda-1}$ are adjacent in P^* by Proposition 2. Let $w_{i-1}=v_{j_\lambda+1}$, $w_i=v_{j_\lambda}$, $w_{i+1}=v_{j_\lambda-1}$. Then the cycle $w_1P^*w_{i-1}w_{i+1}P^*w_rx_1Bx_\lambda w_iw_1$ would be longer than C. Therefore we conclude that $v_{j_\lambda} \notin N(v_{j_1-\alpha_1})$ for $\lambda=2,\ldots,h$ and, by V, $v_{j_1} \notin N(v_{j_1-\alpha_1})$.

XIII. All results of V, IX, XI and XII hold if $\{1,\lambda\}$ is replaced by any pair $\{s,t\} \subseteq \{1,\ldots,h\}$.

Now we can establish the second assertion of III:
$N(v_{j_s-\alpha_s}) \cap N(v_{j_t-\alpha_t}) \neq \emptyset$ contradicts XI;
$N(v_{j_s-\alpha_s}) \cap \{v_{j_1}, \ldots, v_{j_h}\} \neq \emptyset$ contradicts XII;
$N(v_{j_s-\alpha_s}) \cap \{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\} \neq \emptyset$ contradicts IX;
$\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\} \cap \{v_{j_1}, \ldots, v_{j_h}\} \neq \emptyset$ contradicts II; and
$V(B) \cap N(v_{j_s-\alpha_s}) \neq \emptyset$ contradicts VIII.

Hence, $V(B)$, $N(v_{j_1-\alpha_1})$, ..., $N(v_{j_h-\alpha_h})$, $\{v_{j_1}, \ldots, v_{j_h}\}$ and $\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$ are disjoint sets of $V(G)$.

If we let $I'=\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$, then

$$\sum_{v \in I'} d(v) \leq |V(G) \backslash V(B)| - 2h.$$

Let $I=I'\cup\{x\}$ for any $x\in V(B)$. Recall that $|N_C(x)|\le h$ by the definition of h. Then

$$\begin{aligned}\sum_{v\in I} d(v) &\le \sum_{v\in I'} d(v)+(|V(B)\setminus\{x\}|+h)\\ &\le n-h-1\\ &\le n-k-1.\end{aligned}$$

Therefore any $(k+1)$-subset of I will have degree sum less than $n-k$ which contradicts the condition of the theorem. We conclude that G has a Hamilton cycle.

■■■

PART C

LONGEST CYCLES AND THEIR CHORDS

§1. INTRODUCTION

An edge e is called <u>a chord of a cycle</u> if e is not an edge of the cycle and both endvertices of e are in the cycle. Thomassen has conjectured [2, p.466] that *any longest cycle of a 3-connected graph must have a chord.* In this chapter, we shall show the conjecture is true for cubic planar graphs and planar graphs with minimum degree at least four. The conjecture is also true for claw-free graphs.

In addition, some structural results about minimum counterexamples to Thomassen's conjecture will be given. A lower bound on the length of longest cycles in the cyclically 4-edge connected cubic planar graphs will be given. This slightly improves the result obtained by Grünbaum and Malkevitch [9] in 1976.

§2. TERMINOLOGY

Let $G=(V,E)$ be a simple graph with V as the vertex set and E as the edge set. Let C be a cycle of G and let B' be a component of $G\backslash V(C)$. Let B be the union of B' and the edges joining B' and C, that is, $B=B'\cup[B',C]$. A <u>bridge</u> of C is either B or a chord of C. The vertices of $N(B')\cap V(C)$ and the endvertices of a chord are called the attachment of the bridge, B is called a t-<u>attachment bridge</u> if $|N(B')\cap V(C)|=t$, and a chord is a 2-<u>attachment bridge.</u> $A(B)$ denotes the set of attachment vertices of the bridge B and $V(B)$ denotes the set of vertices of the bridge B (excluding the attachment vertices on C).

If U is a subset of $V(G)$ and $G\backslash U$ is disconnected, then U is called a <u>vertex-cut</u> of G. An i-<u>vertex-cut</u> is a vertex-cut containing i vertices. If F is a subset of $E(G)$ and $G\backslash F$ is disconnected, then F is called an <u>edge-cut</u> of G. An i-<u>edge-cut</u> of G is an edge-cut containing i edges. If F is an edge-cut and one of the components of $G\backslash F$ is a singleton, then F is called a <u>trivial edge-cut.</u> If F is an edge-cut and none of the components of $G\backslash F$ is a tree, then F is called a <u>cyclic edge-cut.</u> A graph is called <u>cyclically</u> 4-<u>edge connected</u> if any i-edge-cut of this graph is not a cyclic edge-cut whenever $i\le 3$. By counting the number of the edges, it is easy to show that <u>when the minimum degree of the graph</u> G <u>is at least three, a</u> 3-<u>edge-cut</u> F <u>is cyclic if and only if</u> F <u>is non-trivial.</u>

§3. LEMMAS

TUTTE'S LEMMA. [16, Th.5.2.1.]

Let G be a planar graph, let e be an edge of G, let F and F' be the two faces incident with e and let e' be an edge on the boundary of F and adjacent with e. Then there is a cycle C of G such that

(i) e, $e' \in E(C)$,

(ii) any bridge of C has at most three attachments, and

(iii) any bridge of C intersecting with the boundary of F or F' has at most two attachments.

LEMMA 1.

Let G be a cyclically 4-edge connected cubic planar graph of order n ($n \geq 4$) and let e and e' be a pair of adjacent edges. Then there is a cycle C in G such that

(i) e, $e' \in E(C)$,

(ii) all bridges of C are either a single vertex or a chord,

(iii) C has at least two chords, and

(iv) $|V(C)| \geq (3/4)n+1$.

PROOF.

Let e, e' and e" be three distinct edges incident with the vertex v. Let F be the face with e and e' on its boundary and let F' be the face with e and e" on its boundary. Let

$$e=(v,x),\ e'=(v,y) \text{ and } e''=(v,z).$$

By Tutte's Lemma, there is a cycle C such that each bridge of C has at most three attachments and each bridge intersecting with the boundary of F or F' has at most two attachments. Since G is cyclically 4-edge connected and cubic, each three attachment bridge is a single vertex and each two attachment bridge is a chord.

Here e" must be a chord of C. Let e, f and f' be the three edges incident with vertex x. Two of them must be in C and without loss of generality, let $e,f \in E(C)$. Now f' is on the boundary of either F or F', so f' is a chord too. Thus C has at least two chords.

Let α be the number of chords of C and let β be the number of single vertex bridges of C. Then

$$\beta = |V(G)\backslash V(C)| \text{ and } 2\alpha+3\beta = |V(C)|.$$

Hence,

$$\begin{aligned} |V(C)| &= 2\alpha+3\beta \\ &\geq 4+3[\,|V(G)\backslash V(C)|\,] \\ &= 4+3n-3|V(C)|, \end{aligned}$$

that is, $$4|V(C)| \geq 4+3n.$$

Therefore, $$|V(C)| \geq (3/4)n+1.$$

■■■

LEMMA 2.

Let G be a cyclically 4-edge connected cubic planar graph of order n, and let e and e' be a pair of adjacent edges of G. Then any longest cycle of G containing e and e' must have at least two chords.

PROOF.

Assume C is a longest cycle of G containing e and e', and which has at most one chord. If the bridge B is not a chord of C, then

$$|A(B)| \leq 3|V(B)|.$$

Since G is cubic,

$$\Sigma|A(B)| = |V(C)|,$$

where the summation is over all bridges of C. Moreover,

$$\Sigma|V(B)| = n-|V(C)|.$$

Since C has at most one chord,

$$\Sigma|A(B)| \leq 2+3\Sigma|V(B)|.$$

Hence,

$$|V(C)| = \Sigma|A(B)| \leq 2+3\Sigma|V(B)| = 2+3(n-|V(C)|)$$

which implies that

$$|V(C)| \leq (3/4)n+(1/2).$$

However, this contradicts Lemma 1 and we conclude that C has at least two chords.

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The graph in the following lemma need not be planar.

LEMMA 3.

If G is a minimum counterexample to Thomassen's conjecture restricted to cubic planar graphs and C is a chordless longest cycle of G, then

$$T \cap E(C) \neq \emptyset$$

for any non-trivial 3-edge-cut T of G.

PROOF.

Let T be a non-trivial 3-edge-cut of G which separates V(G) into two disjoint parts V_1 and V_2. Here $|V_1|>1$ and $|V_2|>1$. If $T \cap E(C)=\emptyset$, then C is contained in either $G(V_1)$ or $G(V_2)$. Without loss of generality, let C be contained in $G(V_1)$ and $G(V_2)$ be connected. Let

$$T=\{(x_i,y_i) \mid x_i \in V_1,\ y_i \in V_2\}.$$

Let w be a new vertex not in G. Let

$$G^*=G(V_1) \cup \{(x_i,w) \mid (x_i,y_i) \in T\}.$$

We claim that G^* <u>is a</u> 3-<u>connected graph.</u> If G^* is not 3-connected, assume that U^* is a minimum vertex-cut of G^*, $|U^*| \le 2$ and U^* separates G^* into two parts U' and U''. If $w \notin U^*$, let $w \in U'$ and then $N(w) \setminus U^* \subseteq U'$. In this case, U^* would separate G into two parts $V_2 \cup [U' \setminus w]$ and U'' which contradicts G being 3-connected. Hence, assume that $w \in U^*$. If $\{x_1,x_2,x_3\} \setminus U^* \subseteq U'$, then $N(w) \cap U''=\emptyset$ and $U^* \setminus w$ is also a vertex-cut of G^* which contradicts U^* being minimum. So $|\{x_1,x_2,x_3\} \cap U'| \ge 1$ and $|\{x_1,x_2,x_3\} \cap U''| \ge 1$. Assume that $x_1 \in U'$ and $x_2 \in U''$. Let $U^{**}=U^* \setminus \{w\}$. Since w is a cut-vertex of $G^* \setminus U^{**}$, there is no path joining x_1 and x_2 in $G(V_1) \setminus U^{**}$. Hence, there is no path joining x_1 and x_2 in $G \setminus [U^{**} \cup \{y_1\}]$ and we would have a 2-vertex-cut $U^{**} \cup \{y_1\}$ of G which contradicts G being 3-connected. Therefore our claim holds and G^* is 3-connected.

If C is a longest cycle in G^*, C has a chord in G^* because G is a minimum counterexample. This chord is also a chord of C in G because $w \notin V(C)$. So assume C is not a longest cycle in G^*.

Let C' be a longest cycle in G^*, $|C'|>|C|$. Clearly, $w \in V(C')$. Let (x_1,w) and $(x_2,w) \in C'$. Let $P=y_1 \ldots y_2$ be a path in $G(V_2)$. Then the cycle $x_1 y_1 P y_2 x_2 C x_1$ would be longer than C which is a contradiction. Therefore $T \cap E(C) \neq \emptyset$ must hold.

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§4. MAIN RESULTS

THEOREM 2.1.

Let G be a cubic 3-connected planar graph. Then any longest cycle of G must have a chord.

PROOF.

The proof is by induction on $|V(G)|$ with the induction starting at $|V(G)|=4$ when $G=K_4$. Let C be a longest cycle of G. Assume that C has no chord.

By Lemma 2, G is not cyclically 4-edge connected, so G must have some cyclic 3-edge-cut which is also a non-trivial 3-edge-cut. Choose a non-trivial 3-edge-cut $\{e,e',e''\}$ of G such that $\{e,e',e''\}$ separates V(G) into two parts V' and V" with $|V''|$ as small as possible.

Let $e=(x,x')$, $e'=(y,y')$ and $e''=(z,z')$. Since G is 3-connected, x,y,z are distinct vertices in V' and x',y',z' are distinct vertices in V". By Lemma 3, $E(C)\cap\{e,e',e''\}\neq\emptyset$. Without loss of generality, let $e,e'\in E(C)$.

Let $G''=G(V'')\cup\{(w,x'),(w,y'),(w,z')\}$, where w is a new vertex which was not in V(G). By the minimality of $|V''|$, G" is cyclically 4-edge connected.

Let $C''=[C\cap G(V'')]\cup\{(w,x'),(w,y')\}$. Since C is a longest cycle in G, C" is a longest cycle in G' containing (w,x') and (w,y'). By Lemma 2, C" has at least two chords. Since (w,z') may be a chord of C", at least one of the chords of C" would also be a chord of C in G. This is a contradiction and establishes the result.

■■■

THEOREM 2.2.

Let G be a 3-connected planar graph with minimum degree at least four. Then any longest cycle of G must have a chord.

PROOF.

Let C be a chordless longest cycle of G which satisfies the hypotheses of the Theorem.

V* is called a separating vertex-cut with respect to C if V* separates G into two parts V' and V" such that C intersects both V' and V". And C is called separable if there is a separating 3-vertex-cut with respect to C.

I. If C is not separable.

Let $e=(x,y)$ be an edge of C and F_1, F_2 be two faces incident with e. Since $d(x) \geq 4$, let $(x,x_i)=e_i$ be the edge on the boundary of F_i and $x_i \neq y$ for $i=1,2$. There exists a cycle C' of G obtained by Tutte's Lemma which contains e and e_1. Now e_2 is a chord of C' because G is 3-connected and e_2 is on a 2-attachment bridge.

Since each bridge B of C' has at most three attachments, A(B) is a 3-vertex-cut of G if B is not a chord. Hence, $V(B) \cap V(C) = \emptyset$ because C is not separable. Then V(B) is contained in some bridges of C which implies that $V(G) \backslash V(C')$ is a subset of $V(G) \backslash V(C)$ and V(C) is a subset of V(C'). Now $|V(C)|=|V(C')|$

because C is a longest cycle of G. Let $E'=E(G(V(C)))=E(G(V(C')))$. Each edge of E' must either lie on C (respectively, C') or a chord of C (respectively, C'). Hence, the number of chords of C and C' is $|E'|-|V(C)|=|E'|-|V(C')|$ and the existence of chords of C' guarantees the existence of chords of C. But the edge $e_2=(x,x_2)$ being a chord of C' would contradict C being chordless.

II. Assume that C is separable.

Choose a separable 3-vertex-cut V* with respect to C such that V* separates G into V' and V" with V" as small as possible. Since C must pass through two vertices of V* to enter V" from V', the parts of C in $G(V'\cup V^*)$ and $G(V''\cup V^*)$ are paths. Let $C\cap G(V'\cup V^*)=P'=x\ldots y$ and $C\cap G(V''\cup V^*)=P''=y\ldots x$. Obviously, $x,y\in V^*$. Let $V^*=\{x,y,z\}$. We construct a new graph G* according to the following two cases:

a) If $z\notin V(P')$, let w be a new vertex not in G and

$$G^* = G(V''\cup V^*)\cup\{(w,x),(w,y),(w,z)\}.$$

b) If $z\in V(P')$, let w=z and

$$G^* = G(V''\cup V^*)\cup\{(w,x),(w,y)\}.$$

Let

$$C^*=P''\cup\{(w,x),(w,y)\}.$$

Here C* is a longest cycle of G* containing (w,x) and (w,y).

Let F_1 and F_2 be two faces incident with (x,w). There is a cycle C° of G* obtained by Tutte's Lemma which contains (w,x) and (w,y). Each bridge of C° has at most three attachments by Tutte's Lemma.

i. Notice that the minimum degree of G is at least four, $|V''|\geq 2$. Since V'' is minimum, $|N(v)\cap V''|\geq 2$ for any $v\in V^*$. Let (x,x_i) be the edge lying on the boundary of F_i and $(x,x_i)\neq(x,w)$ for $i=1,2$. Obviously, $(x,x_1),(x,x_2)\in E(G)$ and $\{x_1,x_2\}\subseteq V(G)$. By Tutte's Lemma, (x,x_i) either lies on C° or is a chord of C°. Hence, $\{x,x_1,x_2\}\subseteq V(C^\circ)$ and one of them must be a chord of C°.

ii. Assume $V(C^\circ)\setminus\{x,y,z,w\}\neq\emptyset$.

First of all, we claim that each non-chord bridge B of C° must be contained in some bridge of C^*. Suppose $V(B)\cap V(C^*)\neq\emptyset$ for some bridge B of C°.

ii-α. Case 1. $w\notin A(B)$ or $w=z$.

If $w\notin A(B)$, then $z\notin V(B)$ because $w\in V(C^\circ)$. If $w=z$, then $w=z\notin V(B)$. Hence, $x,y,z\notin V(B)$ and V' adjacent only with $\{x,y,z\}$ in G will imply that $A(B)$ is a vertex-cut which separates G into $V(B)$ and $V(G)\setminus[A(B)\cup V(B)]$. Since $V(C^\circ)\setminus\{x,y,z,w\}\neq\emptyset$, $V(B)$ would be a proper subset of V''. However, $V(B)$ intersects with C which contradicts the choice of V^* with V'' minimum.

ii-β. Case 2. $w\in A(B)$ and $w\neq z$.

Since $\{x,y,w\}\subseteq V(C^\circ)$ and $w\in A(B)$, $z\in V(B)$ and $(w,z)\in[V(B),A(B)]$. By Tutte's Lemma, B is a 2-attachment bridge because (w,z) lies on the boundary of F_1 or F_2. Since $d(z)\geq 3$ in G^*, $V(B)\setminus\{z\}\neq\emptyset$. Let $A(B)=\{w,u\}$. Then $U^*=\{z,u\}$ is a vertex-cut of G^* because $d(w)=3$ and $w\in A(B)$. U^* would separate G^* into $U''=V(B)\setminus\{z\}$ and $U'=[V(G^*)\setminus(V(B)\cup A(B))]\cup\{w\}$. Since $\{x,y,z,w\}\subseteq U'\cup U^*$, V' only adjacent with $V^*=\{x,y,z\}$ would imply that V' and U'' are disconnected in $G\backslash U^*$. Hence, U^* is a 2-vertex-cut separating G into U'' and $V'\cup U'\backslash\{w\}$ which contradicts G being 3-connected.

Now we conclude our claim in all cases. By the same argument we applied in I, $V(C^*)\subseteq V(C^\circ)$. Moreover, $V(C^\circ)=V(C^*)$ because C° contains (x,w) and (y,w) and C^* is a longest cycle of G^* containing (x,w) and (y,w). Hence, the number of chords of C^* is equal to the number of chords of C°. By i, $\{x,x_1,x_2\}\subseteq V(C^\circ)=V(C^*)$ and one of $\{(x,x_1),(x,x_2)\}$ is a chord of C^* which is also a chord of C in G. This contradicts C being chordless, and therefore $V(C^\circ)\backslash\{x,y,z,w\}\neq\emptyset$ is impossible.

<u>iii.</u> <u>Assume</u> $V(C^\circ)\backslash\{x,y,z,w\}=\emptyset$.

By i, $\{x_1,x_2\}\subseteq\{y,z\}\subseteq V(G)\cap V(C^\circ)\backslash\{x\}$. If $(x,y)\in E(G)$, then (x,y) would be a chord of C because $V(C)\cap V''\neq\emptyset$ and P_2 is a path of length at least two. But $y\notin\{x_1,x_2\}$ would imply that $z=x_1=x_2$ and (x,z) is a multiple edge of G. This contradicts G being simple. Again this is a contradiction, and the proof of the theorem is complete.

■■■

THEOREM 2.3.

<u>Let</u> G <u>be</u> <u>a</u> <u>cyclically</u> 4-<u>edge</u> <u>connected</u> <u>cubic</u> <u>planar</u> <u>graph</u> <u>of</u> <u>order</u> n. <u>Then</u> <u>a</u> <u>longest</u> <u>cycle</u> <u>of</u> G <u>must</u> <u>be</u> <u>of</u> <u>length</u> <u>at</u> <u>least</u> $(3/4)n+1$, <u>and</u> <u>must</u> <u>have</u> <u>two</u> <u>chords.</u>

This theorem improves the result Grünbaum and Malkevitch [9] given in 1976 which states that the length of a longest cycle in a cyclically 4-edge connected cubic planar graph of order n is at least (3/4)n.

The proof of the preceding theorem follows directly from Lemmas 1 and 2.

THEOREM 2.4.

If G is a 3-connected claw-free graph, then a longest cycle of G has a chord.

PROOF

The proof is a simple corollary of Lemma 1 in [12]. For the completeness of this chapter, we will give the proof.

Let $C=v_1 \ldots v_r v_1$ be a longest cycle of G. If C is a Hamilton cycle, all edges not in C are chords of C. So assume that there is a vertex $u \in V(G) \backslash V(C)$ and $u \in N(v_1)$. Since $\{u,v_r,v_2\} \subseteq N(v_1)$ and G is claw-free, one of (u,v_r), (u,v_2) and (v_r,v_2) must be in E(G). If (u,v_r) or $(u,v_2) \in E(G)$, C would not be a longest cycle. Thus, the chord $(v_2,v_r) \in E(G)$.

■■■

THEOREM 2.5.

If G is a minimum counterexample to Thomassen's conjecture and C is a chordless longest cycle of G, then $G \backslash V(C)$ is an independent set.

PROOF.

Suppose that B is a bridge of C and $|V(B)| \geq 2$. Let $A(B)=\{x_1, \ldots, x_t\}$ be the set of attachment vertices of B and

$$G'=G(V \backslash V(B)) \cup \{(x_i,w) \mid x_i \in A(B)\}$$

where w is a new vertex not contained in G.

We claim that G' is 3-connected. Assume that G' has a 2-vertex-cut U* which separates G' into two parts U' and U". If $w \notin U^*$, let $w \in U'$. Then $A(B)\backslash U^*=N(w)\backslash U^* \subseteq U'$ and U* would be a 2-vertex-cut of G which separates G into two parts U" and $[U'\backslash w] \cup V(B)$. So we assume that $w \in U^*$. Then $U^*\backslash w$ would be a cut-vertex of $G(V\backslash V(B))$. Notice that $G(V\backslash V(B))$ is a union of cycle C and all its bridges except for B. Since each of these bridges has at least three attachments, $G(V\backslash V(B))$ must be 2-connected which contradicts $G(V\backslash V(B))$ having a cut-vertex. Therefore G' cannot have a 2-vertex-cut and our claim holds.

Since G is minimum and G' is smaller than G, any longest cycle of G' must have a chord. Hence, C is not a longest cycle in G'. Let C' be a longest cycle in G'. C' is longer than C and must contain w. Let $C'=x_i w x_j \ldots x_i$ and $x_i B x_j$ be a path of B, where $x_i, x_j \in A(B)$. Then the cycle $x_i B x_j \ldots x_i$ would be longer than C in G, that is a contradiction. Therefore each bridge of this chordless cycle C is a single vertex.

■■■

THEOREM 2.6.

<u>A minimum counterexample to Thomassen's conjecture contains no triangles, that is, its girth is at least four.</u>

PROOF.

Suppose G is a minimum counterexample to Thomassen's conjecture and G contains a triangle $\{x,y,z\}$. Let C be a chordless longest cycle of G. By Theorem 2.5, at least two of

$\{x,y,z\}$ are in $V(C)$. Without loss of generality, let $x,y \in V(C)$. Then (x,y) is a chord of C if $(x,y) \notin E(C)$. Hence, $(x,y) \in E(C)$. It follows that $z \in V(C)$ because, otherwise, the cycle xzyCx would be longer than C. Now, x,y and $z \in V(C)$. Similar to the argument for (x,y), we must have that (y,z), $(z,x) \in E(C)$. Then C=xyzx. But C can be extended to be longer because G is 3-connected. This contradicts C being a longest cycle and the result follows.

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PART D

BRIDGES OF LONGEST CYCLES

§1 INTRODUCTION

Some graphs contain Hamilton cycles and some do not. How long is a longest cycle in non-hamiltonian graphs? What can be said about the structure of the subgraph outside a longest cycle? These are two problems among many interesting similar problems. Nash-Williams [13] and Bondy [6], [7] have found some structural results about the subgraph outside a longest cycle. It is obvious that the length of a longest cycle and the structure of the subgraph outside a longest cycle are not independent. This chapter will establish a result which gives a relation between the lengths of a longest cycle and its bridges.

DEFINITIONS.

Let C be a subgraph of G. Recall that a *bridge* of C is either a component of $G\backslash V(C)$ together with its attachments on C or a chord of C. A C-*path* is a path of G such that only its endvertices are on C. If B is a bridge of C, let P be a longest C-path contained in B. Then *the length of the bridge* B *is defined as the length of* P.

THEOREM 3.1.

Let G *be a* 3-*connected non-hamiltonian graph and*

$$d(x)+d(y) \geq m$$

for each pair of non-adjacent vertices x *and* y. *Let the length of any longest cycle* C *be* r. *Then the length of any bridge of* C *is at most* r-m+2.

In other words, let C be a longest cycle of G and let p be the length of the longest bridge of C. Then the length of C is at least m+p-2. Hence, the shorter a longest cycle is, the shorter the bridges of the cycle are.

Some examples will show that this theorem is the best possible result. The condition of 3-connectivity cannot be reduced, for example, $3K_t+K_2$ is a 2-connected graph which is constructed by joining all vertices of three vertex disjoint K_t's to two new vertices x and y. This graph contains a longest cycle of length 2t+2 with a bridge of length t+1, but m=2t+2. The inequality of the theorem cannot be reduced, either. One example is the complete bipartite graph $K_{t,t+1}$ which is 3-connected (if $t \geq 3$) and contains a longest cycle of length 2t with a bridge of length 2, but m=2t. Another example is $4K_t+K_3$ which is also 3-connected and contains a longest cycle of length 3t+3 with a bridge of length t+1, but m=2t+4.

This theorem also generalizes the result found by Linial [11] for 3-connected graphs.

LINIAL'S THEOREM

Let G be a 2-connected graph, and

$$d(x)+d(y) \geq m$$

for each pair of non-adjacent vertices x and y. Then G contains either a Hamilton cycle or a cycle of length at least m.

§2 TERMINOLOGY

Let $C=v_1\ldots v_r v_1$. The path $v_i v_{i+1}\ldots v_{j-1}v_j$ will be denoted by $v_i C v_j$ and the path $v_i v_{i-1}\ldots v_{j+1}v_j$ will be denoted by $v_i \bar{C} v_j$ where v_{r+1} is taken to be v_1.

If $P=u_1\ldots u_h$ is a path and T is a subset of its vertices, let

$$T_P^{+1}=\{u_{k+1}\epsilon P \mid u_k \epsilon T\cap P\}, \text{ and } T_P^{-1}=\{u_{k-1}\epsilon P \mid u_k \epsilon T\cap P\}.$$

Sometimes we simply write T^{+1} if no confusion will occur.

Let D be a subgraph of G and $a\epsilon V(D)$. Let $w(a,D)$ denote any vertex b which is the endvertex of a longest path in $V(G)\backslash[V(D)\backslash\{a\}]$ starting at a. Note that if a is not adjacent to any vertex outside D, then $w(a,D)=a$. For example, if $P=a\ldots b$ is a longest path in $G\backslash[V(D)\backslash\{a\}]$ with one specified endvertex a, then we can choose b as $w(a,D)$.

Let $h(a,D) =|N(b)\cap[G\backslash(D\backslash a)]|$ where $b=w(a,D)$. Note that if $h(a,D)=0$, then $w(a,D)=a$ and a is an isolated vertex in $G\backslash[V(D)\backslash\{a\}]$.

Let

$$M(a,D) = \{v\epsilon V(D)\backslash a \mid \text{there is a D-path joining a and v with length at least } h(a,D)+1\}.$$

Let

$$N(a,D)=\{v\epsilon V(D)\backslash a \mid \text{there is a D-path joining a and v}\}.$$

Obviously, $M(a,D)\subseteq N(a,D)$. Note that if $h(a,D)=0$, then $N(a)\subseteq V(D)$ and, hence, $M(a,D)=N(a,D)=N(a)$.

By a***c denote a D-path a...c of D, where $a,c \in V(D)$. Note that a single edge in D is also a D-path according to the definition in §1, because the two endvertices are in D.

§3. LEMMAS

LEMMA 1 (Fournier & Fraisse [8]).

Let D be a subgraph of a 2-connected G with $|V(D)|\geq 2$, and $P=x\ldots y$ be a longest path in $G\backslash[D\backslash\{x\}]$ starting at x. Then there is a D-path starting at x that contains y and all its neighbours in $G\backslash V(D)$.

In other words, if G is 2-connected and D is any subgraph of G satisfying $|V(D)|\geq 2$ and $a\in V(D)$, then $M(a,D)\neq\emptyset$.

LEMMA 2.

If G is 3-connected, then $|M(a,D)|\geq 2$ for any subgraph D of G with $|V(D)|\geq 3$ and $a\in V(D)$.

PROOF.

By Lemma 1, there is $b\in M(a,D)$. Since $G\backslash\{b\}$ is 2-connected, by Lemma 1 we have $|M(a,D\backslash b)|\geq 1$.

■■■

LEMMA 3.

Let $P=x_1\ldots x_t$ be a path and let $y,z\notin V(P)$. If

$$N_P(y)\cap N_P^{\pm 1}(z) = \emptyset,$$

then

$$|N_I(y)|+|N_I(z)| \leq |I|+1$$

for any interval $I=x_i\ldots x_j\subseteq P$.

PROOF.

Since $N_I(y)\cap N_I^{\pm 1}(z) = \emptyset$ and $|N_I(z)| \leq |N_I^{\pm 1}(z)|+1$,

$$|I| \geq |N_I(y)|+|N_I^{\pm 1}(z)| \geq |N_I(y)|+|N_I(z)|-1.$$

■■■

§4. PROOF OF THE THEOREM

Let $C=v_1\ldots v_r v_1$ be a longest cycle of G and p be the length of a longest bridge of C. We assume that $r\le m+p-3$ and will prove the theorem by contradiction.

PART A.

In this part, we will obtain some general propositions which will be used frequently during the proof.

Let $B=v_r{*}{*}{*}v_t$ be a longest C-path. Note that it contains p-1 vertices not in C.

For the sake of convenience, denote $w(v_i,C)$, $h(v_i,C)$, $M(v_i,C)$ and $N(v_i,C)$ by $w(i)$, $h(i)$, $M(i)$ and $N(i)$, respectively, for $i=1,2,\ldots,r$.

Since p is the length of a longest bridge of C, by Lemma 1, we must have that

$$h(i)\le p-1, \qquad \text{\underline{for any} } i. \tag{4.1}$$

<u>And</u>

$$d(w(i))\le h(i)+|M(i)|, \qquad \text{\underline{for any} } i. \tag{4.2}$$

PROPOSITION 1.

We have

$$M(i)\cap\{v_{i-h(i)},\ldots,v_{i+h(i)}\}=\emptyset, \qquad \text{\underline{for any} } i.$$

PROOF.

Otherwise, let $v_j\in M(i)$ and $i-h(i)\le j\le i-1$. Then $v_j{*}{*}{*}v_iCv_j$ would be a cycle longer than C. A similar argument works if

$i+1 \leq j \leq i+h(i)$.

■■■

PROPOSITION 2.

<u>We have</u>

$$t \geq p \quad \underline{\text{and}} \quad r-t \geq p.$$

PROOF.

If $t \leq p-1$, the cycle $v_r B v_t C v_r$ is longer than C. A similar contradiction arises when $r-t<p$.

■■■

PROPOSITION 3.

<u>We have</u>

$$m \geq p+3.$$

PROOF.

If $m \leq p+2$, then $r \leq m+p-3 \leq 2p-1$. It then follows that either $t \leq p-1$ or $r-t \leq p-1$, both of which contradict Proposition 2.

■■■

DEFINITION.

The pair (i,j) is called a <u>summable pair</u> on C if v_i and v_j are not joined by a C-path (which implies that $(w(i),w(j)) \notin E(G)$) and either $M(i) \cap M^{+1}(j)=\emptyset$ or $M(j) \cap M^{+1}(i)=\emptyset$ on any interval of $C \setminus \{v_i, v_j\}$.

During the proof, the basic method will be to get a summable pair (i,j) and to check the sum of $d(w(i))$ and $d(w(j))$. So we need some propositions about summable pairs and the sums of the appropriate degrees.

PROPOSITION 4.

The pairs $(1,t+1)$ and $(t-1,r-1)$ are summable.

PROOF.

Obviously, $v_1 \notin N(t+1)$. Otherwise, the cycle $v_1 C v_t B v_r \bar{C} v_{t+1} *** v_1$ would be longer than C.

Moreover,

$$M(1)\cap M^{+1}(t+1)=\emptyset \qquad \text{in } \{v_2,\ldots,v_t\}$$

and $$M(t+1)\cap M^{+1}(1)=\emptyset \qquad \text{in } \{v_{r+2},\ldots,v_r\}.$$

Otherwise, without loss of generality, let $v_i \in M(1)\cap M^{+1}(t+1)$, $2\le i\le t+1$. Then the cycle $v_1 C v_{i-1} *** v_{t+1} C v_r B v_t \bar{C} v_i *** v_1$ would be longer than C.

The pair $(t-1,r-1)$ is symmetric to $(1,t+1)$.

■■■

PROPOSITION 5.

Let $\{J_\mu \mid \mu\in I\}$ be a collection of pairwise vertex-disjoint intervals of $C\backslash\{v_i,v_j\}$, (i,j) be a summable pair on C, and $M(i)\cup M(j)\subseteq \bigcup_{\mu\in I} J_\mu$. Let

$$I'=\{\mu\in I \mid M(i)\cap J_\mu\neq\emptyset \text{ and } M(j)\cap J_\mu\neq\emptyset\}$$

and $$J=C\backslash[(\bigcup_{\mu\in I} J_\mu)\cup\{v_i,v_j\}].$$

Then

$$\begin{aligned}|J| &\le h(i)+h(j)+p-5+|I'|\\ &\le h(i)+h(j)+p-5+|I|.\end{aligned}$$

PROOF.

Since $w(i)$ and $w(j)$ are non-adjacent, $m\le d(w(i))+d(w(j))$ by the hypotheses of Theorem 3.1. By (4.2), it follows that

$$\begin{aligned}m &\le h(i)+h(j)+|M(i)|+|M(j)|\\ &= h(i)+h(j)+\sum_{\mu\in I}[J_\mu\cap M(i)|+|J_\mu\cap M(j)|]\end{aligned}$$

$\leq h(i)+h(j)+\sum_{\mu\in I'}[|J_\mu|+1]+\sum_{\mu\in I\setminus I'}|J_\mu|$ (by Lemma 3)

$= h(i)+h(j)+|\bigcup_{\mu\in I} J_\mu|+|I'|.$

Since $r\leq m+p-3$ and $|J|+|\bigcup_{\mu\in I} J_\mu|=r-2$,

$$|J| \leq p-5+h(i)+h(j)+|I'|.$$

■■■

The following proposition is the main result of this section. It is a very important part of the proof of the theorem.

PROPOSITION 6.

We have

$M(1)\subseteq\{v_{r-1},v_r,v_{2+h(1)},\dots,v_t\}$,

$M(t-1)\subseteq\{v_r,v_1,\dots,v_{t-2-h(t-1)},v_t,v_{t+1}\}$,

$M(t+1)\subseteq\{v_{t-1},v_t,v_{t+2+h(t+1)},\dots,v_r\}$ and

$M(r-1)\subseteq\{v_t,\dots,v_{r-2-h(r-1)},v_r,v_1\}$.

That is, $M(1)$ does not intersect with $\{v_{t+1},\dots,v_{r-2}\}$, and so on.

PROOF.

Without loss of generality, we may consider $M(t+1)$ and assume that $M(t+1)\cap\{v_1,\dots,v_{t-2}\}\neq\emptyset$. Choose v_k to be the vertex in this intersection with k as big as possible.

I. Case 1: $M(t+1)\cap\{v_1,\dots,v_{1+h(1)}\}=\emptyset$.

i. We claim that if $2+h(1)\leq i<j\leq t$, it is impossible that

$$v_i\in M(t+1) \text{ and } v_j\in N(1).$$

Prove this claim by contradiction, so let

$$v_i \epsilon M(t+1) \text{ and } v_j \epsilon N(1)$$

and choose $j-i$ as small as possible.

Since the cycle $v_1 C v_i {*}{*}{*} v_{t+1} C v_r B v_t \bar{C} v_j {*}{*}{*} v_1$ is not longer than C, $\{v_{i+1}, \ldots, v_{j-1}\}$ must contain at least $p-1+h(t+1)$ vertices. This follows because the C-path $v_i {*}{*}{*} v_{t+1}$ contains at least $h(t+1)$ vertices not in C and $v_r B v_t$ contains $p-1$ vertices not in C.

Let

$J_1=\{v_{2+h(1)}, \ldots, v_i\}$, $J_2=\{v_j, \ldots, v_t\}$ and $J_3=\{v_{t+2}, \ldots, v_r\}$.

Here,

$$M(1) \cup M(t+1) \subseteq J_1 \cup J_2 \cup J_3 \text{ and } I=\{1,2,3\}.$$

Let

$$J=\{v_2, \ldots, v_{1+h(1)}, v_{i+1}, \ldots, v_{j-1}\}$$

when $h(1)>0$, or

$$J=\{v_{i+1}, \ldots, v_{j-1}\}$$

when $h(1)=0$, which contains at least $h(1)+h(t+1)+p-1$ vertices. This is a contradiction of Proposition 5.

<u>ii.</u> By (i) and the assumption of Case 1, $v_{t-1} \notin N(1)$. Hence, $w(1)$ <u>and</u> $w(t-1)$ <u>are a pair of non-adjacent vertices.</u>

We shall consider this pair of vertices. First of all, we wish to show that $(1,t-1)$ <u>is a summable pair on</u> C.

Assume that $v_i \epsilon M(1) \cap M^{+1}(t-1)$. If $t \le i \le r$, then the cycle $v_1 C v_{t-1} {*}{*}{*} v_{i-1} \bar{C} v_t B v_r \bar{C} v_i {*}{*}{*} v_1$ would be longer than C. If $2 \le i \le t-2$, then $i \le k$ by (i). The fact that the cycle $v_1 C v_{i-1} *** v_{t-1} v_t B v_r \bar{C} v_{t+1} *** v_k \bar{C} v_i *** v_1$ is not longer than C implies that

$$J=\{v_{k+1},\ldots,v_{t-2}\}$$

must contain at least $p-1+h(1)+h(t-1)+h(t+1)$ vertices and J does not intersect with M(1) or M(t+1) by the choice of k. Consider the summable pair (1,t+1). Let

$$J_1=\{v_2,\ldots,v_k\},\ J_2=\{v_{t-1},v_t\} \text{ and } J_3=\{v_{t+2},\ldots,v_r\}.$$

Here,

$$M(1) \cup M(t+1) \subseteq J_1 \cup J_2 \cup J_3,\ I=\{1,2,3\}$$

which leads to a contradiction of Proposition 5. Thus (1,t-1) is a summable pair.

<u>iii.</u> <u>If</u> $1 \le i < j \le t-1$, <u>then</u> <u>it</u> <u>is</u> <u>impossible</u> <u>that</u>

$$v_i \in M(t-1) \text{ \underline{and} } v_j \in M(1).$$

We prove this claim by contradiction. Choose j-i as small as possible. (The proof of this claim is quite similar to parts of ii.)

By (i), $j \le k$ and by the choice of k,

$$J=\{v_{i+1},\ldots,v_{j-1},v_{k+1},\ldots,v_{t-2}\}$$

will not intersect with M(1) and M(t+1). Since the cycle $v_1 C v_i *** v_{t-1} v_t B v_r \bar{C} v_{t+1} *** v_k \bar{C} v_j *** v_1$ is not longer than C, J must contain at least $p-1+h(1)+h(t-1)+h(t+1)$ vertices.

But consider the summable pair (1,t+1). Let

$$J_1=\{v_2,\ldots,v_i\},\ J_2=\{v_j,\ldots,v_k\},$$
$$J_3=\{v_{t-1},v_t\} \text{ and } J_4=\{v_{t+2},\ldots,v_r\}.$$

Here, $I=\{1,2,3,4\}$ and $I'\subseteq\{1,2,4\}$ because $M(1)\cap J_3=\emptyset$ by (i) and (ii). This leads to a contradiction of Proposition 5.

<u>iv.</u> <u>If</u> $t\le i<j\le r$, <u>then</u> <u>it</u> <u>is</u> <u>impossible</u> <u>that</u>

$$v_i\in M(1) \qquad \underline{\text{and}} \qquad v_j\in M(t-1).$$

We prove this claim by contradiction. Choose $j-i$ as small as possible. Then

$$J=\{v_{i+1},\ldots,v_{j-1}\}$$

would not intersect with $M(1)$ and $M(t-1)$. Since the cycle $v_1Cv_{t-1}{*}{*}{*}v_jCv_rBv_tCv_i{*}{*}{*}v_1$ is not longer than C, J must contain at least $p-1+h(1)+h(t-1)$ vertices. Now consider the summable pair $(1,t-1)$. Let

$$J_1=\{v_2,\ldots,v_{t-2}\},\ J_2=\{v_t,\ldots,v_i\} \text{ and } J_3=\{v_j,\ldots,v_r\}.$$

Here, $I=\{1,2,3\}$ and again it leads to a contradiction of Proposition 5.

<u>v.</u> By (iii) and (iv), there are integers a and b such that $2\le a\le t-2$, $t\le b\le r$,

$$M(1)\subseteq\{v_b,\ldots,v_r,v_2,\ldots,v_a\}\backslash\{v_{r+1-h(1)},\ldots,v_{1+h(1)}\},$$
$$M(t-1)\subseteq\{v_a,\ldots,v_{t-2},v_t,\ldots,v_b\}\backslash\{v_{t-1-h(t-1)},\ldots,v_{t-1+h(t-1)}\}.$$

We now have enough information to get the final contradiction for this case.

Choose i and j such that $t\le i<j\le r$, $v_i\in M(t-1)\cup\{v_t\}$, $v_j\in M(1)\cup\{v_r\}$, and $j-i$ is as small as possible. Obviously, $i\le b\le j$. Since the cycle $v_1Cv_{t-1}{*}{*}{*}v_i\bar{C}v_tBv_r\bar{C}v_j{*}{*}{*}v_1$ is not longer than C,

$$|\{v_{i+1},\ldots,v_{j-1}\}|\ge p-1+(|v_1{*}{*}{*}v_j|-2)+(|v_{t-1}{*}{*}{*}v_i|-2).$$

α). If $v_j \in M(1)$ and $v_i \in M(t-1)$, let

$$J=\{v_{i+1},\ldots,v_{j-1}\}.$$

Then $|J| \geq p-1+h(1)+h(t-1)$. If we let

$$J_1=\{v_2,\ldots,v_{t-2}\},\ J_2=\{v_t,\ldots,v_i\},$$
$$J_3=\{v_j,\ldots,v_r\} \text{ and } I=\{1,2,3\},$$

we again contradict Proposition 5.

β). If $v_i \notin M(t-1)$ and $v_j \in M(1)$, that is, $v_i=v_t$, then

$$M(t-1) \subseteq \{v_b, v_a, \ldots, v_{t-2-h(t-1)}\}.$$

(By Lemma 2, $|M(t-1)| \geq 2$ which implies that $t-2-h(t-1) \geq a$.)

Let

$$J=\{v_{t-1-h(t-1)},\ldots,v_{t-2},v_t,\ldots,v_{j-1}\}$$

when $h(t-1)>0$, or

$$J=\{v_t,\ldots,v_{j-1}\}$$

when $h(t-1)=0$. Note that $|J| \geq p+h(1)+h(t-1)$ because $|\{v_{t+1},\ldots,v_{j-1}\}| \geq p-1+h(1)$. If

$$J_1=\{v_2,\ldots,v_{t-2-h(t-1)}\},\ J_2=\{v_j,\ldots,v_r\} \text{ and } I=\{1,2\},$$

we again contradict Proposition 5.

Via a symmetric argument, a contradiction follows for $v_i \in M(t-1)$ and $v_j \notin M(1)$.

γ). So we consider $v_i \notin M(t-1)$ and $v_j \notin M(1)$, that is, $v_i=v_t$ and $v_j=v_r$. Let

$$J=K_1 \cup K_2 \cup K_3$$

where
$$K_1=\{v_2,\ldots,v_{1+h(1)}\}$$

when $h(1)>0$ or the empty set when $h(1)=0$,

$$K_2=\{v_{t-1-h(t-1)},\ldots,v_{t-2}\}$$

when $h(t-1)>0$ or the empty set when $h(t-1)=0$ and

$$K_3=\{v_{t+1},\ldots,v_{r-1}\}.$$

By Proposition 2, $|J|\geq p-1+h(1)+h(t-1)$. Since $|M(1)|\geq 2$ and $|M(t-1)|\geq 2$, $2+h(1)\leq a\leq t-2-h(t-1)$. Letting

$$J_1=\{v_{2+h(1)},\ldots,v_{t-2-h(t-1)}\},\ J_2=\{v_t\} \text{ and } J_3=\{v_r\}$$

with $I=\{1,2,3\}$, we again contradict Proposition 5.

The first case of Proposition 6 has now been proved.

<u>II.</u> <u>Case</u> <u>2:</u> $M(t+1)\cap\{v_1,\ldots,v_{1+h(1)}\}\neq\emptyset$.

Let v_i be a vertex of this intersection.

<u>i.</u> Since the cycle $v_iCv_tBv_r\bar{C}v_{t+1}{*}{*}{*}v_i$ is not longer than C, $i\geq h(t+1)+p$. By (4.1), $h(1)\leq p-1$. So $p\geq 1+h(1)\geq i\geq h(t+1)+p$ implies that $h(t+1)=0$, $h(1)=p-1$ and $v_i=v_{1+h(1)}\in M(t+1)$.

<u>ii.</u> Since Case 1 of Proposition 6 has been solved, we have a symmetric result for $M(1)$ which is

$$M(1)\cap\{v_{t+1},\ldots,v_{r-2}\}=\emptyset \text{ if } M(1)\cap\{v_{t+1},\ldots,v_{t+1-h(t+1)}\}=\emptyset.$$

By (i), $h(t+1)=0$ and we have that

$$\{v_{t+1},\ldots\ldots,v_{t+1-h(t+1)}\}=\{v_{t+1}\}$$

with which $M(1)$ does not intersect. Hence,

$$M(1)\cap\{v_{t+1},\ldots,v_{r-2}\}=\emptyset.$$

<u>iii.</u> Since $h(1)=p-1\geq 1$ and $v_r\notin M(1)$, $M(1)\cap\{v_{2+h(1)},\ldots,v_t\}\neq\emptyset$ because $|M(1)|\geq 2$ and by Proposition 1.

Since $v_i\in M(t+1)\cap\{v_1,\ldots,v_{1+h(1)}\}$ and $M(1)\cap\{v_{2+h(1)},\ldots,v_t\}\neq\emptyset$, there are integers k and j, with j-k as small as possible, such that $2\le k<j\le t$, $v_j\in M(1)$ and $v_k\in M(t+1)$. Let

$$J=\{v_{k+1},\ldots,v_{j-1}\}$$

with which neither $M(1)$ nor $M(t+1)$ intersects or else $j-k$ could be chosen smaller. Since the cycle $v_1Cv_k{*}{*}{*}v_{t+1}Cv_rBv_t\bar{C}v_j{*}{*}{*}v_1$ is not longer than C, J contains at least $p-1+h(1)+h(t+1)$ vertices. On the other hand, letting

$$J_1=\{v_2,\ldots,v_k\},\ J_2=\{v_j,\ldots,v_t\} \text{ and } J_3=\{v_{t+2},\ldots,v_r\},$$

$M(1)\cup M(t+1)\subseteq J_1\cup J_2\cup J_3$. With $I=\{1,2,3\}$, Proposition 5 is contradicted and the proof of Proposition 6 is complete.

■■■

PROPOSITION 7.

We have

$$M(1)\cap\{v_2,\ldots,v_t\}\neq\emptyset,\ M(t-1)\cap\{v_r,v_1,\ldots,v_{t-2}\}\neq\emptyset,$$
$$M(t+1)\cap\{v_{t+2},\ldots,v_r\}\neq\emptyset \text{ and } M(r-1)\cap\{v_t,\ldots,v_{r-2}\}\neq\emptyset.$$

PROOF.

Without loss of generality, we consider $M(1)$. If $h(1)=0$, $v_2\in M(1)$. If $h(1)\ge 1$, $v_r\notin M(1)$. Since $M(1)\subseteq\{v_{r-1},v_r,v_2,\ldots,v_t\}$, by the previous proposition, and $|M(1)|\ge 2$, $M(1)\cap\{v_2,\ldots,v_t\}\neq\emptyset$.

■■■

PROPOSITION 8.

We have

$$t\ge 3 \text{ and } r-t\ge 3.$$

PROOF.

If $t\le 2$, $p=2$ and $t=2$ by Proposition 2. By Proposition 6, $M(1)\subseteq\{v_{r-1},v_r,v_2,\ldots,v_t\}=\{v_{r-1},v_r,v_2\}$ and $v_3=v_{t+1}\notin M(1)$. Since $v_1=v_{t-1}$, $M(1)=M(t-1)\subseteq\{v_r,\ldots,v_{t-2},v_t,v_{t+1}\}=\{v_r,v_2,v_3\}$ and $v_{r-1}\notin M(t-1)=M(1)$. So $M(1)=\{v_r,v_2\}$ because $|M(1)|\ge 2$. Now $h(1)=0$, otherwise, $v_r, v_2 \notin M(1)$. But then v_1 is a vertex of degree two which contradicts the 3-connectivity of the graph. Thus, $t \geq 3$ and by symmetry $r-t \geq 3$.

■■■

Now we can get into the main part of the theorem's proof. First, we define a Y-bridge of a longest cycle C.

DEFINITION.

If D is a bridge of C and vertices v_r, $v_{t'}$, $v_{t''}$ of C are distinct attachments of D such that there are two C-paths $v_r{*}{*}{*}v_{t'}$ and $v_r{*}{*}{*}v_{t''}$ of length p contained in D, then D is called a Y-bridge of C.

We shall consider two cases in the proof, namely, with a Y-bridge (Part B) and without a Y-bridge (Part C).

PART B. CASE ONE. C has a Y-bridge.

Propositions 5 and 6 will be the keys to the proof in this case.

Let $B' = v_r{*}{*}{*}v_{t'}$ and $B'' = v_r{*}{*}{*}v_{t''}$ be two C-paths of length p contained in a Y-bridge of C, $t'' > t'$. Obviously, $t'' \geq t'+2$. The index t in all propositions of Part A can be replaced by both t' and t".

I. We claim $t''-t' \leq p-2$, that is, $1 \leq |\{v_{t'+1}, \ldots, v_{t''-1}\}| \leq p-3$.

Let us consider the summable pair $(r-1, t'-1)$. Let

$$K_1 = \{v_{t'-1-h(t'-1)}, \ldots, v_{t'-2}\}$$

when $h(t'-1)>0$ or the empty set when $h(t'-1)=0$,

$$K_2=\{v_{r-1-h(r-1)},\ldots,v_{r-2}\}$$

when $h(r-1)>0$ or the empty set when $h(r-1)=0$, and

$$K_3=\{v_{t'+2},\ldots,v_{t''-1}\}$$

when $t''\geq t'+3$ or the empty set when $t''=t'+2$. Let

$$J = K_1 \cup K_2 \cup K_3.$$

By (4.1) and Proposition 2, $r-t''\geq p\geq h(r-1)+1$ implies that $r-1-h(r-1)\geq t''$. Hence, J contains no coincident vertices. Let

$$J_1=\{v_r,\ldots,v_{t'-2-h(t'-1)}\},$$

$$J_2=\{v_{t'},v_{t'+1}\} \text{ and } J_3=\{v_{t''},\ldots,v_{r-2-h(r-1)}\}.$$

By Proposition 7, $M(t'-1)\cap\{v_r,v_1,\ldots,v_{t'-2-h(t'-1)}\}\neq\emptyset$ and $t'-2-h(t'-1)\geq 0$.

When $t'-2-h(t'-1)\geq 1$, $\{v_r,v_1\}\subseteq J_1$. Hence, by Proposition 6,

$$M(r-1)\subseteq J_1\cup J_3 \text{ and } M(t'-1)\subseteq J_1\cup J_2,$$

$$I=\{1,2,3\} \text{ and } I'\subseteq\{1\}.$$

When $t'-2-h(t'-1)=0$, v_r is the single vertex in $M(t'-1)\cap\{v_r,\ldots,v_{t'-2}\}$ by Proposition 7. If $v_1=v_{t'-1-h(t'-1)}\in M(r-1)$, then $v_1Cv_{t'-1}{*}{*}{*}v_rBv_{t'}Cv_{r-1}{*}{*}{*}v_1$ would be a cycle longer than C. Hence, $v_1\notin M(r-1)$. So we still have that $M(r-1)\subseteq J_1\cup J_3$, $M(t'-1)\subseteq J_1\cup J_2$,

$$I=\{1,2,3\} \text{ and } I'\subseteq\{1\}.$$

By Proposition 5,

$$|K_1|+|K_2|+|K_3| = |J| \leq h(r-1)+h(t'-1)+p-5+|I'|.$$

Since $|K_1|=h(t'+1)$, $|K_2|=h(r-1)$ and $|K_3|=t''-t'-2$, $t''-t'-2=|K_3|\leq p-4$.

II. We claim v_{r-1} and $v_{t'+1}$ are not joined by a C-path.

Otherwise, the cycle $v_r C v_{t'+1} *** v_{r-1} \bar{C} v_{t''} B'' v_r$ would be longer than C because $|\{v_{t'+2}, \ldots, v_{t''-1}\}| \leq p-4$. Hence, $(w(r-1), w(t'+1)) \notin E(G)$.

III. We claim $(r-1, t'+1)$ is summable.

By Proposition 6 and II, we only need to consider the intervals $\{v_{t'+2}, \ldots, v_{r-2}\}$ and $\{v_r, v_1\}$.

If $v_i \in M^{+1}(r-1) \cap M(t'+1)$, $t'+2 \leq i \leq r-2$. (Note that $v_{i-1} \in M(r-1)$ implies $i-1 \geq t''$ by Proposition 6). Then the cycle $v_r C v_{t'+1} *** v_i C v_{r-1} *** v_{i-1} \bar{C} v_{t''} B'' v_r$ would be longer than C because of I.

If $v_1 \in M^{+1}(t'+1) \cap M(r-1)$, then the cycle $v_{r-1} *** v_1 C v_{t'} B' v_r *** v_{t'+1} C v_{r-1}$ would be longer than C. Finally, $v_r \notin M^{+1}(t'+1)$ by II.

IV. If $t'' \leq i < j \leq r-1$, it is impossible that

$$v_j \in M(t'+1) \text{ and } v_i \in M(r-1).$$

Otherwise, choose $j-i$ as small as possible. Since the cycle $v_r C v_{t'+1} *** v_j C v_{r-1} *** v_i \bar{C} v_{t''} B'' v_r$ is not longer than C, $\{v_{i+1}, \ldots, v_{j-1}\} \cup [\{v_{t'+1}, \ldots, v_{t''-1}\} \setminus \{v_{t'+1}\}]$ must contain at least $p-1+h(t'+1)+h(r-1)$ vertices. By I, $\{v_{i+1}, \ldots, v_{j-1}\}$ contains at least $h(t'+1)+h(r-1)+3$ vertices. Let

$$J_1 = \{v_r, v_1\},\ J_2 = \{v_{t'-1}, v_{t'}\},$$
$$J_3 = \{v_{t'+2}, \ldots, v_i\},\ J_4 = \{v_j, \ldots, v_{r-2}\},$$

(note, by Proposition 8, $J_1 \cap J_2 = \emptyset$),

$$J=\{v_2,\ldots,v_{t'-2},v_{i+1},\ldots,v_{j-1}\} \text{ and } I=\{1,2,3,4\}.$$

From above

$$|J| \geq (t'-3)+[h(t'+1)+h(r-1)+3]$$

and by Proposition 2,

$$|J| \geq p+h(t'+1)+h(r-1).$$

This contradicts Proposition 5.

<u>V.</u> By IV and Propositions 6 and 7, there is an integer k such that

$$t'+2+h(t'+1) \leq k \leq r-2-h(r-1)$$

with $M(t'+1) \subseteq \{v_{t'-1},v_{t'},v_{t'+2+h(t'+1)},\ldots,v_k\}$ and

$$M(r-1) \subseteq \{v_k,\ldots,v_{r-2-h(r-1)},v_r,v_1\}.$$

Let

$$J_1=\{v_r,v_1\},\ J_2=\{v_{t'-1},v_{t'}\},$$

$$J_3=\{v_{t'+2+h(t'+1)},\ldots,v_{r-2-h(r-1)}\}$$

where $I=\{1,2,3,\}$ and $I' \subseteq \{3\}$.

Let

$$J=K_1 \cup K_2 \cup K_3$$

where $K_1=\{v_2,\ldots,v_{t'-2}\}$,

$$K_2=\{v_{t'+2},\ldots,v_{t'+1+h(t'+1)}\}$$

when $h(t'+1)>0$ or the empty set when $h(t'+1)=0$, and

$$K_3=\{v_{r-1-h(r-1)},\ldots,v_{r-2}\}$$

when $h(r-1)>0$ or the empty set when $h(r-1)=0$. Here,

$$|J| = (t'-3)+h(t'+1)+h(r-1)$$
$$\geq p-3+h(t'+1)+h(r-1)$$

which contradicts Proposition 5.

Case One now has been solved.

PART C. CASE TWO. C has no Y-bridge.

Let $B=v_r u_1 \ldots u_{p-1} v_t$ be a longest C-path of C.

I. Since G is 3-connected, $G\backslash\{v_r,v_t\}$ is still connected. Let $\Phi = \{q \mid$ there is a (BUC)-path $P=u_q \ldots v_{q'}$ joining B and C in $G\backslash\{v_r,v_t\}\}$.

Obviously, $\Phi \subseteq \{2,\ldots,p-2\}$. Otherwise, there would be a Y-bridge of C.

II. In the proof of the previous case, we paid more attention to the cycle C. In the proof of this case, we will pay more attention to the bridge B.

For the sake of convenience, denote $w(u_i,BUC)$, $h(u_i,BUC)$ and $M(u_i,BUC)\cap V(B)$ by w_i, h_i and M_i, respectively. Here, we have that

$$d(w_i) \leq h_i+d_C(w_i)+|M_i|$$

by Lemma 1.

III. SUBCASE 1. Assume there is a q in Φ such that

$$d_C(w_1)+d_C(w_{q+1})\leq 3 \quad \text{or} \quad d_C(w_{p-1})+d_C(w_{q-1})\leq 3.$$

Without loss of generality, let $q\in\Phi$ and $d_C(w_1)+d_C(w_{q+1})\leq 3$. And let $P=v_{q'}{*}{*}{*}u_q$ be a (BUC)-path joining u_q and $v_{q'}$ $(q'\neq r,\ t)$.

The pair of vertices w_1,w_{q+1} have some properties similar to a "summable pair" on B which was considered in Case one. This similarlity will be considered and exploited in this subcase.

i. We claim that there is no (BUC)-path joining $v_{q'}$ and u_1. Otherwise. either the C-path $v_{q'}{*}{*}{*}u_1Bu_{p-1}v_r$ would be longer than B or C would have a Y-bridge.

We claim that there is no (BUC)-path joining u_1 and u_{q+1}. Otherwise, the (BUC)-path $u_1{*}{*}{*}u_{q+1}$ would not intersect with P, and then either the C-path $v_{q'}Pu_q\bar{B}u_1{*}{*}{*}u_{q+1}Bu_{p-1}v_r$ would be longer than B or C would have a Y-bridge. *Hence,* w_1 *and* w_{q+1} *are a pair of non-adjacent vertices.*

We claim that u_q and u_{q+1} is not joined by a (BUC)-path of length at least 2. Otherwise, B would not be a longest C-path.

Hence, if $u_i \epsilon N(u_1,BUC)$ *and* $u_j \epsilon N(u_q,BUC)$, *then the three* (BUC)-*paths* $u_i{*}{*}{*}u_1$, $u_j{*}{*}{*}u_{q+1}$ *and* P *are internally disjoint.*

ii. We claim $M_1 \cap M_{q+1}^{+1} = \emptyset$ *in* $\{u_2,\ldots,u_q\}$. Otherwise, let u_i be a vertex in this intersection. By i,
$v_{q'}Pu_q\bar{B}u_i{*}{*}{*}u_1Bu_{i-1}{*}{*}{*}u_{q+1}Bu_{p-1}v_t$ is a C-path. This path is either longer than B or else there is a Y-bridge of C both of which are contradictions.

Similarly, $M_{q+1} \cap M_1^{+1} = \emptyset$ *in* $\{u_{q+2},\ldots,u_{p-1}\}$. Now we can use Lemma 3 on M_1 and M_{q+1}.

iii. Let's get a general inequality similar to Proposition 5 for M_1 and M_{q+1}. Let

$$M_1 \cup M_{q+1} \subseteq \bigcup_{\mu \epsilon I} J_\mu$$

where $\{J_\mu \mid \mu \in I\}$ is a collection of pairwise-disjoint subintervals of $\{u_2,\ldots,u_q\}$ or $\{u_{q+2},\ldots,u_{p-1}\}$. By Lemma 3,

$$|M_1 \cap J_\mu| + |M_{q+1} \cap J_\mu| \le |J_\mu|+1$$

for any $\mu \in I$. So

$$|M_1|+|M_{q+1}| \le \sum_{\mu \in I} |J_\mu| + |I|.$$

Let

$$J = [\{u_2,\ldots,u_q\} \cup \{u_{q+2},\ldots,u_{p-1}\}] \setminus [\bigcup_{\mu \in I} J_\mu].$$

We have that

$$|M_1|+|M_{q+1}| \le p-3-|J|+|I|.$$

Hence, by Proposition 3,

$$\begin{aligned} p+3 &\le m \\ &\le d(w_1)+d(w_{q+1}) \\ &\le h_1+h_{q+1}+d_C(w_1)+d_C(w_{q+1})+|M_1|+|M_{q+1}| \\ &\le h_1+h_{q+1}+3+(p-3-|J|+|I|), \end{aligned}$$

that is,

$$|J| \le h_1+h_{q+1}-3+|I|. \tag{4.3}$$

This inequality will be used frequently in this subcase.

<u>iv. If</u> $2 \le i < j \le q$, <u>it is impossible that</u>

$$u_i \in M_{q+1} \text{ and } u_j \in M_1.$$

If not, choose $j-i$ as small as possible. By i, $v_{q'}Pu_q\bar{B}u_j{*}{*}{*}u_1Bu_i{*}{*}{*}u_{q+1}Bu_{p-1}v_t$ is a C-path and is either longer than B or produces a Y-bridge if

$$J=\{u_{i+1},\ldots,u_{j-1}\}$$

contains fewer than $h_1+h_{q+1}+1$ vertices. Hence, let

$$J_1=\{u_2,\ldots,u_i\},\ J_2=\{u_j,\ldots,u_q\} \text{ and } J_3=\{u_{q+2},\ldots,u_{p-1}\}$$

with $$I=\{1,2,3\}$$
which leads to a contradiction of (4.3) of iii.

<u>v.</u> <u>If</u> $q+2\le i<j\le p-1$, <u>it is impossible that</u>
$$u_j\in M_{q+1} \ \underline{\text{and}}\ u_i\in M_1.$$

If not, choose j-i as small as possible. As in the previous cases, $v_q' P u_q \bar{B} u_1 \ast\ast\ast u_i \bar{B} u_{q+1} \ast\ast\ast u_j B u_{p-1} v_t$ is a C-path. This path is either longer than B or C has a Y-bridge unless
$$J=\{u_{i+1},\dots,u_{j-1}\}$$
contains at least $h_1+h_{q+1}+1$ vertices. Hence, letting
$$J_1=\{u_2,\dots,u_q\},\ J_2=\{u_{q+2},\dots,u_i\},\ J_3=\{u_j,\dots,u_{p-1}\}$$
and $$I=\{1,2,3\},$$
we contradict (4.3) of iii.

<u>vi.</u> By iv and v, there are integers a and b such that,
$$2\le a\le q,\quad q+2\le b\le p-1,$$
$$M_1\subseteq\{u_2,\dots,u_a,u_b,\dots,u_{p-1}\}$$
and $$M_{q+1}\subseteq\{u_a,\dots,u_b\}.$$
Because the maximum length of a C-path is p, we must have that
$$M_1\subseteq\{u_2,\dots,u_a,u_b,\dots,u_{p-1}\}\backslash\{u_1,\dots,u_{1+h_1}\}$$
and $$M_{q+1}\subseteq\{u_a,\dots,u_b\}\backslash\{u_{q+1-h_{q+1}},\dots,u_{q+1+h_{q+1}}\}.$$

<u>vii.</u> <u>We claim</u> $M_1\ne\emptyset$ <u>and</u> $M_{q+1}\ne\emptyset$.

If $h_1=0$, $u_2\in M_1$. If $h_1\ge 1$, $C\cap M(u_1,B\cup C)\subseteq\{v_t\}$ since B is a longest C-path and C has no Y-bridges. So $|M_1|\ge 1$ because $|M(u_1,B\cup C)|\ge 2$ by Lemma 2.

If $M_{q+1}=\emptyset$, then $M(u_{q+1},B\cup C)$ contains at least two vertices of C. Let

$$i=\min\{\mu\geq q+2 \mid u_\mu\in M_1\}$$

when $M_1\cap\{u_{q+2},\ldots,u_{p-1}\}\neq\emptyset$, or

$$i=\min\{\mu \mid u_\mu\in M_1\}$$

when $M_1\cap\{u_{q+2},\ldots,u_{p-1}\}=\emptyset$.

When $i\geq q+2$, let $v_s\in M(u_{q+1},B\cup C)\setminus\{v_t\}$. Since the C-path $v_s{*}{*}{*}u_{q+1}\bar{B}u_1{*}{*}{*}u_iBu_{p-1}v_t$ is not longer than B,

$$J=V(B)\setminus[\{u_1,\ldots,u_{q+1}\}\cup\{u_i,\ldots,u_{p-1}\}]$$

must contain at least $h_{q+1}+h_1$ vertices. Letting

$$J_1=\{u_2,\ldots,u_q\},\ J_2=\{u_i,\ldots,u_{p-1}\}$$

and
$$I=\{1,2\}$$

we contradict (4.3) of iii.

When $i\leq q$, let $v_s\in M(u_{q+1},B\cup C)\setminus\{v_r\}$. Since the C-path $v_s{*}{*}{*}u_{q+1}\bar{B}u_i{*}{*}{*}u_1v_r$ is not longer than B,

$$J=\{u_2,\ldots,u_{i-1},u_{q+2},\ldots,u_{p-1}\}$$

must contain at least h_1+h_{q+1} vertices. (Note that $i\geq 2$ because $M_1\neq\emptyset$.) Letting

$$J_1=\{u_i,\ldots,u_q\} \text{ and } I=\{1\},$$

we again contradict (4.3) of iii.

<u>viii.</u> Suppose that $M_1\cap\{u_2,\ldots,u_q\}\neq\emptyset$.

If $M_{q+1}\cap\{u_2,\ldots,u_q\}\neq\emptyset$, let

$$J=\{u_1,\ldots,u_{1+h_1},u_{q+1-h_{q+1}},\ldots,u_{q+1}\}\setminus\{u_1,u_{q+1}\}$$

which contains h_1+h_{q+1} vertices. (By vi, $1+h_1<a<q+1-h_{q+1}$ because both M_1 and M_{q+1} are not empty in $\{u_2,\ldots,u_q\}$.) Let

$$J_1=\{u_{2+h_1},\ldots,u_{q-h_{q+1}}\},\ J_2=\{u_{q+2},\ldots,u_{p-1}\}$$

and

$$I=\{1,2\}.$$

Since $M_1 \cup M_{q+1} \subseteq J_1 \cap J_2$, we have a contradiction of (4.3) of iii.

If $M_{q+1} \cap \{u_2,\ldots,u_q\}=\emptyset$, then by v and vi, M_1 and M_{q+1} would not intersect with $\{u_{q+1},\ldots,u_{q+1+h_{q+1}}\}$ and

$$M_{q+1} \subseteq \{u_{q+2+h_{q+1}},\ldots,u_{p-1}\}.$$

(Note that $M_{q+1}\neq\emptyset$ implies that $p-1\geq q+2+h_{q+1}$.) Let

$$J=\{u_1,\ldots,u_{1+h_1},u_{q+1},\ldots,u_{q+1+h_{q+1}}\}\setminus\{u_1,u_{q+1}\},$$

$$J_1=\{u_{2+h_1},\ldots,u_q\},\ J_2=\{u_{q+2+h_{q+1}},\ldots,u_{p-1}\}$$

and

$$I=\{1,2\}.$$

Here, $M_1 \cup M_{q+1} \subseteq J_1 \cup J_2$ from which follows a contradiction of (4.3) of iii.

So we will assume that $M_1 \cap \{u_2,\ldots,u_q\}=\emptyset$, that is,

$$M_1 \subseteq \{u_{q+2},\ldots,u_{p-1}\}.$$

<u>ix.</u> Let

$$i=\min\{\mu \mid u_\mu \epsilon M_{q+1}\}\ \underline{\text{and}}\ j=\max\{\mu \mid u_\mu \epsilon M_1\}.$$

Recall that $M_1\neq\emptyset$, $M_{q+1}\neq\emptyset$ by vii. By vi and viii, $M_1 \cup M_{q+1} \subseteq \{u_i,\ldots,u_j\}$.

When $i\leq q$, let

$$J_1=\{u_i,\ldots,u_q\},\ J_2=\{u_{q+2},\ldots,u_j\}$$

and

$$I=\{1,2\}.$$

Since the C-path $v_r u_1 *** u_j \bar{B} u_{q+1} *** u_i B u_q \bar{P} v_{q'}$ is not longer than B and C has no Y-bridge,

$$J=V(B)\setminus[J_1 \cup J_2 \cup \{u_1,u_{q+1}\}]$$

must contain at least $h_1+h_{q+1}+1$ vertices. This again contradicts

(4.3) of iii.

When $i \geq q+2$, then $M_{q+1} \cap \{u_2,\ldots,u_q\}=\emptyset$ and $i \geq q+2+h_{q+1}$ so that

$$J_1=\{u_{q+2+h_{q+1}},\ldots,u_j\},$$

contains all vertices of M_1 and M_{q+1}. Since the C-path $v_r u_1{*}{*}{*}u_j \bar{B} u_q \bar{P} v_q$, is not longer than B and C has no Y-bridge, $\{u_2,\ldots,u_q,u_{j+1},\ldots,u_{p-1}\}$ must contain at least h_1+1 vertices. Hence, $J=V(B)\backslash[J_1 \cup \{u_1,u_{q+1}\}]$ contains at least $h_1+h_{q+1}+1$ vertices. We again contradict (4.3) of iii.

<u>This completes the proof of Subcase 1.</u>

<u>IV. SUBCASE 2. We may assume</u>

$d_C(w_1)+d_C(w_{q+1}) \geq 4$ <u>and</u> $d_C(w_{p-1})+d_C(w_{q-1}) \geq 4$, <u>for any</u> $q \in \Phi$.

<u>i.</u> In order to avoid a Y-bridge,

$V(C) \cap M(u_1,B \cup C) \subseteq \{v_t,v_r\}$ and $V(C) \cap M(u_{p-1},B \cup C) \subseteq \{v_t,v_r\}$.

Hence,

$$d_C(w_1),\ d_C(w_{p-1}) \leq 2,$$

and therefore, $d_C(w_{q+1})$ and $d_C(w_{q-1}) \geq 2$

for any $q \in \Phi$.

<u>ii.</u> If $h_1 \geq 1$, then $M(u_1,B \cup C) \subseteq V(B) \cup \{v_t\}$ in order to avoid a C-path of length greater than p joining v_r and v_t. So $d_C(w_1) \leq 1$. But we can choose q as the greatest element of Φ. Then

$$M(u_{q+1},B \cup C) \cap [V(C)\backslash\{v_r,v_t\}]=\emptyset,$$

and hence, $d_C(w_{q+1}) \leq 2$. Thus, $d_C(w_1)+d_C(w_{q+1}) \leq 3$ which contradicts the hypotheses of Subcase 2.

So we conclude that $h_1=0$ and symmetrically, that $h_{p-1}=0$.

<u>iii.</u> Choose q as the greatest element in Φ . Since $q+1\notin\Phi$,

$$M(u_1,BUC)\cap V(C) = M(u_{q+1},BUC)\cap V(C) = \{v_r,v_t\}.$$

Let $v_{q'}\in N(u_q,BUC)$ such that $q'\neq r,t$. Since B is a longest C-path, $u_q{***}v_{q'}$ and $u_{q+1}{***}v_r$ are disjoint (BUC)-paths.

<u>iv.</u> <u>We</u> <u>claim</u> $(u_1,u_{p-1})\notin E(G)$. Otherwise, the C-path $v_r{***}u_{q+1}Bu_{p-1}u_1Bu_q{***}v_{q'}$ either is longer than B or has the same length as B and C would have a Y-bridge.

<u>v.</u> Since $h_1=h_{p-1}=0$, $N_B(u_1)=M_1$ and $N_B(u_{p-1})=M_{p-1}$. We have that

$$M_1\cap M_{p-1}^{+1}\cap\{u_2,\ldots,u_{p-2}\} = \emptyset$$

If not, let u_i be in this set. If $i\neq q+1$, then without loss of generality assume $i\leq q$. The C-path $v_r{***}u_{q+1}Bu_{p-1}u_{i-1}\bar{B}u_1u_iBu_q{***}v_{q'}$ either is longer than B or there is a Y-bridge of C. If $i=q+1$, then the C-path $v_tu_{p-1}\bar{B}u_{q+1}u_1Bu_q{***}v_{q'}$ either is longer than B or there is a Y-bridge of C.

<u>vi.</u> By iv and v, The pair of vertices u_1,u_{p-1} behaves similar to a "summable pair" on B. We have that

$$\begin{aligned} m &\leq d(u_1)+d(u_{p-1}) \\ &\leq d_C(u_1)+d_C(u_{p-1})+|M_1|+|M_{p-1}| \\ &\leq 4+(|\{u_2,\ldots,u_{p-2}\}|+1) \qquad \text{(by lemma 3)} \\ &= p+2 \end{aligned}$$

which contradicts Proposition 3.

This completes the proof of the theorem.

■■■

PART E

SOME RESULTS ABOUT DIRECTED GRAPHS

In addition to the preceding work done on undirected graphs, I have obtained some results about directed graphs. Most of them have been published already and thus will be surveyed here. The proofs will not be included.

An oriented graph is a directed graph in which each pair of vertices is joined by at most one arc. Let k be the minimum indegree and outdegree of the oriented graph. B.Jackson [10; 2, p465] conjectured that *the length of a longest directed cycle in an oriented graph is at least* 2k+1. He was able to prove that *an oriented graph contains a directed path of length* 2k+1. This result can be improved as follows.

THEOREM 4.1 [22].

<u>Let</u> D <u>be</u> <u>an</u> <u>oriented</u> <u>graph</u> <u>in</u> <u>which</u> <u>the</u> <u>indegree</u> <u>and</u> <u>the</u> <u>outdegree</u> <u>is</u> <u>at</u> <u>least</u> k <u>for</u> <u>each</u> <u>vertex.</u> <u>Then</u> D <u>contains</u> <u>either</u> <u>a</u> <u>directed</u> <u>cycle</u> <u>of</u> <u>length</u> 2k+1 <u>or</u> <u>a</u> <u>directed</u> <u>path</u> <u>of</u> <u>length</u> 2k+2.

B.Jackson [10] showed that *if* D *is an oriented graph of order* 2k+2, k≥2, *then* D *contains a directed Hamilton cycle.* The following improves the latter result.

THEOREM 4.2 [24].

<u>If</u> D <u>is</u> <u>an</u> <u>oriented</u> <u>graph</u> <u>of</u> <u>order</u> 2k+3, <u>where</u> k <u>is</u> <u>the</u> <u>minimum</u> <u>indegree</u> <u>and</u> <u>outdegree</u> <u>and</u> k≥3, <u>then</u> D <u>contains</u> <u>a</u> <u>directed</u> <u>Hamilton</u> <u>cycle.</u>

The main part of the following theorem is a the corollary of Theorem 4.2, but the nice structure of tournaments allows a simpler proof and a better lower bound on the degrees.

THEOREM 4.3 [20].

<u>A regular tournament of order at least</u> 5 <u>contains two arc-disjoint directed Hamilton cycles.</u>

B.Jackson [10] also showed that *if* D *is a bipartite oriented graph of order at most* 4k, *then* D *contains a directed Hamilton cycle.* The following theorem removes the constraint on the order of the graphs.

THEOREM 4.4 [21].

<u>If</u> D <u>is an bipartite oriented graph with the indegree and the outdegree at least</u> k <u>for each vertex, then</u> D <u>contains either a directed cycle of length at least</u> 4k <u>or a directed path of length</u> 4k+1.

An oriented graph D=(V,A) is called a <u>multipartite tournament</u> if V(D) is a union of disjoint parts $V_1,\ldots,V_t$ and each part V_i is an independent set of D with exactly one arc joining each pair of vertices from different parts of V(D). C.Thomassen [4, 17] conjectured that *an oriented graph of order at most* 3k *contains a directed Hamilton cycle.* Although some counterexamples to this conjecture have been found [4, p9], it still may be true if the graph is tripartite. It is easy to see that a tripartite oriented graph of order at most 3k is a regular tripartite tournament. Furthermove, it is possible that every regular multipartite tournament contains a directed Hamilton cycle. B.Jackson's theorem about bipartite oriented graphs actually deals with regular bipartite tournaments, because 4k is the lower bound on the order of the bipartite oriented graphs. The result I obtained about the multipartite tournament is the following.

THEOREM 4.5 [27].

A regular multipartite tournament of order n contains a directed cycle of length at least n-1.

The following result is an immediate consequence.

COROLLARY 4.5 [27].

A regular multipartite tournament contains a directed Hamilton path.

For the general directed graphs, Nash-Williams [14; 3, p201] showed that *if* D *is a directed graph of order* n *with minimum indegree and outdegree at least* n/2, *then* D *contains a directed Hamilton cycle.* He also conjectured that *such a directed graph contains two arc-disjoint directed Hamilton cycles.* Nincak [15] found a counterexample of order 6, but this conjecture is still open for $n \geq 7$. I obtained the following results on this problem.

THEOREM 4.6 [23].

If D is a directed graph of order n in which the indegree and the outdegree of every vertex is at least n/2 and $n \geq 9$, then D contains two arc-disjoint directed cycles. One is a directed Hamilton cycle and the other is of length at least n-1.

The following result is almost a corollary of Theorem 4.6, but it has a better lower bound.

THEOREM 4.7 [25].

If D is a directed graph of order n in which the indegree and the outdegree of every vertex is at least n/2 and $n \geq 5$, then D contains an arc-disjoint pair consisting of a directed Hamilton cycle and a directed

Hamilton path.

The directed graph D is said to have the arc-pancyclic property if each arc of D is contained in directed cycles of lengths h, $h=3,\ldots,|V(D)|$. The arc-pancyclic property of tournaments was a quite interesting problem. Since Alspach [1] obtained the first result for regular tournaments, many different families of tournaments have been studied by mathematicians. Suggested by Y.Zhu [28], a necessary and sufficient condition for this property in tournaments has been found.

THEOREM 4.8 [19].

If each arc of the tournament D is contained in a directed cycle of length 3, then D has the arc-pancyclic property unless D belongs to one of two certain families of tournaments.

The structure of these two families of tournaments has been described in [19]. Although these exceptions exist, we still can obtain the following result.

COROLLARY 4.8 [19].

If each arc of the tournament D is contained in a directed cycle of length three, then except for at most one arc of D, every arc is contained in directed cycles of lengths h, $h=3,\ldots,|V(D)|$.

All these properties about the existence of directed Hamilton cycles, being pan-cyclic, vertex-pancyclic or arc-pancyclic in tournaments can be determined in polynomial time. Surprisingly, the complexity of the determination of the property of whether or not a tournament is arc-hamiltonian is still unknown. The preceding property seems "easier" than the arc-pancyclic property. (A tournament is arc-hamiltonian if each arc lies in a directed Hamilton cycle.)

A tournament D is said to be domination orientable if there is a labeling of its vertices such that v_i dominates $v_{i+1}, v_{i+2}, \ldots, v_{i+od(v_i)}$ for every $v_i \epsilon V(D)$, modulo $|V(D)|$. A result about a property similar to the pan-path-connectivity property has been obtained for this class of tournaments.

THEOREM 4.9 [26].

If D is a domination orientable tournament, then D contains directed paths of each length p, $p=5, \ldots, |V(D)|-1$, from vertex x to vertex y for any pair of vertices x and y in V(D), except for one pair of vertices at most.

Alspach conjectured that among all tournaments, domination orientable tournaments are those which contain the maximum number of directed Hamilton cycles. Using Theorem 4.9, we can get a recursive method for counting the number of directed Hamilton cycles in domination orientable tournaments which will yield the following result.

THEOREM 4.10 [26].

If D is a domination orientable tournament of order n with every indegree and outdegree at least δ, then D contains h(D) directed Hamilton cycles, where

$$h(D) \geq \begin{cases} 3(2^{n-6}) & \text{if } \delta\geq 2, \\ 17(2^{n-8}) & \text{if } \delta\geq 3, \\ 140(2^{n-11}) & \text{if } \delta\geq 4. \end{cases}$$

In the original paper, the lower bound fomula of h(D) in terms of δ and n looks quite complicated. Here, Theorem 4.10 is only a simplified corollary of it.

PART F

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