

**CHIRAL SYMMETRY BREAKING IN QCD-LIKE THEORIES**

by

**Joseph Obi Otu**

**B.Sc., University of Nigeria, 1976**

**M.Sc., University of Nigeria, 1982**

**THESIS SUBMITTED IN PARTIAL FULFILLMENT OF  
THE REQUIREMENTS FOR THE DEGREE OF  
MASTER OF SCIENCE  
in the Department  
of  
Physics**

©

**Joseph Obi Otu, 1986**

**SIMON FRASER UNIVERSITY**

**October 1986**

**APPROVAL**

**Name:** Joseph Obi Otu  
**Degree:** Master of Science  
**Title of Thesis:** Chiral Symmetry Breaking in QCD-Like Theories

**Examining Committee:**

Chairman: M. Plischke

K.S. Viswanathan  
Senior Supervisor

R.H. Enns

D.H. Boal

G.W. Semenoff  
External Examiner  
Department of Physics  
University of British Columbia

Date Approved: October 20, 1986

PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis, project or extended essay (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this work for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this work for financial gain shall not be allowed without my written permission.

Title of Thesis/Project/Extended Essay

Chiral Symmetry Breaking in OCD-like Theories

---

---

---

---

Author:

(signature)

Joseph Obi Otu

(name)

Nov. 6, 1986

(date)

## ABSTRACT

Chiral symmetry breaking in Quantum Chromodynamics (QCD) is studied using a composite operator scheme developed by Cornwall, Jackiw and Tomboulis (CJT). A modified effective potential originally due to Casalbuoni, De Curtis, Dominici and Gatto (CDDG) is used which is equivalent to the conventional scheme in that it contains the same physics as the original CJT formulation, but it is shown to be free of defects in the CJT effective potential.

It is shown here that the modified effective potential is bounded below and is stable against fluctuations to the stationarity condition, namely, the Schwinger Dyson equation. Furthermore, it is found that only in asymptotically free field theories such as QCD, chiral symmetry occurs for sufficiently large coupling constant.

Two approaches to studying chiral symmetry breaking are used. In the first one, explicit chiral symmetry breaking is studied by analyzing the Schwinger Dyson equation where the concept of constituent and current quark mass is exploited to estimate the constituent masses of the light quarks. The second approach is the Rayleigh-Ritz variational scheme where the effective potential is minimized with respect to  $m$  (the vacuum expectation value of the fermion bilinear) and the finite value of  $m$  that minimizes the effective potential is estimated for various values of the infrared cut-off parameter  $p_c$ . For example, when  $p_c/\Lambda = 1.3$ , the minimum occurs at  $m = 0.67\Lambda_{\text{QCD}}$ .

## ACKNOWLEDGEMENTS

I am very grateful to my supervisor, K.S. Viswanathan, for his help and support. Vish suggested this problem and helped see it through to completion.

I would also like to express my sincere thanks to Steve Kloster for the time he took to patiently explain how to do numerical analysis on a computer.

I am especially pleased to acknowledge many helpful conversations with Dan Kay, Bev McLeod and Ed Levinson. I also want to thank S. Maheswaran and Rudolf Bauchspiess for showing me how to plot graphs on a computer.

I deeply appreciate the company of my friends, M. Hikam, Alan Steele, S. Charbonneau and D. Jiang.

I would like to thank my family, Imma, Mike and Brigit; they were a real source of encouragement to me.

Finally, I would like to thank Doreen Young, who carefully typed this manuscript.

## TABLE OF CONTENTS

APPROVAL . . . . .	ii
ABSTRACT . . . . .	iii
ACKNOWLEDGEMENTS . . . . .	iv
TABLE OF CONTENTS . . . . .	v
LIST OF TABLES . . . . .	vi
LIST OF FIGURES . . . . .	vi
CHAPTER 1 . . . . .	1
1.1 Introduction and Motivation . . . . .	1
CHAPTER 2 . . . . .	11
2.1 Chiral Symmetries in QCD-Like Theories . . . . .	11
2.2 The Effective Potential Formalism . . . . .	20
CHAPTER 3 . . . . .	32
3.1 The Schwinger-Dyson Equation . . . . .	32
3.2 Solutions of the SD Equation . . . . .	35
3.3 Explicit Chiral Symmetry Breaking . . . . .	46
CHAPTER 4 . . . . .	51
4.1 Spontaneous Chiral Symmetry Breaking: A Variational Scheme . . . . .	51
4.2 Spontaneous Chiral Symmetry Field Breaking for Free Field Theories . . . . .	64
CHAPTER 5 . . . . .	
5.1 Stability of Chiral Symmetry Breaking Solutions . . . . .	70
CHAPTER 6 . . . . .	79
APPENDIX A . . . . .	81
REFERENCES . . . . .	86

### LIST OF TABLES

Table 5.1. Eigenvalues of the operator (I-WD).

### LIST OF FIGURES

Figure 2.1 Fermion bilinear with vacuum quantum numbers.

Figure 2.2 Two loop contribution to effective potential.

Figure 3.1 A truncated QCD running coupling constant.

Figure 3.2 Dependence of constituent quark mass  $M(0)$  on current quark mass  $m_R$ .

\*Figure 3.3 Dependence of constituent quark mass on infrared cut-off  $t_c$  at exact chiral limit  $m_R = 0$ .

Figure 4.1 Dependence of the effective potential  $V_{CDDG}$  on the vacuum expectation value (vev)  $m$  for  $p_C/\Lambda = 1.3$ .

Figure 4.2 Dependence of  $V_{CDDG}$  on  $m$  for  $p_C/\Lambda = 1.687$ .

Figure 4.3 Dependence of  $V_{CDDG}$  on  $m$  for  $p_C/\Lambda = 2.0$ .

Figure 4.4 Dependence of the effective potential  $V_{CJT}$  on  $m$  for  $p_C/\Lambda = 1.3$ .

Figure 4.5 Dependence of  $V_{CDDG}$  on  $m$  when the solution of the SD equation is used for  $\Sigma(p)$  in the  $\lg(S_0^{-1} + \Sigma(p))$  term of  $V_{CDDG}$  for  $p_C/\Lambda = 1.3$ .

Figure 4.6 Dependence of the effective potential  $\bar{V}$  on  $m$  for  $p_C = 1.3$ .

† Figure 4.7 Dependence of  $V_{CDDG}$  on  $m$  for for  $p_C/\Lambda = 1.3$ ,  
 $\epsilon_0 = 0.01/9$ .

Figure 4.8 Dependence of  $V_{CDDG}$  on  $m$  for  $p_C/\Lambda = 1.3$ ,  $\epsilon_0 = 8/9$ .

Figure 4.9 Dependence of  $V_{CJT}$  on  $m$  for  $p_C/\Lambda = 1.3$ ,  $\epsilon_0 = 0.01/9$ .

Figure 4.10 Dependence of  $V_{CJT}$  on  $m$  for  $p_C/\Lambda = 1.3$ ,  $\epsilon_0 = 8/9$ .

\* All figures from here on are for three colors ( $N = 3$ ), three flavors ( $n_f = 3$ ) and a running coupling constant.

† All figures from here on are for ( $N = 3$ ), ( $n_f = 3$ ) and a fixed coupling constant.



## CHAPTER 1

### 1.1 INTRODUCTION AND MOTIVATION

Quantum Chromodynamics (QCD) proposed as a theory of strong interactions is based on three fundamental ideas [1].

1. That all hadrons (strongly interacting particles) are made up of fundamental constituents called quarks which possess fractional electric charge and are spin  $1/2$  Fermi fields.
2. The existence of a particular kind of quantum number called "color". There are three kinds of color (Red Green Blue) and the corresponding symmetry of nature is exact. With three colors also, the statistics of the baryons come out right.
3. The symmetry transformation may be coordinate dependent, and the derivatives are absorbed in gauge transformations of a set of massless vector fields, the gluons. Yang and Mills [26] proposed such a theory in another context in 1954.

2 and 3 differ from Quantum Electrodynamics (QED) only in one point: three colors instead of one electric charge. However, this very important difference is of crucial importance. Unlike  $U(1)$  transformations in QED, the  $SU(3)$  transformations do not commute with each other. So gluons, unlike photons, are themselves charged. Because of the non-Abelian nature of the QCD symmetry transformations and the demand that this symmetry be exact, QCD exhibits confinement and asymptotic freedom which are not there in QED.

The Fermi spin 1/2 fields (quarks) are represented by the Dirac spinor fields  $q_{\alpha}^m(x)$  (the four spinor components are implicit) where  $m$  and  $\alpha$  label internal degrees of freedom. The index  $m$  refers to flavor and corresponds to observed degrees of freedom of existing hadrons. At present five flavors are known and the sixth has been predicted. The index  $\alpha$  refers to the color degrees of freedom. It is assumed that there are exactly three colors so that the statistics of the baryons come out just right, other phenomenological support for fixing the number of colors at three are found in [1]. QCD is diagonal in flavor index, ie. the flavor index has no dynamical role. The weak interaction current will couple to flavor.

With the number of color fixed at three, the only candidates for a simple Lie group of color are  $SU(3)$  and  $S_0(3)$ . With  $S_0(3)$  the desired property of asymptotic freedom is lost when the number of flavors exceeds two, while for  $SU(3)$  we can have up to 16 flavors. Since six flavors are already known,  $SU(3)$  is the only viable possibility for the color group of QCD. Another negative point is that with  $S_0(3)$ , a di-quark system could be a color singlet and such objects are not observed.

The main idea in QCD is to make the  $SU_c(3)$  symmetry a local, rather than just a global symmetry. Global gauge invariance implies the existence of conserved currents. Local gauge invariance produces massless vector gauge bosons, prescribes (or at least restricts) the form of the interaction of gauge bosons with sources, and generates interactions among the gauge bosons if the symmetry is non-Abelian.

The local  $SU_C(3)$  gauge fields are the colored gluons,  $A(x)$ ,  $a = 1, 2, \dots, 8$ , which transform as an adjoint representation of  $SU_C(3)$ . They do not carry flavor. The minimally locally gauge invariant Lagrangian density implied by this  $SU_C(3)$  symmetry is

$$\mathcal{L}_1(x) = i\bar{q}_\alpha^m \not{\partial}_{\alpha\beta} q_\beta^m - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \quad (1.1)$$

$\alpha, \beta = 1, 2, 3; a = 1, 2, \dots, 8; m = 1, 2, \dots, N.$

where the covariant derivative is defined by

$$\not{\partial}_{\alpha\beta} q_\beta^m \equiv (\delta_{\alpha\beta} \partial_\mu - ig\lambda_{\alpha\beta}^a A_\mu^a / 2) \gamma^\mu q_\beta^m \quad (1.2)$$

and the gauge field strength tensor is

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \quad (1.3)$$

The  $\lambda_{\alpha\beta}^a$  are  $SU(3)$  matrices satisfying

$$[\lambda^a, \lambda^b] = 2if^{abc} \lambda^c, \quad (1.4)$$

and  $f^{abc}$  are the structure constants of  $SU_C(3)$ . The second term in (1.1) is the Yang-Mills Lagrangian for self-interacting  $SU_C(3)$  non-Abelian vector gauge fields. Equation (1.1) is invariant under the local gauge transformations

$$\begin{aligned} A_\mu(x) &= A_\mu^a(x) \lambda^a / 2 + U(x) A_\mu(x) U^{-1}(x) + \frac{i}{g} U(x) \partial_\mu U^{-1}(x) \\ q^m(x) &\rightarrow U(x) q^m(x) \\ U(x) &= \exp(i\theta^a(x) \frac{\lambda^a}{2}) \end{aligned} \quad (1.5)$$

where  $\theta^a(x)$  is the space-time dependent parameter of the local  $SU_C(3)$  gauge transformation  $U(x)$ . (Matrix notation is used in 1.5.)

For  $N$ -flavors, the flavor group is  $SU(N)$ . That is,  $(q_\alpha^1, q_\alpha^2, \dots, q_\alpha^N)$  for  $\alpha = 1, 2, 3$  form the fundamental representations of  $SU(N)$ .  $N = 3$  gives the well known  $SU(3)$  hadronic spectrum, while  $N = 4$  allows for the charm flavor, though these flavor symmetries are broken in nature. Since the Lagrangian density  $\mathcal{L}_1(x)$  describes massless quarks, it is actually invariant under the larger chiral group  $SU(N) \times SU(N) \times U_B(1) \times U_A(1)$  with

$$q_{R-L} = \frac{1}{2} (1 \pm \gamma_5) q \quad (1.6)$$

where  $\gamma_5$  is defined in (2.6).

$U_B(1)$  corresponds to baryon number conservation and  $U_A(1)$  to axial-baryon number conservation. The  $U_A(1)$  symmetry is not observed or probably does not exist in the first place. The chiral  $SU(N) \times SU(N)$  flavor symmetry is not exact in the real world. It is easily broken in QCD by adding to  $\mathcal{L}_1(x)$  a quark mass term  $\mathcal{L}_M$  which is an  $SU_C(3)$  singlet but transforms like a member of the  $(N, N^*) \times (N^*, N)$  representation of the chiral group.

$$\begin{aligned} \mathcal{L}_M &= \bar{q}_\alpha^m M_0^{mn} q_\alpha^n \\ M_0^{mn} &= \delta^{mn} M_0^n \end{aligned} \quad (1.7)$$

$M_0^{mn}$  is the quark mass matrix. This is the only possible flavor breaking term one can have without introducing new fields or losing renormaliz-

ability. The origin of this flavor breaking term is in the weak and electromagnetic interactions where the flavor index plays a dynamical role.

For a consistent quantum field theory involving non-Abelian vector gauge fields, we must add to the Lagrangian ( $\mathcal{L}_1(x) + \mathcal{L}_M(x)$ ) a gauge fixing term  $\mathcal{L}_{g.f}$  which ensures a proper quantization procedure, and a Faddeev-Popov ghost term  $\mathcal{L}_{gh}$  which depends on how the gauge is fixed and preserves unitarity. Therefore

$$\mathcal{L}_{QCD} = \mathcal{L}_1 + \mathcal{L}_M + \mathcal{L}_{g.f} + \mathcal{L}_{gh} \quad (1.8)$$

is the complete Lagrangian density for QCD.

The Lagrangian density (1.8) when used in naive perturbation theory in the coupling constant  $g$ , seems to describe a world of interacting quarks and colored gluons in which asymptotically free quarks and gluons could exist. Since free quarks and colored gluons are not observed in nature, the picture may not be a proper description of the world. In addition, much of the low energy phenomenology of the hadronic spectrum has been understood to a degree via PCAC (Partially Conserved Axial Currents) and any viable strong interaction theory should therefore exhibit this feature. To understand the real hadronic world, QCD has to undergo at least two phase transitions, so that the physical world resulting from  $\mathcal{L}_{QCD}$  is nonperturbative. The confinement phase transition results in physical states that are color  $SU_c(3)$  singlets and are integrally charged. For a more qualitative treatment of confinement see [1].

The second phase transition, and which will be of interest to us, is the PCAC phase transition. While the confinement phase transition involves the local color symmetry, PCAC phase transition involves the global flavor chiral symmetry  $SU(N) \times SU(N)$  for  $N$  flavors. Without the quark mass term  $\mathcal{L}_M$ , the chiral symmetry of  $\mathcal{L}_{QCD}$  would imply that each hadronic state has to be parity doubled. The nucleon, for example, would have a partner of opposite parity. If this is not the case, then the chiral  $SU(N) \times SU(N)$  symmetry must be spontaneously broken to  $SU(N)$  in the ground state so that the vacuum has no more left-right symmetry, i.e., the vacuum is not  $\gamma_5$  invariant. Then the Goldstone theorem gives us  $N^2 - 1$  pseudoscalar bosons. All hadronic states are representations of just the vacuum  $SU(N)$  symmetry corresponding to the usual supermultiplets. This PCAC phase accommodates existing phenomenology. The pseudoscalars are the lightest hadrons and all hadrons form approximately  $SU(N)$  multiplets. The mass of the pseudoscalars and the lifting of the  $SU(N)$  degeneracy is to be accounted for by including the quark mass matrix. In the PCAC phase, the pion appears as a collective excitation of a quark anti-quark pair.

The major theoretical problem has been to show that QCD is in the confinement and PCAC phases. Since these are purely non-perturbative phenomena, field theoretic methods have been developed to study the problems non-perturbatively, one of which is the effective potential formalism for composite operators due to Cornwall, Jackiw and Tomboulis [9].

In this thesis, I will be primarily concerned with the PCAC phase transition, i.e., the breaking of chiral symmetry, and nothing else will be said about confinement phase transition.

Chiral symmetry breaking has been studied by many authors [10,11,16] by analyzing an effective potential for chiral symmetry breaking. However, in each case, the treatment has not been very exhaustive. This has led to some results that are not very conclusive. In Ref. [11] also, the authors did not use the effective potential in a consistent way.

The motivation to study chiral symmetry breaking is to do a more self-consistent and exhaustive analysis of dynamical chiral symmetry breaking. We will use a modified effective potential (which is a variant of the Cornwall, Jackiw, Tomboulis (CJT) effective potential) due to Casalbuoni, DeCurtis, Dominici and Gatto [11]. I now briefly review the work that has been done so far, some of the results and state what more is required to be done.

Since the formulation of the effective potential for composite operators by CJT in 1974, it has been used by many to study dynamical symmetry breaking. Higashijima (1984) used the stationarity condition of the CJT effective potential, namely, the Schwinger Dyson (SD) equation, converted it to a differential equation and solved it numerically in the special case when all first and higher order derivatives of the running coupling  $\varepsilon(t)$  ( $\varepsilon(t) = \varepsilon_0/t$ ,  $t = \ln(p^2/\Lambda^2)$ ,  $p_\mu$  is the four momentum and  $\Lambda$  is the QCD scale) with respect to  $t$  are set to zero. The qualitative results showing explicit breaking of chiral symmetry were

quite good. For example, he found that for the infrared cutoff parameter  $t_c < 0.88$ , chiral symmetry is broken spontaneously.

P. Castorina and So-Young Pi (1985) used a variational method in the CJT effective potential. They used as a variational ansatz the asymptotic solution of the SD equation for the generated mass, also derived by D. Politzer (1976) using an operator product expansion, in the CJT effective potential. This parameter ( $t_c$ )-dependent ansatz reduced the effective potential to just a function of an order parameter  $m$  for various values of  $t_c$ . In their numerical study, they found that a minimum exists and chiral symmetry is spontaneously broken for  $p_c < 1.5$ . The effective potential is bounded below. However, for field theories without logarithmic behavior in the coupling and in the generated mass, they found that for the coupling constant  $\epsilon_0 < 2/3$ ,  $m = 0$  was the global minimum indicating that chiral symmetry is not broken and, for  $\epsilon_0 > 2/3$ , chiral symmetry is broken spontaneously, but a stable minimum does not exist. This is inconclusive since it has been argued that chiral symmetry is broken only in asymptotically free field theories [2,17].

R. Casalbuoni, S. DeCurtis, D. Dominici and R. Gatto (1984, 1985) derived the modified CJT effective potential (which we call  $V_{CDDG}$ ) which was found to be satisfactory because it gave the SD equation on minimization. However, they used this modified form inconsistently in doing numerical computation. The form they used in their numerical study does not give the SD equation on minimization. They used a different variational scheme and claimed that direct comparisons, even for



a similar variational ansatz for the generated mass in both  $V_{CJT}$  and  $V_{CDDG}$  are not easy to make. We will use  $V_{CDDG}$  in a consistent way in the original CJT variational scheme and show that the comparison using a similar ansatz is excellent. We will also show that with our variational ansatz, the form they used in their numerical study gives wrong results.

R. Haymaker and T. Matsuki (1986) solved the SD equation for the general case where the first and higher order derivatives of  $\epsilon(t)$  are not set to zero. They found the same qualitative results as Higashijima. However, in their own case, chiral symmetry is broken for values of  $t_c < 1.78$  (This general case was also done by us independently in early 1986.). Since they used the CJT effective potential to get the SD equation, they studied, as a more complete treatment, the nature of the chiral symmetry breaking solutions using  $V_{CJT}$ . They found that using the CJT effective potential, the chiral symmetry breaking solutions correspond to saddle-point instabilities. We will take up this stability question in the case of  $V_{CDDG}$  and show that its solutions are stable.

We will also do a numerical study of  $V_{CDDG}$  for field theories with constant coupling and show that with  $V_{CDDG}$  chiral symmetry is not broken. In other words, chiral symmetry breaking occurs only for theories that exhibit asymptotic freedom behavior such as QCD and this seems to correspond to what is observed in the realm of strong interactions. In summary, these are the things I will be studying in the rest of this thesis.

In Chapter 2, I will discuss the notion of chiral symmetry in QCD-like theories. Then I review the effective potential formalism for doing a quantitative study of dynamical symmetry breaking.

In Chapter 3 the stationarity condition of the effective potential, namely the Schwinger Dyson (SD) integral equation, is derived. This is converted into a nonlinear differential equation with boundary conditions and its asymptotic solutions found. Two approaches to studying chiral symmetry breaking are used. In the first one, the SD nonlinear differential equation is analyzed. The concept of constituent and current quark masses is used to study explicit chiral symmetry breaking and to estimate the constituent quark masses of the light quarks.

In Chapter 4, a second approach, namely spontaneous chiral symmetry breaking, is used. Here the effective potential is analyzed directly using a Rayleigh-Ritz variational method. The effective potential is minimized with respect to  $m$  (the vacuum expectation value of the fermion bilinear) and the finite value of  $m$  that minimizes the effective potential is estimated. Spontaneous chiral symmetry breaking is also studied for theories with fixed coupling constant.

In Chapter 5, stability of the symmetry breaking solutions due to the effective potential  $V_{\text{CDDG}}$  is studied, i.e., whether the symmetry breaking solutions correspond to stable or unstable points of the effective potential.

I end this thesis with a Conclusion followed by an Appendix.

## CHAPTER 2

### 2.1 CHIRAL SYMMETRIES IN QCD-LIKE THEORIES

Chiral symmetries are introduced as formal symmetries of the massless Dirac Lagrangian.

$$\begin{aligned} \mathcal{L} &= i\bar{\psi}\not{\partial}\psi \\ \not{\partial} &= \gamma^\mu(\partial_\mu - igA_\mu), \quad \bar{\psi} = \psi^\dagger\gamma^0 \end{aligned} \quad (2.1)$$

There is one obvious symmetry, i.e. a transformation of the fields  $\psi$ ,  $\bar{\psi}$  which leaves the Lagrangian  $\mathcal{L}$  invariant.

$$\begin{aligned} \psi &\rightarrow e^{i\alpha}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}e^{-i\alpha} \end{aligned} \quad (2.2)$$

The  $\alpha$  is a real parameter. We substitute (2.2) into  $\mathcal{L}$  (2.1) to have

$$\mathcal{L} \rightarrow \mathcal{L}' = i(\bar{\psi}e^{-i\alpha})\gamma^\mu(\partial_\mu - igA_\mu)(e^{i\alpha}\psi).$$

The  $\alpha$  commutes with all the  $\gamma$ -matrices giving

$$\mathcal{L}' = i\bar{\psi}\gamma^\mu(\partial_\mu - igA_\mu)\psi = \mathcal{L}$$

We see that the transformations (2.2) have retained the original form of  $\mathcal{L}$ . This corresponds to fermion number conservation. There is also another symmetry

$$\psi \rightarrow e^{i\alpha\gamma^5}\psi \quad (2.3)$$

To obtain  $\bar{\psi}$  we use the definition (2.1)

$$\begin{aligned} \bar{\psi} = \psi^\dagger\gamma^0 &\rightarrow (e^{i\alpha\gamma^5}\psi)^\dagger\gamma^0 \\ &\rightarrow \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 \end{aligned} \quad (2.4)$$

Now we must use the anti-commutation of  $\gamma$ -matrices

$$\{\gamma_5, \gamma^\mu\} = 0$$

so that

$$e^{\gamma_5 \gamma^\mu} = \gamma^\mu e^{-\gamma_5}$$

and (2.4) becomes

$$\bar{\psi} \rightarrow \bar{\psi} e^{i\alpha\gamma^5} . \quad (2.4)$$

Substituting (2.4) and (2.3) into (2.1) for  $\mathcal{L}$  gives

$$\mathcal{L} \rightarrow \mathcal{L}' = i(\bar{\psi} e^{i\alpha\gamma^5}) \gamma^\mu (\partial_\mu - igA_\mu) (e^{i\alpha\gamma^5} \psi) .$$

Using  $\{\gamma_5, \gamma^\mu\} = 0$  finally gives

$$\begin{aligned} \mathcal{L}' &= i\bar{\psi} \gamma^\mu (\partial_\mu - igA_\mu) e^{-i\alpha\gamma^5} (e^{i\alpha\gamma^5} \psi) \\ &= i\bar{\psi} \not{\partial} \psi = \mathcal{L} . \end{aligned}$$

Notice that with a mass term

$$\begin{aligned} m\bar{\psi}\psi &\rightarrow m\bar{\psi} e^{2i\alpha\gamma_5} \psi \\ &\neq m\bar{\psi}\psi . \end{aligned}$$

We see that the Lagrangian with a mass term is not  $\gamma_5$  invariant.

To understand these symmetries physically, we use the Weyl representation of  $\gamma$ -matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 \\ \alpha^i &= \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \end{aligned} \quad (2.6)$$

where  $\sigma^i$ 's are the Pauli spin matrices.

In this representation the Dirac action becomes

$$\begin{aligned}
 H &= \int \bar{\psi} i \not{\partial} \psi d^4x \\
 &= \int \bar{\psi} i \gamma^\mu (\partial_\mu - ig A_\mu) \psi d^4x \\
 &= \int \psi^\dagger \gamma^0 \{ \gamma^0 (i\partial_0 + gA_0) - \vec{\gamma} \cdot (i\vec{\partial} + g\vec{A}) \} \psi d^4x
 \end{aligned} \tag{2.7}$$

where we have used the contraction or scalar product of four vectors

$$A^\mu B_\mu = (A_0 B_0 - \vec{A} \cdot \vec{B}) \quad . \tag{2.8}$$

Using (2.6) in (2.7) gives

$$H = \int \psi^\dagger \{ \vec{\alpha} \cdot (\vec{p} - g\vec{A}) + I(i\partial_0 + gA_0) \} \psi d^4x \tag{2.9}$$

where

$$\vec{\alpha} = -i\vec{\sigma} \quad . \tag{2.10}$$

Our units are  $\hbar = c = 1$ .  $H$  can be split apart, if  $\psi$  is written in terms of a two-component spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad . \tag{2.11}$$

Inserting this in  $H$  gives

$$\begin{aligned}
 H &= \int (\psi_L^\dagger \quad \psi_R^\dagger) \left[ \begin{pmatrix} -\vec{\sigma} \cdot \vec{a} & 0 \\ 0 & \vec{\sigma} \cdot \vec{a} \end{pmatrix} + \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} d^4x \\
 H &= \int \psi_L^\dagger [ -\vec{\sigma} \cdot \vec{a} + I a_0 ] \psi_L d^4x + \int \psi_R^\dagger [ +\vec{\sigma} \cdot \vec{a} + I a_0 ] \psi_R d^4x
 \end{aligned} \tag{2.12}$$

where

$$\vec{a} = \vec{p} - g\vec{A} \quad , \quad a_0 = i\partial_0 + gA_0 \tag{2.13}$$

The positive energy states of  $\psi_R$  have positive helicity and the positive-energy states of  $\psi_L$  have negative helicity.  $\psi_R$  and  $\psi_L$  describe, respectively, right- and left-handed massless fermions. The fermion numbers of  $\psi_R$  and  $\psi_L$  are (formally) separately conserved; this is the origin of the extra symmetry.

The two pieces of (2.13) are not actually of a different form. We can write  $\psi_R$  as a second form of  $\psi_L$  by applying charge conjugation

$$\psi_{L2} = \sigma^2 \psi_R^* \quad (2.14)$$

We first perform an integration by parts of the second term of (2.12), then use (2.14) and the identity

$$\sigma_2 \sigma_i \sigma_2 = -\sigma_i^* \quad (2.15)$$

so that the second term can be rewritten as

$$\begin{aligned} & \int \psi_R^\dagger \{ \vec{\sigma} \cdot (\vec{p} - g\vec{A}) + I(i\partial_0 + gA_0) \} \psi_R d^4x \\ &= \int \psi_{L2}^\dagger \{ (-\vec{\sigma}) \cdot (\vec{p} + g\vec{A}) + I(i\partial_0 - gA_0) \} \psi_{L2} d^4x \\ &= \int \psi_{L2}^\dagger \{ (-\vec{\sigma}) \cdot (\vec{p} - (-g)\vec{A}) + I(i\partial_0 + (-g)A_0) \} \psi_{L2} d^4x \end{aligned} \quad (2.16)$$

which is just a  $\psi_{L2}$  action with the opposite sign of the charge  $g$ .

This construction can be readily generalized to non-Abelian gauge theories. In the non-Abelian case the Lagrangian is built from the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu^a t_r^a \quad (2.17)$$

where 'a' runs over the generators of the gauge group and the matrices  $t_r^a$  represent these generators in the representation 'r' to which the fermions are assigned. The matrices for complex conjugate representation are replaced by

$$t_{\bar{r}}^a = -(t_r^a)^* \quad . \quad (2.18)$$

This enables us to recast the action for  $\psi_R$  as that of  $\psi_L$  in the representation r

$$\begin{aligned} \int \psi_R^\dagger [\vec{\sigma} \cdot (\vec{p} - g\vec{A}) + I(i\partial_0 + gA_0)] \psi_R d^4x \\ = \int \psi_{L2}^\dagger [(-\vec{\sigma}) \cdot (\vec{p} - g\vec{A}(-t^*)) + I(i\partial_0 + gA_0(-t^*))] \psi_{L2} d^4x \end{aligned} \quad (2.19)$$

In this notation, the most general action coupling massless fermions to gauge fields may be written compactly in the following form

$$H = \sum_{\text{repr. } r} \sum_{i=1}^{n_r} \int \psi_{Lri}^\dagger [(-\vec{\sigma}) \cdot (\vec{p} - g\vec{A}tr) + I(i\partial_0 + gA_0tr)] \psi_{Lri} d^4x \quad (2.20)$$

where  $r = 1, 2$ .

Once H has been cast into this form, it is easy to read off the global symmetries of this system. For each representation r, this action is (formally) invariant under the general unitary transformation

$$\psi_{Lri} \rightarrow U_{ij} \psi_{Lrj} \quad . \quad (2.21)$$

Actually, one of these formal symmetries is illusory. A certain quantum correction to this theory, the Adler-Bell-Jackiw anomaly [3,4], spoils the conservation of the overall charge current

$$J_\mu = \sum_{ri} \bar{\psi}_{Lri} \partial_\mu \psi_{Lri} . \quad (2.22)$$

(In terms of  $\psi_L$  and  $\psi_R$ , this is the axial-vector current.) This symmetry is therefore broken. The full global symmetry of the theory is, therefore,

$$G = \left[ \prod_r U(n_r) \right] / U(1) . \quad (2.23)$$

G is the group of chiral symmetries of such a theory.

As an example, consider the case of the strong interactions, which are described by a set of two almost massless Dirac fermions (quarks) coupled in the triplet representation to an SU(3) gauge group. These fermions may be written as left-handed fermions, two in the 3 and two in the  $\bar{3}$  representations of the colour SU(3). In the limit of zero quark masses, the chiral symmetry of this theory is

$$G = SU(2) \times SU(2) \times U(1) . \quad (2.24)$$

There is considerable evidence that the full group G is a symmetry of the strong interactions; however, hadrons do not form multiplets classified by G, but only by SU(2) × U(1) (isospin × baryon number). A part of G must then be spontaneously broken.



There is a relatively simple intuitive argument which accounts for this spontaneous breaking of chiral symmetry. It is due in its original form to Nambu and Jona-Lasino [5]. The gauge coupling of SU(3) colour is asymptotically free and becomes large at large distances. Let us assume that it is arbitrarily strong; and think about the structure of the vacuum state as the coupling  $g$  is raised from zero. Imagine that we can integrate over the quantum fluctuations of the gauge field; then  $H$  takes the form

$$H = H_d + H_{0-d} \tag{2.25}$$

where  $H_d$  is diagonal in the number of quark-antiquark pairs and  $H_{0-d}$  changes the number of such pairs.  $H_{0-d}$  is of order  $g^2$  and is a small perturbation when  $g$  is small. In this regime it makes sense to approximate  $H$  by  $H_d$ . Diagonalizing  $H_d$  yields a ground state close to the free-field vacuum. Now, slowly increase  $g$ . If the fermions have zero mass and experience attractive interactions,  $H_d$  decreases as  $g$  increases.  $H_{0-d}$ , of course, increases. At some value of  $g$  it becomes appropriate to treat  $H_{0-d}$  as the zeroth-order problem and  $H_d$  as a perturbation. But  $H_{0-d}$  changes the number of pairs, so its ground state has an indefinite number of fermion pairs. We still expect the ground state to be invariant under Lorentz transformations; hence these pairs must have vacuum quantum numbers - zero total momentum and angular momentum. The only pairs one can form from  $3$  and  $\bar{3}$  left-handed (L-h) and their (right-handed) antiparticles which satisfy this condition are of the form

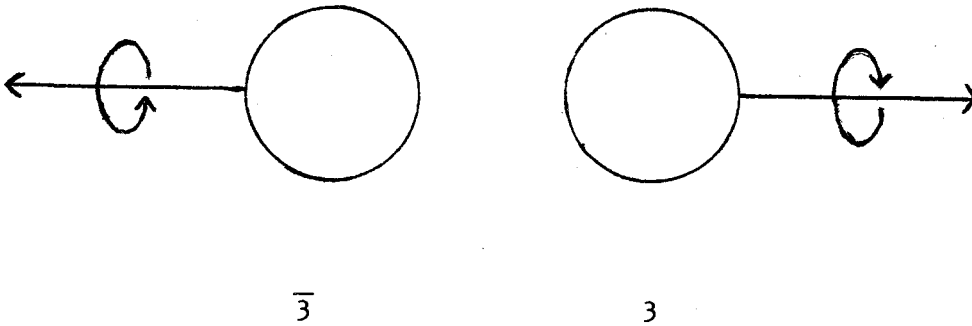


Figure 2.1 Fermion bilinear with vacuum quantum numbers.

and the corresponding pair of antifermions. The pair shown in Figure (2.1) carries a net charge under the transformations:

$$\psi_{L3i} \rightarrow e^{i\alpha} \psi_{L3i} \quad , \quad \psi_{L3i}^- \rightarrow e^{i\alpha} \psi_{L3i}^-$$

$$\psi_{L3i} \rightarrow U_{ij} \psi_{L3j} \quad , \quad \psi_{L3i}^- \rightarrow V_{ij} \psi_{L3j}^-$$

(The indices  $i, j = 1, 2$  are isospin labels.)

The presence of an indefinite number of such pairs in the vacuum breaks these symmetries. More formally, we have found that the ground state  $|\Omega\rangle$  of  $H$  has the property that an operator which destroys a fermion pair has a nonzero vacuum expectation value.

Let us assume that  $|\Omega\rangle$  gives pair annihilation operators the rather simple expectation value

$$\langle \Omega | \psi_{L3i} \psi_{L3j}^- | \Omega \rangle = \Delta \delta_{ij} \quad (2.27)$$

(where  $\Delta \neq 0$ ), corresponding to equal condensation of pairs of each isospin. This expression is preserved by the transformations

$$\begin{aligned} \psi_{L3i} &\rightarrow e^{i\alpha} \psi_{L3i}, & \psi_{L3i}^- &\rightarrow e^{-i\alpha} \psi_{L3i}^- \\ \psi_{L3i} &\rightarrow U_{ij} \psi_{L3j}, & \psi_{L3i}^- &\rightarrow \psi_{L3j}^- U_{ji}^{-1}. \end{aligned}$$

This, however, is an  $SU(2) \times U(1)$  group of unbroken symmetries which corresponds precisely to isospin  $\times$  baryon number. The remaining three symmetry directions of (2.18) must be spontaneously broken symmetries. Goldstone theorem requires that each must generate a massless Goldstone boson. The three  $\pi$  mesons have the right quantum numbers to be identified with these three bosons. (This may be checked by rewriting  $\psi_{L3i}^-$  as  $\psi_R$ ; then the symmetries (2.22) correspond to vector currents and the broken symmetries to axial currents.)

## 2.2 THE EFFECTIVE POTENTIAL FORMALISM

To do a more quantitative computation of chiral symmetry breaking, one needs to know how to test whether the energy of the vacuum is lowered if some fermion bilinear acquires a nonzero vacuum expectation value (vev). If the quantity acquiring a nonzero vacuum expectation value is a scalar field  $\phi$ , one has at one's disposal an object called the effective potential [6,7]. This object is equal to the energy of the vacuum under the constraint that the vev of  $\phi$  has some definite value  $\phi_0$ ; it can be computed straight-forwardly in perturbation theory [8]. One need only compute this effective potential and minimize it with respect to  $\phi_0$  to determine the vacuum value of  $\phi$ .

For chiral symmetry breaking (hereafter  $\chi$ SB) there is a similar construction due to Cornwall, Jackiw and Tomboulis [9]. To produce a vev of a fermion bilinear operator, we must, in principle, turn on some external field (analogous to a magnetic field orienting a potentially ferromagnetic system), construct the ordered vacuum in the presence of this field, and then see if the order in this vacuum survives when we turn off the field.

Cornwall, Jackiw and Tomboulis (CJT) constructed the effective action for Bose fields. We will sketch a derivation of this for Fermi fields.

Consider an  $SU(N)$  gauge theory with massless fermions in the fundamental representation. We introduce a bilocal source  $K(x,y)$  coupled to the fermion bilinear  $\bar{\psi}(x)\psi(y)$  in the generating functional of the theory. In Euclidean space the generating functional is

$$Z(\mathcal{J}, \bar{\mathcal{J}}, K) = e^{-W(\mathcal{J}, \bar{\mathcal{J}}, K)} \quad (2.29)$$

where  $\mathcal{J}(x)$ ,  $\bar{\mathcal{J}}(x)$  are the usual local sources coupled to the fields  $\bar{\psi}(x)$  and  $\psi(x)$  respectively.  $W(\mathcal{J}, \bar{\mathcal{J}}, K)$  is the generating functional for connected Green's functions.

$$\begin{aligned} Z(\mathcal{J}, \bar{\mathcal{J}}, K) = & \int D\psi DA_\mu DC \exp\{-[I(\psi, A_\mu, C) \\ & + \int d^4x \bar{\mathcal{J}}(x)\psi(x) + \int d^4x \bar{\psi}(x)\mathcal{J}(x) \\ & + \int d^4x d^4y \bar{\psi}(x)K(x, y)\psi(y)]\} \end{aligned} \quad (2.30)$$

where the functional measure  $D\psi$  stands for the product of all the  $d\psi$ 's, for example,  $D\psi = d\bar{\psi}d\psi$ . Spinor and flavor indices have been suppressed in (2.30).  $I(\psi, A_\mu, C)$  is the gauge theory action for gauge boson fields  $A_\mu$ , ghost fields  $C$  and the massless Fermi fields  $\psi$ . The local source terms for gauge fields and ghost fields have been omitted.

The effective action  $\Gamma(\varphi, S)$  is a double Legendre transform of  $W(\mathcal{J}, \bar{\mathcal{J}}, K)$ .

$$\frac{\delta W}{\delta \mathcal{J}(x)} = \bar{\varphi}(x) \quad (2.31a)$$

$$\frac{\delta W}{\delta \bar{\mathcal{J}}(x)} = \varphi(x) \quad (2.31b)$$

$$\frac{\delta W}{\delta K(x, y)} = [S(x, y) + \bar{\varphi}(x)\varphi(y)] \quad (2.31c)$$

where  $S(x, y)$  is the full fermion propagator. Then

$$\Gamma(\varphi, S) = W(\mathcal{J}, \bar{\mathcal{J}}, K) - \frac{\delta W}{\delta \mathcal{J}(x)} \mathcal{J}(x) - \bar{\mathcal{J}}(x) \frac{\delta W}{\delta \bar{\mathcal{J}}(x)} - \frac{\delta W}{\delta K(x, y)} K(x, y) \quad (2.32)$$

where integration with respect to  $x$  and  $y$  is understood.

$$\begin{aligned}
 \Gamma(\varphi, S) = & W(\mathcal{J}, \bar{\mathcal{J}}, K) - \int d^4x \bar{\varphi}(x) \mathcal{J}(x) - \int d^4x \bar{\mathcal{J}}(x) \varphi(x) \\
 & - \int d^4x d^4y \varphi(x) K(x, y) \bar{\varphi}(y) \\
 & - \int d^4x d^4y S(x, y) K(y, x)
 \end{aligned} \tag{2.33}$$

It is easy to see that

$$\frac{\delta \Gamma(\varphi, S)}{\delta \varphi(x)} = -\bar{\mathcal{J}}(x) - 2 \int d^4y \bar{\varphi}(y) K(y, x) \tag{2.34a}$$

$$\frac{\delta \Gamma(\varphi, S)}{\delta \bar{\varphi}(x)} = -\mathcal{J}(x) - 2 \int d^4y K(x, y) \varphi(y) \tag{2.34b}$$

$$\frac{\delta \Gamma(\varphi, S)}{\delta S(x, y)} = -K(x, y) \tag{2.34c}$$

Since physical processes correspond to vanishing sources  $\mathcal{J}$ ,  $\bar{\mathcal{J}}$  and  $K$ , equations (2.34) provide a derivation of the stationarity requirement.

$$\left. \frac{\delta \Gamma}{\delta \varphi} \right|_{\bar{\mathcal{J}}=0} = 0 \quad \left. \frac{\delta \Gamma}{\delta \bar{\varphi}} \right|_{\mathcal{J}=0} = 0 \quad \left. \frac{\delta \Gamma}{\delta S} \right|_{K=0} = 0 \tag{2.35}$$

$\Gamma(\varphi, S)$  is the generating functional in  $\varphi$  for two-particle irreducible (2PI) Green's functions expressed in terms of the propagator  $S(x, y)$ .

$\Gamma_{\text{CJT}}(\varphi, S)$  (where CJT stands for Cornwall, Jackiw, Tomboulis), can be expanded into the following formal series for vanishing values of  $A_\mu$  and  $C$  [9].

$$\begin{aligned}
 \Gamma_{\text{CJT}}(\varphi, S) = & I(\varphi) + \text{Tr} \text{Ln} S^{-1}(x, y) + \text{Tr} S_0^{-1}(x-y) S(x, y) \\
 & + \Gamma_2(\varphi, S) + \text{Const.}
 \end{aligned} \tag{2.36}$$

where integration over  $x$  and  $y$  is understood.  $S_0(x-y)$  is the free fermion propagator given by the classical action  $I(\varphi)$ .  $\Gamma_2(\varphi, S)$  is given by all two-particle irreducible vacuum diagrams evaluated with  $S(x, y)$  as the fermion propagator [27]. A graph is said to be two particle

irreducible if it does not become disconnected upon opening two lines.

From (2.34c) and (2.36) we have

$$\begin{aligned} \frac{\delta \Gamma}{\delta S} &= -K = -S^{-1} + S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \\ K &= S^{-1} - S_0^{-1} - \frac{\delta \Gamma_2}{\delta S} \end{aligned} \quad (2.37)$$

Using equation (2.37) we can rearrange (2.36) as follows

$$\begin{aligned} \Gamma_{\text{CJT}}(\varphi, S) &= I(\varphi) + \text{Tr} \text{Ln} \left( K + S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right) + \text{Tr} \left( 1 - KS - \frac{\delta \Gamma_2}{\delta S} S \right) \\ &+ \Gamma_2(\varphi, S) + \text{Const.} \\ &= I(\varphi) + \text{Tr} \text{Ln} \left[ \left( S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right) \left( 1 + \left\{ S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right\}^{-1} K \right) \right] \\ &+ \text{Tr}(1 - KS) - \text{Tr} \left( \frac{\delta \Gamma_2}{\delta S} S \right) + \Gamma_2(\varphi, S) + \text{Const.} \\ &= I(\varphi) + \text{Tr} \text{Ln} \left( S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right) - \text{Tr} \left( \frac{\delta \Gamma_2}{\delta S} S \right) + \Gamma_2(\varphi, S) \\ &+ \text{Tr} \text{Ln}(1 + [1 - KS]^{-1} SK) - \text{Tr} KS + \text{Const.} \end{aligned}$$

where we have used

$$S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} = (1 - KS) S^{-1}$$

from (2.37). Finally we have

$$\Gamma_{\text{CJT}}(\varphi, S) = \Gamma_{\text{CDDG}}(\varphi, S) + \text{Tr} \text{Ln}(1 + [1 - KS]^{-1} SK) - \text{Tr} KS \quad (2.38)$$

where

$$\begin{aligned} \Gamma_{\text{CDDG}}(\varphi, S) &= I(\varphi) + \text{Tr} \text{Ln} \left( S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right) - \text{Tr} \left( \frac{\delta \Gamma_2}{\delta S} S \right) + \Gamma_2(\varphi, S) \\ &+ \text{Const.} \end{aligned} \quad (3.39)$$

The effective action  $\Gamma_{\text{CDDG}}$  was first introduced by Casalbuoni, DeCurtis, Dominici and Gatto [10] hence the name  $\Gamma_{\text{CDDG}}$ . The reason for introducing  $\Gamma_{\text{CDDG}}$  will be made clear later on after introducing the corresponding effective potentials derived from the effective actions  $\Gamma_{\text{CJT}}$  and  $\Gamma_{\text{CDDG}}$ . Equation (2.36) with the constant properly evaluated gives

$$\Gamma_{\text{CJT}}(\varphi, S) = I(\varphi) + \text{Tr} \text{Ln} S_0^{-1} S + \text{Tr} S_0^{-1} S + \Gamma_2(\varphi, S) - \text{Tr} 1 \quad . \quad (2.40)$$

Equations (2.39) and (2.40) are the CDDG and CJT effective potential I shall be using throughout the rest of this thesis.

It is an easy matter to derive the effective potential from the effective action. The effective potential which is the generating function for 2PI n-point functions at zero momentum is often computed by letting  $\varphi(x)$  be a constant  $\varphi$  in the effective action. Since we are interested in translation invariant solutions, we also let  $S(x, y)$  be a function only of the difference  $x-y$ . A generalization of the effective potential may be defined by

$$\Gamma(\varphi, S) \Big|_{\text{translation invariant}} = -V(\varphi, S) \int d^4x \quad . \quad (2.41)$$

The series for  $V(\varphi, S)$  can be obtained from (2.40) by Fourier transforming the propagators.

$$\begin{aligned} S(x-y) &= \int d^4p \, e^{ip(x-y)} S(p) \\ S_0(x-y) &= \int d^4p \, e^{ip(x-y)} S_0(p) \\ 1 &= \int d^4x d^4y \, \delta^4(x-y) \delta^4(y-x) \\ \delta^4(x-y) &= \frac{1}{(2\pi)^4} \int d^4p \, e^{ip(x-y)} \end{aligned} \quad (2.42)$$



Equation (2.40) is written in full as

$$\begin{aligned}
 \Gamma_{\text{CJT}}(\varphi, S) &= I(\varphi) + \int d^4x d^4y \text{Tr} \text{Ln} S_0^{-1}(x-y)S(y-x) \\
 &+ \int d^4x d^4y \text{Tr} S_0^{-1}(x-y)S(y-x) \\
 &+ \Gamma_2(\varphi, S) - \int d^4x d^4y \text{Tr} \delta^4(x-y) \delta^4(y-x) .
 \end{aligned} \tag{2.43}$$

Substitute (2.42) into (2.43) to have

$$\begin{aligned}
 &\{-V(\varphi, S) + V_2(\varphi, S) + U(\varphi)\} \int d^4x \\
 &= \int d^4x d^4y \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \text{Tr} e^{-ip(x-y)} [\text{Ln} S_0^{-1}(p)S(p')] e^{-ip'(y-x)} \\
 &+ \int d^4x d^4y \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \text{Tr} e^{-ip(x-y)} [S_0^{-1}(p)S(p')-1] e^{-ip'(y-x)} \\
 &= \int d^4y \frac{d^4p}{(2\pi)^4} d^4p' \text{Tr} [S_0^{-1}(p)S(p')-1] e^{i(p-p')y} \delta^4(p-p') \\
 &+ \int d^4y \frac{d^4p}{(2\pi)^4} d^4p' \text{Tr} [S_0^{-1}(p)S(p')-1] e^{i(p-p')y} \delta^4(p-p') \\
 &= \int d^4y \frac{d^4p}{(2\pi)^4} \text{Tr} \text{Ln} S_0^{-1}(p)S(p) \\
 &+ \int d^4y \frac{d^4p}{(2\pi)^4} \text{Tr} [S_0^{-1}(p)S(p)-1]
 \end{aligned}$$

and finally we have

$$\begin{aligned}
 V_{\text{CJT}}(\varphi, S) &= U(\varphi) - \int \frac{d^4p}{(2\pi)^4} \text{tr} \text{Ln} S_0^{-1}(p)S(p) \\
 &- \int \frac{d^4p}{(2\pi)^4} \text{tr} [S_0^{-1}(p)S(p)-1] \\
 &+ V_2(\varphi, S)
 \end{aligned} \tag{2.44}$$

and

$$\begin{aligned}
 V_{\text{CDDG}}(\varphi, S) &= U(\varphi) - \int \frac{d^4p}{(2\pi)^4} \text{tr} \text{Ln} S_0^{-1}(p) + \frac{\delta V_2}{\delta S} \\
 &- \text{Tr} \frac{\delta V_2}{\delta S} S + V_2(\varphi, S) + \text{Const.}
 \end{aligned} \tag{2.45}$$

where  $U(\varphi)$  is the classical potential and  $-V_2(\varphi, S)$  is the sum of all 2PI vacuum graphs. Notice that the trace is no longer functional but applies to component degrees of freedom and  $V(\varphi, S)$  is a function of and a functional of  $S(p)$ .

The stationary requirement (2.35) becomes

$$\frac{\delta V(\varphi, S)}{\delta \varphi} = 0 \quad (2.46)$$

$$\frac{\delta V(\varphi, S)}{\delta S(p)} = 0 \quad (2.47)$$

Hence only (2.47) is a functional derivative. Also note that

$$V_{\text{CJT}} \Big|_{\kappa=0} = V_{\text{CDDS}} \Big|_{\kappa=0} \quad (2.48)$$

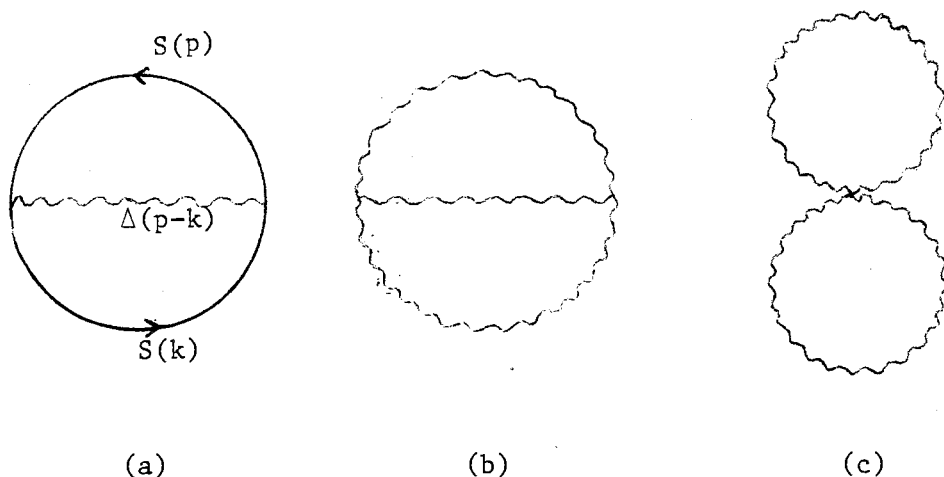
Dropping the classical potential  $U(\varphi)$  from (2.44) and (2.45), we have

$$\begin{aligned} V_{\text{CJT}}(S_0, S) &= - \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln S_0^{-1}(p) S(p) \\ &\quad - \int \frac{d^4 p}{(2\pi)^4} \text{tr} [S_0^{-1}(p) S(p) - 1] \\ &\quad + V_2(\varphi, S) \end{aligned} \quad (2.49)$$

$$\begin{aligned} V_{\text{CDDG}}(S_0, S) &= - \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln \left( S_0^{-1}(p) + \frac{\delta V_2}{\delta S} \right) \\ &\quad - \text{Tr} \left( \frac{\delta V_2}{\delta S} \right) S + V_2(\varphi, S) + \text{Const.} \end{aligned} \quad (2.50)$$

With equations (2.49) and (2.50) we end our sketch of the CJT formalism for Fermi fields.

We have said that  $V_2$  is the sum of all 2PI vacuum diagrams. In the two-loop approximation, the following diagrams contribute, where all internal lines are exact propagators.



**Figure 2.2** Hartree-Fock approximation to two-loop effective potential. The solid line is the full fermion propagator  $S(p)$ ; the wavy line is the full gluon propagator  $\Delta$ .

We use the Hartree-Fock approximation to the effective potential where only one gluon exchange is taken into account. We therefore consider diagram (a) Fig. (2.2) with the full gluon propagator  $\Delta(p-k)$  replaced by the free one  $D(p-k)$  and the full vertex replaced by the bare vertex. This Hartree-Fock approximation which replaces the complete vertex by the free vertex forces us to use the Landau gauge [9]. This

is because in spontaneously broken theories, this vertex has a Goldstone pole, which certainly is not in the bare vertex. The approximation therefore makes sense only in the Landau gauge, since then the Goldstone pole is annihilated in all vacuum graphs. This makes the SD equation finite and no renormalization is needed. Chiral symmetry breaking has also been studied in Coulomb's gauge [28,29], here the SD equation has to be renormalized.

$S(p)$  is the full fermion propagator given by

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) - M(p) \\ S_0(p) &= -i\not{p} \end{aligned} \tag{2.51}$$

In Landau gauge the gluon propagator is

$$D_{\mu\nu}(p) = \frac{1}{p^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \tag{2.52}$$

$$V_2^{(a)}(S_0, S) = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \text{tr} \Gamma_\mu^a S(p) \Gamma_\nu^b S(k) D_{ab}^{\mu\nu}(p-k) \tag{2.53}$$

where

$$\Gamma_\mu = g\gamma_\mu \tag{2.54}$$

is the bare vertex. The unrenormalized coupling constant is  $g$ . However, in a non-Abelian gauge theory like QCD, the coupling constant renormalization is essential, since it insures that phenomena which happen only in the regime of strong coupling do indeed occur at some scale [17]. We replace  $g$  by  $g(p^2, k^2)$ . Using the property that

$$2V_2 = \text{Tr} \left( \frac{\delta V_2}{\delta S} \right) S \tag{2.55}$$

we can write (2.50) as

$$V_{\text{CDDG}}(S_0, S) = - \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln S_0^{-1} + \frac{\delta V_2}{\delta S} - V_2(S_0, S) + \text{Const.} \quad (2.56)$$

Let

$$\frac{\delta V_2}{\delta S} \equiv - \sum(p) \quad (2.57)$$

$$\begin{aligned} V_{\text{CDDG}} &= - \int \frac{d^4 p}{(2\pi)^4} \text{tr} \ln(S_0^{-1}(p) - \sum(p)) \\ &+ \frac{1}{2} C_f(N) \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \text{tr}[S(p)\gamma_\mu S(k)\gamma_\nu] D^{\mu\nu}(p-k) g^2(p^2, k^2) \\ &+ \text{Const.}, \end{aligned} \quad (2.58)$$

where we have carried out the trace in the group space to give  $C_f(N)$ , the quadratic Casimir invariant, which for all  $SU(N)$  representations is given by

$$C_f(N) = (N^2 - 1)/2N \quad (2.59)$$

Substituting  $S$  from (2.51) and performing the remaining trace over spinor and flavor indices gives

$$\begin{aligned} V_{\text{CDDG}}(S_0, S) &= -2n_f \int \frac{d^4 p}{(2\pi)^4} \ln(p^2 + \sum^2(p)) \\ &+ 2n_f \int \frac{d^4 p}{(2\pi)^4} \frac{M(p)}{p^2 + M^2(p)} \int \frac{d^4 k}{(2\pi)^4} 3C_f(N) g^2(p^2, k^2) \frac{M(k)}{k^2 + M^2(k)} \frac{1}{(p-k)^2} \\ &+ \text{Const.} \end{aligned} \quad (2.60)$$

where  $n_f$  is the number of flavors.

Assuming spherical symmetry and, letting  $\Sigma$  and  $M$  be functions of the invariant Euclidean four momentum squared, gives

$$\begin{aligned}
 V_{\text{CDDG}} &= -2n_f \int \frac{d^4 p}{(2\pi)^4} \ln(p^2 + \Sigma^2(p^2)) \\
 &+ 2n_f \int \frac{d^4 p}{(2\pi)^4} \frac{M(p^2)}{p^2 + M^2(p^2)} \int \frac{d^4 k}{(2\pi)^4} \mathfrak{C}_f(N) g^2(p^2, k^2) \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \\
 &+ \text{Const.} \tag{2.61}
 \end{aligned}$$

$g^2(p^2, k^2)$  is the coupling constant of the theory which we shall talk about in detail in subsequent chapters. Following Higashijima [12,13] we adopt the following form for the running coupling constant:

$$g^2(p^2, k^2) = g^2(p^2)\theta(p^2 - k^2) + g^2(k^2)\theta(k^2 - p^2) \tag{2.62}$$

where  $\theta(p^2)$  is a unit step function defined as

$$\theta(p^2) = \begin{cases} 1 & , p^2 > 0 \\ 0 & , p^2 < 0 \end{cases} \tag{2.63}$$

Equation (2.61) can be written in a compact form as

$$\begin{aligned}
 \frac{(2\pi)^4}{2n_f} V_{\text{CDDG}} &= - \int d^4 p \ln(p^2 + \Sigma^2(p^2)) \\
 &+ \int d^4 p \frac{M(p^2)}{p^2 + M^2(p^2)} \int d^4 k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \\
 &+ \text{Const.} \tag{2.64}
 \end{aligned}$$

where

$$A_{p,k} = \frac{\mathfrak{C}_f(N)}{(2\pi)^4} g^2(p^2, k^2) \tag{2.65}$$

and

$$\begin{aligned} \frac{(2\pi)^4}{2n_f} V_{\text{CJT}} &= - \int d^4p \ln \left( 1 + \frac{M^2(p^2)}{p^2} \right) + 2 \int d^4p \frac{M^2(p^2)}{p^2 + M^2(p^2)} \\ &\quad - \int d^4p \frac{M(p^2)}{p^2 + M^2(p^2)} \int d^4k \Lambda_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \end{aligned} \quad (2.66)$$

Equations (2.64) and (2.66) are the Euclidean space CDDG and CJT effective potentials, respectively, which I will be using throughout this thesis.

From (2.57) and (2.53) we have

$$\frac{\delta V_2}{\delta S(p)} = -\Sigma(p^2) = - \int d^4k \Lambda_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \quad (2.67)$$

The reason for introducing the effective potential  $V_{\text{CDDG}}$  will now be explained.

We have seen from (2.48) that

$$V_{\text{CJT}} \Big|_{K=0} = V_{\text{CDDG}} \Big|_{K=0} \quad (2.48)$$

and we shall show also that

$$\frac{\delta V_{\text{CJT}}}{\delta S} \Big|_{K=0} = \frac{\delta V_{\text{CDDG}}}{\delta S} \Big|_{K=0} \quad (2.68)$$

implying that these potentials contain the same information regarding chiral symmetry breaking solutions. However, some of the unpleasant features of the effective potential in the form given by CJT will be shown to be absent in the form given by CDDG.

## CHAPTER 3

### 3.1 THE SCHWINGER-DYSON EQUATION

The SD equation results from requiring the effective potential to be stationary against variations of  $M$ . We saw in Chapter 2 that the stationarity requirement has direct physical consequences which we now identify with the SD integral equation. The solution of this equation furnishes us with one way of studying chiral symmetry breaking.

While the bulk of the original results of this thesis is contained in Chapters 4 and 5, most of the work in this chapter was done independently by the author, although recently similar conclusions have been arrived at by Haymaker and Matsuki.

Taking the first functional derivative of (2.64) with respect to  $M$  gives

$$\begin{aligned} \frac{(2\pi)^4}{2n_f} \frac{\delta V}{\delta M(q^2)} &= - \int d^4 p \frac{2\Sigma(p^2)}{p^2 + \Sigma^2(p^2)} \frac{\delta \Sigma(p^2)}{\delta M(q^2)} \\ &+ 2 \int d^4 p \frac{p^2 - M^2(p^2)}{(p^2 + M^2(p^2))^2} \int d^4 k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{\delta(q^2 - p^2)}{(p-k)^2} \end{aligned} \quad (3.1)$$

Recall from (2.67) that

$$\Sigma(p^2) = \int d^4 k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \quad (2.67)$$

which gives on taking the first functional derivative

$$\frac{\delta \Sigma(p^2)}{\delta M(q^2)} = \int d^4 k A_{p,k} \left[ \frac{k^2 - M^2(k^2)}{(k^2 + M^2(k^2))^2} \right] \frac{\delta(q^2 - k^2)}{(p-k)^2} \quad (3.2)$$



Before continuing I will write down a few of the formulae we will be using for quick reference.

$$\int d^4k \frac{A_{p,k}}{(p-k)^2} f(k^2, p^2) = \pi^2 \int_0^\infty k^2 dk^2 f(k^2, p^2) W(p^2, k^2) \quad (3.3)$$

$$\int d^4k \frac{A_{p,k}}{(p-k)^2} f(k^2, p^2) \delta(s^2 - k^2) = \pi^2 s^2 W(p^2, s^2) f(s^2, p^2) \quad (3.4)$$

$$\int d^4p \int d^4k \frac{A_{p,k}}{(p-k)^2} f(k^2, p^2) \delta(s^2 - p^2) = \int d^4k \pi^2 s^2 \frac{A_{p,k}}{(s-k)^2} f(k^2, s^2) \quad (3.5)$$

$$W(p^2, s^2) = \frac{3c_f(N)}{(2\pi)^4} \left( \frac{g^2(p^2)}{p^2} \theta(p^2 - s^2) + \frac{g^2(s^2)}{s^2} \theta(s^2 - p^2) \right) \quad (3.6)$$

Formula (3.3) has been derived by carrying out the angular integrations.

Note that three angles are involved.

Substituting (3.2) into (3.1) and using (3.4) and (3.5) gives

$$\begin{aligned} \frac{(2\pi)^2}{2n_f} \frac{\delta V}{\delta M(q^2)} &= 2\pi^2 q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \int d^4k \frac{A_{q,k}}{(q-k)^2} \frac{M(k^2)}{k^2 + M^2(k^2)} \\ &\quad - 2\pi^2 \int d^4p \frac{\sum(p^2)}{p^2 + \sum^2(p^2)} W(p^2, q^2) q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \end{aligned} \quad (3.7)$$

We immediately see that

$$\frac{\delta V}{\delta M(q^2)} = 0$$

implies

$$\sum(p^2) = M(p^2) = \int d^4k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \quad (3.8)$$

Eq. (3.8) is the desired SD equation. This is a nonlinear integral equation.

Notice that the same SD equation is obtained [14] by minimizing VCJT Equation (2.66). Thus both VCJT and VCDDG give the SD equation when minimized. To this point in our understanding of the effective potentials, using any of them is a matter of choice.

If we had used in (2.64)  $M(p^2)$  instead of  $\Sigma(p^2)$  in the Log, the first functional derivative with respect to  $M$  would give

$$\begin{aligned} \frac{(2\pi)^2}{2n_f} \frac{\delta V}{\delta M(q^2)} &= - \int d^4p \frac{2M(p^2)}{p^2 + M^2(p^2)} \delta(q^2 - p^2) \\ &+ 2 \int d^4p \frac{p^2 - M^2(p^2)}{(p^2 + M^2(p^2))^2} \int d^4k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{\delta(q^2 - p^2)}{(p-k)^2} \end{aligned} \quad (3.9)$$

and one can not get the SD equation by requiring  $\delta V/\delta M$  to be zero.

### 3.2 SOLUTIONS OF THE SD EQUATION

We have seen that the SD integral equation is highly nonlinear and would be difficult to solve analytically. However, we can change this into a nonlinear differential equation with appropriate boundary conditions and then find its solution numerically.

$$M(p^2) = \int d^4k \Lambda_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} \quad (3.8)$$

Using (3.3) gives

$$M(p^2) = \Pi^2 \int_0^\infty k^2 dk^2 \left[ \frac{3C_f(N)}{(2\pi)^4} \left( \frac{g^2(p^2)}{p^2} \theta(p^2 - k^2) + \frac{g^2(k^2)}{k^2} \theta(k^2 - p^2) \right) \frac{M(k^2)}{k^2 + M^2(k^2)} \right]$$

Getting rid of the step functions explicitly gives

$$M(p^2) = \frac{3\pi^2 C_f(N)}{(2\pi)^4} \frac{g^2(p^2)}{p^2} \int_0^{p^2} dk^2 \frac{M(k^2)}{1 + \frac{M^2(k^2)}{k^2}} + \int_{p^2}^\infty dk^2 \frac{3\pi^2 C_f(N)}{(2\pi)^4} \frac{g^2(k^2)}{k^2} \frac{M(k^2)}{1 + \frac{M^2(k^2)}{k^2}}$$

$$M(x) = \frac{\varepsilon(x)}{x} \int_0^x dy \frac{M(y)}{1 + \frac{M^2(y)}{y}} + \int_x^\infty dy \frac{\varepsilon(y)}{y} \frac{M(y)}{1 + \frac{M^2(y)}{y}} \quad (3.10)$$

where

$$\epsilon(x) = \frac{3\pi^2 C_f(N)}{(2\pi)^4} g^2(x) \quad , \quad x = p^2, y = k^2 \quad . \quad (3.11)$$

At this point we are interested in the form of the running coupling constant. We get this from the renormalization group. The aim of renormalization group is to describe how the dynamics of a system evolves as one changes the scale of the phenomena being observed. One develops methods of constructing an effective Hamiltonian which describes the system at distances which are either much larger or much smaller than the characteristic physical length scale. Thus the renormalization group is a group of transformations of the effective Hamiltonian as the scale length is changed.

For an asymptotically free gauge theory, the running coupling constant obtained from renormalization group analysis for large  $p^2$  (short distance) is [17]

$$g^2(x) = \frac{48\pi^2}{11N-2n_f} \frac{1}{\ln(x/\Lambda^2)} \quad . \quad (3.12)$$

Eq. (3.12) can be substituted into (3.11) to obtain

$$\epsilon(x) = \frac{\epsilon_0}{\ln(x/\Lambda^2)} \quad , \quad \epsilon_0 = \frac{9C_f(N)}{11N-2n_f} \quad . \quad (3.13)$$

Define

$$t \equiv \ln(x/\Lambda^2) \quad (3.14)$$

We see that the behavior of the coupling constant leads to confinement for small  $p^2$ , i.e., the coupling can become arbitrarily large for small

$p^2$ . Since it has been argued [15] that the confinement mechanism does not influence chiral symmetry breaking, we adopt a nonconfining QCD coupling (see Fig. 3.1).

$$\epsilon(t) = \begin{cases} \frac{\epsilon_0}{t_c + t} & t \geq 0 \\ \frac{\epsilon_0}{t_c} & t \leq 0 \end{cases} \quad (3.15)$$

The  $\Lambda$  is a QCD scale parameter and  $t_c$  is the infrared cut off.

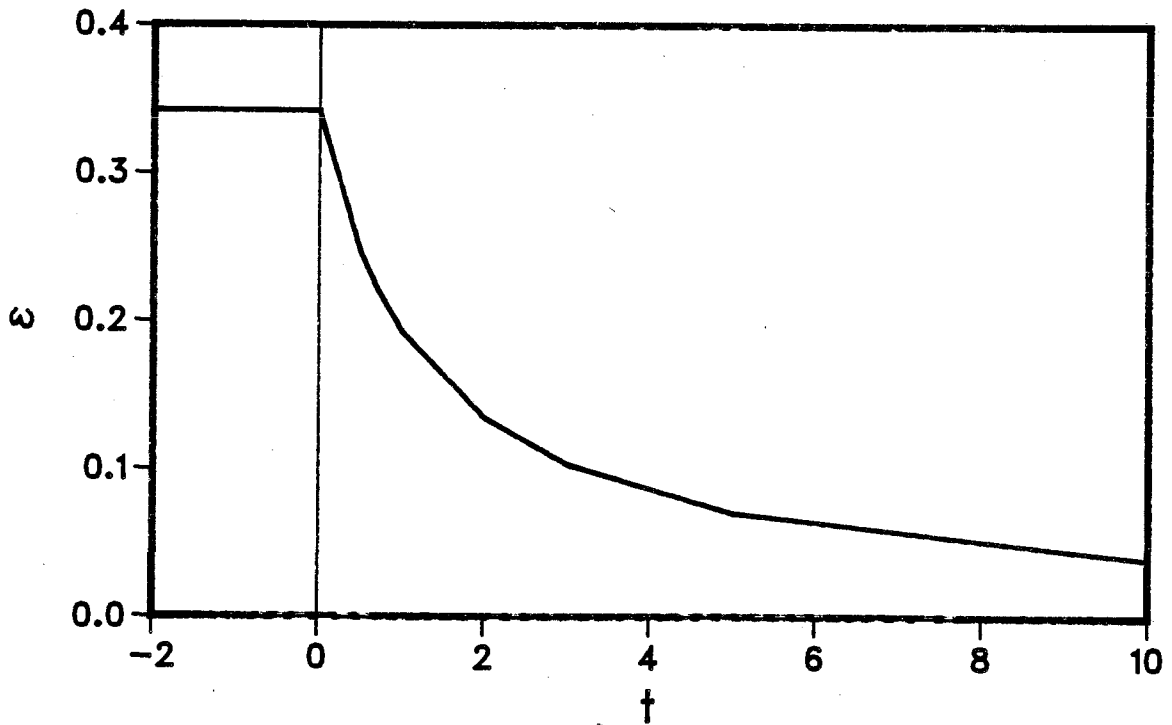


Figure 3.1 Nonconfining QCD coupling.

Returning to equation (3.10) and letting

$$M(x) = \sqrt{x} C(x) \quad (3.16)$$

we have

$$\sqrt{x} C(x) = \frac{\epsilon(x)}{x} \int_0^x dy \frac{sy C(y)}{1 + C^2(y)} + \int_x^\infty dy \frac{\epsilon(y)}{y} \frac{sy C(y)}{1 + C^2(y)} \quad (3.17)$$

Then using (3.14)

$$\begin{aligned} t &= \ln(x/\Lambda^2) \\ s &= \ln(y/\Lambda^2) \\ dy &= \Lambda^2 e^s ds \end{aligned} \quad (3.18)$$

substitute (3.18) into (3.17) to have.

$$\begin{aligned} C(t) &= \epsilon(t) e^{3t/2} \int_{-\infty}^t ds e^{3s/2} \frac{C(s)}{1 + C^2(s)} \\ &+ e^{-t/2} \int_t^\infty ds e^{s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} \end{aligned} \quad (3.19)$$

We now convert (3.19) to a differential equation with boundary conditions. We first multiply across by  $e^{t/2}$  and then take the derivative with respect to  $t$ .

$$\begin{aligned} \frac{d}{dt} \left[ e^{t/2} C(t) \right] &= \left[ \frac{d}{dt} \left( \epsilon(t) e^{-t} \right) \right] \int_{-\infty}^t ds e^{3s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} \\ &+ \left( \epsilon(t) e^{-t} \right) \left( e^{3t/2} \epsilon(t) \frac{C(t)}{1 + C^2(t)} \right) \\ &- e^{t/2} \epsilon(t) \frac{C(t)}{1 + C^2(t)} \end{aligned}$$

$$\frac{d}{dt} \left[ e^{t/2} C(t) \right] = \left[ \frac{d}{dt} \left( \epsilon(t) e^{-t} \right) \right] \int_{-\infty}^t ds e^{3s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} \quad (3.20)$$

which gives

$$\int_{-\infty}^t ds e^{3s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} = \frac{\frac{d}{dt} \left[ e^{t/2} C(t) \right]}{\frac{d}{dt} \left[ \epsilon(t) e^{-t} \right]} \quad (3.21)$$

Taking the derivative of (3.20) and using (3.21) gives straightforwardly

$$\begin{aligned} \ddot{C}(t) + 2\dot{C}(t) + \frac{3}{4} C(t) - \left( \frac{\ddot{\epsilon}(t) - \dot{\epsilon}(t)}{\dot{\epsilon}(t) - \epsilon(t)} \right) \left( \frac{1}{2} C(t) + \dot{C}(t) \right) \\ = \left( \dot{\epsilon}(t) - \epsilon(t) \right) \frac{C(t)}{1 + C^2(t)} \quad (3.22) \end{aligned}$$

The dot above any letter stands for derivative with respect to  $t$ . Next we find the boundary conditions.

From Equation (3.21) we have

$$\begin{aligned} \frac{e^{t/2} \left[ \dot{C}(t) + \frac{1}{2} C(t) \right]}{e^{-t} \left[ \dot{\epsilon}(t) - \epsilon(t) \right]} &= \int_{-\infty}^t ds e^{3s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} \\ \lim_{t \rightarrow \infty} e^{3t/2} \left[ C(t) + \frac{1}{2} C(t) \right] &= \lim_{t \rightarrow \infty} \left[ \frac{-\epsilon_0}{\left( \frac{t+t}{c} \right)^2} - \frac{\epsilon_0}{t+t_c} \right] \int_{-\infty}^t ds e^{3s/2} \epsilon(s) \frac{C(s)}{1 + C^2(s)} \\ \lim_{t \rightarrow -\infty} e^{3t/2} \left[ \frac{d}{dt} + \frac{1}{2} \right] C(t) &= 0 \quad (3.23) \end{aligned}$$

To get the second boundary condition we multiply (3.19) across by  $\frac{e^{3t/2}}{\epsilon(t)}$  and take the first derivative to have

$$\frac{d}{dt} \left[ \frac{e^{3t/2} C(t)}{\varepsilon(t)} \right] = \left[ \frac{d}{dt} \left( \frac{e^t}{\varepsilon(t)} \right) \right] \int_t^\infty ds e^{s/2} \varepsilon(s) \frac{C(s)}{1 + C^2(s)}$$

$$\frac{\frac{d}{dt} \left[ \frac{e^{3t/2} C(t)}{\varepsilon(t)} \right]}{\frac{d}{dt} \left[ \frac{e^t}{\varepsilon(t)} \right]} = \int_t^\infty ds e^{s/2} \varepsilon(s) \frac{C(s)}{1 + C^2(s)} \quad (3.24)$$

which gives

$$\lim_{t \rightarrow \infty} e^{t/2} \left( \frac{d}{dt} + \frac{3}{2} \right) C(t) = 0 \quad (3.25)$$

Because of the way we have defined  $\varepsilon(t)$  (Equation (3.15)), Equation (3.22) separates into three regions ( $t \begin{matrix} < \\ > \end{matrix} 0$ ).

$$1. \quad t < 0 \quad \left( \varepsilon = \frac{\varepsilon_0}{t_c}, \quad \dot{\varepsilon} = \ddot{\varepsilon} = 0 \right)$$

Then (3.22) becomes

$$\ddot{C}(t) + 2\dot{C}(t) + \frac{3}{4} C(t) + \frac{\varepsilon_0}{t_c} \frac{C(t)}{1 + C^2(t)} = 0 \quad (3.26)$$

$$2. \quad t > 0$$

$$\varepsilon(t) = \frac{\varepsilon_0}{(t+t_c)} = \lambda \varepsilon_0$$

$$\dot{\varepsilon}(t) = \frac{-\varepsilon_0}{(t+t_c)^2} = -\lambda^2 \varepsilon_0$$

$$\ddot{\varepsilon}(t) = \frac{2\varepsilon_0}{(t+t_c)^3} = 2\lambda^3 \varepsilon_0$$

where

$$\lambda = \frac{1}{t+t_c} \quad (3.27)$$



Equation (3.22) becomes

$$\ddot{C}(t) + \frac{(2+3\lambda+2\lambda^2)}{(1+\lambda)} \dot{C}(t) + \frac{\frac{3}{4} + \frac{5}{4}\lambda + \lambda^2}{(1+\lambda)} C(t) + \epsilon_0 \lambda(1+\lambda) \frac{C(t)}{1 + C^2(t)} = 0 \quad (3.28)$$

3. At  $t = 0$

Here the derivative of  $\epsilon(t)$  is discontinuous which makes  $\dot{C}(t)$  discontinuous too.

The discontinuity is calculated [Appendix A] to give at  $t = 0$

$$\dot{C}(0+) = \left(1 + \frac{1}{t_c}\right) \dot{C}(0-) + \frac{C(0)}{2t_c} \quad (3.29)$$

To be able to do numerical calculations we need to know the behavior of the solution of (3.22) as  $t \rightarrow -\infty$ . In this region corresponding to large distances or small momentum transfer ( $p^2 \rightarrow 0$ ), the quarks are bound very strongly together to form hadrons because the coupling is large. The quark mass here  $M(0)$  is called the constituent mass. On the other hand, at small distances or large momentum transfer, the quarks are free states. In this region the quark mass  $m_R$  is called current quark mass and it is independent of the renormalization point. The asymptotic solutions are:

1.  $t \rightarrow -\infty$  , we solve (3.26) to have

$$C(t) = \begin{cases} \frac{M(0)}{\Lambda} e^{-t/2} & (3.30) \\ \frac{M(0)}{\Lambda} e^{-3t/2} & (3.31) \end{cases}$$

Note that (3.31) in this region is excluded because the boundary (3.23) forces us to choose the solution (3.30). The solution must also satisfy  $C(t)(\Lambda e^{t/2}) = M(t) \rightarrow \text{constant}$  ( $t \rightarrow -\infty$ ) which gives the constituent quark mass.

2.  $t \rightarrow +\infty$  , there are two acceptable solutions because both satisfy boundary condition (3.25) as  $t \rightarrow +\infty$ . In this region we solve (3.28).

We do this by assuming a solution of the form

$$C(t) = A e^{-\beta t} t^\alpha \quad (3.32)$$

$$\dot{C}(t) = -\beta C(t) + \frac{\alpha}{t} C(t) \quad (3.33)$$

$$\ddot{C}(t) = -\beta \left[ -\beta C + \frac{\alpha}{t} C \right] - \frac{\alpha}{t^2} C + \frac{\alpha}{t} \left[ -\beta C + \frac{\alpha}{t} C \right] \quad (3.34)$$

Since we are interested in the solutions as  $t \rightarrow +\infty$ , we neglect all terms having quadratic or higher powers of  $t$  in the denominator and

$$\ddot{C}(t) = +\beta^2 C - \frac{2\beta\alpha}{t} C \quad (3.35)$$

Recall from (3.28) that

$$\ddot{C}(1+\lambda) + (2+3\lambda+2\lambda^2)\dot{C} + \left(\frac{3}{4} + \frac{5}{4}\lambda + \lambda^2\right)C + \epsilon_0 \lambda(1+\lambda)^2 C = 0 \quad (3.36)$$

where we have dropped  $C^2(t)$  in the denominator of the last term of (3.28). Note that

$$\lambda = \frac{1}{t+t_c} \xrightarrow{t \rightarrow \infty} \frac{1}{t} . \quad (3.37)$$

Substituting (3.32), (3.33), (3.35) and (3.37) into (3.36) and neglecting quadratic or higher powers of  $t$  in the denominator gives

$$\left(\beta^2 - 2\beta + \frac{3}{4}\right)C + \frac{1}{t} (\beta^2 - 2\beta\alpha + 2\alpha - 3\beta + \frac{5}{4} + \epsilon_0) = 0 . \quad (3.38)$$

We thus have two equations

$$\beta^2 - 2\beta + \frac{3}{4} = 0 \quad (3.39)$$

$$2\alpha - \beta(2\alpha + 3 - \beta) + \frac{5}{4} + \epsilon_0 = 0 . \quad (3.40)$$

From Equation (3.39) we have

$$\beta = \frac{1}{2}, \frac{3}{2}$$

and using (3.40) gives for

$$\beta = \frac{1}{2}, \quad \alpha = -\epsilon_0 \quad (3.41)$$

$$\beta = \frac{3}{2}, \quad \alpha = \epsilon_0 - 1$$

Since

$$C(t) = Ae^{-\beta t} t^\alpha \quad (3.42)$$

$$C(t) \xrightarrow{t \rightarrow +\infty} \left\{ \begin{array}{l} A e^{-3t/2} t^{\epsilon_0 - 1} \\ \frac{mR}{\Lambda} e^{-t/2} t^{-\epsilon_0} \end{array} \right\} \quad (3.43)$$

It is easy to verify that (3.42) and (3.43) satisfy the boundary condition (3.25).

We will now try to understand the meaning of  $m_R$  and  $\Lambda$ , the coefficients of (3.43) and (3.42) respectively in a more physical way.

In the light of the transformation (3.16), i.e.

$$M(t) = \Lambda e^{t/2} C(t) \quad (3.16)$$

we see that (3.43), which dominates as  $t \rightarrow \infty$ , becomes

$$\begin{aligned} M(t) &= \Lambda e^{t/2} \frac{m_R}{\Lambda} e^{-t/2} t^{-\epsilon_0} \\ &= m_R t^{-\epsilon_0} = m_R \left[ \frac{1}{\ln(p^2/\Lambda^2)} \right]^{\epsilon_0} \end{aligned} \quad (3.44)$$

$M(t)$  gets a correction which is purely logarithmic and is a consequence of asymptotic freedom. So as  $t \rightarrow \infty$  (in the ultraviolet region)  $m_R$  is the current quark mass when quarks are free states. Thus the solution (3.43) will be used in the numerical study of the SD nonlinear differential equation itself to study explicit chiral symmetry breaking.

However, the actual mechanism of mass generation happens at lower momentum transfer and so it is necessary for  $M$  to have more than just logarithmic dependence. At this lower momentum transfer region (known as the infrared region) we use (3.42)

$$M(t) = \Lambda e^{t/2} \Lambda e^{-3t/2} t^{\epsilon_0-1} = \Lambda \left( \frac{\Lambda^3}{p^2} \right) \left[ \frac{1}{\ln(p^2/\Lambda^2)} \right]^{1-\epsilon_0} \quad (3.45)$$

The correction to  $M$  now has  $(1/p^2)$  dependence. This is in agreement with the result of the operator product expansion obtained by D. Politzer.

The coefficient  $A$  plays the role of an order parameter which determines the phase transition from a chiral symmetric phase to a spontaneously broken phase. " $A$ " will be identified with the vacuum expectation value of the fermion bilinear. The solution (3.42) will be used as a variational ansatz in the effective potential to study dynamical chiral symmetry breaking. The effective potential will be expressed as a function of  $A$  to give explicit phase transition curves.

Equation (3.42) is known in the literature as Regular Solution while (3.43) is known as Irregular Solution.

From the preceding discussion, we see that there emerge in a natural way, two approaches for studying chiral symmetry breaking. Explicit  $\chi$ SB involves using the SD differential and the irregular solution to study dynamical chiral symmetry breaking. This has been used, for example, by Higashijima. Spontaneous  $\chi$ SB involves using a variational scheme to study spontaneous  $\chi$ SB. Here one uses the effective potential and the regular solution as a variational ansatz. This has been used, for example, by Castorina and Pi.

In the next section I will deal with explicit chiral breaking and leave spontaneous  $\chi$ SB to Chapter 4 where most of the original work has been done.

### 3.3 EXPLICIT CHIRAL SYMMETRY BREAKING

We have seen in Section (3.2) that the SD differential equation has two solutions - the regular solution and the irregular solution. The irregular solution dominates as  $t \rightarrow \infty$ . One can therefore use the irregular solution to study  $\chi$ SB. One simply gets the numerical data of the SD differential equation and matches this with the irregular solution. In this way a relationship between the constituent quark mass  $M(0)$  and the current quark mass  $m_R(M(0))$ , is obtained graphically. These curves provide a way of seeing how chiral symmetry is broken explicitly. Explicit  $\chi$ SB was first used by Higashijima [13].

We write down the SD differential equation again derived in Section (3.2).

For  $t < 0$

$$\ddot{C} + 2\dot{C} + \frac{3}{4}C + \frac{\epsilon_0}{t_c} \frac{C}{1+C^2} = 0 \quad (3.26)$$

at  $t = 0$

$$\ddot{C}(0+) = \left(1 + \frac{1}{t_c}\right) \dot{C}(0-) + \frac{1}{2t_c} C(0) \quad (3.29)$$

and

for  $t > 0$

$$\ddot{C} + \frac{2+3\lambda+2\lambda^2}{1+\lambda} \dot{C} + \frac{\frac{3}{4} + \frac{5}{4}\lambda + \lambda^2}{1+\lambda} C + \epsilon_0 \lambda (1+\lambda) \frac{C}{1+C^2} = 0 \quad (3.28)$$

The asymptotic solutions we will need are

$$\lim_{t \rightarrow -\infty} C(t) = \frac{M(0)}{\Lambda} e^{-t/2} \quad (3.30)$$

$$\lim_{t \rightarrow +\infty} C(t) = \frac{m_R}{\Lambda} e^{-t/2} t^{-\epsilon_0} \quad (\text{irregular solution}) \quad (3.43)$$

To solve the SD equation numerically, the boundary condition (3.30) is used as  $t \rightarrow -\infty$ , say  $t = -20$ , with  $M(0)/\Lambda$  given as input, integrate to  $t = 0$ , impose the discontinuity condition (3.29) and then integrate to  $t \rightarrow +\infty$ . The numerical data are the values of  $C(t)$  for each value of  $t$ . Then for  $t > 0$ , we divide  $C(t)$  by  $[\exp(-t/2)]t^{\epsilon_0}$  to get  $\frac{m_R}{\Lambda} (M(0))$ .

The expression  $\frac{m_R}{\Lambda} (M(0))$  is a single-valued odd function of  $M(0)$ . The curves for  $M(0)/\Lambda$  vs  $\frac{m_R}{\Lambda} (M(0))$  are obtained for various values of the parameter  $t_c$ , see Fig. (3.2).

The curves in Figure 3.2 show a nontrivial relationship between the constituent quark mass and the current quark mass for each value of  $t_c$ .

## RESULTS

From Figure 3.2 we see that for the parameter  $t_c < 1.6$ , the constituent quark mass  $M(0)$  clearly remains nonvanishing in the chiral limit  $m_R = 0$ ; that is, chiral symmetry is broken spontaneously. For  $t_c > 1.6$ , the constituent quark mass vanishes in the chiral limit  $m_R = 0$ , indicating chiral symmetry is not broken.

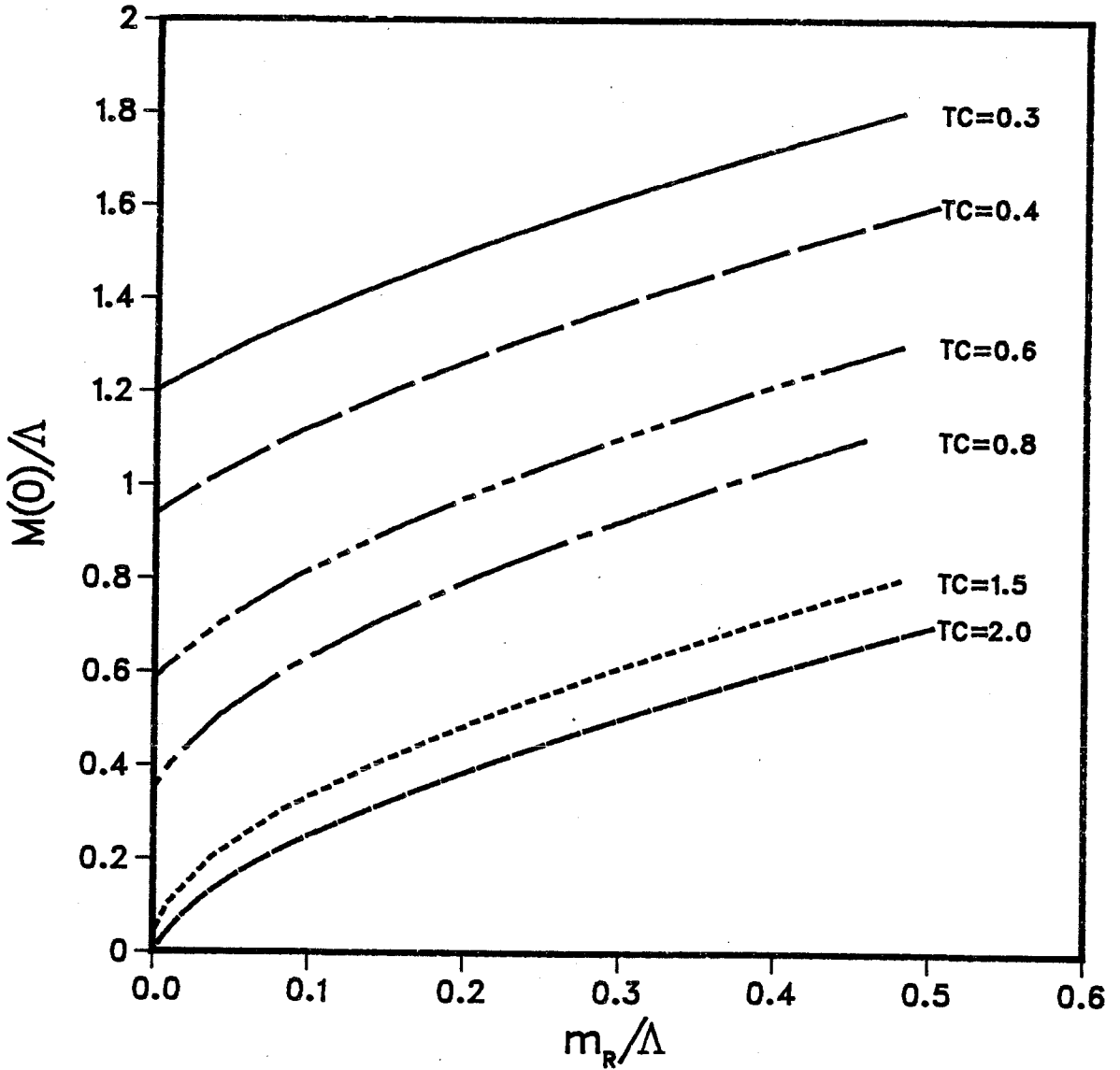


Figure 3.2. Dependence of constituent quark mass  $M(0)$  on the current quark mass  $m_R$  for three colors and three flavors.

So for  $t_c < 1.6$ , the attractive one gluon exchange between quarks and antiquarks ( $q\bar{q}$ ) causes the perturbative vacuum to decay by spontaneous creation of  $q\bar{q}$  pairs. This indicates that the true vacuum must be a

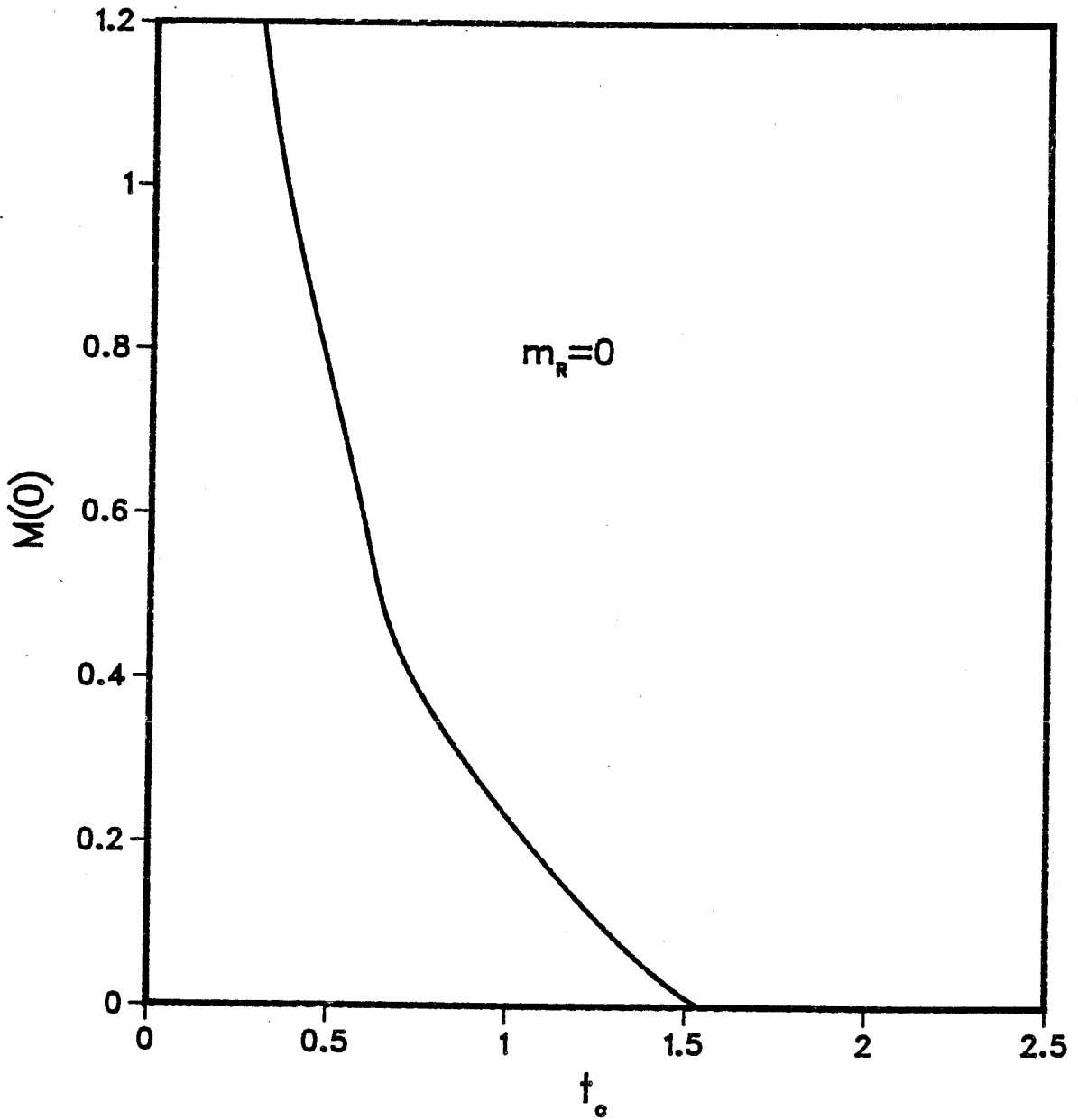


condensate of such pairs. The symmetry breaking can be explained [13] by observing that for  $t_c < 1.6$ , the critical value determined by the coupling constant  $\epsilon(t_c) = \epsilon_0/t_c \approx 0.3$ , the coupling may become strong enough to bind quark antiquark into a bound state. Since a bound state of massless particles is necessarily a tachyon in nonconfining theories like the one we are using, the chiral-symmetric vacuum becomes unstable. To cure this instability, the quark acquires a dynamical mass  $M(0)$  and the bound state becomes a massless Nambu-Goldstone boson. On the other hand, for  $t_c > 1.6$ , ie.  $\epsilon(t_c) < 0.3$ , the attractive force between  $q\bar{q}$  is not strong enough to make a bound state. Therefore, the chiral-symmetric vacuum is stable and the quark remains massless in this case.

Other interesting features we can read from Figure 3.2 are estimates of the constituent masses of the light quarks for various values of  $t_c < 1.6$  given a typical QCD scale  $\Lambda$  of a few hundred MeV. We take the case when  $t_c = 0.4$ . Here  $m_R$  is about 5 MeV. We find  $M(0) \approx 386$  MeV for  $\Lambda = 400$  MeV. However, the constituent quark mass  $M(0)$  crucially depends on the choice of infrared cut off  $t_c$ .

The dependence of  $M(0)$  on the infrared cut off parameter  $t_c$  in the exact chiral limit  $m_R = 0$  is plotted. See Figure 3.3.

Observe that if we take the confining force to correspond to a coupling constant  $\epsilon(t_c) > 1$ , it may suggest that since the  $\chi$ SB corresponds to  $\epsilon(t_c) > 0.3$ , the confining force is not necessary for  $\chi$ SB. For  $\epsilon(t_c) > 1$ , gluon condensation effects may not set in. These gluon condensation effects may affect the detailed structure of the quark condensate, but as we have seen, are not directly responsible for chiral symmetry breaking.



**Figure 3.3** Dependence of constituent quark mass  $M(0)$  on  $t_c$  when the chiral limit is exact, ie.  $m_R = 0$ .

## CHAPTER 4

### 4.1 SPONTANEOUS CHIRAL SYMMETRY BREAKING: A VARIATIONAL SCHEME

In the last chapter we were primarily concerned with analyzing the stationarity condition of the effective potentials - the SD equation, to understand chiral symmetry breaking. We saw that minimizing  $V_{CJT}$  or  $V_{CDDG}$  gave the SD equation and we did not have to worry about the difference between the two.

In this chapter we analyze the effective potential itself and since the two are slightly structurally different we must pay particular attention to possible differences between them.

Our aim here, as was stated at the end of Chapter 1, is to do a selfconsistent computation with  $V_{CDDG}$  using a Rayleigh-Ritz variational method. Though a different variational scheme was used [11] to do computations with  $V_{CDDG}$ , their scheme was shown in the previous chapter to be not selfconsistent, ie., the form of  $V_{CDDG}$  they used will not give the SD equation on minimization. This will also demonstrate how a Rayleigh-Ritz scheme is successful in studying spontaneous  $\chi_{SB}$  with  $V_{CDDG}$  which they (CDDG) had claimed was not the case. Castorina and Pi have also used the Raleigh-Ritz scheme to study spontaneous  $\chi_{SB}$  with  $V_{CJT}$  which we will also review here for comparison. And lastly, we will also study the behavior of the effective potentials for theories in which the coupling constant does not run. We find a significant result. Chiral symmetry remains exact in such theories. This is understandable since without a running coupling constant there is no dimensionful parameter to set the scale of symmetry breaking. The study of SD equation yields information about the effective potential at the stationary point.

We start with equation (2.64).

$$\begin{aligned} \frac{(2\pi)^4}{2n_f} V_{\text{CDDG}} &= - \int d^4p \ln(p^2 + \sum^2(p^2)) \\ &+ \int d^4p \frac{M(p^2)}{p^2 + M^2(p^2)} \int d^4k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} + \text{const.} \end{aligned} \quad (2.64)$$

We now evaluate  $V$  with a specific parameter dependent ansatz for  $M(p^2)$  and vary these parameters. The regular solution of the SD equation readily lends itself to an ansatz. Remembering that  $M(p^2) = p C(p^2)$ , substitute (3.42), the regular solution for  $C(p^2)$  to obtain

$$p^2 \xrightarrow{p \rightarrow +\infty} M(p^2) = A_p \left(\frac{\Lambda}{p}\right)^3 \left[ \ln\left(\frac{p^2}{\Lambda^2}\right) \right]^{\epsilon_0 - 1} \quad (4.1a)$$

This is consistent with the solution obtained by D. Politzer [21] using operator product expansion.

For continuity purposes, we use the following ansatz which is also used by Castorina and Pi in their study of spontaneous  $\chi$ SB with the CJT effective potential.

$$M(p^2) = \left\{ \begin{array}{ll} m & \text{for } p^2 \leq p_c^2 \\ m \frac{p_c^2}{p^2} \left[ \ln\left(\frac{p_c^2}{\Lambda^2}\right) \right]^{1-\epsilon_0} \left[ \ln\left(\frac{p^2}{\Lambda^2}\right) \right]^{\epsilon_0 - 1} & \text{for } p^2 \geq p_c^2 \end{array} \right\} \quad (4.1b)$$

For  $g^2(p^2)$  (the coupling), we use our nonconfining form

$$g^2(p^2) = a \lambda^2(p^2)$$

where

$$a = \frac{48\pi^2}{11N-2n_f}$$

(4.2)

$$\lambda^2(p^2) = \left\{ \begin{array}{l} \frac{1}{\ln\left(\frac{p_c^2}{\Lambda^2}\right)} \quad p^2 \leq p_c^2 \\ \frac{1}{\ln\left(\frac{p^2}{\Lambda^2}\right)} \quad p^2 \geq p_c^2 \end{array} \right\} \quad (4.3)$$

We first simplify (2.64) into a manageable form. Using the definition of  $\Sigma(p^2)$  (2.67), we rewrite (2.64) as

$$\frac{(2\pi)^4}{2n_f} V = - \int d^4p \ln(p^2 + \Sigma^2(p^2)) + \int d^4p \frac{M(p^2)}{p^2 + M^2(p^2)} \Sigma(p^2) + \text{const.} \quad (4.4)$$

Then using (3.3) and (3.6) gives

$$\Sigma(x) = \frac{3C_f(N)}{16\pi^2} \left[ \frac{g^2(x)}{x} \int_0^x y dy + \int_0^\infty \frac{g^2(y)}{y} y dy \right] \frac{M(y)}{y + M^2(y)} \quad (4.5)$$

Define

$$M(x) = mB(x), \quad (4.6)$$

Substituting (4.6) and (4.2) into (4.5) we find,

$$\Sigma(x) = \frac{3C_f(N)}{16\pi^2} am \left[ \frac{\lambda^2(x)}{x} \int_0^x y dy + \int_x^\infty \lambda^2(y) dy \right] \frac{B(y)}{y + m^2 B^2(y)} \quad (4.7)$$

Substituting (4.7) into (4.4), one finds

$$\begin{aligned} \frac{(2\pi)^4}{2n_f} V &= -\pi^2 \int_0^\infty dx x \ln \left( 1 + \frac{\sum^2(x)}{x} \right) - \pi^2 \int_0^\infty dx x \ln x \\ &+ \pi^2 \int_0^\infty dx x \frac{mB(x)}{x+m^2B^2(x)} \sum(x) + \text{const.} \end{aligned}$$

and finally we have

$$\begin{aligned} \frac{8\pi^2}{n_f} V &= - \int_0^\infty dx x \ln \left( 1 + \frac{\sum^2(x)}{x} \right) \\ &+ \int_0^\infty dx x \frac{B(x)}{x+m^2B^2(x)} \sum(x) \end{aligned} \quad (4.8)$$

An irrelevant constant has been subtracted in (4.8). Using the definition of  $\sum(x)$  given by (4.7) we find

$$\begin{aligned} \frac{8\pi^2}{n_f} V &= - \int_0^\infty dx x \ln \left\{ 1 + \left( \frac{3C_f(N)a}{16\pi^2} \right)^2 m^2 \times \right. \\ &\left. \left( \left[ \frac{\lambda^2(x)}{x} \int_0^x y dy + \int_x^\infty \lambda^2(y) dy \right] \frac{B(y)}{y+m^2B^2(y)} \right)^2 \right\} \\ &+ m^2 \left( \frac{3C_f(n)}{16\pi^2} a \right) \int_0^\infty dx x \frac{B(x)}{x+m^2B^2(x)} \left( \left[ \frac{\lambda^2(x)}{x} \int_0^x y dy \right. \right. \\ &\left. \left. + \int_x^\infty \lambda^2(y) dy \right] \times \frac{B(y)}{y+m^2B^2(y)} \right). \end{aligned} \quad (4.9)$$

Then we make the following change of variables

$$\begin{aligned} y &= \Lambda^2 y' \\ x &= \Lambda^2 x' \end{aligned} \quad (4.10)$$

and defining

$$\frac{\mu}{\Lambda} = b \tag{4.11}$$

$$\frac{3C_f(N)}{16\pi^2} a \equiv \epsilon_0$$

We rewrite (4.9) as follows

$$\frac{8\pi^2}{n_f} \frac{V}{\Lambda^4} = - \int_0^\infty x dx \ln F(x) + \epsilon_0 \cdot b^2 \int_0^\infty x dx G(x) \tag{4.12}$$

where

$$F(x) = \left( 1 + \frac{\epsilon_0^2 \cdot b^2}{x} \left[ \left( \frac{\lambda^2(x)}{x} \int_0^x y dy + \int_x^\infty \lambda^2(y) dy \right) \frac{B(y)}{y+b^2B^2(y)} \right]^2 \right) \tag{4.13}$$

$$G(x) = \frac{B(x)}{x+b^2B^2(x)} \left[ \left( \frac{\lambda^2(x)}{x} \int_0^\infty y dy + \int_x^\infty \lambda^2(y) dy \right) \frac{B(y)}{y+b^2B^2(y)} \right] \tag{4.14}$$

and we have replaced  $x'$  by  $x$  since it is a dummy index.

To start the numerical computation on (4.12) we make one more change of variable

$$x = \frac{10-t}{t} \tag{4.16}$$

then (4.12) becomes

$$\frac{8\pi^2}{10n_f} \frac{V}{\Lambda^4} = \int_0^{10} dt \frac{(10-t)}{t^3} \left( \epsilon_0 b^2 G\left(\frac{10-t}{t}\right) - \ln F\left(\frac{10-t}{t}\right) \right) . \tag{4.17}$$

Going through the same steps, the corresponding expression for  $V_{CJT}$  is

$$\frac{8\pi^2}{10n_f} \frac{V_{CJT}}{\Lambda^4} = \int_0^{10} \frac{(10-t)}{t^3} \left( -\epsilon_0 b^2 G\left(\frac{10-t}{t}\right) - \ln H\left(\frac{10-t}{t}\right) + 2y\left(\frac{10-t}{t}\right) \right) dt \quad (4.18)$$

where

$$H(x) = \left( 1 + \frac{b^2}{x} B^2(x) \right) \quad (4.19)$$

$$Y(x) = b^2 \frac{B^2(x)}{x + b^2 B^2(x)} \quad (4.20)$$

and  $G(x)$  is (4.14). Eqs. (4.17) and (4.18) are the equations we solve numerically.

There are integration packages in the NAG library for solving one- and multi-dimensional integrals. Our integrals are too complicated to simply use the package for double integration. We have instead used a double precision one-dimensional recursive application, which is always recommended to check the accuracy of the other methods.

With (4.1) and (4.3) as ansatz for  $B(x)$  and  $g^2(x)$  respectively, we find that our effective potential  $V/\Lambda^4$  is a function of  $m/\Lambda$  ( $=b$ ) for various values of the parameter  $\rho_C/\Lambda$  ( $x = p^2/\Lambda^2$ ).  $m$  is the renormalization-point-independent vacuum expectation value  $\langle 0 | \bar{\psi}\psi | 0 \rangle$  and plays the role of an order parameter (as magnetization for a ferromagnetic system).

From both equations (4.17) and (4.18) we see that  $M(p^2) \equiv 0$  is always a solution corresponding to exact chiral symmetry ( $m_R = 0$ ). If chiral symmetry is broken spontaneously,  $M(p^2)$  no longer vanishes identically but is dynamically generated, the behaviour of which is given by the regular solution.



RESULTS

With  $V_{\text{CDDG}}$ , we find that for  $SU_c(3)$  with three flavors ( $n_f = 3$ ), dynamical symmetry breaking occurs when  $p_c/\Lambda < 1.7$ . For  $p_c/\Lambda = 1.3$ , a stable minimum occurs at  $m/\Lambda = 0.67$ . See Fig. (4.1).

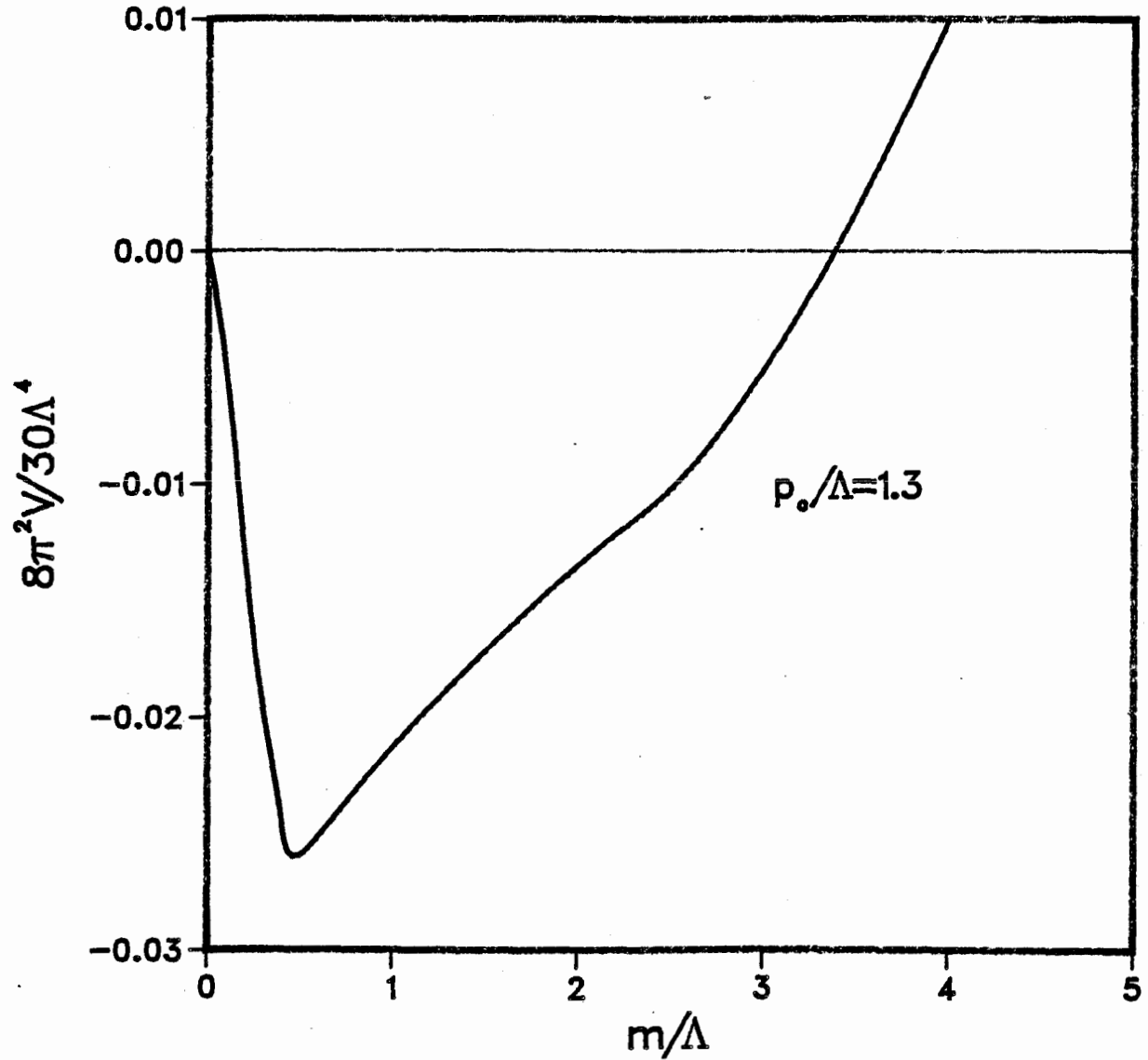


Figure 4.1 Effective potential  $V_{\text{CDDG}}$  for  $p_c/\Lambda = 1.3$ ,  $N = 3$ ,  $n_f = 3$ .

For  $p_c/\Lambda = 1.687$ , a stable minimum occurs at  $m/\Lambda = 0.05$ .

See Fig. (4.2).

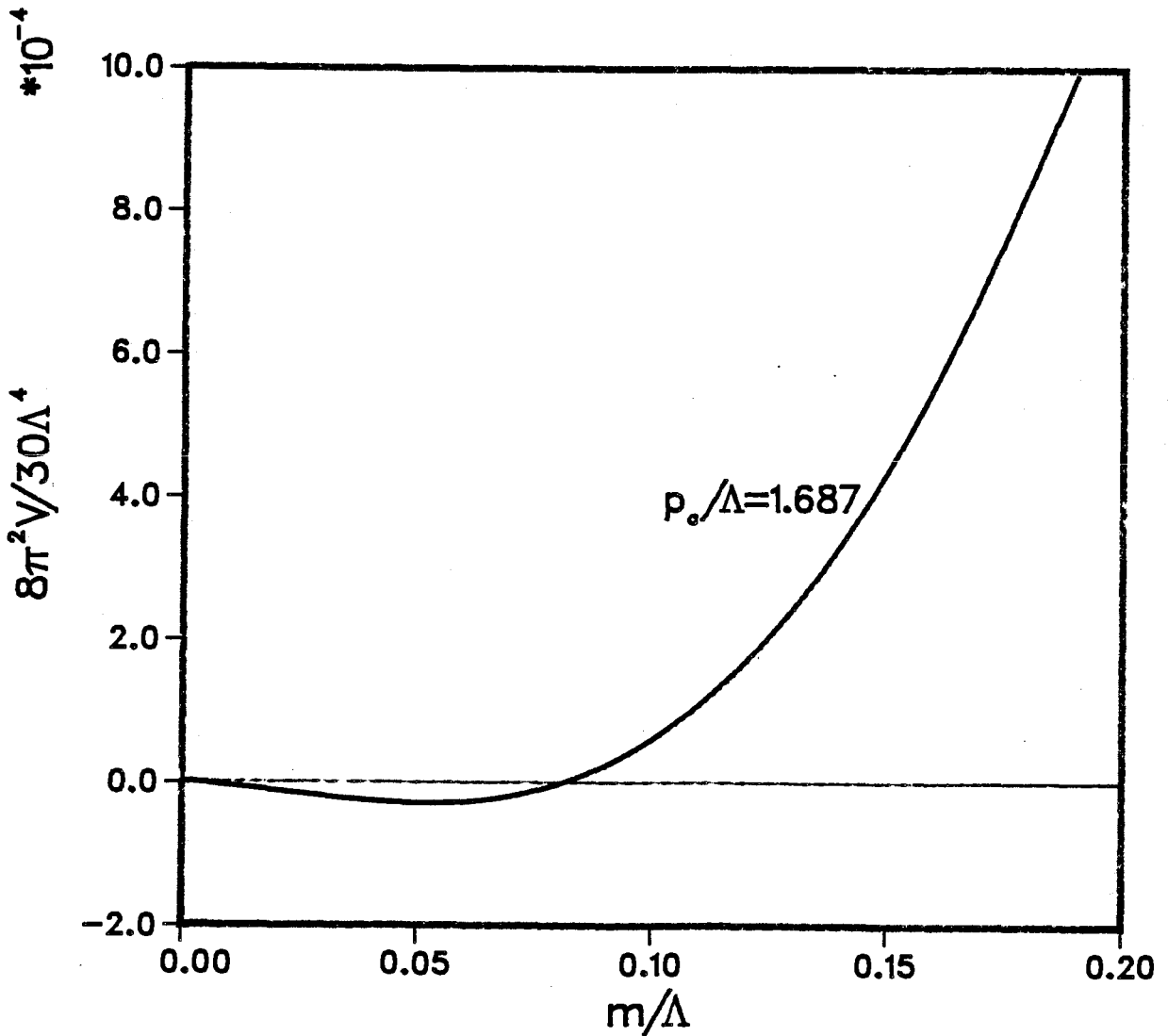


Figure 4.2 Effective potential  $V_{\text{CDDG}}$  for  $p_c/\Lambda = 1.687$ ,  $N = 3$ ,  $n_f = 3$ .

For  $p_c/\Lambda = 2$ , a stable minimum occurs at  $m/\Lambda = 0$ . In fact, for all  $p_c/\Lambda > 1.7$ ,  $m = 0$  is always the global minimum. See Fig. (4.3).

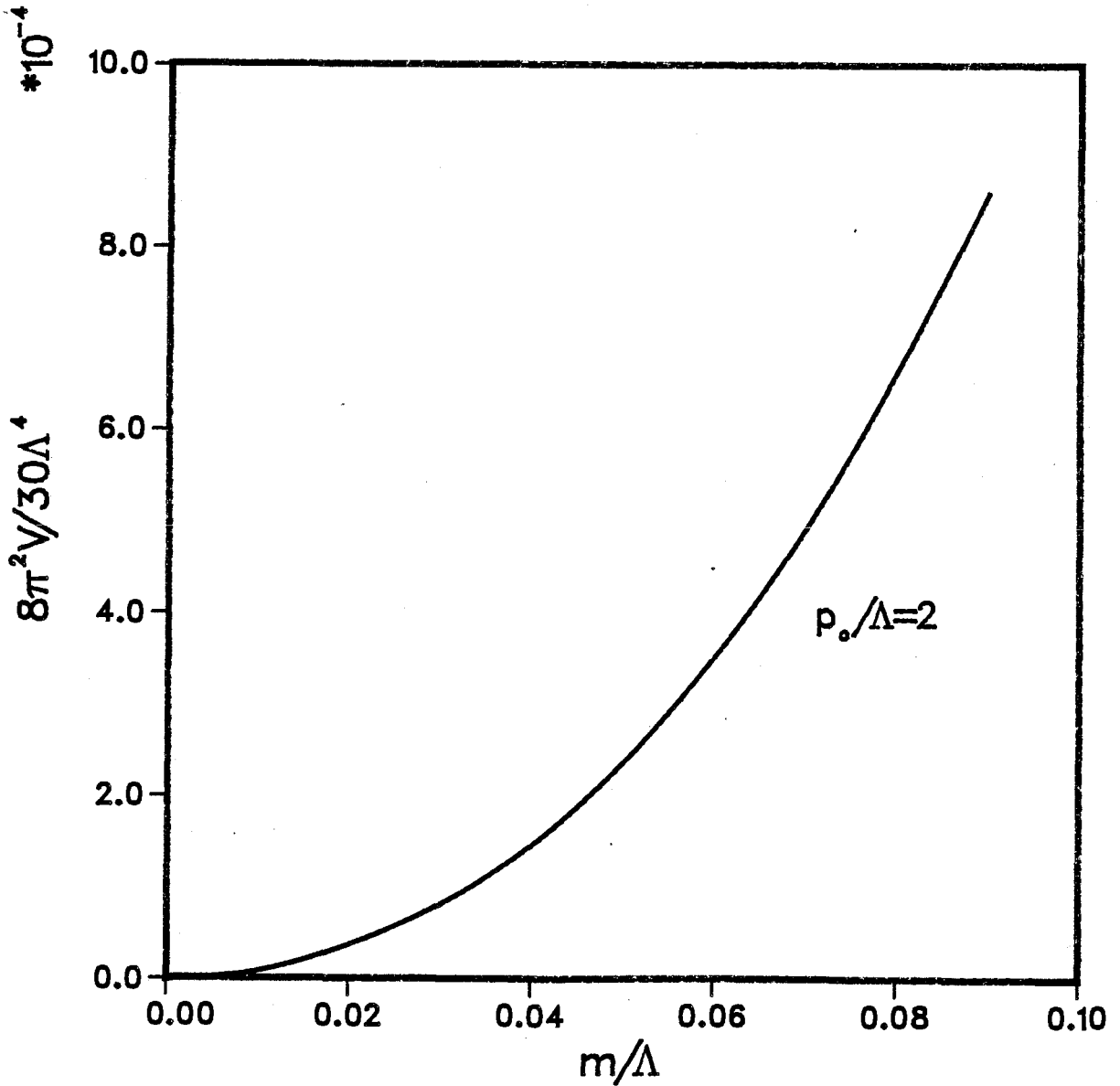


Figure 4.3 Effective potential  $V_{\text{CDDG}}$  for  $p_c/\Lambda = 2$ ,  $N = 3$ ,  $n_f = 3$ .

Using  $V_{CJT}$  we refer the reader to the work done by Castorina and Pi [16]. However, we show here the case for  $p_c/\Lambda = 1.3$  in Fig. (4.4). A stable minimum occurs at  $m/\Lambda = 0.55$ .

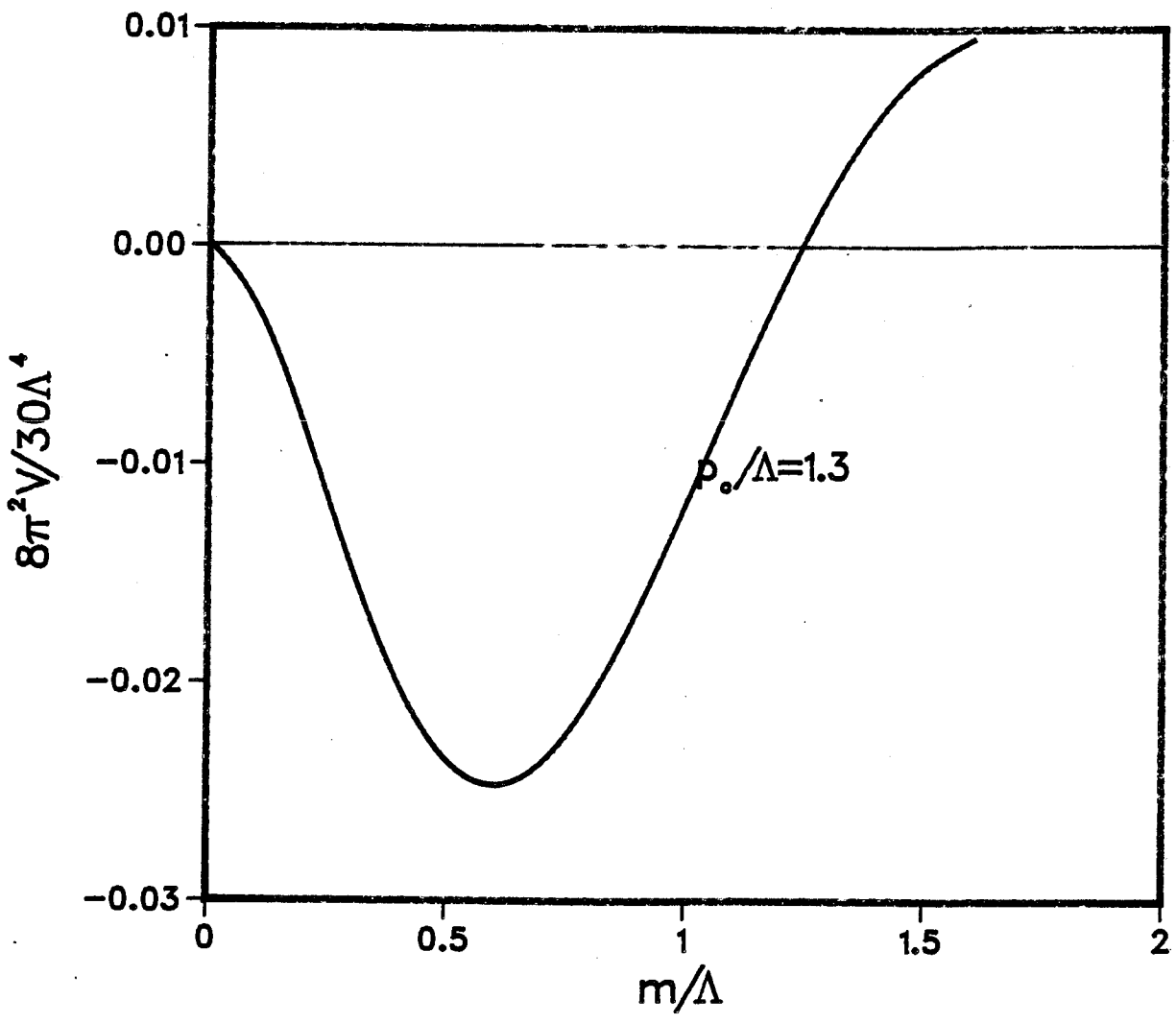


Figure 4.4 Effective potential  $V_{CJT}$  for  $p_c/\Lambda = 1.3$ ,  $N = 3$ ,  $n_f = 3$ .

With  $V_{CJT}$ , dynamical symmetry breaking occurs when  $p_c/\Lambda < 1.5$ , while with  $V_{CDDG}$ , dynamical symmetry breaking occurs when  $p_c/\Lambda < 1.7$ .

Before going further, we would like to comment on the successful comparisons using the same ansatz for both  $V_{CJT}$  and  $V_{CDDG}$ . The qualitative results are the same and both are seen to be bounded below. Casalbuoni, DeCurtis, Dominici and Gatto [11] had claimed that direct

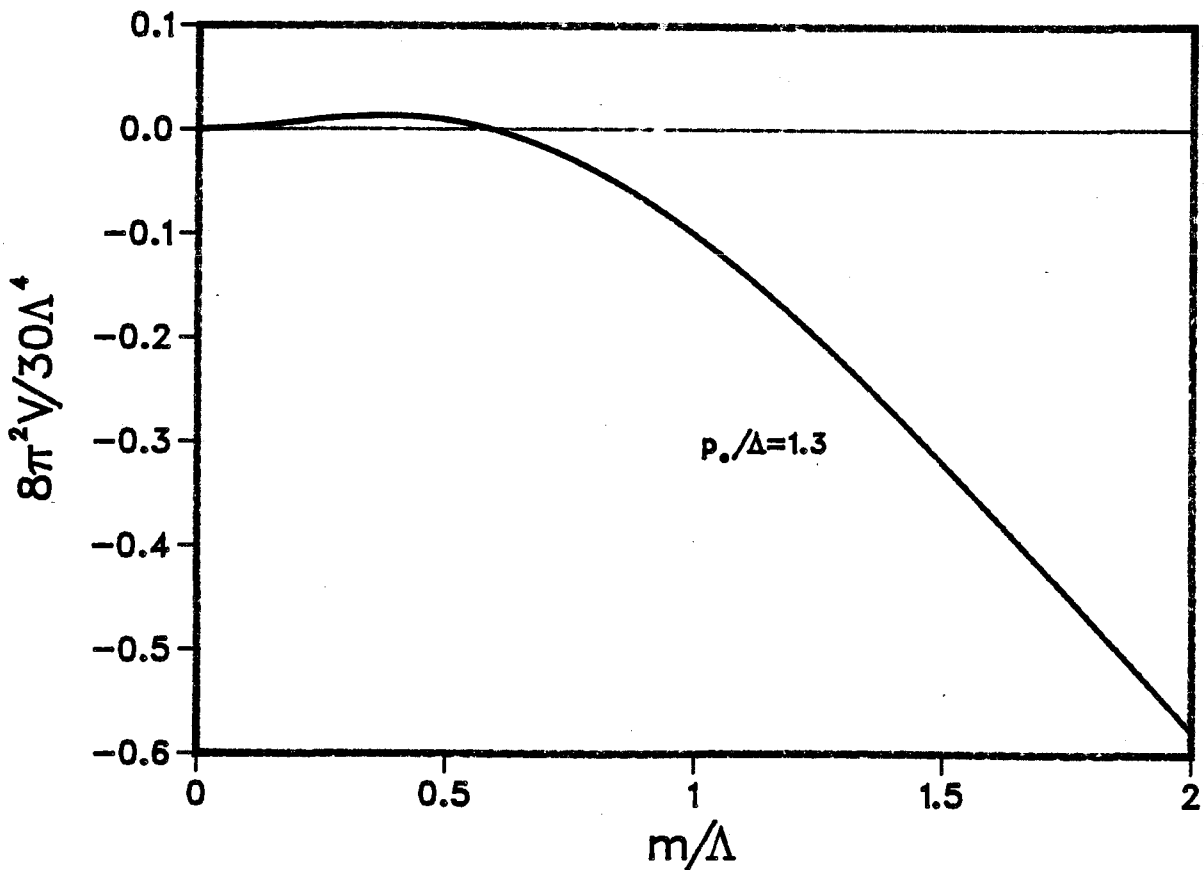


Figure 4.5 The effective potential when the solution of the SD equation  $M(p^2)$  is used for  $\Sigma(p^2)$  in the  $\lg(S_0^{-1} + \Sigma(p))$  term of  $V_{CDDG}$ . This case is for  $p_c/\Lambda = 1.3$ ,  $N = 3$  and  $n_f = 3$ .

comparisons, even for similar ansatz, are not easy to make. They substituted the solution of the SD equation  $M(p^2)$  for  $\sum(p^2)$  into the  $\ln\left(1 + \frac{\sum^2(p^2)}{p^2}\right)$  term in the effective potential and directly used their ansatz for  $M(p^2)$ . Note that their  $\sum(p^2)$  is equivalent to  $M(p^2)$  we are using here. We have shown that this form does not yield the SD equation on minimization. Fig. (4.7) shows this form in the original CJT variation principle we are using.

The effective potential shows a maximum and is unbounded below. A related effective potential is that which replaces all  $\sum(p^2)$  by  $M(p^2)$ , i.e., the effective potential for every value of  $m$  which is a solution of the SD equation.

Note that at the stationary point

$$\sum(p^2) = M(p^2) = \int d^4k A_{p,k} \frac{M(k^2)}{k^2 + M^2(k^2)} \frac{1}{(p-k)^2} . \quad (3.8)$$

Substitute this into (2.64) and (2.66) to have

$$\frac{(2\pi)^4}{2n_f} \bar{V}_{\text{CDDG}} = - \int d^4p \ln \left( 1 + \frac{M^2(p^2)}{p^2} \right) + \int d^4p \frac{M^2(p^2)}{p^2 + M^2(p^2)} \quad (4.21)$$

$$\frac{(2\pi)^4}{2n_f} \bar{V}_{\text{CJT}} = - \int d^4p \ln \left( 1 + \frac{M^2(p^2)}{p^2} \right) + \int d^4p \frac{M^2(p^2)}{p^2 + M^2(p^2)} \quad (4.22)$$

Equations (4.21) and (4.22) are exactly the same. Fig. (4.8) shows  $\bar{V}$  for  $p_C/\Lambda = 1.3$ ,  $N = 3$ ,  $n_f = 3$ . Notice that  $\bar{V}$  is unbounded below. However, this does not mean there is an instability,  $\bar{V}$  does not determine the minimum. Minimization is carried out on  $V$  which is bounded below and  $\bar{V}$  is the value of  $V$  at the stationary point.

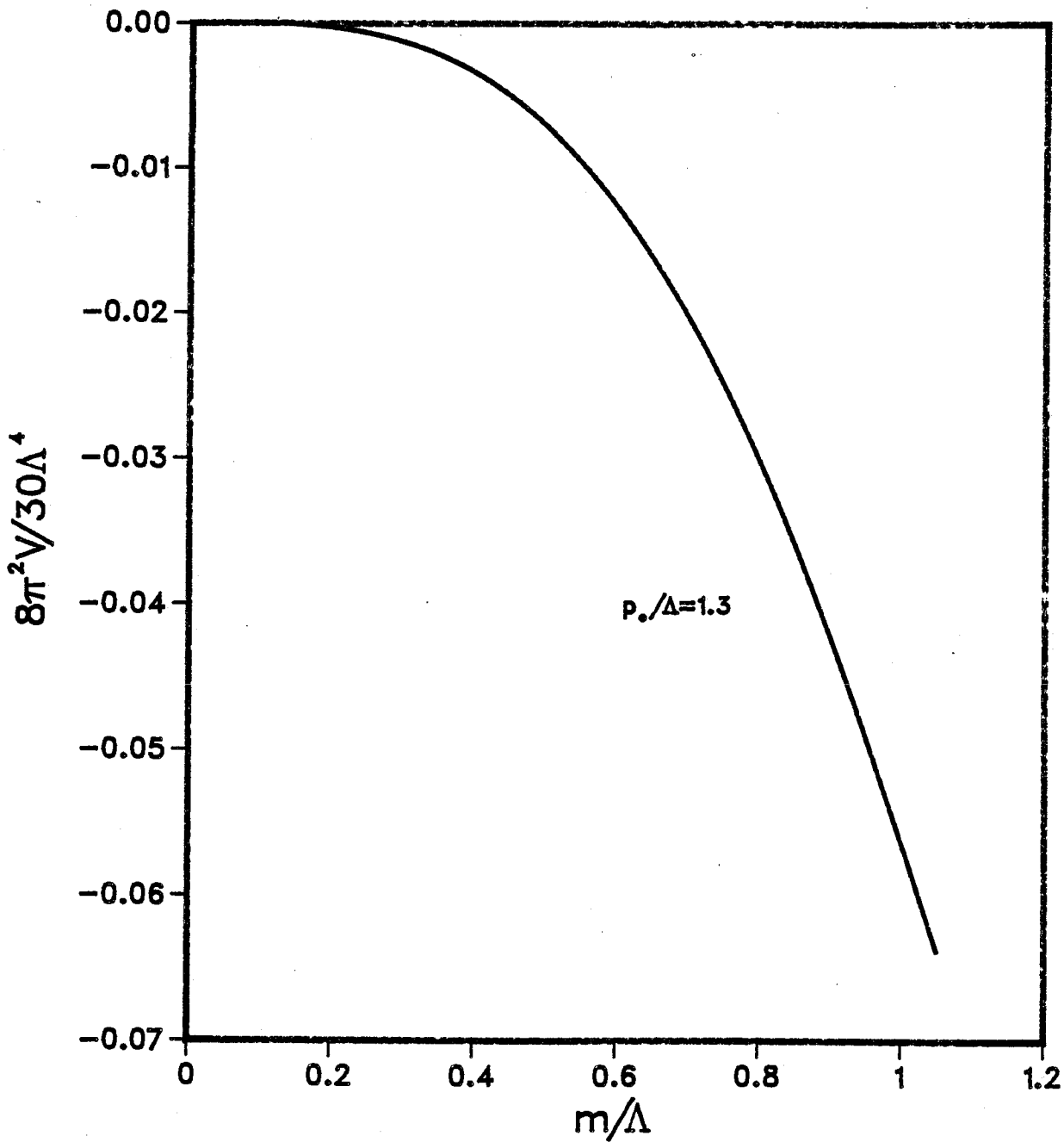


Figure 4.6 Effective potential  $\bar{V}$  for  $p_c/\Lambda = 1.3$ ,  $N = 3$ ,  $n_f = 3$ .

#### 4.2 SPONTANEOUS CHIRAL SYMMETRY BREAKING IN FIELD THEORIES WITH FIXED COUPLING CONSTANT

So far, we have seen that asymptotically free field theories allow chiral symmetry breaking. The deeper question is: is it only in asymptotically free field theories that  $\chi$ SB occurs? In Chapter 3, end of Section 3.1, we saw that the logarithmic dependence of  $g^2(p^2)$  and  $M(p^2)$  are features of asymptotic freedom as determined by the renormalization group. We therefore suppress this logarithmic dependency in  $g^2(p^2)$  and in  $M(p^2)$  and carry out the numerical calculation again to see whether chiral symmetry is still broken or not.

Recall that

$$\epsilon(p^2) = \frac{\epsilon_0}{\ln\left(\frac{p^2}{\Lambda^2}\right)}$$

$$\epsilon_0 = \frac{9C_f(N)}{11N-2n_f} = \frac{9(N^2-1)}{(11N-2n_f)2N}, \quad N = n_f = 3.$$

So without the logarithmic dependence, the coupling  $\epsilon(p^2)$  is a constant  $\epsilon_0$ , ie.

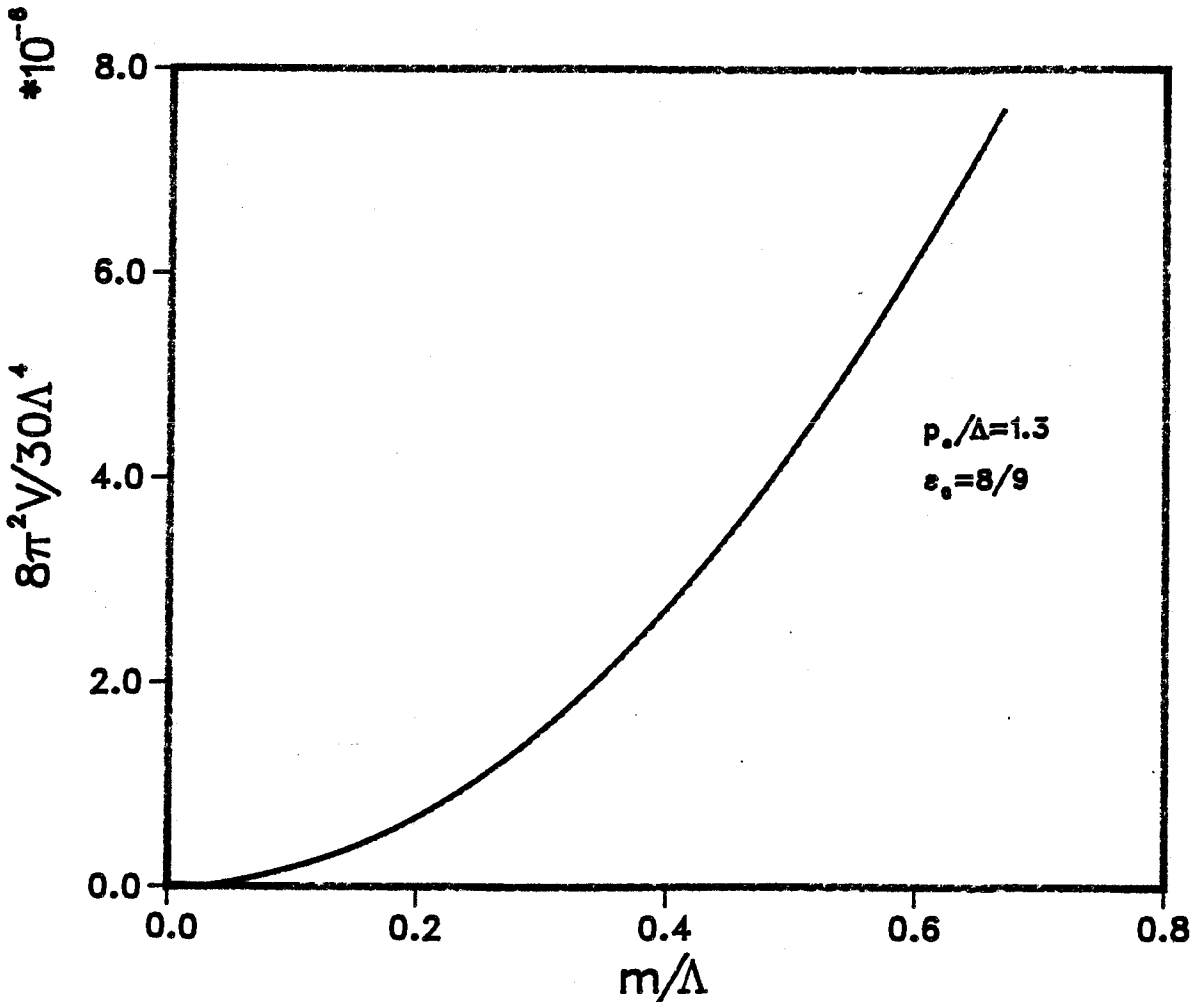
$$\epsilon(p^2) = \epsilon_0 = \frac{4}{9} \quad (\text{without logarithmic dependence}).$$

We now vary the value of  $\epsilon_0$  and see what happens to the effective potentials.

We start with  $V_{\text{CDDG}}$ . Figures (4.7) and (4.8) show the cases when  $\epsilon_0 = 0.01/9$  and  $\epsilon_0 = 8/9$ . We find that  $m = 0$  is always the global minimum. We find, in general, that for any value of  $\epsilon_0$  used,  $m = 0$  remains the global minimum. This shows that with the modified effective



potential  $V_{\text{CDDG}}$  chiral symmetry is never broken for non-asymptotically free field theories. This point is worth noting because it emphasizes the importance of logarithmic corrections in asymptotically free field theories and seems to further justify the claim that only in asymptotically free gauge theories chiral symmetry breaking occurs at reasonable momentum scales.



**Figure 4.7** Effective potential  $V_{\text{CDDG}}$  for field theories with a fixed coupling constant:  $p_c/\Lambda = 1.3$ ,  $N = n_f = 3$ ,  $\epsilon_0 = 8/9$ .

If we replace  $\sum(p^2)$  by  $M(p^2)$  in the  $\ln(S_0^1 + \sum(p))$  term in  $V_{CDDG}$ , we obtain the same result. However, this has no physical significance since the corresponding expression with logarithmic behavior is not an effective potential for  $\chi_{SB}$ .

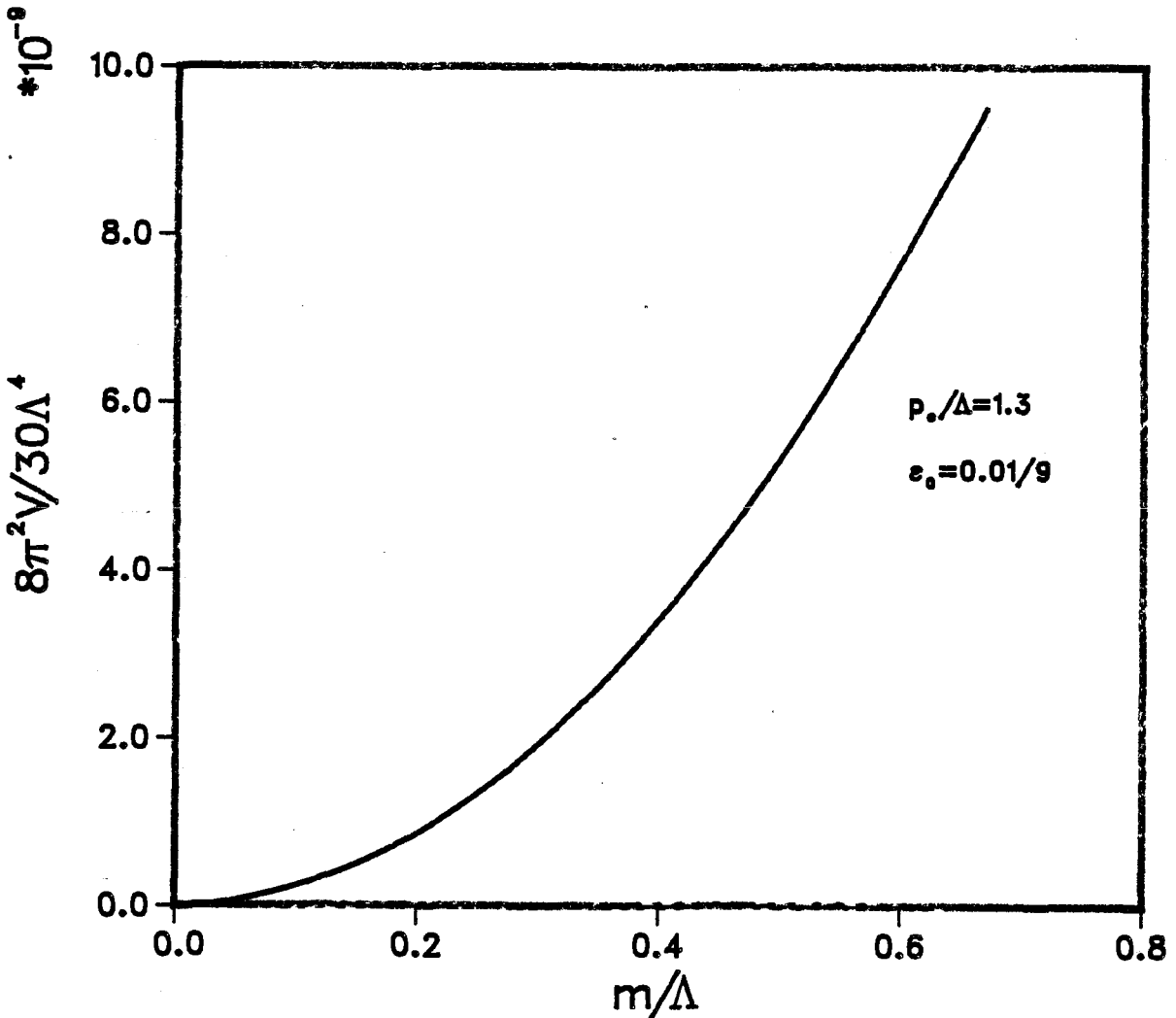


Figure 4.8 Effective potential  $V_{CDDG}$  for field theories with a fixed coupling constant:  $p_c/\Lambda = 1.3$ ,  $N = n_f = 3$ ,  $\epsilon_0 = 0.01/9$ .

With  $V_{\text{CJT}}$  for field theories with a coupling constant that does not run, see Figures (4.9) and (4.10), we find that for  $\epsilon_0 < 2/3$ ,  $m = 0$  is a stable minimum, indicating no chiral symmetry breaking. For  $\epsilon_0 > 2/3$ , dynamical symmetry breaking occurs but a stable minimum does not exist; the effective potential decreases monotonically as  $m$  increases

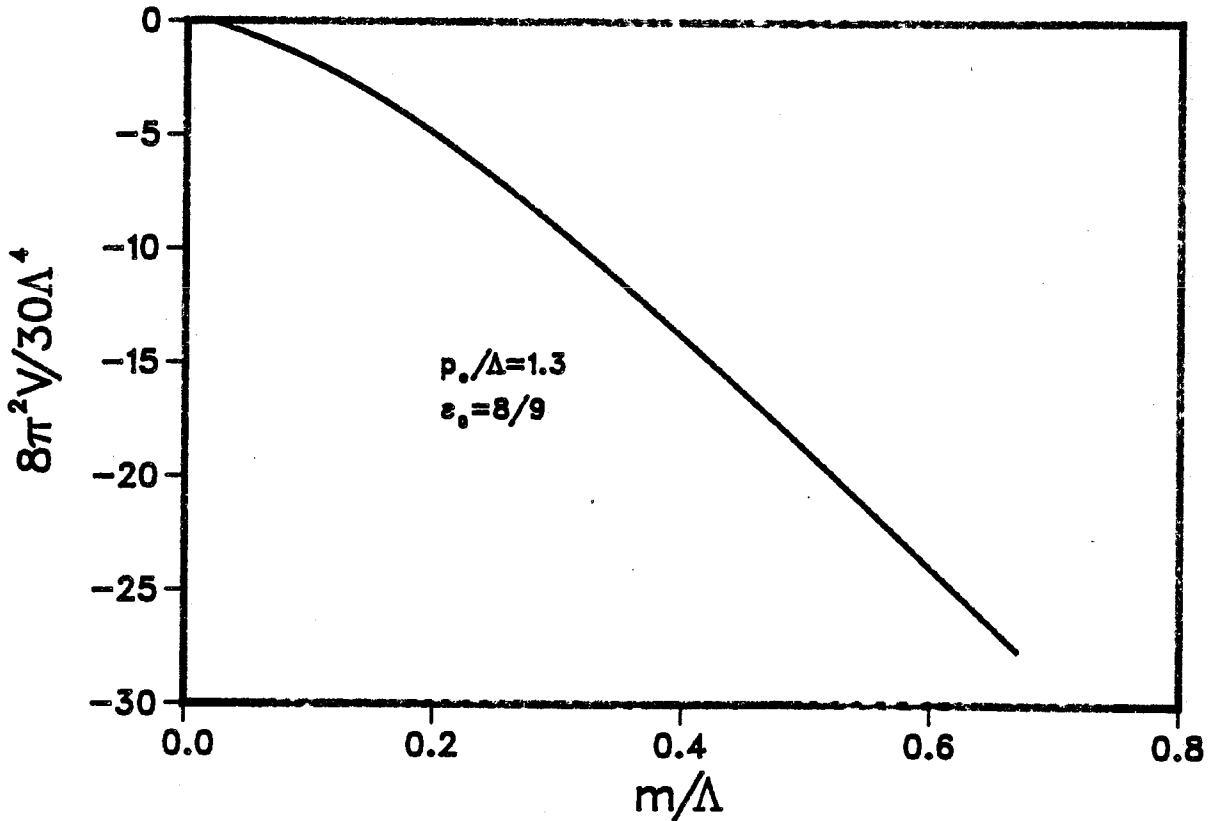


Figure 4.9 Effective potential  $V_{\text{CJT}}$  for field theories with a fixed coupling constant:  $p_c/\Lambda = 1.3$ ,  $N = n_f = 3$ ,  $\epsilon_0 = 8/9$ .

from zero and is unbounded from below. This feature is not very pleasing, since we know that only in asymptotically free field theories chiral symmetry is broken. This is the first caution in using  $V_{CJT}$  because in both asymptotically free field theories and non-asymptotically free field theories, chiral symmetry is broken except that in the one case, namely non-asymptotically free field theories, a stable minimum does not exist.

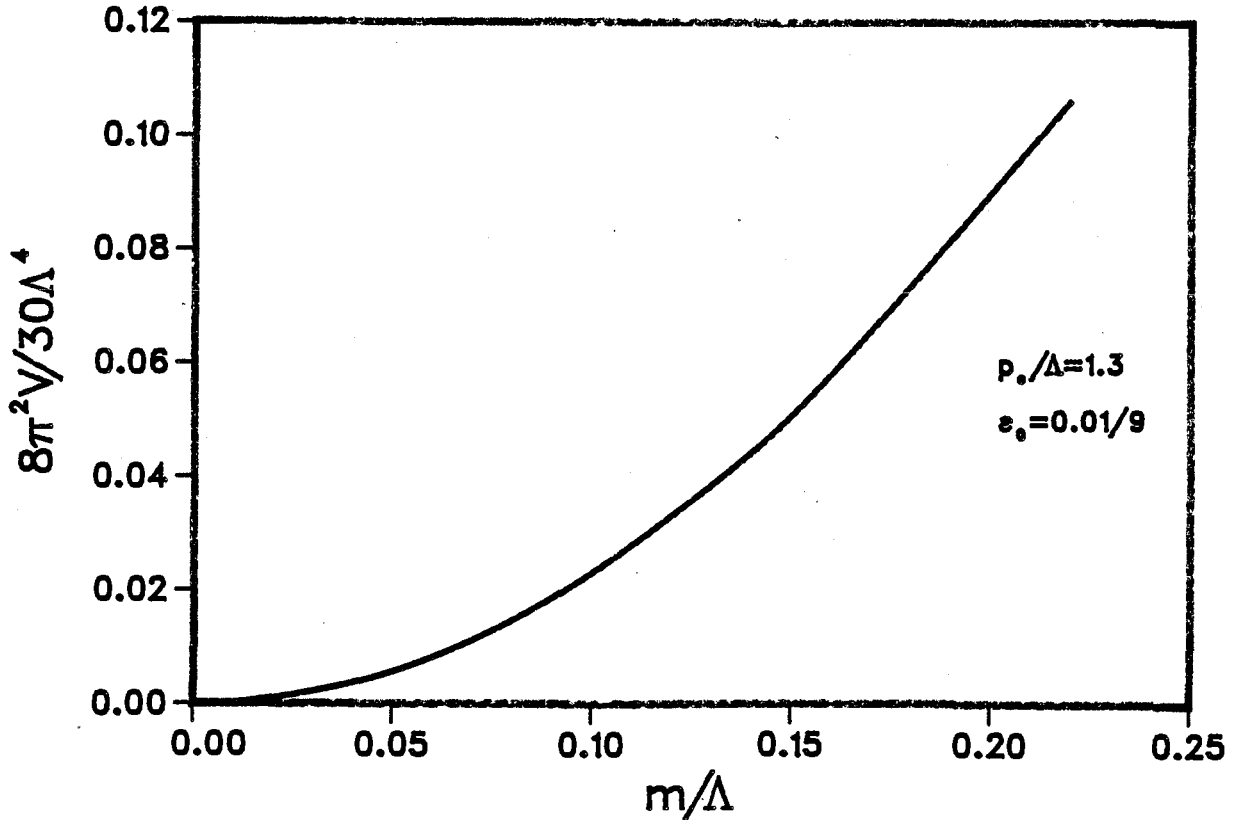


Figure 4.10 Effective potential  $V_{CJT}$  for field theories with a fixed coupling constant:  $p_c/\Lambda = 1.3$ ,  $N = n_f = 3$ ,  $\epsilon_0 = 0.01/9$ .

**REMARKS:** The critical coupling  $\epsilon(t_c)$  for chiral symmetry breaking obtained by using the two approaches - explicit and spontaneous  $\chi$ SB - is in good agreement. Recall

$$\epsilon(t_c) = \frac{e_0}{t_c}$$

for explicit  $\chi$ SB       $\epsilon(1.6) \approx 0.3$

for spontaneous  $\chi$ SB       $\epsilon(1.06) \approx 0.4$

In summary, we have in this chapter shown the effectiveness of using a Raleigh-Ritz variational method in doing straightforward computations with any selfconsistent effective potential for chiral symmetry breaking. We have also shown that with the modified effective potential  $V_{\text{CDDG}}$  applied to theories where the coupling constant does not run, there is no evidence for chiral symmetry breaking. While this result has been speculated upon in the literature in the past, our calculations demonstrate it using the effective potential scheme in a satisfactory manner. As remarked earlier, this result is consistent, since in such theories there is no dimensionful parameter to set the scale of chiral symmetry breaking.

## CHAPTER 5

### 5.1 STABILITY OF CHIRAL SYMMETRY BREAKING SOLUTIONS

It was observed by R.W. Haymaker and J. Perez-Mercader [18] that the chiral symmetry breaking solutions correspond to saddle-point instabilities when the CJT effective potential is used. This was formerly proved by Haymaker and Matsuki in 1986 [14]. Instability here means that under small arbitrary variations, the symmetry breaking solutions diverge from the stationary point. This feature can be seen when one expands the effective potential about the stationary point. The symmetry breaking solution is  $M(x)$  and an expansion of  $V$  about this is as follows

$$\begin{aligned} V[M+\delta M] &= V[M] + \int_0^\infty \frac{\delta V}{\delta M(x)} \delta M(x) dx \\ &+ \frac{1}{2} \int_0^\infty dx dy \delta M(y) \frac{\delta^2 V}{\delta M(y) \delta M(x)} \delta M(x) . \end{aligned} \quad (5.1)$$

The stationarity conditions  $\delta V / \delta M(x) = 0$  sets to zero the second term on the right hand side giving

$$V[M+\delta M] = V[M] + \frac{1}{2} \int_0^\infty dx dy \delta M(y) \frac{\delta^2 V}{\delta M(y) \delta M(x)} \delta M(x) . \quad (5.2)$$

Now the problem reduces to finding the eigenvalues of the second functional derivative of  $V$ , because the second term can be positive, negative or zero depending on the eigenvalues of the second functional derivative. We call this term on the right hand side the expectation value of the second functional derivative. If the expectation value is

positive, then the symmetry breaking solutions are stable. If the expectation value is negative, the symmetry breaking solutions are unstable. The expectation value can also be zero but we do not consider this case here.

Any viable effective potential for  $\chi$ SB must exhibit this stability requirement. In this chapter we study the stability of  $\chi$ SB solutions using CDDG effective potential [25].

We start by taking the second functional derivative of  $V_{\text{CDDG}}$  w.r.t  $M(p^2)$ . Recall equation (3.7).

$$\begin{aligned} \frac{(2\pi)^4}{2n_f} \frac{\delta V}{\delta M(q^2)} &= 2\pi^2 q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \int d^4k \frac{\Lambda_{q,k}}{(q-k)^2} \frac{M(k^2)}{k^2 + M^2(k^2)} \\ &\quad - 2\pi^2 \int d^4p \frac{\Sigma(p^2)}{p^2 + \Sigma^2(p^2)} W(p^2, q^2) q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \end{aligned} \quad (3.7)$$

Now we take the second functional derivative

$$\begin{aligned} \frac{(2\pi)^4}{4\pi^2 n_f} \frac{\delta^2 V}{\delta M(q^2) \delta M(S^2)} &= \left[ \frac{\delta}{\delta M(S^2)} \left( q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \right) \right] \times \\ &\quad \int d^4k \frac{\Lambda_{q,k}}{(q-k)^2} \frac{M(k^2)}{k^2 + M^2(k^2)} \\ &\quad - \int d^4p \frac{\Sigma(p^2)}{p^2 + \Sigma^2(p^2)} W(p^2, q^2) \left[ \frac{\delta}{\delta M(S^2)} \left( q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \right) \right] \\ &\quad + q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \left[ \frac{\delta}{\delta M(S^2)} \int d^4k \frac{\Lambda_{q,k}}{(q-k)^2} \frac{M(k^2)}{k^2 + M^2(k^2)} \right] \\ &\quad - \left[ \frac{\delta}{\delta M(S^2)} \int d^4p \frac{\Sigma(p^2)}{p^2 + \Sigma^2(p^2)} W(p^2, q^2) \right] q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \end{aligned} \quad (5.3)$$

Since we need the solution at  $\sum = M$ , the first two terms cancel and we have

$$\begin{aligned} \frac{4\pi^2}{n_f} \frac{\delta^2 V}{\delta M(q^2) \delta M(S^2)} &= q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \left[ \int d^4 k \frac{A_{q,k}}{(q-k)^2} \times \right. \\ &\quad \left. \left( \frac{k^2 - M^2(k^2)}{(k^2 + M^2(k^2))^2} \right) \delta(S^2 - k^2) \right] \\ &- \int d^4 p \left( \frac{1}{p^2 + \sum^2(p^2)} - \frac{2\sum(p^2)}{(p^2 + \sum^2(p^2))^2} \right) W(p^2, q^2) q^2 \frac{q^2 - M^2(q^2)}{(q^2 + M^2(q^2))^2} \frac{\delta \sum(p^2)}{\delta M(S^2)} \end{aligned} \quad (5.4)$$

Making use of equations (2.2) to (2.6) gives

$$\begin{aligned} \frac{4\pi^2}{n_f} \frac{\delta^2 V}{\delta M(q^2) \delta M(S^2)} &= \pi^2 \frac{1 - M^2(q^2)/q^2}{(1 + M^2(q^2)/q^2)^2} W(q^2, S^2) \frac{1 - M^2(S^2)/S^2}{(1 + M^2(S^2)/S^2)^2} \\ &- \pi^4 \int dp^2 \frac{1 - M^2(q^2)/q^2}{(1 + M^2(q^2)/q^2)^2} W(q^2, p^2) \frac{1 - \sum^2(p^2)/p^2}{(1 + \sum^2(p^2)/p^2)^2} \times \\ &\quad W(p^2, s^2) \frac{1 - M^2(s^2)/s^2}{(1 + M^2(s^2)/s^2)^2} \end{aligned}$$

and setting  $\sum = M$  gives (in operator language)

$$\frac{4}{n_f} \frac{\delta^2 V}{\delta M(q^2) \delta M(s^2)} = D(q^2) [W(q^2, s^2) - B(q^2, s^2)] D(s^2) \quad (5.5)$$

where

$$D(q^2) = \frac{1 - M^2(q^2)/q^2}{(1 + M^2(q^2)/q^2)^2} \quad (5.6)$$

$$B(q^2, s^2) = \pi^2 \int dp^2 W(q^2, p^2) D(p^2) W(p^2, s^2) . \quad (5.7)$$

Letting  $q^2 = x$ ,  $s^2 = y$ ,  $p^2 = z$ , gives

$$\frac{4}{n_f} \frac{\delta^2 V}{\delta M(x) \delta M(y)} = D(x) [W(x, y) - B(x, y)] D(y) \quad (5.8)$$



Then using the identity

$$W(x,y)D(y) = \pi^2 \int dz \delta(z-y)W(x,z)D(z) \quad (5.9)$$

we rewrite (5.5) as

$$\begin{aligned} \Lambda &\equiv D(x) \left[ \int dz \delta(z-y)W(x,z)D(z) - \int dz W(x,z)D(z)W(z,y)D(y) \right] \\ &= \int dz D(x)W(x,z)D(z) \left[ \delta(z-y) - W(z,y)D(y) \right] \end{aligned} \quad (5.10)$$

$$= \frac{4}{\pi^2 n_f} \frac{\delta^2 V}{\delta M(x) \delta M(y)} \quad (5.11)$$

An expansion of the effective potential about a stationary point gives  
(5.2)

$$V[M+\delta M] \approx V[M] + \frac{1}{2} \int_0^\infty dx dy \delta M(y) \frac{\delta^2 V}{\delta M(y) \delta M(x)} \delta M(x) \quad (5.2)$$

Using matrix notation and (5.11) in (5.2) gives

$$\begin{aligned} V[M+\delta M] &= V[M] + \frac{\pi^2 n_f}{4} \delta M^T (DWD - DWDWD) \delta M \\ V[M+\delta M] &= V[M] + \frac{\pi^2 n_f}{4} \delta M^T [DWD(I-WD)] \delta M \end{aligned} \quad (5.12)$$

where the superscript T on  $\delta M$  stands for transpose of matrix. The reduced eigenvalue problem

$$(I-WD)\psi_i = \lambda_i \psi_i \quad (5.13)$$

has been shown by Haymaker and Matsuki [14] to give correct stability information.

Equation (5.13) expressed in terms of  $t$  variables using (2.15) and

$$\psi(x) = \sqrt{x} \phi(x) \quad (5.14)$$

gives

$$\begin{aligned} \phi(t) - \epsilon(t)e^{-3t/2} \int_{-\infty}^t ds e^{3s/2} D(s)\phi(s) \\ - e^{-t/2} \int_t^{\infty} ds \epsilon(s) e^{s/2} D(s)\phi(s) = \lambda\phi(t) \end{aligned} \quad (5.15)$$

where  $\lambda$  here stands for eigenvalue, and

$$D(t) = \frac{1-C^2(t)}{(1+C^2(t))^2} \quad (5.16)$$

$C(x)$  has its usual meaning ( $C(x) = \frac{1}{\sqrt{x}} M(x)$ ). Equation (5.15) is converted into a differential equation and then solved for the possible eigenvalues.

The orthogonality condition for eigenfunctions is calculated as follows: using (5.13)

$$\int dy [\delta(x-y) - W(x,y)D(y)] \psi_m(y) = \lambda_m \psi_m(x) \quad (5.17)$$

$$\int dx [\delta(x-y) - W(y,x)D(x)] \psi_n(x) = \lambda_n \psi_n(y) \quad (5.18)$$

Multiply (5.17) by  $\psi_n(x)D(x)W(x,y)D(y)$  and integrate over  $x$ ; multiply (5.18) by  $\psi_m(y)D(y)W(y,x)D(x)$  and integrate over  $y$  to have

$$\begin{aligned} \int dx dy \psi_n(x)D(x)W(x,y)D(y) [\delta(x-y) - W(x,y)D(y)] \psi_m(y) \\ = \lambda_m \int dx dy \psi_n(x)D(x)W(x,y)D(y) \psi_m(y) \end{aligned} \quad (5.19a)$$

$$\begin{aligned} \int dx dy \psi_m(y)D(y)W(y,x)D(x) [\delta(x-y) - W(y,x)D(x)] \psi_n(x) \\ = \lambda_n \int dx dy \psi_m(y)D(y)W(y,x)D(x) \psi_n(x) \end{aligned} \quad (5.19b)$$

Subtract (5.19) from (5.18) to get

$$(\lambda_m - \lambda_n) \int dx dy \psi_n(x) D(x) W(x,y) D(y) \psi_m(y) = 0$$

$$\int dx dy \psi_n(x) D(x) W(x,y) D(y) \psi_m(y) = \delta_{mn} \quad (5.20)$$

(5.20) is the orthogonality integral for eigenfunctions.

Using (5.15) and (5.20) Haymaker and Matsuki found eigenvalues which are classified into two parts depending on the regions only in which the eigenvalues oscillate - infrared or ultraviolet region. The point which separates them is given by a vanishing point of the function  $D(x)$ . The eigenvalues are tabulated as follows.

TABLE (5.1)

Number of Nodes	$\lambda_n$ (Infrared Region)
0	1.20
1	1.05
2	1.05
.	.
.	.
.	.
.	.
Number of Nodes	$\lambda_n$ (Ultraviolet Region)
.	.
.	.
.	.
.	.
2	0.84
1	0.76
0	0.56

(Cf. Haymaker and Matsuki [14].)

Notice that eigenvalues with peaks in the infrared region are greater than unity, and those with peaks in the ultraviolet region are less than unity. The spectrum of eigenvalues above is for the parameter  $t_c = 0.3$ .

With the table above, we now go ahead and calculate the expectation value of the 2nd functional derivative of  $V_{CDDG}$  w.r.t  $M$ .

Expanding  $\delta M(x)$  as

$$\delta M(x) = \sum_n a_n \psi_n(x) \quad . \quad (5.21)$$

We recall that the second term on the r.h.s. of equation (5.2) is

$$\frac{1}{2} \int dx dy \delta M(y) \frac{\delta^2 V}{\delta M(y) \delta M(x)} \delta M(x) \quad (5.22)$$

Eq. (5.22) is the expectation value of the curvature matrix which we have said may be negative, positive or zero depending on the eigenvalues of the 2nd functional derivative of  $V_{CDDG}$  already given in Table (5.1). The expectation value  $\bar{\Lambda}$  is

$$\bar{\Lambda} = \int dx dy \delta M(y) \Lambda \delta M(x) \quad (5.23)$$

$$\bar{\Lambda} = \frac{4}{\pi^2 n_f} \int dx dy \delta M(y) \frac{\delta^2 V}{\delta M(y) \delta M(x)} \delta M(x) \quad (5.24)$$

where we have used (5.11). Again using (5.8) and (5.21) in (5.24) gives

$$\bar{\Lambda} = \int dx dy dz \sum_{m,n} a_m \psi_m(x) [D(x)W(x,z)(\delta(z-y)-W(z,y)D(y))] a_n \psi_n(y)$$

Use (5.17) to have

$$\bar{\Lambda} = \int dx dz \sum_{m,n} a_m \psi_m(x) D(x)W(x,z)D(z) \lambda_n \psi_n(z) a_n$$

and finally, using (5.20) the orthogonality condition, we find

$$\boxed{\bar{\Lambda} = \sum_n a_n^2 \lambda_n} \quad (5.25)$$

The analogous expression for  $\bar{\Lambda}$  using  $V_{CJT}$  is [14]

$$\bar{\Lambda}_{CJT} = \int dx dy \sum_{m,n} a_m \psi_m(x) D(x) [\delta(x-y) - W(x,y) D(y)] a_n \psi_n(y) \quad (5.26)$$

using (5.17) to have

$$\bar{\Lambda} = \int dx \sum_{m,n} a_m \psi_m(x) D(x) \psi_n(x) a_n \lambda_n \quad (5.27)$$

Before using the orthogonality condition (5.20), notice from (5.26) and (5.27) that

$$\begin{aligned} & \int dx dy a_m \psi_m(x) D(x) [\delta(x-y) - W(x,y) D(y)] a_n \psi_n(y) \\ &= \lambda_n \int dx a_m \psi_m(x) D(x) a_n \psi_n(x) \end{aligned}$$

giving

$$\begin{aligned} \int dx \psi_m(x) D(x) [1 - \lambda_n] \psi_n(x) &= \int dx dy \psi_m(x) D(x) W(x,y) D(y) \psi_n(y) \\ \int dx \psi_m(x) D(x) \psi_n(x) &= \frac{1}{1 - \lambda_n} \int dx dy \psi_m(x) D(x) W(x,y) D(y) \psi_n(y) \\ \int dx \psi_m(x) D(x) \psi_n(x) &= \frac{1}{1 - \lambda_n} \delta_{mn} \end{aligned} \quad (5.28)$$

Then, substituting (5.28) into (5.27), we find

$$\bar{\Lambda}_{CJT} = \sum_n a_m^2 \frac{\lambda_m}{1-\lambda_m} \quad (5.29)$$

Expressed in this way,  $\bar{\Lambda}$  and  $\bar{\Lambda}_{CJT}$  are obviously different from each other in the following ways.

1. In the infrared region (Table 5.1) all eigenvalues are larger than unity.  $\bar{\Lambda}$  is positive while  $\bar{\Lambda}_{CJT}$  is negative, implying that with  $V_{CJT}$  the symmetry breaking solutions are unstable saddle points.  $\bar{\Lambda}$  is positive and hence the symmetry breaking solutions are stable.
2. In the ultraviolet region all eigenvalues are less than unity (Table 5.1). Here both  $\bar{\Lambda}$  and  $\bar{\Lambda}_{CJT}$  are positive and hence the solutions are stable.

We emphasize here that it is in the infrared region that one looks for chiral symmetry breaking and therefore the saddle point instability associated with  $V_{CJT}$  makes it not quite satisfactory as an effective potential for dynamical chiral symmetry breaking.

A new effective potential which includes auxiliary fields has been formulated by F. Cooper, T. Matsuki and R. Haymaker [19], which is bounded below and gives stable chiral symmetry breaking solutions.

## CHAPTER 6

### CONCLUSIONS

In this work, we have exhaustively studied chiral symmetry breaking in QCD like theories.

The study was motivated in the introduction by observing that any viable theory of the strong interactions must undergo at least two phase transitions so that the physical world resulting from the QCD Lagrangian is non-perturbative. The phases are the confinement and PCAC phases. The PCAC phase transition is associated with  $\chi$ SB and most of the low energy phenomenology of the hadronic spectrum has been understood to a degree through PCAC.

The effective potential for composite operators due to Cornwall, Jackiw and Tomboulis was used. However, since the original form of the effective potential due to CJT has been shown to give unstable  $\chi$ SB solutions, we have used the modified form of  $V_{CJT}$  due to CDDG. We have shown [25] that  $V_{CDDG}$  gives chiral symmetry breaking solutions which are stable, the proof of which forms the contents of Chapter 5.

In Chapter 4, we used  $V_{CDDG}$  self-consistently in the Raleigh-Ritz scheme to study spontaneous  $\chi$ SB. Here a solution which tends to the regular solution of the SD equation in the asymptotic limit was used as a variational ansatz in  $V_{CDDG}$ . The vacuum expectation value of the fermion bilinear was estimated. This treatment also highlighted the success of using a Raleigh-Ritz scheme in studying spontaneous  $\chi$ SB with  $V_{CDDG}$ , contrary to the claim by CDDG that this was not possible.

We also studied, in Chapter 4, spontaneous  $\chi$ SB for field theories with a fixed coupling constant using  $V_{\text{CDDG}}$ . While it has been shown by Castorina and Pi that with  $V_{\text{CJT}}$  chiral symmetry is broken spontaneously for values of the constant coupling constant  $\epsilon(t_c) > 2/3$ , we found that for all values of  $\epsilon(t_c)$  the minimum of  $V_{\text{CDDG}}$  occurs at  $m = 0$ , indicating chiral symmetry is not broken. This feature is consistent with the claim that only asymptotically free field theories allow  $\chi$ SB and, emphasizes the importance of logarithmic behavior (which is the hallmark of asymptotic freedom) in dynamical  $\chi$ SB.

In Chapter 3, we studied explicit  $\chi$ SB in a very general way using the irregular solution of the SD equation; this agrees with recent work by Haymaker and Matsuki. The concept of constituent quark mass and current quark mass was exploited to estimate the constituent quark masses of the light quarks. The value of the critical coupling  $\epsilon(t_c)$  for phase transition to take place, ie., from the chiral-symmetric phase to the spontaneously-broken phase, agrees very well in both schemes - explicit  $\chi$ SB and the variational approach.

And finally, an Appendix is included for calculating the discontinuity condition encountered in solving the SD nonlinear differential equation.



A P P E N D I X A

Given the coupling constant in the form we have used it

$$\epsilon(t) = \left\{ \begin{array}{ll} \frac{\epsilon_0}{t+t_c} & t \geq 0 \\ \frac{\epsilon_0}{t_c} & t \leq 0 \end{array} \right\} \quad (\text{A.1})$$

one finds that  $\epsilon(t)$  is continuous at  $t = 0$  but the derivative  $\dot{\epsilon}(t)$  is not, which makes the second derivative  $\ddot{\epsilon}(t)$  a delta function. As a consequence  $C(t)$  has the same property.

We rewrite (A.1) as

$$\epsilon(t) = \frac{\epsilon_0}{t_c} \theta(-t) + \frac{\epsilon_0}{t+t_c} \theta(t) \quad (\text{A.2})$$

$$\dot{\epsilon}(t) = -\frac{\epsilon_0}{(t+t_c)^2} \theta(t) \quad (\text{A.3})$$

$$\ddot{\epsilon}(t) = \frac{2\epsilon_0}{(t+t_c)^3} \theta(t) - \frac{\epsilon_0}{(t+t_c)^2} \delta(t) \quad (\text{A.4})$$

where we have used the fact that the derivative of a step function is a delta function.  $\theta(t)$  is the step function whose integral representation is

$$\theta(t) = \left\{ \begin{array}{ll} 1 & t > 0 \\ 0 & t < 0 \end{array} \right\} = \lim_{\delta \rightarrow 0} \left( \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t)}}{\omega + i\delta} \right) . \quad (\text{A.5})$$

To calculate the discontinuity of  $C(t)$ , we integrate the differential equation from  $(0-\delta)$  to  $(0+\delta)$  where  $\delta$  is a small positive number and

then let  $\delta$  go to zero.

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt [\ddot{C}(t) + 2\dot{C}(t) + \frac{3}{4}C(t)] - \lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \left( \frac{\ddot{\epsilon}(t) - \dot{\epsilon}(t)}{\dot{\epsilon}(t) - \epsilon(t)} \right) \\ \times \left( \frac{1}{2}C(t) + \dot{C}(t) \right) = \lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \left( \dot{\epsilon}(t) - \epsilon(t) \right) \frac{C(t)}{1+C^2(t)} \end{aligned} \quad (A.5)$$

We do term by term integration of (A.5).

$$\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \ddot{C}(t) = \lim_{\delta \rightarrow 0} [\dot{C}(0+\delta) - \dot{C}(0-\delta)] = [\dot{C}(0+) - \dot{C}(0-)] \quad (A.6)$$

$$\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \dot{C}(t) = \lim_{\delta \rightarrow 0} [C(0+\delta) - C(0-\delta)] = 0$$

because  $C(0+) = C(0-)$ , ie.  $C(t)$  is continuous.

$$\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt C(t) = 0$$

because  $C(t)$  is analytic. The third term of (A.5) becomes

$$-\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \left( -\dot{\epsilon}(t) + \delta(t) \right) \frac{C(t)}{1+C^2(t)} .$$

Split the integration region into two to have

$$\begin{aligned} = -\lim_{\delta \rightarrow 0} \int_{0-\delta}^0 dt \left( \frac{\epsilon_0}{t_c} + \frac{\epsilon_0}{t_c^2} \theta(t) \right) \frac{C(t)}{1+C^2(t)} + 0(\delta) \\ -\lim_{\delta \rightarrow 0} \int_0^{0+\delta} dt \left( \frac{\epsilon_0}{t+t_c} + \frac{\epsilon_0}{(t+t_c)^2} \theta(t) \right) \frac{C(t)}{1+C^2(t)} + 0(\delta) \end{aligned}$$

where we have used (A.2) and (A.3). Then use the definition of the unit step function  $\theta(t)$  to give

$$= \lim_{\delta \rightarrow 0} \int_{0-\delta}^0 dt \left[ \frac{\varepsilon_0}{t_c} \frac{C(t)}{1+C^2(t)} + \int_0^{0+\delta} dt \left( \frac{\varepsilon_0}{t+t_c} + \frac{\varepsilon_0}{(t+t_c)^2} \right) \frac{C(t)}{1+C^2(t)} \right] + 0$$

The second term of (A.5) becomes

$$\begin{aligned} & -\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} \left( \frac{\ddot{\varepsilon}(t) - \dot{\varepsilon}(t)}{\varepsilon(t) - \varepsilon(t)} \right) \left( \frac{1}{2} C(t) + \dot{C}(t) \right) \\ & = -\lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \frac{\ddot{\varepsilon}(t)}{\dot{\varepsilon}(t) - \varepsilon(t)} \left( \frac{1}{2} C(t) + \dot{C}(t) \right) + o(\delta) \\ & + \lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \frac{\dot{\varepsilon}(t)}{\dot{\varepsilon}(t) - \varepsilon(t)} \left( \frac{1}{2} C(t) + \dot{C}(t) \right) + o(\delta) . \end{aligned}$$

Using (A.2), (A.3) and (A.4) we have

$$\begin{aligned} & = + \lim_{\delta \rightarrow 0} \int_{0-\delta}^{0+\delta} dt \frac{\frac{\varepsilon_0}{(t+t_c)^2} \delta(t)}{\varepsilon(t) - \varepsilon(t)} \left( \frac{1}{2} C(t) + \dot{C}(t) \right) + o(\delta) \\ & - \lim_{\delta \rightarrow 0} \int_0^{0+\delta} dt \frac{\frac{2\varepsilon_0}{(t+t_c)^3}}{\frac{-\varepsilon_0}{(t+t_c)^2} - \frac{\varepsilon_0}{t+t_c}} \left( \frac{1}{2} C(t) + \dot{C}(t) \right) + o(\delta) \\ & + \lim_{\delta \rightarrow 0} \int_0^{0+\delta} dt \frac{\frac{-\varepsilon_0}{(t+t_c)^3}}{\frac{-\varepsilon_0}{(t+t_c)^2} - \frac{\varepsilon_0}{t+t_c}} \left( \frac{1}{2} C(t) + \dot{C}(t) \right) + o(\delta) \end{aligned}$$

The last two terms vanish because of the analyticity of  $C(t)$  and  $\dot{C}(t)$  in that integration range. The first term integrates out easily because of the delta function giving

$$\frac{\varepsilon_0}{t_c^2} \frac{1}{\varepsilon(0) - \varepsilon(0)} \left( \frac{1}{2} C(0) + \dot{C}(0) \right). \quad (\text{A.7})$$

We are left with only (A.6) and (A.7) from integration of (A.5) viz.

$$\dot{C}(0+) - \dot{C}(0-) = - \frac{\varepsilon_0}{t_c^2} \frac{1}{\dot{\varepsilon}(0) - \varepsilon(0)} \left[ \frac{1}{2} C(0) + \dot{C}(0) \right]. \quad (\text{A.8})$$

But  $\dot{C}(0)$  is not continuous because  $\varepsilon(t)$  is discontinuous and we must define these as

$$\ddot{\varepsilon}(0) = \frac{1}{2} [\dot{\varepsilon}(0+) + \dot{\varepsilon}(0-)] = - \frac{1}{2} \frac{\varepsilon_0}{(t+t_c)^2} \quad (\text{A.9})$$

$$\dot{C}(0) = \frac{1}{2} [\dot{C}(0+) + \dot{C}(0-)]. \quad (\text{A.10})$$

Substitute (A.10) into (A.8) to have

$$\dot{C}(0+) - \dot{C}(0-) = - \frac{\varepsilon_0}{2t_c^2} \frac{1}{\varepsilon(0) - \varepsilon(0)} [C(0) + \dot{C}(0+) + \dot{C}(0-)]$$

$$\dot{C}(0+) [2t_c^2 \dot{\varepsilon}(0) - 2t_c^2 \varepsilon(0) + \varepsilon_0] - \dot{C}(0-) [2t_c^2 \dot{\varepsilon}(0) - 2t_c^2 \varepsilon(0) - \varepsilon_0] + \varepsilon_0 C(0) = 0 \quad (\text{A.11})$$

Finally substitute (A.9) into (A.11) (ie.  $\varepsilon(0)$ ) to have

$$\dot{c}(0+) - c(0-) \left[ \frac{1}{t_c} + 1 \right] - \frac{1}{2t_c} c(0) = 0$$

$$\dot{c}(0+) = \left[ 1 + \frac{1}{t_c} \right] \dot{c}(0-) + \frac{1}{2t_c} c(0) \quad (\text{A.12})$$

This is the discontinuity condition.

## REFERENCES

1. W. Marciano and H. Pagels, Phys. Rep. 36, 137 (1978).
2. D.J. Gross and F. Wilczek, Phys. Rev. 8, 3633 (1973).
3. J.S. Bell and R. Jackiw, Nuovo Cimento A60, 47 (1969).
4. S.L. Adler, Phys. Rev. 177, 2426 (1969).
5. Y. Nambu and G. Jona-Lasino, Phys. Rev. 122, 345 (1961).
6. G. Jona-Lasino, Nuovo Cimento 34, 2426 (1969).
7. S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).
8. R. Jackiw, Phys. Rev. D9, 1686 (1974).
9. J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10, 2428 (1974).
10. R. Casalbuoni, S. DeCurtis, D. Dominici and R. Gatto, Phys. Lett. 14B, 357 (1985).
11. R. Casalbuoni, S. DeCurtis, D. Dominici and R. Gatto, Phys. Lett. 150B, 295 (1985).
12. K. Higashijima, Phys. Lett. 124B, 257 (1983).
13. K. Higashijima, Phys. Rev. D29, 1228 (1984).
14. R.W. Haymaker and T. Matsuki, Phys. Rev. D33, 1137 (1986).
15. J. Kogut, M. Stone, H.W. Wyld, J. Shigemitsu, S.H. Shenker and D.K. Sinclair, Phys. Rev. Lett. 48, 1140 (1982).
16. P. Castorina and S.Y. Pi, Phys. Rev. D31, 411 (1985).
17. D.J. Gross, in Methods in Field Theory, R. Balian and J. Zinn-Justine, eds. (North Holland, Amsterdam, 1976).
18. R.W. Haymaker and J. Perez-Mercader, Phys. Rev. D27, 1353 (1983).

19. T. Matsuki, F. Cooper and R. Haymaker, Preprint 1986.
20. M. Peskin, in Recent Advances in Field Theory and Statistical Mechanics, Proceedings of the Les Houches Summer School, Session XXXIX, J.B. Zuber and R. Stora, eds. (North-Holland, Amsterdam, 1984).
21. H.D. Politzer, Nucl. Phys. B117, 397 (1976).
22. R. Jackiw, Phys. Rev. D9, 1686 (1974).
23. R. Fukuda and T. Kugo, Nucl. Phys. B117, 250 (1976).
24. P. Ramond, Field Theory (The Benjamin/Cummings Company, Inc., Reading, Massachusetts, 1981), Chapters 3 and 4.
25. J. Otu and K.S. Viswanathan, Phys. Rev. D 34 (1986).
26. C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954).
27. A.N. Vasil'ev and A.K. Kazanskii, Theor. Mat. Fiz. 12, 352 (1972).
28. S.L. Adler and A.C. Davis, Nucl. Phys. B244, 469 (1984).
29. J.R. Finger and J.E. Mandula, Nucl. Phys. B199, 168 (1982).
30. C. Itzykson and J. Zuber, Quantum Field Theory (McGraw-Hill Inc., 1980), Chapter 10.