CYCLES AND CONNECTIVITY IN GRAPHS

by

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ABSTRACT

In this thesis we discuss theorems about connectivity and cycles in graph theory.

The first three chapters are concerned with connectivity. Menger's Theorem and Perfect's Theorem are given as well as several theorems about reductions which preserve 3-connectivity.

The last two chapters use the connectivity results to prove theorems about cycles. Chapter 4 gives existence theorems for cycles of given parity through specified edges in 3-connected graphs. Chapter 5 examines cycles through specified vertices in planar, 3-connected graphs.

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PREFACE

This thesis follows the notation and terminology of J.A. Bondy and U.S.R. Murty [3]. Additional notation and terminology are also needed.

Let X and Y be sets of vertices. An (X,Y)-path is a path with origin in X, terminus in Y, and no internal vertices in $X \cup Y$. If u and v are vertices, a $(\{u\},\{v\})$-path will be called a (u,v)-path.

A set of paths is openly disjoint if the paths have a common origin and no other common vertices.

Let S, $\{u\}$, and $\{v\}$ be disjoint subsets of the vertex set of graph G. Then S separates u and v if every (u,v)-path in G contains a vertex in S.

Let G be a graph, V be a set of vertices, and E be a set of edges. Then $G-V$ is the induced subgraph of G with vertex set $V(G)-V$, $G+V$ is the graph with vertex set $V(G) \cup V$ and edge set $E(G)$, $G-E$ is the graph with vertex set $V(G)$ and edge set $E(G)-E$, and $G+E$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup E$.

If B and C are graphs then $B \Delta C$ is the graph with vertex set $V(B) \cup V(C)$ and edge set $E(B) \Delta E(C)$.

Let F be a subset of the edge set of graph G. A cycle C of G is even (odd) with respect to F if $C \cap F$ contains an even (odd) number of edges.

An edge e is a chord of a cycle C if both ends of e are in $V(C)$ and e is not in $E(C)$.

A branchvertex is a vertex of degree greater than two.

A colour class of a bipartite graph is a set of vertices with the same colour in a proper 2-vertex-colouring of the graph.

A graph G is critically n-connected if for every edge e, G-{e} is not n-connected.

Let $e=x_1x_2$ be an edge of graph G. Then G-e* is G-{e} unless the degree of some x_i in G-{e} is two, $i\in\{1,2\}$, in which case G-e* is obtained from G-{e} by replacing each such x_i and the two edges incident with it by a single edge.

If e is an edge in a graph G then the graph obtained by contracting e will be denoted by $G^\circ e$.

I. Chapter 1

The fundamental theorem on connectivity in graphs was discovered by K. Menger [9]. The proof given here is due to the author.

Theorem 1.1. If no set of fewer than n vertices separates nonadjacent vertices u and v in a graph G, then there are n internally disjoint (u,v)-paths.

Proof. The proof uses induction on n. The theorem is trivial for n=1. Suppose u and v are not separated by any set having less than n+1 vertices ($n\geq 1$). By the induction hypothesis there are n internally disjoint (u,v)-paths $P_1,\ldots,P_n$. Since the set of second vertices of $P_1,\ldots,P_n$ does not separate u and v, there is a (u,v)-path P whose initial edge is not on P_i, i=1,...,n. Let x be the first vertex after u which is both on P and on some P_i, $1\leq i\leq n$. Let P_{n+1} be the (u,x)-section of P. Suppose $P_1,\ldots,P_n,P_{n+1}$ have been chosen so that the distance in G-{u} between x and v is the minimum. If x=v we are done, so assume not.

In G-{x} there are n internally disjoint (u,v)-paths $Q_1,\ldots,Q_n$, again by the induction hypothesis. Choose $Q_1,\ldots,Q_n$ using the minimum number of edges in $B=E(G)-\bigcup_{i=1}^{n+1}E(P_i)$. Let H be the graph consisting of the vertices and arcs of

$Q_1, \dots, Q_n$ together with the vertex x. Choose some P_k, $1 \le k \le n+1$, whose initial edge is not in $E(H)$. Let y be the first vertex after u which is on P_k and in $V(H)$. If $y=v$ we are done, so assume not.

If $y=x$ then let R be the shortest (x,v)-path in $G-\{u\}$. Let z be the first vertex of R on some Q_j, $1 \le j \le n$. Then the distance in $G-\{u\}$ between z and v is less than the distance between x and v. This contradicts our choice of $P_1, \dots, P_n, P_{n+1}$.

If y is on some Q_i, $1 \le i \le n$, then the (u,y)-section of Q has an edge in B. Otherwise, two paths in $\{P_1, \dots, P_n, P_{n+1}\}$ intersect at a vertex other than u, v, or x. Now if we replace the (u,y)-section of Q_i by the (u,y)-section of P_k we get n internally disjoint (u,v)-paths in $G-\{x\}$ using less edges in B than $Q_1, \dots, Q_n$. This is a contradiction.

Menger's theorem has the following two standard corollaries.

Corollary 1.1. If $\{x\}$ and $Y=\{y_1, \dots, y_n\}$ are disjoint sets of vertices in an n-connected graph G, then there are n openly disjoint $(\{x\},Y)$-paths in G.

Proof. Let $H=G+\{z\}+\{y_i z \mid i=1,\dots,n\}$, where z is not a vertex of G. Since G is n-connected, no set of fewer than n vertices separates x and z. In addition, x and z are nonadjacent, so by Theorem 1.1 H has n internally disjoint (x,z)-paths $P_1, \dots, P_n$.

Each vertex in Y must necessarily be on exactly one such path, so $P_i - \{z\}$, $i=1,\ldots,n$, are the required paths in G.

<u>Corollary 1.2.</u> If X and Y are disjoint sets of vertices in an n-connected graph G such that both have at least n vertices, then there are n disjoint (X,Y)-paths.

<u>Proof.</u> Let $H=G+\{w,z\}+\{wx \mid x \in X\}+\{zy \mid y \in Y\}$, where w and z are not vertices of G. Now w and z are nonadjacent edges in an n-connected graph H, so there are n internally disjoint (w,z)-paths, $P_1,\ldots,P_n$, in H. We can assume P_i contains only one vertex in each of X and Y, $i=1,\ldots,n$. Then $P_i - \{w,z\}$, $i=1,\ldots,n$, are the required paths in G.

II. Chapter 2

H. Perfect [10] proved the following theorem.

<u>Theorem 2.1.</u> For a graph G, let $\{x\}$ and S be disjoint subsets of $V(G)$. Suppose $P_1,\ldots,P_n$ are openly disjoint $(\{x\},S)$-paths with termini $y_1,\ldots,y_n$, respectively, and $Q_1,\ldots,Q_{n+1}$ are openly disjoint $(\{x\},S)$-paths. Then there are $n+1$ openly disjoint $(\{x\},S)$-paths with termini $y_1,\ldots,y_n,v$, for some v in S.

The following proof was discovered independently by the author but it can also be found in L. Lovász [8-p.44].

<u>Proof.</u> Let $E(P)=\bigcup_{i=1}^{n} E(P\ \ )$ and $E(Q)=\bigcup_{i=1}^{n+1} E(Q\ )$. Choose n openly disjoint $(\{x\},S)$-paths, $R_1,\ldots,R_n$, with termini $y_1,\ldots,y_n$, respectively, using only edges in $E(P) \cup E(Q)$ and using a minimum number of edges in $E(P)-E(Q)$. Choose some Q_i, $1 \le i \le n+1$, having an initial edge different from the initial edges of R_j, $j=1,\ldots,n$.

If Q_i does not intersect some R_j, $j=1,\ldots,n$, at a vertex other than x, then we are done. If not, then let z be the first vertex after x which is on Q_i and on some R_j, $1 \le j \le n$. Then the (x,z)-section of R_j has an edge in $E(P)-E(Q)$. Otherwise, two paths in $\{Q_1,\ldots,Q_{n+1}\}$ intersect at a vertex other than x. Now by replacing the (x,z)-section of R_j by the (x,z)-section of

Q_i we get n openly disjoint $(\{x\},S)$-paths with termini $y_1,\ldots,y_n$ using only edges in $E(P)\cup E(Q)$ and using less edges in $E(P)-E(Q)$ than $R_1,\ldots,R_n$. This is a contradiction.

III. Chapter 3

D.W. Barnette and B. Grünbaum [1] and V.K. Titov [13] independently proved the following theorem.

Theorem 3.1. If G is a 3-connected graph of order at least five, then G contains an edge e such that $G-e^*$ is 3-connected.

C. Thomassen [12] proved the following result.

Theorem 3.2. If G is a 3-connected graph of order at least five, then G contains an edge e such that $G^o e$ is 3-connected.

In the chapter we present variations of these theorems.

Theorem 3.3 Let $e=x_1x_2$ be an edge in a 3-connected graph G. Suppose there exist y and z in $V(G)-\{x_1,x_2\}$ such that $G-\{e\}-\{y,z\}$ has components H_1 and H_2, where x_i is in $V(H_i)$, i=1,2. If H_1 and H_2 each have at least two vertices, then $G^o e$ is 3-connected.

Proof. If $G^o e$ is not 3-connected, then $\{x_1,x_2\}$ is contained in a 3-vertex cut of G. Thus, it suffices to show that $G-\{x_1,x_2,u\}$ is connected for any u in $V(G)-\{x_1,x_2\}$. There are essentially two cases.

Suppose u=y. We now show that every vertex v in $V(G)-\{x_1, x_2, u\}$ is in the same component as z. Without loss of generality, let v be in $V(H_1)-\{x_1\}$. By Corollary 1.1 there are three openly disjoint $(\{v\},\{x_1, y, z\})$-paths in G. Since any (x_2, v)-path includes a vertex in $\{x_1, y, z\}$, x_2 is not on the (v,z)-path.

Suppose u is in $V(H_1)-\{x_1\}$. Let w_1 be in $V(H_1)-\{x_1, u\}$ and w_2 be in $V(H_2)-\{x_2\}$. Since there are three openly disjoint $(\{w_2\},\{x_2, y, z\})$-paths in G, the vertices w_2, y, and z are in the same component of $G-\{x_1, x_2, u\}$. Since there are three openly disjoint $(\{w_1\},\{x_1, y, z\})$-paths in G, there is a (w_1, y)-path or a (w_1, z)-path in $G-\{x_1, x_2, u\}$. Because the choice of w_1 and w_2 was arbitrary, $G-\{x_1, x_2, u\}$ is connected.

The following theorem is found in P.J. Slater [11].

Theorem 3.4. Every vertex x of degree three in a 3-connected graph G of order at least five is incident with an edge e such that $G^{o}e$ is 3-connected.

Proof. Let x be incident with edges $e_i = xy_i$, i=1,2,3. Suppose $G-\{x, y_3, z\}$ is disconnected for some z in V(G). Let y_i be in H_i, i=1,2, where H_1 and H_2 are the components of $G-\{x, y_3, z\}$. If $V(H_1)$ has at least two vertices, then the components of $G-\{e_1\}-\{y_3, z\}$, H_1 and $H_2+\{x\}+\{e_2\}$, both have at least two vertices, so Theorem 3.3 implies $G^{o}e_1$ is 3-connected.

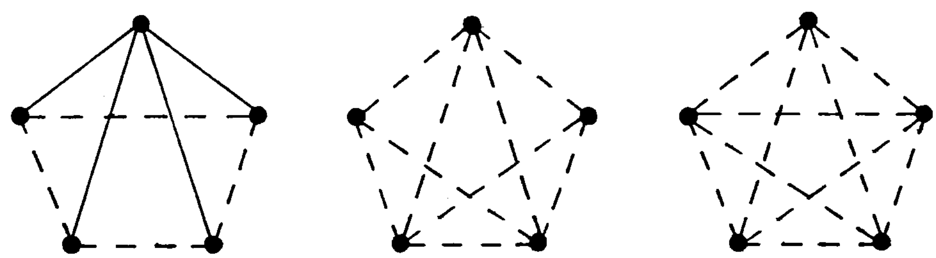

Figure 3.1. G^oe is 3-connected for each dashed edge e.

Similarly, if $V(H_2)$ has at least two vertices, then G^oe is 3-connected. If $V(H_1)$ and $V(H_2)$ both have one vertex, then G has order 5. It is easy to show that the result holds for the three 3-connected graphs of order five (figure 3.1.).

The following theorem is also in L. Lovász [8-p.46].

<u>Theorem 3.5.</u> If e is an edge with both ends of degree at least four in a critically 3-connected graph G of order at least five, then G^oe is 3-connected.

<u>Proof.</u> G is critically 3-connected, so there are vertices x and y in V(G) such that $G-\{e\}-\{x,y\}$ is disconnected. Since both ends

of e have degree at least four, neither component of $G-\{e\}-\{x,y\}$ has just one vertex. Thus, Theorem 3.3 implies $G^{o}e$ is 3-connected.

Theorem 3.6. For any edge $e=x_1x_2$ in a 3-connected graph G of order at least five, $G^{o}e$ or $G-e^*$ is 3-connected.

Proof. The result is easily checked when G has order five, so assume G has order at least 6.

Suppose $G-e^*$ is not 3-connected. Since $|V(G-e^*)|\geq 4$, there are vertices w_1 and w_2 in $V(G-e^*)$ which are in different components of $(G-e^*)-\{y,z\}$, where $\{y,z\}$ is a 2-vertex cut. Therefore, $G-\{e\}-\{y,z\}$ has two components, H_1 and H_2, where x_i and w_i are in $V(H_i)$, $i=1,2$.

Now H_i has at least two vertices, $i=1,2$. If $x_i \neq w_i$, we are done. If $x_i=w_i$, then x_i must have degree at least four in G to be a vertex in $G-e^*$. Therefore, x_i is adjacent to some other vertex in H_i. Hence, $G^{o}e$ is 3-connected by Theorem 3.3.

IV. Chapter 4

In this chapter we examine the question of when two edges in a 3-connected graph lie on a common even cycle and when they lie on a common odd cycle.

First we give some related theorems.

Theorem 4.1. (G.A. Dirac [4]) Any two edges and any k-2 vertices in a k-connected graph lie on a common cycle.

Theorem 4.2. (R. Häggkvist and C. Thomassen [5]) Any k-1 pairwise nonadjacent edges in a k-connected graph lie on a common cycle.

Theorem 4.3. (J.A. Bondy and L. Lovász [2]) In a k-connected graph any k-1 vertices lie on a common odd cycle if the graph is not bipartite, and any k vertices lie on a common even cycle.

To prove the main theorem we need a lemma.

Lemma 4.1. If X is a set of four vertices in a 3-connected graph G of order at least six, then there is an edge e with at most one end in X such that $G^{\circ}e$ is 3-connected.

Proof. We may assume G is critically 3-connected. The result holds for the three critically 3-connected graphs of order six (figure 4.1), so suppose G has order at least seven.

Suppose e=yz does not have an end in X. If y and z both have degree at least four, then Theorem 3.5 implies $G^\circ e$ is 3-connected. If y or z has degree three, then Theorem 3.4 implies there is an edge f incident with y or z such that $G^\circ f$ is 3-connected.

Suppose every edge of G has at least one end in X. If some vertex in V(G)-X has degree three, then we apply Theorem 3.4. If all vertices in V(G)-X have degree four, then every vertex in X is adjacent to at least three vertices in V(G)-X. If x in X has degree three then we apply Theorem 3.4, and if x in X has degree at least four then we apply Theorem 3.5.

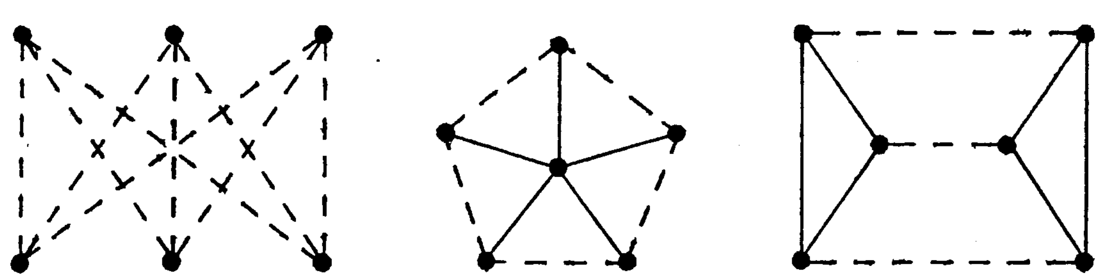

Figure 4.1. $G^\circ e$ is 3-connected for every dashed edge e.

Theorem 4.4. Let G be a simple 3-connected graph. Suppose $f=wx$ and $g=yz$ are nonadjacent edges and F is a subset of E(G). Then $G-\{f,g\}$ contains an odd cycle with respect to F if and only if there are even and odd cycles with respect to F containing both f and g.

Proof. The theorem is proven by induction on $|V(G)|$. The theorem is easily verified when G has order four or five.

Suppose G has order at least six. Then by Lemma 4.1 there is an edge $e=uv$ such that e has at most one end in $\{w,x,y,z\}$ and $G^\circ e$ is 3-connected. Suppose $G-\{f,g\}$ contains an odd cycle C with respect to F. There are three cases.

In the first case we assume that e is not in F and that there is no cycle of length three whose edges consist of an edge in $\{f,g\}$, an edge h on C, and the edge e.

If e is in E(C), then $C^\circ e$ is an odd cycle with respect to F in $(G^\circ e)-\{f,g\}$. If e is a chord of C, then $(G^\circ e)-\{f,g\}$ contains an even and an odd cycle with respect to F with one common vertex. If u or v is not in V(C), then C is an odd cycle with respect to F in $(G^\circ e)-\{f,g\}$. Thus, $(G^\circ e)-\{f,g\}$ contains an odd cycle C' with respect to F.

Suppose $|V(C')|\geq 3$. Then we remove an edge from each double edge in $G^\circ e$ so as not to destroy C'. Let G' be the resulting graph. Since e has at most one end in common with f and g, f and g are nonadjacent in G'. Now we apply the induction hypothesis to G' to obtain an odd and an even cycle with respect to F which

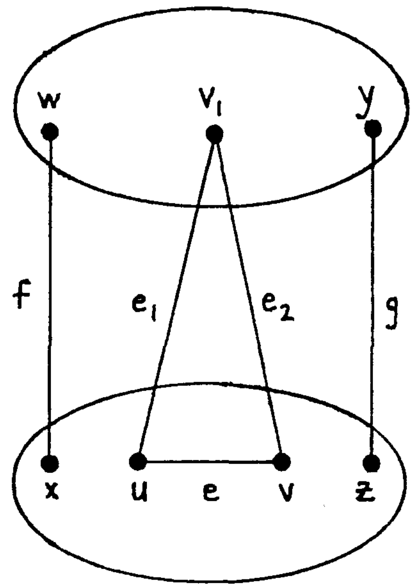

Figure 4.2.

both contain f and g. These cycles correspond to cycles in G with the same parities with respect to F as in G' because e is not in F.

Suppose $|V(C')|=2$ and $C'=v_1 e_1 v_2 e_2 v_1$. If v_1 and v_2 are on a cycle in $(G^\circ e)-\{f,g\}$ of length at least three, then we can remove an edge from each double edge to obtain a simple graph G' suitable for applying the induction hypothesis. If v_1 and v_2 are not on a cycle in $(G^\circ e)-\{f,g\}$ of length at least three, then $v_1 v_2$ disconnects $(G^\circ e)-\{f,g\}$. Hence, $\{f,g,v_1 v_2\}$ is an edge cut of $G^\circ e$. Thus, G has the form shown in figure 4.2, where we assume e is in F and e is not. Since $G-\{v_1\}$ is 2-connected, Theorem 4.1 implies it contains a cycle B with e and f in E(B).

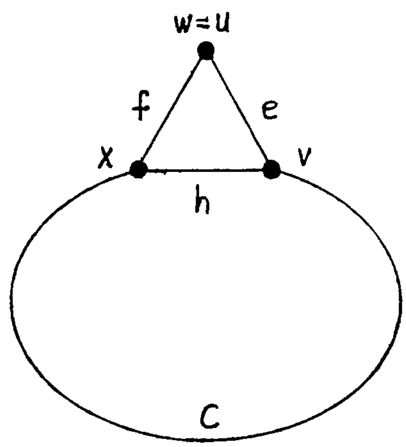

Figure 4.3.

The cycle B must necessarily also contain g. Now we are done since B and $(B-\{e\})+\{v_1\}+\{e_1,e_2\}$ have opposite parities with respect to F.

In the second case we assume that e is not in F and that there is a cycle of length three whose edges consist of an edge in {f,g}, an edge h in E(C), and the edge e (figure 4.3).

If there is an odd cycle with respect to F in G-{f,g,h,} then we have the first case. Therefore, we can assume that in G-{f,g} all odd cycles with respect to F include h.

G-{x} is 2-connected, so Corollary 1.2 implies it contains two disjoint ({y,z},{w,v})-paths P and Q. Let $D_1=(P\cup Q)+\{x\}+\{h,f,g\}$. Since G-{v} is 2-connected, Theorem 4.1

implies it has a cycle D_2 containing f and g.

$D_1 \Delta D_2$ is the union of edge-disjoint cycles in G-{f,g}, and one of these cycles contains h. Thus, $D_1 \Delta D_2$ consists of one cycle which is odd with respect to F and possibly other cycles which are all even with respect to F. Therefore, D_1 and D_2 have opposite parities with respect to F, so they are the required cycles.

In the third case we assume e is in F. Let E' be the set of edges incident with u. Then each cycle in G has the same parity with respect to F and $F \Delta E'$. Now we have one of the first two cases, since e is not in $F \Delta E'$.

Conversely, suppose C and D are even and odd cycles with respect to F which both contain f and g. Then $C \Delta D$ is the union of edge-disjoint cycles in G-{f,g}. Since $E(C \Delta D)$ and F have a odd number of edges in common, one of the cycles of $C \Delta D$ is odd with respect to F.

<u>Theorem 4.5.</u> Let G be a simple 3-connected graph. Suppose e=xy and f=xz are adjacent edges and F is a subset of E(G). Then G-{x} contains an odd cycle C with respect to F if and only if there are even and odd cycles in G with respect to F containing both e and f.

<u>Proof.</u> Suppose G-{x} contains an odd cycle C with respect to F. Since G-{x} is 2-connected, Corollary 1.2 implies it contains two disjoint ({y,z},V(C))-paths P and Q. If y or z is on C then

P or Q has zero length. Let B and D be the cycles in the subgraph $(P \cup Q \cup C)+\{e,f\}$ which contain e and f. Since C is odd with respect to F, B and D have opposite parity with respect to F.

The converse is proven as in Theorem 4.4.

<u>Corollary 4.1.</u> Let G be a simple 3-connected graph. Suppose e and f are nonadjacent edges, and g and h are adjacent edges with common end x. Then $G-\{e,f\}$ contains an odd cycle if and only if there are even and odd cycles containing both e and f, and $G-\{x\}$ contains an odd cycle if and only if there are even and odd cycles containing both g and h.

<u>Proof.</u> Let $F=E(G)$ and apply Theorems 4.4 and 4.5.

L. Lovász has conjectured that for any set L of k pairwise nonadjacent edges in a k-connected graph G, where $G-L$ is connected if k is odd, there is a cycle containing all the edges of L. He has verified the conjecture for $k=3$ and the author has verified the conjecture for $k=4$. Theorem 4.4 allows an easy proof when $k=3$.

<u>Corollary 4.2.</u> If $\{e,f,g\}$ is a set of pairwise nonadjacent edges in a 3-connected graph G, where $G-\{e,f,g\}$ is connected, then there is a cycle containing e, f, and g.

Proof. Let $F=\{g\}$. The subgraph $G-\{e,f\}$ contains a cycle C through g, since otherwise $G-\{e,f,g\}$ is disconnected. The cycle C is odd with respect to F, so by Theorem 4.4 G contains an odd cycle B with respect to F which contains both e and f. Since B is odd with respect to F it must necessarily contain g.

V. Chapter 5

G.A. Dirac [4] proved the following result.

Theorem 5.1. There is a cycle containing any n vertices in an n-connected graph.

This is the best possible, since $K_{n,n+1}$ is n-connected while the n+1 vertices in the larger colour class do not lie on a common cycle.

If we restrict ourselves to planar graphs, we can make improvements.

Theorem 5.2. (W.T. Tutte [14]) Any planar 4-connected graph is hamiltonian.

Kel'mans and Lomonosov [6] have announced the following result.

Theorem 5.3. Let G be a planar 3-connected graph. Then:

(i) Any five vertices in V(G) lie on a common cycle.

(ii) If v_1, v_2, v_3, v_4, v_5, and v_6 are in V(G) and do not lie on a common cycle, then G contains a subdivision of the Herschel graph in which v_1, v_2, v_3, v_4, v_5, and v_6 are the branchvertices corresponding to the larger colour class (figure 5.1).

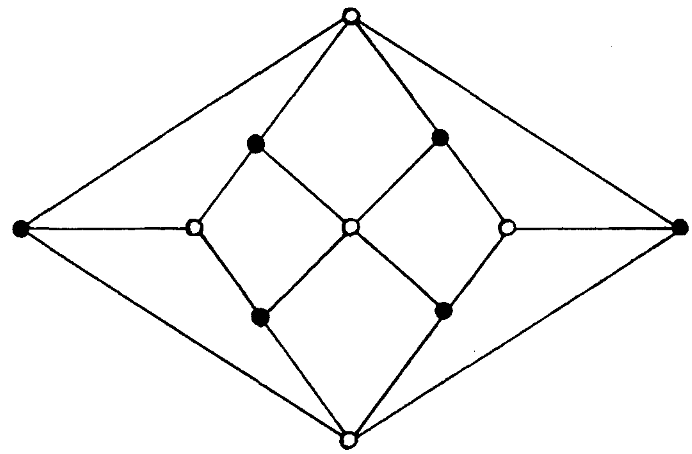

Figure 5.1. The Herschel graph.

We present a proof of Theorem 5.3 due to the author which uses the following theorem.

<u>Theorem 5.4.</u> (K. Kuratowski [7]) A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

<u>Proof of Theorem 5.3.(i).</u> Suppose G is a counterexample of minimum order, where $W=\{v_1, v_2, v_3, v_4, v_5\}$ is a set of vertices not on a common cycle. We may assume G is critically 3-connected.

We first prove that v_1, v_2, v_3, and v_4 are on a common cycle. By Theorem 5.1, v_1, v_2, and v_3 are on a common cycle B. If v_4 is not on B, then Corollary 1.1 implies there are three

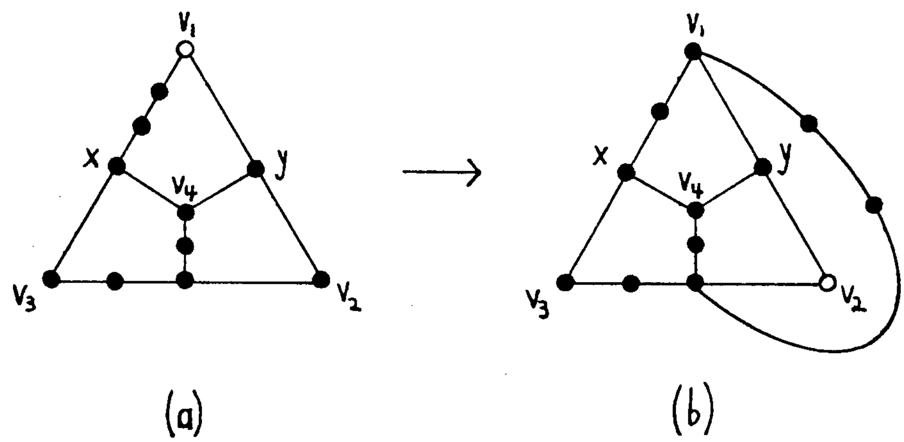

Figure 5.2

openly disjoint $(\{v_4\}, V(B))$-paths. Then we obtain the subgraph G_1 shown in figure 5.2.a or a cycle containing v_1, v_2, v_3, and v_4. Let G_2 be the maximal 2-connected subgraph of $G_1 - \{v_1\}$. Since G is 3-connected, Theorem 2.1 implies there are three openly disjoint $(\{v_1\}, V(G_2))$-paths where x and y are the termini of two of these paths. Then we obtain the subgraph G_3 shown in figure 5.2.b or a cycle containing v_1, v_2, v_3, and v_4. Let G_4 be the maximal 2-connected subgraph of $G_3 - \{v_2\}$. Now we apply Theorem 2.1 to v_2 and G_4. Considering all cases, we either get a subdivision of $K_{3,3}$ which contradicts the planarity of G, or we get a cycle containing v_1, v_2, v_3, and v_4.

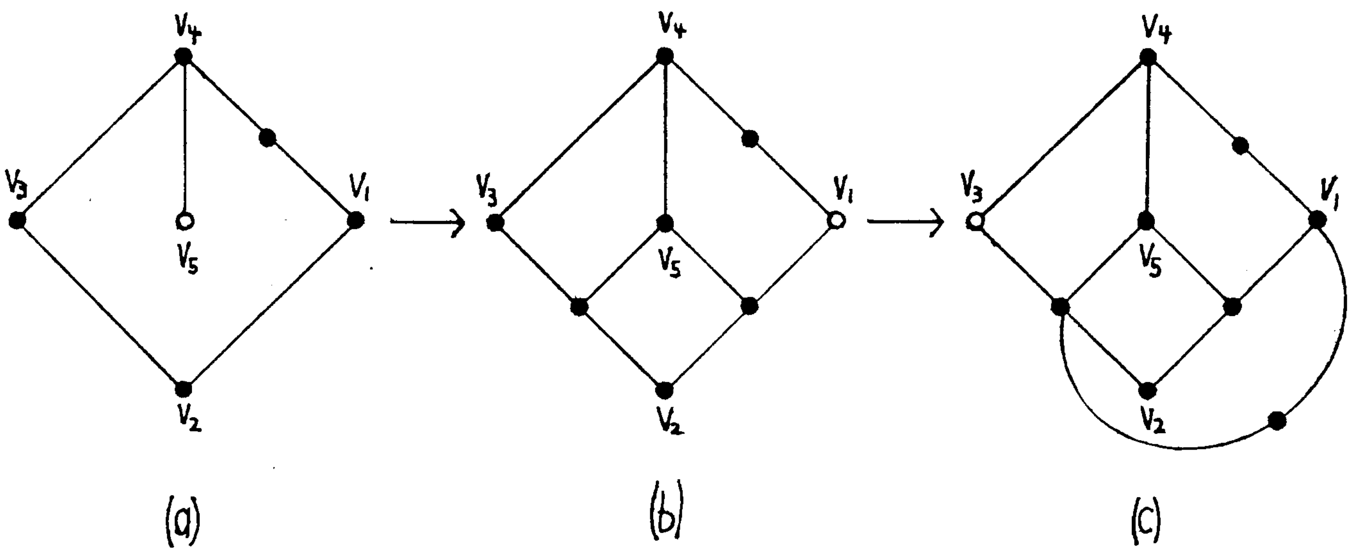

Figure 5.3.

Suppose two vertices in W are adjacent. We may assume $v_4 v_5$ is in E(G). Thus, we have the subgraph in figure 5.3.a. Now we apply Theorem 2.1 three times as shown in figure 5.3. In the application to the subgraph G', we apply Theorem 2.1 to the empty vertex v_i and the maximal 2-connected subgraph of $G'-\{v_i\}$. The subgraph to the right of any G' is the only case which does not immediately result in a contradiction, that is, a subdivision of $K_{3,3}$ or a cycle containing v_1, v_2, v_3, v_4, and v_5. The last application (figure 5.3.c) results in a contradiction in all cases. Thus, W is an independent set.

Suppose some e in E(G) does not have an end in W. By Theorem 3.6, G-e* or G°e is 3-connected. Then G°e or G-e*

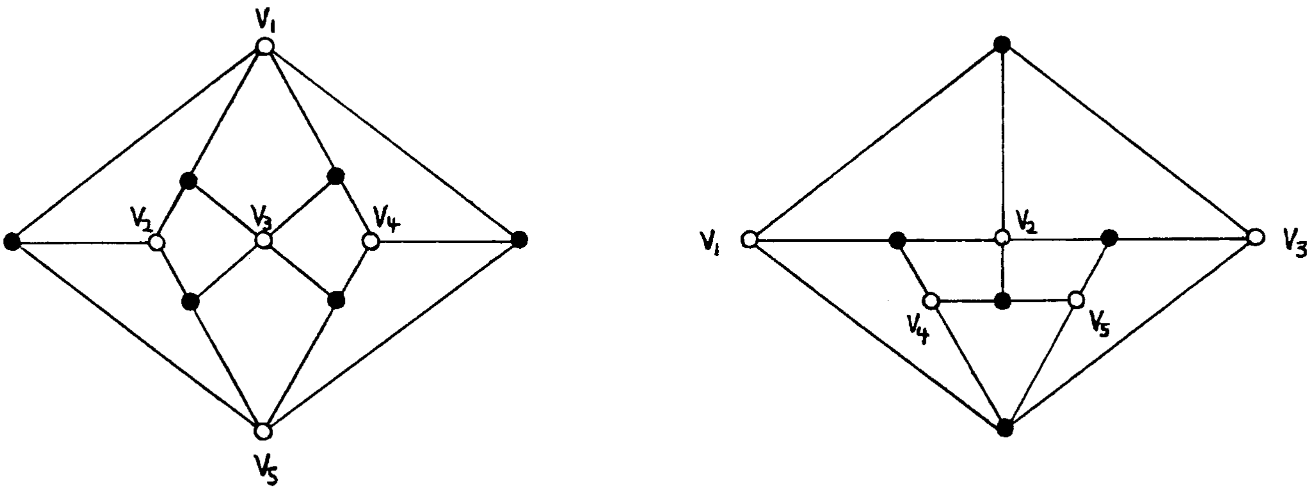

Figure 5.4.

contains a cycle through v_1, v_2, v_3, v_4, and v_5 because G is a counterexample of minimum order. But this implies there is a cycle through v_1, v_2, v_3, v_4, and v_5 in G.

Thus, G is a bipartite graph with colour class W. It is easy to show that G is then one of the graphs in figure 5.4. But in all these graphs there is a cycle containing v_1, v_2, v_3, v_4, and v_5, so we have a contradiction.

<u>Proof of Theorem 5.3.(ii).</u> Suppose G is a counterexample of minimum order, where $W=\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a set of vertices which are not on a common cycle, and are not the branchvertices corresponding to the larger colour class in a subdivision of the

Herschel graph. We may assume G is critically 3-connected.

Suppose $v_5 v_6$ is in E(G). By Theorem 5.3.i there is a cycle C containing v_1, v_2, v_3, v_4, and v_5. By our assumption on G, v_6 is not on C. On applying Theorem 2.1 to v_6 and C we get the three cases in figures 5.5.a, 5.6.a, and 5.7.a. We now apply Theorem 2.1 several times as shown in figures 5.5, 5.6, and 5.7. Each time Theorem 2.1 is applied to the empty vertex w of G' and the maximal 2-connected subgraph of G'-{w} we attempt to avoid a subdivision of $K_{3,3}$ and a cycle containing v_1, v_2, v_3, v_4, v_5, and v_6. In addition, for the case in figure 5.6 we attempt to avoid the subgraph in figure 5.5.a, and for the case in figure 5.7 we attempt to avoid the subgraphs in figures 5.5.a and 5.6.a. Since all cases eventually lead to a contradiction, W is an independent set.

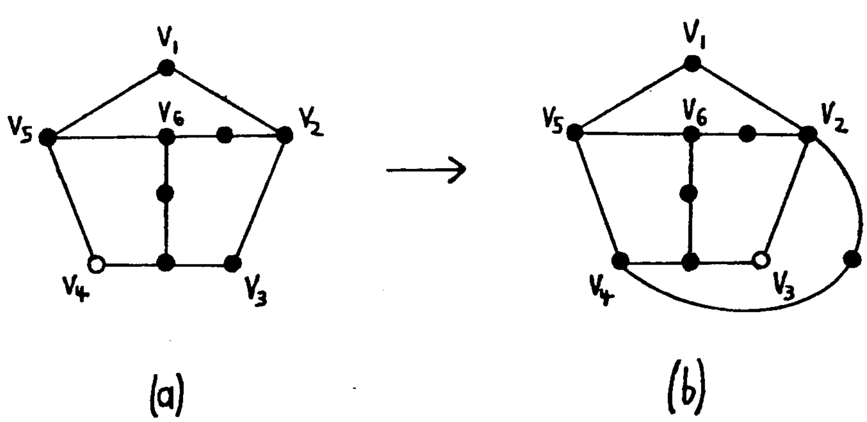

Figure 5.5.

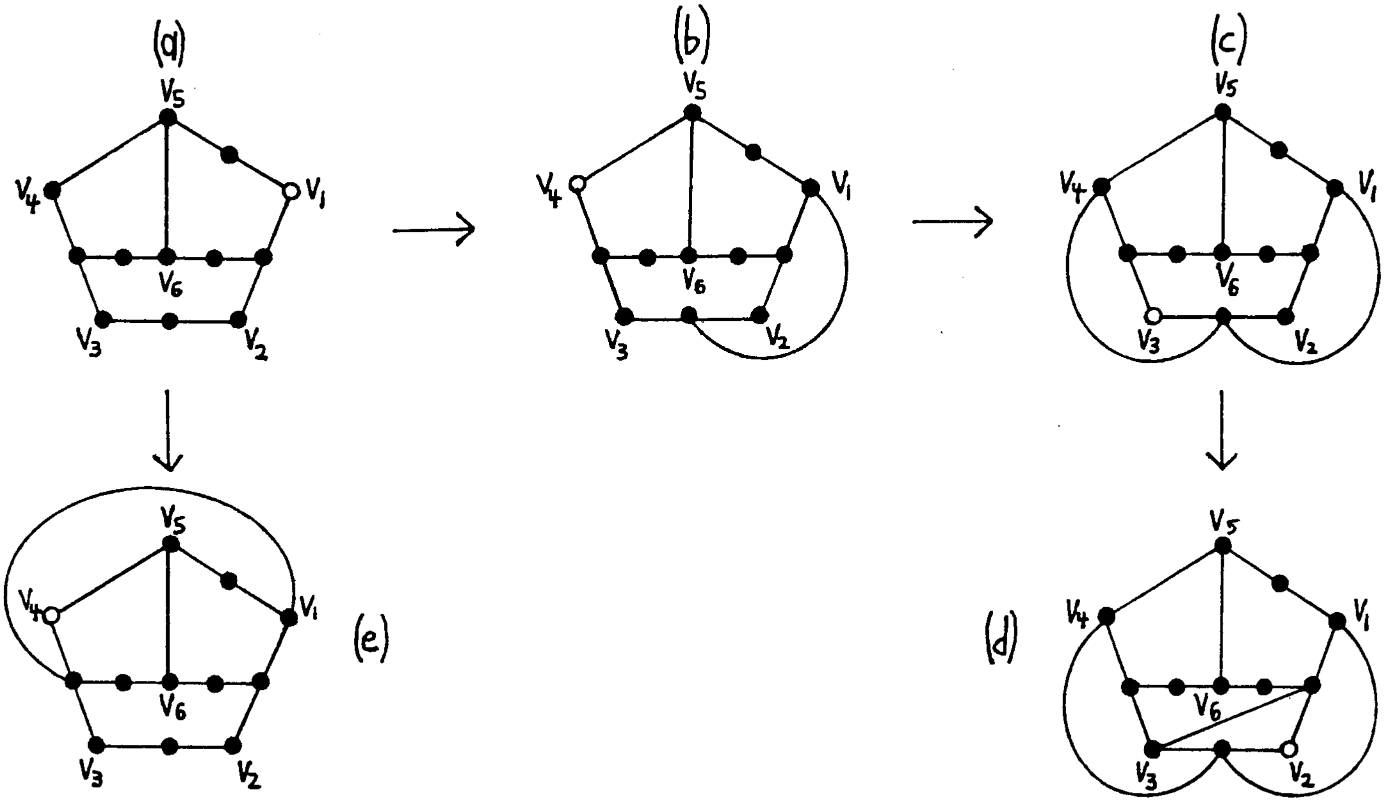

Figure 5.6.

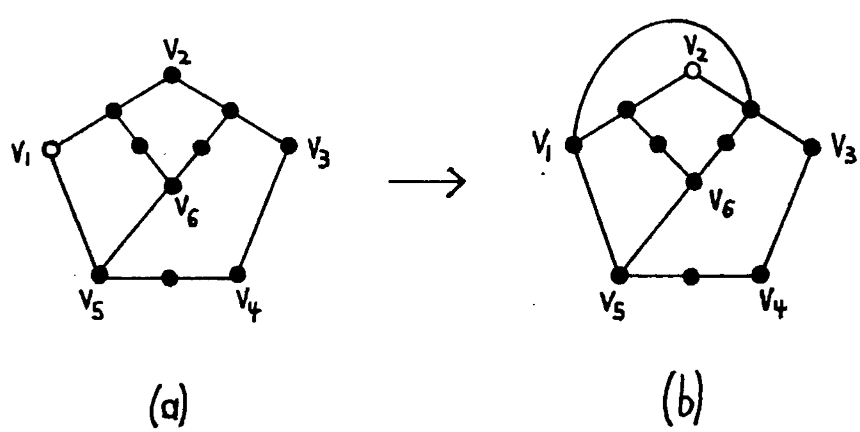

Figure 5.7.

Suppose some e in E(G) does not have an end in W. By Theorem 3.6, G-e* or G°e is 3-connected. If G-e* or G°e has a cycle containing v_1, v_2, v_3, v_4, v_5, and v_6, then so does G. Thus, G-e* or G°e contains a subdivision H' of the Herschel graph in which v_1, v_2, v_3, v_4, v_5, and v_6 are the branchvertices corresponding to the larger colour class, since G is a counterexample of minimum order. Then G also has such a subgraph unless H' corresponds to a subgraph H of G which is a subdivision of one of the graphs shown in figure 5.8. But v_1, v_2, v_3, v_4, v_5, and v_6 are on a common cycle in H, and hence in G.

Thus, G is a bipartite graph with colour class W. It is easy to show that G is then one of the graphs in figure 5.9. But in all cases v_1, v_2, v_3, v_4, v_5, and v_6 are on a common cycle.

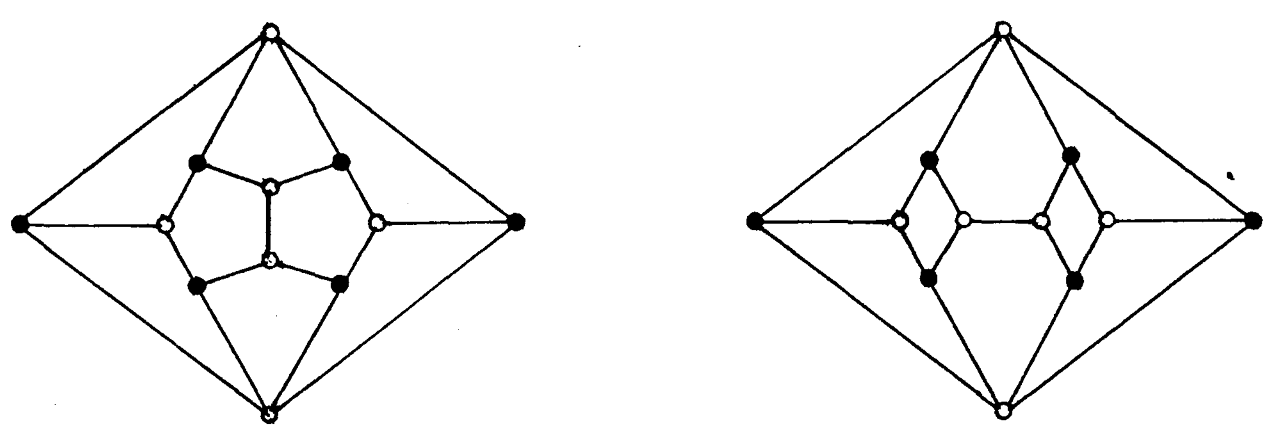

Figure 5.8.

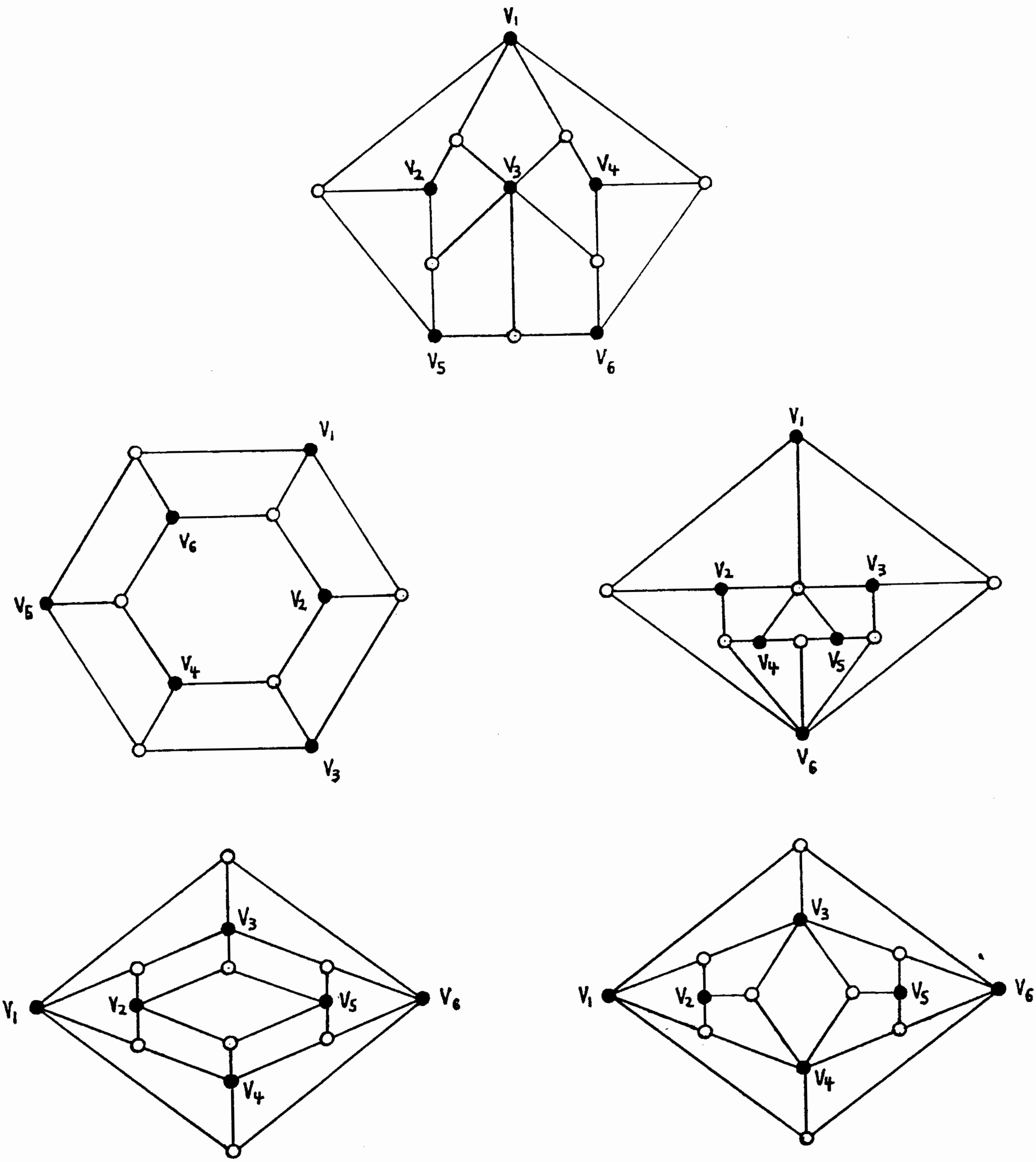
V1
V2
V3
V4
V5
V6
V1
V6
V5
V2
V4
V3
V1
V2
V3
V4
V5
V6
V3
V1
V2
V5
V6
V4
V3
V1
V2
V5
V6
V4

Figure 5.9.

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