GRAPH THEORETIC CONTROLLED ROUNDING

by

Brenda Yuk-Yee Li

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APPROVAL

Name : Brenda Yuk-Yee Li

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Examining Committee :

Chairman : Dr. Lou Hafer

Senior Supervisor : Dr. Pavol Hell

Dr. Tiko Kameda

Dr. Arthur Lee Liestman

External Examiner : Dr. Kathy Heinrich
Department of Mathematics

Date Approved : 13 March 1986

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Graph Theoretic Controlled Rounding

Author: ______________________

(signature)

Brenda Yuk-Yee Li

(name)

16 April 1986

(date)

ABSTRACT

The *Matrix Controlled Rounding* (MR) problem is the problem of rounding all real number entries of a given matrix, all row and column totals, and the grand total, to integer multiples of a positive base B subject to some constraints. In many applications the matrices may be symmetric, and it would be desirable to ensure that the rounded matrices are also symmetric. This motivates another class of the Controlled Rounding problem called the *Graph Theoretic Controlled Rounding* (GR) problem, which is the problem of rounding all edge weights of a given graph. We show that the MR problem as studied in [CE82] is a special case of our GR problem in the sense that it is linearly equivalent to the GR problem restricted to bipartite graphs. We also prove the existence of roundings of various kinds for different types of graphs. These results are useful for solving a stronger version of the Matrix Controlled Rounding problem, namely the problem of a *symmetric* rounding of a symmetric matrix.

The previously known algorithm for the MR problem appeals to the algorithms for the Capacitated Transportation problem which are not guaranteed to run in polynomial time. In this thesis we present two algorithms of time complexity $O(|E|^{3/2})$ where $|E|$ is the number of edges in the graph. They solve not only the MR problem, but also the GR problem for different types of graphs. The solutions are obtained by solving a certain Degree Constrained Subgraph and a certain Euler Tour problem on undirected graphs.

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CHAPTER 1

INTRODUCTION

The *Matrix Controlled Rounding* (abbreviated **MR**) problem is the problem of rounding all real number entries of a given tabular array (or matrix), all row and column totals, and the grand total, to integer multiples of a positive integer base B subject to the following constraints:

1.1 Each entry (or total) in the matrix is rounded to an adjacent integer multiple of B, i.e. an entry (or total) a is rounded to either $B(\lfloor \frac{a}{B} \rfloor + 1)$ or $B(\lfloor \frac{a}{B} \rfloor)$.

1.2 The sum of the rounded values along any row (respectively column) of the matrix is equal to the rounded value of the corresponding row (respectively column) total entry.

1.3 The total sum of all rounded values is equal to the rounded value of the total sum of all values in the matrix.

Matrix Controlled Rounding has several applications. For example, one may use it to uniformize the data values in a matrix for analysis or to reduce the unnecessary fine numerical details to a desirable level. It can even be used to control statistical disclosure in tabular representation of frequency count data. Small magnitude frequency count data may reflect a small, perhaps identifiable subset of respondent population. Under some circumstances this may result in disclosure of data obtained under pledges of confidentiality. Thus the data releaser may want to modify these frequency counts so that they become imprecise. But he must be careful to do so without introducing unnecessary disruption to the relationships between data items such as the additive structure of totals [Fe75]. The Matrix Controlled Rounding problem with the additional constraint that integer multiples of the base B must be rounded to themselves is said to be 0-restricted. It has important application in the area of survey design, especially in selection of sampling units [Er81].

There are some obvious simple rounding methods. One is to conventionally round the entries to the fixed integer base, e.g. to round a data value a to $B(\lfloor \frac{a}{B} + 0.5 \rfloor)$. Another is to round the data entries randomly. However these two methods often fail to satisfy the additive structure of the totals (constraints 1.2 and 1.3). In 1979, Causey developed an heuristic rounding procedure which worked satisfactorily in a large percentage of cases but has been shown to fail the constraints 1.2 and 1.3 in some particular examples [Ca79]. Then the problem was investigated by Cox and Ernst in 1982. They have shown that a rounding solution satisfying constraints 1.1, 1.2 and 1.3 always exists by modeling the problem as a Capacitated Transportation problem, and also developed some algorithms which depended on Transportation problem algorithms [CE82].

In many applications the matrices are symmetric, and it would be desirable to ensure that the rounded matrices are also symmetric. This motivates another class of controlled rounding problems called *Graph Theoretic Controlled Rounding* (abbreviated **GR**) problems which cannot be solved by the algorithms developed by Cox and Ernst. The Graph Theoretic Controlled Rounding problem is the problem of rounding all edge weights of a given graph, as well as vertex totals (sum of the weights of all edges incident at a vertex) and grand total (sum of the weights of all edges in the given graph), which satisfies the following constraints:

1.4 The weight of each edge (or total) in the graph is rounded to an adjacent integer multiple of the base B, i.e. a weight (or total) w is rounded to either $B(\lfloor \frac{w}{B} \rfloor + 1)$ or $B(\lfloor \frac{w}{B} \rfloor)$.

1.5 The sum of the rounded weights of the edges incident at any vertex is equal to the rounded value of the sum of the weights of all edges incident at the corresponding vertex.

1.6 The sum of all the rounded weights is equal to the rounded value of the total weight of all edges in the graph.

The *Directed Graph Theoretic Controlled Rounding* (abbreviated **DGR**) problem can be defined similarly if the given graph is directed. Instead of having a vertex total for each vertex, we have two vertex totals, one for the incoming edges and the other for the outgoing edges. Therefore constraints 1.5 will be replaced by the following two constraints for DGR problems:

1.5a The sum of the rounded weights of the edges going into any vertex is equal to the rounded value of the sum of the weights of all edges going into that vertex.

1.5b The sum of the rounded weights of the edges going out from any vertex is equal to the rounded value of the sum of the weights of all edges going out from that vertex.

The notation, formal statement of the problems and problem simplification are established in Chapter 2. In Chapter 3, we will show that MR can be viewed as a special case of GR, namely the rounding of bipartite graphs, and give some insights and known results of the existence of a solution. Chapter 4 is concerned with the rounding problem of undirected graphs which is equivalent to the rounding of symmetric matrices with the additional constraint that the resulting matrices are also symmetric. The solution is obtained by solving a certain *Degree Constrained Subgraph* (abbreviated **DCS**) problem and a certain Euler tour problem for undirected graphs. Chapter 5 presents some algorithms for solving the DCS problem, with which the GR problem can be solved efficiently.

CHAPTER 2

DEFINITIONS AND TERMINOLOGY

As stated in Chapter 1, the first constraint (constraint 1.1) of the Matrix Controlled Rounding problem requires that each entry in the matrix be rounded to an adjacent integer multiple of the base B. Some simplifying assumptions can be made immediately. By dividing each of the entries in the matrix by B, an equivalent Controlled Rounding Problem with base $B=1$ is obtained. Next, replacing each internal entry λ_{ij} by its fractional part, i.e. $\lambda_{ij}-\lfloor\lambda_{ij}\rfloor$ and adjusting all the totals accordingly, we have the condition $0\leqslant\lambda_{ij}<1$. An example is illustrated in Figure 2.1. This is also true in the Graph Theoretic version. Therefore we may assume that

$B=2$

$$\left[\begin{array}{cc|c} 0.7 & 4.0 & 4.7 \\ 2.5 & 1.3 & 3.8 \\ \hline 3.2 & 5.3 & 8.5 \end{array}\right]$$

$B=1$

$$\left[\begin{array}{cc|c} 0.35 & 2.0 & 2.35 \\ 1.25 & 0.65 & 1.90 \\ \hline 1.6 & 2.65 & 4.25 \end{array}\right]$$

$0\leq\lambda_{ij}<1$

$$\left[\begin{array}{cc|c} 0.35 & 0 & 0.35 \\ 0.25 & 0.65 & 0.90 \\ \hline 0.60 & 0.65 & 1.25 \end{array}\right]$$

Figure 2.1 : Example of an MR problem

the simplifying conditions hold when considering a Controlled Rounding problem.

Now we will introduce the terminology that is going to be used throughout this thesis and give a formal statement on the rounding problem.

A two dimensional tabular array A is an $(m+1) \times (n+1)$ array of real number entries with $m \times n$ internal entries λ_{ij}, where $0 \leqslant \lambda_{ij} < 1$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, m row total entries $\lambda_{i\cdot}$, where $1 \leqslant i \leqslant m$, n column total entries $\lambda_{\cdot j}$, where $1 \leqslant j \leqslant n$, and a grand total entry $\lambda_{\cdot\cdot}$. An MR (α,β,γ) is a function Λ which maps each internal entry λ_{ij} of A to Λ_{ij} which is either 0 or 1, row total entry $\lambda_{i\cdot}$ to $\Lambda_{i\cdot} = \sum_{j=1}^{n} \Lambda_{ij}$, column total entry $\lambda_{\cdot j}$ to $\Lambda_{\cdot j} = \sum_{i=1}^{n} \Lambda_{ij}$, and the grand total entry $\lambda_{\cdot\cdot}$ to $\Lambda_{\cdot\cdot} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Lambda_{ij}$, satisfying the following constraints :

2.1 $\quad |\Lambda_{ij} - \lambda_{ij}| < \alpha$

2.2a $\quad |\Lambda_{i\cdot} - \lambda_{i\cdot}| < \beta$

2.2b $\quad |\Lambda_{\cdot j} - \lambda_{\cdot j}| < \beta$

2.3 $\quad |\Lambda_{\cdot\cdot} - \lambda_{\cdot\cdot}| < \gamma$

An SMR (α,β,γ) (symmetric MR (α,β,γ)) is an MR (α,β,γ) of a symmetric matrix with the additional constraint that the resulting matrix be symmetric. If any of the parameters α,β or γ has an underbar, then the corresponding constraints would be non-strict inequalities rather than strict inequalities. For example, an MR $(\alpha,\underline{\beta},\underline{\gamma})$ is the function Λ that satisfies the following constraints :

2.4 $\quad |\Lambda_{ij} - \lambda_{ij}| < \alpha$

2.5a $\quad |\Lambda_{i\cdot} - \lambda_{i\cdot}| \leqslant \beta$

2.5b $\quad |\Lambda_{\cdot j} - \lambda_{\cdot j}| \leqslant \beta$

2.6 $\quad |\Lambda_{\cdot\cdot} - \lambda_{\cdot\cdot}| \leqslant \gamma$

The MR (α,β,γ) problem asks for an MR (α,β,γ) of a matrix A. Similarly, The SMR (α,β,γ) problem asks for an SMR (α,β,γ) of a symmetric matrix A.

A weighted undirected graph G is an undirected graph with vertex set V (where $|V|=n$), edge set E (where $|E|=m$) and a weight function $\lambda:E\rightarrow\{w:0<w<1\}$. An integer weight function $\Lambda:E\rightarrow\{0,1\}$ is defined to be a GR (α,β,γ) of the graph G if it satisfies the following constraints :

2.7 $\quad |\Lambda(e)-\lambda(e)| < \alpha \quad \forall e$

2.8 $\quad |\Lambda(v)-\lambda(v)| < \beta \quad \forall v\in V$

2.9 $\quad |\Lambda(E)-\lambda(E)| < \gamma$

where $\Lambda(v)$ ($\lambda(v)$ respectively) is the sum of the weights $\Lambda(e)$ ($\lambda(e)$ respectively) of all edges e incident with v,

and $\Lambda(E)$ ($\lambda(E)$ respectively) ~~is the~~ sum of the weights $\Lambda(e)$ ($\lambda(e)$ respectively) of all edges e in the edge set E.

For convenience, we define $\lambda(uv)=0$ and $\Lambda(uv)=0$ if $uv\notin E$ and $u,v\in V$. A directed edge from vertex i to vertex j is denoted by (i,j) and an undirected edge joining vertices i and j is denoted by ij. Let $f:A\rightarrow B$ be a function and $S\subseteq A$ be a set. Then $f(S) = \sum_{a\in S} f(a)$.

A BGR (α,β,γ) (Bipartite GR (α,β,γ)) is a GR (α,β,γ) on a bipartite graph G. If G is directed, then a DGR (α,β,γ) (Directed GR (α,β,γ)) can be defined similarly. Instead of having one constraint for the vertex totals, we would have two constraints, one for the incoming edges and the other for the outgoing edges. For example, constraint 2.8 would be replaced by the following two constraints for a DGR (α,β,γ) of G :

2.8a $\quad |\Lambda(v_{in})-\lambda(v_{in})| < \beta \quad \forall v\in V$

2.8b $\quad |\Lambda(v_{out})-\lambda(v_{out})| < \beta \quad \forall v\in V$

where $\Lambda(v_{in})$ ($\lambda(v_{in})$ respectively) is the sum of the weight $\Lambda(e)$ ($\lambda(e)$ respectively) of all incoming edges e of the vertex v,

and $\Lambda(v_{out})$ ($\lambda(v_{out})$ respectively) is the sum of the weight $\Lambda(e)$ ($\lambda(e)$ respectively) of all outgoing edges e of the vertex v.

A rounding in which integers are always rounded to themselves is said to be 0-restricted. For example, MR(1,1,1) and GR(1,1,1) are 0-restricted. Given a graph G and a weight function λ as in Figure 2.2a, a GR(1,1,1) is shown in Figure 2.2b. The edges that are missing in G can be conceptually viewed as edges having weights 0 and the rounding to 0 or 1 is only done on the edges that are present in G. Therefore the missing edges still have weights 0 after the rounding is done. If 0 can be rounded to 1, then they should be present in the graph. This amounts to rounding in complete graphs whose edges may have weights 0. We will refer to this kind of rounding as 0-relaxed rounding. For example, MR(1,1,1) and GR (1,1,1) are 0-relaxed.

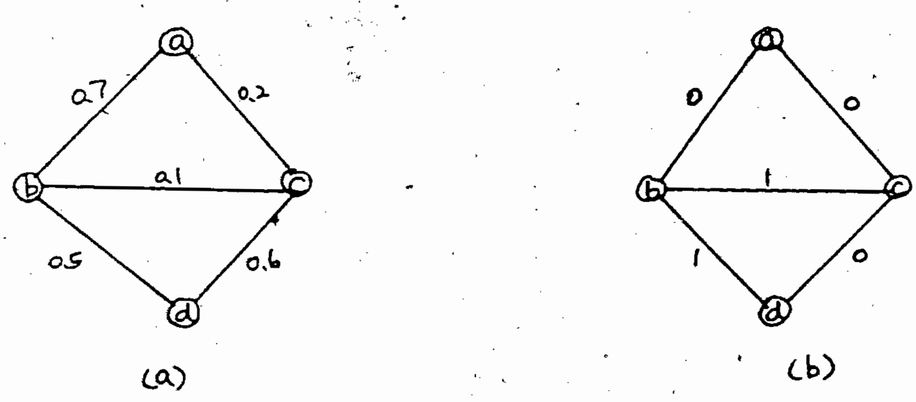

Figure 2.2 : Example of a GR problem

Mostly we are only interested in roundings in which all the parameters α, β and γ are equal to 1 or all the parameters $\underline{\alpha}, \underline{\beta}$ and $\underline{\gamma}$ are equal to $\underline{1}$. Thus, for simplicity, we will refer to the roundings with $\alpha=\beta=\gamma=1$ as 0-restricted and the roundings with $\underline{\alpha}=\underline{\beta}=\underline{\gamma}=\underline{1}$ as 0-relaxed.

Now we introduce another problem called *Degree Constrained Subgraph* (abbreviated **DCS**) problem which can be reduced from the GR problem. Let G be a graph in which each vertex i has an associated integer called the prescribed degree $p(i)$. A DCS is a spanning subgraph of G in which each vertex has degree equal to the prescribed degree. The DCS problem asks for a DCS of G.

Let us define the linear equivalence relation between two problems P_1 and P_2. Suppose that Φ is an algorithm that transforms each instance I_1 of problem P_1 to an instance I_2 of problem P_2, and Φ' is an algorithm that transforms each solution S_2 of the instance I_2 of P_2 to a solution S_1 of the instance I_1 of P_1 such that S_2 is a solution of $I_2=\Phi(I_1)$ if and only if $\Phi'(S_2)=S_1$ is a solution of I_1. If Φ and I_2 are linear in the size of I_1, S_2 is linear in the size of I_2, and if Φ' and S_1 are linear in the size of S_2, then we say that P_1 is **linearly reducible** to P_2, denoted by $P_1 \rightarrow P_2$. P_1 and P_2 are said to be **linearly equivalent**, denoted by $P_1 \leftrightarrow P_2$, if $P_1 \rightarrow P_2$ and $P_2 \rightarrow P_1$.

CHAPTER 3

GR ON DIRECTED GRAPHS

In this chapter we concentrate on the GR problem for directed graphs, which plays an important role in the GR problems. We will first show the relationships between the problems MR, DGR (GR on directed graphs) and BGR (GR on bipartite graphs). Then we will discuss the existence of a rounding for each of the problems.

3.1. Relation between MR and DGR

In Figure 3.1 we have two examples of controlled rounding, one is MR(1,1,1) and the other is DGR(1,1,1). The reader may notice that there are some similarities or correspondences between them.

$$\left[\begin{array}{ccc|c} 0.5 & 0.7 & 0.9 & 2.1 \\ 0.6 & 0.1 & 0.5 & 1.2 \\ \hline 1.1 & 0.8 & 1.4 & 3.3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 3 \end{array}\right]$$

Figure 3.1 : Correspondence between MR and DGR

Theorem 3.1 : The problems MR(1,1,1) and DGR(1,1,1) are linearly equivalent.

Proof :

(1) The MR(1,1,1) problem is linearly reducible to the DGR(1,1,1) problem.

Given a two dimensional matrix A , we construct a directed graph $\bar{G}=(\bar{V},\bar{E})$ such that any rounding DGR(1,1,1) of $\bar{G}$ will give a rounding MR(1,1,1) of A .

Let A be an $m \times n$ matrix and λ_{ij} be the entry of A in row i and column j. Without loss of generality, we may assume $m \leqslant n$. Let $\bar{G}=(\bar{V},\bar{E})$ be a directed graph with vertex set $\bar{V}=\{1,2,3,\cdots,n\}$, edge set $\bar{E}$ and weight function $\bar{\lambda}$ which are defined as follows:

$$\bar{E}=\{e=(i,j) \mid \lambda_{ij}>0\}$$

$$\bar{\lambda}:\bar{E}\rightarrow\{w:0<w<1\} \text{ so that } \bar{\lambda}(e)=\lambda_{ij} \text{ where } e=(i,j)\in\bar{E}.$$

Obviously the construction and the size of the DGR(1,1,1) instance are linear in the size of the given MR(1,1,1) instance. Suppose we are given a DGR(1,1,1) $\bar{\Lambda}$ of $\bar{G}$. The MR(1,1,1) Λ of the matrix A is obtained as follows. the rounding of the weight of the directed edge joining vertex i to vertex j of $\bar{G}$ gives the corresponding rounding of the entry λ_{ij}, i.e.,

$$\Lambda_{ij} = \begin{cases} \bar{\Lambda}(e) & \text{if } e=(i,j)\in\bar{E} \\ 0 & \textit{otherwise} \end{cases}$$

Since $\bar{\Lambda}$ is a DGR(1,1,1), it is easy to see that Λ will satisfy the constraints for being an MR(1,1,1) of A.

(2) The DGR(1,1,1) problem is linearly reducible to the MR(1,1,1) problem.

Given a directed graph $\bar{G}=(\bar{V},\bar{E})$ and a weight function $\bar{\lambda}$, we construct a two dimensional matrix A such that any rounding MR(1,1,1) of A will give a rounding DGR(1,1,1) of $\bar{G}$.

Let $\bar{V}=\{1,2,3,\ldots,n\}$ and let

$\bar{\lambda}(i,j)$ be the weight of the directed edge from vertex i to vertex j in $\bar{G}$.

Then A is a square matrix of dimension n, and the entry in row i and column j, denoted by λ_{ij}, is defined as follows:

$$\lambda_{ij} = \begin{cases} \bar{\lambda}(i,j) & \text{if } (i,j) \in \bar{E} \\ 0 & \textit{otherwise} \end{cases}$$

Obviously the construction and the size of the MR(1,1,1) instance are linear in the size of the given DGR(1,1,1) instance. Suppose we are given an MR(1,1,1) Λ of A. The corresponding DGR(1,1,1) $\bar{\Lambda}$ of $\bar{G}$ is obtained as follows, the rounded entry Λ_{ij} in row i and column j gives the corresponding rounded weight $\bar{\Lambda}(i,j)$ of the directed edge from vertex i to vertex j in $\bar{G}$, i.e.,

$$\bar{\Lambda}(e) = \Lambda_{ij} \quad \text{where } e=(i,j).$$

Since Λ is an MR(1,1,1), it is easy to see that $\bar{\Lambda}$ will satisfy the constraints for being a DGR(1,1,1) of $\bar{G}$.

Corollary 3.2 : The problems MR (1,1,1) and DGR (1,1,1) are linearly equivalent.

Proof : The proof is essentially the same as that of Theorem 3.1 except that the DGR (1,1,1) instance is always a complete directed graph of n vertices.

3.2. Relation between DGR and BGR

As in many graph theoretic problems, Controlled Rounding on directed graphs is linearly equivalent to that on bipartite graphs. An example is shown in Figure 3.2.

Theorem 3.3 : The problems DGR(1,1,1) and BGR(1,1,1) are linearly equivalent.

Proof :

(1) The DGR(1,1,1) problem is linearly reducible to the BGR(1,1,1) problem.

Given a directed graph $\bar{G}=(\bar{V},\bar{E})$ and a weight function $\bar{\lambda}$, we construct a weighted bipartite graph $G=(X \cup Y,E)$ such that any rounding BGR(1,1,1) of G will give a rounding DGR(1,1,1) of $\bar{G}$.

Let $\bar{V}=\{1,2,3,\ldots,n\}$ and let

$\bar{\lambda}(i,j)$ be the weight of the directed edge from vertex i to vertex j in $\bar{G}$.

Then $G=(X \cup Y,E)$ is the corresponding bipartite graph instance where $X = \{ x_1, x_2, \ldots, x_n \}$ and $Y = \{ y_1, y_2, \ldots, y_n \}$. A vertex i of $\bar{G}$ corresponds to two vertices x_i and y_i of G. The edge set E and weight function λ are defined as follows:

$$E=\{ e = x_i y_j \mid (i,j) \in \bar{E}\} ,$$

$$\lambda : E \rightarrow \{w : 0<w<1\} \text{ so that } \lambda(e)=\bar{\lambda}(\bar{e}) \text{ where } e=x_i y_j \text{ and } \bar{e}=(i,j).$$

Clearly the construction and the size of the BGR(1,1,1) instance are linear in the size of the given DGR(1,1,1) instance. Suppose Λ is a BGR(1,1,1) of G. Then the rounded weight of the edge joining vertices x_i and y_j will give the rounded weight of the directed edge from vertex i to vertex j in $\bar{G}$. Thus the rounding $\bar{\Lambda}$ of $\bar{G}$ is obtained as follows.

$$\overline{\Lambda(i,j)} = \Lambda(x_i y_j).$$

Since the rounding Λ is a BGR(1,1,1), it is easy to see that $\bar{\Lambda}$ will satisfy the constraints for being a DGR(1,1,1) of $\bar{G}$.

(2) The BGR(1,1,1) problem is linearly reducible to the DGR(1,1,1) problem.

Given a bipartite graph $G=(X \cup Y,E)$ and a weight function λ, we construct a weighted directed graph $\bar{G}=(\bar{V},\bar{E})$ such that any rounding DGR(1,1,1) of $\bar{G}$ will give a rounding

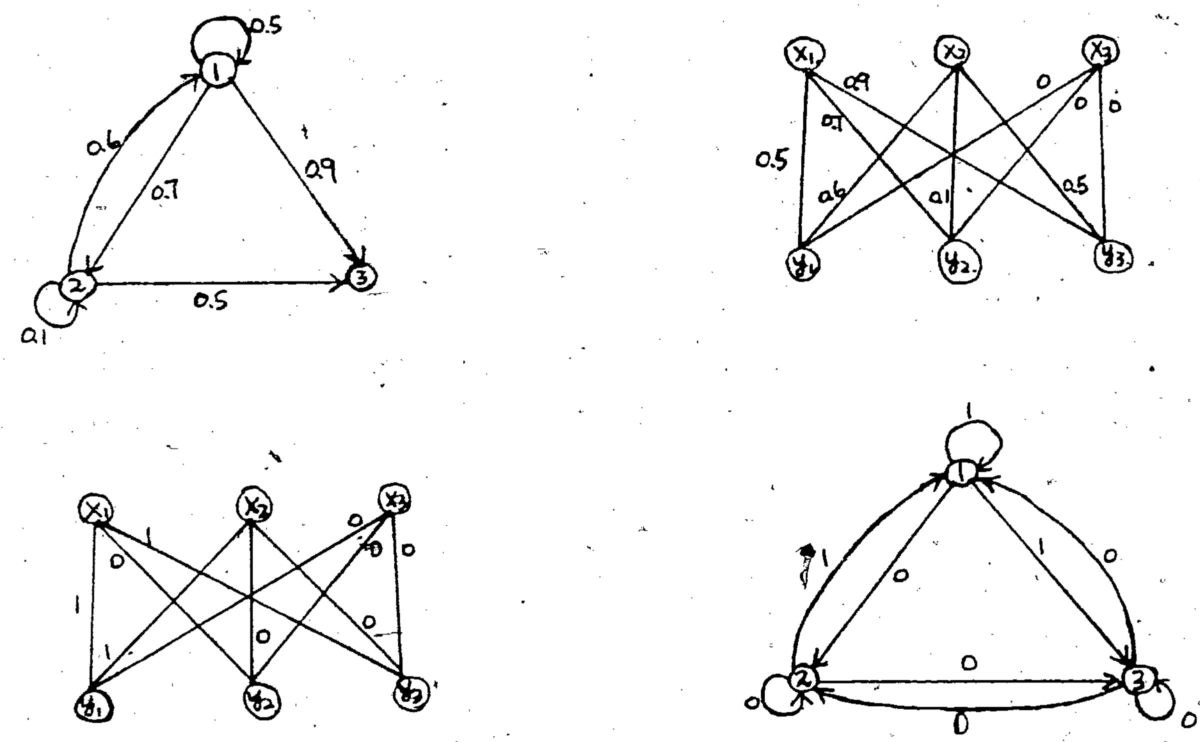

Figure 3.2 : Correspondence between DGR and BGR

BGR(1,1,1) of G.

Let $X = \{ x_1, x_2, \ldots, x_{n_1} \}$, $Y = \{ y_1, y_2, \ldots, y_{n_2} \}$ and let

$\bar{\lambda}(x_i y_j)$ be the weight of the edge joining vertices x_i and y_j in G.

Then the weighted directed graph instance $\bar{G}$ of the DGR(1,1,1) problem is defined as follows:

$\bar{V} = \{ 1, 2, \ldots, n \}$ where $n = \max(n_1, n_2)$,

$\bar{E} = \{ e=(i,j) \mid x_i y_j \in E \}$ and

$\bar{\lambda}:\bar{E}\rightarrow\{\, w : 0 < w < 1 \,\}$ so that $\bar{\lambda}(i,j) = \lambda(x_i y_j)$

Clearly the construction and the size of the DGR(1,1,1) instance are linear in the size of the given BGR(1,1,1) instance. Suppose $\bar{\Lambda}$ is a DGR(1,1,1) of $\bar{G}$. Then the rounded weight of the directed edge from vertex i to vertex j will give the corresponding rounded weight of the edge joining vertices x_i and y_j in G. Thus the rounding Λ of G is obtained as follows,

$$\Lambda(x_i y_j) = \bar{\Lambda}(i,j).$$

Since the rounding $\bar{\Lambda}$ is a DGR(1,1,1), it is easy to see that Λ will satisfy the constraints for being a BGR(1,1,1) of G.

Corollary 3.4 : The problems DGR (1,1,1) and BGR (1,1,1) are linearly equivalent.

Proof : The proof is the same as that of Theorem 3.3 with all the instances being complete graphs.

In some applications we may want some edges to be 0-restricted and others to be 0-relaxed. That is, some 0's must be rounded to 0 and others may be rounded to either 0 or 1. In this case we can delete the edges that are 0-restricted and treat the problem as a GR (1,1,1) problem. As illustrated in Figure 3.3, the 0-restricted edges are dotted and the 0-relaxed edges are solid. Now we delete all the dotted edges and construct the bipartite graph as in Figure 3.3b. Once we obtain a BGR (1,1,1) for the bipartite graph as in Figure 3.3c, we can get a DGR (1,1,1) to the directed graph. We see in Figure 3.3d that all the 0-restricted edges are rounded to 0 while some of the 0-relaxed edges are rounded to 1.

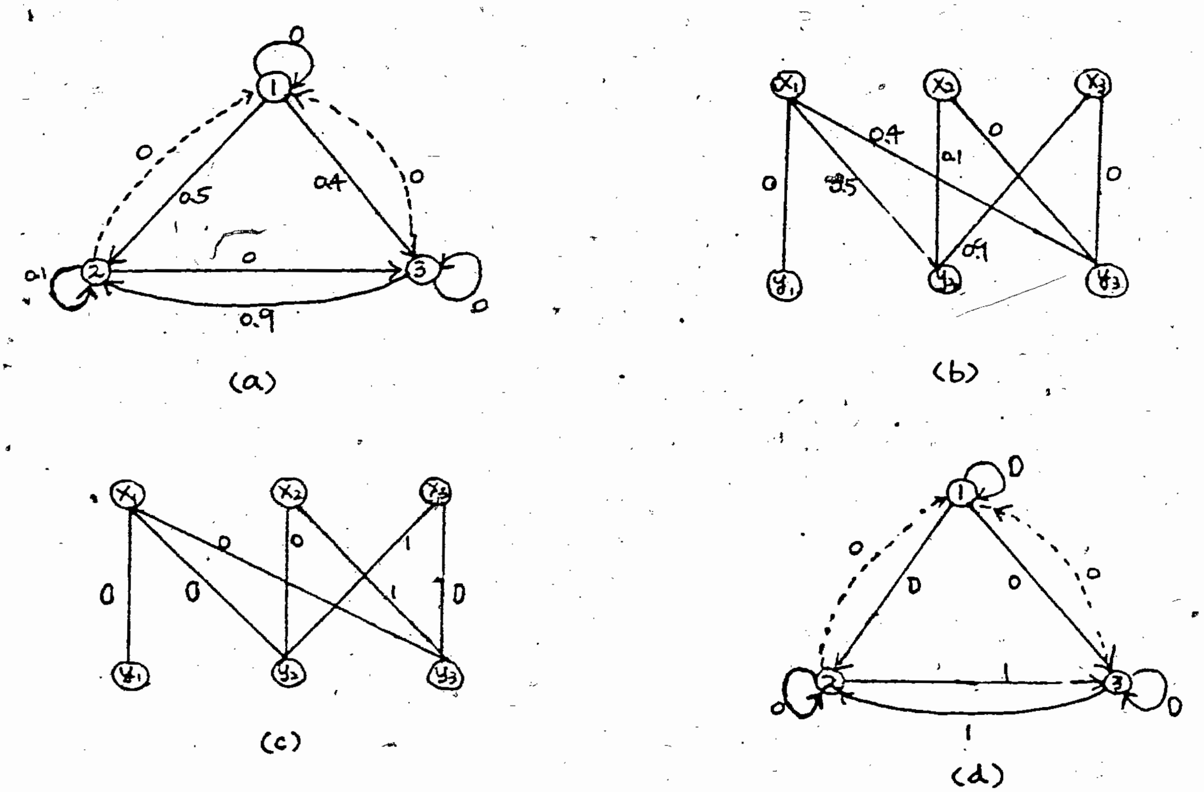

Figure 3.3 : Example of a GR problem with some 0-relaxed edges

3.3. Existence of a Rounding

Cox and Ernst have shown that an MR(1,1,1) always exists by modeling the problem as a Capacitated Transportation problem [CE82]. Recalling the linear equivalence relations shown in Sections 3.1 and 3.2, since we know that an MR(1,1,1) always exists, a DGR(1,1,1) and a BGR(1,1,1) always exist. So we have the following result for the Graph Theoretic Controlled Rounding problem on directed graphs.

Theorem 3.5 : A DGR(1,1,1) always exists.

An alternative direct (graph theoretic) proof will be given in Chapter 4 after we have discussed the relation between the GR and DCS problems.

In summary, Figure 3.4 depicts the linear equivalence relations of rounding problems. The MR(1,1,1) problem is linearly equivalent to the DGR(1,1,1) problem, which is linearly equivalent to the BGR(1,1,1) problem. The MR (1,1,1) problem is linearly equivalent to the DGR (1,1,1) problem, which is linearly equivalent to the BGR (1,1,1) problem. If we have an algorithm for solving the BGR(1,1,1) problem, the MR(1,1,1) and DGR(1,1,1) problems can also be solved by essentially the same algorithm.

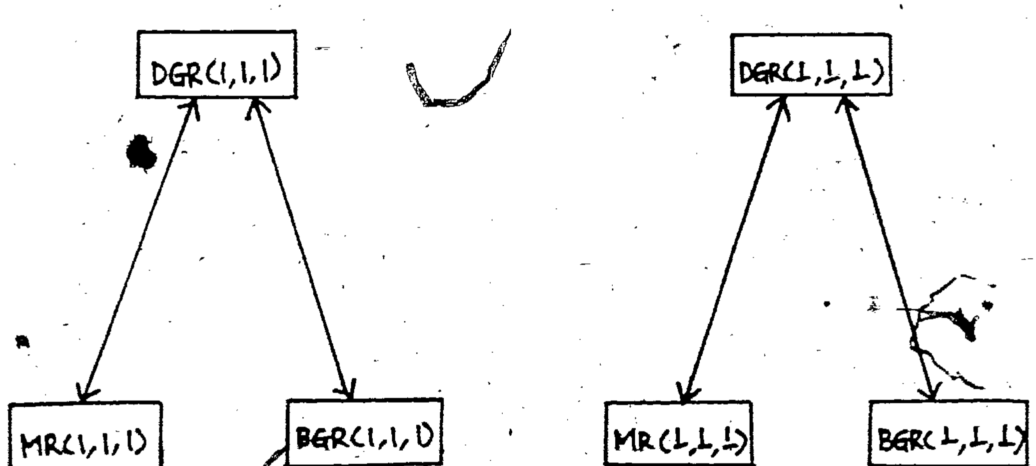

Figure 3.4 : Linear equivalence relations of rounding problems

CHAPTER 4

GR ON UNDIRECTED GRAPHS

Before showing the existence of a rounding for the GR problem on undirected graphs, we discuss the relationship between the problems SMR (Symmetric MR), DCS (Degree Constrained Subgraph) and GR on undirected graphs.

4.1. Relation between SMR and GR

We have seen in Chapter 3 that the MR(1,1,1) problem is linearly equivalent to the DGR(1,1,1) problem. The reader may suspect that MR is also related to GR on undirected graphs and it is not difficult to find such a relation. A weighted undirected edge joining vertices i and j in a graph can be viewed conceptually as two directed edges of the same weight in the graph, one from vertex i to vertex j, and the other from vertex j to vertex i, corresponding to two entries λ_{ij} and λ_{ji} of the same value in a matrix. However the correspondence is not the same between the loop at vertex i and the diagonal entry λ_{ii} with positive value, because the weight of the loop contributes twice to the vertex sum while the value of λ_{ii} contributes only once to the row or column sum. An example is shown in Figure 4.1.

Theorem 4.1 : The SMR $(\alpha,\beta,2\gamma)$ problem on symmetric matrices with all diagonal entries being 0 is linearly equivalent to the GR (α,β,γ) problem on loopless undirected graphs.

Proof :

(1) The SMR $(\alpha,\beta,2\gamma)$ problem on symmetric matrices with all diagonal entries being 0 is linearly reducible to the GR (α,β,γ) problem on loopless undirected graphs.

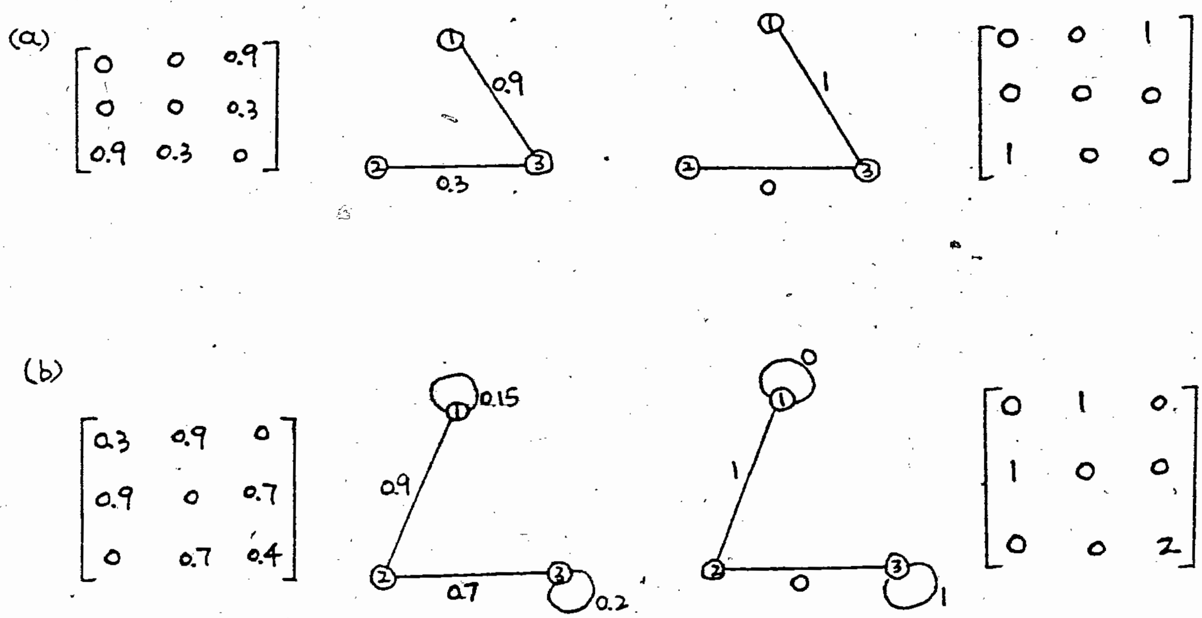

Figure 4.1 : Correspondence between SMR and GR on undirected graphs

Given a symmetric matrix A with all diagonal entries being 0, we construct a loopless undirected graph $G=(V,E)$ such that any rounding GR (α,β,γ) of G will produce a symmetric rounding SMR $(\alpha,\beta,2\gamma)$ of A.

Let A be the given symmetric matrix of dimension n and let

λ_{ij} be the entry of A in row i and column j.

Then $G=(V,E)$ is an undirected graph with vertex set $V=\{1,2,3,\cdots,n\}$, edge set E and weight function λ_G which are defined as follows :

$E = \{ e=ij \mid \lambda_{ij} > 0 \}$

$\lambda_G : E \to \{ w : 0<w<1 \}$ so that $\lambda_G(e) = \lambda_{ij}$ where $e=ij \in E$

Clearly the construction and the size of the GR instance are linear in the size of the given SMR instance. Suppose Λ_G is a GR (α,β,γ) of G. Then the symmetric rounding Λ of the matrix A is obtained as follows, the rounding of the weight of the edge joining vertices i and j gives the corresponding rounding of the entries λ_{ij} and λ_{ji}. i.e.

$$\Lambda_{ij} = \begin{cases} \Lambda_{ji} & \text{if } i>j \\ \Lambda_G(e) & \text{if } i<j \text{ and } e=ij \in E \\ 0 & \textit{otherwise} \end{cases}$$

The rounded row and column sums of A are equal to the rounded vertex sums of G, and the sum of all rounded values of A is equal to twice of the sum of the rounded edge weights of G. Since Λ_G is a GR (α,β,γ), it is easy to see that the symmetric rounding Λ constructed above will satisfy the constraints for being an SMR $(\alpha,\beta,2\gamma)$.

(2) The GR (α,β,γ) problem on loopless undirected graphs is linearly reducible to the SMR $(\alpha,\beta,2\gamma)$ problem on symmetric matrices with all diagonal entries being 0.

Given a loopless undirected graph $G=(V,E)$ and a weight function λ_G, we construct a symmetric matrix A with all diagonal entries being 0 such that any symmetric rounding SMR $(\alpha,\beta,2\gamma)$ of A will give a rounding GR (α,β,γ) of G.

Let $V = \{ 1, 2, \ldots, n \}$ and let

$\lambda_G(ij)$ be the weight of the edge joining vertices i and j in G.

Then A is a symmetric matrix of dimension n, and the entry in row i and column j, denoted by λ_{ij}, is defined as follows :

$$\lambda_{ij} = \begin{cases} \lambda_G(e) & \text{if } e=ij\in E \\ 0 & \textit{otherwise} \end{cases}$$

Obviously the construction and the size of the SMR $(\alpha,\beta,2\gamma)$ instance are linear in the size of the given GR (α,β,γ) instance. Since G is loopless, all the diagonal entries of A are 0. Suppose Λ is an SMR $(\alpha,\beta,2\gamma)$ of A. Then the corresponding rounding Λ_G of the graph G is obtained as follows. the value of the rounded entry Λ_{ij} in row i and column j of A gives the rounded weight $\Lambda_G(ij)$ of the edge joining vertices i and j in G, i.e.,

$$\Lambda_G(e) = \Lambda_{ij} \quad \text{where } e=ij.$$

The rounded vertex totals of G are equal to the rounded row sums and column sums of A, and the sum of the rounded weights of G is equal to half of the sum of the rounded values of A. Since Λ is an SMR $(\alpha,\beta,2\gamma)$, it is easy to see that the rounding Λ_G constructed above will satisfy the constraints for being a GR (α,β,γ) of the loopless undirected graph G.

Theorem 4.2 : The SMR $(2\alpha,\beta,2\gamma)$ problem on arbitrary symmetric matrices is linearly reducible to the GR (α,β,γ) problem on arbitrary undirected graphs.

Proof : The proof is similar to that of Theorem 4.1. However the weight of a loop contributes twice to the vertex sum while the value of a diagonal entry contributes only once to the row and column sum. So in order to make the row (or column) sums equal to the vertex sums, the weight of a loop in the GR instance should be equal to half of the value of the corresponding diagonal entry. Then a rounding GR (α,β,γ) of the constructed graph G will give a symmetric rounding of the corresponding matrix, in which the difference between the original and rounded value of a diagonal entry may be close to 2γ, and thus an SMR $(2\alpha,\beta,2\gamma)$.

4.2. Relation between GR and DCS

The linear equivalence relations we have seen so far are all on controlled rounding problems. Now we are going to show the relation between the problems GR and DCS (Degree Constrained Subgraph).

Theorem 4.3 : The GR(1,1,1) problem is linearly reducible to the DCS problem.

Proof : Given a graph $G=(V,E)$ and a weight function $\lambda : E \rightarrow \{ w : 0<w<1 \}$, we define a DCS instance $G'=(V',E')$ with weight function $\lambda' : E' \rightarrow \{ w : 0<w<1 \}$ and prescribed degrees p as follows:

$$V' = V \cup \{z\} \quad (\text{where } z \notin V)$$

$$E' = E \uplus \{ e=zv \mid v \in V \text{ and } \lambda(v) \text{ is not integral}\}$$

$$\cup \{ e=zz \mid \lambda(E) \text{ is not integral}\}.$$

$$\lambda'(e) = \begin{cases} \lambda(e) & \text{if } e \in E \\ \lceil \lambda(v) \rceil - \lambda(v) & \text{if } e=zv \text{ and } v \in V \\ \lambda(E) - \lfloor \lambda(E) \rfloor & \text{if } e=zz \end{cases}$$

The prescribed degree of each vertex is the sum of the weight of all edges incident with that vertex (the weight of a loop contributes twice to the prescribed degree).

$$p(v) = \lambda'(v) \quad \forall\ v \in V'-\{z\}$$

$$= \lambda(v) + \lceil \lambda(v) \rceil - \lambda(v) = \lceil \lambda(v) \rceil$$

$$p(z) = \lambda'(z)$$

$$= 2(\lambda(E) - \lfloor \lambda(E) \rfloor) + \sum_{v \in V} \{\lceil \lambda(v) \rceil - \lambda(v)\}$$

$$= 2\lambda(E) - 2\lfloor \lambda(E) \rfloor + (\sum_{v \in V} \lceil \lambda(v) \rceil) - 2\lambda(E) = \sum_{v \in V} \lceil \lambda(v) \rceil - 2\lfloor \lambda(E) \rfloor$$

Clearly the construction and the size of G' are bounded linearly in the size of G as $|V'|=|V|+1$ and $|E'|\leqslant|E|+|V|+1$. Now we are going to describe an algorithm for obtaining a GR(1,1,1) of G when given a DCS of G'.

Given a DCS $\bar{G}=(\bar{V},\bar{E})$ of $G'=(V',E')$, the corresponding rounding Λ of $G=(V,E)$ is obtained as follows :

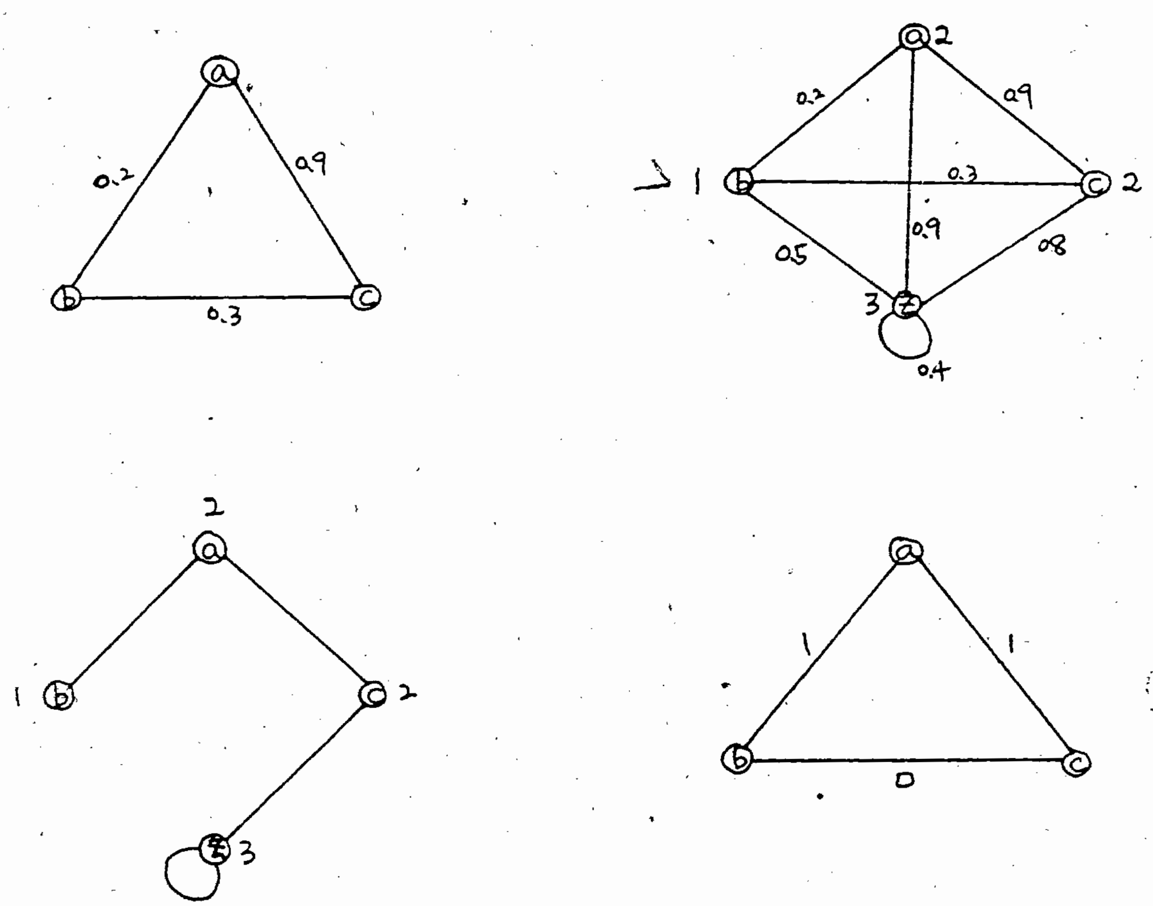

Figure 4.2 : Correspondence between GR and DCS on undirected graphs

$\Lambda : E \to \{0,1\}$ where

$$\Lambda(e) = \begin{cases} 1 & \text{if } e \in \bar{E} \\ 0 & \textit{otherwise} \end{cases}$$

Obviously the construction is linear and it remains to show that Λ satisfies the conditions for being a GR(1,1,1).

(a) $|\Lambda(e) - \lambda(e)| < 1 \quad \forall e$

Trivial.

(b) $|\Lambda(v) - \lambda(v)| < 1 \quad \forall\ v \in V$

Let the degree of v in $\bar{G}$ be $\bar{d}(v)$.

Since $\bar{G}$ is a DCS, the degree of every vertex is precisely the prescribed degree, i.e. $\bar{d}(v) = p(v) = \lceil \lambda(v) \rceil$. In other words, the number of edges incident with vertex v in $\bar{G}$ is $\lceil \lambda(v) \rceil$. Among these $\lceil \lambda(v) \rceil$ edges, there may be the edge $e{=}vz$. Thus $\Lambda(v) = \lceil \lambda(v) \rceil - 1$ or $\Lambda(v) = \lceil \lambda(v) \rceil$ depending on whether e is in $\bar{G}$ or not. In case $\lambda(v)$ is an integer, the edge e does not exist in G', so $\Lambda(v) = \lceil \lambda(v) \rceil = \lambda(v)$.

Therefore $|\Lambda(v) - \lambda(v)| < 1$.

(c) $|\Lambda(E) - \lambda(E)| < 1$

Let the degree of the vertex z in $\bar{G}$ be $\bar{d}(z)$.

Let $S_1 = \{\ v \in V : \Lambda(v) = \lceil \lambda(v) \rceil\ \}$ and

$$S_2 = \{\ v \in V : \Lambda(v) = \lceil \lambda(v) \rceil - 1\ \} = \{\ v \in V : \Lambda(v) < \lambda(v)\ \}.$$

Since $\bar{G}$ is a DCS, $\bar{d}(z) = p(z)$, and we have

$$\bar{d}(z) = \Big(\sum_{v \in V} \lceil \lambda(v) \rceil\Big) - 2\lfloor \lambda(E) \rfloor$$

$$= \sum_{v\in S_1}\lceil\lambda(v)\rceil + (\sum_{v\in S_2}\lceil\lambda(v)\rceil) - 2\lfloor\lambda(E)\rfloor$$

$$= \sum_{v\in S_1}\Lambda(v) + (\sum_{v\in S_2}\Lambda(v)+1) - 2\lfloor\lambda(E)\rfloor$$

$$= \sum_{v\in S_1}\Lambda(v) + \sum_{v\in S_2}\Lambda(v) + (\sum_{v\in S_2}1) - 2\lfloor\lambda(E)\rfloor$$

$$= \sum_{v\in V}\Lambda(v) + |S_2| - 2\lfloor\lambda(E)\rfloor$$

$$= 2\Lambda(E) - 2\lfloor\lambda(E)\rfloor + |\{v\in V : \Lambda(v)<\lambda(v)\}|$$

The loop at the vertex z contributes 2 to the degree $\bar{d}(z)$ if it is in the DCS $\bar{G}$. Recall that every vertex v has at most one edge incident with vertex z and its existence in $\bar{G}$ determines whether $\Lambda(v)$ is less than $\lambda(v)$ or not. If the loop is not in $\bar{G}$, then the degree $\bar{d}(z)$ must come from all edges joining vertex z and vertices of V only. In other words, $\bar{d}(z) = |\{v\in V : \Lambda(v)<\lambda(v)\}|$, i.e., $2\Lambda(E)-2\lfloor\lambda(E)\rfloor = 0$, which implies $\Lambda(E) = \lfloor\lambda(E)\rfloor$. If the loop is in $\bar{G}$, then $\bar{d}(z) - 2$ is the sum of all edges joining vertex z to vertices of V in $\bar{G}$. In other words, $\bar{d}(z)-2 = |\{v\in V : \Lambda(v)<\lambda(v)\}|$, i.e., $2\Lambda(E)-2\lfloor\lambda(E)\rfloor = 2$, which implies $\Lambda(E)-\lfloor\lambda(E)\rfloor = 1$ or $\Lambda(E) = \lfloor\lambda(E)\rfloor+1$. When $\lambda(E)$ is an integer, the loop does not exist in G' and so $\Lambda(E) = \lfloor\lambda(E)\rfloor$.

Therefore $|\Lambda(E)-\lambda(E)| < 1$.

Hence the constructed rounding Λ is a GR(1,1,1) of G.

If the given graph G is bipartite, we would prefer the corresponding instance of DCS to be bipartite too (because bipartite DCS problems are easier to solve). This can be done by introducing two vertices z_x and z_y instead of one vertex z in the graph G', and we have the following Theorem.

Theorem 4.4 : The BGR(1,1,1) problem is linearly reducible to the DCS problem on bipartite graphs.

Proof : Given a bipartite graph $G = (X \cup Y, E)$ and a weight function $\lambda : E \rightarrow \{w : 0 < w < 1\}$, we define a bipartite DCS instance $G' = (X' \cup Y', E')$ with weight function $\lambda' : E' \rightarrow \{w : 0 < w < 1\}$ and prescribed degrees p as follows :

$G' = (X' \cup Y', E')$

$X' = X \cup \{z_x\}$ (where $z_x \notin X$)

$Y' = Y \cup \{z_y\}$ (where $z_y \notin Y$)

$E' = E \cup \{\, e = z_x y \mid y \in Y$ and $\lambda(y)$ is not integral$\}$

$\cup \{\, e = x z_y \mid x \in X$ and $\lambda(x)$ is not integral$\}$

$\cup \{\, e = z_x z_y \mid \lambda(E)$ is not integral$\}$

$$\lambda'(e) = \begin{cases} \lambda(e) & \text{if } e \in E \\ \lceil \lambda(y) \rceil - \lambda(y) & \text{if } e = z_x y \text{ and } y \in Y \\ \lceil \lambda(x) \rceil - \lambda(x) & \text{if } e = x z_y \text{ and } x \in X \\ \lambda(E) - \lfloor \lambda(E) \rfloor & \text{if } e = z_x z_y \end{cases}$$

The prescribed degree of each vertex in G' is the sum of the weight of all edges incident with that vertex:

$p(v) = \lambda'(v) \qquad \forall\ v \in X \cup Y$

$= \lceil \lambda(v) \rceil$

$p(z_x) = \lambda'(z_x)$

$$= \lambda(E) - \lfloor \lambda(E) \rfloor + \sum_{y \in Y} (\lceil \lambda(y) \rceil - \lambda(y))$$

$$= \lambda(E) - \lfloor \lambda(E) \rfloor + \Big(\sum_{y \in Y} \lceil \lambda(y) \rceil\Big) - \lambda(E) = \Big(\sum_{y \in Y} \lceil \lambda(y) \rceil\Big) - \lfloor \lambda(E) \rfloor$$

$$p(z_y) = \lambda'(z_y)$$

$$= \lambda(E) - \lfloor \lambda(E) \rfloor + \sum_{x \in X} (\lceil \lambda(x) \rceil - \lambda(x))$$

$$= \lambda(E) - \lfloor \lambda(E) \rfloor + (\sum_{x \in X} \lceil \lambda(x) \rceil) - \lambda(E) = (\sum_{x \in X} \lceil \lambda(x) \rceil) - \lfloor \lambda(E) \rfloor$$

Clearly the construction and the size of G' are bounded linearly in the size of G as $|V'|=|V|+2$ and $|E'| \leqslant |E|+|V|+1$. Now we are going to describe an algorithm for obtaining a BGR(1,1,1) of G when given a DCS of G'.

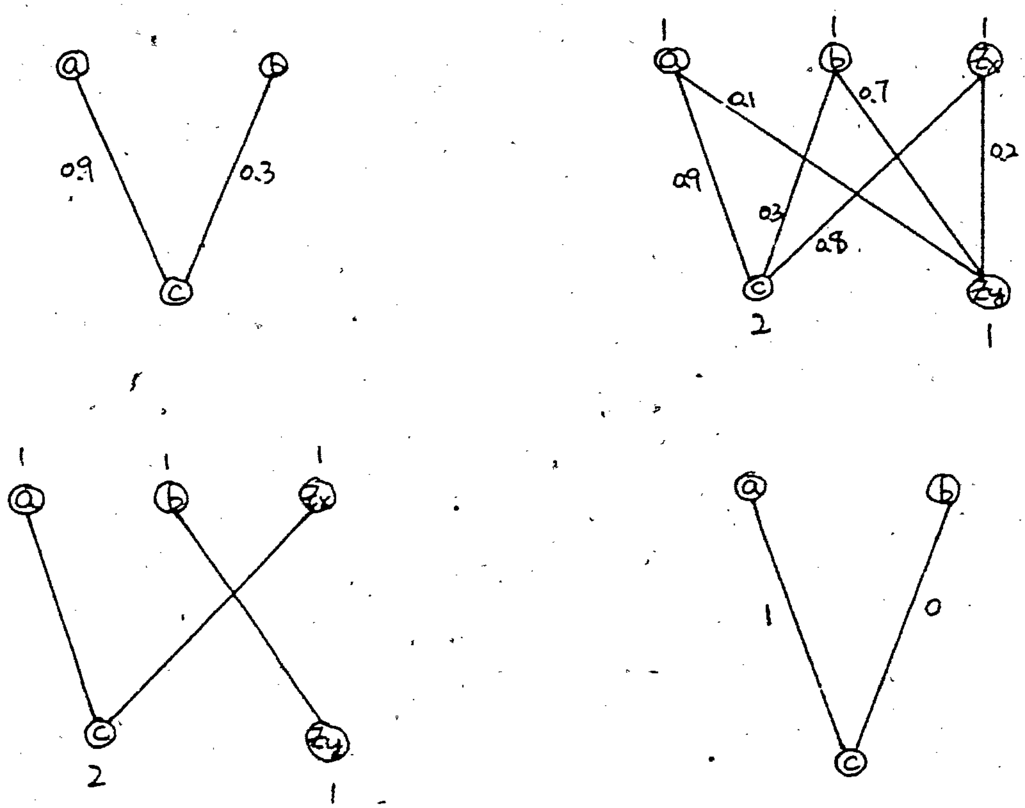

Figure 4.3 : Correspondence between BGR and DCS on bipartite graphs

Given a DCS $\bar{G} = (\bar{X} \cup \bar{Y}, \bar{E})$ of $G' = (X' \cup Y', E')$, the corresponding rounding Λ of $G=(X \cup Y, E)$ is constructed as follows :

$$\Lambda : E \rightarrow \{0,1\}$$

$$\Lambda(e) = \begin{cases} 1 & \text{if } e \in \bar{E} \\ 0 & \textit{otherwise} \end{cases}$$

Obviously the construction is linear and it remains to show that Λ satisfies the conditions for being a BGR(1,1,1).

(a) $|\Lambda(e) - \lambda(e)| < 1 \quad \forall e$

Trivial.

(b) $|\Lambda(v) - \lambda(v)| < 1 \quad \forall\ v \in X \cup Y$

The proof is the same as that of Theorem 4.3.

(c) $|\Lambda(E) - \lambda(E)| < 1$

Let the degree of vertex z_x in $\bar{G}$ be $\bar{d}(z_x)$.

Let $S_1 = \{y \in Y : \Lambda(y) = \lceil \lambda(y) \rceil\}$ and

$S_2 = \{y \in Y : \Lambda(y) = \lceil \lambda(y) \rceil - 1\} = \{y \in Y : \Lambda(y) < \lambda(y)\}$.

Since $\bar{G}$ is a DCS, $\bar{d}(z_x) = p(z_x)$, and we have

$$\bar{d}(z_x) = \left(\sum_{y \in Y} \lceil \lambda(y) \rceil\right) - \lfloor \lambda(E) \rfloor$$

$$= \sum_{y \in S_1} \lceil \lambda(y) \rceil + \left(\sum_{y \in S_2} \lceil \lambda(y) \rceil\right) - \lfloor \lambda(E) \rfloor$$

$$= \sum_{y \in S_1} \Lambda(y) + \left(\sum_{y \in S_2} \Lambda(y) + 1\right) - \lfloor \lambda(E) \rfloor$$

$$= \sum_{y \in S_1} \Lambda(y) + \sum_{y \in S_2} \Lambda(y) + \left(\sum_{y \in S_2} 1\right) - \lfloor \lambda(E) \rfloor$$

$$= (\sum_{y \in Y} \Lambda(y)) + |S_2| - \lfloor \lambda(E) \rfloor$$

$$= \Lambda(E) - \lfloor \lambda(E) \rfloor + |\{y \in Y : \Lambda(y) < \lambda(y)\}|$$

Among these $\bar{d}(z_x)$ edges, there is at most one edge which is not incident with any vertex in Y, namely the edge joining vertices z_x and z_y. Recalling from the construction of Λ that $\Lambda(y)$ is less than $\lambda(y)$ when the edge joining vertex $y \in Y$ and vertex z_x is in the DCS $\bar{G}$. If the edge $z_x z_y$ is not in $\bar{G}$, then the degree $\bar{d}(z_x)$ must come from all edges joining vertex z_x and vertices of Y. In other words $\bar{d}(z_x) = |\{y \in Y : \Lambda(y) < \lambda(y)\}|$ which implies $\Lambda(E) - \lfloor \lambda(E) \rfloor = 0$ or $\Lambda(E) = \lfloor \lambda(E) \rfloor$. If the edge $z_x z_y$ is in $\bar{G}$, then $\bar{d}(z_x) - 1 = |\{y \in Y : \Lambda(y) < \lambda(y)\}|$ which implies $\Lambda(E) - \lfloor \lambda(E) \rfloor = 1$ or $\Lambda(E) = \lfloor \lambda(E) \rfloor + 1$. When $\lambda(E)$ is an integer, the edge $z_x z_y$ does not exist in the graph G', so in this case $\Lambda(E) - \lfloor \lambda(E) \rfloor = 0$ or $\Lambda(E) = \lfloor \lambda(E) \rfloor$.

Therefore $|\Lambda(E) - \lambda(E)| < 1$.

Hence the rounding Λ constructed above is a BGR(1,1,1) of G.

4.3. Existence of a Rounding for the BGR problem

Before getting into the theorems, let us first give some notation that is used in this section. Given a weighted graph G with vertex set V. If V_1 and V_2 are subsets of V, then

$m_G(V_1, V_2)$ denotes the number of edges joining vertices in V_1 with vertices in V_2 and

$\lambda_G(V_1, V_2)$ denotes the sum of the weights of all edges joining vertices in V_1 with vertices in V_2.

We know from Chapter 3 that the MR(1,1,1) problem is linearly equivalent to the DGR(1,1,1) problem, which is linearly equivalent to the BGR(1,1,1) problem. Since an MR(1,1,1) is shown to exist in [CE82], we also have a theorem on the existence of a BGR(1,1,1).

We give a different proof of this fact based on the following necessary and sufficient condition for a bipartite graph to have a DCS with prescribed degrees.

Theorem 4.5 (L. Lovász [Lo79]) : A bipartite graph $G=(X \cup Y,E)$ has a DCS with prescribed degrees p if and only if :

(1) $p(X) = p(Y)$

(2) $p(X') \leqslant p(Y') + m_G(X',Y-Y')$ for every $X' \subseteq X$ and $Y' \subseteq Y$

Theorem 4.6 : A BGR(1,1,1) always exists.

Proof :

Let $G=(X \cup Y,E)$ be an arbitrary bipartite graph with weight function $\lambda: E \rightarrow \{w : 0<w<1\}$ for which we seek a BGR(1,1,1), and let $G'=(X' \cup Y',E')$ be the corresponding DCS instance with prescribed degrees p defined as follows:

$$X' = X \cup \{z_x\} \quad (\text{where } z_x \notin X)$$

$$Y' = Y \cup \{z_y\} \quad (\text{where } z_y \notin Y)$$

$$E' = E \cup \{ z_x y \mid y \in Y \text{ and } \lambda(y) \text{ is not integral}\}$$

$$\cup \{ x z_y \mid x \in X \text{ and } \lambda(x) \text{ is not integral}\}$$

$$\cup \{ z_x z_y \mid \lambda(E) \text{ is not integral}\}$$

$$p(v) = \lceil \lambda(v) \rceil \quad \forall v \in X \cup Y$$

$$p(z_x) = \Big(\sum_{y \in Y} \lceil \lambda(y) \rceil\Big) - \lfloor \lambda(E) \rfloor$$

$$p(z_y) = \Big(\sum_{x \in X} \lceil \lambda(x) \rceil\Big) - \lfloor \lambda(E) \rfloor$$

By Theorem 4.4 a DCS with prescribed degrees p of the graph G' will give a BGR(1,1,1) of the bipartite graph G. So now instead of showing the existence of a BGR(1,1,1) for G, we will show the existence of a DCS for G', i.e. we show that the conditions of Theorem 4.5 are met.

(1) $p(X') = p(Y')$

Note that $p(X') = p(X) + p(z_x) = \sum_{x\in X}\lceil\lambda(x)\rceil + \sum_{y\in Y}\lceil\lambda(y)\rceil - \lfloor\lambda(E)\rfloor$

$$p(Y') = p(Y) + p(z_y) = \sum_{y\in Y}\lceil\lambda(y)\rceil + \sum_{x\in X}\lceil\lambda(x)\rceil - \lfloor\lambda(E)\rfloor$$

Thus $p(X') = p(Y')$.

(2) $p(X'') \leqslant p(Y'') + m_{G'}(X'', Y' - Y'')$ for every $X'' \subseteq X'$ and $Y'' \subseteq Y'$

Let $X'' \subseteq X'$ and $Y'' \subseteq Y'$.

Recalling from Theorem 4.4 that $p(v)$ is the sum of the weights of all edges incident with v (for each $v \in X' \cup Y'$), we have

$$p(X'') = \lambda_{G'}(X'', Y') = \lambda_{G'}(X'', Y'' \cup Y' - Y'')$$

$$p(Y'') = \lambda_{G'}(Y'', X') = \lambda_{G'}(Y'', X'' \cup X' - X'')$$

$$p(X'') - p(Y'') = \lambda_{G'}(X'', Y' - Y'') - \lambda_{G'}(Y'', X' - X'')$$

$$\leqslant \lambda_{G'}(X'', Y' - Y'') \leqslant m_{G'}(X'', Y' - Y'') \quad \text{(since } \lambda(e) < 1\text{)}.$$

Thus $p(X'') \leqslant p(Y'') + m_{G'}(X'', Y' - Y'')$ for every $X'' \subseteq X'$ and $Y'' \subseteq Y'$.

Therefore a DCS always exists for G' and hence a BGR(1,1,1) always exists for a bipartite graph G.

4.4. Existence of a Rounding for the GR problem

We know that a BGR(1,1,1) always exists for bipartite graphs, but this is not true for general graphs as illustrated in Figure 4.4. We can see the rounding in Figure 4.4b or 4.4c is the best we can do, and thus a GR(1,1,1) may not exist for an arbitrary graph.

Given an arbitrary undirected graph G of n vertices, and a weight function $\lambda : E \rightarrow \{w : 0<w<1\}$, we create a weighted bipartite graph $\bar{G}$ of $2n$ vertices. A vertex i in G corresponds to two vertices x_i and y_i in $\bar{G}$, and an edge ij of weight $\lambda(ij)$ in G corresponds to two edges $x_i y_j$ and $x_j y_i$ in $\bar{G}$, both of weight $\lambda(ij)$. Since $\bar{G}$ is bipartite, a BGR(1,1,1) always exists. Thus we can obtain the BGR(1,1,1) $\bar{\Lambda}$ of $\bar{G}$ and convert the 0—1 rounding $\bar{\Lambda}$ in $\bar{G}$ to a 0—½—1 rounding Λ (i.e. a rounding with weights either 0,½, or 1) in G as follows.

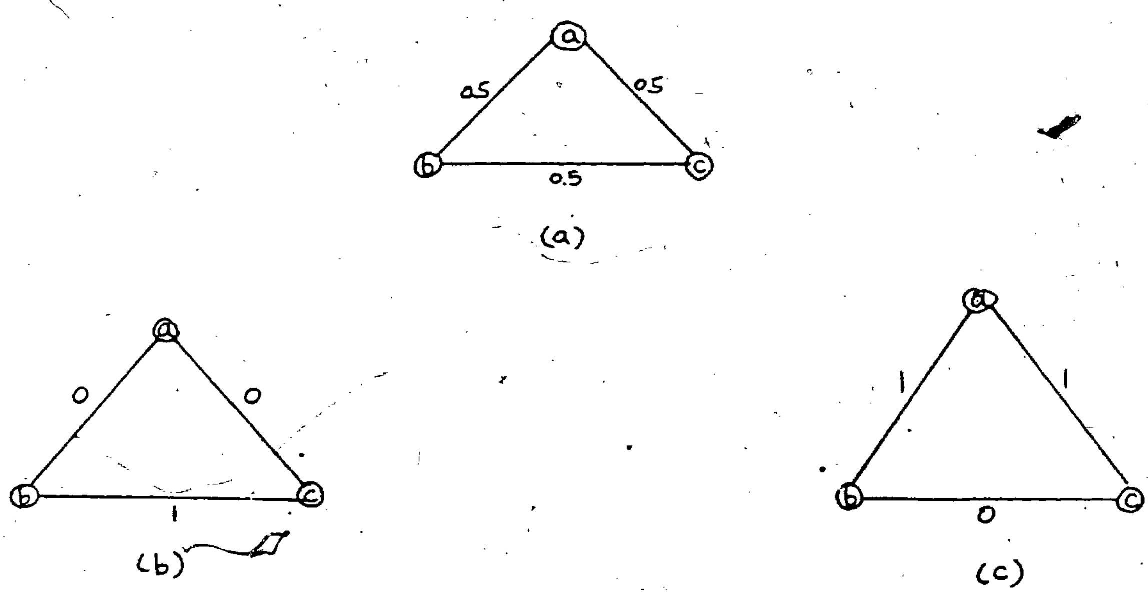

Figure 4.4 : Example of a rounding in general graphs

Let e be the edge ij in G and e_1, e_2 be the edges x_iy_j and x_jy_i respectively in $\bar{G}$.

Then $\Lambda(e) = \dfrac{\bar{\Lambda}(e_1) + \bar{\Lambda}(e_2)}{2}$.

Since $2\Lambda(E)=\bar{\Lambda}(\bar{E})$, $2\lambda(E)=\bar{\lambda}(\bar{E})$ and $|\bar{\Lambda}(\bar{E})-\bar{\lambda}(\bar{E})| < 1$, the 0—½—1 rounding Λ will satisfy the constraint $|\Lambda(E)-\lambda(E)| < ½$. Let G_* be a subgraph of G consisting of all the ½—edges of G. There is an even number of vertices of odd degree in each of the connected components of G_*. We are going to eliminate these odd vertices in pairs as follows. Find a trail (with repeated vertices allowed) joining two vertices of odd degree in the component and alternately add and subtract ½ to the edges of the trail. If a vertex v has odd degree in G_*, then $\Lambda(v)$ is half-integral and lies between $\lfloor \lambda(v) \rfloor$ and $\lfloor \lambda(v) \rfloor + 1$, i.e. $\Lambda(v) = \lfloor \lambda(v) \rfloor + ½$, and so must satisfy the constraint $|\Lambda(v) - \lambda(v)| < ½$. After adding or subtracting ½ to one of its incident edges, then $\Lambda(v) = \lfloor \lambda(v) \rfloor + 1$ or $\lfloor \lambda(v) \rfloor$, and the constraint $|\Lambda(v) - \lambda(v)| < 1$ is satisfied. Since the addition or subtraction of ½ is arbitrary for the starting edge of the trail, it is easy to see that the vertices of odd degree can all be eliminated in such a way that the following two constraints are satisfied :

(4.1) $\quad |\Lambda(v) - \lambda(v)| < 1 \quad \forall v \in V$

(4.2) $\quad |\Lambda(E) - \lambda(E)| \leqslant 1/2$

If $\Lambda(E) \leqslant \lambda(E)$ (or $\Lambda(E) > \lambda(E)$), then we will add (or subtract respectively) ½ to the starting edge. Now the degree of every vertex v in G_* is even and thus every connected component of G_* is Eulerian. If a component has an even number of edges, it can be eliminated (without affecting the vertex sum $\Lambda(v)$ and total sum $\Lambda(E)$) by alternately assigning 0 and 1 to the edges of the Eulerian tour. Similarly all even length cycles can be eliminated in the same way leaving G_* a union of odd cycles. Moreover, two odd cycles with any vertices in common can be decomposed into one or two even length closed trails which can also be eliminated. If they have only one vertex in common, then the union of the two cycles is certainly an even length closed trail.

If they have two vertices, say v_i and v_j, in common, then the union of the odd length paths between v_i and v_j and the union of the even length paths between v_i and v_j of the two cycles are the two even length closed trails. Therefore we may assume that $G_\cdot$ consists of vertex disjoint odd cycles and the rounding Λ that we obtain from $\overline{\Lambda}$ satisfies the constraints 4.1 and 4.2. In the proofs of the following Theorems we will show how to assign 0 and 1 to the edges of the odd cycles to obtain different roundings.

Theorem 4.7 : A GR $(1,1,(n+1)/2)$ always exists for an arbitrary undirected graph of n vertices.

Proof : Let G be the given arbitrary undirected graph of n vertices,

let $\overline{G}$ be the corresponding bipartite graph of $2n$ vertices,

let $\overline{\Lambda}$ be a BGR(1,1,1) of $\overline{G}$ and

let Λ be a 0—½—1 rounding of G satisfying constraints 4.1 and 4.2 such that $G_\cdot$, the subgraph of G consisting of all ½—edges, has the fewest possible edges.

Recall that $G_\cdot$ consists of vertex disjoint odd cycles. For each of the odd cycles we alternately assign 0 and 1 to the edges in the following way, the starting edge of the cycle is assigned 0 when the starting vertex v is rounded up, i.e. $\Lambda(v) > \lambda(v)$, and assigned 1 otherwise. It is easy to see that the constraint $|\Lambda(v) - \lambda(v)| \leqslant 1$ is satisfied for all $v \in V$, since we only decrease 1 to $\Lambda(v)$ if $\Lambda(v) > \lambda(v)$ and increase 1 if $\Lambda(v) \leqslant \lambda(v)$ for the starting vertex v of each cycle. Now we estimate the overall change to the total sum $\Lambda(E)$. For each of the odd cycles $\Lambda(E)$ is increased or decreased by ½, thus the worst case will happen when $\Lambda(E)$ is either increased or decreased for all the cycles. There are at most n vertex disjoint odd cycles possible (a loop is a cycle of length 1). Therefore the overall change to $\Lambda(E)$ is less than or equal to $n \times \frac{1}{2} = \frac{n}{2}$.

Hence $|\Lambda(E) - \lambda(E)| \leqslant \frac{1}{2} + \frac{n}{2} = \frac{n+1}{2}$.

Theorem 4.8 : A GR $(1,1,(n+3)/6)$ always exists for an arbitrary loopless undirected graph of n vertices.

Proof : From the proof of Theorem 4.7, we know that the difference between $\Lambda(E)$ and $\lambda(E)$ is less than or equal to ½ plus ½ times the maximum number of vertex disjoint odd cycles possible in the graph G_*. If loops are not allowed in the given graph, then G_* can have $n/3$ cycles (the smallest cycle has at least 3 vertices for loopless graphs).

Hence $|\Lambda(E) - \lambda(E)| \leqslant \frac{1}{2} + \frac{n}{6} = \frac{n+3}{6}$.

Theorem 4.9 : A GR $(1,2,1/2)$ always exists for an arbitrary undirected graph.

Proof : Let G be the given arbitrary undirected graph of n vertices,

let $\bar{G}$ be the corresponding bipartite graph of $2n$ vertices,

let $\bar{\Lambda}$ be a BGR$(1,1,1)$ of $\bar{G}$ and

let Λ be a 0−½−1 rounding of G satisfying constraints 4.1 and 4.2 such that G_*, the subgraph of G consisting of all ½−edges, has the fewest possible edges.

Recall that G_* consists of vertex disjoint odd cycles. For each of the cycles we alternately assign 0 and 1 to the edges in such a way that the constraint $|\Lambda(E) - \lambda(E)| \leqslant$ ½ is not violated. The number of incident edges assigned 1 is the same as that assigned 0 for every vertex of a cycle except one, namely the starting vertex v of the cycle. Then the vertex sum $\Lambda(v)$ and total sum $\Lambda(E)$ will be increased (or decreased) by 1 and ½ respectively depending on whether the starting edge is assigned 1 (or 0). It is not difficult to see that the assignment can always be made so that $|\Lambda(E) - \lambda(E)| \leq ½$. If there is an even number of odd cycles. We can have the same number with starting edges assigned 1 as there are those with starting edges assigned 0. If the number of odd cycles is odd, then the starting edge of the last cycle is assigned 0 when $\Lambda(E) > \lambda(E)$ and assigned 1 when $\Lambda(E) \leq \lambda(E)$.

Since the 0—½—1 rounding of G satisfies the constraint $|\Lambda(v) - \lambda(v)| < 1$ for all $v \in V$, and only the starting vertex of each odd cycle is increased or decreased by 1. Therefore the 0—1 rounding Λ we obtain will satisfy constraint $|\Lambda(v) - \lambda(v)| < 2$ for all $v \in V$.

Theorem 4.10 : A GR (1,4/3,1/2) always exists for an arbitrary loopless undirected graph.

Proof : Let $G=(V,E)$ be a weighted loopless graph with weight function λ.

From Theorem 4.8, we know that a rounding that satisfies the constraint $|\Lambda(v) - \lambda(v)| < 4/3$ for all $v \in V$ always exists. We let Λ be such a rounding which gives the minimum value of $|\Lambda(E) - \lambda(E)|$. If $-½ \leq \Lambda(E) - \lambda(E) \leq ½$, then we are done since Λ is indeed a GR (1,4/3,1/2). So we assume that $\Lambda(E) - \lambda(E) < -½$ (for the other case where $\Lambda(E) - \lambda(E) > ½$ the proof is similar).

For any such rounding Λ we can partition the vertices of G into three sets :

(1) $V^{+} = \{v \in V \mid \Lambda(v) - \lambda(v) \geq 1/3\}$

(2) $V^{-} = \{v \in V \mid -2/3 \leq \Lambda(v) - \lambda(v) < 1/3\}$

(3) $V^{--} = \{v \in V \mid \Lambda(v) - \lambda(v) < -2/3\}$

According to our assumptions, the configurations (a), (b) and (c) of Figure 4.5 in which the edges having rounded weights 1 and 0 are denoted as double and single edges respectively are impossible. Indeed, in any of the three cases the roles of the double and single edges can be

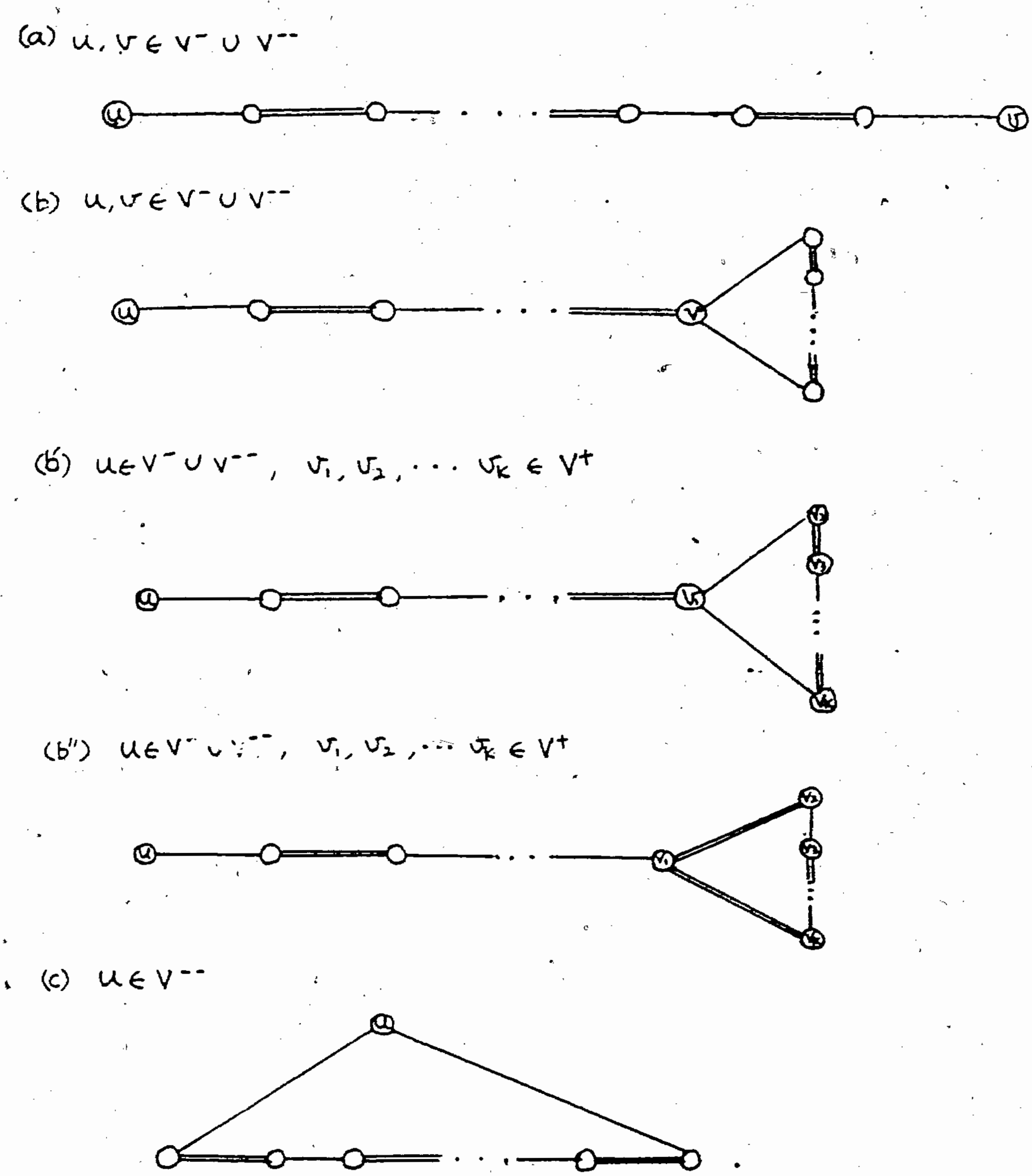

Figure 4.5 : Example of the Forbidden Configurations

interchanged producing another rounding that still satisfies the constraint $|\Lambda(v)-\lambda(v)| < 4/3$ for all $v \in V$ and increases $\Lambda(E)$ by 1, which contradicts the minimality of $|\Lambda(E)-\lambda(E)|$.

An alternating cycle (relative to the rounding Λ) is an odd cycle

$$v_1 - e_1 - v_2 - e_2 - \cdots - v_k - e_k - v_1$$

with one vertex, say v_1, in V^-, and all other vertices in V^+, and such that the edges $e_1, e_3, \ldots, e_k$ have rounded weights 0 and the edges $e_2, e_4, \ldots, e_{k-1}$ have rounded weights 1.

We will assume that Λ, in addition to all previous constraints, also maximizes the number of vertices that belong to alternating cycles. Then the configurations (b') and (b'') of Figure 4.5 are also impossible. Otherwise we can interchange the roles of all the double and single edges in case (b'') and those between vertex u and vertex v_1 in case (b') to produce another rounding that still satisfies all the previous constraints and has more vertices belong to alternating cycles.

Now we are going to partition the graph G into three subgraphs G_1, G_2 and G_3 as follows.

(1) $G_1 = (V_1, E_1)$ where

$V_1 = \{v \mid v \text{ belongs to some alternating cycle of } G\}$

$E_1 = \{\, uv \in E \mid u,v \in V_1\}$

It is easy to see that in G_1 no vertex v in V^+ belongs to two alternating cycles which have $x,y \in V^-$ where $x \neq y$ because of the forbidden case (a) in Figure 4.5. Thus the alternating cycles are all vertex disjoint except at the vertices belonging to V^-. Consider any connected component of G_1: it contains exactly one vertex in V^- and at least two vertices in V^+. If we sum up $\Lambda(v) - \lambda(v)$ for all vertices in each component, then the sum is non-negative.

Therefore $\Lambda(V_1) \geqslant \lambda(V_1)$.

(2) $G_2 = (V_2, E_2)$

The vertices in G_2 consist of levels L_i where

$L_0 = \{v \mid v \in (V^- \cup V^{--}) - V_1\}$

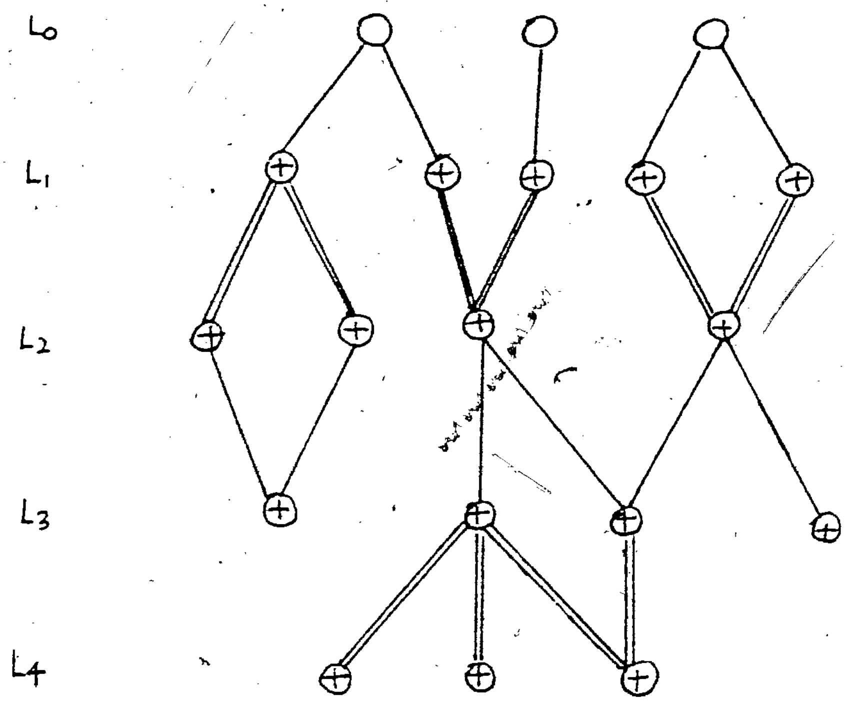

Figure 4.6 : Example of a Graph G_2

$E^i = \{\, e=uv \in E \mid u \in L_i,\ v \notin V_1 \cup L_0 \cup L_1 \cup \cdots \cup L_{i-1} \text{ and } \Lambda(e)=0\}$ $\qquad i = 0,2,4,\cdots$

$E^i = \{\, e=uv \in E \mid u \in L_i,\ v \notin V_1 \cup L_0 \cup L_1 \cup \cdots \cup L_{i-1} \text{ and } \Lambda(e)=1\}$ $\qquad i = 1,3,5,\cdots$

$L_{i+1} = \{\, v \mid \text{for some u, } uv \in E^i\}$

Let $i^* = \min\,\{i \mid L_{i+1}=\emptyset\}$,

and let $V_2 = L_0 \cup L_1 \cup \cdots \cup L_{i^*}$,

$$E_2 = E^0 \cup E^1 \cup \cdots \cup E^{i^*}.$$

Note that there are no edges $e=uv$ with $u,v \in L_i$. This follows from the absence of the configurations (a), (b), (b'), (b'') and (c) of Figure 4.5. Also as in Figure 4.6, all vertices in $V_2 - L_0$ belong to V^+, so the sum of $\Lambda(v) - \lambda(v)$ for all vertices v on odd levels, $L_{odd} = L_1 \cup L_3 \cup L_5 \cup \cdots$ is non-negative.

Thus $\Lambda(L_{odd}) \geqslant \lambda(L_{odd})$.

(3) $G_3 = (V_3, E_3)$ where

$V_3 = V - V_1 - V_2$ and

$E_3 = \{ e=uv \in E \mid u,v \in V_3 \}$

The sum of $\Lambda(v) - \lambda(v)$ of all vertices v in G_3 is non-negative, i.e. $\Lambda(V_3) \geqslant \lambda(V_3)$, since they all belong to V^+.

Since $\Lambda(V_1) \geqslant \lambda(V_1)$, $\Lambda(L_{odd}) \geqslant \lambda(L_{odd})$ and $\Lambda(V_3) \geqslant \lambda(V_3)$, the sum

$$S = \Lambda(V_1) + 2\Lambda(L_{odd}) + \Lambda(V_3) - \lambda(V_1) - 2\lambda(L_{odd}) - \lambda(V_3)$$

is non-negative. If we can show that $2(\Lambda(E) - \lambda(E)) = \Lambda(V) - \lambda(V) \geqslant S$, then $\Lambda(E) - \lambda(E) \geqslant 0$ or $\Lambda(E) \geqslant \lambda(E)$, which contradicts our assumption that $\Lambda(E) - \lambda(E) < -\frac{1}{2}$. Now to show that $2(\Lambda(E) - \lambda(E)) \geqslant S$, we will show that $\Lambda(e) - \lambda(e)$ for every double (or single) edge e is counted at most (at least, respectively) twice in the sum S.

Consider an edge $e = uv \in E$.

Case 1 : $e \in E_1 \cup E_2 \cup E_3$

It is obvious that $\Lambda(e) - \lambda(e)$ is counted exactly twice in $\Lambda(V_1) - \lambda(V_1)$ or $\Lambda(V_3) - \lambda(V_3)$ if e belongs to E_1 or E_3 respectively. As we have observed, no edge of E_2 joins two vertices on the same level and thus every edge in E_2 is incident with one vertex in the odd levels and one vertex in the even levels. Therefore $\Lambda(e) - \lambda(E)$ is counted exactly twice in $2(\Lambda(L_{odd}) - \lambda(L_{odd}))$ too if e belongs to E_2.

Case 2 : $e \notin E_1 \cup E_2 \cup E_3$ and both u, v belong to V_2.

- If $\Lambda(e) = 1$, then the edge e cannot join two vertices of L_{odd} by an earlier comment. Thus e is incident with at most one vertex belonging to the odd levels and therefore $\Lambda(e)-\lambda(e)$ is counted at most twice in $2(\Lambda(L_{odd})-\lambda(L_{odd}))$.

- If $\Lambda(e) = 0$, then as in the case where $\Lambda(e)=1$, e is incident with at least one vertex of L_{odd} and therefore $\Lambda(e)-\lambda(e)$ is counted at least twice in $2(\Lambda(L_{odd})-\lambda(L_{odd}))$.

Case 3 : $e \notin E_1 \cup E_2 \cup E_3$, $u \in V_1$ and $v \in V_3$

Since e is incident with one vertex in V_1 and another in V_3, $\Lambda(e)-\lambda(e)$ is counted once in $\Lambda(V_1)-\lambda(V_1)$ and once in $\Lambda(V_3)-\lambda(V_3)$ and therefore twice in the sum $\Lambda(V_1)+\Lambda(V_3)-\lambda(V_1)-\lambda(V_3)$.

Case 4 : $e \notin E_1 \cup E_2 \cup E_3$, $u \in V_2$ and $v \in V_1 \cup V_3$

- If $\Lambda(e) = 1$, then the vertex u does not belong to L_{odd} because of the forbidden cases (a) and (b) in Figure 4.5 and the definition of G_2. Thus the edge e is incident with one vertex in the even levels and another in V_1 or V_3 and therefore $\Lambda(e)-\lambda(e)$ is counted only once in the sum S.

- If $\Lambda(e) = 0$, then as in the case where $\Lambda(e)=1$, the edge e is incident with one vertex in L_{odd} and another in V_1 or V_3. Thus $\Lambda(e)-\lambda(e)$ is counted twice in $2(\Lambda(L_{odd})-\lambda(L_{odd}))$ and once in $\Lambda(V_1)+\Lambda(V_3)-\lambda(V_1)-\lambda(V_3)$ and therefore three times in the sum S.

In all of the above cases, $\Lambda(e)-\lambda(e)$ is counted at most twice for a double edge e (with rounded weight 1) and at least twice for a single edge e (with rounded weight 0) in the sum S. Thus we have

$$2(\Lambda(E)-\lambda(E)) \geqslant \Lambda(V)-\lambda(V)$$

$$\geqslant \Lambda(V_1) + 2\Lambda(L_{odd}) + \Lambda(V_3) - \lambda(V_1) - 2\lambda(L_{odd}) - \lambda(V_3) \geqslant 0$$

which contradicts our assumption that $\Lambda(E)-\lambda(E)<-\frac{1}{2}$, and therefore $\Lambda(E)-\lambda(E)\geqslant-\frac{1}{2}$.

Hence a GR $(1,4/3,\underline{1/2})$ always exists for a loopless undirected graph.

4.5. Existence of a Rounding for the SMR problem

Recall from Section 4.1 the linear reducibility of the SMR problems to the GR problems. Thus the existence of a symmetric rounding for the SMR problems can be derived from the existence of a rounding for the GR problems on undirected graphs.

Corollary 4.11 : An SMR $(2,\underline{1},\underline{n+1})$ always exists for an arbitrary symmetric matrix of dimension n.

Corollary 4.12 : An SMR $(1,\underline{1},\underline{(n+3)/3})$ always exists for a symmetric matrix of dimension n with all diagonal entries being 0.

Corollary 4.13 : An SMR $(2,2,\underline{1})$ always exists for an arbitrary symmetric matrix of dimension n.

Corollary 4.14 : An SMR $(1,4/3,\underline{1})$ always exists for a symmetric matrix of dimension n with all diagonal entries being 0.

4.6. Some Important Examples Showing Our Bounds are Best Possible

A GR(1,1,1) may not exist for an arbitrary graph, yet a GR $(1,\underline{1},\underline{(n+1)/2})$ and a GR $(1,2,\underline{1/2})$ always exist. In this section we will give some examples showing that the bounds in the constraints for GR $(1,\underline{1},\underline{(n+1)/2})$, GR $(1,\underline{1},\underline{(n+3)/6})$, GR $(1,2,\underline{1/2})$ and GR $(1,4/3,\underline{1/2})$ are best possible in the following sense. For an arbitrary $\epsilon>0$, there exist examples so that any rounding Λ which satisfies the constraints $|\Lambda(e)-\lambda(e)|<1$ and $|\Lambda(v)-\lambda(v)| \leqslant 1$ will have $|\Lambda(E)-\lambda(E)|$ asymptotically close to $(n+3)/6$ for loopless connected graphs, and asymptotically close to $(n+1)/2$ for arbitrary connected graphs. There also exist examples so that any rounding which satisfies the constraints $|\Lambda(e)-\lambda(e)|<1$ and $|\Lambda(E)-\lambda(E)| \leqslant 1/2$ will have $\max|\Lambda(v)-\lambda(v)| = 4/3-\epsilon$ for loopless connected graphs, and $\max|\Lambda(v)-\lambda(v)| = 2-\epsilon$ for arbitrary connected graphs.

For the graph given in Figure 4.7a, $\lambda(E)=k/2$ and $\lambda(v)=1-\epsilon$ or $k\epsilon$. If we choose $k\epsilon<1$, then $\Lambda(v)=0$ or 1 for all vertices v. Thus the rounded weights of the loops can only be 0 for otherwise $\Lambda(v)$ would be equal to 2 for the loop at vertex v. Note that the rounding typified in

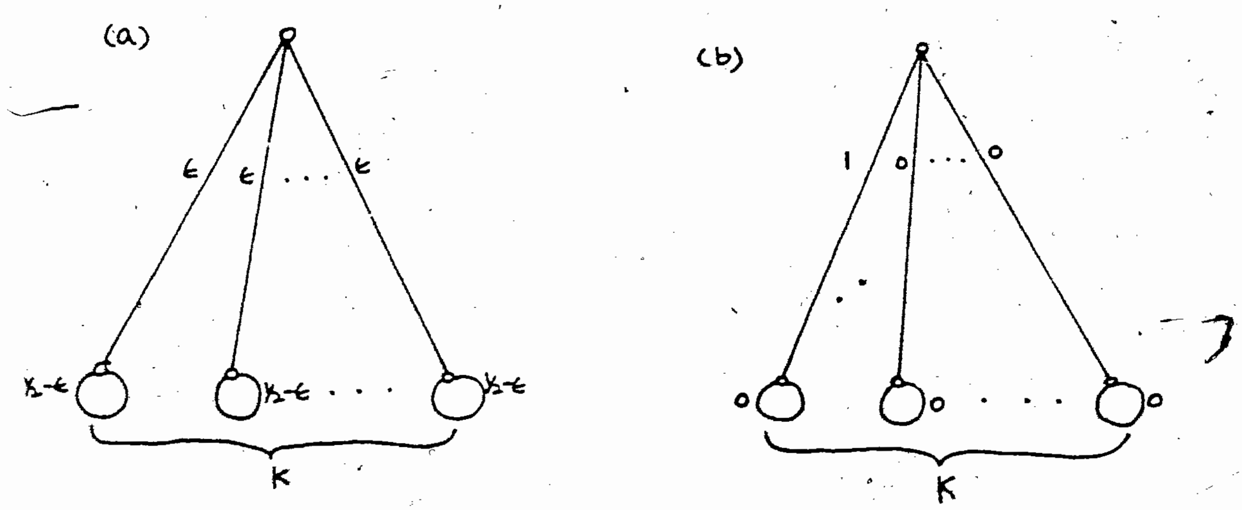

Figure 4.7 : Example of a GR $(1,\underline{1},\underline{(n+1)/2})$

Figure 4.7b is then best possible and it has $\Lambda(E)=1$.

Therefore $|\Lambda(E)-\lambda(E)| = \frac{k}{2}-1 = \frac{n-1}{2}-1 = \frac{n-3}{2}$.

For the loopless graph given in Figure 4.8a, $\lambda(E)=(3/2)k-k\epsilon$ and $\lambda(v)=1-\epsilon$ or $k\epsilon$. If we choose $k\epsilon<1$, then $\Lambda(v)$ can only be 0 or 1 for all vertices v. Thus at least two edges of each triangle have to be rounded to 0. The rounding typified in Figure 4.8b is then best possible as we try to maximize $\Lambda(E)$ and it has $\Lambda(E)=k+1$.

Therefore $|\Lambda(E)-\lambda(E)| = \frac{k}{2}-1-k\epsilon > \frac{k}{2}-2 = \frac{n-1}{6}-2 = \frac{n-13}{6}$.

Now we can see that the bounds in the constraints for a GR $(1,\underline{1},\underline{(n+1)/2})$, and a GR $(1,\underline{1},\underline{(n+3)/6})$ are tight in the sense that there exist examples of connected graphs achieving our

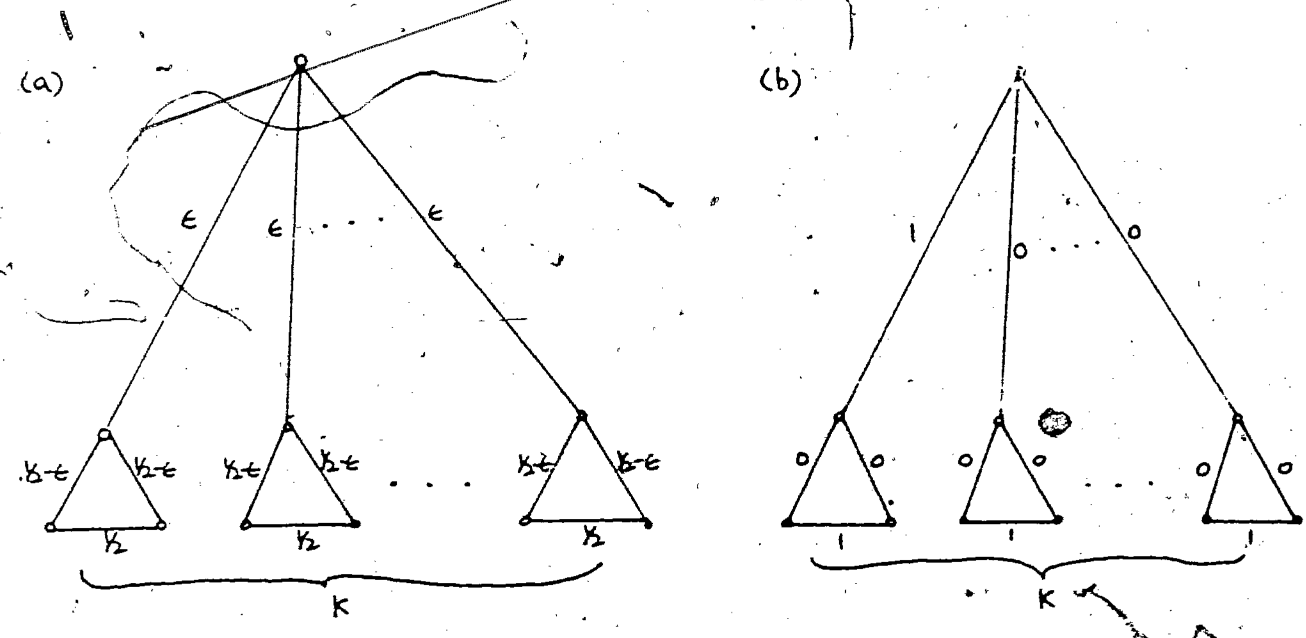

Figure 4.8 : Example of a GR $(1,\underline{1},\underline{(n+3)/6})$ for Loopless Graphs

bounds asymptotically.

For the graph given in Figure 4.9a, $\lambda(E)=k-(k\epsilon/3)$ and $\lambda(v)=2-\epsilon$ or $k\epsilon/3$. If we choose $k\epsilon>3/2$, then $\lambda(E)<k-1/2$. Thus in any rounding which satisfies the constraint $|\Lambda(E)-\lambda(E)|\leqslant 1/2$, we have $\Lambda(E)\leqslant k-1$ and hence there must exists a branch with rounded weights 0 on all edges. In other words, there must exists a loop at a vertex v with rounded weight 0 as in Figure 4.9b. Thus the difference between $\Lambda(v)$ and $\lambda(v)$ is $2-\epsilon$. Therefore for an arbitrary ϵ, we can construct a graph with k loops as shown where $k>3/(2\epsilon)$ so that in every rounding which satisfies the above constraint, we get that for some vertex v $|\Lambda(v)-\lambda(v)|$ is as close to 2 as we want.

The example given in Figure 4.10a shows that for an arbitrary ϵ, we can construct a loop-less graph with $k>1/(2\epsilon)$ so that $\max|\Lambda(v)-\lambda(v)|$ will be as close to 4/3 as we want.

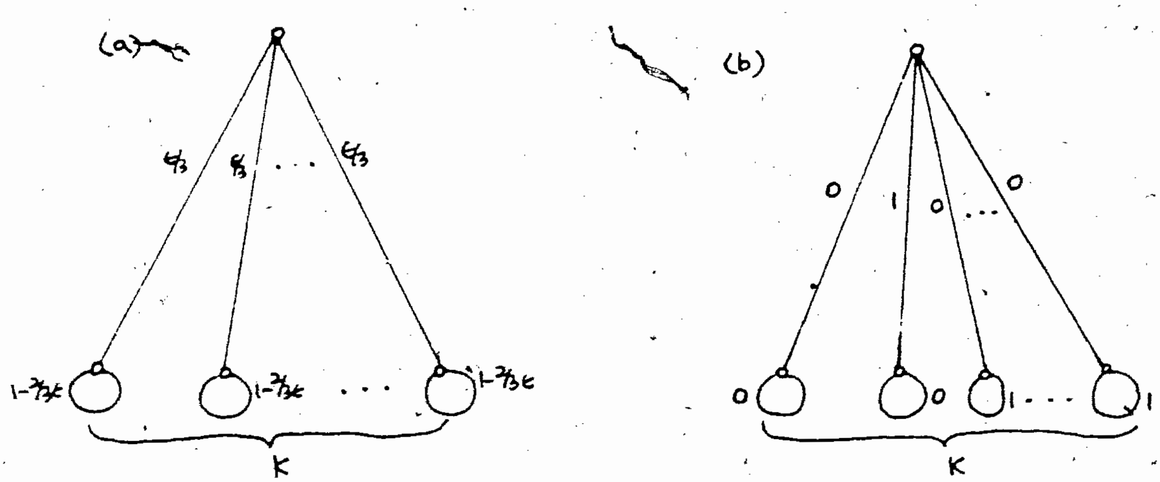

Figure 4.9 : Example of a GR (1,2,1/2)

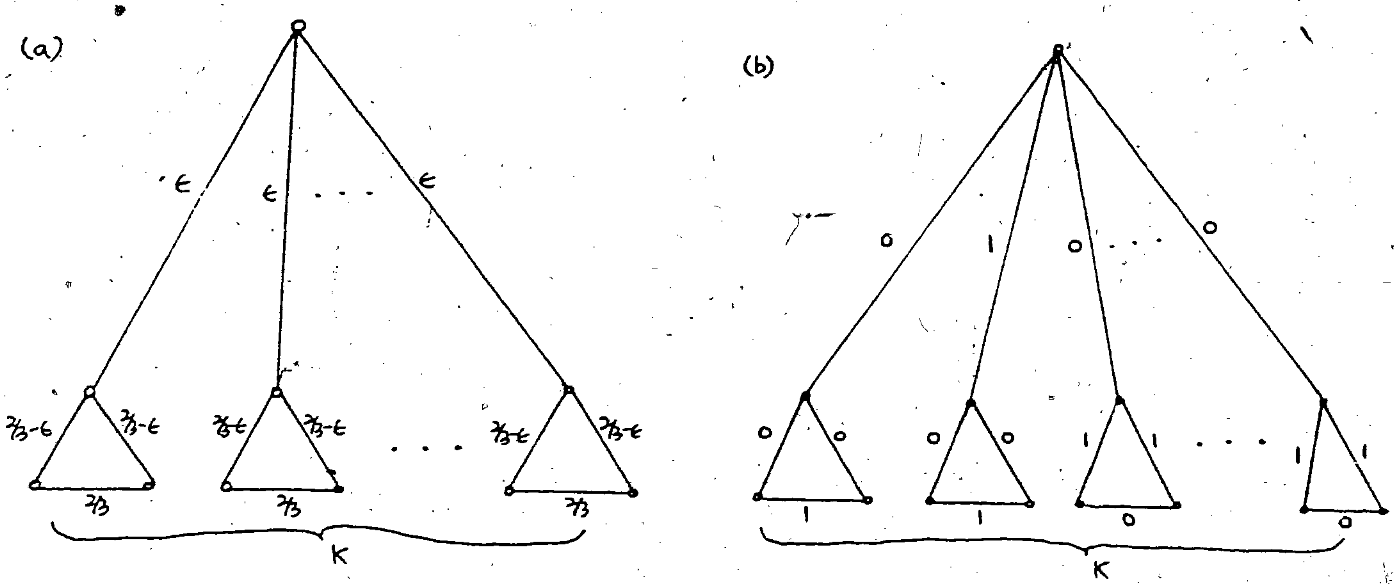

Figure 4.10 : Example of a GR $(1,4/3,\underline{1/2})$ for Loopless Graphs

Again the bounds in the constraints for a GR $(1,2,\underline{1/2})$ and a GR $(1,4/3,\underline{1/2})$ are tight in the sense that there exist examples of connected graphs achieving these bounds.

CHAPTER 5

ALGORITHMS FOR THE DCS PROBLEM

We have seen that all of the GR problems can be eventually reduced to the DCS problem for bipartite graphs. In this chapter we discuss the algorithms for finding a DCS with prescribed degrees for bipartite graphs only. However if the graph is complete (bipartite or not), we have simpler linear algorithms.

5.1. DCS for Complete Graphs

If the given graph is complete, then finding a DCS is equivalent to joining the vertices by edges so that the prescribed degrees are met. There are known algorithms for doing this on bipartite graphs and on arbitrary graphs without loops. We will modify the latter one to allow for loops.

5.1.1. Gale's Algorithm for Bipartite Graphs

Let $G' = (X' \cup Y', E')$ be a DCS instance with prescribed degrees p, which corresponds to a complete weighted bipartite graph $G = (X \cup Y, E)$ with weight function λ. Recall that

$X' = X \cup \{z_x\} = \{x_1, x_2, \ldots, x_m, z_x\}$,

$Y' = Y \cup \{z_y\} = \{y_1, y_2, \ldots, y_n, z_y\}$,

$E' = E \cup \{\, e = z_x y \mid y \in Y \text{ and } \lambda(y) \text{ is not integral} \}$

$\quad \cup \{\, e = x z_y \mid x \in X \text{ and } \lambda(x) \text{ is not integral} \}$

$\quad \cup \{\, e = z_x z_y \mid \lambda(E) \text{ is not integral} \}$.

$$p(v) = \lceil \lambda(v) \rceil \quad \forall\, v \in X \cup Y,$$

$$p(z_x) = (\sum_{y \in Y} \lceil \lambda(y) \rceil) - \lfloor \lambda(E) \rfloor \text{ and}$$

$$p(z_y) = (\sum_{x \in X} \lceil \lambda(x) \rceil) - \lfloor \lambda(E) \rfloor.$$

We are going to find a subgraph $\overline{G}$ of G' such that the above degree constraints are satisfied. Let A be the adjacency matrix of the subgraph $\overline{G}$ of dimension $(m+1)\times(n+1)$ and $A(i,j)$ be the entry of A in row i and column j. We assume that

$$\lambda(x_1) \geqslant \lambda(x_2) \geqslant \cdots \geqslant \lambda(x_m) \quad \text{and} \quad \lambda(y_1) \geqslant \lambda(y_2) \geqslant \cdots \geqslant \lambda(y_n).$$

The ith row of A corresponds to the vertex x_i, where $i=1,2,\ldots,m$, the $(m+1)st$ row of A corresponds to the vertex z_x, the jth column of A corresponds to the vertex y_j, where $j=1,2,\ldots,n$, and the $(n+1)st$ column of A corresponds to the vertex z_y. First we fill out the entries of the last row and column of A as follows:

$$A(m+1,n+1) = 0$$

$$A(i,n+1) = p(x_i) - (\lfloor \sum_{k=1}^{i} \lambda(x_k) \rfloor - \lfloor \sum_{k=1}^{i-1} \lambda(x_k) \rfloor)$$

$$A(m+1,j) = p(y_j) - (\lfloor \sum_{k=1}^{j} \lambda(y_k) \rfloor - \lfloor \sum_{k=1}^{j-1} \lambda(y_k) \rfloor)$$

Before defining the other entries of A, we make some observations:

(1) Let $R_i = \lfloor \sum_{k=1}^{i} \lambda(x_k) \rfloor - \lfloor \sum_{k=1}^{i-1} \lambda(x_k) \rfloor$ for $i=1,2,\ldots,m$ and

$$C_j = \lfloor \sum_{k=1}^{j} \lambda(y_k) \rfloor - \lfloor \sum_{k=1}^{j-1} \lambda(y_k) \rfloor \text{ for } j=1,2,\ldots,n.$$

Since $\lfloor a \rfloor + \lfloor b \rfloor \leqslant \lfloor a+b \rfloor \leqslant \lfloor a \rfloor + \lfloor b \rfloor + 1$, each R_i is either $\lfloor \lambda(x_i) \rfloor$ or $\lfloor \lambda(x_i) \rfloor + 1$ and each C_j is either $\lfloor \lambda(y_j) \rfloor$ or $\lfloor \lambda(y_j) \rfloor + 1$. If $\lambda(x_i)$ is integral, then $R_i = \lfloor \lambda(x_i) \rfloor = \lambda(x_i)$. Similarly if $\lambda(y_j)$ is integral, then $C_j = \lambda(y_j)$.

(2) Since $p(x_i)=\lceil\lambda(x_i)\rceil$ for $i=1,2,\ldots,m$ and $p(y_j)=\lceil\lambda(y_j)\rceil$ for $j=1,2,\ldots,n$, each entry in the last row and column of A is either 0 or 1.

(3) Since $\sum_{i=1}^{s}R_i=\sum_{i=1}^{s}(\lfloor\sum_{k=1}^{i}\lambda(x_k)\rfloor-\lfloor\sum_{k=1}^{i-1}\lambda(x_k)\rfloor)=\lfloor\sum_{i=1}^{s}\lambda(x_i)\rfloor$ for $1\leqslant s\leqslant m$ and

$$\sum_{j=1}^{t}C_j=\sum_{j=1}^{t}(\lfloor\sum_{k=1}^{j}\lambda(y_j)\rfloor-\lfloor\sum_{k=1}^{j-1}\lambda(y_j)\rfloor)=\lfloor\sum_{j=1}^{t}\lambda(y_j)\rfloor \text{ for } 1\leqslant t\leqslant n,$$

we have $\sum_{i=1}^{m}R_i=\lfloor\sum_{i=1}^{m}\lambda(x_i)\rfloor=\lfloor\lambda(E)\rfloor=\lfloor\sum_{j=1}^{n}\lambda(y_j)\rfloor=\sum_{j=1}^{n}C_j$.

(4) For $k<l$

$$\lfloor\sum_{i=1}^{k}\lambda(x_i)\rfloor-\lfloor\sum_{i=1}^{k-1}\lambda(x_i)\rfloor\geqslant\lfloor\lambda(x_k)\rfloor+\lfloor\sum_{i=1}^{k-1}\lambda(x_i)\rfloor-\lfloor\sum_{i=1}^{k-1}\lambda(x_i)\rfloor$$

$$=\lfloor\lambda(x_k)\rfloor\geqslant\lfloor\lambda(x_l)\rfloor$$

$$=\lfloor\lambda(x_l)\rfloor+\lfloor\sum_{i=1}^{l-1}\lambda(x_i)\rfloor-\lfloor\sum_{i=1}^{l-1}\lambda(x_i)\rfloor$$

$$\geqslant\lfloor\sum_{i=1}^{l}\lambda(x_i)\rfloor-\lfloor\sum_{i=1}^{l-1}\lambda(x_i)\rfloor-1$$

Thus $R_k\geqslant R_l-1$, and similarly $C_k\geqslant C_l-1$.

Now it remains to fill out the internal entries $A(i,j)$ where $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. This is equivalent to finding an $m\times n$ 0-1 matrix with row sums R_i and column sums C_j. According to D. Gale [Ga57], such a matrix exists if and only if for any $J\subseteq\{1,2,\ldots,n\}$

(*) $$\sum_{j\in J}C_j\leqslant\sum_{i=1}^{m}\min\{\,|J|,\,R_i\,\}.$$

Theorem 5.1 : There exists an $m \times n$ 0-1 matrix with row sums and column sums defined as follows:

$$R_i = \lfloor \sum_{k=1}^{i} \lambda(x_k) \rfloor - \lfloor \sum_{k=1}^{i-1} \lambda(x_k) \rfloor \quad \text{for } i = 1,2,\dots,m \text{ and}$$

$$C_j = \lfloor \sum_{k=1}^{j} \lambda(y_j) \rfloor - \lfloor \sum_{k=1}^{j-1} \lambda(y_j) \rfloor \quad \text{for } j = 1,2,\dots,n.$$

Proof : We will prove first that (*) is true for any J of the form $J=\{1,2,\dots,t\}$, and then for any $J \subseteq \{1,2,\dots,n\}$.

Suppose that (*) is false for some $J=\{1,2,\dots,t\}$, i.e.

$$\sum_{j=1}^{t} C_j \geqslant 1 + \sum_{i=1}^{m} \min(t,R_i).$$

Then $R_i > t$ for some i, for otherwise

$$\sum_{i=1}^{m} R_i = \sum_{j=1}^{n} C_j \geqslant \sum_{j=1}^{t} C_j \geqslant 1 + \sum_{i=1}^{m} \min(t,R_i) = 1 + \sum_{i=1}^{m} R_i,$$ which is a contradiction.

Let s be the largest subscript such that $R_s > t$ and by Observation (4) $R_i \geqslant R_s - 1$ for $i=1,2,\dots,s-1$, i.e. $R_i > t-1$ or $R_i \geqslant t$ for $i=1,2,\dots,s-1$. Thus

$$\sum_{j=1}^{t} C_j \geqslant 1 + \sum_{i=1}^{m} \min(t,R_i) = 1 + \sum_{i=1}^{s} t + \sum_{i=s+1}^{m} R_i$$

$$= 1 + st + \sum_{i=s+1}^{m} R_i$$

$$\sum_{i=1}^{m} R_i = \sum_{j=1}^{n} C_j \geqslant \sum_{j=1}^{t} C_j + \sum_{i=1}^{s} \sum_{j=t+1}^{n} A(i,j)$$

$$= \sum_{j=1}^{t} C_j + \sum_{i=1}^{s} (R_i - \sum_{j=1}^{t} A(i,j))$$

$$\geqslant \sum_{j=1}^{t} C_j + \sum_{i=1}^{s} (R_i - t)$$

$$\geq \sum_{j=1}^{t} C_j + \sum_{i=1}^{s} R_i - st$$

$$\geq 1 + st + \sum_{i=s+1}^{m} R_i + \sum_{i=1}^{s} R_i - st$$

$$\geq 1 + \sum_{i=1}^{m} R_i \qquad \text{(contradiction)}$$

Therefore (*) is true for any J of the form $J=\{1,2,...,t\}$.

Suppose (*) is false for some $J \subseteq \{1,2,...,n\}$. We assume that J is the first one in lexicographic order among all the subsets of $\{1,2,...,n\}$ for which (*) fails. Also let J contain $1,2,...,u$ (possibly $u=0$) but not $u+1$, $J \subseteq \{1,2,...,u+v+w\}$, $|J|=u+w$ and $u+v+w \in J$.

Let $C_{u+v+w}=x$. Then by Observation (4) $C_{u+1} \geq x-1$. If $C_{u+1} \geq x$, and we consider the subset $J' = J-\{u+v+w\} \cup \{u+1\}$ that appears before J in the lexicographic order, we have

$$\sum_{j\in J'} C_j = \sum_{j\in J} C_j + C_{u+1} - C_{u+v+w} \geq \sum_{j\in J} C_j \geq 1 + \sum_{i=1}^{m} \min\{|J|, R_i\}$$

which contradicts our assumptions. Therefore $C_{u+1}=x-1$ and by a similar argument $C_j=x-1$ for $u+1 < j < u+v+w$ and $j \notin J$. If we consider the subset $J' = J-\{j\} \cup \{u+1\}$ for $j \in J-\{1,2,...,u\}$ that appears before J in the lexicographic order, then since $u+1 < j \leq u+v+w$, again by Observation (4) $x-1 \geq C_j-1$ and $C_j \geq x-1$, i.e. $x-1 \leq C_j \leq x$. If $C_j=x-1$, then

$$\sum_{k\in J'} C_k = \sum_{k\in J} C_k - C_j + C_{u+1}$$

$$= \sum_{k\in J} C_k \geq 1 + \sum_{i=1}^{m} \min\{J, R_i\}$$

which again contradicts our assumptions. Therefore $C_j=x$ for $j \in J-\{1,2,...,u\}$.

Let P_k denotes the number of i such that $R_i \geq k$ and let $U=\sum_{j=1}^{u} C_j$. Then $P_1 \geq P_2 \geq P_3 \geq \cdots$ and we have verified that $U = \sum_{j=1}^{u} C_j \leq \sum_{i=1}^{m} \min(u, R_i)$.

Thus $U \leqslant \sum_{i=1}^{m} \min(u, R_i) = P_1 + P_2 + \cdots + P_u$.

Since (*) fails for the subset J, i.e. $\sum_{j \in J} C_j > \sum_{i=1}^{m} \min\{\,|J|, R_i\,\}$, and we have

$$\sum_{j \in J} C_j = \sum_{j=1}^{u} C_j + \sum_{j \in J - \{1,2,\ldots,u\}} C_j = U + wx$$

$$> \sum_{i=1}^{m} \min\{|J|, R_i\} = P_1 + P_2 + \cdots + P_{u+w}$$

$$\geqslant U + P_{u+1} + \cdots + P_{u+w} \geqslant U + wP_{u+w}$$

Thus $U + wx > U + wP_{u+w}$ which implies $x > P_{u+w}$ or $P_{u+w} \leqslant x-1$.

Now we consider the subset $J^* = \{1, 2, \ldots, u, u+1, \ldots, u+v+w\}$.

$$\sum_{j \in J^*} C_j = \sum_{j=1}^{u+v+w} C_j = \sum_{j=1}^{u} C_j + \sum_{j=u+1}^{u+v+w} C_j = U + wx + v(x-1)$$

$$\leqslant \sum_{i=1}^{m} \min\{u+v+w, R_i\}$$

$$= P_1 + P_2 + \cdots + P_{u+w} + P_{u+w+1} + \cdots + P_{u+v+w}$$

$$\leqslant P_1 + P_2 + \cdots + P_{u+w} + v(x-1)$$

$$< U + wx + v(x-1) \qquad \text{(contradiction)}$$

Hence (*) is true for any $J \subseteq \{1, 2, \ldots, n\}$ and the proof is complete.

Corollary 5.2 : There exists a subgraph of G' with prescribed degrees defined as follows:

$$p(v) = \lceil \lambda(v) \rceil \qquad \forall\, v \in X \cup Y$$

$$p(z_x) = \Big(\sum_{y \in Y} \lceil \lambda(y) \rceil \Big) - \lfloor \lambda(E) \rfloor$$

$$p(z_y) = \Big(\sum_{x \in X} \lceil \lambda(x) \rceil \Big) - \lfloor \lambda(E) \rfloor$$

Now we apply Gale's algorithm [Ga57] to find a 0-1 matrix with prescribed row sums R_i and column sums C_j.

<u>ALGORITHM A</u> (Gale's Algorithm for complete bipartite graphs)

For each $j = 1, 2, \ldots, n$,

$A(i,j) = 1$ for a set of C_j rows i whose values of $R_i - \sum_{k=1}^{j-1} A(i,k)$ are largest and

$A(i,j) = 0$ for all other i.

5.1.2. Modification of Hakimi's Algorithm for Arbitrary Graphs

A sequence $p = (p_1, p_2, \cdots, p_r)$ where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r > 0$ is called realizable if there exists a graph G with degree sequence p. Before presenting the linear algorithm that produces the graph G when given a realizable sequence p, we first prove the following Theorem which constitutes the basic step of our algorithm.

Theorem 5.3 : There exists a graph with degrees $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ if and only if there exists one with degrees $p_2', p_3', \ldots, p_n'$ where

$$p_i' = \begin{cases} p_i - 1 & \text{for } i = 2,3,\ldots,\delta+1 \\ p_i & \text{for } i = \delta+2,\delta+3,\ldots,n \end{cases}$$

$\delta = p_1$ if $p_1 < 2$ and $\delta = p_1 - 2$ otherwise.

Proof : Suppose there exists a graph with degrees $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$.

Note that the theorem is trivially true if $p_1 < 2$.

Let v_i be the vertex with degree p_i. We claim that there exists a graph G with degree sequence $p = (p_1, p_2, \ldots, p_n)$ in which the vertex v_1 has a loop if $p_1 \geqslant 2$. Suppose not. Then v_1 must be adjacent to $p_1 \geqslant 2$ vertices in $\{ v_2, v_3, \ldots, v_n \}$ and all these p_1 vertices are adjacent to each other. For otherwise if any two of them, say v_s and v_t, are not adjacent, Figure 5.1a shows that we can

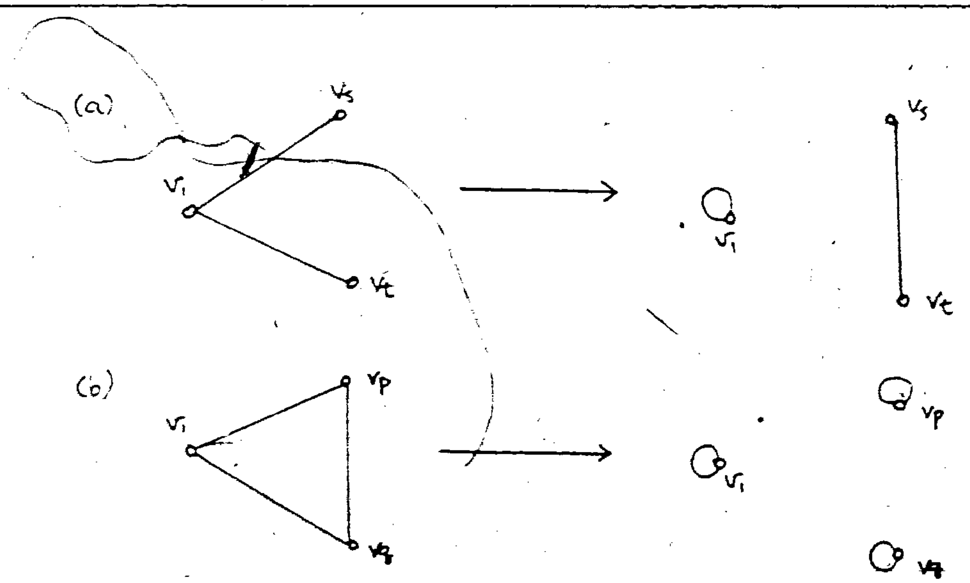

Figure 5.1 : Existence of a graph in which vertex v_1 has a loop

delete the edges v_1v_s and v_1v_t, but add v_sv_t and v_1v_1 to obtain a graph with the same degree sequence and a loop at vertex v_1. Furthermore, all of the p_1 vertices except at most one must have loops. Figure 5.1b shows that if two of them, say v_p and v_q, do not have loops, then we can delete the edges v_1v_p, v_1v_q and v_pv_q, but add the loops v_1v_1, v_pv_p and v_qv_q. This again is a contradiction. Since $p_1 \geqslant 2$, there is at least one vertex, say v_r, among all the vertices adjacent to v_1, which has a loop and is adjacent to all the other adjacent vertices of v_1. Thus $p_r = p_1+2 > p_1$, which contradicts our assumption that p_1 is the largest in the sequence p.

Now let us choose G to be a realization of p in which v_1 has a loop if $(p_1 \geqslant 2)$ and also v_1 is adjacent to as many vertices in $S = \{v_2, v_3, ..., v_{\delta+1}\}$ as possible. Suppose v_1 is not adjacent to one vertex, say v_i, where $2 \leqslant i \leqslant \delta+1$. Then v_1 must be adjacent to a vertex v_j where $\delta+2 \leqslant j \leqslant n$. If v_i does not have a loop or if both of v_i and v_j have loops, there must exist a vertex v_k adjacent to v_i but not to v_j since $p_i \geqslant p_j$. As shown in Figure 5.2a, we can delete the edges v_1v_j and v_iv_k but add v_1v_i and v_jv_k. If v_i has a loop but not v_j, then depending on whether vertices v_i and v_j are adjacent or not, we can delete the edges v_1v_j and v_iv_j but add v_1v_i and v_jv_j, or delete the edges v_1v_j and v_iv_i but add v_1v_i and v_iv_j (Fig 5.2b, Fig 5.2c). In each of these cases, we obtain a graph with the same degree sequence and v_1 adjacent to more vertices in $\{v_2, v_3, ..., v_{\delta+1}\}$. Therefore we have a graph G with degrees $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$ in which vertex v_1 has a loop if $p_1 \geqslant 2$ and is adjacent to all vertices $v_2, ..., v_{\delta+1}$. Removing v_1, we obtain a graph with degrees $p_2', ..., p_n'$.

Conversely, suppose there is a graph with degrees $p_2', ..., p_n'$. We can then add the vertex v_1 trivially to obtain a graph with degrees $p_1, p_2, ..., p_n$.

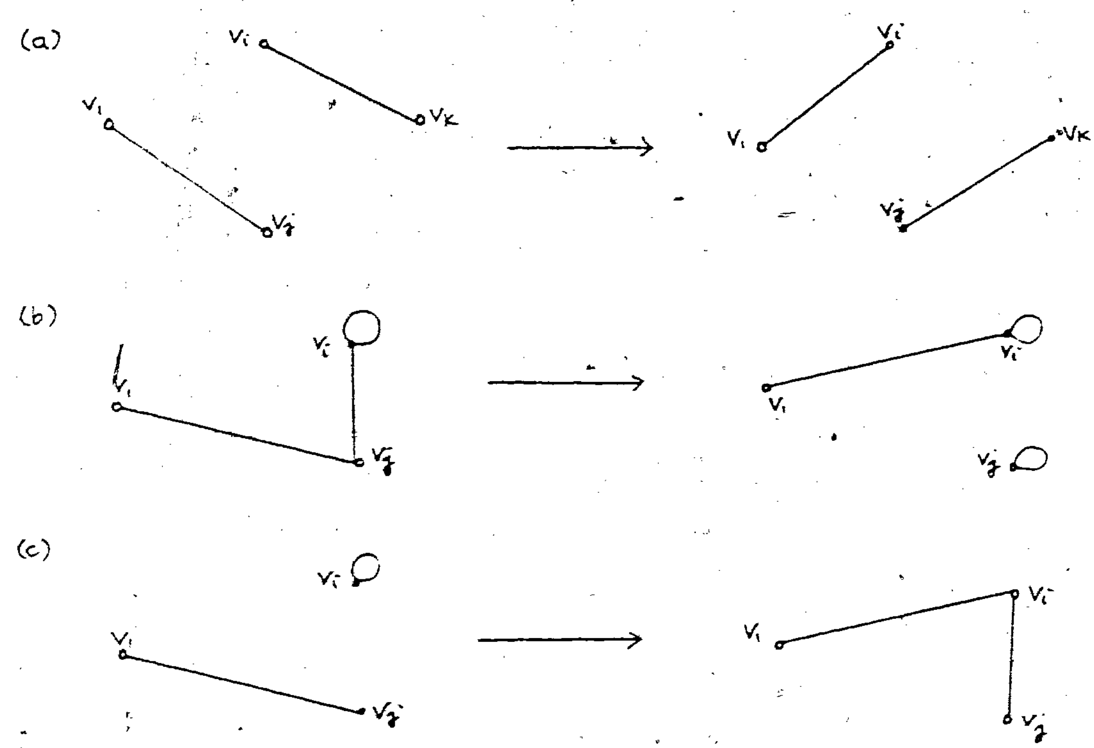

Figure 5.2 : Existence of a graph in which v_1 is adjacent to $v_2, \ldots, v_{\delta+1}$

The following is a modification of Hakimi's Algorithm [Ha62], which constructs a graph with degrees $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 0$.

ALGORITHM B (Modification of Hakimi's Algorithm for complete graphs with loops allowed)

Let $v(p_i)$ be the vertex of degree p_i where $i=1,2,\ldots,n$ and

k be the number of non-negative terms in the sequence p.

STEP 0 : $k \leftarrow n$; $v(p_i) \leftarrow i$.

STEP 1 : IF $k=0$ THEN a DCS has been constructed; GOTO STEP 2.

IF $p_1 > k+1$ THEN a DCS does not exist; GOTO STEP 2.

IF $p_1 \geqslant 2$ THEN { create a loop at vertex $v(p_1)$; $p_1 \leftarrow p_1 - 2$ }.

For $j=2$ TO p_1+1 DO

{create an edge between vertices $v(p_1)$ and $v(p_j)$; $p_j \leftarrow p_j - 1$}

$p_1 \leftarrow 0$.

Reorder the updated sequence p and the corresponding vertices $v(p_1),\ldots,v(p_k)$ so that p is non-increasing; update k.

GOTO STEP 1.

STEP 2 : HALT.

Corollary 5.4 : ALGORITHM B finds a graph of n vertices with degree sequence $p = (p_1, p_2, \ldots, p_n)$ if $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_r > 0$ and p is realizable, and reports the non-existence if p is not realizable.

5.2. DCS for Bipartite Graphs

There are a number of ways to solve the DCS problem for bipartite graphs. Here we present two algorithms; one uses the maximum network flow algorithm of Dinic [Di70] and the other generalizes the maximum cardinality matching algorithm of Hopcroft and Karp [HK73]. Both of them run in $O(|E|^{3/2})$ time where $|E|$ is the number of edges in the bipartite graph. As a matter of fact the latter one will perform better in general for its time complexity is indeed $O(\sqrt{P}|E|)$ where P is the sum of the prescribed degrees of all vertices. These two algorithms are substantial improvements over the previously known algorithm for the MR problem which is not guaranteed to run in polynomial time.

5.2.1. Maximum Network Flow Algorithm

A network is a directed graph $N=(V,A)$ with two distinguished vertices, a source s and a sink t, and a positive capacity $c(x,y)$ associated with every arc $(x,y)\in A$ (for convenience we will abbreviate $c((x,y))$ by $c(x,y)$ for the arc $(x,y)\in A$). A flow in a network N is a function $f:A\rightarrow R$ which satisfies the following conditions:

(1) For every $v\in V-\{s,t\}$, $\sum_{a\in \omega_{in}(v)} f(a) = \sum_{a\in \omega_{out}(v)} f(a)$, where $\omega_{in}(v)$ and $\omega_{out}(v)$ are the sets of arcs in A entering and emanating from vertex v respectively.

(2) For every arc $(v,w)\in A$, $0 \leq f(v,w) \leq c(v,w)$.

The value of the flow f, denoted by $|f|$, is the flow out of the source vertex s, namely $\sum_{(s,v)\in A} f(s,v)$. The maximum flow problem asks for a flow f in N that maximizes $|f|$.

Given a bipartite graph $G=(X\cup Y,E)$ with prescribed degrees $p(v)$ on all vertices $v\in X\cup Y$. Then $N(G) = (V,A)$ is a network constructed as follows:

$V = \{s,t\} \cup X \cup Y$,

$A = \{(s,x) \mid x\in X\} \cup \{(y,t) \mid y\in Y\} \cup \{(x,y) \mid xy\in E\}$ and

$$c(a) = \begin{cases} 1 & \text{if } a=(x,y)\in A,\ x\in X \text{ and } y\in Y \\ p(x) & \text{if } a=(s,x)\in A \text{ and } x\in X \\ p(y) & \text{if } a=(y,t)\in A \text{ and } y\in Y \end{cases}$$

For example, consider the bipartite graph G shown in Figure 5.3a. Its corresponding network $N(G)$ is shown in Figure 5.3b.

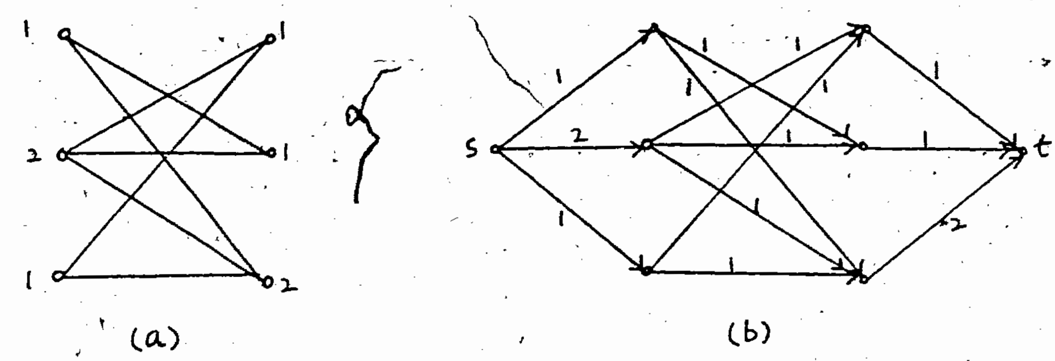

Figure 5.3 : Example of a bipartite graph and its corresponding network

Lemma 5.5 : The bipartite graph G has a subgraph whose vertices have degrees equal to the prescribed degrees if and only if the corresponding network $N(G)$ has an integral flow of value $P/2$, where P is the sum of the prescribed degrees of all vertices in G.

Proof : Let $p(v)$ be the prescribed degree of vertex v in G.

Suppose $\bar{G} = (X \cup Y, \bar{E})$ is a subgraph of G whose vertices have degrees equal to the prescribed degrees.

Let f be an integral flow of the network $N(G)$ defined as follows:

$f(s,x) = p(x)$ for all arcs $(s,x) \in A$ and $x \in X$,

$f(y,t) = p(y)$ for all arcs $(y,t) \in A$ and $y \in Y$,

$f(x,y) = 1$ for all arcs $(x,y) \in A$, $xy \in \bar{E}$, $x \in X$ and $y \in Y$ and

$f(x,y) = 0$ for all arcs $(x,y) \in A$, $xy \notin \bar{E}$, $x \in X$ and $y \in Y$.

The value of the flow f is the sum of the flow out of the source s, which is equal to

$\sum_{(s,x) \in A} f(s,x) = p(X) = p(Y) = P/2$.

Suppose $N(G)$ has an integral flow f of value $P/2$.

Consider the subgraph $\bar{G} = (X \cup Y, \bar{E})$ where

$\bar{E} = \{xy \mid (x,y) \in A, x \in X, y \in Y \text{ and } f(x,y) = 1\}$.

Since the flow f has value $P/2 = p(X) = p(Y)$, the amount of flow on all the arcs emanating from the source s and entering the sink t is equal to the capacities. The degree of every vertex $x \in X$ is then equal to $\sum_{(x,y) \in A} f(x,y) = f(s,x) = p(x)$. Similarly, the degree of every vertex $y \in Y$ is equal to $\sum_{(x,y) \in A} f(x,y) = f(y,t) = p(y)$. Therefore the vertices in the subgraph $\bar{G}$ have degrees equal to the prescribed degrees.

The proof indicates how the network flow solution can yield a DCS solution. Now we apply the maximum flow algorithm of DINIC [Di70] to find a DCS of a bipartite graph G with prescribed degrees. Since all capacities are integers, Dinic's algorithm will produce an integral maximum flow.

ALGORITHM C (Extension of Dinic's Algorithm for Bipartite Graphs)

STEP 0: Construct the network $N(G)$ corresponding to the bipartite graph G.

STEP 1: Find a maximum flow f of $N(G)$ by Dinic's Algorithm.

STEP 2: IF $|f| = P/2$, THEN construct the DCS $\bar{G}$ from the flow f. OTHERWISE a DCS does not exist.

STEP 3: HALT.

The construction of the network $N(G)$ takes $O(|E|)$ time where $|E|$ is the number of edges in the bipartite graph G. The construction of the DCS $\bar{G}$ from the flow f of value $P/2$ also takes $O(|E|)$ time. In order to get the complexity for the entire algorithm, we need to find out how much time STEP 1 takes to find a maximum flow f of $N(G)$. In general Dinic's algorithm runs in $O(|V|^2|E|)$ time where $|V|$ and $|E|$ are the number of vertices and edges of the network respectively. However if the network is of special type, the time complexity may be improved. A 0-1 network is a network in which the capacity of all the arcs is one. Our network $N(G)$ constructed from G is very similar to a 0-1 network except that the capacities of the arcs emanating from s and entering t may not be all one. Fortunately, we can replace each of those arcs of capacity greater than one by a number of arcs of capacity one to obtain a 0-1 network and yet the maximum flows on both networks are the same. This is shown in Figure 5.4, where the arc $a=(v,w)$ of capacity $k>1$ has been replaced by k arcs of capacity 1 from vertex v to vertex w. Clearly the newly constructed network, denoted by $N'(G) = (V',A')$, is a 0-1 network where $|V'| = |V|$ and $|A'| = |A|+P-|X \cup Y|$. It is not difficult to see that the values of the maximum flows in the networks $N(G)$ and $N'(G)$ are the same. Moreover, any flow in $N'(G)$ corresponds to a flow in $N(G)$ of the same value. Thus we may work on the 0-1 network $N'(G)$ instead of $N(G)$ to find the maximum flow of value $P/2$. This can be done by Dinic's algorithm in $O(|A'|^{3/2})$ for the 0-1 network $N'(G)$. Since $|A'| = |A|+P-|X \cup Y| = |E|+P = O(|E|)$, an execution of STEP 1 of ALGORITHM C takes $O(|E|^{3/2})$ time. Hence the entire algorithm is of time complexity $O(|E|^{3/2})$.

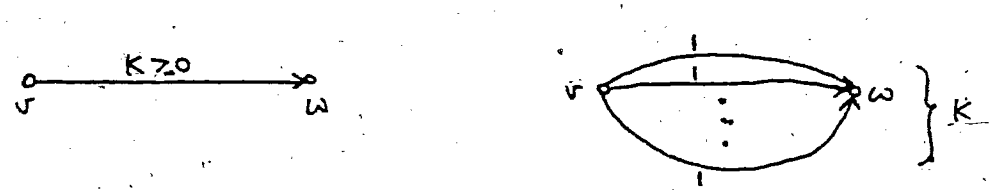

Figure 5.4 : Replacing an arc by multiple arcs to obtain a 0–1 network

5.2.2. Generalized Maximum Matching Algorithm

Let $G=(X \cup Y, E)$ be a bipartite graph having vertex set $X \cup Y$ and edge set E. A set $M \subseteq E$ is called a matching if no vertex $v \in X \cup Y$ is incident with more than one edge in M. A perfect matching M has every vertex incident with exactly one edge of M. In other words, a perfect matching is the edge set of a DCS of G with all the prescribed degrees equal to one. In this section we are going to generalize the maximum matching algorithm of Hopcroft and Karp [HK73] to solve the DCS problem with prescribed degrees $p(v)$ on all vertices v in the bipartite graph G.

Let M be a p-subgraph of G, i.e., a subgraph of G in which all vertices have degrees less than or equal to the prescribed degrees. A maximum p—subgraph is one that has the maximum number of edges. A simple path $Q = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ in G is an augmenting path relative to M if the end vertices v_1 and v_{2k} have degrees less than the prescribed degrees $p(v_1)$ and $p(v_{2k})$, all intermediate vertices $v_2, \ldots, v_{2k-1}$ have degrees equal to the prescribed degrees $p(v_2), \ldots, p(v_{2k-1})$, and the edges are alternatively in $E-M$ and in M with starting edge in $E-M$.

When there is no ambiguity, we let Q denote the set of edges in the augmenting path as well as the path itself and let M denote the set of edges in the p-subgraph as well as the subgraph itself. If S and T are sets, then $S \oplus T$ denotes the symmetric difference of S and T, $S-T$ denotes the set of elements in S which are not in T, and $|S|$ denotes the number of elements in S if S is finite. If M and N are graphs, then $M \oplus N$ denotes the symmetric difference of the sets of edges of M and N, $M-N$ denotes the set of edges in M which are not in N, and $|M|$ denotes the number of edges in the graph M. An augmentation of M along the augmenting path Q is achieved by taking the symmetric difference of the edge sets of M and Q. Clearly $M \oplus Q$ is also a p-subgraph and $|M \oplus Q| = |M|+1$.

Theorem 5.6 : Let M and N be p-subgraphs of G. If $|M|=m$, $|N|=n$ and $n \geq m$, then $M \oplus N$ contains at least $n-m$ edge disjoint augmenting paths relative to M.

Proof : Consider the graph $(M \oplus N)^{sep}$ formed by replicating each vertex $v \in M \oplus N$ $p(v)$ times. Let v be a vertex in $M \oplus N$, let $v_1, v_2, \ldots, v_{p(v)}$ be the replicated vertices arising from v in $(M \oplus N)^{sep}$, and let v have r edges from $M-N$ and s edges from $N-M$. Then the incidence of the $r+s$ edges with vertices $v_1, v_2, \ldots, v_{p(v)}$ is defined as follows. Each of the vertices $v_1, v_2, \ldots, v_r$ is incident with one of the r edges from $M-N$ and each of the vertices $v_1, v_2, \ldots, v_s$ is incident with one of the s edges from $N-M$. Thus each vertex in $(M \oplus N)^{sep}$ is incident with at most one edge from $N-M$ and at most one edge from $M-N$. Hence each connected component of $(M \oplus N)^{sep}$ is either

(1) an isolated vertex,

(2) a cycle of even length with edges alternatively in $M-N$ and in $N-M$, or

(3) a path whose edges are alternatively in $M-N$ and in $N-M$.

In any of the connected components, the number of edges belonging to N is either one more than, one less than, or equal to, the number of edges belonging to M. A component has one more edge belonging to N than to M if and only if it is an alternating path with starting and ending edges in $N-M$. There are at least $n-m$ of these components since $|N-M| - |M-N| = |N| - |M| = n-m$. Hence there exist at least $n-m$ vertex disjoint "alternating" paths, $Q_1^{sep}, Q_2^{sep}, \ldots, Q_{n-m}^{sep}$, in $(M \oplus N)^{sep}$ with starting and ending edges in $N-M$. Consider an end vertex of any of these paths; it is incident with one edge in $N-M$. By the definition of $(M \oplus N)^{sep}$, the corresponding vertex, say v, in $M \oplus N$ must be incident with more edges in $N-M$ than in $M-N$. In other words, the degree of the vertex v in the p-subgraph N is more than that in the p-subgraph M. But the vertices in both M and N have degrees less than or equal to the prescribed degrees, so the degree of the end vertex v in M must be less than the prescribed degree.

Now we claim that the vertex disjoint alternating paths. $Q_1^{sep}, Q_2^{sep}, ..., Q_{n-m}^{sep}$, in $(M \oplus N)^{sep}$ correspond to paths $Q_1, Q_2, \ldots, Q_{n-m}$ in $M \oplus N$ which are edge disjoint. and such that each Q_i contains an augmenting path relative to M. Let Q_i be the path in $M \oplus N$ consisting of all the edges of the alternating path Q_i^{sep}. Then all the paths $Q_1, Q_2, ..., Q_{n-m}$ are edge disjoint. This follows from the fact that every edge in $M \oplus N$ appears exactly once in $(M \oplus N)^{sep}$. It remains to show that each Q_i contains an augmenting path relative to M. Since Q_i^{sep} has odd length, Q_i is an odd length trail (possibly self intersecting). But $M \oplus N \subseteq G$ is bipartite, so Q_i contains an odd length simple subpath Q_i' which is evidently also alternating and has the same set of vertices. Recall that the end vertices of the paths Q_i have degrees (in M) less than the prescribed degrees. If all of the intermediate vertices of Q_i' have degrees (in M) equal to the prescribed degrees, then Q_i' is in fact an augmenting path relative to M. Otherwise by parity, we can always find an odd length subpath in Q_i' so that the end vertices have degrees (in M) less than the prescribed degrees. Repeatedly, we must find a subpath whose end vertices have degrees (in M) less than the prescribed degrees and all the intermediate vertices have degrees (in M) equal to the prescribed degrees, and thus an augmenting path relative to M. Hence $M \oplus N$ contains at least $n-m$ edge disjoint augmenting paths relative to M.

The following Theorems and Corollaries are similar to [HK73].

Corollary 5.7 : M is a maximum p–subgraph if and only if there is no augmenting path relative to M.

Corollary 5.8 : Let M be a p–subgraph. Suppose $|M|=m$, and suppose that the number of edges in a maximum p–subgraph is n, $n>m$. Then there exists an augmenting path relative to M of length $\leq 2\lfloor \frac{m}{n-m} \rfloor+1$.

Proof : Let N be a maximum p–subgraph. Then by Theorem 5.6, $M\oplus N$ contains $n-m$ edge disjoint augmenting paths relative to M. Altogether these contain at most m edges from M, so one of them must contain at most $\lfloor \frac{m}{n-m} \rfloor$ edges from M, and hence at most $2\lfloor \frac{m}{n-m} \rfloor+1$ edges together.

Let M be a p–subgraph. The augmenting path Q is called shortest relative to M if the length of Q is the shortest among all the augmenting paths relative to M.

Corollary 5.9 : Let M be a p–subgraph, Q a shortest augmenting path relative to M, and Q' an augmenting path relative to $M\oplus Q$. Then $|Q'| \geq |Q| + 2|Q\cap Q'|$.

Proof : Let $N = M\oplus Q\oplus Q'$. Then N is a p–subgraph and $|N| = |M|+2$, so by Theorem 5.6, $M\oplus N$ contains two edge disjoint augmenting paths relative to M: call them Q_1 and Q_2. Since $M\oplus N = Q\oplus Q'$ and $|M\oplus N| \geq |Q_1|+|Q_2|$, $|Q\oplus Q'| \geq |Q_1|+|Q_2|$. But $|Q_1|\geq|Q|$ and $|Q_2|\geq|Q|$, since $|Q|$ is a shortest augmenting path. So $|Q\oplus Q'| \geq |Q_1| + |Q_2| \geq |Q| + |Q| = 2|Q|$, and also we have the identity $|Q\oplus Q'| = |Q| + |Q'| - 2|Q\cap Q'|$. Hence

$$|Q'| = |Q\oplus Q'| - |Q| + 2|Q\cap Q'| \geq 2|Q| - |Q| + 2|Q\cap Q'| = |Q| + 2|Q\cap Q'|.$$

Now we apply the following scheme of computation : starting with a p–subgraph $M_0=\emptyset$, compute a sequence $M_0, M_1, M_2, \ldots, M_i, \ldots$, where Q_i is a shortest augmenting path relative to M_i, and $M_{i+1} = M_i\oplus Q_i$.

Corollary 5.10 : $|Q_i| \leqslant |Q_{i+1}|$.

Corollary 5.11 : For all i and j such that $|Q_i|=|Q_j|$, Q_i and Q_j are edge disjoint.

Proof : Suppose that $|Q_i|=|Q_j|$, $i<j$, and Q_i and Q_j are not edge disjoint. Then there must exist k and l, $i \leqslant k < l \leqslant j$, so that Q_k and Q_l are not edge disjoint, and for each m, $k<m<l$, Q_m is edge disjoint from Q_k and Q_l. Then Q_l is an augmenting path relative to $M_k \oplus Q_k$, so $|Q_l| \geqslant |Q_k|+2|Q_k \cap Q_l|$. But $|Q_l|=|Q_k|$, so $|Q_k \cap Q_l|=0$, which is a contradiction.

Theorem 5.12 : Let n be the number of edges in a maximum p-subgraph. The number of distinct integers in the sequence

$|Q_0|, |Q_1|, \cdots, |Q_i|, \cdots$ is less than or equal to $2\lfloor \sqrt{n} \rfloor + 2$.

Proof : Let $m = \lfloor n-\sqrt{n} \rfloor$, and let M_m be the p-subgraph with m edges. Then by Corollary 5.8,

$$|Q_m| \leqslant 2\left\lfloor \frac{\lfloor n-\sqrt{n} \rfloor}{n-\lfloor n-\sqrt{n} \rfloor} \right\rfloor + 1 = 2\left\lfloor \frac{n-\lceil \sqrt{n} \rceil}{\lceil \sqrt{n} \rceil} \right\rfloor + 1$$

$$= 2\left\lfloor \frac{n}{\lceil \sqrt{n} \rceil} - \frac{\lceil \sqrt{n} \rceil}{\lceil \sqrt{n} \rceil} \right\rfloor + 1 \leqslant 2\left\lfloor \frac{n}{\lceil \sqrt{n} \rceil} \right\rfloor + 1$$

$$\leqslant 2\lfloor \sqrt{n} \rfloor + 1$$

Thus for each $i<m$, $|Q_i|$ is one of the $\lfloor \sqrt{n} \rfloor+1$ positive odd integers less than or equal to $2\lfloor \sqrt{n} \rfloor+1$. Also $|Q_{m+1}|, \ldots, |Q_n|$ contribute at most $n-m = n-\lfloor n-\sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$ distinct integers, hence the total number of distinct integers is less than or equal to $\lfloor \sqrt{n} \rfloor + 1 + \lceil \sqrt{n} \rceil \leqslant 2\lfloor \sqrt{n} \rfloor + 2$.

In view of Corollaries 5.10 and 5.11 and Theorem 5.12, the computation of the sequence $M_0, M_1, ..., M_t, \cdots$ breaks into at most $2\lfloor\sqrt{n}\rfloor+2$ phases, and within each all the augmenting paths found are edge disjoint and of the same length. Since these paths may fail to be vertex disjoint, an augmenting path relative to the p—subgraph with which the phase is begun need not be one of the augmenting paths within the phase. So we have to be cautious when we are going to find all the augmenting paths of the same length within a phase at the time the phase is begun. The following is an alternative way of describing the computation of a maximum p—subgraph, and hence a DCS.

ALGORITHM D (Generalization of Hopcroft and Karp's Algorithm for Bipartite Graphs)

STEP 0: $M \leftarrow \varnothing$.

STEP 1: Let $l(M)$ be the length of a shortest augmenting path relative to M.

Find a maximal set of paths $\{Q_1^M, Q_2^M, ..., Q_k^M\}$ with the properties that

(1a) For each i, Q_i^M is an augmenting path relative to M such that

$M \oplus Q_1^M \oplus \cdots \oplus Q_i^M \oplus \cdots Q_k^M$ is a p—subgraph and $|Q_i^M| = l(M)$.

(1b) The Q_i^M are edge disjoint.

IF no such paths exist, THEN GOTO STEP 3.

STEP 2: $M \leftarrow M \oplus Q_1^M \oplus Q_2^M \oplus \cdots \oplus Q_k^M$; GOTO STEP 1.

STEP 3: IF $|M|=P/2$, THEN M is a DCS. OTHERWISE a DCS does not exist.

STEP 4: HALT.

Corollary 5.13 : If the number of edges in a maximum p—subgraph is n, then ALGORITHM D constructs a maximum p—subgraph within $2\lfloor\sqrt{n}\rfloor+2$ executions of STEP 1.

Corollary 5.14 : ALGORITHM D constructs a DCS with prescribed degrees within $2\lfloor \sqrt{P/2} \rfloor + 2$ executions of STEP 1 if one exist, where P is the sum of the prescribed degrees of all vertices.

Let M be a p–subgraph of the bipartite graph $G=(X\cup Y,E)$, $p(v)$ be the prescribed degree of vertex $v\in X\cup Y$, and let $d(v)$ be the degree of vertex v in M. Let $X_< = \{x \mid d(x)<p(x)$ and $x\in X\}$ and $X_= = \{x \mid d(x)=p(x)$ and $x\in X\}$. The sets $Y_<$ and $Y_=$ are defined similarly. We discuss the efficient implementation of STEP 1 of ALGORITHM D in which a maximal set of augmenting paths satisfying properties (1a) and (1b) is found. First we assign directions to the edges of the graph so that the augmenting paths become directed paths. This is done by directing each edge in $E-M$ from a vertex in X to a vertex in Y, and each edge in M from a vertex in Y to a vertex in X. The resulting graph is described as follows:

$G_M = (V_M,E_M)$ where

$V_M = X\cup Y$ and

$E_M = \{(x,y) \mid xy\in E-M,\ x\in X \text{ and } y\in Y\} \cup \{(y,x) \mid xy\in M,\ x\in X \text{ and } y\in Y\}$.

Next we extract a subgraph G'_M of G_M in which the directed paths with starting vertex in $X_<$, ending vertex in $Y_<$ and all intermediate vertices in $X_=\cup Y_=$ correspond one-to-one to the shortest augmenting paths relative to M. This is done as follows.

Let $L_0 = X_<$ and let

$E_i = \{(u,v) \mid (u,v)\in E_M,\ u\in L_i, \text{ and } v\notin L_0\cup L_1\cup\cdots\cup L_i\}$ for $i = 0,1,2,3,\ldots$

$L_{i+1} = \{v \mid \text{ for some } u,\ (u,v)\in E_i\}$ for $i = 0,1,2,3,\ldots$

Let $i^* = \min\{i \mid L_i\cap Y_< \neq\emptyset\}$.

Then $G'_M = (V'_M,E'_M)$ where

$V'_M = L_0 \cup L_1 \cup \cdots \cup (L_{i^*} \cap Y_<)$ and

$E'_M = E_0 \cup E_1 \cup \cdots \cup E_{i^*-2} \cup \{(u,v) \mid (u,v) \in E_{i^*-1}, u \in L_{i^*-1} \text{ and } v \in Y_<\}$.

All directed paths in G'_M from a vertex in $X_<$ to a vertex in $Y_<$ are the shortest augmenting paths relative to the p-subgraph M. However for every vertex $x \in X_<$ there are only at most $p(x)-d(x)$ of them that start at vertex x and need to be augmented. Also for every vertex $y \in Y_<$ there are only at most $p(y)-d(y)$ of them that end at vertex y and need to be augmented. Therefore we adjoin to G'_M two new vertices s and t, $p(x)-d(x)$ edges from s to every vertex x in $X_<$, and $p(y)-d(y)$ edges to t from every vertex y in $Y_<$. A maximal set of edge disjoint directed paths from s to t in G'_M is then a maximal set of edge disjoint augmenting paths $\{Q_1^M, Q_2^M, \ldots, Q_k^M\}$ so that $M \oplus Q_1^M \oplus Q_2^M \oplus \cdots \oplus Q_k^M$ is still a p-subgraph of G. The mechanism for finding a maximal set of edge disjoint directed paths from s to t in G'_M is straightforward depth first search, which takes $O(\text{number of edges in } G'_M) = O(|E'_M|+P) = O(|E|+P) = O(|E|)$ time. Hence the execution of STEP 1 of ALGORITHM D runs in time $O(|E|)$ and the entire algorithm has time complexity $O(\sqrt{P}\,|E|)$ or $O(|E|^{3/2})$.

CHAPTER 6

CONCLUSIONS

The MR problem was previously studied by Cox and Ernst in 1982. They showed that an MR(1,1,1) always exists by modeling the problem as a Capacitated Transportation problem. The algorithm for obtaining an MR(1,1,1) appeals to the algorithms for solving the Capacitated Transportation problem which are not guaranteed to run in polynomial time. In this thesis we investigate the GR problem which is more general than the MR problem as we have shown that the MR problem is linearly equivalent to the GR problem on a special type of graphs, namely the bipartite graphs. An alternative (graph theoretic) proof on the existence of a GR(1,1,1) for bipartite graphs is given. Yet a GR(1,1,1) may not exist for arbitrary graphs. Fortunately with the constraints being relaxed, a GR(1,2,1/2) and a GR(1,1,$(n+1)/2$) always exist for an arbitrary undirected graph of n vertices; a GR(1,4/3,1/2) and a GR(1,1,$(n+3)/6$) always exist for a loopless undirected graph of n vertices. The bounds in the constraints for these roundings are indeed tight in the sense that there are examples of connected graphs achieving our bounds asymptotically. We have constructed examples in which any rounding Λ that satisfies the constraints $|\Lambda(e)-\lambda(e)| < 1$ and $|\Lambda(v)-\lambda(v)| \leqslant 1$ will have $|\Lambda(E)-\lambda(E)|$ asymptotically close to $(n+1)/2$ for an arbitrary undirected graph, and asymptotically close to $(n+3)/6$ for a loopless undirected graph. For an arbitrary $\epsilon>0$, there are examples in which any rounding Λ that satisfies the constraints $|\Lambda(e)-\lambda(e)| < 1$ and $|\Lambda(E)-\lambda(E)| \leqslant 1/2$ will have $\max |\Lambda(v)-\lambda(v)| = 2-\epsilon$ for an arbitrary undirected graph, and $\max |\Lambda(v)-\lambda(v)| = 4/3-\epsilon$ for a loopless undirected graph.

The different roundings for the GR problem can be obtained by solving a certain DCS and a certain Euler Tour problem on undirected graphs. We have presented two algorithms for solving the DCS problem: one is the extended Maximum Network Flow algorithm of Dinic, and the other is the generalized Maximum Cardinality Matching algorithm of Hopcroft and Karp. Both of them have time complexity $O(|E|^{3/2})$ in the worse case analysis where $|E|$ is the number of edges in the graph. In general the latter one will perform better as it runs in $O(\sqrt{P}\,|E|)$ time where P is the sum of the prescribed degrees of all vertices in the graph. This is a ~~substantial~~ improvement over the previously known algorithm developed by Cox and Ernst [CE82] which may have exponential behavior. If the graphs are complete, i.e., if the rounding problems are 0-relaxed, then we have simpler linear algorithms.

From our results on the existence of a rounding for undirected graphs, the existence of a symmetric rounding for symmetric matrices is derived. An SMR(2,2,$\underline{1}$) and an SMR(2,$\underline{1}$,$\underline{n+1}$) always exist for an arbitrary symmetric matrix of dimension n; an SMR(1,4/3,$\underline{1}$) and an SMR(1,$\underline{1}$,$\underline{(n+3)/3}$) always exist for a symmetric matrix of dimension n with all diagonal entries being 0. These roundings can also be obtained by our algorithms for the GR problem.

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