CERTAIN WEAKLY MINIMAL THEORIES

by

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CERTAIN WEAKLY MINIMAL THEORIES

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ABSTRACT

A weakly minimal structure is one in which any non-algebraic strong type has a unique non-algebraic extension over any set. If p is a non-algebraic strong type in a saturated weakly minimal structure M, and aEb if and only if a and b are algebraic in each other, we can give $p(M)/E$ the structure inherited from M. We suppose:

(*) In the structure $p(M)/E$ the algebraic closure operation is non-trivial, but the algebraic closure of a finite set is finite.

Many known examples of weakly minimal theories have strong types satisfying (*). We find in this case that there is an almost 0-definable equivalence relation θ and formula $\varphi(x) \in p$ such that $\varphi(M)/\theta$ has formulas giving it the structure of either an affine or projective space over a finite field. We determine exactly what further structure is possible on $\varphi(M)/\theta$.

If we further assume that M has no two orthogonal types, we find θ as above and get a global structure theorem for M/θ.

ACKNOWLEDGEMENTS

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INTRODUCTION

In [Mo], Morley proved Łoś's conjecture that a countable theory T categorical in one uncountable cardinal is categorical in every uncountable cardinal. Shelah, in [Sh, 1], generalized this to uncountable T, proving that if T is categorical in some $\kappa > |T|$, it is categorical in every $\kappa > |T|$. To do this he invented the notion of weakly minimal sets. In retrospect, the study of weakly minimal sets is quite natural, as they are precisely the superstable structures of U-rank 1. U-rank is due to Lascar [Las]. Here we investigate a subclass of the class of weakly minimal sets.

Strongly minimal sets, a special case of weakly minimal sets, were investigated earlier than general weakly minimal sets first by Marsh [Mar], and later by Baldwin and Lachlan [BL]. In the latter, strongly minimal sets are used to show that a countable $\aleph_1$-categorical theory has either one or $\aleph_0$ countable models up to isomorphism. Even more special are the strictly minimal sets which in the $\aleph_0$-categorical case have been classified by Zil'ber [Z1, Z2] and Cherlin [CHL] as affine or projective spaces over finite fields, or disintegrated sets. Strictly minimal sets are widely used in [CHL] to study $\aleph_0$-categorical, $\aleph_0$-stable structures. The weakly minimal sets studied here have a strong resemblance to $\aleph_0$-categorical strictly minimal sets.

General weakly minimal sets have been investigated by Andler [A], and rather more recently and deeply by Buechler [Bu 1-8]. Buechler has considered and exploited the geometrical nature of these sets — if we take as points the algebraic closures of singletons, the algebraic closure operation naturally yields a geometry on any weakly minimal set. Usually this geometry is restricted to the collection of realizations of some strong type. Here we concern ourselves with weakly minimal sets M such that there is a non-algebraic strong type p such that:

(*) For any $N \succ M$ and finite $F \subseteq p(N)$ there is a finite $G \subseteq p(N)$ such that $ac\ell(F) \cap p(N) = \bigcup_{g \in G} ac\ell\{g\} \cap p(N)$.

That is, the geometry described above is locally finite. Equivalently, the geometry is either disintegrated or an affine or projective space over a finite field. See 3.4(1).

Perhaps Buechler's crucial result is that if p(N) is as above, either p has Morley rank 1 or is locally modular. Local modularity means that for any closed X and Y with $X \cap Y \neq \phi$, $\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$. This gives, between any two non-orthogonal strong types, an algebraic relation after naming at most one element realizing each.

The following comes essentially from [Bu 7]: Suppose M is weakly minimal and locally modular, Th(M) contains only countably many k-types over ϕ for each $k \in \omega$, and p is a strong type such that p(M) is not contained in a strongly minimal set. Then p satisfies (*). So if we have the above hypotheses, the work that follows here applies.

Work of Hrushovski [Hr] also has a direct bearing on weakly minimal sets. While his work is not in final form, at the time of writing it seems safe to say that his results imply the following:

Suppose M is saturated, weakly minimal, uni-dimensional, Th(M) is countable but M is not $\aleph_0$-stable. Then there is a strong type p and E, an almost 0-definable equivalence relation with finite classes, such that $p(M)/E$ has an almost 0-definable abelian group structure. As in Section 6 of this paper, the group structure exists on $\varphi(M)/E$ for some $\varphi(x) \in p$. The result of the present paper can be seen as a special case of this, although in our case we get more specific information.

The importance of weakly minimal sets in the study of uni-dimensional superstable theories is indicated by the following theorem of Buechler:

If M is an LMFR structure, $\{a\} \cup A \subseteq M$ and $a \notin acl(A)$, then for some $N \succ M$ and $c \in N^{eq}$, $U(c/A) = 1$ and $c \in acl(Aa)$.

An LMFR structure has every weakly minimal type locally modular (LM), and every type of finite U-rank (FR). N^{eq} is a structure canonically associated with N having names for definable subsets. Every non-$\aleph_0$-stable, superstable, uni-dimensional structure is LMFR. This result is similar to the Coordinatization Theorem of [CHL], which gives considerable information about $\aleph_0$-categorical, $\aleph_0$-stable structures.

The naturalness and amenability to study of superstable uni-dimensional structures, at least those without the omitting types order property, is indicated by the following consequence of Shelah's general classification machinery:

For a complete theory T, there is a cardinal κ such that in any cardinal λ T has $\leq \kappa$ non-isomorphic models of size λ iff T is superstable, uni-dimensional and has NOTOP. For countable such T, the minimum such κ is either 1 or $\beth_2$.

The standard example of a uni-dimensional, weakly minimal, non-$\aleph_0$-stable structure is the following, due essentially to Morley:

$M = (F(n)^{\omega}; +, \cdot\alpha, P_{k,\alpha}: k \in \omega, \alpha \in F(n))$.
Here $F(n)$ is the finite field of n elements, $(F(n)^{\omega}; +, \cdot\alpha: \alpha \in F(n))$ is a vector space over $F(n)$ with operations defined pointwise, and $P_{k,\alpha}(\eta)$ iff $\eta(k) = \alpha$ for $\eta \in F(n)^{\omega}$. If $N \succ M$ is saturated and p is a non-algebraic strong type over ϕ, $p(N)$ is an affine or projective space over $F(n)$. Also, the algebraic closure of a finite set is finite. This is naturally lost if infinitely many points of N are named, but our condition (*) remains. We will find that in some sense this is the only example of a weakly minimal, uni-dimensional theory with a strong type satisfying (*).

Another example of a weakly minimal, uni-dimensional, non-$\aleph_0$-stable structure is $(Z,+)$, the integers as an abelian group. Andler credits recognition of this example to Harnik. If N is a saturated elementary extension of $(Z,+)$ and p is a strong type, $p(N)$ is an affine or projective space over the rationals. A third example, due to Hrushovsky, is somewhat similar to Morley's except that in a strong type there is an affine or projective space over $\bigcup_{k\in\omega} F(p^k)$, where p is some prime. This example seems somehow intermediate to the previous two.

SECTION 1

PRELIMINARIES AND NOTATION

In this section we will define the basic notions and fix notation for the rest of the paper. Much of the section is devoted to a description of affine and projective spaces over finite fields. While these are well-understood, the usual description of them is as two-sorted structures, and we want a one-sorted description.

We will assume the reader has some familiarity with stability theory, although little of the deep theory will in fact be used here. Indeed, except for quoting the Classification Theorem for $\aleph_0$-categorical strictly minimal sets in section 3, most of the proofs will use nothing deeper than the Compactness Theorem.

We will require from stability theory the notions of strong type, algebraic closure, M^{eq}, and of course weak minimality. We will denote models as well as their underlying sets by M or N with various decorations, and in one case by K. Aut(M) denotes the group of automorphisms of M. Our structures will always be infinite.

M^{eq} is the structure obtained from M by adjoining, for each $k \in \omega$ and each 0-definable equivalence relation E on M^k, a point for each class of M^k/E. In M^{eq} we have, for each $k \in \omega$ and E as above, a 0-definable function taking $(a_1,\ldots,a_k)$ to $(a_1,\ldots a_k)/E$ for $a_1,\ldots,a_k \in M$. For further details see [Sh,2] or [Ma]. Much of our notation comes from the latter.

We write $\varphi(M,\bar{a})$ for the set of realizations in M of a formula $\varphi(x,\bar{a})$. Similarly $p(M)$ if p is a collection of formulas. $|M|$ is the cardinality of M.

Suppose $A \subseteq M$; a type over A is <u>algebraic</u> if it is realized by only finitely many elements in any $N \succ M$. $acl(A)$, the <u>algebraic closure</u> of A, is the set of realizations of all algebraic types over A — $acl(A)$ will usually be taken in M^{eq}. A <u>strong type</u> over A is a complete type over $acl(A)$ taken in M^{eq}. The term "strong type" will refer to a non-algebraic strong 1-type over ϕ unless otherwise indicated. If $\bar{a},\bar{b}$ realize the same type (strong type) over C, we will write $\bar{a} \equiv \bar{b}(C)$ ($\bar{a} \equiv {}^s\bar{b}(C)$). Often we will ignore the distinction between sets, sequences and singletons, writing such things as $a \equiv {}^s b(c)$ and $a \in acl(\bar{c})$. An equivalent definition of $\bar{a} \equiv {}^s\bar{b}(C)$ is that for every C-definable equivalence relation E with finitely many classes, $E(\bar{a},\bar{b})$.

<u>Definition 1.1</u>: (1) A structure M is <u>weakly minimal</u> if for any $N \succ M$, $a,b \in N$ and $C \subseteq N$ with $a \equiv^s b$ and $a,b \notin acl(C)$, we have $a \equiv^s b(C)$.

(2) M is <u>strongly minimal</u> if it is weakly minimal and for any $N \succ M$ and $a,b \in N \setminus acl(\phi)$, $a \equiv^s b$.

(3) M is <u>strictly minimal</u> if it is strongly minimal, Aut(M) is transitive on M, and for each $a \in M$, $acl\{a\} \cap M = \{a\}$.

<u>Remark</u>: Weak minimality is equivalent to being superstable and of U-rank 1. Strong minimality is equivalent to being superstable and of U-rank and multiplicity 1.

<u>Proposition 1.2</u>: M is weakly minimal if and only if the following holds:

For any formula $\varphi(x,\bar{y})$ without parameters there are unary $\theta_\varphi^1(x),\ldots,\theta_\varphi^k(x)$ with parameters from $acl(\phi)$ taken in M^{eq}, such that $\theta_\varphi^1(M),\ldots,\theta_\varphi^k(M)$ partition M and such that:

For all $N \succ M$, $\bar{a} \in N^{eq}$ and $1 \le i \le k$, either $\theta_\varphi^i(N) \cap \varphi(N,\bar{a})$ or $\theta_\varphi^i(N) \setminus \varphi(N,\bar{a})$ is finite.

<u>Proof</u>: ($\Leftarrow$) Suppose the condition holds, $N \succ M$, $a,b \in N$, $C \subseteq N$ and $a \equiv^s b$ but $a,b \notin acl(C)$. Suppose $N \models \varphi(a,\bar{c})$ where $\bar{c} \subseteq acl(C)$. Choose $\theta_\varphi^1,\ldots,\theta_\varphi^k$ as given by the condition and suppose $N \models \theta_\varphi^i(a)$. Since $a \notin acl(\bar{c})$, $\theta_\varphi^i(N) \cap \varphi(N,\bar{c})$ is infinite, so $\theta_\varphi^i(N) \setminus \theta(N,\bar{c})$ is finite. Since $b \notin acl(\bar{c})$ and $N \models \theta_\varphi^i(b)$, $N \models \varphi(b,\bar{c})$. Since φ was arbitrary, $a \equiv^s b(C)$, and since a,b,C and N were arbitrary, M is weakly minimal.

($\Rightarrow$) Suppose M is weakly minimal and let $\varphi(x,\bar{y})$ be a formula without parameters, and $N \succ M$ be saturated. For any strong type p and $\bar{a} \subseteq N^{eq}$, either $p(N) \cap \varphi(N,\bar{a})$ or $p(N) \setminus \varphi(N,\bar{a})$ is a subset of $acl(\bar{a})$ by the weak minimality, and hence finite by the saturation. So we can find by compactness $\theta_{\varphi,p,\bar{a}} \in p$ such that $\theta_{\varphi,p,\bar{a}}(N) \cap \varphi(N,\bar{a})$ or $\theta_{\varphi,p,\bar{a}}(N) \setminus \varphi(N,\bar{a})$ is finite. Again using the saturation, for each p the following collection of formulas is inconsistent:

$$\{\exists^{\ge m}x(\theta_{\varphi,p,\bar{a}}(x) \wedge \varphi(x,\bar{y})) \wedge \exists^{\ge m}x(\theta_{\varphi,p,\bar{a}}(x) \wedge \neg\varphi(x,\bar{y})) : m \in \omega,\ \bar{a} \subseteq N^{eq}\}$$

Here $\exists^{\ge m}x$ abbreviates "there are at least m x's".

Using compactness we find a finite subset of the above set inconsistent. Let $\theta_{\varphi,p}(x) \in p$ be the conjunction of the $\theta_{\varphi,p,\bar{a}}$'s occurring in this subset. For any $\bar{a} \subseteq N^{eq}$ we have either

$\theta_{\varphi,p}(N) \cap \varphi(N,\bar{a})$ or $\theta_{\varphi,p}(N) \setminus \varphi(N,\bar{a})$ finite. By compactness, we can choose a finite set of $\theta_{\varphi,p}$'s covering N and then appropriate Boolean combinations of these yield the required θ^i_φ's. As N is saturated, the proof is complete.

Remarks: (1) Full saturation of N is not really needed in the above proof, but it is direct from weak minimality that in any cardinal $\kappa \geq 2^{|Th(M)|}, |M|$ there is $N \succ M$ saturated of size κ.

(2) From this characterization it is clear that if M is weakly minimal and $\varphi(x,\bar{y})$ is an M-formula, we cannot find $\{\bar{a}_i : i \in \omega\}$ with each $\varphi(M,\bar{a}_i)$ infinite and $\varphi(M,\bar{a}_i) \cap \varphi(M,\bar{a}_j) = \phi$ for $i \neq j \in \omega$.

(3) The original definition of weak minimality is similar to the above condition, except that the θ^i_φ's were allowed parameters from M. That this is equivalent to the above condition follows from a normalization argument.

(4) We will use the condition of this proposition without mention at various points of the paper.

On a weakly minimal structure, the algebraic closure operation has the following nice geometric properties, of which only exchange requires weak minimality:

(1) $A \subseteq acl(A)$;

(2) (transitivity) If $a \in acl(B)$ and $B \subseteq acl(C)$, then $a \in acl(C)$;

(3) (monotonicity) If $A \subseteq B$, $acl(A) \subseteq acl(B)$;

(4) (finite basis) If $a \in acl(A)$, there is finite $B \subseteq A$ such that $a \in acl(B)$;

(5) (exchange, a.k.a. forking symmetry) If $a \in acl(A \cup \{b\}) \setminus acl(A)$, then $b \in acl(A \cup \{a\})$.

We will sometimes write $a \not\downarrow_A b$ for $a \in acl(A \cup \{b\}) \setminus acl(A)$ and $a \downarrow_A b$ for the negation. The above properties allow us to meaningfully introduce the notion of independent sets — $A \subseteq M$ is independent if for no $a \in A$ is $a \in acl(A \setminus \{a\})$ — and of basis, spanning set and well-defined dimension.

We now consider a general method of obtaining other structures from a given one.

Definition 1.3: If M is a structure, $A \subseteq M$ and E is an equivalence relation on A, the structure induced by formulas (of M) on A/E is the structure with universe A/E and, for each formula $\psi(x_1,\ldots x_k)$ of M without parameters, a predicate P_ψ.
$A/E \models P_\psi(a_1/E,\ldots,a_k/E)$, where $a_1,\ldots,a_k \in A$, if and only if there are $a_i^* \in A$ with $a_i^* E a_i$ for $1 \le i \le k$ and $M \models \psi(a_1^*,\ldots,a_k^*)$.

Remarks: (1) If the terminology "structure induced by formulas" is used and no E is mentioned, it is assumed to be the identity.
(2) If both A and E are definable, this construction is harmless. More interesting uses come when at least one is not definable.

In fact, the first thing done in Section 3 is to apply the above where $A = p(M)$, for some strong type p, and aEb if and only if $a \in acl\{b\}$. In addition to assuming M is weakly minimal, we will be assuming that p satisfies the following:
(*) For any $N \succ M$ and $G \subseteq p(N)$ finite, there is $H \subseteq p(N)$ finite such that $acl(G) \cap p(N) = \bigcup_{h \in H} acl\{h\} \cap p(N)$.

Any type we subject to scrutiny will in fact be assumed to satisfy (*). When we take the structure induced by formulas on $p(M)/E$ we find we have an $\aleph_0$-categorical strictly minimal structure, which brings us to:

Theorem 1.4 (The Classification Theorem for $\aleph_0$-categorical Strictly Minimal Sets): If M is an $\aleph_0$-categorical strictly minimal structure, then $(M, ac\ell)$, where $ac\ell$ is the algebraic closure operation, is isomorphic to one of the following:

(1) The degenerate geometry on M;

(2) An affine geometry of infinite dimension over a finite field;

(3) A projective geometry over a finite field.

This is Theorem 2.1 of [CHL].

Here the degenerate geometry has $ac\ell(A) = A$ for any $A \subseteq M$; in this case M effectively cannot have any structure at all. We will rarely consider this case.

The isomorphism here is a geometric one; if $(M, ac\ell)$ is isomorphic to an affine geometry we mean there is an affine structure on the set M and for any $A \subseteq M$, $ac\ell(A) = \langle A \rangle$ where the latter notation means the affine closure of A. Similarly in the projective case.

Definition 1.5: (1) An affine structure over $F(n)$, where $F(n)$ is the finite field of n elements, is a structure M with a ternary predicate R and a 4-ary predicate Q satisfying the following axioms:

#0 $R(x,x,y) \rightarrow y = x$

#1 $\exists x,y \exists^{!n} z R(x,y,z)$

(Here $\exists^{!n} z$ abbreviates "there are exactly n z".)

#2 $R(x,y,x)$

#3 $R(x,y,z) \rightarrow R(y,x,z)$

#4 $R(x,y,z) \wedge R(x,y,w) \wedge x \neq z \rightarrow R(x,z,w)$

#5 $\exists! w Q(x,y,z,w)$

#6 $Q(x,y,x,y)$

#7 $Q(x,y,z,w) \rightarrow Q(x,z,y,w) \wedge Q(y,x,w,z)$

#8 $Q(x,y,z,w) \wedge R(x,y,u) \wedge Q(x,z,u,v) \rightarrow R(z,w,v)$

#9 $Q(x,y,z,x) \rightarrow R(x,y,z)$

#10 $Q(x,y,z,w) \wedge Q(x,y,u,v) \rightarrow Q(z,w,u,v)$

#11 $Q(x,y,z,w) \wedge R(x,y,u) \wedge u \neq x \rightarrow \exists v[R(u,z,v) \wedge R(y,w,v)]$

(2) A <u>projective structure over</u> $F(n)$ is a structure M with a ternary predicate S satisfying the following axioms:

#0 $S(x,x,y) \rightarrow y = x$

#1 $\exists x,y \exists^{!n+1} z S(x,y,z)$

#2 $S(x,y,x)$

#3 $S(x,y,z) \rightarrow S(y,x,z)$

#4 $S(x,y,z) \wedge S(x,y,w) \wedge x \neq z \rightarrow S(x,z,w)$

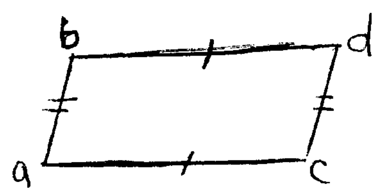
#5 $S(x,y,z) \wedge S(x,u,v)$ and x,y,z,u,v distinct

$\rightarrow \exists w(S(y,u,w) \wedge S(z,v,w))$.

<u>Remark:</u> In the above definitions, we make no restrictions on what other structure M may have.

We shall justify that these axioms give the usual notions of affine, respectively projective, space in an appendix. For now we content ourselves with describing what they mean. $R(a,b,c)$ and $S(a,b,c)$ should both be read "a,b and c are on the same line". $Q(a,b,c,d)$ is best read as "b plus c equals d, with a as zero"; in fact when we have an affine structure we will often write $b +_a c = d$ for $Q(a,b,c,d)$. Any structure M with a 4-ary Q satisfying #'s 5,6,7 and 10 has an abelian group structure on it; pick any $a \in M$ and let $b + c = d$ iff $Q(a,b,c,d)$ as above. #'s 5,6,7 and 10 imply that this gives an abelian group operation with a as zero. Similarly, we will sometimes write "$c = \alpha b + (1-\alpha)a$ for some $\alpha \in F(n)$" instead of $R(a,b,c)$ or just "$c = \alpha b$" if a is taken as zero, since there is little difference between an affine and a vector space over $F(n)$. Abusing terminology, we will call the structure with universe M and predicate Q an abelian group.

Geometrically, for a,b,c not collinear, $Q(a,b,c,d)$ means d "completes the parallelogram":

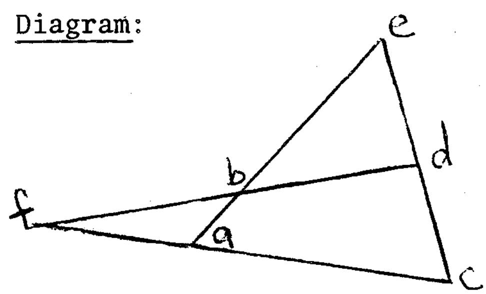


Here $\overline{ab} \| \overline{cd}$ and $\overline{ac} \| \overline{bd}$.

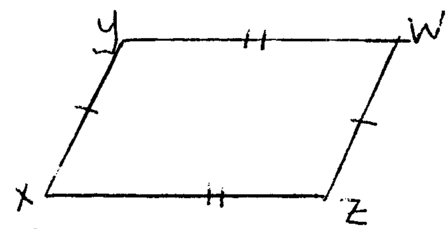
Axiom #1 says that some line has exactly n, respectively n+1, points in the affine, respectively projective, case. What #'s 2 and 3 say is obvious. #4 says that a line is determined by any two distinct points on it.

In the projective case, #5 says that no two lines are parallel — if a,b,c and d are in the same plane in the sense that there is e with $S(a,b,e)$ and $S(e,c,d)$ then the line through a and c meets the line through b and d.

Diagram:

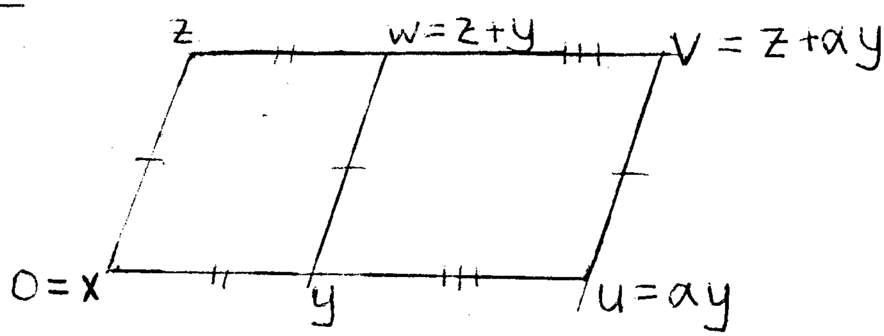


In the affine case, what #'s 5 and 6 say is clear. #7's meaning is clear from the diagram



#8 is also best understood by a

Diagram:



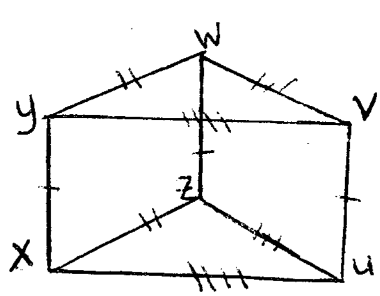
#8 can also be understood as saying $z + \alpha y = \alpha(z+y) + (1-\alpha)z$.

#9 says that $y, -y$ and zero are collinear.

#10 can be understood as saying (with x as zero) that

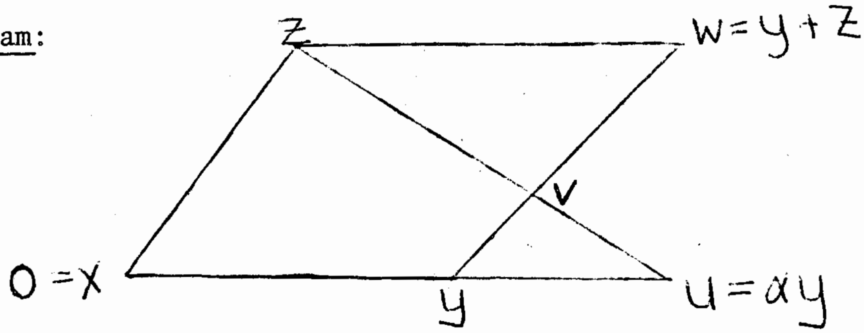
$y+z=w \wedge y+u=v \rightarrow z+v=w+u$, or that if $\overline{xy}||\overline{zw}$, $\overline{xy}||\overline{uv}$, $\overline{xz}||\overline{yw}$ and $\overline{xu}||\overline{yv}$, then $\overline{zw}||\overline{uv}$ and $\overline{zu}||\overline{wv}$.

Diagram:



#11 is again best understood by a

Diagram:



#11 can also be understood algebraically as saying
$\alpha^{-1}(\alpha y) + (1-\alpha^{-1})z = \alpha^{-1}y + (1-\alpha^{-1})(y+z)$.

Where we have an affine or projective structure, we will feel free to use any theorems true about them. For more details, the reader is referred to such combinatorial works as [Hir] and [Ha].

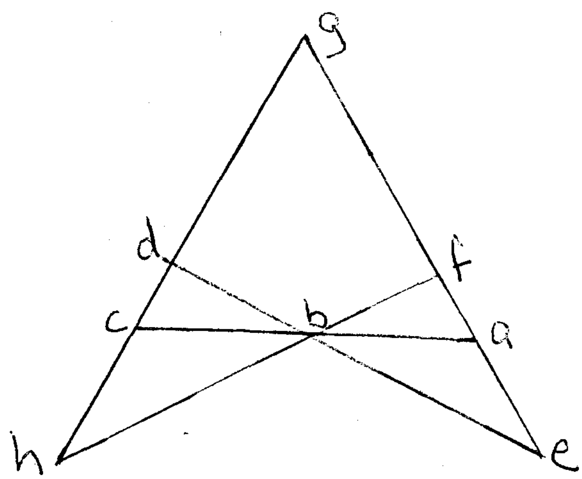
For $A \subseteq M$, $\langle A \rangle$ is the closure under Q and R of A if M is an affine space, $\langle A \rangle$ is the closure under S if M is projective. We will use the following frequently:

Proposition 1.6: (1) If M is an affine structure, and $C \neq \phi$, then $a \in \langle\{b\} \cup C\rangle$ if and only if either there is $c \in \langle C \rangle$ with $R(c,b,a)$ or there are $d,c \in \langle C \rangle$ with $Q(d,c,b,a)$, for any a,b,C in M. (2) If M is a projective structure, and $C \neq \phi$, $a \in \langle\{b\} \cup C\rangle$ if and only if there is $c \in \langle C \rangle$ such that $S(c,b,a)$.

Proof: (1) Pick any point in C and label it 0. Then $\langle\{b\} \cup C\rangle = \{\alpha b + c : \alpha \in F(n)$ and $c \in \langle C \rangle\}$. Let $a \in \langle\{b\} \cup C\rangle$; find $\alpha \in F(n)$ and $c \in \langle C \rangle$ with $a = \alpha b + c$. If $\alpha = 1$, $Q(0,c,b,a)$. If $\alpha \neq 1$, $a = \alpha b + (1-\alpha)(1-\alpha)^{-1}c$ and $(1-\alpha)^{-1}c \in C$. In the second case, $R(b,(1-\alpha)^{-1}c,a)$. The converse is clear.

(2) If $c \in \langle C \rangle$ and $S(c,b,a)$, certainly $a \in \langle \{b\} \cup C \rangle$. For the converse, we show that $\{a: \exists c \in \langle C \rangle S(b,c,a)\}$ is closed under S, which suffices. So suppose $c,d \in \langle C \rangle$, $S(b,c,a) \wedge S(b,d,e) \wedge S(a,e,f)$. We assume a,b,c,d and e are distinct; the other cases are similar. Find, by #5 of 1.5(2), g such that $S(c,d,g) \wedge S(a,e,g)$. Then $g \in \langle C \rangle$, and #3 gives $S(a,g,f)$. Now $S(a,f,g) \wedge S(a,b,c)$ give, by #5, h with $S(f,b,h) \wedge S(c,g,h)$. So $h \in \langle C \rangle$ and $S(h,b,f)$.

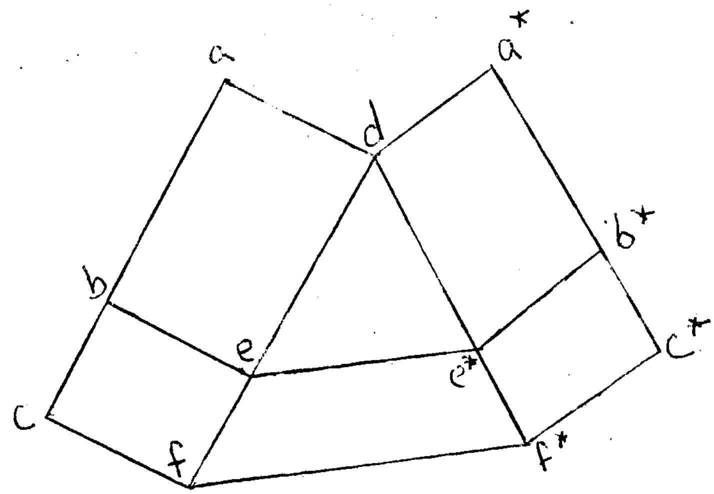
Diagram:



Remarks: (1) We can prove (1) above directly from #'s 0 to 11 of 1.5(1) as we did (2) above, but there would be several cases to check. (2) In the above proof, and throughout the paper, if $\psi(\bar{x})$ is an M-formula, $\bar{a} \subseteq M$ and M is understood, we write just $\psi(\bar{a})$ for $M \models \psi(\bar{a})$.

As is well known, the field underlying an affine or projective space can be canonically identified from the space. Suppose M is an affine space, $a,b,a^*,b^* \in M$ with $\langle a,b,a^*,b^* \rangle$ of dimension 4 and $R(a,b,c)$. Pick $d \notin \langle a,b,a^*,b^* \rangle$ and then e and f with $Q(a,b,d,e) \wedge Q(a,c,d,f)$. Then pick e^* with $Q(a^*,b^*,d,e^*)$ and then f^* with $R(d,e^*,f^*)$ and the line through e and e^* parallel to the line through f and f^*; then choose c^* with $Q(d,a^*,f^*,c^*)$.

Diagram:

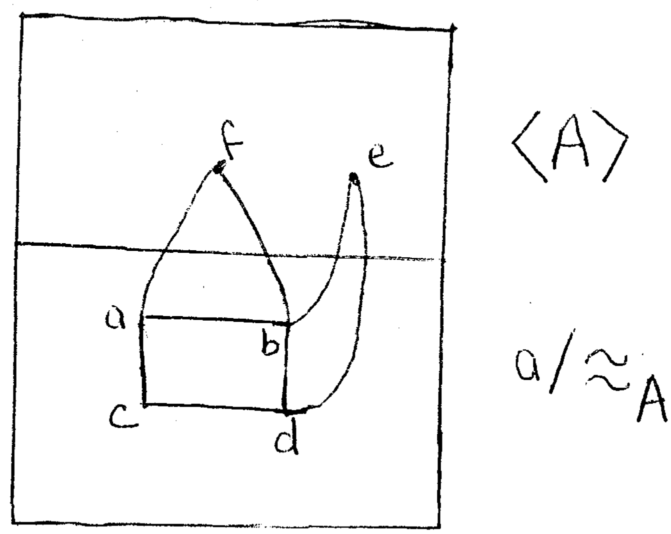


We will have $R(a^*,b^*,c^*)$ and c^* uniquely determined by a,b,c,a^* and b^* — the choice of d is irrelevant. The above construction gives an equivalence relation δ on triples (a,b,c) with $R(a,b,c) \wedge a \neq b$. It has n classes, and is 0-definable.

Notation 1.7: (1) The equivalence classes of the δ just described will be called the field elements. They will be in $\mathrm{acl}(\phi)$ taken in M^{eq}.
(2) There is a similar equivalence relation on 4-tuples (a,b,c,d) with a,b,c distinct and $S(a,b,c) \wedge S(a,b,d)$ in a projective space. The classes will again be called the field elements.

Given a projective space M and any $A \subseteq M$, there is a natural equivalence relation on $M \setminus \langle A \rangle$ called the localization of M at A and denoted $\approx_A$. We have, for $a,b \in M \setminus \langle A \rangle$, $a \approx_A b$ iff $S(a,c,b)$ for some $c \in \langle A \rangle$. S naturally induces a projective structure on $(M \setminus \langle A \rangle)/\approx_A$. Each class of $\approx_A$ is an affine space if we set R to be the restriction of S to the class and let $Q(a,b,c,d)$ iff for some $e,f \in \langle A \rangle$, $S(a,e,b) \wedge S(c,e,d)$ and $S(a,f,c) \wedge S(b,f,d)$ on the class.

Diagram:



For any subspace A of M, the $\approx_A$-classes will be referred to as "affine subspaces" of M, abusing terminology.

Given an affine structure M, there is a canonical way of deriving a projective structure from it.

Definition 1.8: If M is an affine structure over F(n), let $M' = [M^2 \setminus \{(a,a): a \in M\}]/\sim$ where $(a,b) \sim (c,d)$ iff there is e with R(c,d,e) and Q(a,b,c,e). ~ is a 0-definable equivalence relation, so $M' \subseteq M^{eq}$. Let $F: M^2 \setminus \{(a,a): a \in M\} \to M'$ be the projection. M' is called the projective companion of M, as it is a projective structure over F(n). We have $S((a,b)/\sim,(a,c)/\sim,(a,d)/\sim)$ if and only if $d \in \langle a,b,c \rangle$ in M.

In fact, if M is affine, M ∪ M' is naturally a projective space; we extend S to M ∪ M' as follows:

For a,b,c ∈ M', S(a,b,c) in M ∪ M' iff S(a,b,c) in M'. For a,b,c ∈ M, S(a,b,c) iff R(a,b,c). For a,b ∈ M, c ∈ M', S(a,b,c) iff F(a,b) = c. This essentially takes care of all possible cases. M' is a maximal subspace of M' ∪ M.

There is no canonical way of creating an affine space from a projective one, but the preceding several paragraphs indicate a non-canonical way. Suppose M is a maximal subspace of the projective space N. Then $N \setminus M$ consists of a single $\approx_M$-class, so is an affine space. In the structure $(N;M)^{eq}$ — here $(N;M)$ has a predicate for M — there is a 0-definable bijection between M and $(N \setminus M)'$. As this construction is not canonical, care must be exercised when it is applied in Section 5.

The notation above will be used later. So $(M;A)$ refers to the structure M with a predicate for the set A. (M,A) will have constants for the elements of A — as these are quite similar notationally, we will remind the reader whenever the former is used.

If p and q are non-algebraic types (or strong types) over some set A, we say p and q are <u>non-orthogonal</u> if there are $B \subseteq p(N)$, $C \subseteq q(N)$ each independent over A, such that $B \cup C$ is not independent over A. We say our structure M is <u>uni-dimensional</u> if for any two types p and q, p and q are non-orthogonal.

We record the following [CHL, Corollary 2.5]:

<u>Proposition 1.9</u>: If H_0, H_1 are 0-definable non-orthogonal strictly minimal sets in an $\aleph_0$-categorical structure M such that each is either disintegrated or projective over a finite field, there is in M a 0-definable bijection between them.

SECTION 2

Our purpose here is simply to state the results proved in the present paper and to give an outline of the proofs.

We start with M, a weakly minimal structure, and p, a strong type satisfying our condition (*) but having non-trivial dependence relation. The main result here is that there is θ, an almost 0-definable equivalence relation on M with finite classes, and $\varphi(x) \in p$ such that $\varphi(M)/\theta$ is an affine or projective structure over a finite field. That is, either there are almost 0-definable predicates Q and R on $\varphi(M)/\theta$ satisfying #'s 0 through 11 of 1.5(1), or there is almost 0-definable S on $\varphi(M)/\theta$ satisfying #'s 0 through 6 of 1.5(2).

Furthermore, we can specify what further structure is possible on $\varphi(M)/\theta$. For simplicity, assume $acl^{M^{eq}}(\phi)$ is named. Suppose $\varphi(M)/\theta$ is affine. Let M_1 be the structure with universe $\varphi(M)/\theta$, predicates Q and R and a unary predicate for each 0-definable subspace of $\varphi(M)/\theta$. Then for some $C \subseteq M_1^{eq}$, (M_1,C) is interdefinable with $\varphi(M)/\theta$ under the full structure inherited from M. If $\varphi(M)/\theta$ is projective, let M_1 have predicates S and a predicate for each 0-definable subspace X and each $\approx_X$-class. Recall that $a \approx_X b$ iff $S(a,c,b)$ for some $c \in X$. Again, there is $C \subseteq M_1^{eq}$ such that $\varphi(M)/\theta$ and (M_1,C) are interdefinable.

If we make the further assumption that M is uni-dimensional, then we can find an almost 0-definable equivalence relation θ such that we can give a global structure theorem for M/θ. Again assume for simplicity that $acl^{M^{eq}}(\phi)$ is named. Then we can find X_i $(i < k \in \omega)$, Y and

$G \subseteq M^{eq}$ such that:

(i) M/θ is either $G \cup \bigcup_{i<k} X_i$ or $G \cup Y \cup \bigcup_{i<k} X_i$;

(ii) Each X_i is an affine space with structure as specified for $\varphi(M)/\theta$ above;

(iii) Y is a projective space as specified above;

(iv) For each $i < k$, there is a bijection between $X_i^!$ and Y;

(v) G is finite;

(vi) All of the above is 0-definable.

The proof of the above results goes in several steps, which we now outline.

First, we assume our structure M is saturated, and consider the structure induced by formulas on $p(M)/E$, where aEb iff $a \in acl\{b\}$. (*) on p implies that this is an $\aleph_0$-categorical strictly minimal set, and the non-triviality of the dependence relation tells us it is not disintegrated. By the Classification Theorem for $\aleph_0$-categorical strictly minimal sets, $p(M)/E$ is then either an affine or projective space over a finite field. Next we find M-formulas inducing the affine or projective structure on $p(M)/E$. Next we show that $p(M)$ in the affine case, or $p(M)$ plus some algebraic points in the projective case, is a strongly minimal set in the structure induced by formulas of M. This is all done in Section 3.

The next step is to take a transitive strongly minimal set M such that M/E (= $p(M)/E$ in the previous paragraph) is an affine $\aleph_0$-categorical strictly minimal set, and show that the affine structure lifts to M/θ where θ has finite classes and is almost 0-definable;

this is perhaps the most difficult part of the paper. We find predicates Q and R on M that induce the affine structure on M/E, and after a little adjustment find that the additive structure lifts. Using this, we show that any definable binary predicate B with B(a,M) and B(M,a) finite for $a \in M$ has a definable transitive closure. We know that while Q and R may not quite give M an affine structure, they must be "close" as they induce the structure on M/E. So on an appropriate factor M/θ they induce affine structure. This is in Section 4.

Also in Section 4 we show that M/θ has little other structure. Specifically, if M/θ is strongly minimal and has an affine structure over a finite field on it which induces the dependence relation, we will see that effectively only some points of M/θ or $(M/\theta)'$ can be named. There's no room for other structure.

The fifth section is devoted to doing much the same thing for the projective case. We assume M is strongly minimal and if p is the non-algebraic strong type, p(M)/E is a strictly minimal projective space over a finite field. We find an M-formula S inducing the projective structure. The basic result here is that there is θ, a 0-definable equivalence relation on M with finite classes, and a finite 0-definable set G such that S induces projective structure on $(M \setminus G)/\theta$. To do this we find $N \succ M$ such that $N = acl(\{a\} \cup M)$ for $a \in N \setminus M$, and consider the structure (N;M) with a predicate for M. We find this has rank 1 and finite multiplicity, and any strongly minimal piece of it besides M is precisely what we started with in the previous section. So a suitable factor K of it is an affine space, and the relationship S gives between K and M implies there is a factor of M (less a finite set) which is an almost 0-definable bijection with K'. Section 5 concludes with the result that any strongly minimal projective structure arising as above can have essentially no other structure except possibly some set of constants named. Again we go through $N = acl(\{a\} \cup M)$ as above.

Returning to the situation where our structure M is assumed weakly rather than strongly minimal, we assume that there is a strong type p such that either $p(M)$ carries an affine structure induced by M-formulas, or that a set P consisting of $p(M)$ and some algebraic points carries a projective structure induced by M-formulas. In the second case P is the projective closure of $p(M)$. We find that for any $\lambda(x) \in p$ there is $\varphi(x) \in p$ such that the affine or projective structure extends to $\varphi(M)$, and $\varphi(M) \setminus \lambda(M)$ consists of a finite set of algebraic points. In each case the proof is through a series of successive approximations. In the affine case, we find that the strong types in $\varphi(M)$ are essentially affine subspaces which are translates of $p(M)$. In the projective case, if P is as above, the strong types are essentially the classes of the relation $\approx_p$.

In the seventh section, we study what further structure is possible on $\varphi(M)$ as in the previous paragraph. Actually we consider the projective case as the affine case can then be read off by considering $\varphi(M) \cup (\varphi(M))'$. We assume then that M is a weakly minimal projective structure with a strong type p such that $\langle p(M) \rangle = P$ is a projective subspace on which we know the complete structure as in Section 5. Section 6 tells us that any point in every definable subspace is in P, and so that if a and b are non-algebraic and $a \approx_X b$ for each 0-definable subspace $X \supseteq P$, then $a \equiv^s b$. From this and our knowledge of P, we then find that M can only have structure as specified for $\varphi(M)/\theta$ at the beginning of this section.

In Section 8, we first notice that if p and q are non-orthogonal strong types in a weakly minimal set and p satisfies (*), so does q. Using this, we consider the case where M, in addition to being weakly minimal and having a strong type satisfying (*), is uni-dimensional. From the above and compactness we can find $Z_1,\ldots,Z_k$ and $\theta_1,\ldots,\theta_k$ such that $M \setminus \bigcup Z_i$ is finite and each Z_i/θ_i is like $\varphi(M)/\theta$ described at the beginning of this section. From this, it is largely a matter of "straightening out" to get the final result.

SECTION 3

In this section we get off the ground. We take M, a saturated weakly minimal structure, and p a strong type on M satisfying our condition (*). Recall that we mean p is a non-algebraic strong 1-type over ϕ. Then if we factor out the algebraic closure of points in p(M) and transfer the structure from M, we find we have an $\aleph_0$-categorical strictly minimal set. We choose formulas on M inducing the structure on this factor of p(M) and devote the bulk of the section to showing that p(M) along with a collection of algebraic points is a strongly minimal set.

To begin, let M be a saturated weakly minimal structure of size $> |Th(M)|$ and p be a strong type. For $a,b \in p(M)$ let aEb if and only if $a \in ac\ell\{b\}$; this gives an Aut(M)-invariant equivalence relation on p(M). We place the following structure on p(M)/E: For each M-formula $\psi(x_1,\ldots,x_k)$ without parameters, we have a predicate $P_\psi(x_1,\ldots,x_k)$ on p(M)/E; if $a_i \in p(M)$ $(i=1,\ldots,k)$ let $a_i' = a_i/E$ and by definition $p(M)/E \models P\ (a_1',\ldots,a_k')$ if and only if $M \models \psi(a_1^*,\ldots,a_k^*)$ for some $a_i^*Ea_i (i=1,\ldots,k)$. We will use the notation a' for a/E where $a \in p(M)$ from now on; also if $\bar{a} = \langle a_1,\ldots,a_k\rangle$, $\bar{a}' = \langle a_1/E,\ldots,a_k/E\rangle$, etc. This is the structure induced by formulas on p(M)/E.

<u>Proposition 3.1</u>: Any $\alpha \in Aut(M)$ fixing p(M) setwise induces an automorphism of p(M)/E.

<u>Proof</u>: Suppose $p(M)/E \models P_\psi(\bar{a}')$ and $\alpha \in Aut(M)$. Then for some $\bar{a}^*E\bar{a}$, $M \models \psi(\bar{a}^*)$. So $M \models \psi(\alpha(\bar{a}^*))$ and so $p(M)/E \models P_\psi(\alpha(\bar{a}^*)')$.

Since $\alpha(\bar{a}^*)' = \alpha(\bar{a}')$, we are done.

Corollary 3.2 (1) If $\bar{a} \equiv \bar{b}$ in M, then $\bar{a}' \equiv \bar{b}'$ in $p(M)/E$ for $\bar{a},\bar{b} \subseteq p(M)$.

(2) If $a \notin acl(\bar{b})$ in M, $a' \notin acl(\bar{b}')$ in $p(M)/E$.

Proof: (1) is immediate from 3.1. For (2) we can find $\{a_i : i < |M|\} \subseteq p(M)$ with $a_i \not{E} a_j$ if $i \neq j$ such that each $a_i \equiv a(\bar{b})$ if $a \notin acl(\bar{b})$. But then in $p(M)/E$, $a_i' \equiv a'(\bar{b}')$ and $a_i' \neq a_j'$ if $i \neq j$. So $a' \notin acl(\bar{b}')$.

Until further notice, we assume the following condition on p:

(*) If $N \equiv M$ and $F \subseteq p(N)$ is finite, then there is $G \subseteq p(N)$ finite such that $acl(F) \cap p(N) = \cup\{acl\{g\} : g \in G\} \cap p(N)$.

Remark: Since M is saturated, we can replace N by M in the above. It is stated as it is so that (*) holding on p depends only on p and Th(M), and not on M.

Given (*) on p, we get the converse of 3.2(2).

Lemma 3.3: For $a,\bar{b}$ in $p(M)$, if $a \in acl(\bar{b})$ in M then $a' \in acl(\bar{b}')$ in $p(M)/E$.

Proof: We may assume $\bar{b}$ is independent, as a counter-example with $\bar{b}$ of shortest possible length has $\bar{b}$ independent. Choose $\psi(x,\bar{y})$ with $M \models \psi(a,\bar{b})$ witnessing $a \in acl(\bar{b})$ and $G \subseteq p(M)$ finite with $acl(\bar{b}) \cap p(M) = \bigcup_{g \in G} acl\{g\} \cap p(M)$. Now $p(M)/E \models P_\psi(a',\bar{b}')$ and if $p(M)/E \models P_\psi(c',\bar{b}')$ there is c^*Ec, $\bar{b}^*E\bar{b}$ with $M \models \psi(c^*,\bar{b}^*)$. Since $\bar{b}$ is independent, so is $\bar{b}^*$, so $c^* \in acl(\bar{b}^*) = acl(\bar{b})$. Thus

$c' = (c^*)' \in \{g' : g \in G\}$ and so $a' \in acl(\bar{b}')$.

Lemma 3.4(1): $p(M)/E$ is an $\aleph_0$-categorical strictly minimal structure.

(2) Suppose $p_1, \ldots, p_k$ are distinct strong types of M, all of which satisfy (*), and we define E on $p_1(M) \cup \ldots \cup p_k(M)$ by aEb iff $a \equiv^s b$ and $a \in acl\{b\}$. Then $(p_1(M) \cup \ldots \cup p_k(M))/E$ under the structure induced by formulas is $\aleph_0$-categorical and the union of k strictly minimal sets. For $a, \bar{b}$ in $(p_1(M) \cup \ldots \cup p_k(M))$, $a \in acl(\bar{b})$ iff $a/E \in acl(\bar{b}/E)$.

Proof: (1) If $b' \not\equiv c'(\bar{a}')$ then in M, $b \not\equiv c(\bar{a})$ by 3.2(1). So $b \in acl(\bar{a})$ or $c \in acl(\bar{a})$. So by 3.3, $b' \in acl(\bar{a}')$ or $c' \in acl(\bar{a}')$. Further $b' \in acl(\bar{a}')$ implies $b \in acl(\bar{a}) \cap p(M) = \cup\{acl\{g\} : g \in G\} \cap p(M)$ for a finite $G \subseteq p(M)$; so $b' \in \{g' : g \in G\}$ and $acl(\bar{a}')$ is finite. This implies that there are only finitely many k-types in $p(M)/E$ for any $k \in \omega$, so $p(M)/E$ is $\aleph_0$-categorical. Then the above also tells us $p(M)/E$ is strongly minimal. Also, if $b' \in acl\{a'\}$, $b \in acl\{a\}$ so $b' = a'$.

(2) This is similar to 3.2, 3.3 and (1) above, once we have the following

Claim: If $F \subseteq p_1(M) \cup \ldots \cup p_k(M)$ is finite, there is $G \subseteq p_1(M) \cup \ldots \cup p_k(M)$ finite such that

$$acl(F) \cap [\bigcup_{1 \le i \le k} p_i(M)] = \cup\{acl\{g\} : g \in G\} \cap [\bigcup_{1 \le i \le k} p_i(M)]$$

Proof of Claim: Pick $F_i \subseteq acl(F) \cap p_i(M)$ a maximal independent set. From the superstability of M, each F_i is finite, and $acl(F) \cap p_i(M) = acl(F_i) \cap p_i(M)$. Since (*) holds on p_i, choose

$G_i \subseteq p_i(M)$ finite with $acl(F_i) \cap p_i(M) = \cup\{acl\{g\}: g \in G_i\} \cap p_i(M)$. Then take $G = \bigcup_{1 \le i \le k} G_i$.

Note: This lemma implies that $p(M)/E$ has an essentially countable language, even though no restriction was made on the size of the language of M.

Now we apply the Classification Theorem for $\aleph_0$-categorical strictly minimal structures. See 1.14. We say that p is of affine, projective or disintegrated character respectively if $p(M)/E$ is an affine or projective space over a finite field $F(n)$, or disintegrated respectively.

Suppose p is of affine character over $F(n)$, and let $a,b \in p(M)$ be independent. Then $acl\{a,b\} \cap p(M)$ consists of precisely n E-classes by 3.2(2) and 3.3 since in $p(M)/E$ $acl\{a',b'\}$ has size n. Pick $R(x,y,z)$ an M-formula such that $R(a,b,M)$ consists of a finite set of elements from each of these E-classes, except a' and b'. We assume that for any $c,d,e \in M$, $R(c,d,e)$ implies that each of c,d,e is algebraic in the other two. So P_R defines the lines on $p(M)/E$. Let $a,b,c \in p(M)$ be independent and pick $d \in p(M)$ such that in $p(M)/E$ $b' +_{a'} c' = d'$; if $d^* \in p(M)$ and $d \equiv d^*(\{a,b,c\})$ then $(d^*)' = d'$ since $abcd \equiv abcd^*$ implies $a'b'c'd' \equiv a'b'c'(d^*)'$. Choose $Q(x,y,z,w)$ an M-formula with $Q(a,b,c,M)$ a non-empty subset of d'. We assume $Q(d,e,f,g)$ implies each of d,e,f,g algebraic in the other three for any d,e,f,g in M. Later we will redefine R and Q.

Now assume p is of projective character over $F(n)$; we will keep this assumption until after Corollary 3.16, where we find a set consisting of $p(M)$ plus some algebraic points which is strongly minimal under the structure induced by formulas.

We call a binary M-formula $\chi(x,y)$ an <u>algebraic formula</u> if for any $d,e \in M$, $\chi(d,e)$ implies $d \in \mathrm{ac}\ell\{e\}$ and $e \in \mathrm{ac}\ell\{d\}$. For the remainder of this section, we will assume any algebraic formula also satisfies $\chi(d,M) \cup \chi(M,d) \subseteq p(M)$ for $d \in p(M)$.

Let $a,b \in p(M)$ be independent; then $\mathrm{ac}\ell\{a,b\} \cap p(M)$ contains n-1 E-classes besides a' and b'. Pick an M-formula $S(x,y,z)$ such that $S(a,b,M)$ consists of a finite number of points from each of these E-classes, and such that for any $c,d,e \in p(M)$, $S(c,d,e)$ implies each of c,d,e is algebraic in the other two. We may further assume $S(a,M,b)$ and $S(M,a,b)$ are subsets of $p(M)$ and that $M \models S(x,y,z) \rightarrow S(y,x,z) \wedge S(z,y,x)$; for this, replace $S(x,y,z)$ by $\bigvee_{\sigma \in \mathrm{Sym}\{x,y,z\}} S(\sigma x,\sigma y,\sigma z)$.

We can find an algebraic $\chi(x,y)$ such that if $a,b,c \in p(M)$ are independent, then:

$$(\dagger)\ \forall x,y[S(a,b,x) \wedge S(a,c,y) \rightarrow$$

$$\exists z,z^*(S(b,c,z) \wedge S(x,y,z^*) \wedge \chi(z,z^*))].$$

For pick such a,b,c; for any d,e with $S(a,b,d) \wedge S(a,c,e)$, $d,e \in p(M)$ are independent. Also $P_S(a',b',d') \wedge P_S(a',c',e')$ in $p(M)/E$. So, since $p(M)/E$ is a projective space, there is $f' \in p(M)/E$ with $P_S(b',c',f') \wedge P_S(d',e',f')$. Since $b \perp c$ and $d \perp e$, there are $f^*, f^{**} \in f'$ with $S(b,c,f^*) \wedge S(d,e,f^{**})$; f^*Ef^{**} can be witnessed by an algebraic formula. There are finitely many choices for d and e and hence for f^* and f^{**}, so a single $\chi(x,y)$ can be chosen as claimed.

By compactness, we can find $\varepsilon(x,y)$ algebraic such that if $a,b,c \in p(M)$, $\neg\varepsilon(a,b)$ and $c \notin acl\{a,b\}$, then (†) for a,b,c. Let $S_0(x,y,z) \longleftrightarrow S(x,y,z) \wedge \bigwedge_{u \neq v \in \{x,y,z\}} \neg\varepsilon(u,v)$.

<u>Lemma 3.5</u>: For $a,b \in p(M)$, $S_0(a,b,c)$ implies $c \in p(M)$ or c is algebraic.

<u>Proof</u>: If $a \perp b$, $c \in p(M)$. Suppose $a \not\perp b$. Choose $d \in p(M) \setminus acl\{a\}$. Since $\neg\varepsilon(a,b)$ we can find, for any e with $S(a,d,e)$, f and f^* with $S(b,d,f) \wedge S(c,e,f^*) \wedge \chi(f,f^*)$. $e \in p(M)$ and $f \in p(M)$ since $d \perp b$, so $f^* \in p(M)$. If $f^* \notin acl\{e\}$, $S(f^*,e,c)$ gives $c \in p(M)$. If $f^* \not\perp e$, $c \in acl\{e\}$. But $e \notin acl\{c\}$ since otherwise $d \in acl\{a,e\}$ implies $d \in acl\{a,c\} \subseteq acl\{a,b\} = acl\{a\}$ contrary to our choice. So c is algebraic.

Now if $a,b \in p(M)$ are independent, then for some algebraic $\chi_0(x,y)$,

$$(\dagger\dagger)\ \forall y,z[S_0(a,b,y) \wedge S_0(a,b,z) \rightarrow$$

$$\chi_0(y,z) \vee \exists z^*(S_0(b,y,z^*) \wedge \chi_0(z,z^*))].$$

So by compactness, there is $\delta(x) \in p$ such that if $\delta(a^*)$ and $b \in p(M) \setminus acl\{a^*\}$, then (††) holds for a^*,b. Let $S_1(x,y,z) \longleftrightarrow S_0(x,y,z) \wedge \delta(x) \wedge \delta(y) \wedge \delta(z)$.

<u>Lemma 3.6</u>: Suppose $S_1(a,b,c) \wedge S_1(a,b,d)$, $b \in p(M) \setminus acl\{a\}$ and $c \in p(M)$. Then $d \in p(M)$.

Proof: This is clear if $a \in p(M)$, so by 3.5 we can assume $a \in acl(\phi)$. Also $\delta(a)$, so by (††) either $\chi_0(c,d)$ or there is e with $S_0(b,d,e) \wedge \chi_0(c,e)$. If $\chi_0(c,d)$, $d \in p(M)$. If $S_0(b,d,e) \wedge \chi_0(c,e)$, $e \in p(M)$ and by 3.5 either $d \in p(M)$ or $d \in acl(\phi)$. Since $b \in acl\{a,d\} = acl\{d\}$, $d \notin acl(\phi)$.

$$(\dagger\dagger\dagger)\ \forall x,y[S_1(a,b,x) \wedge S_1(a,c,y) \rightarrow$$

$$\exists z,z^*(S_1(b,c,z) \wedge S_1(x,y,z^*) \wedge \chi(z,z^*)] \quad .$$

holds for $a,b,c \in p(M)$ independent. Find $\varepsilon_1(x,y)$ algebraic such that if $a^*,b^* \in p(M)$, $\neg\varepsilon_1(a^*,b^*)$ and $c \in p(M) \setminus acl\{a^*,b^*\}$ then (†††) holds for a^*,b^*,c. Pick $\delta_1(x) \in p$ such that $\delta_1(a^*)$ and $b \in p(M) \setminus acl\{a^*\}$ imply $\neg\varepsilon_1(a^*,b) \wedge \neg\varepsilon_1(b,a^*)$. Also $\delta_1(a^*) \wedge \delta_1(b^*) \wedge \neg\varepsilon_1(a^*,b^*)$ and $c \in p(M) \setminus acl\{a^*,b^*\}$ imply (†††) for a^*,b^*,c.

Let $S_2(x,y,z) \longleftrightarrow S_1(x,y,z) \wedge \bigwedge_{u \neq v \in \{x,y,z\}} (\neg\varepsilon_1(u,v) \wedge \delta_1(u))$.

Definition 3.7: Let $p^*(M)$ denote $\{a \in M: \exists b \in p(M) \setminus acl\{a\} \exists c \in p(M)\ S_2(a,b,c)\}$.

Lemma 3.8: (1) $p(M) \subseteq p^*(M) \subseteq p(M) \cup acl(\phi)$.

(2) There is a finite set A of algebraic elements of M such that $a,b \in p^*(M)$ and $S_2(a,b,c)$ implies $c \in p^*(M) \cup A$.

Proof: (1) Since $S_2(a,b,c) \rightarrow S_0(a,b,c)$ this follows from 3.5.

(2) Pick any $a \in p(M)$ and let $A = [\cup\{S_1(a,b,M): \varepsilon_1(a,b)\} \cap acl(\phi)] \setminus p^*(M)$. This finite set does not depend on the choice of $a \in p(M)$. Now suppose $a,b \in p^*(M)$ and

$S_2(a,b,c)$; choose $d \in p(M) \setminus acl\{a,b\}$. By definition there are $e,f \in p(M)$ with $S_2(a,d,e) \wedge S_2(b,d,f)$. Also $\neg\varepsilon_1(a,b) \wedge \delta_1(a) \wedge \delta_1(b)$. By (†††) applied to a,b,d there are g and g^* with $S_1(b,d,g) \wedge S_1(c,e,g^*) \wedge \chi(g,g^*)$. Now $S_1(b,d,g) \wedge S_1(b,d,f)$, $d \in p(M) \setminus acl\{b\}$, and $f \in p(M)$ gives $g \in p(M)$ by 3.6. So $g^* \in p(M)$. By 3.5 and $S_0(c,e,g^*)$, $c \in p(M) \cup acl(\phi)$. We can assume $c \notin p(M)$. If $\neg S_2(c,e,g^*)$, by $\delta_1(c)$ and $e,g^* \in p(M)$ we have $\neg\varepsilon_1(c,e) \wedge \neg\varepsilon_1(c,g^*) \wedge \neg\varepsilon_1(e,c) \wedge \neg\varepsilon_1(g^*,c)$, so $\varepsilon_1(e,g^*) \vee \varepsilon_1(g^*,e)$. But then $S_1(c,e,g^*)$ gives $c \in A \cup p^*(M)$. If $S_2(c,e,g^*)$, $c \in p^*(M)$ by definition.

Let $S_3(x,y,z) \longleftrightarrow S_2(x,y,z) \wedge x \notin A \wedge y \notin A \wedge z \notin A$.

Corollary 3.9: $p^*(M)$ is closed under S_3.

Proof: Immediate.

S_3 is symmetric and $P_{S_3} = P_S$ so we forget about the original S and call S_3 S from now on.

Lemma 3.10: There is $\eta(x,y)$ algebraic such that if $a,b \in p(M)$, $a \not\perp b$ and $\neg\eta(a,b)$ then there is $c \in p^*(M) \setminus p(M)$ with $S(a,b,c)$.

Proof: There is η_0 algebraic such that if $a,b,c \in p(M)$ are independent,

$$(**)\ \forall x,y[S(c,a,x) \wedge S(c,b,y) \rightarrow \exists z,z^*(S(a,b,z) \wedge S(x,y,z^*) \wedge \eta_0(z,z^*))].$$

Let η be algebraic such that if $\neg\eta(a,b)$, $a,b,c \in p(M)$ and $c \notin acl\{a,b\}$, then (**) holds. Now suppose $a,b \in p(M)$, $\neg\eta(a,b)$ but $a \not\perp b$. Pick $c \in p(M) \setminus acl\{a,b\}$ and then d with $S(c,a,d)$. In

$p(M)/E$, $P_S(c',a',d')$ and $b'=a'$, so there is eEd with $S(c,b,e)$. By (**) find f and f^* with $S(a,b,f) \wedge S(d,e,f^*) \wedge \eta_0(f,f^*)$. Then $f \in acl\{a\}$; also $f \in acl\{f^*\} \subseteq acl\{d\}$. Since $d \notin acl\{a\}$, $f \in acl(\phi)$.

Lemma 3.11: Suppose $\sigma(x,\bar{a})$ is such that for $b \in p(M) \setminus acl(\bar{a})$, $M \models \sigma(b,\bar{a})$. Then for all but finitely many $b \in p^*(M)$, $M \models \sigma(b,\bar{a})$. In particular, there is a set of formulas p^* over $acl(\phi)$ with $a \in p^*(M)$ iff a realizes each formula of p^*.

Proof: Pick $\tau(x) \in p$ such that for any $\bar{c} \subseteq M$ either $\tau(M) \wedge \sigma(M,\bar{c})$ or $\tau(M) \wedge \neg\sigma(M,\bar{c})$ is finite. By assumption $\tau(M) \wedge \sigma(M,\bar{a})$ is not finite, so $\tau(M) \wedge \neg\sigma(M,\bar{a})$ is. Thus it suffices to prove that $p^*(M) \setminus \tau(M)$ is finite. If $a,b \in p(M)$, $a \perp b$ and $S(a,b,c)$, then $c \in p(M)$ so $\tau(c)$. By compactness, find $\gamma(x,y)$ algebraic such that for $a,b \in p(M)$, $\neg\gamma(a,b) \wedge S(a,b,c) \rightarrow \tau(c)$. The finite set $C = \{c \in M: S(a,b,c)$ for some b with $\gamma(a,b)\} \cap acl(\phi)$ does not depend on the choice of $a \in p(M)$. But if $c \in p^*(M) \setminus \tau(M)$ then $c \in acl(\phi)$; pick any $a,b \in p(M)$ with $S(a,b,c)$. Then since $\neg\tau(c)$, $\gamma(a,b)$, so $c \in C$.

Now for any $\tau(x) \in p$, let $\tau^*(x) \longleftrightarrow \tau(x) \vee x \in p^*(M) \setminus \tau(M)$. By the above, $p^*(M) \setminus \tau(M)$ is a finite algebraic set, so τ^* is almost 0-definable. Clearly $a \in p^*(M)$ iff a realizes each τ^* for $\tau \in p$.

We define formulas $\pi_k(x,\bar{y})$ with $\ell h(\bar{y}) = k$ by induction on $k \geq 1$. $\pi_1(x,y)$ is $x=y$. If $\pi_k(x,\bar{y})$ is defined, let

$$\pi_{k+1}(x,\bar{y}^\frown\langle y_0\rangle) \longleftrightarrow \exists z[\pi_k(z,\bar{y}) \wedge (S(x,y_0,z) \vee x=y_0 \vee x=z)].$$

Lemma 3.12: If $\bar{a} \subseteq p^*(M)$ and $\pi_{\ell h(\bar{a})}(b,\bar{a})$, then $b \in p^*(M)$.

Proof: Immediate from the definition and 3.9.

Lemma 3.13: If $a, \bar{b} \subseteq p(M)$ and $a \in acl\{\bar{b}\}$, then there is $c \in p(M)$ with $\pi_{\ell h(\bar{b})}(c,\bar{b})$ and $a \in acl\{c\}$.

Proof: We go by induction on $\ell h(\bar{b})$, the result being obvious if this is 1. If $a \in acl(\bar{b})$ then in $p(M)/E$, $a' \in acl(\bar{b}')$ by 3.3. Suppose $\bar{b} = \bar{d}\hat{\ }\langle e\rangle$; we can assume $a \notin acl(\bar{d})$, so for some $f \in acl(\bar{d})$, $P_S(f',e',a')$ since $p(M)/E$ is a projective space. See 1.6(2). By induction we can assume $\pi_{\ell h(\bar{d})}(f,\bar{d})$; also since $f \perp e$, $S(f,e,M)$ intersects a', so for some cEa, $S(f,e,c)$. Then $\pi_{\ell h(\bar{b})}(c,\bar{b})$.

Lemma 3.14: There is a finite set $B \subseteq acl(\phi) \setminus p^*(M)$ such that if $a \in p^*(M)$ and $\eta(a,b)$ then $b \in p^*(M) \cup B$.

Proof: Suppose $c,d \in p(M)$ are independent and $S(c,d,e) \wedge \eta(e,f)$; then $P_S(c',d',e')$ so $P_S(c',d',f')$ and $c \perp f$, so there is gEd with $S(c,g,f)$. Given c and d, there are finitely many choices for e, f and g, so we can find $\eta_1(x,y)$ algebraic such that $\chi(c,d)$ where

$$\chi(x,y) \longleftrightarrow \forall z,w[S(x,y,z) \wedge \eta(z,w) \rightarrow \exists v(S(x,v,w) \wedge \eta_1(y,v))].$$

Let $A = \{e \in p^*(M) \cap acl(\phi) : \exists y(S(c,y,e) \wedge \neg\chi(c,y))\}$ for some $c \in p(M)$. There are finitely many $d \in p(M)$ with $\neg\chi(c,d)$, so this set is finite, and does not depend on the choice of $c \in p(M)$. Let $B = \cup\{\eta(a,M) \setminus p^*(M) : a \in A\}$. If $a \in p^*(M) \setminus A$ and $b \in p(M) \setminus acl\{a\}$ choose $c \in p(M)$ with $S(a,b,c)$. We have $\chi(b,c)$ so if $\eta(a,d)$ there is e with $\eta_1(c,e) \wedge S(d,b,e)$; so $e \in p(M)$ and $d \in p^*(M)$. This suffices.

Lemma 3.15: For any $\varphi(\bar{y})$ 0-definable on $p^*(M)$ in the structure induced by formulas, there is $\varphi^*(\bar{y})$ almost 0-definable on M such that for $\bar{a} \subseteq p^*(M)$, $M \models \varphi^*(\bar{a})$ if and only if $p^*(M) \models \varphi(\bar{a})$.

Proof: By induction on $\varphi(\bar{y})$, the only difficult case being $\exists x\theta(x,\bar{y})$ where by hypothesis we can assume $\theta(x,\bar{y})$ is an M-formula almost without parameters. Choose $\sigma(x) \in p$ such that for any $\bar{b} \subseteq M$, either $\sigma(M) \wedge \theta(M,\bar{b})$ or $\sigma(M) \wedge \neg\theta(M,\bar{b})$ is finite; by 3.11 we can assume $\sigma(a)$ for all $a \in p^*(M)$. Now for each k, $0 \leq k \leq \ell h(\bar{y})$ we find $\psi_k(x)$ and $\rho_k(x,\bar{y})$ M-formulas almost without parameters such that $\psi_k(M) \cap p^*(M) = \phi$, $\rho_k(M,\bar{a}) \subseteq p^*(M)$ for all $\bar{a} \subseteq p^*(M)$, and if $\bar{a} \subseteq p^*(M)$ has an independent subset of size $\geq k$, then either $\sigma(M) \wedge \neg\theta(M,\bar{a})$ is finite or $\tau_k(\bar{a})$ where

$$\tau_k(\bar{y}) \longleftrightarrow \forall x(\sigma(x) \wedge \theta(x,\bar{y}) \longrightarrow [\psi_k(x) \vee \rho_k(x,\bar{y})]).$$

We start at $k = \ell h(\bar{y})$. Pick $\langle a_1,\ldots,a_k\rangle = \bar{a}$ independent in $p(M)$; if $\sigma(M) \wedge \neg\theta(M,\bar{a})$ is finite we let $\psi_k(x) \longleftrightarrow x \neq x$ and $\rho_k(x,\bar{y}) \longleftrightarrow x \neq x$ and there is nothing to check. Suppose $\sigma(M) \wedge \neg\theta(M,\bar{a})$ is not finite, so $\sigma(M) \wedge \theta(M,\bar{a})$ is. Let $\{b_i : i<\ell\}$ list $p^*(M) \cap \sigma(M) \cap \theta(M,\bar{a})$ and $\{c_i : i<m\}$ list $\sigma(M) \cap \theta(M,\bar{a}) \setminus p^*(M)$. Using 3.11, find $\psi_k^i(x)$ with $\psi_k^i(M) \cap p^*(M) = \phi$ and $M \models \psi_k^i(c_i)$ for each $i<m$; $\psi_k(x)$ is $\bigvee_{i<m} \psi_k^i(x)$. For each $i<\ell$, we find $\rho_k^i(x,\bar{y})$ isolating $tp(b_i/\bar{a} \cup ac\ell(\phi))$; if $b_i \in p^*(M) \setminus p(M)$ we let $\rho_k^i(x,\bar{y}) \longleftrightarrow x = b_i$. Suppose $b_i \in p(M)$. Then by 3.13 there is c_i with $\pi_k(c_i,\bar{a})$ and $c_i \not\perp b_i$. By 3.10, either $\eta(c_i,b_i)$ or there is $d \in ac\ell(\phi)$ with $S(d,c_i,b_i)$. So we can assume either $\rho_k^i(x,\bar{y}) \to \exists z(\pi_k(z,\bar{y}) \wedge \eta(x,z))$ or $\rho_k^i(x,\bar{y}) \to \exists z(\pi_k(z,\bar{y}) \wedge S(d,z,x))$

for some $d \in p^*(M) \setminus p(M)$. We let $\rho_k(x,\bar{y}) \longleftrightarrow \bigvee_{i<\ell} \rho_k^i(x,\bar{y}) \wedge x \notin B$, where B comes from 3.14. By 3.12 and 3.14, for any $\bar{b} \subseteq p^*(M)$, $\rho_k(M,\bar{b}) \subseteq p^*(M)$. Clearly also $\psi_k(M) \cap p^*(M) = \phi$. Since any independent sequence of length k realizes the same strong type as $\bar{a}$, we just need to check $\tau_k(\bar{a})$. But this is clear since any solution to $\sigma(x) \wedge \theta(x,\bar{a})$ is either a c_i, and $\psi_k^i(c_i)$, or a b_i, and $\rho_k^i(b_i,\bar{a}) \wedge b_i \notin B$.

Now suppose we have defined ψ_k and $\rho_k(x,\bar{y})$ as above for all k, $j < k \leq \ell h(\bar{y})$. Pick $a_1,\ldots,a_j \in p(M)$ independent and consider the set

$$\Gamma_j = \{\bar{a} \subseteq p^*(M) : \ell h(\bar{a}) = \ell h(\bar{y}),\ \bar{a} \supseteq \{a_1,\ldots,a_j\}$$
$$\text{and } \sigma(M) \wedge \neg\theta(M,\bar{a}) \text{ is infinite and } \neg\tau_{j+1}(\bar{a})\}.$$

This set is definable by a collection of formulas, so if it was infinite, by the saturation of M it would have a sequence with $j+1$ independent elements, contrary to our supposition. We let $\{c_i : i<m\}$ list $\bigcup\{\sigma(M) \cap \theta(M,\bar{a}) : \bar{a} \in \Gamma_j\} \setminus p^*(M)$ and $\{b_i : i<\ell\}$ list $\bigcup\{\sigma(M) \cap \theta(M,\bar{a}) : \bar{a} \in \Gamma_j\} \cap p^*(M)$. We pick $\psi_j^i(x)$ and $\rho_j^i(x,\bar{y})$ precisely as in the case where $j = k$; $\rho_j^i(x,\bar{a})$ isolates $tp(b_i/\bar{a})$ for any $\bar{a} \in \Gamma_j$ with $\sigma(b_i) \wedge \theta(b_i,\bar{a})$, for $i<\ell$. We let $\psi_j(x) \longleftrightarrow \bigvee \psi_j^i(x) \vee \psi_{j+1}(x)$ and $\rho_j(x,\bar{y}) \longleftrightarrow (\bigvee \rho_j^i(x,\bar{y}) \vee \rho_{j+1}(x,\bar{y})) \wedge x \notin B$; again, checking that ψ_j and ρ_j are as claimed is routine.

Now let $(\exists x\theta(x,\bar{y}))^*$ be $\exists x(\sigma(x) \wedge \theta(x,\bar{y}) \wedge \neg\psi_0(x))$. If $\bar{a} \subseteq p^*(M)$ and $p^*(M) \models \exists x\theta(x,\bar{a})$ choose $c \in p^*(M)$ with $\theta(c,\bar{a})$; $M \models \sigma(c) \wedge \theta(c,\bar{a}) \wedge \neg\psi_0(c)$. Now suppose $\bar{a} \subseteq p^*(M)$ and $M \models \exists x(\sigma(x) \wedge \theta(x,\bar{a}) \wedge \neg\psi_0(x))$. If $\sigma(M) \wedge \neg\theta(M,\bar{a})$ is finite, then for all but finitely many $c \in p^*(M)$, $M \models \theta(c,\bar{a})$ so $p^*(M) \models \exists x\theta(x,\bar{a})$. If $\sigma(M) \wedge \neg\theta(M,\bar{a})$ is infinite, then

$$M \models \forall x(\sigma(x) \wedge \theta(x,\bar{a}) \rightarrow (\psi_0(x) \vee \rho_0(x,\bar{a}))),$$

so for some c, $\theta(c,\bar{a}) \wedge \rho_0(c,\bar{a})$. So $c \in p^*(M)$ and $p^*(M) \models \exists x\theta(x,\bar{a})$.

Corollary 3.16: $p^*(M)$ is saturated and strongly minimal.

Proof: Suppose $A \subseteq p^*(M)$ with $|A| < |p^*(M)| = |M|$, and $r \in S^1(A)$ taken in $p^*(M)$ is non-algebraic. For each $\psi(x,\bar{a}) \in r$ find $\psi^*(x,\bar{y})$ from 3.15 with $M \models \psi^*(b,\bar{a})$ if and only if $p^*(M) \models \psi(b,\bar{a})$ for $\bar{a},b \in p^*(M)$. Then $p(x) \cup \{\psi^*(x,\bar{a}): \psi(x,\bar{a}) \in r\}$ is consistent as an M-type and so is realized in M, since M is saturated. So r is realized in $p^*(M)$, which is therefore saturated.

Now if $\theta(x,\bar{y})$ is 0-definable on $p^*(M)$, choose $\theta^*(x,\bar{y})$ from 3.15. For any $\bar{b} \subseteq p^*(M)$, let $a \in p(M) \setminus acl(\bar{b})$. If $M \models \theta^*(a,\bar{b})$ then using 3.11 $\neg\theta^*(M,\bar{b}) \cap p^*(M)$ is finite, so $\neg\theta(p^*(M),\bar{b})$ is finite, otherwise $\theta(p^*(M),\bar{b})$ is finite. Since $p^*(M)$ is saturated, it is therefore strongly minimal.

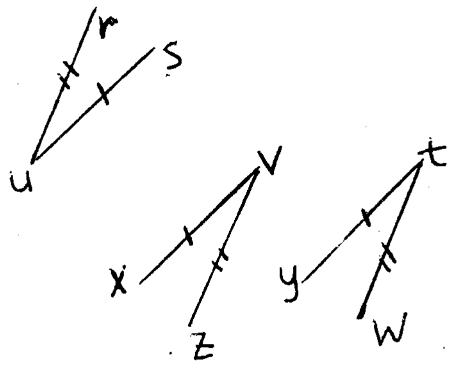
Lemma 3.17: Suppose p is of affine character. Then $p(M)$, under the structure induced from M, is saturated and strongly minimal. Also for any $\varphi(\bar{x})$ a $p(M)$-formula without parameters, there is $\varphi^*(\bar{x})$ an M-formula almost without parameters such that if $\bar{a} \subseteq p(M)$, $M \models \varphi^*(\bar{a})$ if and only if $p(M) \models \varphi(\bar{a})$.

Proof: Recall that we have R and Q such that for $a,b,c \in p(M)$ independent, $R(a,b,M)$ intersects all the E-classes except a' and b' of $acl\{a,b\} \cap p(M)$ and $Q(a,b,c,M)$ intersects $b' +_{a'} c' \in p(M)/E$. So P_R and P_Q generate the algebraic closure operation on $p(M)/E$. Now we can replace Q and R by Q_0 and R_0 still having this property such that $p(M)$ is closed under Q_0 and R_0. Once we do this, we can mimic 3.13, 3.15 and 3.16 virtually unchanged, except there is no mention of algebraic elements. We give the description of Q_0 and R_0 and omit the rest of the proof.

Let $Q_1(x,y,z,w;u,v) \longleftrightarrow$

$$\exists r,s,t[Q(x,u,v,s) \wedge Q(u,y,s,t) \wedge Q(z,u,v,r) \wedge Q(r,u,t,w)].$$

Diagram:



Let $Q_0(x,y,z,w)$ abbreviate

$$\exists u,v[u \notin acl\{x,y,z\} \wedge v \notin acl\{x,y,z,u\} \wedge Q_1(x,y,z,w;u,v)].$$

Suppose $a,b,c \in p(M)$, $d \in p(M) \setminus acl\{a,b,c\}$ and $e \in p(M) \setminus acl\{a,b,c,d\}$.

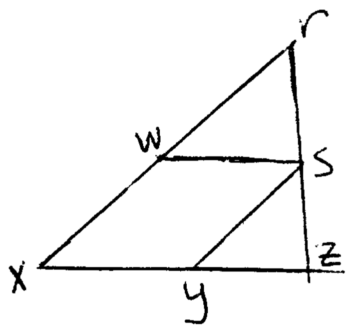
a,d and e are independent so there is f with $Q(a,d,e,f)$ and any such f is in $p(M)$ and satisfies $f' = d' +_{a'} e'$; d,b and f are independent since $e \in acl\{a,d,f\}$, so for any g with $Q(d,b,f,g)$, $g \in p(M)$ and $g' = b' +_{d'} f' = b' +_{a'} f' -_{a'} d' = b' +_{a'} e'$. c,d and e

are independent, so if $Q(c,d,e,h)$, $h \in p(M)$ and $h' = d' +_{c'} e' = d' +_{a'} e' -_{a'} c'$. Also h,d and g are independent for otherwise either $h' = \alpha d' +_{a'} (1-\alpha)g'$ in $p(M)/E$ or $g' = \alpha d'$ for some $\alpha \in F(n)$. If $g' = \alpha d'$, $\alpha d' = b' +_{a'} e'$, so $e \in acl\{a,b,d\}$; if $h' = \alpha d' + (1-\alpha)g'$, $\alpha d' + (1-\alpha)(b'+e') = d' + e' - c'$. (We omit the a''s.) Then if $\alpha \neq 0$, $(\alpha-1)d' + (1-\alpha)b' + c' = \alpha e'$ gives $e \in acl\{a,b,c,d\}$; if $\alpha = 0$, it gives $d \in acl\{a,b,c\}$. In any case, we get a contradiction. So for any i with $Q(h,d,g,i)$, $i \in p(M)$ and $i' = d' +_{h'} g' = d' +_{a'} g' -_{a'} h' = d' +_{a'} b' +_{a'} e' -_{a'} (d' +_{a'} e' -_{a'} c') = b' +_{a'} c'$. This shows $Q_1(a,b,c,i;d,e)$ for some i, and for any such i, $i' = b' +_{a'} c'$. In particular, $i \in acl\{a,b,c\}$, so for any $d^*,e^* \in p(M)$ with $d^* \notin acl\{a,b,c\}$ and $e^* \notin acl\{a,b,c,d^*\}$ we have $Q_1(a,b,c,i;d,e) \longleftrightarrow Q_1(a,b,c,i;d^*,e^*)$. Thus $Q_0(x,y,z,w)$ is almost 0-definable.

Now let $R_1(x,y,z;w)$ if and only if

$\exists r,s[R(x,w,r) \wedge Q_0(x,w,y,s) \wedge R(r,s,z)]$

Diagram:



Let $R_0(x,y,z) \longleftrightarrow R_1(x,y,z;w) \wedge R_1(x,y,z;v)$ for some (any) $w \in p(M) \setminus acl\{x,y\}$, $v \in p(M) \setminus acl\{x,y,w\}$. The choice of w and v is irrelevant, and R_0 is as desired.

Note: Whether p is of projective or affine character, it is clear that for $\bar{a} \subseteq p^*(M)(p(M))$, $ac\ell(\bar{a}) \cap p^*(M)$ $(ac\ell(\bar{a}) \cap p(M))$ is the same whether computed in M or in $p^*(M)$ $(p(M))$.

The following example shows that we may not be able to replace $p^*(M)$ by $p(M)$ in 3.15 and 3.16.

Example 3.19: Let M be a projective space over a finite field $F(n)$, with constants for every element of a subspace A. Let $|A| = \aleph_0$, $|M| = \aleph_1$. Then the type p of an element of $M \setminus A$ is of projective character, but $p(M)$ under the structure induced by formulas does not satisfy 3.17. For consider the following formula of $p(M) — \exists^{!n} zS(x,y,z)$. $p(M) \models \exists^{!n} zS(a,b,z)$ exactly if $a \in ac\ell\{b\} = \langle\{b\} \cup A\rangle$ in M, recalling that $\langle C\rangle$ is the projective span of C. This formula is not induced by any M-formula, and $\exists^{!n} zS(a,y,z)$ defines an infinite, co-infinite set.

SECTION 4

In this section we assume M is a strongly minimal transitive structure such that (*) holds on M, so for any $N \succ M$ and finite $F \subseteq N$, there is a finite $G \subseteq N$ with $acl(F) = \bigcup\{acl\{g\}: g \in G\}$. We also assume M is of affine character. This includes the structures arising from 3.17. Our aim is to prove the following:

<u>Theorem 4.1</u>: There is a 0-definable equivalence relation θ on M with finite classes such that M/θ, under the obvious structure, is an affine space over $F(n)$.

We also determine what other structure is possible on M/θ. The crucial lemma for 4.1 is

<u>Lemma 4.2</u>: Suppose $\rho(x,y)$ is an algebraic M-formula; i.e. for $a \in M$, $\rho(a,M)$ and $\rho(M,a)$ are both finite. Then $\bar{\rho}$, the transitive closure of ρ, is defined by an algebraic formula $\bar{\rho}(x,y)$.

An equivalent statement is that any such ρ can be included in a 0-definable equivalence relation with finite classes.

We may assume M is saturated and $|M| > |Th(M)|$.

To prove the lemma, we use Q; the crucial property of Q, we recall, is that for $a,b,c \in M$ independent $P_Q(a',b',c',d')$ is the same as $b' +_{a'} c' = d'$ in M/E. We may assume that $Q(x,y,z,w) \rightarrow Q(y,x,w,z) \wedge Q(x,z,y,w)$ since if we replace $Q(x,y,z,w)$ by

$$Q(x,y,z,w) \vee Q(x,z,y,w) \vee Q(y,x,w,z) \vee Q(y,w,x,z)$$
$$\vee\, Q(z,x,w,y) \vee Q(z,w,x,y) \vee Q(w,y,z,x) \vee Q(w,z,y,x)$$

we do not change the above property. Let $Q'(x,y,z,w)$ abbreviate "$Q(x,y,z,w)$ and $\dim\{x,y,z,w\} = 3$" and $\theta_1(x,y) \longleftrightarrow \forall u,v,z(Q'(u,v,z,x) \longleftrightarrow Q'(u,v,z,y))$. θ_1 gives an equivalence relation on M and if $\theta_1(a,b)$ we have $b \in acl\{a\}$. So $\theta_1(a,b)$ if and only if, for some (any) $c \notin acl\{a\}$, $d \notin acl\{a,c\}$, $\forall z[Q(c,d,z,a) \longleftrightarrow Q(c,d,z,b)]$. So θ_1 is 0-definable.

M/θ_1 is strongly minimal, transitive, Q induces a relation on M/θ_1 with the same symmetry, and $P_Q(x',y',z',w')$ is still the same as $y' +_{x'} z' = w'$ in $(M/\theta_1)/E$. From the definition and the symmetry of Q it is clear that θ_1 is a congruence with respect to Q'. So in M/θ_1 we have: $\forall x,y,z(Q'(x,y,z,w) \longleftrightarrow Q'(x,y,z,w^*)) \rightarrow w = w^*$.

A counter-example to 4.2 in M remains a counter-example in M/θ_1, so we assume θ_1 is trivial. Thus in M if a,b,c are independent and $\langle d_0\rangle{}^\frown\bar{d}$ lists $Q(a,b,c,M)$, then $\bigwedge_{d\in\langle d_0\rangle^\frown\bar{d}} Q(d,b,c,e) \longleftrightarrow e = a$ and $\bigwedge_{d\in\langle d_0\rangle^\frown\bar{d}} Q(b,a,d,e) \longleftrightarrow e = c$. As $Q(a,b,c,d^*)$ implies $b' +_{a'} c' = (d^*)'$, $\bar{d} \subseteq acl\{d_0\}$; let $\rho(x,\bar{y})$ isolate the type of $\bar{d}$ over d_0.

Let

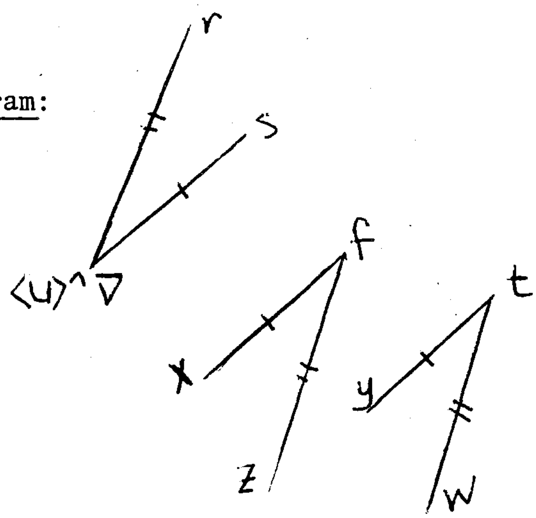
$$Q_0(x,y,z,w;u,\bar{v}) \longleftrightarrow \rho(u,\bar{v}) \wedge$$

$$\exists s,t,r \bigwedge_{v^*\in\langle u\rangle^\frown\bar{v}} (Q(x,f,v^*,s) \wedge Q(v^*,s,y,t) \wedge Q(z,f,v^*,r) \wedge$$

$$Q(r,t,v^*,w))$$

for some $f \notin acl\{x,y,z,u\}$. This is similar to the Q_0 defined in 3.17.

Diagram:



As in 3.17, if $a,b,c \in M$ and $e_0 \notin acl\{a,b,c\}$ and $\rho(e_0,\bar{e})$ we get $Q_0(a,b,c,d,e_0,\bar{e})$ implies $d' = b' +_{a'} c'$. Thus the choice of f is irrelevant; the choice of $\langle e_0\rangle^\frown\bar{e}$ is also irrelevant as we will see later. First we note some properties of Q_0.

Lemma 4.3: Suppose $e_0 \notin acl\{a,b,c,a^*\}$, and $\rho(e_0,\bar{e})$. Then

(1) There is a unique d with $Q_0(a,b,c,d;e_0,\bar{e})$.

(2) If $Q_0(a,b,c,d;e_0,\bar{e})$, then $Q_0(c,d,a,b;e_0,\bar{e})$ and $Q_0(d,c,b,a;e_0,\bar{e})$.

(3) If $Q_0(a,b,c,d;e_0,\bar{e}) \wedge Q_0(c,d,a^*,b^*,e_0,\bar{e})$, then $Q_0(a,b,a^*,b^*;e_0,\bar{e})$.

(4) If $Q_0(a,b,c,d;e_0,\bar{e}) \wedge Q_0(b,a^*,d,c^*;e_0,\bar{e})$, then $Q_0(a,a^*,c,c^*;e_0,\bar{e})$.

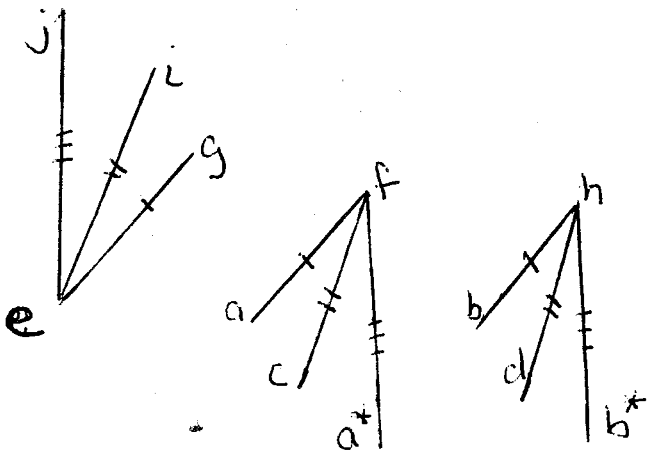
(5) If $Q_0(a,b,c,d;e_0,\bar{e})$, then $Q_0(a,c,b,d;e_0,\bar{e})$.

Proof: (1) Pick an $f \notin acl\{a,b,c,e_0\}$; then there is a unique g with $\bigwedge_{e\in\langle e_0\rangle^\frown\bar{e}} Q(a,f,e,g)$. Since e,g and b are independent, there is a unique h with $\bigwedge_{e\in\langle e_0\rangle^\frown\bar{e}} Q(e,g,b,h)$. We finish as in 3.17 except keeping uniqueness throughout.

(2) Pick $f \notin acl\{a,b,c,e_0\}$. We have g,h and i with $Q(a,f,e,g) \wedge Q(e,g,b,h) \wedge Q(c,f,e,i) \wedge Q(i,h,e,d)$ for all $e \in \langle e_0\rangle^\frown\bar{e}$. By the symmetry of Q, $Q(c,f,e,i) \wedge Q(e,i,d,h) \wedge Q(a,f,e,g) \wedge Q(g,h,e,b)$ for $e \in \langle e_0\rangle^\frown\bar{e}$, witnessing $Q_0(c,d,a,b;e_0,\bar{e})$. Also, $h \notin acl\{a,b,c,e_0\}$

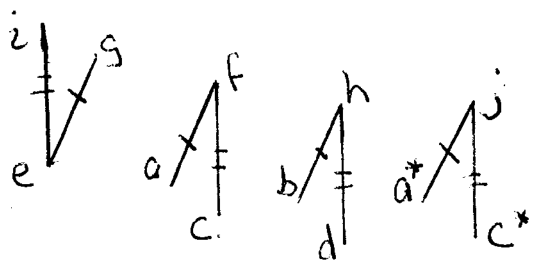
and $Q(d,h,e,i) \wedge Q(e,i,c,f) \wedge Q(b,h,e,g) \wedge Q(g,f,e,a)$ witnessing $Q_0(d,c,b,a;e_0,\bar{e})$.

(3) Diagram:



Pick $f \notin ac\ell\{a,b,c,a^*,e_0\}$. Find g,h,i with $Q(a,f,e,g) \wedge Q(e,g,b,h) \wedge Q(c,f,e,i) \wedge Q(i,h,e,d)$ and then j with $Q(a^*,f,e,j)$ (for all $e \in \langle e_0\rangle\hat{}\bar{e}$). Since $Q_0(c,d,a^*,b^*;e_0,\bar{e})$, $Q(c,f,e,i) \wedge Q(e,i,d,h) \wedge Q(a^*,f,e,j)$ we must have $Q(j,h,e,b^*)$. Now $Q(a,f,e,g) \wedge Q(e,g,b,h) \wedge Q(a^*,f,e,j) \wedge Q(j,h,e,b^*)$ witnesses $Q_0(a,b,a^*,b^*,e_0,\bar{e})$.

(4) Diagram:



Pick $f \notin ac\ell\{a,b,c,a^*,e_0\}$. Find g,h,i with $Q(a,f,e,g) \wedge Q(e,g,b,h) \wedge Q(c,f,e,i) \wedge Q(i,h,e,d)$ and then j with $Q(e,a^*,g,j)$, for all $e \in \langle e_0\rangle\hat{}\bar{e}$. $h \notin ac\ell\{b,d,a^*,e\}$, and so $Q(b,h,e,g) \wedge Q(e,a^*,g,j) \wedge Q(d,h,e,i)$ along with $Q_0(b,a^*,d,c^*,e_0,\bar{e})$ give $Q(i,j,e,c^*)$. Now $Q(a,f,e,g) \wedge Q(e,g,a^*,j) \wedge Q(c,f,e,i) \wedge Q(i,j,e,c^*)$

witnesses $Q_0(a,a^*,c,c^*;e_0,\bar{e})$.

(5) For any $f \notin acl\{e_0\}$, let f^* be such that for some a,b independent over $\{f,e_0\}$ there is c with $Q_0(a,b,c,f;e_0,\bar{e}) \wedge Q(a,c,b,f^*;e_0,\bar{e})$. By (1) and (2), c is uniquely determined by a,b,f and $\langle e_0\rangle^\frown\bar{e}$ so f^* is uniquely determined by a,b, $\langle e_0\rangle^\frown\bar{e}$ and f. Also in M/E, $f' = b' +_{a'} c' = c' +_{a'} b' = (f^*)'$, so $f^* \in acl\{f\}$, which implies that the choice of a and b is irrelevant; f^* is determined by f and $\langle e_0\rangle^\frown\bar{e}$.

Now suppose a,b and e_0 are independent and $Q_0(a,b,c,d;e_0,\bar{e})$. Pick $a_0 \notin acl\{a,b,c,e_0\}$ and b_0 with $Q_0(a,b,a_0,b_0;e_0,\bar{e})$. By definition $Q_0(a,a_0,b,b_0^*;e_0,\bar{e})$. By (2) and (3) above, $Q_0(c,d,a_0,b_0;e_0,\bar{e})$. Since c,d,a_0 and e_0 are independent, $Q_0(c,a_0,d,b_0^*;e_0,\bar{e})$. From (2), $Q_0(a_0,c,b_0^*,d;e_0,\bar{e})$ and from (4) and $Q_0(a,a_0,b,b_0^*;e_0,\bar{e})$, $Q_0(a,c,b,d;e_0,\bar{e})$.

We now remove the assumption $a \perp b$, still assuming that $Q_0(a,b,c,d;e_0,\bar{e})$ and $e_0 \notin acl\{a,b,c\}$. Pick $f \notin acl\{a,b,c,e_0\}$ and g with $Q_0(a,f,c,g;e_0,\bar{e})$. By the above $Q_0(a,c,f,g;e_0,\bar{e})$ and by (4) and (2) $Q_0(b,f,d,g;e_0,\bar{e})$. b,f and e_0 are independent, so $Q_0(b,d,f,g;e_0,\bar{e})$. From this and $Q_0(a,c,f,g;e_0,\bar{e})$, (2) and (3) we get $Q_0(a,c,b,d;e_0,\bar{e})$.

<u>Lemma 4.4</u>: Suppose $\rho(e_i,\bar{e}_i)$ for $i = 0,1$ and that $e_i \notin acl\{a,b,c\}$. Then $Q_0(a,b,c,d;e_0,\bar{e}_0) \longleftrightarrow Q_0(a,b,c,d;e_1,\bar{e}_1)$.

<u>Proof</u>: Picking $e_2 \notin acl\{a,b,c,e_0,e_1\}$ and then $\bar{e}_2$ with $\rho(e_2,\bar{e}_2)$, we may assume that e_0 and e_1 are independent over $\{a,b,c\}$. By 4.3(1) we find d_i, $i = 0,1$, with $Q_0(a,b,c,d_i;e_i,\bar{e}_i)$ and these are uniquely determined. Choose $f \notin acl\{a,b,c,d_0,d_1,e_0,e_1\}$ and g_i, $i = 0,1$, with

$Q_0(a,b,f,g_i;e_i,\bar{e}_i)$. By 4.3(3) and (2), $Q_0(c,d_i,f,g_i;e_i,\bar{e}_i)$. Now $acl\{f,g_0,g_1,e_0,\bar{e}_0,e_1,\bar{e}_1\} = acl\{f,g_0,e_0,e_1\}$. Our assumptions give $c,a \notin acl\{f,e_0,e_1\}$, so if a or c is in $acl\{f,g_0,e_0,e_1\}$ then g_0 is in $acl\{a,f,e_0,e_1\}$ or $acl\{c,f,e_0,e_1\}$ respectively. $Q_0(a,b,f,g_0;e_0,\bar{e}_0)$ gives $b \in acl\{a,f,g_0\}$, so $a \in acl\{f,g_0,e_0,e_1\}$ then gives $b \in acl\{a,f,e_0,e_1\}$. If $b \in acl\{a\}$, $g_0 \in acl\{f\}$ and then $a \in acl\{f,e_0,e_1\}$ which is impossible; but if $b \notin acl\{a\}$, $\{a,b,e_0,e_1,f\}$ is independent by choice so $a \notin acl\{f,g_0,e_0,e_1\}$. If $c \in acl\{f,g_0,e_0,e_1\}$, $d_0 \notin acl\{c\}$ so again $\{c,d_0,e_0,e_1,f\}$ is independent. But $d_0 \in acl\{c,f,g_0\} \subseteq acl\{c,f,e_0,e_1\}$, again a contradiction. So $a,c \notin acl\{f,g_0,g_1,e_0,\bar{e}_0,e_1,\bar{e}_1\}$ so $a \equiv c(\{f,g_0,g_1,e_0,\bar{e}_0,e_1,\bar{e}_1\})$. Since by 4.3(1) and (2), $Q_0(a,x,f,g_0;e_0,\bar{e}_0)$ defines b and $Q_0(c,x,f,g_0;e_0,\bar{e}_0)$ defines d_0, $ab \equiv cd_0(\{f,g_0,g_1,e_0,\bar{e}_0,e_1,\bar{e}_1\})$. Similarly, $ab \equiv cd_1(\{f,g_0,g_1,e_0,\bar{e}_0,e_1,\bar{e}_1\})$. But then $cd_0 \equiv cd_1(\{f,g_0,e_0,\bar{e}_0\})$ so $Q_0(c,d_1,f,g_0;e_0,\bar{e}_0)$. By 4.3(1) and (2), $d_0 = d_1$.

Let

$$Q_1(x,y,z,w) \longleftrightarrow \exists u,\bar{v}(u \notin acl\{x,y,z\} \wedge \rho(u,\bar{v}) \wedge Q_0(x,y,z,w;u,\bar{v})).$$

4.4 allows us to replace $u \notin acl\{x,y,z\}$ by a suitable first-order formula, so Q_1 is 0-definable. As Q_1 satisfies 4.3, it gives M an abelian group structure. In fact the group is of bounded exponent, for otherwise by saturation we can find a and $b_i (i\in\omega)$ all distinct with $Q_1(a,b_0,b_i,b_{i+1})$ for all $i\in\omega$. Let $c \perp d_0$; if $Q_1(c,d_0,d_i,d_{i+1})$ for $i < n-1$ then $c \not\perp d_{n-1}$ for in M/E we have $d'_{i+1} = d'_i +_{c'} d'_0$ so $d'_{n-1} = c'$. Let $\tau_0(x,y)$ witness $c \not\perp d_{n-1}$ and $\tau_1(x,y)$ be algebraic such that

$$M \models [\neg\tau_1(x,y_0) \wedge \exists y_1,\dots,y_{n-1} \bigwedge_{i<n-1} Q_1(x,y_0,y_i,y_{i+1})] \rightarrow \tau_0(x,y_{n-1}).$$

We can pick $k\in\omega$ s.t. $\neg\tau_0(a,b_i) \wedge \neg\tau_1(a,b_i)$ for all $i\geq k$. Yet $\bigwedge_{i<n-1} Q_1(a,b_k,b_{ik},b_{(i+1)k})$. This is a contradiction, and it establishes that the group is of bounded exponent.

Proof of Lemma 4.2: If not, we can find $b_i (i\in\omega)$ distinct with $\rho(b_i,b_{i+1})$ for all $i\in\omega$, where ρ is algebraic. Pick $a \notin ac\ell\{b_0\}$ and let $\{c_j: j<r\}$ list $\exists x(\rho(b_0,x) \wedge Q_1(a,M,b_0,x))$. Now $b_i \equiv b_0(a)$, so for each $i\in\omega$ there is $j<r$ with $Q_1(a,c_j,b_i,b_{i+1})$. But then the set generated by the finite set $\{a,b_0\} \cup \{c_j: j<r\}$ under Q_1 is infinite. This is impossible in an abelian group of bounded exponent.

We now head toward the proof of 4.1. First we deal with the case $n=2$, which does not require 4.2. But we do need the following:

Lemma 4.6: (1) Let $Q_1(a,b,c,d) \wedge Q_1(a,e,f,d)$ in M. Then $Q_1(b,e,f,c)$.

(2) If $n=2$ (n = size of field), $M \models Q_1(x,y,y,x)$.

Proof: (1) Let $Q_1(a,b,c,d) \wedge Q_1(a,e,f,d)$. Pick h with $Q_1(a,b,e,h)$. Using 4.3 repeatedly, $Q_1(c,d,e,h) \wedge Q_1(b,h,f,d)$, so $Q_1(h,d,e,c) \wedge Q_1(b,f,h,d)$, so $Q_1(b,f,e,c)$.

(2) Let $a,b \in M$; pick $c \notin ac\ell\{a,b\}$ and d with $Q_1(c,a,a,d)$. Since $n=2$, $d \in ac\ell\{c\}$ so $a \equiv b\{c,d\}$ and $Q_1(c,b,b,d)$. By (1), $Q_1(a,b,b,a)$.

We can now finish:

Proof of Theorem 4.1 [Case n=2]: Let Q be Q_1 and $R(x,y,z) \longleftrightarrow x=z \vee y=z$. Running through #0 to #11 of 1.5(1) is trivial; occasionally use 4.3 and for #9 ($Q(x,y,z,x) \longrightarrow R(x,y,z)$) use 4.6(2). So the long-forgotten θ_1 is the θ for our original M.

Actually, we won't ever return to our original Q, so we drop the "1" from Q_1. We assume $n \neq 2$; we will later redefine R, but we need to keep the original for a while. We assume that $R(x,y,z) \to R(y,x,z) \wedge R(z,y,x)$ by replacing it by $\bigvee_{\sigma \in \mathrm{Sym}\{x,y,z\}} R(\sigma x, \sigma y, \sigma z)$. This doesn't affect the crucial property of R, which is that if $a \downarrow b$, $R(a,b,M)$ intersects every E-class in $acl\{a,b\} \setminus [acl\{a\} \cup acl\{b\}]$ in a finite non-empty set; i.e. P_R defines the lines on M/E.

Now let

$$R_0(x,y,z;w) \longleftrightarrow \exists r,s(R(x,w,r) \wedge Q(x,w,y,s) \wedge R(r,s,z)),$$

and

$$R_1(x,y,z) \longleftrightarrow R_0(x,y,z;a) \wedge R_0(x,y,z;b)$$

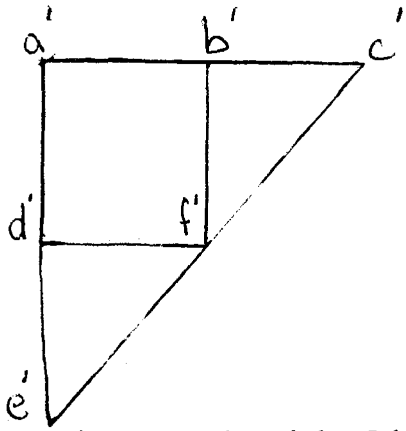
for any a,b independent over $\{x,y\}$. The choice of a and b is irrelevant since for any c,d,e with $R_1(c,d,e)$, $e \in acl\{c,d\}$. This is the same as in 3.17.

<u>Lemma 4.7</u>: If $a \downarrow b$, $R_1(a,b,M)$ meets the same E-classes as $R(a,b,M)$.

<u>Proof</u>: Suppose $R_1(a,b,c)$; so $c \in acl\{a,b\}$. Pick $d \notin acl\{a,b\}$. There are then e,f with $R(a,d,e) \wedge Q(a,d,b,f) \wedge R(e,f,c)$. Since $d \notin acl\{a,b\}$, in M/E, b' is not on the line through a' and d'. The line through a' and d' is parallel to the one through b' and f', and meets the line through f' and c' at e'; so $b' \neq c'$. d' is on the line through a' and e' but not the one through c' and e', so $c' \neq a'$. Thus $c \notin acl\{a\} \cup acl\{b\}$.

Now suppose $R(a,b,c)$ and $a \perp b$. Pick any $d \notin acl\{a,b\}$ and f with $Q(a,d,b,f)$. Since $c' \notin \{a',b'\}$ is on the line through a' and b', the line through a' and d' meets the one through c' and f', at say e'.

Diagram:



$e' \notin \{d',a'\}$, so there is $e \in e'$ with $R(a,d,e)$. $e' \notin \{f',c'\}$ so there is $e^* \in e'$ with $R(e^*,f,c)$. Thus there is $c^* \in c'$ with $R(e,f,c^*)$. So $R(a,d,e) \wedge Q(a,d,b,f) \wedge R(e,f,c^*)$ and since d was arbitrary, $R_1(a,b,c^*)$.

Now let $R_2(x,y,z) \longleftrightarrow R_1(x,y,z) \vee R_1(y,x,z) \vee x = z \vee y = z$.

Lemma 4.8: There is an algebraic $\tau(x,y)$ without parameters such that:

(1) If $a \perp b$, $R(a,b,c) \wedge R_2(a,b,d)$ and $c \not\perp d$, then $\tau(c,d)$.

(2) If $a \perp b$, $R_2(a,b,c) \wedge R_2(a,b,d)$ and $c \not\perp d$, then $\tau(c,d)$.

(3) If $R_2(a,b,c) \wedge R_2(a,b,d) \wedge a \neq c$, then there is d^* with $\tau(d,d^*) \wedge R_2(a,c,d^*)$.

(4) If $Q(a,b,c,d) \wedge Q(a,b,e,f) \wedge R_2(a,c,e)$, then there is f^* with $\tau(f,f^*) \wedge R_2(b,d,f^*)$.

(5) If $Q(a,b,c,a)$, there is c^* with $\tau(c,c^*) \wedge R_2(a,b,c^*)$.

(6) If $Q(a,b,c,d) \wedge R_2(a,c,e) \wedge a \neq e$, then there are f and f^* with

$\tau(f,f^*) \wedge R_2(b,e,f) \wedge R_2(c,d,f^*)$.

Proof: Doing each of (1) - (6) separately is sufficient. For (1), given $a \downfree b$ there are finitely many c and d with $R(a,b,c) \wedge R_2(a,b,d)$; for each pair with $c \not\downfree d$ pick a formula witnessing this. Take the disjunction of these. (2) is similar. (3) to (6) are all similar to each other; we just do (6) to illustrate. First suppose $a \downfree c$, $Q(a,b,c,d) \wedge R_2(a,c,e) \wedge a \neq e$. In M/E, $d' = b' +_{a'} c'$ and e', c' and a' are collinear with $a' \neq e'$ so the line through e' and b' meets the line through c' and d', say in f'. So we can find $f, f^* \in f'$ with $R_2(e,b,f) \wedge R_2(c,d,f^*)$. If $a \not\downfree c$, $Q(a,b,c,d) \wedge R_2(a,c,e) \wedge e \neq a$, then $a' = c' = e'$ and $b' = d'$ so the line through e' and b' is the line through c' and d', so again we have f and f^* with $R_2(e,b,f) \wedge R_2(c,d,f^*)$ and $f \not\downfree f^*$.

Now find τ_i $(i=1,2,3)$ algebraic such that if $\{a,b,c\}$ has $\geq i$ independent elements, then $\gamma_i(a,b,c)$ where

$$\gamma_i(x,y,z) \longleftrightarrow \forall w,u[Q(x,y,z,w) \wedge R_2(x,z,u) \wedge u \neq x \rightarrow$$

$$\exists v,v^*(\tau_i(v,v^*) \wedge R_2(y,u,v) \wedge R_2(z,w,v^*))].$$

Let a,b,c be independent; there are finitely many choices for d,e,f and f^* with

$$Q(a,b,c,d) \wedge R_2(a,c,e) \wedge e \neq a \wedge R_2(b,e,f) \wedge R_2(c,d,f^*) \text{ and } f \not\downfree f^*.$$

For each such pair find a formula witnessing $f \not\downfree f^*$. τ_3 is the disjunction of these. Now if $a \downfree b$, there are finitely many c with $\bigvee_{\sigma \in \mathrm{Sym}\{a,b,c\}} \neg\gamma_3(\sigma a, \sigma b, \sigma c)$. For each such c we can find finitely many

d,e,f,f^* as above, and pick a formula witnessing $f^* \not\perp f$ for each. τ_2 is the disjunction of these and τ_3. τ_1 is picked similarly. This does (6).

By 4.2, we can assume τ from the above lemma defines an equivalence relation. Now we can find $\varepsilon(x,y,z;u,v,w)$ inducing the relation δ of 1.7(1) on M/E. We may assume $\varepsilon(x,y,z;u,v,w) \to R_2(x,y,z) \wedge R_2(u,v,w)$ and if $a \perp b$, $d \perp e$ and $R_2(a,b,c)$ then $\varepsilon(a,b,c;d,e,M)$ is $R_2(d,e,M) \cap f'$ for some f with $R_2(d,e,f)$. Similarly for $\varepsilon(d,e,M;a,b,c)$. Let $\theta^*(z,w) \longleftrightarrow \exists x,y(\varepsilon(a,x,z;a,y,w) \wedge \tau(x,y))$ for some (any) $a \notin \mathrm{ac}\ell\{z\}$. $\theta^*(b,c)$ implies $b \not\perp c$ so the choice of a is irrelevant. So $\theta^*(b,c)$ if and only if for some d,e with $\tau(d,e)$ and $a \notin \mathrm{ac}\ell\{b\}$, $R_2(a,d,b) \wedge R_2(a,e,c)$ and $b \not\perp c$. If we take the transitive closure of θ^*, we get by 4.2 an 0-definable equivalence relation θ with finite classes, as θ^* is reflexive and symmetric. θ is the formula required for 4.1.

We have $M \models \tau(x,y) \to \theta(x,y)$ as $\varepsilon(a,b,b;a,c,c)$ always. The crucial property of θ is:

Lemma 4.9: (1) $R_2(a,b,c)$ and $b\theta b_0$ implies there is $c_0\theta c$ with $R_2(a,b_0,c_0)$.
(2) $Q(a,b,c,d) \wedge c\theta c_0$ implies there is $d_0\theta d$ with $Q(a,b,c_0,d_0)$
(3) $Q(a,b,c,d) \wedge a\theta a_0$ implies there is $d_0\theta d$ with $Q(a_0,b,c,d_0)$.

Proof: We first do (1) in case $a \perp b$, then (2), and finally return to (1). We finish with (3). So suppose $a \perp b$, $R_2(a,b,c)$ and $\theta^*(b,b_0)$. If $c=a$, let $c_0 = a$; so suppose $c \neq a$ and so $a \perp c$. By the definition of θ^* find d and d_0 with $\tau(d,d_0)$ and $\varepsilon(a,d,b;a,d_0,b_0)$. So $R_2(a,d,b)$; since $R_2(a,d,d)$ and $a \neq b$, by 4.8(3) there is d^* with

$\tau(d,d^*)$ and $R_2(a,b,d^*)$. From $R_2(a,b,d^*) \wedge R_2(a,b,c)$ and $a \neq d^*$ find c^* with $\tau(c,c^*) \wedge R_2(a,d^*,c^*)$. Since $a \downarrow d^*, d_0$ find c_0^* with $\varepsilon(a,d^*,c^*;a,d_0,c_0^*)$. $\tau(d^*,d_0)$, so $\theta^*(c^*,c_0^*)$. From $R_2(a,d_0,b_0) \wedge R_2(a,d_0,c_0^*) \wedge a \neq b_0$ and 4.8(3) find c_0 with $R_2(a,b_0,c_0) \wedge \tau(c_0^*,c_0)$. $\tau(c,c^*) \wedge \theta^*(c^*,c_0^*) \wedge \tau(c_0,c_0^*)$ gives $\theta(c,c_0)$; this suffices for (1) if $a \downarrow b$.

(2) Again, first suppose a,b and c are independent, and $Q(a,b,c,d) \wedge c\theta c_0$. Find d_0 with $Q(a,b,c_0,d_0)$. Pick $e \notin \{a,b\}$ with $R_2(a,b,e)$. $Q(a,c,b,d) \wedge R_2(a,b,e) \wedge e \neq a$ give, by 4.8(6) f and f^* with $R_2(c,e,f) \wedge R_2(b,d,f^*) \wedge \tau(f,f^*)$. Similarly there are f_0, f_0^* with $R_2(c_0,e,f_0) \wedge R(b,d_0,f_0^*) \wedge \tau(f_0,f_0^*)$. By $e \downarrow c$, $c\theta c_0$ and the above, $f\theta f_0$ as $R_2(e,c,f) \wedge R_2(e,c_0,f_0)$ and $f \not\downarrow f_0$. Use also 4.8(2). So $f^*\theta f_0^*$. $R_2(b,d,f^*) \wedge R_2(b,d,d) \wedge b \neq f^*$ gives $R_2(b,f^*,d^*)$ with $\tau(d,d^*)$; similarly $R_2(b,f_0^*,d_0^*)$ with $\tau(d_0,d_0^*)$. If $b \not\downarrow f^*$, then $b \not\downarrow d^*$; then $Q(a,c,b,d) \wedge \tau(d,d^*)$ give $a \not\downarrow c$ contrary to assumption. So $b \downarrow f^*$ and so $f^*\theta f_0^*$ gives $d^*\theta d_0^*$. Thus $d\theta d_0$. Now suppose $a \downarrow b$, $c \in acl\{a,b\}$, $Q(a,b,c,d) \wedge Q(a,b,c_0,d_0) \wedge c\theta c_0$. Choose $e \notin acl\{a,b\}$ and f with $Q(a,b,e,f)$. Then $Q(e,f,c,d) \wedge Q(e,f,c_0,d_0)$; in M/E c' is on the line through a' and b', so not on the line through e' and f' so $c \notin acl\{e,f\}$. So e,f,c are independent and $c\theta c_0$, so by the above $d\theta d_0$. Now just suppose $Q(a,b,c,d) \wedge Q(a,b,c_0,d_0) \wedge c\theta c_0$. Pick $e \notin acl\{a,b,c\}$ and f,f_0 with $Q(a,e,c,f) \wedge Q(a,e,c_0,f_0)$. We have $f\theta f_0$. Also from 4.3, $Q(e,b,f,d) \wedge Q(e,b,f_0,d_0)$, so $d_0\theta d$.

To finish (1), suppose $R_2(a,b,c)$, $b\theta b_0$ and $a \not\downarrow b$. If $c=a$, let $c_0=a$; if $c=b$, let $c_0=b$. So we can assume $R_1(a,b,c) \vee R_1(b,a,c)$. First suppose $R_1(a,b,c)$; pick $d \notin acl\{a\}$, e and f with

$R(a,d,e) \wedge Q(a,b,d,f) \wedge R(e,f,c)$, by the definition of R_1. Find f_0 with $Q(a,b_0,d,f_0)$; by (2), $f\theta f_0$. $e \downarrow f$ since otherwise $e \in acl\{c\} = acl\{a\}$, so there is c_0 with $R(e,f_0,c_0)$ and $c_0 \not\downarrow c$. Since $c_0 \in acl\{a_0,b_0\}$, $R_2(a,b_0,c_0)$. We can by 4.7 find $c^*, c_0^* \not\downarrow c$ with $R_2(e,f,c^*) \wedge R_2(e,f_0,c_0^*)$; by 4.8(1), $\tau(c,c^*) \wedge \tau(c_0,c_0^*)$ and by $f\theta f_0$ and (1) in the independent case, $c^*\theta c_0^*$. So $c\theta c_0$. If $R_1(b,a,c)$, pick $d \notin acl\{a,b\}$, e and f with $Q(b,d,a,e) \wedge R(b,d,f) \wedge R(f,e,c)$. Then find d_0 with $Q(b_0,d_0,a,e)$, so by (2) $d\theta d_0$. Choose $f_0 \not\downarrow f$ with $R(b_0,d_0,f_0)$ and then $c_0 \not\downarrow c$ with $R(f_0,e,c_0)$; we then have $R_1(b_0,a,c_0)$. Pick $f_1 \not\downarrow f$ with $R(b,d_0,f_1)$; as in the last paragraph, $f_1\theta f$. Now $R(d_0,b,f_1) \wedge R(d_0,b_0,f_0) \wedge b\theta b_0$, so $f_0\theta f_1$. So $f_0\theta f$ and similarly $c\theta c_0$.

(3) First suppose a,b and c are independent, $Q(a,b,c,d) \wedge Q(a_0,b,c,d_0) \wedge a\theta a_0$. Pick $e \notin \{a,b\}$ with $R_2(a,b,e)$ and $e_0\theta e$ with $R_2(a_0,b,e_0)$. By 4.8(6) find f,f^*,f_0,f_0^* with $\tau(f,f^*) \wedge \tau(f_0,f_0^*)$ and $R_2(e,c,f) \wedge R_2(b,d,f^*) \wedge R_2(e_0,c,f_0) \wedge R_2(b,d_0,f_0^*)$. Now $f\theta f_0$, so $f^*\theta f_0^*$. $b \downarrow f^*$ for otherwise $e \in acl\{b,c\} \cap acl\{a,b\} = acl\{b\}$ implying $e=b$, so $R_2(b,d,f^*)$ gives d^* with $\tau(d,d^*)$ and $R_2(b,f^*,d^*)$; similarly find d_0^* with $\tau(d_0,d_0^*) \wedge R_2(b,f_0^*,d_0^*)$. $f^*\theta f_0^*$ implies $d^*\theta d_0^*$ yielding $d\theta d_0$.

We can finish (3) like (2).

Now we forget about the original R, and let R denote R_2. In the following, Q and R also denote the relations they induce on M/θ.

<u>Proof of Theorem 4.1 ($n\neq 2$)</u>: We run through #'s 0 to 11 of 1.5(1). Actually, we alter R on M/θ a little further by replacing $R(x,y,z)$ by

$(R(x,y,z) \wedge x \neq y) \vee (x=y=z)$ to get #0. This doesn't affect anything else. Since θ is a congruence for Q, the following are immediate from the same in M:

#5 $\exists ! w Q(x,y,z,w)$.

#6 $Q(x,y,x,y)$.

#7 $Q(x,y,z,w) \rightarrow Q(x,z,y,w) \wedge Q(y,x,w,z)$.

#10 $Q(x,y,z,w) \wedge Q(x,y,u,v) \rightarrow Q(z,w,u,v)$.

The next two are also clear from the same in M:

#2 $R(x,y,x)$.

#3 $R(x,y,z) \rightarrow R(y,x,z)$.

#1 $\exists x,y \exists^{!n} z R(x,y,z)$.

Pick $a \mathop{\smile\!\!\!\!\!\!|}\, b$ in M. Then in M, $R(a,b,M)$ meets n E-classes, so in M/θ, $R(a/\theta, b/\theta, M/\theta)$ has size at least n. Now suppose $R(a,b,c) \wedge R(a^*,b^*,d)$ in M, with $a\theta a^*$, $b\theta b^*$ and $c \not\mathop{\smile\!\!\!\!\!\!|}\, d$. Pick e with $R(a,b^*,e) \wedge c\theta e$, by 4.9(1). Now $R(b^*,a,e)$, so by 4.9(1) again find f with $R(b^*,a^*,f) \wedge e\theta f$. Also $R(b^*,a^*,d)$, $b^* \mathop{\smile\!\!\!\!\!\!|}\, a^*$ and $f \not\mathop{\smile\!\!\!\!\!\!|}\, d$, so by 4.8(2), $d\theta f$. So $c\theta d$, which suffices.

#4 $R(x,y,z) \wedge R(x,y,w) \wedge x \neq z \rightarrow R(x,z,w)$.

Suppose $R(a/\theta,b/\theta,c/\theta) \wedge R(a/\theta,b/\theta,d/\theta) \wedge a/\theta \neq c/\theta$. Pick a,b,c, a^*,b^*,d with $a\theta a^*$, $b\theta b^*$ and $R(a,b,c) \wedge R(a^*,b^*,d)$. Apply 4.9(1) twice to find d^* with $R(a,b,d^*)$ and $d\theta d^*$. By 4.8(3), find d^{**} with $d^*\theta d^{**}$ and $R(a,c,d^{**})$; then $d\theta d^{**}$, so $R(a/\theta,c/\theta,d/\theta)$.

#8 $Q(x,y,z,w) \wedge R(x,y,u) \wedge Q(x,z,u,v) \rightarrow R(z,w,v)$.

Suppose $Q(a/\theta,b/\theta,c/\theta,d/\theta) \wedge R(a/\theta,b/\theta,e/\theta) \wedge Q(a/\theta,c/\theta,e/\theta,f/\theta)$. Pick $a\theta a^*\theta a^{**}$, $b\theta b^*$, $c\theta c^*$, $e\theta e^*$ with $Q(a,b,c,d) \wedge R(a^*,b^*,e) \wedge Q(a^{**},c^*,e^*,f)$ in M. By 4.9(1), find $e^{**}\theta e$ with $R(a,b,e^{**})$ and then using 4.9(2) and (3) find f^* with $Q(a,c,e^{**},f^*)$. So by $Q(a,b,c,d) \wedge R(a,b,e^{**}) \wedge Q(a,c,e^{**},f^*)$ and 4.8(4) find $f^{**}\theta f^*$ with $R(c,d,f^{**})$. So $M/\theta \models R(c/\theta,d/\theta,f/\theta)$.

#9 $Q(x,y,z,w) \rightarrow R(x,y,z)$.

Clear from 4.8(5).

#11 $Q(x,y,z,w) \wedge R(x,z,u) \wedge u \neq x \rightarrow \exists v(R(u,y,v) \wedge R(z,w,v))$.

This follows like #8, using 4.8(6) instead of 4.8(4).

Our next task is to determine the possible other structure on M/θ. For the rest of this section we assume θ is the identity. We will prove:

Theorem 4.10: Suppose M is strongly minimal, transitive, of affine character, Q and R give an affine structure on M, and P_Q and P_R give the affine structure on M/E. Let M_1 be the structure with universe M and the predicates Q and R. Then there is $C \subseteq M_1^{eq}$ with M and (M_1,C) interdefinable.

We prove a few lemmas first. Recall M', the projective companion of M as mentioned in 1.8, and $F: M^2 \setminus \{(a,a): a \in M\} \rightarrow M'$. Both are 0-definable in M^{eq}. We keep the notation and assumptions of 4.10 for the remainder of this section. Again, we also assume M is saturated.

Lemma 4.11: If a,b are distinct in M, then $a \in acl\{b\}$ iff $F(a,b) \in acl(\phi)$.

Proof: ($\Leftarrow$) Direct from $a \in acl\{b,F(a,b)\}$.

($\Rightarrow$) Suppose $a \in acl\{b\}$; pick $c \in M \setminus acl\{a\}$ and d with $Q(a,b,c,d)$. In M/E, $P_Q(a',b',c',d')$ and $a' = b'$, so $c' = d'$.

$F(a,b) = F(c,d) \in acl\{a\} \cap acl\{c\} = acl(\phi)$.

Lemma 4.12: If $a \in acl(B)$ where $\{a\} \cup B \subseteq M$, then $a \in \langle B \cup acl\{b\}\rangle$ for any $b \in B$. Recall $\langle C\rangle$ is the affine span of C, i.e. the closure of C under Q and R.

Proof: We may assume B is finite, and proceed by induction on $|B|$, the case $|B| = 1$ being clear. Suppose $b,c \in B$ are distinct, $a \in acl(B)$ and without loss of generality that $c \notin acl(B\setminus\{c\})$. In M/E, $a' \in acl(\{c'\} \cup \{d' : d \in B\setminus\{c\}\})$ so either $a' = c' +_{d'} e'$ or a' is on the line through c' and d' for some $d,e \in acl(B\setminus\{c\})$. So $Q(d,c,e,a^*)$ or $R(d,c,a^*)$ where $a^* \mathrel{E} a$. By the induction hypothesis, e and d are in $\langle B\setminus\{c\} \cup acl\{b\}\rangle$, so $a^* \in \langle B \cup acl\{b\}\rangle$. Now choose b^* with $Q(b,b^*,a,a^*)$; $b^* \in acl\{b\}$ so $a \in \langle B \cup acl\{b\}\rangle$.

We let $racl(\phi)$ (the "relevant algebraic closure") be the definable closure taken in M_1^{eq} of the following sets in M_1^{eq}:

(1) $acl(\phi) \cap M'$;

(2) the field elements - see 1.7(1);

(3) for each $\ell \in acl(\phi) \cap M'$, $Q(x,y,z,w) \wedge F(x,y) = \ell$ yields an almost 0-definable equivalence relation with finitely many classes on the set defined by $F(x,y) = \ell$, so we include the corresponding points of $acl(\phi)$.

We will use different notions of $racl(\phi)$ in different contexts later.

Lemma 4.13: If $a \in acl(B)$ in M, then a is definable over B in $(M_1, racl(\phi))$.

Proof: We may assume B is finite, and proceed by induction on $|B|$.

If $a \in acl\{b\}$, either $a=b$ or $F(a,b) \in acl(\phi)$ by 4.11. If $F(a,b) = F(a_1,b) = \ell \in acl(\phi)$, then if $ab \equiv a_1 b$ in $(M_1, racl(\phi))$, $Q(a,b,a_1,b)$ so $a=a_1$. This takes care of $|B| = 1$. Suppose $|B| > 1$; pick $b,c \in B$ distinct. By 4.12, if $a \in acl(B)$, $a \in \langle\{c\} \cup B\setminus\{c\} \cup acl\{b\}\rangle$ so either $Q(d,e,c,a)$ or $R(d,c,a)$ where $d,e \in acl(B\setminus\{c\})$. By induction, d and e are definable over $B\setminus\{c\}$. If $Q(d,e,c,a)$, clearly a is definable over B. If $R(d,c,a)$ and $a \equiv a_0(\{c,d\})$ in $(M_1,racl(\phi))$, then $\delta(d,c,a;d,c,a_0)$ implies $a = a_0$, so again a is definable over B.

Proof of Theorem 4.10: Let $C \subseteq racl(\phi)$ be the collection of points of M_1^{eq} definable from M. Then clearly (M_1,C) is definable in M. For the converse it suffices to show M is definable in $(M_1,racl(\phi))$.

If T is a relation 0-definable on M but not on $(M_1, racl(\phi))$ then since M is saturated there are $\bar{a},\bar{b} \in M$ realizing the same type in M_1, with $T(\bar{a}) \wedge \neg T(\bar{b})$. Towards a contradiction pick $\bar{a}$ and $\bar{b}$ of shortest possible length such that $\bar{a} \equiv \bar{b}$ in $(M_1,racl(\phi))$ but not in $(M,acl(\phi))$ where the last $acl(\phi)$ is in M^{eq}. If $\bar{a} = \langle a_1\rangle^\frown\bar{a}_0$ and $\bar{b} = \langle b_1\rangle^\frown\bar{b}_0$ then $\bar{a}_0 \equiv \bar{b}_0$ in $(M,acl(\phi))$ so find a strong automorphism of M taking $\bar{b}_0$ to $\bar{a}_0$; say it takes b_1 to a_2. So in $(M,acl(\phi))$, $a_1 \not\equiv a_2(\bar{a}_0)$ and so either $a_1 \in acl(\bar{a}_0)$ or $a_2 \in acl(\bar{a}_0)$. From 4.11 and 4.12, $acl(\bar{a}_0)$ is the same in M and in $(M_1,racl(\phi))$. But then $a_1 \equiv a_2(\bar{a}_0)$ in $(M_1,racl(\phi))$ implies $a_1 = a_2$ by 4.13. This contradiction finishes the proof.

SECTION 5

In this section we do the same as in the last, except for a structure of projective character. So we assume M is strongly minimal and if $p \in S^1(\phi)$ is the non-algebraic type, p satisfies (*) and is of projective character over $F(n)$. We prove

Theorem 5.1: (1) There is A, a 0-definable finite set, and a 0-definable equivalence relation χ on $M \setminus A$ with finite classes such that $(M \setminus A)/\chi$ has a 0-definable ternary relation S giving it a projective structure, such that S induces the projective structure on $p(M)/E$.

(2) If M, A, χ and S are as in (1) and we let M_1 be the structure with universe $(M \setminus A)/\chi$ and predicate S, there is some $C \subseteq M_1^{eq}$ with $(M \setminus A)/\chi$ and (M_1, C) interdefinable.

Again we assume M is saturated and $|M| > |Th(M)|$. One possible approach to (1) is to prove the analogue of 4.2, but we take a different tack. Essentially we construct an affine structure with M attached to it, apply 4.1 to it, and use this to induce the projective structure on M.

We start with a few more general facts.

Lemma 5.2: Let N be a saturated strongly minimal structure and $N_1 \succ N$. For any $\bar{a}, \bar{b}$ in N, if $\bar{a} \equiv \bar{b}$ in N, then $\bar{a} \equiv \bar{b}$ in the structure $(N_1; N)$ with a predicate for N.

Proof: It suffices to find an automorphism of N_1 fixing N setwise and taking $\bar{a}$ to $\bar{b}$. If $acl(\bar{a}) \neq acl(\bar{a}^\frown\bar{b})$, pick $c \in acl(\bar{a}^\frown\bar{b}) \setminus acl(\bar{a})$ and then $d \in acl(\bar{a}^\frown\bar{b})$ with $\bar{a}^\frown\langle c\rangle \equiv \bar{b}^\frown\langle d\rangle$;

$\bar{b}$ has the same dimension as $\bar{a}$, so any $d \in acl(\bar{a}^\frown\bar{b}) \setminus acl(\bar{b})$ will do. By iterating this, we can assume $acl(\bar{a}) = acl(\bar{b})$. Then find B a basis for N over $\bar{a}$ and complete it to a basis C for N_1 over $\bar{a}$. Then $\bar{a}C \equiv \bar{b}C$, so we can find an automorphism of N_1 fixing C pointwise and taking $\bar{a}$ to $\bar{b}$. It will take $acl(\bar{a} \cup B)$ to $acl(\bar{b} \cup B)$; that is, it fixes N setwise.

Note: By essentially the same proof, we can get this result in more generality.

Corollary 5.3: If N, N_1 as in 5.2, then any formula definable on $(N_1;N) \upharpoonright N$ is definable on N.

Proof: Take $(N_1^*;N^*) \succ (N_1;N)$ $|Th(N)|^+$ - saturated. If $\varphi(\bar{x})$ is a counter-example, we have $\bar{a},\bar{b} \in N^*$ realizing the same type in N^* but $\varphi(\bar{a}) \wedge \neg\varphi(\bar{b})$, contradicting 5.2.

Lemma 5.4: Suppose N is a structure, $N = N_1 \dot{\cup} N_2$ where N_1 is 0-definable in N and strongly minimal. Suppose there is an N-formula $\pi(x,y)$ such that: (1) if $a \in N_1 \setminus acl(\phi), \pi(a,N)$ is a subset of N_2 of size $k<\omega$, and (2) $\cup\{\pi(a,N): a \in N_1, |\pi(a,N)| \leq k\}$ covers all but finitely many points of N_2.

Then N has rank 1 and multiplicity $\leq 1+k$.

Proof: Suppose not. So for some $N^* \succ N$ with $|N^*| > |Th(N)|$ and $\varphi_i(x,\bar{a}_i)$ $(i \leq 1+k)$ formulas with parameters from $N^* = N_1^* \dot{\cup} N_2^*$ we have $\varphi_i(N^*,\bar{a}_i)$ infinite and if $i \neq j$ $\varphi_i(N^*,\bar{a}_i) \cap \varphi_j(N^*,\bar{a}_j) = \phi$. (2) implies $tp(\bar{a}_i|N_1^*)$ is algebraic and so isolated, say by $\psi_i(\bar{x},\bar{c}_i)$, where $\bar{c}_i \subseteq N_1^*$. Then $\exists\bar{y}(\varphi_i(x,\bar{y}) \wedge \psi_i(\bar{y},\bar{c}_i))$ has the same solutions in N_1^* as $\varphi_i(x,\bar{a}_i)$

does. This implies that $\varphi_i(x,\bar{a}_i) \wedge x \in N_1^*$ is N_1^*-definable, so at most one $\varphi_i(N^*,\bar{a}_i)$ intersects N_1^* in an infinite set. So we can assume that for $i \le k$ each $\varphi_i(N^*,\bar{a}_i)$ is an infinite subset of N_2^*. Let $\gamma_i(x,\bar{a}_i) \longleftrightarrow \exists y[\varphi_i(y,\bar{a}_i) \wedge \pi(x,y)]$; for each $i \le k$ ψ_i defines an infinite subset of N_1^*. As above, choose $\bar{c}_i \subseteq N_1^*$ with γ_i $\bar{c}_i$-definable, and let $a \in N_1^* \setminus \mathrm{ac}\ell\{\bar{c}_0{}^\frown \ldots {}^\frown \bar{c}_k\}$. For $i \le k$, $N^* \models \psi_i(a,\bar{a}_i)$ so we can find b_i with $\varphi_i(b_i,\bar{a}_i) \wedge \pi(a,b_i)$; the b_i's are disinct so $|\pi(a,N^*)| \ge k+1$, contradicting $N^* \succ N$ and (1).

Now we assume M is saturated of size $>|\mathrm{Th}(M)|$ and strongly minimal of projective character. As in section 3, we pick a ternary S inducing the lines on $p(M)/E$; so we assume that for any $a,b \in p(M)$ independent, $S(a,b,M)$, $S(a,M,b)$ and $S(M,a,b)$ each contain a finite non-empty subset of each E-class of $\mathrm{ac}\ell\{a,b\} \setminus [\mathrm{ac}\ell\{a\} \cup \mathrm{ac}\ell\{b\}]$. Now let $N \succ M$ and $N = \mathrm{ac}\ell(M \cup \{a\})$ for some $a \in N \setminus M$.

5.3 implies that M and $(N;M) \restriction M$ are essentially the same, i.e. any predicate definable on the set M in $(N;M)$ is definable in the structure M. $(N;M)$ has a predicate for M.

<u>Lemma 5.5</u>: (1) $(N;M)$ has rank 1 and finite multiplicity, and is saturated.

(2) For any $A \subseteq N$, $\mathrm{ac}\ell(A)$ is the same in $(N;M)$ as it is in N.

<u>Proof</u>: (1) For any $a,b \in N \setminus M$, there is an automorphism of N fixing M pointwise and taking a to b. Hence fixing a point $a \in N \setminus M$ does not affect M, and clearly if $((N;M),a)$ has rank 1 and finite multiplicity so does $(N;M)$. So we fix some (any) $a \in N \setminus M$. If $b \in M \setminus \mathrm{ac}\ell\{a\}$, $S(b,a,N)$ is a finite subset of $N \setminus M$. If

$c \in (N \setminus M) \setminus acl\{a\}$ in N, then in $p(N)/E$ $c' \in acl(\{a'\} \cup \{m' : m \in M\})$ so there is $m \in M$ with m', a' and c' collinear. Hence $S(N,a,c) \cap M \neq \phi$. Also for all but finitely many points $c \in acl\{a\}$ we have by 3.10 that $S(N,a,c)$ intersects $acl(\phi) \subseteq M$. We can assume for $b \in acl(\phi)$, $|S(b,a,N)| \leq k = |S(d,a,N)|$ where $a \perp d$. So we apply 5.4 to $((N;M),a)$ with $S(x,a,y) \wedge x \in M$ taking the part of π, M the part of N_1 and $N \setminus M$ the part of N_2. This does (1), as the saturation follows from $|(N;M)| > |Th(N;M)|$.

(2) Clearly if $b \in acl(A)$ in N, then $b \in acl(A)$ in $(N;M)$. If not the converse, pick a counter-example A,b with $|A|$ as small as possible. Then $|A|$ is independent in $(N;M)$ and so in N. For $A \subseteq M$, $acl(A)$ is the same in N as in $(N;M)$ by 5.3 and the fact that for any $a,b \in N \setminus M$ there is an N-automorphism fixing M pointwise taking a to b. So let $a \in A \setminus M$. For each $c \in A \setminus M$, $c \neq a$, choose $c^* \in M$ with $S(c^*,a,c)$ and let $A^* = (A \cap M) \cup \{a\} \cup \{c^* : c \in A \setminus M\}$. Then $b \in acl(A^*)$ in $(N;M)$ but not in N since in both c^* and c are algebraic in each other over a. If $b \notin M$ find $b^* \in M$ with $S(b^*,a,b)$; $b^* \in acl(A^*)$ in $(N;M)$ but not in N. So in $(N;M)$, $b^* \in acl(A^*) \setminus acl(A^* \setminus \{a\})$, so $a \in acl(A^* \setminus \{a\} \cup \{b^*\})$. But $A^* \setminus \{a\} \cup \{b^*\} \subseteq M$, a contradiction.

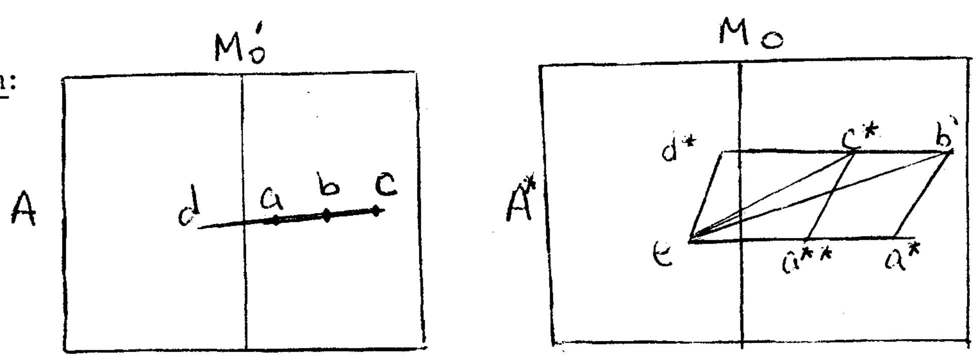
Now suppose M_0 satisfies 4.10. So it is a transitive, strongly minimal affine structure under Q and R, and P_Q and P_R give the canonical affine structure on M_0/E. We will show that a quotient of M can be identified with M_0' for some such M_0 later. First we notice a few things about M_0'. It is clear that M_0' is of projective character and that S_0, the ternary relation on it induced by Q and R, gives it a projective structure and induces the projective structure on $q(M_0')/E$

where q is the type of any non-algebraic element.

Lemma 5.6: (1) If $a,b \in M_0'$ and $a \in ac\ell\{b\}$ there is $f \in ac\ell(\phi) \cap M_0'$ with $S_0(b,f,a)$.
(2) Suppose A is a maximal subspace of M_0' containing the algebraic elements; $a,b,c \in M_0' \setminus A$, $a \neq b,c$ and $S_0(a,b,c)$. Then there is an automorphism of M_0' fixing $A \cup \{a\}$ pointwise taking b to c.
(3) Suppose U is a 0-definable binary algebraic relation on $q(M_0')$; then there is a finite $C \subseteq ac\ell(\phi) \cap M_0'$ such that C is 0-definable and for $a,b \in q(M_0')$ distinct $M_0' \models U(a,b) \longleftrightarrow \exists x \in C\ S_0(a,x,b)$. If U defines an equivalence relation, C is closed under S_0.

Proof: (1) We can assume $a \neq b$ and $a \notin ac\ell(\phi)$. Pick any $c \in M_0' \setminus ac\ell\{a\}$ and $d \notin \{a,c\}$ with $S_0(a,c,d)$. Then $d \notin ac\ell\{a\}$. In $q(M_0')/E$, a',c' and d' are collinear and $a' = b'$, so there is $e \in M_0'$ with eEd and $S_0(b,c,e)$. $S_0(c,a,d) \wedge S_0(c,b,e)$ and a,b,c,d and e are distinct so there is $f \in M_0'$ with $S_0(a,b,f) \wedge S_0(d,e,f)$. So $f \in ac\ell\{a\} \cap ac\ell\{d\} = ac\ell(\phi)$ and $S_0(b,f,a)$.
(2) Pick any $e \in M_0$; corresponding to e and A there is a maximal subspace A^* of M_0; recall $F: M_0^2 \setminus \{(d,d): d \in M_0\} \longrightarrow M_0'$. There is $d \in A$ with $S_0(d,a,b)$. We can find $a^* \in M_0$ with $F(e,a^*) = a$ and $b^*,d^* \in M_0$ with $F(e,b^*) = b$, $F(e,d^*) = d$ and $Q(e,d^*,a^*,b^*)$.

Diagram:



Then the line through d^* and b^* meets the line through e corresponding to c at, say, c^*. Let a^{**} be such that $Q(e,d^*,a^{**},c^*)$. Now $a^*,a^{**} \notin A^* = \mathrm{acl}(A^*)$, so there is an automorphism of M_0 fixing A^* pointwise taking a^* to a^{**}. It then takes b^* to c^*. This induces an automorphism of M_0' fixing $A \cup \{a\}$ pointwise and taking b to c.

(3) If $a,b \in q(M_0')$ and $U(a,b)$, then from (1) there is $c \in \mathrm{acl}(\phi)$ with $S_0(a,c,b)$. Fix $a \in q(M_0')$ and for each b with $U(a,b)$ choose such a c, and let C be the collection of these c's. Clearly C is 0-definable. From (2) it follows that any $d \in q(M_0')$ with $S_0(a,c,d)$ has $ab \equiv ad$. So for any a,d distinct in $q(M_0')$, $M_0' \models U(a,d) \longleftrightarrow \exists x \in C\ S_0(a,x,d)$. Now suppose $c_0,c_1 \in C$ and $S_0(c_0,c_1,c_2)$. We may assume $c_0 \neq c_1$. Pick $a,b_0,b_1 \in q(M_0')$ with $S_0(a,c_0,b_0) \wedge S_0(a,c,b_1)$, $U(a,b_0) \wedge U(a,b_1)$. From $S_0(c_0,c_1,c_2) \wedge S_0(c_0,a,b_0)$ find d with $S_0(a,c_1,d) \wedge S_0(b_0,c_2,d)$. $U(a,d)$ as $d \in q(M_0')$. If U is an equivalence relation, $U(b_0,d)$. So $c_2 \in C$.

<u>Proof of 5.1(1)</u>: Consider $(N;M)$. It has rank 1 and finite multiplicity; one of the strongly minimal sets is M and the others are subsets of $N\setminus M$. Let K be one of them. If r is the strong type of an element of K, certainly r satisfies (*) by 5.5(2). Clearly also r is non-orthogonal to p. In $[(N;M)\setminus \mathrm{acl}(\phi)]/E$ where we recall $a\,E\,b$ iff $a \in \mathrm{acl}\{b\}$ and $a \equiv^s b$ we have $p(M)/E$ non-orthogonal to K/E; P_S witnesses this. Since both are, by 3.4, strictly minimal $\aleph_0$-categorical sets, K/E is either affine or projective over the same field $F(n)$ over which $p(M)/E$ is projective. Now by 5.5(2), if

$a,b \in K$ are independent, $\mathrm{ac}\ell\{a,b\}$ is the same in $(N;M)$ as in N. In N, $\mathrm{ac}\ell\{a,b\}$ intersects $n+1$ different E-classes, one of which is in $p(M)$. So in K, $\mathrm{ac}\ell\{a,b\}$ intersects at most n E-classes. So K is of affine character over $F(n)$. By 4.1, there is θ 0-definable on K such that K/θ is an affine structure over $F(n)$.

Consider the structure $M_1 = M \dot{\cup} K/\theta \dot{\cup} (K/\theta)'$. This consists of three strongly minimal sets, and $[M_1 \setminus \mathrm{ac}\ell(\phi)]/E$ has three strictly minimal $\aleph_0$-categorical sets. $p(M)/E$ and $q(M_1)/E$ are both projective and they are non-orthogonal, where q is the type of an element of $(K/\theta)'$. So by 1.9, in $[M_1 \setminus \mathrm{ac}\ell(\phi)]/E$ there is a 0-definable bijection between them. So in M_1 there is a formula $\tau(x,y)$ with $\tau(a,M_1)$ a finite subset of $q(M_1)$ for $a \in p(M)(= p(M_1))$ and $\tau(M_1,b)$ a finite subset of $p(M)$ for $b \in q(M_1)$.

Let $\tau^*(x,y) \longleftrightarrow \exists z(\tau(z,x) \wedge \tau(z,y))$; this gives an algebraic relation on $q(M_1)$. Find $C \subseteq (K/\theta)'$ as in 5.6(3). Then consider the localization of $(K/\theta)'$ at C; recall that this means the equivalence relation χ_C on $(K/\theta)' \setminus \langle C\rangle$ with $\chi_C(a,b)$ iff $\exists c \in \langle C\rangle S_0(a,c,b)$. Here S_0 is the ternary relation giving the lines on $(K/\theta)'$ and $\langle C\rangle$ is the closure of C under S_0. Then τ induces a function from M minus a finite set A onto $[(K/\theta)' \setminus \langle C\rangle]/\chi_C$ minus a finite set — by expanding C we assume the latter is empty. So if we take the appropriate equivalence relation χ on $M \setminus A$, τ induces a 0-definable bijection between $(M \setminus A)/\chi$ and the projective space $[(K/\theta)' \setminus \langle C\rangle]/\chi_C$.

S_0 transfers in a natural manner to a ternary relation on $((K/\theta)' \setminus \langle C\rangle)/\chi_C$. The pre-image S of this certainly induces the lines on $p(M)/E$. Since χ is on M, it is 0-definable on M, as are A

and S. This completes the proof of 5.1(1).

Towards the proof of 5.1(2) let us start with (M,S) a projective space over $F(n)$ such that S induces the lines on $p(M)/E$; all as above except we assume $A = \phi$ and χ is the identity. Then take $N \succ M$, $N = acl(M \cup \{a\})$ for some (any) $a \in N \setminus M$, as above. Again, consider $(N;M)$. 5.5 holds, and in this case:

<u>Lemma 5.7</u>: $(N;M) \setminus M = M_0$ is a strongly minimal structure satisfying 4.10. M_0' is M; strictly speaking, there is in $(M;N)^{eq}$ a 0-definable bijection between M_0' and M.

<u>Proof</u>: We know from 5.5 that $(N;M)$ has rank 1 and finite multiplicity. Suppose M_0 is not strongly minimal. For any $a,b \in M_0$ independent, from 5.5 $acl\{a,b\}$ intersects M, and since S induces the lines on N/E, there is $c \in M$ with $S(a,b,c)$; $c \notin acl(\phi)$. We could have chosen a,b in different strongly minimal pieces of M_0, so for any $a \in M_0$, $c \in p(M)$, there is b in a different strongly minimal piece of M_0 than a with $S(a,b,c)$. Let C be the strongly minimal piece containing $a \in M_0$, and $c,d \in p(M)$ be independent. We count $\langle c,d,a\rangle \cap C$ in two different ways. Let $b \in M_0 \setminus C$ be such that $S(a,b,c)$ and $k = |\{f: S(a,c,f)\} \cap C| = |\{f: S(b,c,f)\} \cap C|$. For each $e \in \langle c,d\rangle \subseteq p(M)$, $ea \equiv ca$ and $eb \equiv cb$ in $(N;M)$, so $|\{f: S(a,e,f)\} \cap C| = k = |\{f: S(b,e,f)\} \cap C|$. Now

$$\langle c,d,a\rangle = \langle c,d,b\rangle = \bigcup_{e\in\langle c,d\rangle} S(a,e,N) = \bigcup_{e\in\langle c,d\rangle} S(b,e,N).$$

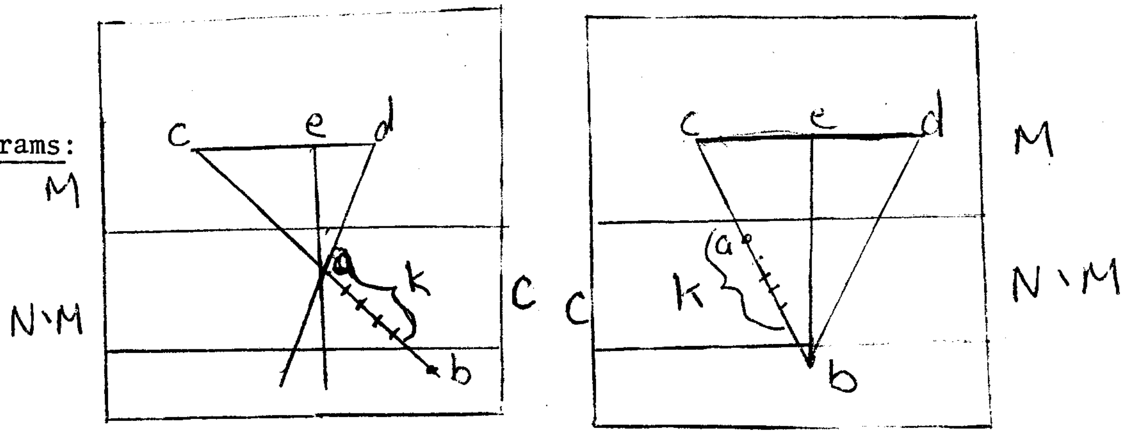
For distinct $e_0,e_1 \in \langle c,d\rangle$, the lines $S(a,e_0,N)$ and $S(a,e_1,N)$ meet at $a \in C$, but the lines $S(b,e_0,N)$ and $S(b,e_1,N)$ meet at $b \notin C$.

So $|\bigcup_{e\in\langle c,d\rangle} S(a,e,N) \cap C| = (n+1)(k-1) + 1$. But

$|\bigcup_{e\in\langle c,d\rangle} S(b,e,N) \cap C| = k(n+1)$, a contradiction. So M_0 is strongly

minimal.

Diagrams:



Now there is a natural way to define R and Q on M_0 giving it an affine structure. We let $R(a,b,c) \longleftrightarrow S(a,b,c)$ for $a,b,c \in M_0$ and for $a,b,c,d \in M_0$, let

$$Q(a,b,c,d) \longleftrightarrow \exists x,y \in M(S(a,b,x) \wedge S(c,d,x) \wedge S(a,c,y) \wedge S(b,d,y)).$$

See just after 1.8. It is easily checked that Q and R give M_0 an affine structure; we know M_0 is strongly minimal and transitive, and 5.5 tells us M_0 is of affine character. The fact that S induces the lines on $p(N)/E$ and 5.5 imply that R induces the lines on M_0/E. Since the addition on M_0/E is derived from the lines on N/E in the same manner as Q was derived from S, Q induces the addition. So M_0 satisfies 4.10. That M_0' is M is also clear.

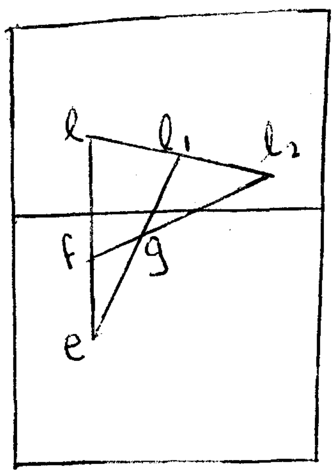
Proof of 5.1(2): Let M_1 be the structure with universe M and predicate S. In this situation, let $\mathrm{rac}\ell(\phi) \subseteq M_1^{eq}$ consist of the definable closure taken in M_1^{eq} of the algebraic points of M and the field elements. As in the proof of 4.10, it suffices to show that if

$a \equiv b(\bar{c})$ in $(M_1, \mathrm{rac}\ell(\phi))$ then $a \equiv b(\bar{c})$ in $(M, \mathrm{ac}\ell(\phi))$; the last $\mathrm{ac}\ell(\phi)$ is taken in M^{eq}.

Let N be formed from M as usual; let N_1 be obtained from $(M_1, \mathrm{rac}\ell(\phi))$ in the same manner; we can assume N and N_1 have the same universe. If $a \equiv b(\bar{c})$ in $(M_1, \mathrm{rac}\ell(\phi))$, there is an automorphism γ of $(N_1; M_1)$ fixing $\bar{c}$ pointwise and taking a to b. For any $e \in N_1 \setminus M_1$ we can assume this automorphism fixes e. Now pick $\ell \in M \cap \mathrm{ac}\ell(\phi)$ if this is not empty and $f \neq e$ in $N_1 \setminus M_1$ with $S(e,\ell,f)$. By 5.6(2) there is an automorphism β of N_1 fixing $M \cup \{e\}$ pointwise and taking $\gamma(f)$ to f. If $M \cap \mathrm{ac}\ell(\phi) = \phi$, let β be the identity.

$\beta\gamma$ is an automorphism of $(N_1; M_1)$; it fixes $\mathrm{ac}\ell(\phi) \cap M$ and the field elements as well as e. Further, it fixes the classes of the relation $Q(x,y,z,w) \wedge F(x,y) = \ell$ on $N_1 \setminus M_1$. Let $\ell_1 \neq \ell$ be any point of $M \cap \mathrm{ac}\ell(\phi)$, and $g \in N_1 \setminus M_1$ be such that $e \neq g$ and $S(e,\ell_1,g)$. Pick ℓ_2 with $S(\ell,\ell_1,\ell_2) \wedge S(f,g,\ell_2)$; this comes from $S(e,f,\ell) \wedge S(e,g,\ell_1)$. Now $\ell_2 \in M \cap \mathrm{ac}\ell(\phi)$, so is fixed by $\beta\gamma$, as are e,f and ℓ_1. So g is fixed by $\beta\gamma$.

<u>Diagram:</u>



Thus for any $\ell_1 \in M \cap ac\ell(\phi)$ the classes of the relation $Q(x,y,z,w) \wedge F(x,y) = \ell_1$ on $N \setminus M$ are fixed by $\beta\gamma$.

Applying 4.10 we get that if $M_0 = (N;M) \setminus M$, then $(M_0, ac\ell(\phi))$ is interdefinable with $(M_2, rac\ell(\phi))$ where M_2 has universe $N \setminus M$, predicates Q and R, and here $rac\ell(\phi)$ is the definable closure of $M_2' \cap ac\ell(\phi) = M \cap ac\ell(\phi)$, the field elements, and the classes of relations $Q(x,y,z,w) \wedge F(x,y) = \ell$ for $\ell \in M \cap ac\ell(\phi)$. So the restriction of $\beta\gamma$ to $N \setminus M$ is an automorphism of $(M_0, ac\ell(\phi))$. It extends uniquely to an automorphism f of $(M_0, ac\ell(\phi))'$ which is $(M, ac\ell(\phi))$. For any a,b distinct in M_0, we must have $f(F(a,b)) = F(\beta\gamma(a), \beta\gamma(b)) = \beta\gamma(F(a,b))$, so $\beta\gamma \restriction M$ is an automorphism of $(M, ac\ell(\phi))$. It fixes $\bar{c}$ pointwise and takes a to b. This completes the proof of 5.1(2).

SECTION 6

Our aim in this section is to extend the affine or projective structure found to hold on a strong type (in the projective case on a strong type plus some algebraic points) to a formula in a weakly minimal structure. We assume M is a saturated weakly minimal structure of size $>|Th(M)|$, p is a strong 1-type over ϕ which is non-algebraic.

If we assume there are almost 0-definable formulas of M giving an affine or projective structure on $p(M)$, we show here that there is an almost 0-definable subset of M on which the same formulas give us an affine or projective structure. We begin with the affine case. We first prove:

Lemma 6.1: Suppose there is an almost 0-definable predicate Q which acts as an abelian group operation on $p(M)$ (i.e. its restriction to $p(M)$ satisfies #5,6,7 and 10 of the axioms for an affine space), and $\lambda(x) \in p$. Then there is $\sigma(x) \in p$ with $M \models \sigma(x) \rightarrow \lambda(x)$ such that Q acts as an abelian group operation on $\sigma(M)$.

Towards the proof of 6.1, we first pick by compactness $\sigma_0(x) \in p$ with $M \models \sigma_0(x) \rightarrow \lambda(x)$ and if $a,b,c,d,e,f \in \sigma_0(M)$, then
(1) $\exists^{!w} Q(a,b,c,w)$, (2) $Q(a,b,a,b)$,
(3) $Q(a,b,c,d) \rightarrow Q(a,c,b,d) \wedge Q(b,a,d,c)$ and
(4) $Q(a,b,c,d) \wedge Q(a,b,e,f) \rightarrow Q(c,d,e,f)$. σ_0 may not be the formulas we seek since it may not be closed under Q. But any $\sigma(x) \in p$ with

$$M \models (\sigma(x) \rightarrow \sigma_0(x)) \wedge (\sigma(x) \wedge \sigma(y) \wedge \sigma(z) \wedge Q(x,y,z,w) \rightarrow \sigma(w))$$

will do. Pick $\sigma_1(x) \in p$ with $M \models \sigma_1(x) \to \sigma_0(x)$ and

$$M \models \sigma_1(x) \wedge \sigma_1(y) \wedge \sigma_1(z) \wedge Q(x,y,z,w) \to \sigma_0(w).$$

Define $\sim$ on $\sigma_1(M)$ by $c \sim d$ iff for all $a \in p(M)$, there is $b \in p(M)$ with $Q(a,b,c,d)$.

<u>Lemma 6.2</u>: (i) $c \sim d$ iff for some $a \in p(M)$, $b \in p(M)$, $Q(a,b,c,d)$.

(ii) $\sim$ is an equivalence relation on $\sigma_1(M)$.

<u>Proof</u>: (i) ($\Rightarrow$) obvious. ($\Leftarrow$) Pick any $e \in p(M)$, and f with $Q(e,f,c,d)$. By (1) and (3) we can, and $M \models \sigma_0(f)$. $Q(a,b,c,d) \wedge Q(e,f,c,d)$ give, from (3) and (4), $Q(a,b,e,f)$; since $a,b,e \in p(M)$, $f \in p(M)$ and we are done.

(ii) From (2) and (3), $Q(a,a,b,b)$, so reflexivity is clear. Suppose $c \sim d$ and $a \in p(M)$. Pick $b \in p(M)$ with $Q(a,b,c,d)$; then (3) gives $Q(b,a,d,c)$ so (i) gives $d \sim c$. Now suppose $c \sim d$ and $d \sim e$, and $a \in p(M)$. Pick $b \in p(M)$ with $Q(a,b,c,d)$ and then $f \in p(M)$ with $Q(b,f,d,e)$. Then by (3), $Q(b,d,a,c) \wedge Q(b,d,f,e)$ so by (4), $Q(a,c,f,e)$ and by (3) again, $Q(a,f,c,e)$. So $c \sim e$ by (i). So $\sim$ is transitive.

<u>Lemma 6.3</u>: Suppose $\sigma_1(c) \wedge \sigma_1(d)$ and neither c nor d is algebraic. Then $c \equiv^s d$ iff $c \sim d$.

<u>Proof</u>: ($\Rightarrow$) Let $a \in p(M) \setminus acl\{c,d\}$. We can find $f \sim c$ with $f \notin acl\{a,c,d\}$ for as b varies through $p(M)$, $Q(a,b,c,M)$ varies through $c/\sim$ and by (1) and (3) distinct b's give distinct f's with $Q(a,b,c,f)$; so there are $|M|$ f's with $c \sim f$. Pick $b \in p(M)$

with $Q(a,b,c,f)$. Since $c \equiv^s d$ and $c,d \notin acl\{a,f\}$ there is a strong automorphism of M fixing a and f taking c to d. It takes b to some $g \in p(M)$ with $Q(a,g,d,f)$. So $d \sim f$ and so $c \sim d$.

($\Leftarrow$) Suppose $c \sim d$. We can assume $c \perp d$ as we can compare both to an element of $c/\sim \setminus acl\{c,d\}$. Choose $a \in p(M) \setminus acl\{c,d\}$, $e \equiv^s c$, $e \notin acl\{a,c,d\}$. Pick $b \in p(M)$ with $Q(a,b,c,d)$ and, using the above, $f \in p(M)$ with $Q(a,f,c,e)$. $f,b \notin acl\{a,c\}$ and $f \equiv^s b$, so $abc \equiv^s afc$. So $d \equiv^s e \equiv^s c$.

Choose $\sigma_2(x) \in p$ with $M \models \sigma_2(x) \to \sigma_1(x)$ and $M \models \sigma_2(x) \wedge \sigma_2(y) \wedge \sigma_2(z) \wedge Q(x,y,z,w) \to \sigma_1(w)$. 6.3 implies that if $\sigma_2(c) \wedge \neg\sigma_2(d)$, $a,b \in p(M)$ and $Q(a,b,c,d)$ or $Q(a,b,d,c)$ then either c or d is algebraic. By compactness, find I and J finite with $I \cup J$ minimal such that $I \cup J \subseteq acl(\phi)$ and $\sigma_2(a) \wedge \neg\sigma_2(b)$ and $a \sim b$ implies $a \in I$ or $b \in J$. Let $\sigma_3(x) \longleftrightarrow (\sigma_2(x) \vee x \notin I) \vee x \in J$. $\sigma_3(x) \in p$.

<u>Lemma 6.4</u>: If $\sigma_1(a) \wedge \sigma_1(b)$ and $a \sim b$, then $\sigma_3(a) \longleftrightarrow \sigma_3(b)$.

<u>Proof</u>: Since we can compare both a and b to a non-algebraic element of $a/\sim$, we can assume $a \notin acl(\phi)$. If $\sigma_3(a)$ then $\sigma_2(a)$ and $a \notin I$, so any $c \sim a$ with $\neg\sigma_2(c)$ is in J. If $d \sim a$ and $\sigma_2(d)$, any $c \sim d$ has $c \sim a$, so either $\sigma_2(c)$ or $c \in J$; by the minimality of $I \cup J$, $d \notin I$. So $b \in (\sigma_2(M) \cup J) \setminus I = \sigma_3(M)$. Similarly if $\neg\sigma_3(a)$.

Now let

$$\sigma_4(x,y) \longleftrightarrow \sigma_3(x) \wedge \forall z,w[\sigma_3(z) \wedge (Q(x,y,z,w) \vee Q(x,y,w,z)) \to \sigma_3(w)].$$

Note that if $a,b \in p(M)$, $M \models \sigma_4(a,b)$.

<u>Lemma 6.5</u>: If $a \in p(M)$, $\sigma_4(b,a) \wedge \sigma_4(c,a)$ and $Q(a,b,c,d) \vee Q(a,b,d,c)$, then $\sigma_4(d,a)$.

<u>Proof</u>: Since $\sigma_4(b,a) \wedge \sigma_3(c)$, $\sigma_3(d)$. So to show $\sigma_4(d,a)$, we pick e,f with $\sigma_3(e) \wedge (Q(a,d,e,f) \vee Q(a,d,f,e))$ and we show $\sigma_3(f)$. Notice $\sigma_0(f)$. We have four cases:

(i) $Q(a,b,c,d) \wedge Q(a,d,e,f)$. Pick g with $Q(a,b,e,g)$; since $\sigma_4(b,a) \wedge \sigma_3(e)$, $\sigma_3(g)$. Also $Q(b,d,g,f)$, so $Q(a,c,g,f)$; since $\sigma_4(c,a) \wedge \sigma_3(g)$, $\sigma_3(f)$.

(ii) $Q(a,b,c,d) \wedge Q(a,d,f,e)$. Pick g with $Q(a,b,g,e)$; $\sigma_3(g)$. Pick h with $Q(a,d,g,h)$; $\sigma_0(h)$, so $Q(b,d,e,h) \wedge Q(f,g,e,h)$. So $Q(b,d,f,g)$, so $Q(a,c,f,g)$. Since $\sigma_4(c,a) \wedge \sigma_3(g)$, $\sigma_3(f)$.

(iii) $Q(a,b,d,c) \wedge Q(a,d,e,f)$. Pick g with $Q(a,b,g,e)$; $\sigma_3(g)$. So $Q(d,c,g,e)$. Pick h with $Q(a,d,h,g)$, so $Q(e,f,h,g) \wedge Q(a,h,c,e)$, so $Q(a,c,g,f)$. So $\sigma_3(f)$.

(iv) $Q(a,b,d,c) \wedge Q(a,d,f,e)$. Pick g with $Q(a,b,e,g)$; $\sigma_3(g)$. $Q(d,c,e,g)$ so $Q(a,c,f,g)$ so $\sigma_3(f)$.

<u>Lemma 6.6</u>: If $a,b \in p(M)$, then $M \models \sigma_4(x,a) \longleftrightarrow \sigma_4(x,b)$. Hence there is $\sigma(x) \in p$ such that for any $a \in p(M)$ $M \models \sigma(x) \longleftrightarrow \sigma_4(x,a)$.

<u>Proof</u>: If $\sigma_4(c,a) \wedge \neg\sigma_4(c,b)$ there must be d and e with $\sigma_3(d) \wedge \neg\sigma_3(e) \wedge (Q(b,c,d,e) \vee Q(b,c,e,d))$. If $Q(b,c,d,e)$, pick f with $Q(a,b,f,d)$, so $\sigma_3(f)$. But also $Q(a,c,f,e)$; since $\sigma_4(c,a)$, $\sigma_3(e)$. If $Q(b,c,e,d)$, pick f with $Q(a,c,f,d)$, so $\sigma_3(f)$. But then $Q(a,b,f,e)$ gives $\sigma_3(e)$. This contradiction establishes the lemma.

Proof of 6.1: Suppose $\sigma(a) \wedge \sigma(b) \wedge \sigma(c) \wedge Q(a,b,c,d)$. Pick $e \in p(M)$ and f with $Q(a,b,e,f)$. By the two previous lemmas, $\sigma(f)$. Also $Q(e,f,c,d)$ so again $\sigma(d)$.

We now take care of the affine case:

Lemma 6.7: Suppose there are ternary R and 4-ary Q almost 0-definable on M whose restrictions give an affine space structure over $F(n)$, say, on $p(M)$. Then for any $\lambda(x) \in p$ there is $\tau(x) \in p$ such that Q and R give an affine space structure over $F(n)$ on $\tau(M)$ and $\tau(M) \subseteq \lambda(M)$.

Proof: First find $\tau_0(x) \in p$ such that #'s 0 through 11 hold on $\tau_0(M) \subseteq \lambda(M)$, as do the following affine space theorems:

#12 $Q(x,y,z,w) \wedge R(x,w,v) \rightarrow \exists r,s[R(x,y,r) \wedge R(x,z,s) \wedge Q(x,r,s,v)]$

#13 $Q(x,y,z,w) \wedge R(x,y,v) \rightarrow \exists r,s[R(x,z,r) \wedge R(x,w,s) \wedge Q(x,r,s,v)]$

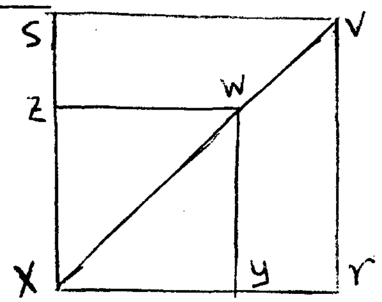
#14 $R(y,z,w) \rightarrow \exists r,s[R(x,y,r) \wedge R(x,z,s) \wedge Q(x,r,s,w)]$.

For example, if $a,b,c,d,e \in \tau_0(M)$, then

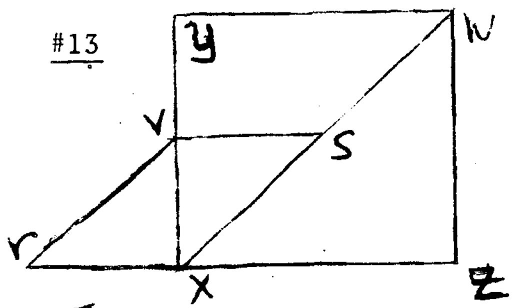
$Q(a,b,c,d) \wedge R(a,d,e) \rightarrow \exists r,s[R(a,b,r) \wedge R(a,c,s) \wedge Q(a,r,s,e)]$.

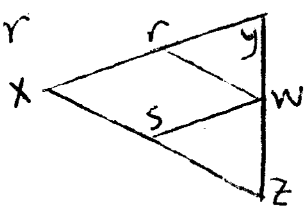
Diagrams:

#12



#13





#14

Then find $\tau_1(x) \in p$ such that $M \models (\tau_1(x) \rightarrow \tau_0(x)) \wedge (\tau_1(x) \wedge \tau_1(y) \wedge R(x,y,z) \rightarrow \tau_0(z))$. Applying 6.1, we find $\sigma(x) \in p$ with $M \models \sigma(x) \rightarrow \tau_1(x)$ and $\sigma(M)$ closed under Q. Let $\tau(x,y) \longleftrightarrow \sigma(x) \wedge \sigma(y) \wedge \forall z(R(x,y,z) \longrightarrow \sigma(z))$. Note that if $a,b \in p(M)$, $M \models \tau(a,b)$.

Now suppose $a \in p(M)$, $\tau(b,a) \wedge \tau(c,a) \wedge Q(a,b,c,d)$. Then $\tau(d,a)$; for consider any e with $R(a,d,e)$. $\tau_0(e)$, so by #12, find f and g with $R(a,b,f) \wedge R(a,c,g) \wedge Q(a,f,g,e)$. By $\tau(b,a)$, $\sigma(f)$; by $\tau(c,a)$, $\sigma(g)$ and by $Q(a,f,g,e)$, $\sigma(e)$.

Suppose $a \in p(M)$, $\tau(b,a) \wedge \tau(c,a) \wedge Q(a,b,d,c)$. Using #13, $\tau(d,a)$.

Now suppose $a \in p(M)$, $\tau(b,a) \wedge \tau(c,a) \wedge \tau(d,a) \wedge Q(b,c,d,e)$. Then $\tau(e,a)$, for pick f with $Q(a,b,f,c)$; then $\tau(f,a)$ and also $Q(a,f,d,e)$, so $\tau(e,a)$ by the two previous arguments.

Suppose $a \in p(M)$, $\tau(b,a) \wedge \tau(c,a) \wedge R(b,c,d)$. From #14, find e and f with $R(a,b,e) \wedge R(a,c,f) \wedge Q(a,e,f,d)$. So $\sigma(d)$. Suppose $R(a,d,g)$. From #12, find h,i with $R(a,e,h) \wedge R(a,f,i) \wedge Q(a,h,i,g)$. If $a=e$, $a=h$; if $a \neq e$, $R(a,b,h)$ from $R(a,b,e) \wedge R(a,e,h)$. Similarly $R(a,c,i)$. So $\sigma(h) \wedge \sigma(i)$ and so $\sigma(g)$. This shows $\tau(d,a)$. Thus for any $a \in p(M)$, $\tau(M,a)$ is closed under Q and R; if we show that $M \models \tau(x,a) \longleftrightarrow \tau(x,b)$ for any $a,b \in p(M)$ we can then find $\tau(x) \in p$ such that $M \models \tau(x) \longleftrightarrow \tau(x,a)$ and this τ is our formula. Suppose $\tau(c,a) \wedge R(b,c,d)$ where $a,b \in p(M)$; then $\tau(b,a)$, so $\tau(d,a)$ by the above argument. So $\sigma(d)$ and so $\tau(c,b)$. This completes the proof.

Now we turn to the projective case. So we assume M has an almost 0-definable ternary relation S such that $p^*(M)$ is a projective space over $F(n)$ under the restriction of S. Here $p^*(M)$ is the closure of $p(M)$ under S, and $p^*(M) \setminus p(M) \subseteq acl(\phi)$. Also, as in 3.11, p^* is a possibly incomplete type over $acl(\phi)$.

Lemma 6.8: For any $\lambda(x) \in p^*$, there is $\sigma(x) \in p^*$ such that $\sigma(M) \subseteq \lambda(M)$ is a projective space over $F(n)$ under S.

As a first approximation, pick $\sigma_0(x) \in p^*$ with $\sigma_0(M) \subseteq \lambda(M)$ such that if $a,b,c,d,e \in \sigma_0(M)$, then:

(1) $a \neq b \rightarrow \exists^{!n+1} z S(a,b,z)$

(2) $S(a,b,a)$; $S(a,a,b) \rightarrow b = a$

(3) $S(a,b,c) \rightarrow S(b,a,c)$

(4) $S(a,b,c) \wedge a \neq c \rightarrow S(a,c,b)$

(5) $S(a,b,c) \wedge S(a,b,d) \wedge a \neq c \rightarrow S(a,c,d)$

(6) $S(a,b,c) \wedge S(a,d,e) \wedge a,b,c,d,e$ distinct $\rightarrow \exists u(S(b,d,u) \wedge S(c,e,u))$.

If we can find $\sigma(x) \in p^*$ with

$$M \models (\sigma(x) \rightarrow \sigma_0(x)) \wedge (\sigma(x) \wedge \sigma(y) \wedge S(x,y,z) \rightarrow \sigma(z))$$

we will be done. Pick $\sigma_1(x) \in p^*$ with

$$M \models (\sigma_1(x) \rightarrow \sigma_0(x)) \wedge (\sigma_1(x) \wedge \sigma_1(y) \wedge S(x,y,z) \rightarrow \sigma_0(z)).$$

Let $\sigma_2(x,y) \longleftrightarrow \sigma_1(x) \wedge \sigma_1(y) \wedge \forall z(S(x,y,z) \rightarrow \sigma_1(z))$. Note $\sigma_2(a,b)$ if $a,b \in p^*(M)$. If $\sigma_2(c,a)$ for some $a \in p(M) \setminus acl\{c\}$, then $\sigma_2(c,b)$ for any $b \in p(M) \setminus acl\{c\}$, so there is $\sigma_3(x) \in p^*$ with $\sigma_3(c) \longleftrightarrow \sigma_2(c,a)$ for any $a \in p(M) \setminus acl\{c\}$.

<u>Lemma 6.9</u>: If $p^*(a)$, $\sigma_3(b) \wedge \neg\sigma_3(c) \wedge S(a,b,c)$, then either b or c is algebraic.

<u>Proof</u>: First suppose $b \notin acl(\phi)$, $a \in p(M) \setminus acl\{b\}$, $\sigma_3(b) \wedge S(a,b,c)$. Certainly $\sigma_1(c)$, and if $c \neq a$, $a \notin acl\{c\}$. This is because $a \in acl\{c\}$ implies $c \in acl\{a\}$, giving $b \in acl\{a\}$ and so $a \in acl\{b\}$. So if not $\sigma_3(c)$, there is d with $S(a,c,d) \wedge \neg\sigma_1(d)$. Note that $\sigma_0(d)$ as $\sigma_1(a) \wedge \sigma_1(c)$. But $S(a,b,c) \wedge S(a,b,d) \wedge a \neq c$ gives $S(a,b,d)$ and this contradicts $\sigma_2(a,b)$. So $\sigma_3(c)$.

Now suppose $a \in p^*(M)$, $\sigma_3(b) \wedge S(a,b,c)$ and $b,c \notin acl(\phi)$. If $\neg\sigma_3(c)$, we have $a \in acl\{b\}$ by the above. Hence $b \not\downarrow c$. Pick $d \in p(M) \setminus acl(b)$ and $e \notin \{b,d\}$ with $S(b,d,e)$; $\sigma_3(e)$ by the above. a,b,c,d and e must be distinct, so from (6) and $S(b,a,c) \wedge S(b,d,e)$ find f with $S(a,d,f) \wedge S(c,e,f)$. $S(a,d,f)$ implies $f \in p^*(M)$. Now $f \notin acl\{e\}$, for if $f = e$, then $S(b,d,e) \wedge S(a,d,e)$ give $S(a,b,d)$ and $d \in acl\{b\}$; if $f \neq e$ and $f \not\downarrow e$, then $c \in acl\{e\}$, so $b \in acl\{e\}$ and $d \in acl\{b\}$ contrary to assumption. But the previous paragraph shows that $\sigma_2(e,f) \wedge S(f,e,c)$ implies $\sigma_3(c)$.

So we can find I,J finite sets of algebraic elements such that $p^*(a)$ and $\sigma_3(b) \wedge \neg\sigma_3(c) \wedge S(a,b,c)$ imply $b \in I$ or $c \in J$; pick them with $I \cup J$ minimal. Let $\sigma_4(x) \longleftrightarrow (\sigma_3(x) \wedge x \notin I) \vee x \in J$. Then $\sigma_4(x) \in p^*$ for if $a,b \in p^*(M)$ and $S(a,b,c)$, $c \in p^*(M)$, so $\sigma_3(c)$; thus $b \in \sigma_3(M)$ and $b \notin I$.

Lemma 6.10: If $p^*(a)$, $\sigma_4(b) \wedge S(a,b,c)$, then $\sigma_4(c)$.

Proof: Towards a contradiction let a,b,c yield a counter-example. If $\sigma_3(b) \wedge \neg\sigma_3(c)$, either $b \in I$ or $c \in J$, a contradiction. Three cases remain:

Case 1: $\sigma_3(b) \wedge \sigma_3(c)$. So $c \in I$. By the minimality of $I \cup J$ there is $e \in p^*(M)$ and $d \notin J$ with $\neg\sigma_3(d) \wedge S(e,c,d)$. Either $e = a$ or a,b,c,d,e are distinct; if $e = a$, $S(a,b,c) \wedge S(a,c,d)$ gives $S(a,b,d) \wedge \sigma_3(b) \wedge \neg\sigma_3(d) \wedge b \notin I \wedge d \notin J$, a contradiction. If $e \neq a$, there is f with $S(a,e,f) \wedge S(b,d,f)$ from $S(c,a,b) \wedge S(c,e,d)$. So $f \in p^*(M)$ but $\sigma_3(b) \wedge \neg\sigma_3(d) \wedge b \notin I \wedge d \notin J$, again a contradiction.

Case 2: $\neg\sigma_3(b) \wedge \neg\sigma_3(c)$. So $b \in J$. There are d and e with $e \in p^*(M)$, $d \notin I$, $S(e,d,b) \wedge \sigma_3(d)$. Again either $e = a$ or a,b,c,d and e are distinct. $e = a$ implies $S(a,d,c) \wedge \sigma_3(d) \wedge \neg\sigma_3(c)$; since $d \notin I$, $c \in J$, a contradiction. $e \neq a$ implies there is f with $S(a,e,f) \wedge S(c,d,f)$; $f \in p^*(M)$, so again we have a contradiction.

Case 3: $\neg\sigma_3(b) \wedge \sigma_3(c)$. So $b \in J$ and $c \in I$. By the minimality of $I \cup J$ and $c \in I$ there is $e \in p^*(M)$, $d \notin \sigma_3(M) \cup J$ with $S(e,d,c)$; note $\sigma_0(d)$. If $e = a$, $S(a,b,d)$. If $e \neq a$, a,b,c,d and e are distinct, so there is f with $S(a,e,f) \wedge S(b,d,f)$. In either case there is $f \in p^*(M)$ with $S(f,b,d)$. Since $b \in J$, there is $h \in p^*(M)$ and $i \in \sigma_3(M) \setminus I$ with $S(h,i,b)$. $S(b,f,d) \wedge S(b,h,i)$ and either $f = h$ or b,f,h,d,i are distinct. If $f = h$, $S(f,d,i)$; otherwise there is k with $S(f,h,k) \wedge S(d,i,k)$. $k,f \in p^*(M)$, $d \notin \sigma_3(M) \cup J$ and $i \in \sigma_3(M) \setminus I$, so $S(f,d,i) \vee S(k,d,i)$ gives a contradiction.

Proof of Lemma 6.8: Let

$$\sigma(x) \longleftrightarrow \sigma_4(x) \wedge \forall y,z(\sigma_4(y) \wedge S(x,y,z) \rightarrow \sigma_4(z)).$$

The previous lemma gives $\sigma(x) \in p^*$. Suppose $\sigma(a) \wedge \sigma(b) \wedge S(a,b,c)$. Certainly $\sigma_4(c)$. If not $\sigma(c)$, there are d and e with $S(c,d,e) \wedge \sigma_4(d) \wedge \neg\sigma_4(e)$. Either d=a or d=b or a,b,c,d and e are distinct; if d=a, $S(a,c,b) \wedge S(a,c,e)$ give $S(a,b,e)$ contradicting $\sigma(a) \wedge \sigma_4(b)$. Similarly if d=b. Otherwise there is f with $S(a,e,f) \wedge S(b,d,f)$. Since $\sigma(b) \wedge \sigma_4(d)$, $\sigma_4(f)$. Since $\sigma(a) \wedge \sigma_4(f)$, $\sigma_4(e)$. This completes the proof.

SECTION 7

In this section we concentrate on determining the structure of an affine or projective structure like the ones arising from 6.7 or 6.8. We will show that this structure has some collection of almost 0-definable subspaces of finite co-dimension, some collection of algebraic points, and essentially no other structure. In the projective case, most of the "subspaces" will of course be affine.

We begin the projective case, since once this is decided, there is little trouble reading off the affine case. So until 7.7 we will assume M is weakly minimal, saturated and of size $>|Th(M)|$, and that there is a ternary S 0-definable on M giving it a projective structure over $F(n)$. Further, we will assume there is some strong type p with $p^*(M)$ as in 5.1 and 6.8. Specifically, we have $P \subseteq M$ such that:

1) There is a strong type p with $p(M) \subseteq P \subseteq p(M) \cup ac\ell(\phi)$ and p is of projective character;

2) P is closed under S;

3) S induces the lines on $p(M)/E$;

4) $P = p^*(M)$, where p^* is a possibly incomplete type over $ac\ell(\phi)$.

Lemmas 3.11, 3.15 and 3.16 apply to P as do 6.8 and 5.1(2), so we know the full structure induced on P by formulas. There is some $C \subseteq rac\ell(\phi)$ such that P as a structure is interdefinable with the structure (P_1,C) where P_1 has universe P and predicate S. Recall that in this context $rac\ell(\phi)$ is the definable closure of the algebraic points in P along with the field elements.

If $A \subseteq M$ is any subspace, we recall that $a \approx_A b$ iff there is $c \in A$ with $S(a,c,b)$; this gives an equivalence relation on $M \setminus A$. Each class of $\approx_A$ has an affine structure on it, and we will abuse terminology by calling it an "affine subspace" of M.

Lemma 7.1: Suppose $a,b \in M \setminus (P \cup \mathrm{ac}\ell(\phi))$. Then $a \equiv^S b$ iff $a \approx_P b$.

Proof: ($\Rightarrow$) We can assume $a \neq b$. Clearly $a/{\approx_P}$ has size $|M|$, so choose $c \approx_P a$, $c \notin \mathrm{ac}\ell\{a,b\}$. Then pick $d \in P$ with $S(a,d,c)$. There is a strong automorphism of M fixing c and taking a to b. It takes d to e where $S(b,e,c)$ and $e \in P$. So $b \approx_P c$ and $a \approx_P b$.

($\Leftarrow$) We can assume $a \perp b$ by picking an element of $a/{\approx_P}$ independent from both a and b and comparing both to that element. Choose $c \in P$ with $S(a,c,b)$ and some $d \in P \setminus \mathrm{ac}\ell\{a,b\}$. Now a,c,d and b,c,d are independent triples, so for any e with $S(c,d,e)$, $e \in P$ and $e \notin \mathrm{ac}\ell\{a\} \cup \mathrm{ac}\ell\{b\}$. So $ae \equiv^S ac$ and $be \equiv^S bc$ for any such e. Also $\langle a,c,d \rangle = \langle b,c,d \rangle$. If $a \not\equiv^S b$, we can repeat the argument of 5.7, counting $\langle a,c,d \rangle \cap q(M)$ in two ways where q is the strong type of a. Using a, we get $|\langle a,c,d \rangle \cap q(M)| = (k-1)(n+1) + 1$ and using b we get $|\langle b,c,d \rangle \cap q(M)| = k(n+1)$, a contradiction. See 5.7 for more details.

Lemma 7.2: (1) Suppose $P \subseteq X$, an almost 0-definable subspace of M. Then X is of finite co-dimension in M and the $\approx_X$-classes are almost 0-definable.

(2) $a \approx_P b$ iff $a \approx_X b$ for all X as in (1).

(3) If A is a definable infinite subspace of M, then $P \subseteq A$ and A is almost 0-definable.

Proof: (1) If $a \in M \setminus X$, then $a/\approx_X$ is defined by $\exists y(y \in X \wedge S(a,y,x))$. This does not depend on the choice of a in its $\approx_X$-class, so not upon the choice of a in its strong type. So the class is almost 0-definable. If X had infinite co-dimension, there would be infinitely many $\approx_X$-classes and then $\exists y(y \in X \wedge S(a,y,x))$ would, for various choices of a, give infinitely many pairwise disjoint infinite sets. This would contradict weak minimality. See remark (3) following 1.2.

(2) As $P \subseteq X$, $a \approx_P b$ implies $a \approx_X b$. Suppose $a \not\approx_P b$. Then $p^*(x) \cup \{S(a,x,b)\}$ is inconsistent as M is saturated. Choose $\lambda(x) \in p^*$ with $\lambda(x) \cup \{S(a,x,b)\}$ inconsistent. By 6.8 we can find $\sigma(x) \in p^*$ with $\sigma(M) \subseteq \lambda(M)$ and $\sigma(M)$ an almost 0-definable subspace of M. So $a \not\approx_{\sigma(M)} b$.

(3) Suppose A is $\bar{d}$-definable; choose $a \in A \setminus acl(\bar{d})$. Then choose b, $b \equiv^S a(\bar{d})$ and $b \underset{\bar{d}}{\perp} a$. By 7.1 there is $c \in P$ with $S(a,c,b)$. So $c \notin acl(\bar{d})$ and $c \in A$. Since some $c \in P \setminus acl(\bar{d})$ is in A, $P \setminus acl(\bar{d}) \subseteq A$ and since A is closed, $P \subseteq A$. By 3.11 pick $\lambda(x) \in p^*$ with $\lambda(M) \setminus A$ finite. By 6.8 find $\sigma(M)$ an almost 0-definable subspace with $P \subseteq \sigma(M) \subseteq \lambda(M)$. Since $\sigma(M)$ and A are subspaces, A is infinite and $\sigma(M) \setminus A$ is finite, $\sigma(M) \subseteq A$. So A is the union of $\sigma(M)$ and some $\approx_{\sigma(M)}$-classes, and so A is almost 0-definable.

Now we determine the algebraic closure operation.

Lemma 7.3: Let A be a $\approx_P$-class of $M \setminus P$. Then for any $B \subseteq P \cup A$, $acl(B) \cap (P \cup A) = \langle B \cup (acl(\phi) \cap (P \cup A))\rangle$.

<u>Proof</u>: Clearly $P \cup A$ is closed under S, so $\langle B \cup (acl(\phi) \cap (P \cup A))\rangle \subseteq acl(B) \cap (P \cup A)$. Now assume $a,b \in P \cup A$ and $a \in acl\{b\}$. We can assume $a \neq b$ and $a \notin acl(\phi)$, and in fact that $a \notin P$, as we know from 5.1(2) that if $B \subseteq P$, $acl(B) \cap P = \langle B \cup (acl(\phi) \cap P)\rangle$. First suppose $b \in P$; pick c with $c \equiv^s a$ and $c \perp a$ and then d with $cd \equiv^s ab$. $c \approx_P a$ so find $e \in P$ with $S(a,c,e)$. $e \in acl\{b,d\}$ and $e,b,d \in P$ so $e \in \langle\{b,d\} \cup (acl(\phi) \cap P)\rangle$, so we can find $d^* \in \langle\{d\} \cup (acl(\phi) \cap P)\rangle \subseteq acl\{d\}$ with $S(b,d^*,e)$; $d^* \notin \{e,b\}$. Now $S(a,c,e) \wedge S(b,d^*,e)$ give f with $S(a,b,f) \wedge S(c,d^*,f)$, so $f \in acl\{a,b\} \cap acl\{c,d^*\} = acl\{a\} \cap acl\{c\} = acl(\phi)$.

Now suppose $b \in A$; since $a \approx_P b$ find $c \in P$ with $S(a,b,c)$. Either $c \in acl(\phi)$ or $a \in acl\{c\}$ and we can apply the previous paragraph. So if $B = \{b\}$ we are done.

For the general case, we can assume first that B is finite and then by induction that $acl(B \setminus \{b\}) \cap (P \cup A) = \langle B \setminus \{b\} \cup (acl(\phi) \cap (P \cup A))\rangle$, where $b \in B$. Then we apply the case of a singleton to $(M, B \setminus \{b\})$ to conclude.

<u>Lemma 7.4</u>: For any $B \subseteq M$, $acl(B) = \langle B \cup acl(\phi)\rangle$.

<u>Proof</u>: As in 7.3, it suffices to do the case where $B = \{b\}$. Suppose $a \in acl\{b\} \setminus acl(\phi)$, $a \neq b$. Pick $c \equiv^s a$, $c \perp a$ and d with $cd \equiv^s ab$ and then $e \in P$ with $S(a,c,e)$. $b \approx_P d$ and $e \in acl\{b,d\}$ so by 7.3, $e \in \langle\{b,d\} \cup acl(\phi)\rangle$ so there is $d^* \in acl\{d\}$ with $S(b,d^*,e)$. So there is f with $S(a,b,f) \wedge S(c,d^*,f)$; $f \in acl\{a\} \cap acl\{c\} = acl(\phi)$.

Now let M_1 be the structure with universe M, ternary predicate S, and a unary predicate for each 0-definable projective subspace of M. Our aim is to show

Theorem 7.5: There is $C \subseteq acl(\phi) \cap M_1^{eq}$ with (M_1,C) and M interdefinable. Here $acl(\phi)$ is taken in M.

We first notice that any almost 0-definable (projective) subspace of M is almost 0-definable in M_1. For suppose X is an almost 0-definable subspace of M; let Y be the intersection of all the conjugates of X. Y is a 0-definable subspace, and the $\approx_Y$-classes are almost 0-definable in M_1 as $\approx_Y$ on $M \setminus Y$ is defined by $\exists z(z \in Y \wedge S(x,z,y))$. Since X is the union of Y and finitely many of these classes, it is almost 0-definable in M_1.

As in the proofs of 4.10 and 4.1(2), we have a notion of $racl(\phi)$. Here $racl(\phi)$ is the definable closure of the following sets in M_1^{eq}:

1) $acl(\phi) \cap M$;

2) the classes of $\approx_X$ for each 0-definable subspace X of M;

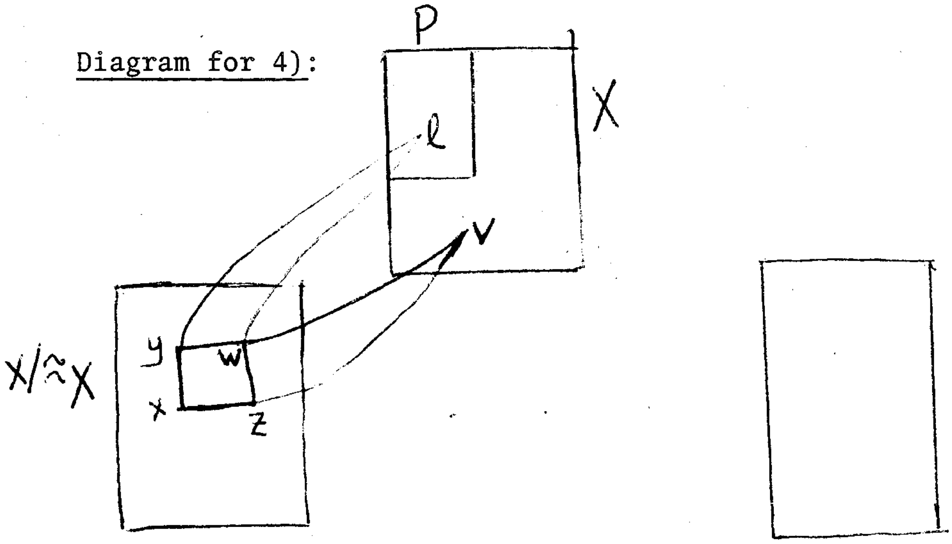
3) the field elements;

4) for each $\ell \in P \cap acl(\phi)$ and X a 0-definable subspace, we have the following equivalence relation on pairs of distinct points in each $\approx_X$-class —

$E_\ell(x,y,z,w) \longleftrightarrow S(x,y,\ell) \wedge S(z,w,\ell) \wedge \exists v(v \in X \wedge S(x,z,v) \wedge S(y,w,v))$.

This has finitely many classes, and these go into $racl(\phi)$.

Diagram for 4):



In 4), the dependence on X is illusory. If $a,b,c,d \notin P$ and $E_\ell(a,b,c,d)$ with respect to X, and Y is another 0-definable subspace with a,b,c,d in the same $\approx_Y$-class we have $E_\ell(a,b,c,d)$ with respect to Y. The effect of naming the equivalence classes of E_ℓ is to distinguish between (a,b) and (a,c) if a,b,c are distinct in $M \setminus P$ and all on the same line through ℓ.

Lemma 7.6: If $a \in M \setminus P$ and $c \in M_1^{eq}$ is in $racl(\{a\})$, then c is definable over $\{a\} \cup racl(\phi)$. Here $racl(\{a\})$ means $racl(\phi)$ taken in (M,a).

Proof: The field elements remain the same whether a is named or not. By 7.2(3), the $\{a\}$-definable subspaces are almost 0-definable, so 2) and 3) of the definition of $racl(\phi)$ for (M,a) take care of themselves. If $b \in acl\{a\} \cap M$ there is by 7.4 some $c \in acl(\phi)$ with $S(a,c,b)$. If $c \in P$, 4) implies b is definable over $\{a\} \cup racl(\phi)$. If $c \notin P$ we can find an almost 0-definable subspace X with $a,c \notin X$ and $a \approx_X c$. For choose any almost 0-definable subspace Y with $a,c \notin Y$; if not $a \approx_Y c$ pick and $d \notin \{a,c\}$ with $S(a,c,d)$ (we can assume $a \neq c$), and let $X = Y \cup d/\approx_Y$. Now there is a unique point of X

on the same line as a and c, so it is definable over a and c. Since the field elements are in $racl(\phi)$, every point on the line through a and c is definable over $\{a\} \cup racl(\phi)$; recall $c \in racl(\phi)$. This takes care of 1). It also takes care of 4). For if $\ell \in acl\{a\} \cap P$, then the points on the line through a and ℓ are all definable over $\{a\} \cup \{b\}$, where $b \in acl(\phi) \cap M$ with $S(a,\ell,b)$, and this implies the E_ℓ-classes are definable over $\{a\} \cup racl(\phi)$.

7.6 is false for $a \in P$.

Proof of 7.5: If 7.5 is false, we can as in 4.10 and 5.1(2) find $\bar{a},\bar{b}$ in M with $\bar{a} \equiv \bar{b}$ in $(M_1, racl(\phi))$ but not in $(M, acl(\phi))$. We pick a counter-example M with $\bar{a},\bar{b}$ as above of shortest possible length in any counter-example. If $\bar{a}$ contains an element of P, without loss $\bar{a} = \langle a_0\rangle^\frown\bar{c}$, $\bar{b} = \langle b_0\rangle^\frown\bar{d}$ with a_0 and hence b_0 in P. If $\bar{a}$ has no element of P, write $\bar{a} = \langle a_0\rangle^\frown\bar{c}$ and $\bar{b} = \langle b_0\rangle^\frown\bar{d}$ for any a_0 in $\bar{a}$. There is by choice of $\bar{a},\bar{b}$ an automorphism of $(M, acl(\phi))$ taking $\bar{d}$ to $\bar{c}$; say it takes b_0 to a_1. So $a_0 \equiv a_1(\bar{c})$ in $(M_1, racl(\phi))$ but not in $(M, acl(\phi))$ and either $a_0, a_1 \in P$ or $\bar{c}$ contains no elements of P.

But by choice of $\bar{a},\bar{b}$ and M, $\bar{c}$ contains no elements of $M \setminus P$. For 7.6 implies that if $c \in \bar{c} \setminus P$, we can replace M by (M,c) and then $a_0 \equiv a_1(\bar{c} \setminus \{c\})$ in $((M_1,c), racl(\{c\}))$. But $a_0 \not\equiv a_1(\bar{c} \setminus \{c\})$ in $((M,c), acl(\{c\}))$.

Now if $a_0 \equiv a_1$ in $(M_1, racl(\phi))$, then $a_0 \equiv^s a_1$ in M since either $a_0 = a_1 \in acl(\phi)$ or a_0, a_1 are in $P \setminus acl(\phi)$ or in the same $\approx_P$-class. We use 7.2(2) for this and then 7.1 applies. So we must have a_0, a_1 and $\bar{c}$ all in P. But 5.1(2) holds on P, giving us a contradiction.

As promised, we can now read off the affine case.

Lemma 7.7: Let M be a weakly minimal structure with predicates Q and R giving it an affine structure. Suppose there is a strong type p of affine character such that:

(1) p(M) is closed under Q and R; and

(2) P_Q and P_R give the affine structure on p(M)/E — the notation is from section 3.

Let M_1 be the structure with universe M, predicates Q and R and a predicate for each 0-definable equivalence relation of M partitioning M into conjugate subspaces. Then there is $C \subseteq M_1^{eq}$ with M and (M_1, C) interdefinable.

Proof: Apply 7.5 to the projective space $M \cup M'$. The set P is the image of $p(M)^2$ under $F: M^2 \setminus \{(a,a): a \in M\} \to M'$. Our assumptions on p(M) assure us that P satisfies 1) - 4) in the beginning of this section. M is a 0-definable subset of $M \cup M'$, and 7.5 gives us the structure of $M \cup M'$.

We now summarize what we have done to this point.

Theorem 7.8: Suppose M is a weakly minimal structure and p is a strong type satisfying (*) of either projective or affine character. Then there is θ an almost 0-definable equivalence relation with finite classes, and $\sigma(x) \in p$ such that $\sigma(M)/\theta$ is either a projective or affine space over a finite field. There is some collection of algebraic points of $\sigma(M)/\theta$, or $(\sigma(M)/\theta)'$ in the affine case, and some collection of almost 0-definable subspaces of $\sigma(M)/\theta$, and $\sigma(M)/\theta$ has essentially no other structure.

Proof: Combine 4.1, 4.10, 5.1, 6.7, 6.8, 7.5 and 7.7.

SECTION 8

In this section we consider the case where our structure M, in addition to being weakly minimal and having a strong type satisfying (*) but not of disintegrated character, is uni-dimensional. That is, no two types are orthogonal. We will show that this implies (*) holds on every strong type.

Using this, we show that we can find pairwise disjoint almost 0-definable sets $G, X_0, \ldots, X_{k-1}, Y \subseteq M^{eq}$, θ an almost 0-definable equivalence relation with finite classes on $M \cup Y$, and $F(n)$ a finite field, such that:

(i) $M = G \cup X_0 \cup \ldots \cup X_{k-1}$ or $M = G \cup X_0 \cup \ldots \cup X_{k-1} \cup Y$;

(ii) G is finite;

(iii) for each $i<k$ there are Q_i, R_i almost 0-definable giving X_i/θ an affine structure over $F(n)$;

(iv) there is an almost 0-definable relation S giving Y/θ a projective structure over $F(n)$;

(v) for each $i<k$, there is an almost 0-definable bijection between $(X_i/\theta)'$ and Y/θ; and this bijection takes the relation on $(X_i/\theta)'$ induced by Q_i and R_i to S;

(vi) Y/θ satisfies the conclusion of 7.8. To be precise, let Y^* be the structure with universe Y/θ, ternary predicate S and a unary predicate for each $\{Y/\theta, S\}$-definable subspace of Y/θ. Here we recall $\{Y/\theta, S\} \subseteq acl(\phi)$ in M^{eq}. Then Y/θ, with the point $S \in M^{eq}$ named and full structure inherited from M, is interdefinable with (Y^*, C) for some $C \subseteq (Y^*)^{eq}$.

(vii) Similarly for each X_i/θ.

We will finish the section with the easy observation that if M is weakly minimal, uni-dimensional and has a strong type of disintegrated character, then M is categorical in power $>|Th(M)|$. Bradd Hart has pointed out to the author that the assumption of weak minimality can be weakened considerably.

We begin with:

Lemma 8.1: If q and r are strong types in weakly minimal M, q satisfies (*) and q and r are non-orthogonal, then r satisfies (*).

Proof: We may assume M is saturated. Pick $A \cup \{a_0\} \subseteq q(M)$, $B \cup \{b_0\} \subseteq r(M)$, with first $A \cup \{a_0\}$ and then $B \cup \{b_0\}$ minimal, such that each of $A \cup \{a_0\}$, $B \cup \{b_0\}$ is independent but their union is dependent. By choice of $A \cup \{a_0\}$, for any $C \subseteq r(M)$, $acl(A \cup C) \cap r(M) = acl(C) \cap r(M)$.

In $(M, A \cup B)$, $stp(a_0|A \cup B) = q|(A \cup B)$ satisfies (*) since for any finite $D \subseteq q(M)$, $acl(D \cup A \cup B) \cap q(M)$ has dimension $\leq |D| + |A| + |B|$ and q satisfies (*). So $stp(b_0|A \cup B)$ satisfies (*) as there is an algebraic relation between it and $stp(a_0|A \cup B)$ in $(M, A \cup B)$. As naming A does not affect the dependence relation on $r(M)$, $stp(b_0|B) = r|B$ satisfies (*). Choose $B_0 \subseteq r(M)$ minimal such that $r|B_0$ satisfies (*). If $B_0 = \phi$ we are done. If not, by shifting to $(M, B_0 \setminus \{b\})$ and considering $r|B_0 \setminus \{b\}$ for some $b \in B_0$, we can assume $B_0 = \{b\}$. So for any $c \in r(M)$, $r|c$ satisfies (*), but r does not.

So for any finite $C \subseteq r(M)$ and $e \in C$, $ac\ell(C) \cap r(M) = \bigcup\{ac\ell\{e,d\}: d \in D\} \cap r(M)$ for some finite $D \subseteq r(M)$ and so for $c,d \in r(M)$ independent $ac\ell\{c,d\} \cap r(M)$ contains an infinite set $\{f_i: i<\omega\}$ of pairwise independent elements of $r(M)$. Let $c,d,e \in r(M)$ be independent and find $g_1,\ldots,g_k \in r(M)$ with $ac\ell\{c,d,e\} \cap r(M) = \bigcup_{1\le j\le k} ac\ell\{e,g_j\} \cap r(M)$. Let $\{f_i: i<\omega\} \subseteq r(M) \cap ac\ell\{c,d\}$ be pairwise independent. Then if $i \neq j$, $f_i \notin ac\ell\{e,f_j\}$. So find i with $ac\ell\{e,f_i\} \neq ac\ell\{e,g_j\}$ for all $1\le j\le k$. Then $f_i \in r(M) \cap ac\ell\{c,d,e\} \setminus \bigcup_{1\le j\le k} ac\ell\{e,g_j\}$, a contradiction establishing the lemma.

For the rest of this section we will assume M is uni-dimensional, so every strong type satisfies (*). Until 8.3 we will also suppose that some strong type is not of disintegrated character; hence no strong type is of disintegrated character. As usual, M will also be assumed saturated of size $>|Th(M)|$ and weakly minimal.

We can also suppose there is a strong type p, which we will fix until further notice, of projective character; this is harmless as if q is of affine character find $\sigma(x) \in q$ and θ_0 with $\sigma(M)/\theta_0$ an affine space as in 7.8. Then $(\sigma(M)/\theta_0)' \subseteq M^{eq}$ is almost 0-definable, and the strong type of a non-algebraic element of $F((q(M)/\theta_0)^2 \setminus \{(a,a): a \in q(M)/\theta_0\})$ is of projective character. We choose $\sigma(x) \in p$ and θ_1 for p as in 7.8; in particular $\sigma(M)/\theta_1$ is a projective space over $F(n)$ for some $n \in \omega$. Shifting to M/θ_1 we may assume θ_1 is the identity. Y will be a subspace of $\sigma(M)$.

Suppose $\sigma(x) \notin q$, q some strong type. Give $(p(M) \cup q(M))/E$

the structure induced by formulas, where we recall aEb if and only if $a \equiv^s b$ and $a \in \mathrm{ac}\ell\{b\}$. By 3.4(2) this is an $\aleph_0$-categorical structure consisting of two strictly minimal sets. These are non-orthogonal, since there is an M-formula $\varphi(\bar{x},\bar{y})$ witnessing that p and q are non-orthogonal; its projection P_φ witnesses the non-orthogonality in $(p(M) \cup q(M))/E$. So if q is of projective character there is in this quotient an almost 0-definable bijection between $p(M)/E$ and $q(M)/E$ by 1.9; if q is of affine character the bijection is between $p(M)/E$ and $(q(M)/E)'$.

Suppose q is of projective character; 3.4(2) and the previous paragraph imply there is T_q an almost 0-definable binary algebraic relation with $T_q(a,M) \subseteq p(M)$ for $a \in q(M)$ and $T_q(M,b) \subseteq q(M)$ for $b \in p(M)$. The relation $\exists z(T_q(z,x) \wedge T_q(z,y))$ defines an algebraic relation on $p(M)$, so if we factor $p(M)$ by an appropriate almost 0-definable equivalence relation χ_q with finite classes, T_q gives a function from $q(M)$ onto $p(M)/\chi_q$. By 5.6(3) χ_q is the localization of $\sigma(M)$ at some $C_q \subseteq p^*(M) \setminus p(M)$. If we factor $q(M)$ by the appropriate equivalence relation θ_q, T_q gives a bijection on the factors. So we can find $\tau_q(x) \in p$ and $\sigma_q(x) \in q$ such that T_q gives a bijection between $\tau_q(M)/\chi_q$ and $\sigma_q(M)/\theta_q$. Without loss, $\sigma_q(M) \cap \sigma(M) = \phi$. Using 6.8 and 3.11 we may assume $\tau_q(M)$ is a subspace of $\sigma(M)$ less a finite set of algebraic points, and we may assume this finite set is a subset of C_q.

It is perhaps not so clear that we can do something similar if q is of affine character.

Lemma 8.2: Let $p, \sigma(x) \in p$ be as above and q be a strong type of affine character with $\sigma(x) \notin q$. Then there is $\sigma_q(x) \in q$ and θ_q an almost 0-definable equivalence relation with finite classes such that $\sigma_q(M) \cap \sigma(M) = \phi$, and $\sigma_q(M)/\theta_q$ is an affine structure as in 7.8. Further, there is a subspace $\tau_q(M)$ of $\sigma(M)$, C_q a finite algebraic subset of $\tau_q(M)$, and T_q an almost 0-definable bijection between $(\sigma_q(M)/\theta_q)'$ and the localization of $\tau_q(M)$ at C_q.

Proof: First choose $\gamma(x) \in q$ and χ such that $\gamma(M) \cap \sigma(M) = \phi$ and $\gamma(M)/\chi$ is an affine space over $F(n)$, as in 7.8. Without loss we may assume χ is the identity. Let r be the strong type of a non-algebraic element of $F[q(M)^2 \setminus \{(a,a)\colon a \in q(M)\}] \subseteq \gamma(M)'$. r is of projective character, so as in the preceding paragraphs find $\sigma_r(x) \in r$, θ_r, T_r, $\tau_r(x) \in p$, χ_r and C_r such that $T_r\colon \sigma_r(M^{eq})/\theta_r \to (\tau_r(M) \setminus C_r)/\chi_r$ is an almost 0-definable bijection. We can assume $\sigma_r(M^{eq}) \subseteq \gamma(M)'$.

θ_r, acting on $r(M^{eq})$, is by 5.6(3) a localization with respect to the projective structure of $\gamma(M)'$. So $r(M^{eq})/\theta_r$ has two competing ternary predicates giving it plus some algebraic points a projective structure over $F(n)$. Let S_0 be inherited from the projective structure on $\gamma(M)'$ and let S_1 be transferred by T_r^{-1} from $(\tau_r(M) \setminus C_q)/\chi_q$. S_0 and S_1 both induce the lines on $r(M^{eq})/E$, so if $a,b \in r(M^{eq})/\theta_r$ are independent, we can list $S_0(a,b,M^{eq})$ and $S_1(a,b,M^{eq})$ as $\{c_0,\ldots,c_n\}$, $\{d_0,\ldots,d_n\}$ respectively so that $c_i \not\perp d_i$ for each $i<n+1$. Find an algebraic formula $\psi(x,y)$ with $\psi(c_i,d_i)$ for each $i<n+1$.

The following formula holds of a,b independent in $r(M^{eq})/\theta_r$:

$$\exists z_0,\ldots,z_n,w_0,\ldots,w_n[\bigwedge_{i<n+1}(S_0(x,y,z_i)\wedge S_1(x,y,w_i)\wedge\psi(z_i,w_i))\wedge$$

$$\bigwedge_{i\neq j<n+1}(z_i\neq z_j\wedge w_i\neq w_j)]$$

We can choose $\eta(x,y)$ algebraic such that if $a,b \in r(M^{eq})/\theta_r$ either $\eta(a,b)$ or the above formula holds of a,b. Using 5.6, we can find an almost 0-definable equivalence relation ξ with finite classes such that $M^{eq} \models \psi(x,y) \vee \eta(x,y) \to \xi(x,y)$, and $\xi \restriction r(M^{eq})/\theta_r$ is an S_1-localization at, say, D_1. By 5.6(3), regarding ξ as acting on $r(M^{eq})$, $\xi \restriction r(M^{eq})$ is an S_0-localization at, say, D_0.

Let ξ_0 be the S_0-localization of the projective space $\gamma(M) \cup \gamma(M)'$ at D_0 and ξ_1 be the S_1-localization of $\sigma_r(M^{eq})/\theta_r$ at D_1. It is clear that S_0 and S_1 induce the same relation on $r(M^{eq})/\xi$, i.e. for $a,b \in r(M^{eq})/\xi$, $S_0(a,b,M^{eq}) = S_1(a,b,M^{eq})$. So the S_0- and S_1-closures of $r(M^{eq})/\xi$ are the same, and it is easy to see that S_0 and S_1 on this closure $\langle r(M^{eq})/\xi\rangle$ are the same. We know if $a,b \in r(M^{eq})/\xi$ that $S_0(a,b,M^{eq}) = S_1(a,b,M^{eq})$; if $a \in r(M^{eq})/\xi$ and $b \in \langle r(M^{eq})/\xi\rangle \setminus r(M^{eq})/\xi$, then for any $c \neq a,b$ with $S_0(a,b,c)$ we have $S_0(c,a,b)$ so $S_1(c,a,b)$ as $a,c \in r(M^{eq})/\xi$ and so $S_1(a,b,c)$. Suppose $S_0(a,b,c)$ where $a,b,c \in \langle r(M^{eq})/\xi\rangle \setminus r(M^{eq})/\xi$. Pick any $d \in r(M^{eq})/\xi$ and $e \neq a,d$ with $S_0(a,d,e)$. Then find f with $S_0(b,d,f) \wedge S_0(c,e,f)$. As $d,e,f \in r(M^{eq})/\xi$ we have $S_1(d,a,e) \wedge S_1(d,b,f)$ so there is c^* with $S_1(a,b,c^*) \wedge S_1(e,f,c^*)$. c^* is algebraic and c is the only algebraic element in $S_0(e,f,M^{eq}) = S_1(e,f,M^{eq})$, so $c = c^*$ and $S_1(a,b,c)$.

T_r induces a bijection between $\sigma_r(M^{eq})/\xi_1$ and the localization χ_q of $\tau_r(M)$ at $C_q = \langle C_r \cup T_r(D_1)\rangle$. So T_r gives a bijection between $\langle r(M^{eq})/\xi_0\rangle$ and $\langle p(M)/\chi_q\rangle$. Find $\delta(x) \in r$, $\lambda(x) \in p$ such that $\delta(M^{eq}) \subseteq \sigma_r(M^{eq})$, $\lambda(M) \subseteq \tau_r(M)$ and T_r gives a bijection between $\delta(M^{eq})/\xi_0$ and $\lambda(M)/\chi_q$ and $S_0 = S_1$ on $\delta(M^{eq})/\xi_0$. Using 3.11, we can assume $\langle r(M^{eq})/\xi_0\rangle \subseteq \delta(M^{eq})/\xi_0$ and then by 6.8 that $\delta(M^{eq})/\xi_0$ is a subspace of $[(\gamma(M)' \cup \gamma(M))\setminus D_0]/\xi_0$. So $\lambda(M)$ is $\tau_q(M)\setminus C_q$ for some subspace $\tau_q(M)$ of $\sigma(M)$. If we let $\sigma_q(x) \in q$ be such that $\sigma_q(M)/\xi_0$ is a $\approx_{\delta(M^{eq})/\xi_0}$-class, θ_q be $\xi_0 \restriction \sigma_q(M)$ and T_q be the bijection between $\delta(M^{eq})/\xi_0 = (\sigma_q(M)/\theta_q)'$ and $(\tau_q(M)\setminus C_q)/\chi_q$ induced by T_r, we have what is required.

Now by compactness we can cover all but a finite subset of $M\setminus\sigma(M)$ by a finite set of these $\sigma_q(M)$'s. So we have a finite A of types, and for each $q \in A$ we have $\sigma_q(x) \in q$, θ_q, T_q, $\tau_q(x) \in p$, C_q and χ_q such that:

(i) $\sigma_q(M) \cap \sigma(M) = \phi$;

(ii) $M\setminus[\sigma(M) \cup \bigcup_{q\in A} \sigma_q(M)]$ is finite;

(iii) $\tau_q(M)$ is a subspace of $\sigma(M)$, and χ_q is the localization of $\tau_q(M)$ at C_q;

(iv) θ_q is an equivalence relation with finite classes on $\sigma_q(M)$;

(v) T_q is a bijection between either $\sigma_q(M)/\theta_q$ (if projective) or $(\sigma_q(M)/\theta_q)'$ and $(\tau_q(M)\setminus C_q)/\chi_q$;

(vi) $\sigma_q(M)/\theta_q$ is as in 7.8;

(vii) all of the above are almost 0-definable.

In fact, we can choose the above so that the $\sigma_q(M)$'s are pairwise disjoint. For let $<$ well-order A, and consider

$H = [\sigma_q(M) \setminus \bigcup_{r<q} \sigma_r(M)]/\theta_q$. This is an almost 0-definable subset of $\sigma_q(M)/\theta_q$, so it follows from 7.8 that there is J, a finite union of pairwise disjoint almost 0-definable subspaces of $\sigma_q(M)/\theta_q$ such that $H \Delta J$ is finite; if $\sigma_q(M)/\theta_q$ is projective these subspaces may be affine. T_q induces a bijection between either K or K' and a subspace of $(\tau_q(M) \setminus C_q)X_q$ if K is one of these subspaces. Basically, we ignore the points of $H \setminus J$ and replace $\sigma_q(M)$ by the subsets of $\sigma_q(M)$ which give these subspaces K after factoring by θ_q. However, if K contains points of $J \setminus H$, we should take a further localization of K or $K \cup K'$, and then a further localization of $\tau_q(M)$. If we do this, we can add to the list (i) - (vii) above

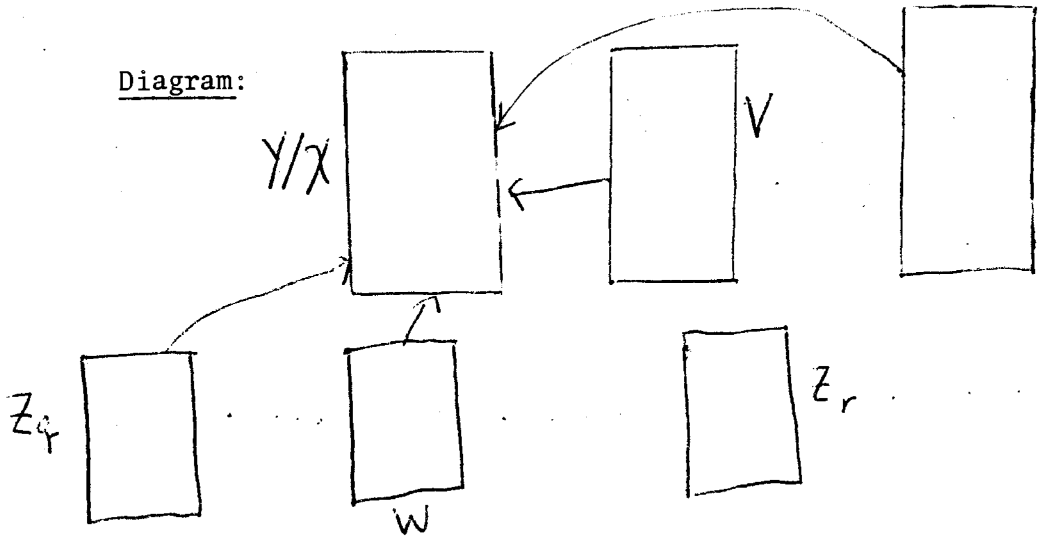
(viii) if $q_1 \neq q_2 \in A$, $\sigma_{q_1}(M) \cap \sigma_{q_2}(M) = \phi$.

Now we are very close to the situation promised at the beginning of the section. Using the notation of (i) - (viii), let $Y = \bigcap_{q \in A} \tau_q(M) \setminus C$ where $C = \langle \bigcup_{q \in A} C_q \rangle$. T_q induces a bijection between a localization of either $\sigma_q(M)/\theta_q$ or $(\sigma_q(M)/\theta_q)'$ and $(\tau_q(M) \setminus C)/X$ where X is the localization of $\sigma(M)$ at C. Replacing θ_q and perhaps deleting a finite set from $\sigma_q(M)/\theta_q$ (if projective), we may assume the first of these localizations is included in θ_q.

Consider $Z_q = T_q^{-1}(Y/X)$, a subspace of either $\sigma_q(M)/\theta_q$ or $(\sigma_q(M)/\theta_q)'$. T_q of course gives a bijection between Z_q and Y/X. T_q also induces a bijection between W' and Y/X for any $\approx_{Z_q}$-class W in $\sigma_q(M)/\theta_q$ or $\sigma_q(M)/\theta_q \cup (\sigma_q(M)/\theta_q)'$. There is also a natural bijection between V' and Y/X for each $\approx_{Y/X}$-class V of $\sigma(M)$. So replacing $\sigma_q(M)$ by the subsets of M which map to Z_q and each W

as above after factoring by θ_q, and similarly replacing $\sigma(M)$ brings us to the following picture:

Diagram:



where each Z_q, W and V, or its prime, is in bijection with Y/X. Restricting to those sets above (expect possibly Y) which are actually in M, we can take a further quotient and identify all the projective spaces above. Let $\theta = \bigcup_{q \in A} \theta_q \cup X$ and $\{X_i : i<k\}$ list the affine spaces as above. This yields the promised situation.

The following is all we will say about the case where we have a type of disintegrated character.

Proposition 8.3: Suppose M is weakly minimal, uni-dimensional and has a strong type of disintegrated character. Then any model N of Th(M) is determined up to isomorphism by dim(q(N)), the number of independent realizations of the strong type q in N. dim(q(M)) is the same for every non-algebraic strong type q. In particular, Th(M) is categorical in any $\kappa > |Th(M)|$.

Proof: First suppose M is saturated; then for any two strong types q and r, there is in $(q(M) \cup r(M))/E$ an almost 0-definable bijection between $q(M)/E$ and $r(M)/E$, so in M there is an algebraic binary relation $T_{q,r}$ with $T_{q,r}(a,M) \subseteq r(M)$ for $a \in q(M)$, and $T_{q,r}(M,b) \subseteq q(M)$ for $b \in r(M)$. Then the $T_{q,r}$'s ensure there are $\leq |Th(M)|$ strong types and that for any $N \models Th(M)$ and q,r strong types, $\dim(q(N)) = \dim(r(N))$. From this the categoricity is immediate.

APPENDIX 1

Our purpose here is to justify the claim in Section 1 that the axioms #0 through #11 given in 1.5(1) yield the usual notion of an affine space over F(n). From the discussion after 1.5(1) it is clear that #0 through #11 hold in an affine space over F(n). We refer the reader to [Ha, p.167] for the projective case.

The following comes from [Hir, p.39-40]. "We now give a set of axioms for AG(n,q), n>2, in which n is not specified but q is." Here AG(n,q) denotes the affine space of dimension n over F(q); since we will no longer be using q for a type we will revert to this notation; as n is unbounded, the fact that the list below is intended for affine spaces of finite dimension n whereas ours are of infinite dimension is not problematic. We continue quoting. "Let L be an incidence structure with an equivalence relation *parallelism* on its lines (blocks).

(i) Any two points P_1, P_2 are incident with exactly one line P_1P_2.

(ii) For every point P and line ℓ, there is a unique ℓ' parallel to ℓ containing P.

(iii) If P_1P_2 and P_3P_4 are parallel lines and P is a point on P_1P_3 distinct from P_1 and P_3, then there is a point P' on PP_2 and P_3P_4 (see Fig.2).

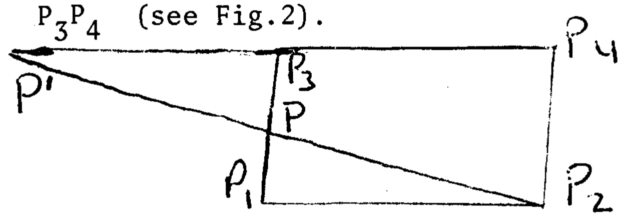


FIG. 2

(iv) If no line contains more than two points and P_1, P_2, P_3 are distinct points, then the line ℓ_3 through P_3 parallel to P_1P_2 and the line ℓ_2 through P_2 parallel to P_1P_3 have a point P in common (see Fig.3).

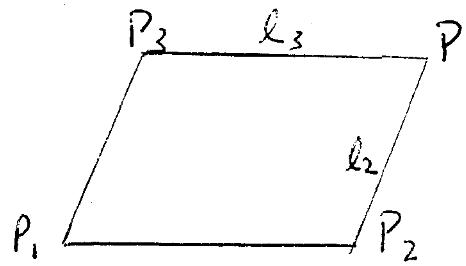


FIG. 3

(v) Some line contains exactly $q \geq 2$ points.

(vi) There exist two lines neither parallel nor with a common point.

Then $L = AG(n,q)$ for some $n \geq 3$."

Our plan is to take a structure M with predicates R and Q satisfying #'s 0 through 11. We define a line to be any set $R(a,b,M)$ for a,b distinct in M. For lines ℓ_1, ℓ_2 we say ℓ_1 is parallel to ℓ_2 iff there are $a,b,c,d \in M$ such that $\ell_1 = R(a,b,M)$, $\ell_2 = R(c,d,M)$ and $Q(a,b,c,d)$.

We intend to show that with the above definitions, we get an incidence structure satisfying (i) through (vi) above; the incidence relation is set membership.

First we show parallelism is an equivalence relation. From $Q(x,y,x,y)$ it is reflexive, and $Q(x,y,z,w) \rightarrow Q(z,w,x,y)$ which follows from #7 gives the symmetry.

Before proving transitivity, we notice that R(x,y,x) and R(x,y,z) $\rightarrow$ R(y,x,z) tell us that a,b $\in$ R(a,b,M) = R(b,a,M) and #4 tells us that any two distinct points of a line ℓ determine it. This gives (i). Now suppose ℓ_1 and ℓ_2 are parallel, and ℓ_2 and ℓ_3 are parallel. Choose $a \neq b \in \ell_1$, $c \neq d \in \ell_2$, $e \neq f \in \ell_2$ and $g \neq h \in \ell_3$ with Q(a,b,c,d) $\wedge$ Q(e,f,g,h). Find, by #5, i with Q(e,g,c,i) and j with Q(e,g,d,j). By #7 and #5, $i \neq j$. From #8 and Q(e,f,g,h) $\wedge$ R(e,f,c) $\wedge$ Q(e,g,c,i) we get R(g,h,i). Similarly R(g,h,j). So ℓ_3 = R(i,j,M). Now #10 gives, from Q(e,g,c,i) $\wedge$ Q(e,g,d,j), that Q(c,i,d,j), so by #7, Q(c,d,i,j). #7 and #10 then give Q(a,b,i,j). So ℓ_1 and ℓ_3 are parallel, and parallelism is transitive.

For (ii), let ℓ = R(a,b,M) be a line and c be a point. #5 yields d with Q(a,b,c,d), so R(c,d,M) is parallel to ℓ. Suppose R(c,e,M) is parallel to ℓ; then by transitivity, R(c,d,M) and R(c,e,M) are parallel.

By #4, it suffices to find any point $\neq$ c of R(c,e,M) which is in R(c,d,M) to show R(c,e,M) = R(c,d,M). Pick f,g $\in$ R(c,d,M) distinct and h,i in R(c,e,M) distinct with Q(f,g,h,i). We may assume $f \neq h$. Pick c*,c** with Q(f,h,c,c*), Q(h,f,c,c**); then by #5, #6 and #7, $c \neq$ c*,c**. By #8, Q(f,h,c,c*) $\wedge$ R(f,g,c) $\wedge$ Q(f,g,h,i) gives R(h,i,c*). #7 gives Q(h,i,f,g), so by #8 and Q(h,f,c,c**) $\wedge$ R(h,i,c), R(f,g,c**). So c** $\in$ R(c,d,M) and c* $\in$ R(c,e,M). From #7, Q(f,h,c**,c), so from #10, Q(c,c*,c**,c). #9 gives R(c,c*,c**), so c* $\in$ R(c,c**,M) = R(c,d,M). So the parallel is unique.

For (iii), suppose $R(a,b,M)$ and $R(c,d,M)$ are parallel and $e \neq a \wedge R(a,c,e)$. Using #8 and adjusting d, we may assume $Q(a,b,c,d)$. So $Q(a,c,b,d) \wedge Q(a,c,e) \wedge e \neq a$. By #11 there is f such that $R(e,b,f) \wedge R(c,d,f)$. This is exactly what we need.

(iv) comes immediately from $Q(x,y,z,w) \rightarrow Q(x,z,y,w)$.

(v) is immediate from #1 and #0.

(vi) is clear as M is infinite.

APPENDIX 2

OPEN PROBLEMS

The following is a list of open problems in the area and things I would just like to know:

1. The ultimate problem is to classify the superstable, uni-dimensional structures, at least those with NOTOP. The results proved here, as well as those mentioned in the introduction, may be regarded as the beginnings of an answer. Obviously there is much more to do.

2. More specifically, what can be said of superstable uni-dimensional structures of rank greater than 1 if the associated rank 1 types satisfy (*)? An interesting example of Saffe ([Sa], pp.18-20) may well be as archetypal for such structures of rank 2 as Morley's is for those of rank 1. This example provides a particularly clear example of Buechler's Coordinatization Theorem (see the Introduction here). [A caveat, however; Saffe incorrectly counts the number of models, which should be $\leq \beth_2$ for every cardinality.]

3. Returning to weakly minimal structures, the known non-$\aleph_0$-stable uni-dimensional examples in the Introduction here all have, as the dependence geometry on a strong type, an affine or projective space over some field. Must this always occur? If not, what can occur?

4. In a similar vein, Hrushovski's work indicates that under fairly general conditions we have an abelian group structure on some strong type. What groups can occur? Given a particular group, what other structure can there be on the strong type? On the model? We know

the abelian group structure extends to a formula. Likely the work of Pillay and Prest [PP] has application here.

5. Here we assume the dependence geometry on a strong type is locally finite, and so if not trivial either an affine or projective space over a finite field. What can be said about weakly minimal structures if we assume the dependence geometry is affine or projective over $\bigcup_{k\in\omega} F(p^k)$ (p a prime) as in Hrushovski's example? Over Q, as in (Z,+)? Other possibilities?

6. Getting even more specific, and returning to structures as in Section 8, there are still a few things to be cleared up. For instance, what further structure is possible on M/θ? We know what Y/θ and each X_i/θ can look like, so this is a question about the interaction of these "boxes". Given the similarity of the structures here to $\aleph_0$-categorical structures of Morley rank 1, the work of Martin [Mr] could well be relevant here.

Also, what are the possibilities for M, given that M/θ is as in Section 8? Or $\varphi(M)$, given that $\varphi(M)/\theta$ is as in Section 7? Perhaps the work of Ahlbrandt and Zeigler ([Ah], [AZ]) is relevant here.

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