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UNSTABLE MODES, ZERO MODES, AND PHASE TRANSITIONS IN QCD

by

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UNSTABLE MODES, ZERO MODES, AND

PHASE TRANSITIONS IN QCD

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ABSTRACT

The vacuum structure of Quantum Chromodynamics (QCD) is probed by computing the one-loop effective action, using the background field technique. This is done in several circumstances at finite, as well as zero, temperature.

A supersymmetric generalization of QCD is considered with the covariantly-constant background "magnetic" field Ansatz for the gauge groups SU(2), SU(3), and SU(4). Such a configuration is seen to be less asymptotically free than the non-supersymmetric case. Further, while there is some partial cancellation of quantum corrections within the vector multiplet, there is none in a chiral multiplet. Moreover, there is no cancellation of the negative eigenvalues arising in the gauge-field fluctuation spectrum. The complete cancellation of quantum corrections found in some models is seen not to be a general feature of supersymmetry.

The extension to finite temperature is not straightforward because of the unstable gauge-field modes, corresponding to negative eigenvalues. Previous authors have either ignored these modes, or treated them in a way which is not very meaningful. Here, two rather different approximations are used to take them into account. Only the SU(2) case is considered.

In the first case, a finite-temperature Ansatz is constructed for exciting the unstable modes. These modes are treated semi-classically through use of the running coupling constant. It is seen that this system undergoes a first-order phase transition to a state of zero "magnetic" field.

The second approach takes the unstable modes into account by going beyond one loop. An "all orders" contribution from the unstable modes is included, neglecting cross terms between the different gauge-invariant subsets. (Flory has shown that, at zero temperature, this does not change the renormalized expression for the real part of the effective action, and gives no imaginary part.) First, an alternative derivation of Flory's result is presented. Second, a finite temperature calculation is made. In this case, the phase transition to zero field is seen to be second order.

Finally, the finite-temperature effects of fermions are considered. As the temperature is increased from zero, fermion fluctuations initially increase the stability of the system. Eventually this trend is reversed, but the phase transition occurs at a higher temperature than in the pure gauge field case. It is shown that this behavior is due to the existence of zero-energy modes in the fermion spectrum.

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CHAPTER 1

Introduction

Quantum chromodynamics (QCD) is widely considered to provide the correct description of the strong interaction. However, the strength of the interaction and its non-linearity make the theory difficult to unravel. Perturbation theory is useful only at high energies where the interaction becomes sufficiently weak (the regime of asymptotic freedom). Moreover, the bound-state problem is completely unsolved.

Associated with the bound-state problem is a whole constellation of puzzles comprising the structure of the vacuum, color confinement, chiral-symmetry breakdown, and hadronization - the formation of color-singlet bound states. This thesis reports on some investigations into the first of these, the vacuum structure of QCD.

The QCD vacuum is certainly unusual, being opaque to the propagation of colored particles. Quarks and gluons are confined to hadrons, which may be thought of as bubbles of a metastable (except, perhaps, for the proton) state within which quarks and gluons can propagate. This state is often referred to as the "perturbative vacuum." The true vacuum is sometimes called "nonperturbative."

The vacuum structure of QCD has drawn the attention of many researchers following various lines of approach. We shall not attempt a review, but refer the reader to the recent article by Shuryak [1]. The approach we shall follow here employs the background field technique. This involves constructing a simple model for the vacuum (the background field configuration), and computing the effects of quantum fluctuations about this state. A nice

feature of this approach is that it is nonperturbative in the background field, so that strong fields are handled as easily as weak ones.

The background field technique originates with the work of Schwinger [2]. He used it to compute the one-loop effective action for a constant background electromagnetic field. (This was a rederivation of the result of Heisenberg and Euler [3], which had been obtained by other methods and a somewhat different point of view.) Schwinger's formalism was extended to include non-Abelian gauge theories by Duff and Ramón-Medrano [4].

In non-Abelian gauge theories there are two gauge-inequivalent ways to construct a covariantly constant field strength [5], which we can refer to as the Abelian and non-Abelian cases. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] , \tag{1.1}$$

where we have employed matrix notation: $A_\mu(x) = A_\mu{}^a(x)T^a$, and T^a are the group generators. In the Abelian case, $A_\mu{}^a(x)$ is linear in x and nonzero only for a subset of the a's corresponding to mutually commuting generators (generators of the Cartan subalgebra). Only the derivative terms contribute in (1.1). In the non-Abelian case, the vector potential is non-commuting but constant, so that only the commutator in (1.1) contributes.

Our work is concerned with the Abelian case. (See [5] for an introduction to the non-Abelian case.) The vector potential has the form

$$A_\mu{}^a(x) = -\frac{1}{2} F_{\mu\nu}^a x^\nu , \tag{1.2}$$

where $F_{\mu\nu}^a$ is nonzero and constant in the Cartan directions only. The first application of the background field technique to this case was by Batalin, Matinyan, and Savvidy [6] and Savvidy [7]. (Also see [8].) They computed the one-loop corrections to the vacuum energy density for a pure

(gauge fields only) SU(2) gauge theory. The renormalized expression for a constant "magnetic" field ($F_{12}^3 = H$) is

$$\varepsilon = \frac{1}{2} H^2 + \frac{11}{48\pi^2} (gH)^2 \left[\ln \frac{gH}{\mu^2} - \frac{1}{2} \right] , \qquad (1.3)$$

where μ is the renormalization point and $1/2\ H^2$ is the classical energy density. We shall refer to a constant background "magnetic" field as a Savvidy state. (The logarithmic correction term in (1.3) had been computed by Vanyashin and Terentev in 1965, but in a somewhat different context [9]. They considered the change in the Lagrangian of a constant electromagnetic field due to the vacuum polarization of a charged massive vector field.)

The important feature of (1.3) is the minimum at

$$gH = \mu^2 \exp\left(- \frac{24\pi^2}{11 g^2} \right) , \qquad (1.4)$$

which indicates a non-trivial vacuum structure. (Of course, the state $F_{12}^3 = H$ is neither gauge nor Lorentz invariant. However, we may think of it as a basis for a more complicated vacuum structure; i.e. we can consider the vacuum to be a gauge and Lorentz invariant linear combination of such states.) The interesting behavior seen in (1.4) remains qualitatively valid beyond one loop, provided the β-function goes to infinity sufficiently fast for strong coupling [7,10]. In QCD it is difficult to check whether this condition is satisfied. In this thesis we shall make the assumption that it is.

As it turns out, (1.3) is not the whole story. Nielsen and Olesen [11] found, through a more careful treatment of the contour integrals involved in its evaluation, that the vacuum energy density also contains an imaginary part, $- \frac{(gH)^2}{8\pi}$. This implies that a constant "magnetic" field is unstable.

(This is true in the non-Abelian case as well [12].) The eigenvalues of the SU(2) gauge-field fluctuation operator are

$$\lambda = (2N+1+S)gH + k_3^2 + k_4^2 , \qquad (1.5)$$

where $S = \pm 2$, $N = 0,1,2,\ldots$, and k_3 and k_4 are plane-wave eigenvalues. For $N = 0$, $S = -2$, and $k_3^2 + k_4^2 < gH$, λ is negative. This is the source of the imaginary part.

For a constant "electric" field, the energy density also acquires an imaginary part, $(gE)^2/96\pi$, which implies a strong instability. However, unlike the case of the constant "magnetic" field, this is analogous to the situation in QED, where a constant electric field is unstable to the creation of electron-positron pairs.

So we see that, in general, constant fields are unstable. In fact, constant fields are stable only if they are self-dual or anti-self-dual (and, hence, Abelian) in Euclidean space [12]. Although we have done some work on the self-dual case, it lies outside the main development of this thesis. A brief summary of the relevant features of the self-dual effective action is given in Appendix F.

The dynamics of the unstable modes, corresponding to the negative eigenvalues present in (1.5), were developed by Nielsen and Olesen [13] and Ambjørn, Nielsen, and Olesen [14]. They showed that these modes behave as a (1+1)-dimensional Higgs model. However, these authors did not explicitly consider how to excite the unstable modes in order to give a lowering of the energy density of the vacuum Ansatz.

The crucial observation about the instability is that it occurs for small momenta ($k_3^2 + k_4^2 < gH$). It is a large-distance phenomenon. This suggests that it can be removed by cutting space into domains with linear sizes not exceeding some critical distance (of order $1/\sqrt{gH}$).

Nielsen and Ninomiya [15] have put forward a specific Ansatz for exciting the unstable modes. The state arrived at this way, when added to the original Ansatz (1.2), provides a new classical configuration. Within a semi-classical approximation, they show that this new state has a lower energy density than the Savvidy state. It consists of domains of flux tubes in the x^1-x^2 plane with area $2\pi/gH$. The structure is similar to the lattice of vortices occcuring in type II superconductors.

Nielsen and Olesen [16] showed that these domains are not strongly fixed. Quantum fluctuations cause the flux tubes to vibrate, giving them fluid properties. In a liquid, it is possible to have a local crystal structure which is not present globally. Rotational invariance is achieved by tunneling between local structures, and this further lowers the energy density. Gauge invariance is attained in a similar fashion. The resulting state is a disordered "spaghetti" vacuum. (This state is further explored by Ambjørn and Olesen [17,18].)

An alternative treatment of the unstable modes has been given by Flory [19]. He showed that a completely real energy density can be achieved by including the quartic terms in the action for these modes. He treated the stable modes to one-loop order in the usual way, but computed the contribution from the unstable modes to "all orders" (neglecting cross terms between the different gauge-invariant subsets). It is interesting that the energy density calculated this way agrees with the real part of previous computations, (1.3).

Much less work has been done on the Savvidy state at finite temperature. This is an especially important topic in the context of the confinement-deconfinement transition. One expects to see this manifested as a phase

transition from a non-trivial to a trivial (no fields) energy minimum. In two publications [20,21], the authors simply omitted the unstable modes from consideration and calculated the free-energy density of the stable modes. They concluded that the system will undergo a first-order phase transition to zero field. (This approach was also followed by Chakrabarti [22] and Reuter and Dittrich [23], who considered the effects of finite temperature and an external magnetic field on a spontaneously broken gauge theory.) Authors of two others papers [24,25] pointed out that the negative eigenvalues give a complex contribution at finite temperature, and they argued that the real part of this should also be included in the free-energy density. However, doing this gives no phase transition at all. In addition, the imaginary part grows with temperature. It seems that nothing can be concluded from this. (We also point out that fermions were not included in any of these calculations.)

The purpose of this thesis is to make some further inroads into the problem of the vacuum structure of QCD, particularly at finite temperature, by computing the free-energy density in several circumstances. We begin, in Chapter 2, with a self-contained exposition of the effective action formalism. We then apply this formalism to a supersymmetric generalization of the Savvidy state in Chapter 3. We consider supersymmetry at zero temperature only, and treat the gauge groups SU(2), SU(3), and SU(4).

The central part of this thesis is containd in Chapters 4-6, where the effects of finite temperature are considered. The emphasis in Chapters 4 and 5 is on a more rigorous treatment of the unstable modes. In Chapter 4 we construct a finite-temperature Ansatz for exciting the unstable modes, along the lines of Nielsen and Ninomiya. This is then used to calculate the

free-energy density. In Chapter 5 we extend Flory's treatment of the unstable modes to the finite-temperature case. Finally, the finite-temperature effects of fermions are considered in Chapter 6. We conclude with a brief summary and some conclusions in Chapter 7.

CHAPTER 2

The Effective Action

The principal tool used for the investigations reported in this thesis is the effective action. It is the purpose of this chapter to discuss the concept both formally and informally, and to present a scheme for its computation.

The effective action, Γ, was originally introduced by Heisenberg and Euler [3] and by Schwinger [26]. Γ is a functional of a classical field $\phi(x)$, which in turn depends on the space-time coordinates. It is defined in such a way that its extrema determine the physical theory in question. When the field ϕ is time-independent, $\Gamma(\phi)$ has a direct physical significance. By time-translation invariance,

$$\Gamma(\phi)\Big|_{\text{static}} = -E(\phi)\int dt \ ,$$

where $E(\phi)$ has a variational definition [27]. It is the minimum expectation value of the Hamiltonian in a normalized state for which the expectation value of the field operator $\Phi(\vec{x})$ is $\phi(\vec{x})$:

$$E(\phi) = \langle a|H|a\rangle \text{ for } |a\rangle \text{ such that } \delta\langle a|H|a\rangle = 0,$$
$$\text{subject to } \langle a|a\rangle = 1 \text{ and } \langle a|\Phi|a\rangle = \phi.$$

(A generalization of this variational definition of Γ to include time-dependent states has also been given [28].) We see that the effective action provides a useful tool for studying the vacuum structure of a field theory.

2.1 Formalism [29-33]

For notational convenience, we shall consider the example of a single self-coupled scalar field, described by a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$. $\mathcal{L}$ is assumed to be quadratic in $\partial_\mu \phi$.

The amplitude to be in the state $|\phi_1\rangle$ at t_1, having been in the state $|\phi_0\rangle$ at t_0, is

$$\langle \phi_1, t_1 | \phi_0, t_0 \rangle \equiv \langle \phi_1 | e^{-i(t_1 - t_0)H} | \phi_0 \rangle \quad , \tag{2.1}$$

where H is the time-independent Hamiltonian operator of the system. Recall that the Hamiltonian density is

$$H = \pi\dot{\phi} - \mathcal{L} \, , \quad \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \, , \quad \dot{\phi} \equiv \partial_t \phi \, .$$

The amplitude (2.1) can be expressed as the Feynman path integral

$$\begin{aligned} \langle \phi_1, t_1 | \phi_0, t_0 \rangle &= \int_{\phi_0}^{\phi_1} (D\phi) \int (D\pi) \exp\left\{ i \int_{t_0}^{t_1} d^4x \, (\mathcal{L} - \pi\dot{\phi}) \right\} \\ &= N \int_{\phi_0}^{\phi_1} (D\phi) \exp\left\{ i \int_{t_0}^{t_1} d^4x \mathcal{L} \right\} \quad , \end{aligned} \tag{2.2}$$

where the (Gaussian) π integration has been performed, N is a normalization constant, and $\phi(\vec{x}, t_1) = \phi_1$, $\phi(\vec{x}, t_0) = \phi_0$.

A useful mathematical device is to couple the field $\phi(x)$ to an arbitrary external c-number source $J(x)$: $\mathcal{L} \to \mathcal{L} + J(x)\phi(x)$. All dynamical information about the system can be deduced from the response of the vacuum state to the external source.

Consider a source such that $J(x) \xrightarrow[|t| \to \infty]{} 0$. Then the amplitude for a system to be in the vacuum state at $t = +\infty$, when it was known to be in the

vacuum state at $t = -\infty$, is a phase:

$$\langle 0^+|0^-\rangle_J \equiv Z(J) \equiv e^{iW(J)} , \tag{2.3}$$

where the functionals Z and W are defined by this expression. In the interaction picture with respect to $J(x)\phi(x)$, the state of the system at t_0 is

$$|0^-\rangle_J = T[\exp i \int_{-\infty}^{t_0} dt \int d^3x\, J(x)\phi(x)]|0\rangle ,$$

where T denotes time ordering. The state which will evolve into $|0\rangle$ at $t = +\infty$ is

$$|0^+\rangle_J = T[\exp i\int_{-}^{t_0} dt \int d^3x\, J(x)\phi(x)]|0\rangle .$$

Therefore,

$$Z(J) = \langle 0^+|0^-\rangle_J = \langle 0|T[\exp i\int d^4x\, J(x)\phi(x)]|0\rangle . \tag{2.4}$$

This immediately yields

$$\left[\frac{\delta\langle 0^+|0^-\rangle_J}{\delta J(x_1)\ldots\delta J(x_n)}\right]_{J=0} = i^n G_n(x_1,\ldots,x_n) , \tag{2.5}$$

where $G_n(x_1,\ldots,x_n) \equiv \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle$ are the Green's functions of the theory. So Z(J) is the generating functional for the Green's functions. From (2.4) and (2.5), Z has the expansion

$$Z(J) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1\ldots d^4x_n G_n(x_1,\ldots,x_n)J(x_1)\ldots J(x_n),$$

where $G_0 \equiv 1$.

The Green's functions defined here are the full (disconnected) ones. However, only the connected pieces contribute to the S matrix. These are obtained by taking the logarithm of Z [33]:

$$\ln Z(J) = i W(J) . \tag{2.6}$$

W(J) is the generating functional for connected Green's functions G_n^c, and can be expanded as

$$i W(J) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n G_n^c(x_1,\ldots,x_n) J(x_1)\ldots J(x_n).$$

The generating functionals can be represented by the path integral (2.2) by including the external source, and letting $t_0 \to -\infty$, $t_1 \to +\infty$:

$$e^{iW(J)} = N\int (D\phi) e^{i[S(\phi)+(J,\phi)]} , \tag{2.7}$$

where $S(\phi) = \int d^4x \mathcal{L}(x)$ and $(J,\phi) \equiv \int d^4x\, J(x)\phi(x)$. It is easy to see that this satisfies (2.5).

From (2.7) the average field is

$$\bar{\phi}(x) \equiv \frac{\langle 0^+ | \phi(x) | 0^- \rangle_J}{\langle 0^+ | 0^- \rangle_J} = \frac{\delta W(J)}{\delta J(x)} . \tag{2.8}$$

(In the literature $\bar{\phi}$ is often called the classical field. This is because, to zeroth order in $\hbar$, $\bar{\phi}$ solves the classical equation of motion for ϕ, with source J.) The limit of $\bar{\phi}(x)$ as $J \to 0$ is the vacuum expectation value $\langle\phi\rangle$.

Now, W is a functional of J (and, of course, $\bar{\phi}$ depends on J through (2.8)). A functional of $\bar{\phi}$, called the effective action, is constructed by making a functional Legendre transformation:

$$\Gamma(\bar{\phi}) \equiv W(J) - (J,\bar{\phi}) . \tag{2.9}$$

It follows that

$$\frac{\delta\Gamma(\bar{\phi})}{\delta\bar{\phi}(x)} = -J(x) \quad . \tag{2.10}$$

By translational invariance, $\bar{\phi}$ should be a constant for vanishing source. Hence $\langle\phi\rangle$ is the root of

$$\left.\frac{d\Gamma(\bar{\phi})}{d\bar{\phi}}\right|_{\bar{\phi} = \langle\phi\rangle} = 0 \; . \tag{2.11}$$

$\Gamma(\bar{\phi})$ is the generating functional for the one-particle irreducible (1-PI) Green's functions [33]. In the expansion

$$\Gamma(\bar{\phi}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \Gamma_n(x_1,\ldots,x_n)\bar{\phi}(x_1)\ldots\bar{\phi}(x_n) \; ,$$

Γ_n is the n-point 1-PI Green's function. (A diagram, representing a Green's function, is 1-PI if it cannot be split into two disjoined pieces by cutting a single internal line. It is easier to compute 1-PI graphs, and then string them together into trees of 1-PI parts, than it is to compute the connected Green's functions directly.)

An alternative expansion for the effective action is given in terms of $\bar{\phi}$ and its derivatives:

$$\Gamma(\bar{\phi}) = \int d^4x \left[-U(\bar{\phi}(x)) + \frac{1}{2}\,\partial_\mu\bar{\phi}(x)\partial^\mu\bar{\phi}(x)Z(\bar{\phi}(x)) + \ldots\right] . \tag{2.12}$$

$U(\bar{\phi})$ is called the effective potential since, for $\bar{\phi}(x) = \rho$ (constant),

$$\Gamma(\rho) = -\int d^4x\, U(\rho) = -\Omega\, U(\rho) \; , \tag{2.13}$$

where Ω is the total space-time volume. Then, from (2.11), $\langle\phi\rangle$ is the root of

$$\left.\frac{dU(\rho)}{d\rho}\right|_{\rho = \langle\phi\rangle} = 0 \ . \qquad (2.14)$$

(Incidentally, to this order in momenta and renormalizing the field such that $Z(\bar{\phi}) = 1$, (2.12) gives

$$\Box\bar{\phi} = -\frac{dU(\bar{\phi})}{d\bar{\phi}} \ .$$

In the limit of small field variations, the quantum corrections to the classical scalar field equation of motion are included by replacing the potential by the effective potential. This helps justify the notion of effective potential. It represents a kind of quantum field potential energy (density).)

2.2 Computation

In general, the effective action is not known and requires some method of approximation. The most widely used method is the loop expansion: first summing all tree graphs, then those with one closed loop, etc. As is well known, this is an expansion in powers of $\hbar$. It is actually more inclusive than perturbation theory since the set of graphs with n loops or less includes, as a subset, all graphs of n^{th} order or less in the coupling constant.

The loop expansion may be obtained from the functional integral (2.7) by a stationary phase evaluation. Let $\phi = \phi_0$ denote the position of the stationary point. This is a solution to the classical equation of motion with source J:

$$\left.\frac{\delta S(\phi)}{\delta\phi(x)}\right|_{\phi = \phi_0} = -J(x) \ . \qquad (2.15)$$

First we introduce a propagator function $\Delta(\phi_0)$ by

$$\langle x \left| i\Delta^{-1}(\phi_0) \right| y \rangle \equiv \frac{\delta^2 S(\phi)}{\delta\phi(x)\delta\phi(y)} \Bigg|_{\phi = \phi_0} . \tag{2.16}$$

Then we expand the action about ϕ_0 (redefining ϕ as the deviation from ϕ_0):

$$S(\phi_0 + \phi) = S(\phi_0) - (J,\phi) + \frac{1}{2}(\phi, i\Delta^{-1}(\phi_0)\phi) + S_2(\phi_0,\phi), \tag{2.17}$$

where S_2 contains all higher order terms ($O(\hbar^2)$). Then

$$\begin{aligned} e^{iW(J)} &= N\int(D\phi)\exp i[S(\phi_0 + \phi) + (\phi_0 + \phi, J)] \\ &= e^{i[S(\phi_0) + (\phi_0,J)]} \left\{ N\int(D\phi)\exp \frac{i}{2}(\phi, i\Delta^{-1}(\phi_0)\phi) \right\} \cdot \langle e^{iS_2} \rangle , \end{aligned}$$

where $\langle\ \rangle$ denotes functional averaging with weighting function $\exp \frac{i}{2}(\phi, i\Delta^{-1}(\phi_0)\phi)$.

The factor within the curly brackets is, by the usual rules of functional integration, $[\det i\Delta^{-1}(\phi_0)]^{-1/2}$. Therefore

$$W(J) = S(\phi_0) + (\phi_0,J) + \frac{i}{2}\ln\det[i\Delta^{-1}(\phi_0)] + O(\hbar^2) . \tag{2.18}$$

Now, since

$$\bar{\phi}(x) = \phi_0(x) + \phi_1(x), \qquad \phi_1 \sim O(\hbar) , \tag{2.19}$$

then

$$\begin{aligned} S(\phi_0) = S(\bar{\phi} - \phi_1) &= S(\bar{\phi}) - \int d^4x\ \phi_1(x) \frac{\delta S(\phi)}{\delta\phi} \Bigg|_{\phi = \phi_0} + O(\hbar^2) \\ &= S(\bar{\phi}) + (\phi_1,J) + O(\hbar^2) . \end{aligned} \tag{2.20}$$

Finally, (2.18) - (2.20) in (2.9) give the one-loop effective action

$$\Gamma(\bar{\phi}) = S(\bar{\phi}) + \frac{i}{2}\ln\det[i\Delta^{-1}(\bar{\phi})] + O(\hbar^2). \tag{2.21}$$

With $\bar{\phi}(x) = \rho$ (constant), and

$$\Gamma(\rho) = -\Omega U(\rho),$$
$$S(\rho) = \left[\int d^4x \mathcal{L}(x)\right]_{\phi=\rho} = -\Omega V(\rho) ,$$

the one-loop effective potential is

$$U(\rho) = V(\rho) - \frac{i}{2}\Omega^{-1} \ln \det[i\Delta^{-1}(\rho)]$$
$$= V(\rho) - \frac{i}{2}\Omega^{-1} \mathrm{Tr} \ln [i\Delta^{-1}(\rho)] . \qquad (2.22)$$

(The coefficient - 1/2 is specific to a real scalar or vector field. It is -1 for a complex scalar, and +1 for a spinor or ghost.)

We conclude this section with two points about this formalism as applied to the covariantly-constant background field. First, it is the field strength $F_{\mu\nu}$ which is constant, not the gauge field. Therefore we compute the effective Lagrangian (action density) and not the effective potential (see (2.12)). However, this will be of no practical consequence here. Second, gauge theories introduce the complexities of gauge fixing and insuring that spurious degrees of freedom are not included in Γ. All we need to say here is that the contribution from unwanted degrees of freedom is cancelled by including the contribution from ghost fields.

2.3 Finite Temperature Formalism [34,35]

There are two formalisms for describing a quantum field theory at finite temperature, known as real-time and imaginary-time. We shall just consider the imaginary-time formalism here since it is more straightforward, and sufficient for our purpose.

In the amplitude (2.2) we set $t_0 = 0$:

$$\langle \phi_1 | e^{-it_1 H} | \phi_0 \rangle = N \int_{\phi_0}^{\phi_1} (D\phi) \exp \left\{ i \int_0^{t_1} d^4x \mathcal{L} \right\}. \tag{2.23}$$

The partition function is

$$Z \equiv \mathrm{Tr}\, e^{-\beta H} = \sum_{\phi} \langle \phi | e^{-\beta H} | \phi \rangle .$$

The crucial observation is that this can be obtained from (2.23) by the transformation $t = -i\tau$ ($t_1 = -i\beta$), and by restricting the integration range to those paths which are periodic in β (anti-periodic for fermions); $\phi(\beta) = \phi(0)$:

$$\mathrm{Tr}\, e^{-\beta H} = N'(\beta) \int_{\text{periodic}} (D\phi) \exp \left\{ \int_0^{\beta} d\tau \int d^3x \mathcal{L} \right\}. \tag{2.24}$$

(Remember that time derivatives in $\mathcal{L}$ are now multiplied by i.) Notice that the difference between zero and finite temperature is merely a difference in boundary conditions.

The discussion in section 2.2 now goes through with little change. The object to compute is still $\mathrm{Tr} \ln [i\Delta^{-1}(\rho)]$, but the requirement of periodicity over the finie range in τ changes energy integrations into sums (in the trace). The means making the replacements

$$k_0 \to \begin{cases} \dfrac{2\pi n}{\beta} , & \text{bosons,} \\ \dfrac{(2n+1)\pi}{\beta} , & \text{fermions,} \end{cases}$$

$$\int \frac{dk_0}{2\pi} \to \frac{1}{\beta} \sum_n . \tag{2.25}$$

CHAPTER 3

Supersymmetric Savvidy State for SU(2), SU(3), and SU(4)

In this chapter, we shall apply the formalism developed in the previous chapter to a supersymmetric (SUSY) version of QCD, using the Savvidy Ansatz for a covariantly constant background field [36]. (I shall use "SUSY" for either "supersymmetry" or "supersymmetric".) One reason for treating the SUSY case is that it requires, at the one-loop level, only a simple extension of the calculation for ordinary QCD. As we shall see, the results for ordinary QCD are easily extracted from this more general setting. In addition, it helps us to understand something more about the structure of SUSY theories.

In SUSY theories, calculations of quantum corrections are often simplified because of cancellations which may occur between terms arising from a field and its SUSY partner. Mass renormalizations are unnecessary in some models, such as those of Wess-Zumino type, for this reason. Another interesting example is given by D'Adda and Di Vecchia [37]. They compute the one-loop correction to the vacuum energy density around a classical self-dual solution to a SUSY Yang-Mills theory (one vector multiplet). They show that there is a complete cancellation of determinants arising from the positive eigenvalues of the fluctuation operators. (Negative eigenvalues do not occur, but there are zero modes which must be considered separately.)

The conclusions of D'Adda and Di Vecchia are specific to the case of a self-dual classical field. In view of the great interest in SUSY theories, it is worthwhile to consider what happens with a different choice of background field. In particular, we shall see that the cancellations found by D'Adda and Di Vecchia do not occur in the Savvidy case.

3.1 Quantum Chromodynamics

We begin by establishing some notations and conventions. QCD will be taken to mean any SU(N) gauge theory with fermions in the fundamental representation. In particular, we shall be interested in the cases N = 2,3,4 (colors). Regardless of the group, the fermions (ψ) and gauge fields (A_μ) will be referred to as quarks and gluons, respectively.

We adopt the metric convention: $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

For simplicity, we take the quarks to be massless. The QCD Lagrangian density is

$$\mathcal{L}_{QCD} = -\frac{1}{4}(F_{\mu\nu}^{a})^2 + i\bar{\psi}_j\sigma^\mu D_\mu\psi_j \ , \qquad (3.1)$$

where

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a} - g f_{abc} A_\mu^{b} A_\nu^{c} \ ,$$

$$D_\mu = \partial_\mu + igT_R^{a} A_\mu^{a} \ .$$

The f_{abc} are the group structure constants and T_R^{a} are generators for the representation R (a,b,c = 1,...,N^2-1). The Pauli matrices σ^μ indicate that ψ is a Weyl spinor. A Dirac quark is constructed from two Weyl spinors. Therefore, for N_f quark flavors, j = 1,...,$2N_f$.

3.2 Supersymmetric QCD

This section contains a brief summary of the relevant features of supersymmetry. More details may be found in the book by Wess and Bagger [38], and references therein. I have adopted most of their notations and conventions.

Supersymmetry is an extension of the symmetry of the Poincaré group. The SUSY algebra is obtained by adding, to the set of Lorentz rotation generators $M_{\mu\nu}$ and translation generators P_μ, a set of generators $Q_\alpha{}^a$ which transform bosons into fermions, and vice versa. Such generators are obviously fermionic and may be chosen to transform as left-handed Weyl spinors under Lorentz transformations ($\alpha = 1,2$). We shall consider only the so-called $N = 1$ SUSY, which has one generator Q_α (and its hermitian conjugate). In addition to the Poincaré commutation relations, the algebra is given by

$$[Q_\alpha, P_\mu] = 0,$$
$$\{Q_\alpha, Q_{\dot\beta}\} = 2\sigma^{\mu}_{\alpha\dot\beta} P_\mu . \qquad (3.2)$$

Such an algebra, involving commutation and anti-commutation relations, is known as a graded Lie algebra.

There are two SUSY multiplets which are important for constructing particle theories, known as scalar (or chiral) and vector. A chiral multiplet contains a complex scalar S, a Weyl spinor ψ, and an auxiliary complex scalar F. (Auxiliary fields are always necessary to close the group of SUSY transformations. They are called auxiliary because they do not propagate. They yield equations of constraint through which they may be eliminated.)

The structure of a vector multiplet is more complicated. For our purposes, it is sufficent to know that a special SUSY gauge exists (the Wess-Zumino gauge) in which most of the components of the vector multiplet vanish. What remain are a real vector field A_μ, a Weyl spinor Λ, and an auxiliary real scalar D. Further, working in this gauge still allows the usual gauge transformations.

A SUSY QCD needs quarks and gluons. We construct a quark from the Weyl spinors of two chiral multiplets, so we require $2N_f$ chiral multiplets. Therefore, all component fields (S,ψ,F) are in the fundamental representation. (The scalars S are called s-quarks.) The vector field of a vector multiplet provides the gluon. The component fields (A_μ,Λ,D) are in the adjoint representation. (The spinors Λ are known as gluinos.)

The most general Lagrangian density (apart from possible mass terms) for a gauge theory of interacting chiral and vector multiplets, which is supersymmetric and gauge invariant, is (in the Wess-Zumino gauge)

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu}^{a})^2 + i\bar{\Lambda}\sigma^{\mu}D_{\mu}\Lambda + \frac{1}{2}(D^{a})^2 \\ &+ (D^{\mu}S_j)^{\dagger}D_{\mu}S_j + i\bar{\psi}_j\sigma^{\mu}D_{\mu}\psi_j + F_j^{\dagger}F_j \\ &+ g(\epsilon^{\alpha\beta}S_j^{\dagger}\Lambda_{\alpha}^{a}T_R^{a}\psi_{\beta j} + \text{H.c.}) + g(S_j^{\dagger}T_R^{a}S_j)D^{a} \\ &+ \mathcal{L}_{gf} + \mathcal{L}_{gh}\,, \end{aligned} \tag{3.3}$$

where $\epsilon^{\alpha\beta}$ is the completely antisymmetric tensor, and $\mathcal{L}_{gf}$ and $\mathcal{L}_{gh}$ represent gauge-fixing and ghost terms, respectively. The theory given by (3.3), with the Savvidy Ansatz for the background field, is our starting point.

3.3 Fluctuation Operators

As we saw in Chapter 2, to compute the one-loop quantum corrections to the effective action requires evaluation of determinants of those operators, in $\mathcal{L}$, which involve terms quadratic in the fluctuation fields. (Equivalently, we could consider the linearized equations of motion.) First, we note the constraint equations for the auxiliary fields F and D:

$$F_{\dot{j}} = 0,$$

$$D^a = -gS_{\dot{i}}^{\dagger} T_R^a S_{\dot{j}} \ . \tag{3.4}$$

with the use of (3.4), we write (3.3) as

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + i\bar{\Lambda}\sigma^\mu D_\mu \Lambda + (D^\mu S_{\dot{j}})^\dagger D_\mu S_{\dot{j}}$$
$$+ i\bar{\psi}_{\dot{j}}\sigma^\mu D_\mu \psi_{\dot{j}} + \mathcal{L}_{gf} + \mathcal{L}_{gh}$$
$$+ \text{higher order terms.} \tag{3.5}$$

We introduce the quantum gauge field variable $a_\mu^{\ a}$ by

$$A_\mu^{\ a} = \tilde{A}_\mu^{\ a} + a_\mu^{\ a} \ , \tag{3.6}$$

where $\tilde{A}_\mu^{\ a}$ is the background gauge field, and define a derivative covariant in the background field by

$$\nabla_\mu \equiv \partial_\mu + igT_R^a \tilde{A}_\mu^{\ a} \ . \tag{3.7}$$

In this notation, the covariantly constant field condition is

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0, \tag{3.8}$$
$$\nabla_\mu = \partial_\mu + ig[T_R^a \tilde{A}_\mu^{\ a}, \ ] \ .$$

Clearly, the terms in (3.5) quadratic in Λ, S, and ψ are obtained by replacing D_μ by ∇_μ. The linearized eigenvalue equations are

$$\nabla_\mu \nabla^\mu S_{\dot{j}} = \lambda S_{\dot{j}} \ , \tag{3.9}$$

and, where ψ stands for either $\psi_{\dot{j}}$ or Λ,

$$i\sigma^\mu \nabla_\mu \bar{\psi} = \lambda\psi \ ,$$
$$i\bar{\sigma}^\mu \nabla_\mu \psi = \lambda\bar{\psi} \ , \tag{3.10}$$

with $\bar{\sigma}^0 = \sigma^0$, $\bar{\sigma}^i = -\sigma^i$.

To treat the gauge fields, it is necessary to fix the gauge. It is natural to employ the background field method [39], which makes use of the decomposition (3.6). The idea is to choose a gauge (the background field gauge) which breaks the invariance of the quantum field, but retains gauge invariance in terms of the background field.

The background gauge condition is

$$\nabla_\mu a^\mu = 0 ,$$
$$\mathcal{L}_{gf} = -\frac{1}{2}(\nabla_\mu a^\mu)^2 . \qquad (3.11)$$

This fixes the ghost Lagrangian to be

$$\mathcal{L}_{gh} = (\nabla_\mu \eta)^\dagger \nabla^\mu \eta - g\eta_a^* f_{abc} a_\mu^c \nabla^\mu \eta_b . \qquad (3.12)$$

We recall that the ghosts η and η^* are independent scalars in the adjoint representation, which obey Fermi statistics. Note that only the first term in (3.12) is quadratic, and yields the same eigenvalue equation as do the scalars S_j, (3.9).

For the gauge fields, it is easier to start with the free field equation. We use (3.6) - (3.8) to write

$$F^{\mu\nu} = \tilde{F}^{\mu\nu} + \nabla^\mu a^\nu - \nabla^\nu a^\mu + ig[a^\mu, a^\nu] ,$$
$$D_\mu F^{\mu\nu} = \nabla_\mu \nabla^\mu a^\nu - \nabla_\mu \nabla^\nu a^\mu - ig[\tilde{F}^{\mu\nu}, a_\mu]$$
$$+ \text{higher order terms.} \qquad (3.13)$$

Now, we use the commutator $[\nabla_\mu, \nabla_\nu] = ig\tilde{F}_{\mu\nu}$ and the gauge condition (3.11) in (3.13) to get the gauge field eigenvalue equation:

$$\nabla_\mu \nabla^\mu a_\nu - 2ig[\tilde{F}_{\mu\nu}, a^\mu] = \lambda a_\nu . \qquad (3.14)$$

Finally, we recall that funtional integrals are defined by their analytic continuation to Euclidean space. We shall, therefore, want to solve the Euclidean versions of (3.9), (3.10), and (3.14). We obtain these by the substitutions $x_0 \to ix_4$, $A_0 \to iA_4$, $\gamma^0 \to -\gamma^4(\sigma^0 \to -i\sigma^4)$. The metric change is $g_{\mu\nu} \to -\delta_{\mu\nu}$. Then we get

$$\begin{aligned} -\nabla_\mu \nabla_\mu \phi &= \lambda\phi, \\ i\sigma_\mu \overset{E}{\nabla}_\mu \bar{\psi} &= \lambda\psi, \\ i\bar{\sigma}_\mu \overset{E}{\nabla}_\mu \psi &= \lambda\bar{\psi}, \\ -\nabla_\mu \nabla_\mu a_\nu + 2ig[\tilde{F}_{\mu\nu}, a_\mu] &= \lambda a_\nu , \end{aligned} \tag{3.15}$$

where ϕ stands for either the S_j or ghosts, and ψ for either the ψ_j or Λ.

3.4 One-Loop Effective Action

As we saw in Chapter 1, the Savvidy Ansatz is

$$\tilde{A}_\mu^{\ a} = -\frac{1}{2}\tilde{F}_{\mu\nu}^{\ a} x_\nu , \tag{3.16}$$

where $\tilde{F}_{\mu\nu}^{\ a}$ is a constant which differs from zero only for those a's corresponding to mutually commuting generators. Flyvbjerg [40] has shown that the minimum vacuum energy density in SU(N), $N \leq 4$, is obtained with fields of equal magnitude which are orthogonal in real space. So we take for our Ansätze, after a gauge transformation on (3.16),

$$\begin{aligned} &\text{SU(2):} \quad \tilde{A}_1^{\ 3} = -Hx_2, \\ &\text{SU(3):} \quad \tilde{A}_1^{\ 3} = -Hx_2, \quad \tilde{A}_2^{\ 8} = -Hx_3, \\ &\text{SU(4):} \quad \tilde{A}_1^{\ 3} = -Hx_2, \quad \tilde{A}_2^{\ 8} = -Hx_3, \ \tilde{A}_3^{\ 15} = -Hx_1 . \end{aligned} \tag{3.17}$$

The field strengths are $\tilde{F}_{12}^{\ 3} = \tilde{F}_{23}^{\ 8} = \tilde{F}_{31}^{\ 15} = H$, $H > 0$.

With the Ansätze (3.17), the eigenvalues of (3.15) are worked out in Appendix A. They are of the form

$$\lambda = [(2N+1+S)\chi_R gH + k_3^2 + k_4^2]^p , \tag{3.18}$$

where k_3 and k_4 are plane-wave eigenvalues, and $(2N+1)\chi_R gH$ are harmonic oscillator eigenvalues; χ_R is the magnitude of the nonzero weights of the representation R. For fermions, p = 1/2; p = 1 otherwise. For scalars (and ghosts), S = 0; for spinors, S = ±1; and for vectors, S = 0, ±2.

The one-loop effective action is given by

$$\Gamma^{(1)} = -\sum_i q_i \, \mathrm{Tr} \ln \lambda_i , \tag{3.19}$$

where the sum is over the fluctuation operators, and

$$q = \begin{cases} -1/2, & \text{vectors} \\ -1, & \text{scalars} \\ +1, & \text{spinors and ghosts} . \end{cases}$$

Using (3.18) in (3.19), we get the energy density

$$\epsilon^{(1)} = \frac{\Gamma^{(1)}}{\Omega} = -\sum_i q_i p_i M_i \left(\frac{\chi_i gH}{2\pi}\right) \sum_{S_i} \sum_{N=0}^{\infty} \int \frac{dk_3}{2\pi} \int \frac{dk_4}{2\pi} \ln \left\{ \mu^{-2} \times [(2N+1+S_i)\chi_i gH + k_3^2 + k_4^2] \right\} , \tag{3.20}$$

where M is the multiplicity of the eigenvalue, $\chi gH/2\pi$ is the density of oscillator eigenvalues, and μ is a constant of dimension mass which makes the argument of the logarithm dimensionless. (Ω is the space-time volume.)

Of course, (3.20) is infinite. Its regularization is carried out in Appendix B. We use (B.7) - (B.9) to write (3.20) as

$$\mathrm{Re}\,\epsilon^{(1)} = -2\sum_i q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2}\sum_{S_i}\left\{\frac{\mathrm{Res}(S_i)}{\epsilon} + C'_{RF} - \mathrm{Res}(S_i)\,\ln\frac{gH}{\mu^2} + O(\epsilon)\right\}, \tag{3.21}$$

$$\mathrm{Im}\,\epsilon^{(1)}_{(S=-2)} = q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi}\ , \tag{3.22}$$

where

$$\mathrm{Res}(S_i) = \zeta\left(-1, \frac{1+S_i}{2}\right) = \begin{cases} 1/24\ , & S = 0 \\ -1/12\ , & S = \pm 1 \\ -11/24\ , & S = +2\ , \end{cases}$$

$$\mathrm{Res}(-2) = \zeta(-1, 1/2) - 1/2 = -11/24\ ,$$

and C'_{RF} is a real finite constant whose value is unimportant.

We normalize the matrices for the fundamental representation by $\mathrm{Tr}(T^a T^b) = 1/2\ \delta_{ab}$. Then $\chi = 1$ for the adjoint representation of any SU(N). For the fundamental representation, $\chi = 1/2,\ 1/\sqrt{3},\ \sqrt{3/8}$ for SU(2), SU(3), and SU(4), respectively.

Of the N^2-1 states in the adjoint representation, N-1 have zero weight. These do not couple to the background field and don't contribute to (3.21). Thus $M = N^2-N$ for the adjoint. Further, the contribution from the vectors with S = 0 is exactly cancelled by the ghost contribution. Only the $S = \pm 2$ states count for the vectors. For the fundamental, $M = N \times 2N_f$.

We define

$$C_i \equiv 2q_i p_i M_i \chi_i^2 \sum_{S_i} \mathrm{Res}(S_i)\ , \tag{3.23}$$

and write (3.21) as

$$\mathrm{Re}\,\epsilon^{(1)} = \frac{(gH)^2}{8\pi^2}\left\{-\frac{\sum_i C_i}{\epsilon} + C_{RF} + \sum_i C_i \ln\frac{gH}{\mu^2} + O(\epsilon)\right\}. \tag{3.24}$$

The C_i, and the various other parameters, are listed in Table I. The $\sum C_i$ for the different groups are listed in Table II.

TABLE I. Contributions of the various fields to $\varepsilon^{(1)}$ for a constant background magnetic field.

Field	SU(N)	q	p	S	Res	χ	M	C
$a_\mu{}^a$	2	$-\frac{1}{2}$	1	±2	$-\frac{11}{24}$	1	2	$\frac{11}{6}$
	3					1	6	$\frac{11}{2}$
	4					1	12	11
Λ	2	+1	$\frac{1}{2}$	±2	$-\frac{1}{12}$	1	2	$-\frac{1}{3}$
	3					1	6	-1
	4					1	12	-2
ψ	2	+1	$\frac{1}{2}$	±1	$-\frac{1}{12}$	$\frac{1}{2}$	$4N_f$	$-N_f/6$
	3					$1/\sqrt{3}$	$6N_f$	$-N_f/3$
	4					$1/\sqrt{3/8}$	$8N_f$	$-N_f/2$
S	2	1	1	0	$\frac{1}{24}$	$\frac{1}{2}$	$4N_f$	$-N_f/12$
	3					$1/\sqrt{3}$	$6N_f$	$-N_f/6$
	4					$1/\sqrt{3/8}$	$8N_f$	$-N_f/4$

TABLE II. Values of $\sum_i C_i$ for the different groups.

Group	$\sum_i C_i$ Ordinary state	Supersymmetric state
SU(2)	$\frac{11-N_f}{6}$	$\frac{6-N_f}{4}$
SU(3)	$\frac{33-2N_f}{6}$	$\frac{9-N_f}{2}$
SU(4)	$\frac{22-N_f}{2}$	$\frac{3}{4}(12-N_f)$

Renormalization of (3.24) is straightforward. We add to (3.24) the classical energy density, $1/2\ ZH^2$ for each field, where $\sqrt{Z}$ is the field renormalization constant. (We note that the product gH is a renormalization-group invariant in the background gauge [39].)

$$\mathrm{Re}\ \epsilon = \frac{N-1}{2} ZH^2 + \mathrm{Re}\ \epsilon^{(1)} . \tag{3.25}$$

Following Savvidy, we choose the condition

$$\left.\frac{\partial \mathrm{Re}\ \epsilon}{\partial H^2}\right|_{gH=\mu_s^2} = \frac{N-1}{2} . \tag{3.26}$$

This sets $Z = 1$ at the classical level. Acting on (3.25), (3.26) gives

$$Z = 1 - \frac{2}{N-1}\frac{g^2}{8\pi^2}\left\{\frac{1}{2}\sum_i C_i \ln\frac{\mu_s^4}{\mu^4} + \frac{1}{2}\sum C_i - \frac{\sum_i C_i}{\epsilon} + C_{RF} + O(\epsilon)\right\} . \tag{3.27}$$

Finally, by substituting (3.27) into (3.25), we get the renormalized effective energy density

$$\mathrm{Re}\ \epsilon = \frac{N-1}{2} H^2 + \frac{(gH)^2}{8\pi^2}\left(\sum_i C_i\right)\left[\ln\frac{gH}{\mu_s^2} - \frac{1}{2}\right] . \tag{3.28}$$

The minimum of (3.28) occurs for

$$(gH)_{min} \equiv \Lambda_s^2 = \mu_s^2 \exp\left[\frac{-4\pi^2(N-1)}{g^2\sum_i C_i}\right] , \tag{3.29}$$

so that the renormalized coupling constant is

$$\frac{1}{g^2} = \frac{\sum_i C_i}{4\pi^2(N-1)} \ln\frac{\mu_s^2}{\Lambda_s^2} . \tag{3.30}$$

Using this we may write (3.28) as

$$\mathrm{Re}\ \epsilon = \frac{(gH)^2}{8\pi^2}\left(\sum_i C_i\right)\left[\ln\frac{gH}{\Lambda_s^2} - \frac{1}{2}\right] . \tag{3.31}$$

3.5 Discussion

There are several points to be made about the one-loop effective action. First, with our regularization procedure, it contains the imaginary part (3.22):

$$\mathrm{Im}\ \varepsilon \;=\; -\frac{1}{2}\,N(N-1)\,\frac{(gH)^2}{8\pi} \tag{3.32}$$

This contribution comes entirely from the vector fields with $S = -2$, and is the same as in the non-SUSY case. The gluino eigenvalue spectrum is non-negative and, therefore, supersymmetry cannot remove the instability. (However, the instability can be removed by exciting the unstable modes. This will be discussed in Chapter 4.)

Second, the expressions for C_i in Table I show that complete cancellation does not occur among terms arising from non-negative eigenvalues (even without matter fields). In the case of chiral multiplets, the quarks and s-quarks contribute to Γ with the same sign.

Third, for the energy density to be bounded below, we must have $\sum_i C_i \geq 0$. In the case of SU(3), for example, the state will not admit more than 9 quark flavors. This is less than the 16 flavors allowed in ordinary QCD. In this sense, the matter-field corrections have a greater destabilizing influence in the SUSY case.

Finally, we compute the beta function. It satisfies

$$\mu_s \frac{d\Gamma}{d\mu_s} \;\equiv\; \left[\mu_s \frac{\partial}{\partial \mu_s} + \beta(g)\,\frac{\partial}{\partial g} - \gamma H \frac{\partial}{\partial H}\right]\Gamma(g,H,\mu_s) \;=\; 0\ ,$$

$$\beta(g) \;\equiv\; \mu_s \frac{\partial g}{\partial \mu_s}\ ,\quad \gamma \;\equiv\; \frac{1}{2}\,\mu_s \frac{\partial\ \ln Z}{\partial \mu_s}\ , \tag{3.33}$$

where γ is the anomalous dimension of the field and Z is the field renormalization. Since the field and coupling-constant renormalization factors are related ($H_o = \sqrt{Z}\, H$, $g_o = 1/\sqrt{Z}\, g$), it is easy to show that $\beta(g) = g\gamma$. Then we have

$$\left[\mu_s \frac{\partial}{\partial \mu_s} + \beta(g)\left[\frac{\partial}{\partial g} - \frac{H}{g}\frac{\partial}{\partial H}\right]\right] \epsilon(g,H,\mu_s) = 0 \,. \tag{3.34}$$

Applying (3.34) to (3.28), we get

$$\beta(g) = -\left(\frac{2}{N-1}\right)\frac{1}{8\pi^2}\left(\sum_i C_i\right) g^3 \,, \tag{3.35}$$

to one-loop order. Thus asymptotic freedom is assured for $N_f < 9$ (in SU(3)). In the sense of admitting fewer quark flavors, the SUSY state is less asymptotically free than the ordinary state. This is easy to understand since SUSY provides additional fields (gluinos and s-quarks) whose fluctuations further screen the non-Abelian charge.

To summarize, we have analyzed a supersymmetric model for the gauge groups SU(2), SU(3), and SU(4) with a constant Abelian background field. Our results indicate that the kind of cancellations found in Ref. 37 do not occur as a general feature of SUSY theories. In particular, they do not occur for the Savvidy Ansatz. While the theory discussed is asymptotically free, the beta function restricts the number of quark flavors to be less than 6, 9, and 12 for SU(2), SU(3), and SU(4), respectively.

CHAPTER 4

Effective Action at Finite Temperature I - Exciting the Unstable Modes

In this chapter, and the two following, we shall study the constant background color magnetic field at finite temperature. (Only the non-supersymmetric case will be considered.) Were it not for the negative eigenvalues in the gluon fluctuation spectrum, this would be relatively straightforward. However, the existence of the unstable modes, corresponding to these negative eigenvalues, raises questions as to how they should properly be taken into account.

Several publications have presented finite-temperature calculations on the Savvidy state. In two of these [20,21], the authors omitted the unstable modes which give the imaginary part of the energy density at zero temperature. They concluded that the system will undergo a first-order phase transition to a state of zero magnetic field. In two other papers [24,25], these calculations were criticized because the imaginary part of the Savvidy energy density becomes complex at finite temperature. The authors argued that a more meaningful calculation should include the real part of this complex term. The result of doing this is a deepening of the free-energy minimum at nonzero field, with no phase transition at all. However, the authors of Ref. 25 called all of these calculations into question by pointing out that the imaginary part of the free-energy density also grows with temperature, increasing the instability. They conclude that there is nothing to conclude.

In this chapter, we present an improved calculation based on generalizing to finite temperature the Ansatz of Nielsen and Ninomiya [15] for exciting the unstable modes [43]. This Ansatz yields an energy density which is

completely real, and which is lower than that of the Savvidy state. We believe that such a state provides a firmer basis for a finite-temperature investigation.

4.1 Stable Modes

First we summarize the relevant results for the stable modes. We shall restrict ourselves to a pure gauge theory (no fermions), and take the gauge group to be SU(2). From (3.28) and Table I, the energy density is

$$\varepsilon^{(\text{stable})} = \frac{1}{2} H^2 + \frac{11}{48\pi^2} (gH)^2 \left[\ln \frac{gH}{\mu_s^2} - \frac{1}{2} \right] , \tag{4.1}$$

where μ_s is the renormalization point in the Savvidy scheme, (3.26). From (3.30) we can express the renormalized coupling constant as

$$\frac{1}{g^2} = \frac{11}{24\pi^2} \ln \frac{\mu_s^2}{\Lambda_s^2} , \tag{4.2}$$

where Λ_s is the scale parameter in the Savvidy scheme. (We recall that gH is a renormalization-group invariant.)

The contribution of the stable modes to the finite-temperature free-energy density is worked out in Appendix C. Doing the sum over S in (C.4), and rearranging some terms, we get

$$F_\beta^{(\text{stable})} = \frac{gH}{\pi^2 \beta^2} \left\{ \int_{\beta\sqrt{gH}}^{\infty} dx \, \ln\left[1-\exp\left(-\sqrt{x^2-\beta^2 gH}\right) \right] \right.$$

$$\left. + \int_0^{\infty} dx \, \ln\left[1-\exp\left(-\sqrt{x^2+\beta^2 gH}\right) \right] + 2 \sum_{N=1}^{\infty} \int_0^{\infty} dx \, \ln\left[1-\exp\left(-\sqrt{x^2+(2N+1)\beta^2 gH}\right) \right] \right\} . \tag{4.3}$$

The unstable modes are excluded by the lower limit of the first integral.

4.2 Exciting the Unstable Modes

The existence of the unstable modes was first pointed out by Nielsen and Olesen [11]. They computed the vacuum polarization for an SU(2) Yang-Mills theory and showed that the unstable modes exhibit tachyonic (negative mass squared) behavior in 1+1 dimension. In a further paper [13], and one with Ambjørn [14], these authors developed the dynamics of the unstable modes and showed that the instability can be removed by a (1+1) dimensional Higgs mechanism.

To excite the unstable modes means to construct a general linear combination of the appropriate eigenstates, and to treat this as an additional background field. From (3.18) we see that the negative eigenvalues occur for $N = 0$, $S = -2$. $N = 0$ corresponds to the ground state harmonic-oscillator wave function, $\exp[-\frac{1}{2} gH(x_2 - \frac{k_1}{gH})^2]$. (My notation differs from that of Nielsen and Ninomiya [15]. They use the gauge-rotated Ansatz $\tilde{A}_2{}^3 = Hx_1$. Further, they define the covariant derivative with a minus sign, so that their H is my -H. However, these are just conventions.) The eigenstates in x_3 and x_4 are plane waves. The general form for an excitation of the unstable modes is thus

$$W(x) = \left(\frac{\pi}{gH}\right)^{1/4} \int \frac{dk_1 dk_3 dk_4}{(2\pi)^3} \tilde{\phi}(k_1,k_3,k_4) e^{-i(k_1x_1+k_3x_3+k_4x_4)} \times \exp\left[-\frac{1}{2} gH\left(x_2 - \frac{k_1}{gH}\right)^2\right], \qquad (4.4)$$

where we have suppressed group and vector indices (see below). The dynamical field variable is

$$\phi_{k_1}(x_3,x_4) = \left(\frac{\pi}{gH}\right)^{1/4} \int \frac{dk_3 dk_4}{(2\pi)^2} \tilde{\phi}(k_1,k_3,k_4) e^{-i(k_3x_3+k_4x_4)} . \qquad (4.5)$$

The procedure of Nielsen and Ninomiya [15] is to construct a specific Ansatz for $\bar{\phi}$, and use (4.4) as a new background potential, in addition to the one giving rise to the constant field strength. This Ansatz will contain some parameters with which the energy density can be varied to find the optimal choice for lowering the energy density. This excitation of the unstable modes is treated in a semi-classical approximation. This means that the field variable corresponding to these modes is not an integration variable in the functional integral used to compute the effective action. So the effective action for these modes is just their classical action, except that, as we shall see, the coupling constant appears explicitly in this expression (apart from the renormalization-group invariant combination gH). The renormalization group formula for the running coupling constant is then used to replace the classical value.

In terms of the SU(2) component fields, the Yang-Mills Lagrangian in the background gauge is

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}(\nabla_\mu^* W_\nu^- - \nabla_\nu^* W_\mu^-)(\nabla_\mu W_\nu^+ - \nabla_\nu W_\mu^+) - \nabla_\mu^* W_\mu^- \nabla_\nu W_\nu^+ \\ & - \frac{1}{4}(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)^2 - \frac{1}{2}(\partial_\mu W_\mu^3)^2 - ig(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)W_\mu^- W_\nu^+ \\ & + \frac{g^2}{4}(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)^2 \quad , \end{aligned} \tag{4.6}$$

where W_μ^3 and $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2)$ are the group eigenstates obtained from diagonalizing the quadratic operator in (3.14). Diagonalization in the Lorentz indices gives the eigenstates $W_{3,4}^\pm$ and $W_\pm^\pm = \frac{1}{\sqrt{2}}(W_1^\pm \pm i W_2^\pm)$.

Now the unstable modes are W_-^+ and W_+^-. So the Lagrangian for these modes is obtained from (4.6) by setting all other modes to zero:

$$W_-{}^+ = W, \qquad W_+{}^- = W^* ,$$
$$W_\mu{}^3 = 0, \qquad W_3{}^{a_4} = 0 ,$$
$$W_1{}^1 = W_2{}^2, \qquad W_1{}^2 = -W_2{}^1 , \tag{4.7}$$

where W is defined in (4.4), and the last two relations are obtained from $W_+{}^+ = 0$ (or $W_-{}^- = 0$).

We substitute the general form for the unstable mode excitation (4.4) into the Yang-Mills Lagrangian (4.6), with the constraints (4.7), and integrate over x_1 and x_2 to get

$$\begin{aligned} S_u = & \int \frac{dk_1}{2\pi} \int dx_3 dx_4 \left(-\left|\partial_\mu \phi_{k_1}\right|^2 + gH\left|\phi_{k_1}\right|^2\right) \\ & - \frac{g^2}{2} \int \frac{dk_1\, dk_1'\, dk_1''\, dk_1'''}{(2\pi)^4}\, 2\pi\delta(k_1+k_1'-k_1''-k_1''') \left(\frac{gH}{2\pi}\right)^{1/2} \\ & \times \exp\left\{-\frac{gH}{2}\left[k_1{}^2+(k_1')^2+(k_1'')^2+(k_1''')^2-(k_1+k_1')^2\right]\right\} \int dx_3 dx_4 \\ & \times \phi^*_{k_1''} \phi^*_{k_1'''} \phi_{k_1} \phi_{k_1'} \ . \end{aligned} \tag{4.8}$$

We emphasize that we are only considering the self-interaction of the unstable modes, in addition to their interaction with the constant background field.

Now the problem is to construct an Ansatz for ϕ_{k_1} (or, equivalently, $\tilde{\phi}$) which decreases the energy density. As Nielsen and Ninomiya show [15], this is not completely trivial; it is necessary that the contribution to the energy be proportional to the volume. Their Ansatz is

$$\tilde{\phi}(k_1,k_3,k_4) = \Phi\, 2\pi\delta(k_4) 2\pi\delta(k_3) \sum_{m\in Z} 2\pi\delta(k_1-mc) , \tag{4.9}$$

where Φ is a normalization constant and c is the spacing of the δ-functions in the variable k_1. (In Ref. 15 $\delta(k_4)$ is implicit.) With this Ansatz, (4.5) becomes

$$\phi_{k_1}(x_3,x_4) = \left(\frac{\pi}{gH}\right)^{1/4} 2\pi\,\Phi \sum_{m\in Z} \delta(k_1 - mc) \ . \tag{4.10}$$

This expression is then used in (4.8) to yield the classical space-averaged energy density

$$\varepsilon_{cl.} = \frac{1}{2}H^2 - \frac{gH|\Phi|^2}{\sqrt{2\kappa}} + g^2 \frac{\theta_3{}^2(0,\kappa)|\Phi|^4}{4\sqrt{\kappa}} \ , \tag{4.11}$$

where $\kappa \equiv \frac{c^2}{2\pi gH}$, and θ_3 is the third Jacobi θ function defined by

$$\theta_3(\nu,\kappa) = \sum_{n\in Z} e^{-\pi\kappa n^2} e^{2\pi i n\nu} \ .$$

(Some details in arriving at (4.11) are given in Appendix D.)

To (4.11) must be added the quantum correction. It is very difficult to compute this because of the θ_3 functions in the vacuum Ansatz. (As far as I know, this still has not been done.) As in Ref. 15, we shall use the correction given by the second term of (4.1) for a constant field, with a magnitude H equal to the typical order of magnitude of the classical vacuum configuration (i.e. $\hat{H} \approx H - g|\phi|^2$). Adding this to (4.11), we have

$$\varepsilon = \frac{1}{2}H^2 - \frac{gH}{\sqrt{2\kappa}}|\Phi|^2 + g^2 \frac{\theta_3{}^2(0,\kappa)|\Phi|^4}{4\sqrt{\kappa}} + \frac{11}{48\pi^2}(g\hat{H})^2\left[\ln\frac{g\hat{H}}{\mu_s{}^2} - \frac{1}{2}\right] . \tag{4.12}$$

The expression (4.12) is ambiguous since the coupling constant appears explicitly (apart from the invariant gH). The logical thing is to evaluate g at a scale given by the typical field strength $\hat{H}$. With the use of (4.2) we set

$$\frac{1}{g^2} = \frac{11}{24\pi^2} \ln \frac{g\hat{H}}{\Lambda_s^2} \quad . \tag{4.13}$$

Another ambiguity exists due to the choice for $\hat{H}$ in (4.12). We take, as a reasonable guess,

$$\hat{H} \approx H \quad . \tag{4.14}$$

With (4.13) and (4.14), (4.12) must be minimized with respect to Φ and κ, the parameters of the Ansatz (4.10). Minimization with respect to κ gives $\kappa = 1$. We get

$$\varepsilon = \frac{11}{48\pi^2} (gH)^2 \left[\left(1 - \frac{1}{\theta_3(0,1)^2} \right) \ln \frac{gH}{\Lambda_s^2} - \frac{1}{2} \right] \quad . \tag{4.15}$$

Finally, we note that (4.15) has a minimum for

$$\begin{aligned} (gH)_{min} &= \Lambda_s^2 \exp\left[\frac{1}{2(\theta_3(0,1)^2-1)} \right] \\ &\approx 16\Lambda_s^2 \quad , \end{aligned} \tag{4.16}$$

with magnitude

$$\begin{aligned} \varepsilon_{min} &= \frac{-11}{96\pi^2} \Lambda_s^4 \left(1 - \frac{1}{\theta_3(0,1)^2} \right) \exp\left[\frac{1}{\theta_3(0,1)^2-1} \right] \\ &\approx 39 \left[\frac{-11}{96\pi^2} \Lambda_s^4 \right] . \end{aligned} \tag{4.17}$$

This is to be compared with the value $\frac{-11}{96\pi^2} \Lambda_s^4$, as obtained from (4.1).

4.3 Unstable Modes at Finite Temperature

To calculate the finite-temperature free energy, we must decide how to take the unstable modes into account. At zero temperature, as we have just seen, they can be treated semi-classically through use of the relevant running coupling constant (4.13).

The simplest thing would be to ignore any possible temperature dependence for these modes. Then one could estimate the free energy by adding to (4.12) the finite-temperature correction from the stable modes, (4.3). A better approximation may be obtained by postulating some simple temperature dependence for the unstable modes. This is the approach we shall take.

As mentioned in connection with the Ansatz of Nielsen and Ninomiya, we require a configuration which gives a decrease in energy proportional to the volume. This is not trivial since we are limited in momentum space by $k_3^2 + k_4^2 \leq gH$, or $k_3^2 + (\frac{2\pi n}{\beta})^2 \leq gH$ at finite temperature. On physical grounds, a reasonable choice would be to have a gaussian dependence on the allowed momenta. With this choice we must have a gaussian which becomes a delta function in the zero-temperature limit. Thus we are led to postulate the Ansatz

$$\tilde{\phi}(k_1,k_3,\omega_n) = \phi\, 2\pi\delta(k_3) \sum_{m\in Z} 2\pi\delta(k_1-mc)2\pi \frac{\beta}{\sqrt{\pi}} e^{-\beta^2\omega_n^2}, \qquad (4.18)$$

where $\omega_n = \frac{2\pi n}{\beta}$, as usual. In the zero-temperature limit $\beta \to \infty$, and (4.18) $\to$ (4.9). (We recall that a delta function may be defined as $\delta(x) = \lim_{a\to\infty} \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$.)

Using (4.5) we get

$$\phi_{k_1}(x_3,x_4) = \left(\frac{\pi}{gH}\right)^{1/4} (2\pi)^2 \frac{\Phi}{\sqrt{\pi}} \sum_{m\in Z} \delta(k_1 - mc) \sum_n{}' e^{-i\omega_n x_4} e^{-\beta^2\omega_n{}^2}, \tag{4.19}$$

where $\sum'$ denotes a restricted sum: since $k_3 = 0$ (by Ansatz), $|n| \leq \frac{\beta}{2\pi}\sqrt{gH}$. With (4.19) in (4.8) we get the free-energy density

$$\begin{aligned} F = {} & \frac{1}{2}H^2 + \frac{|\Phi|^2}{\sqrt{2\kappa}} \frac{1}{\pi} \sum_n{}' \left[\left(\frac{2\pi n}{\beta}\right)^2 - gH\right] e^{-8\pi^2 n^2} \\ & + \frac{g^2}{2} \frac{|\Phi|^4}{2\sqrt{\kappa}} \theta_3(0,\kappa)^2 \frac{1}{\pi^2} \sum_{n,n',n''}{}' e^{-8\pi^2(n+n')^2} e^{-8\pi^2(n-n'')^2} e^{-8\pi^2(n'-n'')^2} \\ & + \frac{11}{48\pi^2}(g\hat{H})^2 \left[\ln\frac{g\hat{H}}{\mu_s{}^2} - \frac{1}{2}\right] + F_\beta^{(\text{stable})}(g\hat{H},T), \end{aligned} \tag{4.20}$$

where $F_\beta^{(\text{stable})}$ is given by (4.3). Taking $\hat{H} \approx H$, as in Section 4.2, using (4.13), and minimizing with respect to Φ and κ, (4.20) gives

$$\begin{aligned} \frac{F}{\Lambda_s{}^4} = {} & \frac{11}{48\pi^2}\left\{\left(\frac{gH}{\Lambda_s{}^2}\right)^2\left(\ln\frac{gH}{\Lambda_s{}^2} - \frac{1}{2}\right)\right. \\ & \left. - \frac{\left\{\sum_n{}'\left[\left(\frac{2\pi n}{\beta\Lambda_s}\right)^2 - \frac{gH}{\Lambda_s{}^2}\right] e^{-8\pi^2 n^2}\right\}^2 \ln\frac{gH}{\Lambda_s{}^2}}{\theta_3(0,1)^2 \sum_{n,n',n''}{}' e^{-8\pi^2(n+n')^2} e^{-8\pi^2(n-n'')^2} e^{-8\pi^2(n'-n'')^2}}\right\} + \frac{F_\beta^{(\text{stable})}}{\Lambda_s{}^4} \end{aligned} \tag{4.21}$$

Note that we have used $F/\Lambda_s{}^4$, $gH/\Lambda_s{}^2$, and $\beta\Lambda_s$ as natural dimensionless variables.

Expression (4.21) can be evaluated numerically. (Some details are given in Appendix E.) The results are shown in Fig. 4.1. We have actually plotted $\Delta F = F(gH,T) - F(0,T)$ in units of Λ_s^4, versus gH in units of Λ_s^2, where

$$F(0,T) = \frac{-2\pi^2}{45\beta^4}$$

is the free-energy density of the charged modes in zero field. (ΔF is shown for ease in comparing curves for different temperatures.) For comparison, Fig. 4.2 shows a plot of ΔF vs. gH for the stable modes only.

The two cases show the same qualitative behavior. As the temperature is increased from zero, the free-energy minimum becomes shallower. Eventually the minimum at nonzero field becomes degenerate with the one at $H = 0$. This happens at $\beta\Lambda_s = .432$, compared with $\beta\Lambda_s = 1.07$ when the unstable modes are excluded. In either case, this signals a first-order phase transition. The barrier height at the critical temperature is $.264\Lambda_s^4$ in Fig. 4.1, compared with $.002\Lambda_s^4$ in Fig. 4.2. Therefore, this treatment of the unstable modes results in a more stable state (deeper free-energy minimum). a higher critical temperature (smaller β), and a more strongly first-order transition (larger barrier height at T_c).

There are two further points to be made about these results. First, the Ansatz (4.18) does not give any additional contribution to the free-energy density at finite temperature. This was not obvious a priori, but is clear from the final expression (4.21). Since $e^{-8\pi^2} \approx 10^{-34}$, only the $n = 0$ mode effectively contributes, and this gives simply the $T = 0$ result. Of course, if the unstable modes were treated quantum mechanically, their thermal fluctuations would contribute. While this might lead to some quantitative differences, such as a change in the critical temperature, we expect that the

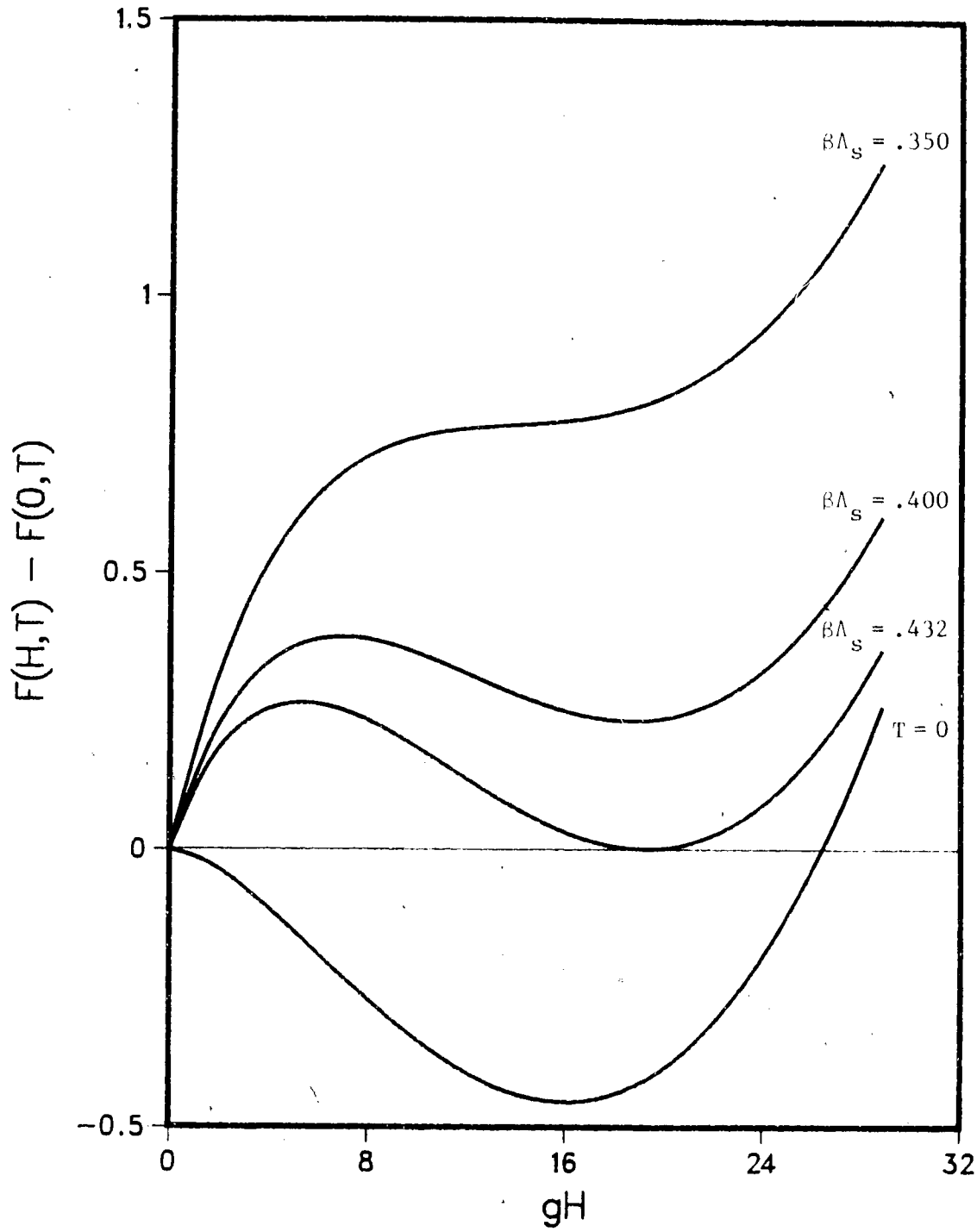

Fig. 4.1 Free-energy density curves for the Copenhagen state.

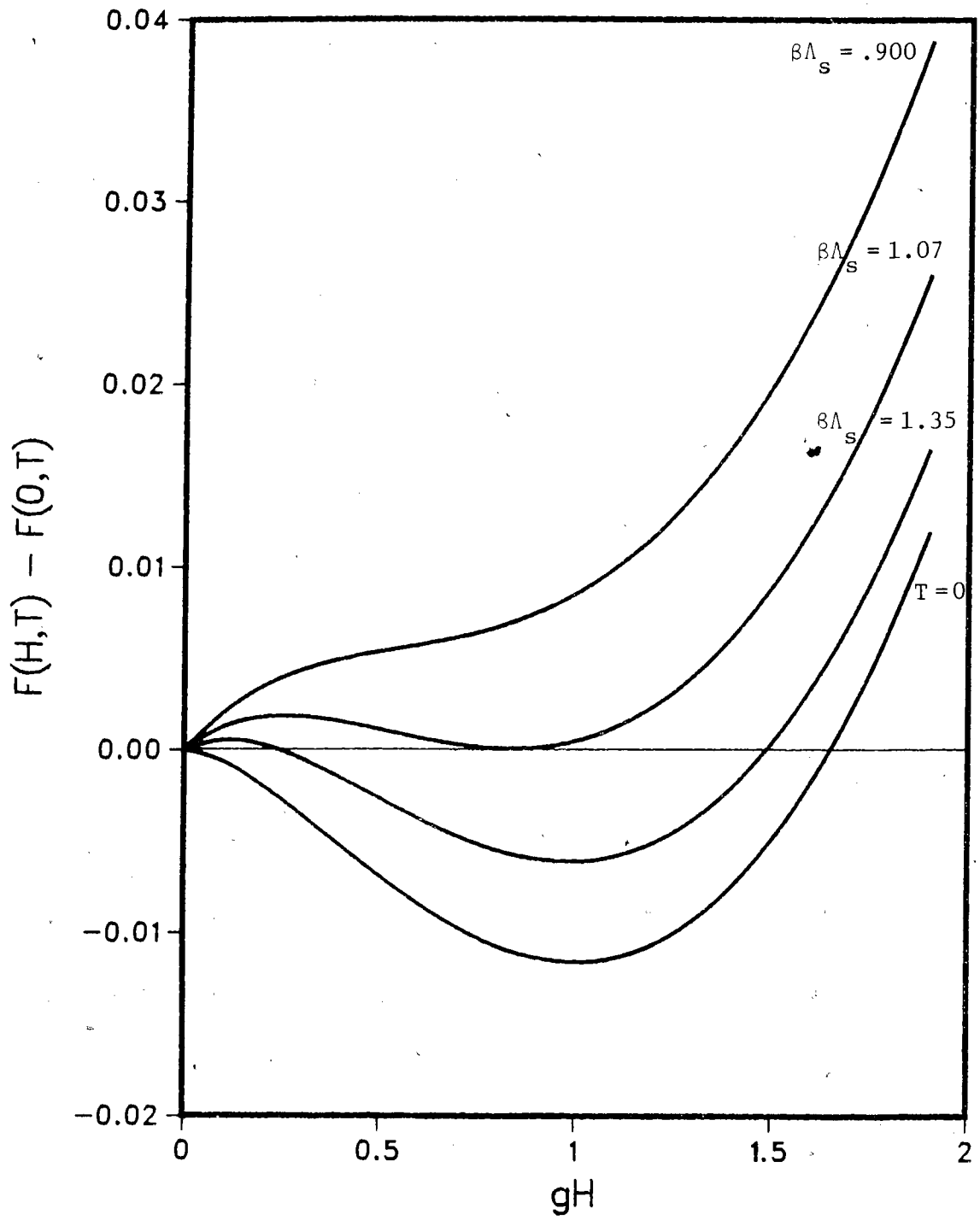

Fig. 4.2 Free-energy density curves for the Savvidy state, excluding the finite-temperature unstable mode contribution.

qualitative behavior of the system (the first-order phase transition) would remain unchanged.

Second, the behavior shown in Fig. 4.1 is slightly peculiar in that the order parameter (gH) increases with temperature, until the phase transition occurs. As we shall see in Chapter 6, this does in fact happen when fermions are included. Here it appears to be an artifact of the approximation in which quantum corrections are computed about an average constant field, rather than the flux-tube configuration determined by (4.9). As we see in Fig. 4.2, this does not happen when the stable modes are considered by themselves.

In conclusion, we have generalized the Ansatz of Nielsen and Ninomiya for exciting the unstable modes of the Savvidy vacuum to the case of finite temperature, and used this to calculate the free-energy density. Within the approximations made, we have shown that the system will undergo a first-order phase transition to a state of vanishing magnetic field.

CHAPTER 5

Effective Action at Finite Temperature II - Unstable Modes Beyond One Loop

As we saw in Chapter 4, the most important consideration in computing the free-energy density for the constant background color magnetic field is an appropriate treatment of the unstable modes. A search of the literature reveals three points of view regarding these modes. We have already discussed the view that the constant background field is unstable, and that a stable vacuum can be achieved by exciting the unstable modes. This provided the basis for the work reported in Chapter 4.

However, it has also been argued that the constant field is stable. This view is taken by researchers following two different lines of thought. In one case, the argument is that there is something wrong with the regularization procedure used by those who obtain an imaginary part in the one-loop effective action [44-46]. We shall not discuss this here, but will return to it in Chapter 7.

In the other case, the argument, due to Flory [19], is that the imaginary part of the effective action is an artifact of the one-loop aproximation. In addition to the one-loop contribution from the stable modes, Flory included an "all orders" contribution from the unstable modes in his calculation. This means using the complete Lagrangian for the unstable-mode field variable, not just the quadratic terms used in the one-loop approximation. This calculation yields the same logarithmic term as the one-loop approximation, a real finite term proportional to $(gH)^2$, and no imaginary part. The vacuum is stabilized by the self-interaction of the unstable modes.

In this chapter, we shall use Flory's approach as the basis for computing the free-energy density for the constant background color magnetic field. (We again restrict ourselves to the gauge group SU(2), and consider gluon fluctuations only.) We shall see that the phase transition to zero field which occurs is second order. Further, the transition temperature is higher than that for the Copenhagen vacuum studied in Chapter 4.

5.1 Zero Temperature

As before, we write the gauge fields as

$$A_\mu^a = \tilde{A}_\mu^a + a_\mu^a ,$$

where $\tilde{A}_\mu^a$ is the background field and a_μ^a is the quantum field variable. The full Euclidean Lagrangian, apart from ghosts, is

$$\mathcal{L}_E = -\frac{1}{4}\tilde{F}_{\mu\nu}^a\tilde{F}_{\mu\nu}^a + \frac{1}{2}a_\mu^a O_{\mu\nu}^{ab} a_\nu^b - g\varepsilon_{acd}a_\mu^c a_\nu^d \nabla_\nu^{ab} a_\mu^b + \frac{g^2}{4}\left[(a_\mu^a a_\mu^a)^2 - (a_\mu^a a_\mu^c)(a_\nu^a a_\nu^c)\right] , \qquad (5.1)$$

where $O_{\mu\nu}^{ab}$ is the operator in (3.14), the determinant of which gives the one-loop correction, and ∇_μ^{ab} is the background field covariant derivative introduced in (3.7).

We want to compute the full action for the unstable modes. (We shall continue to refer to the modes with negative eigenvalues as "unstable modes.") We recall that the unstable-mode eigenvectors are a_-^+ and a_+^-. Using the conditions (4.7), it is easy to show that the cubic term in (5.1) vanishes. (In Chapter 4, we denoted the unstable modes by W_μ^a to distinguish

them from quantum field variables. Here we revert back to our original notation.) The quartic term is

$$\begin{aligned} & \frac{g^2}{2}\,(a_-{}^+ a_+{}^-)^2 \\ &= \frac{g^2}{2}\,\left|a_-{}^+\right|^4 \\ &= \frac{g^2}{4}\,(\left|a_-{}^+\right|^4 + \left|a_+{}^-\right|^4)\ . \end{aligned} \tag{5.2}$$

Combining this with the quadratic term in (5.1) we get

$$\mathcal{L}^u = \frac{1}{2}\,(k_3{}^2 + k_4{}^2 - gH)\left|a_-{}^+\right|^2 + \frac{g^2}{4}\left|a_-{}^+\right|^4 + (+ \leftrightarrow -)\ . \tag{5.3}$$

To compute the action, Flory considered both electric and magnetic background fields. He set the electric field strength to εH, and regained the Savvidy state in the limit $\varepsilon \to 0$. This is computationally convenient since the eigenfunctions then consist solely of a product of harmonic-oscillator wavefunctions, and these are easy to integrate.

However, this procedure is not so convenient at finite temperature in the imaginary-time formalism. This is because of the periodicity requirement on the eigenfunctions in the x_4 direction. In order to apply the imaginary-time formalism we must first provide an alternative derivation of Flory's result, working with only a constant magnetic field from the start.

The normalized unstable-mode eigenfunctions are

$$\phi_{k_3,k_4}(x) = \left(\frac{gH}{2\pi}\right)^{1/2} \exp\left[-\frac{gH}{4}\,(x_1{}^2+x_2{}^2)\right] \frac{1}{\sqrt{L_3 L_4}}\ \exp\left[i(k_3x_3+k_4x_4)\right], \tag{5.4}$$

where we employ a box normalization for the plane waves. (We note that (5.4) differs from the form used in (4.4) by gauge and coordinate transformations.)

Although a_-^+ and a_+^- are complex, they are related by complex conjugation through the conditions (4.7). Further, we see from (5.3) that the action depends only on their magnitudes. Therefore we can take the amplitude of the fluctuations, $c(k_3,k_4)$, to be real. The functional integral for the unstable modes is then

$$Z^u \sim \int [da] e^{-S^u} = \left[\int \prod_{k_3,k_4} dc(k_3,k_4) e^{-\bar{S}^u} \right]^D , \tag{5.5}$$

where $\bar{S}^u$ denotes the action for one of the modes (a_-^+ or a_+^-), and

$$D \equiv 2 \frac{gHL_1L_2}{2\pi} \frac{L_3}{2\pi} \frac{L_4}{2\pi} = \frac{gH}{4\pi^3} \Omega$$

is the degeneracy of states. (The factor of 2 accounts for the 2 modes, a_-^+ and a_+^-.)

The quadratic part of the action is straightforward to evaluate:

$$\begin{aligned} &\frac{1}{2} \int d^4x \; c(k_3,k_4)\phi^* \; 0 \; c(k_3,k_4)\phi \\ &= \frac{1}{2} (k_3^2+k_4^2-gH) \; c(k_3,k_4)^2 \int d^4x \phi^* \phi \\ &= \frac{1}{2} (k_3^2+k_4^2 - gH) c(k_3,k_4)^2 \; . \end{aligned} \tag{5.6}$$

The quartic term is trickier since, for plane-wave states, neglecting cross terms between different gauge-invariant subsets gives no contribution:

$$\psi(x) = \frac{1}{\sqrt{L}} e^{-ikx}$$

$$\int dx (\psi^* \psi)^2 = \frac{1}{L} \; \underset{L \to \infty}{\longrightarrow} \; 0 \; .$$

So we consider a plane wave with wave number k to interact with a small set

of nearby states $k' = k \pm \Delta k$. Then we can take $c(k_3', k_4') \approx c(k_3, k_4)$. Including a density of states factor for the primed states, we get

$$\frac{g^2}{4} \int d^4x \; c(k_3,k_4)^2 \; \phi^* \phi \left[2 \frac{L_3 L_4}{(2\pi)^2} \right] \int dk_3' dk_4' c(k_3',k_4')^2 \phi^* \phi$$

$$\approx \frac{g^3 H}{16\pi^3} c(k_3,k_4)^4 \int dk_3' dk_4'$$

$$= \frac{g^4 H^2}{16\pi^2} c(k_3,k_4)^4 \; , \tag{5.7}$$

where we have used $\int dk_3' dk_4' = \pi g H$ (since $k_3^{\prime 2} + k_4^{\prime 2} \leq gH$).

It is convenient to define dimensionless variables:

$$\hat{c} = \sqrt{gH}\, c \quad \text{and} \quad \hat{k} = \frac{k}{\sqrt{gH}} \; . \tag{5.8}$$

Then (5.6) and (5.7) in (5.5) give

$$Z^u \sim \left[\frac{1}{\sqrt{gH}} \int \prod_{k_3, k_4} d\hat{c} \; \exp\left[\frac{1}{2} (1 - \hat{k}_3^2 - \hat{k}_4^2) \hat{c}^2 - \frac{g^2}{16\pi^2} \hat{c}^4 \right] \right]^D \; . \tag{5.9}$$

Finally, the energy density of the unstable modes is

$$\epsilon^u = -\frac{1}{\Omega} \ln Z^u = \frac{D}{\Omega} \int dk_3 dk_4 \left(\frac{1}{2} \ln gH \right) + (\text{term proportional to } (gH)^2)$$

$$= \frac{(gH)^2}{8\pi^2} \ln gH + (\text{term proportional to } (gH)^2). \tag{5.10}$$

This is as advertised. The logarithmic term is the same as that found in the one-loop appproximation, (B.8). The term proportional to $(gH)^2$ is absorbed through the renormalization condition (3.25). There is no imaginary part. Therefore the complete energy density is given by the same expression as the real part computed to one loop:

$$\varepsilon = \frac{11}{48\pi^2}\left[\ln\frac{gH}{\Lambda_s^2} - \frac{1}{2}\right] . \tag{5.11}$$

5.2 Finite Temperature

We apply the transformations (2.25). The unstable-mode eigenfunctions are now

$$\phi_{k_3,n}(x) = \left(\frac{gH}{2\pi}\right)^{1/2} \exp\left[-\frac{gH}{4}(x_1^2+x_2^2)\right] \frac{1}{\sqrt{L_3\beta}} \exp\left[-i\left(k_3x_3 + \frac{2\pi n}{\beta}x_4\right)\right]. \tag{5.12}$$

The finite-temperature functional integral is

$$Z_\beta^u \sim \left[\int_{k_3,n} \Pi \cdot dc(k_3,n) e^{-S^{-u}}\right]^{D'} , \tag{5.13}$$

where the degeneracy factor is now

$$D' \equiv 2\frac{gHL_1L_2}{2\pi}\frac{L_3}{2\pi} = \frac{gH}{2\pi^2}V ,$$

where V is the spacial volume. (The factor $\beta/2\pi$ is absent since we take the product in (5.13) over n, and not $2\pi n/\beta$.)

The quadratic part of the action is again straightforward:

$$\frac{1}{2}\left[k_3^2 + \left(\frac{2\pi n}{\beta}\right)^2 - gH\right]c(k_3,n)^2 . \tag{5.14}$$

The quartic term is

$$\frac{g^2}{4}\int d^4x\, c(k_3,n)^2\phi^*\phi\left[2\frac{L_3}{2\pi}\right]\int dk_3' \sum_{n'} c(k_3',n')\phi^*\phi$$

$$\approx \frac{g^3H}{8\pi^2}\frac{1}{\beta}c(k_3,n)^4\int dk_3' \sum_{n'} ,$$

where we apply the same argument used in obtaining (5.7). Now

$$\int dk_3 \sum_n = 2\int_0^{\sqrt{gH}} dk_3 \left[2\,\frac{\beta}{2\pi}\,(gH-k_3^2)^{1/2} + 1\right]$$

$$= \frac{\beta gH}{2} + 2\sqrt{gH}\ ,$$

which follows from the constraint $k_3^2 + (\frac{2\pi n}{\beta})^2 \leq gH$. Then the quartic term is

$$\frac{g^4H^2}{16\pi^2}\left(1 + \frac{4}{\beta\sqrt{gH}}\right)c(k_3,n)^4\ . \tag{5.15}$$

We again use the dimensionless variables defined in (5.8), along with (5.14) and (5.15), in (5.13) tb get

$$Z_\beta^{\,u} \sim \left\{\frac{1}{\sqrt{gH}}\int_{k_3,n}^{\mathrm{II}} d\hat{c}\,\exp\left[\frac{1}{2}\left(1-\hat{k}_3^2 - \left(\frac{2\pi n}{\beta\sqrt{gH}}\right)^2\right)\hat{c}^2 - \frac{g^2}{16\pi^2}\left(1+\frac{4}{\beta\sqrt{gH}}\right)\hat{c}^4\right]\right\}D'\ . \tag{5.16}$$

The free-energy density is

$$F_\beta^{\,u} = -\frac{1}{\beta V}\ln Z_\beta^{\,u}$$

$$= -\frac{gH}{2\pi^2}\,\frac{1}{\beta}\int dk_3 \sum_n \left[-\frac{1}{2}\ln gH + \ln I(k_3,n)\right]$$

$$= \frac{(gH)^2}{8\pi^2}\ln gH + \frac{(gH)^{3/2}}{2\pi^2\beta}\ln gH - \frac{(gH)^{3/2}}{2\pi^2\beta}\int d\hat{k}_3 \sum_n \ln I(\hat{k}_3,n), \tag{5.17}$$

where

$$I(\hat{k}_3,n) = \int_{-\infty}^{\infty} d\hat{c}\,\exp\left\{\frac{1}{2}\left[1-\hat{k}_3^2 - \left(\frac{2\pi n}{\beta\sqrt{gH}}\right)^2\right]\hat{c}^2 - \frac{g^2}{16\pi^2}\left(1+\frac{4}{\beta\sqrt{gH}}\right)\hat{c}^4\right\}\ . \tag{5.18}$$

The first term in (5.17) is the zero temperature logarithmic term of (5.10). In the zero-temperature limit the second term in (5.17) vanishes, and the third term reduces to the term proportional to $(gH)^2$ in (5.10). To be consistent with the renormalized zero temperature expression (3.31), we subtract from (5.17)

$$-\frac{(gH)^2}{4\pi^2}\int_0^1 dk^2 \ln I_o(k^2) , \tag{5.19}$$

where $\hat{k}^2 = \hat{k}_3^2 + \hat{k}_4^2$, and

$$I_o(\hat{k}^2) = \int_{-\infty}^{\infty} d\hat{c} \exp\left[\frac{1}{2}(1-\hat{k}^2)\hat{c}^2 - \frac{g^2}{16\pi^2}\hat{c}^4\right] , \tag{5.20}$$

which we recognize from (5.9).

The quartic terms in (5.18) and (5.20) contain the coupling constant explicitly. We evaluate it through the renormalization group equation (3.30). We must choose the scale at which to evaluate g. At finite temperature there are actually three scales in the theory: Λ_s, gH, and β. We take as a reasonable choice $\mu_s^2 = \Lambda_s^2 + gH + 1/\beta^2$. (With this choice for μ_s^2, g^2 is never negative.) Then

$$\frac{1}{g^2} = \frac{11\left[\ln 1 + \frac{gH}{\Lambda_s^2} + \frac{1}{(\beta\Lambda_s)^2}\right]}{24\pi^2} . \tag{5.21}$$

A greater ambiguity exists for the second term of (5.17). Obviously we cannot take the logarithm of a dimensionful quantity. We must have something like $\ln gH/\mu^2$. This is no problem for the zero-temperature logarithm since, as we saw in Chapter 3, an arbitrary μ^2 gets traded, eventually, for the scale parameter Λ_s^2 in the renormalization process.

The source of this ambiguity is obscure, and we have nothing to say about it. (This problem is also discussed in Ref. 22.) The logical procedure is to take $\mu^2 = \Lambda_s^2$, as in the zero temperature logarithm. However, we could also consider choices such as $\mu^2 = \Lambda_s^2 + 1/\beta^2$ or $\mu^2 = \Lambda_s^2 + gH + 1/\beta^2$ (as in (5.21)). The qualitative behavior of the free-energy density turns out to be independent of our choice.

The complete free-energy density is

$$F = \frac{11}{48\pi^2}\left[\ln\frac{gH}{\Lambda_s^2} - \frac{1}{2}\right] + F_\beta^{(\text{stable})} + \frac{(gH)^{3/2}}{2\pi^2\beta}\ln\frac{gH}{\mu^2}$$
$$- \frac{(gH)^{3/2}}{2\pi^2\beta}\int d\hat{k}_3 \sum_n \ln I(\hat{k}_3,n) + \frac{(gH)^2}{4\pi^2}\int_0^1 d\hat{k}^2 \ln I_o(\hat{k}^2), \qquad (5.22)$$

where

$$I(\hat{k}_3,n) = \int_{-\infty}^{\infty} d\hat{c}\, \exp\left\{\frac{1}{2}\left[1-\hat{k}_3^2-\left(\frac{2\pi n}{\beta\sqrt{gH}}\right)^2\right]\hat{c}^2 - \frac{3}{22\ln\left(1+\frac{gH}{\Lambda_s^2}+\frac{1}{(\beta\Lambda_s)^2}\right)}\right.$$
$$\left. \times \left(1+\frac{4}{\beta\sqrt{gH}}\right)\hat{c}^4\right\}, \qquad (5.23)$$

and

$$I_o(\hat{k}^2) = \int_{-\infty}^{\infty} d\hat{c}\, \exp\left[\frac{1}{2}(1-\hat{k}^2)\hat{c}^2 - \frac{3\hat{c}^4}{22\left(\ln 1+\frac{gH}{\Lambda_s^2}\right)}\right]. \qquad (5.24)$$

The first term in (5.22) is the one-loop zero temperature piece, (5.11), and $F_\beta^{(\text{stable})}$ is the finite-temperature contribution ~~from the~~ stable gluon modes given by (4.3).

The expression (5.22) can be evaluated numerically. (Some details are given in Appendix E.) In Figure 5.1 we have plotted $\Delta F = F(gH,T) - F(0,T)$ in units of Λ_s^4, versus gH in units of Λ_s^2, where we have set $\mu^2 = \Lambda_s^2$ in the third term of (5.22). The behavior seen in the curves contrasts

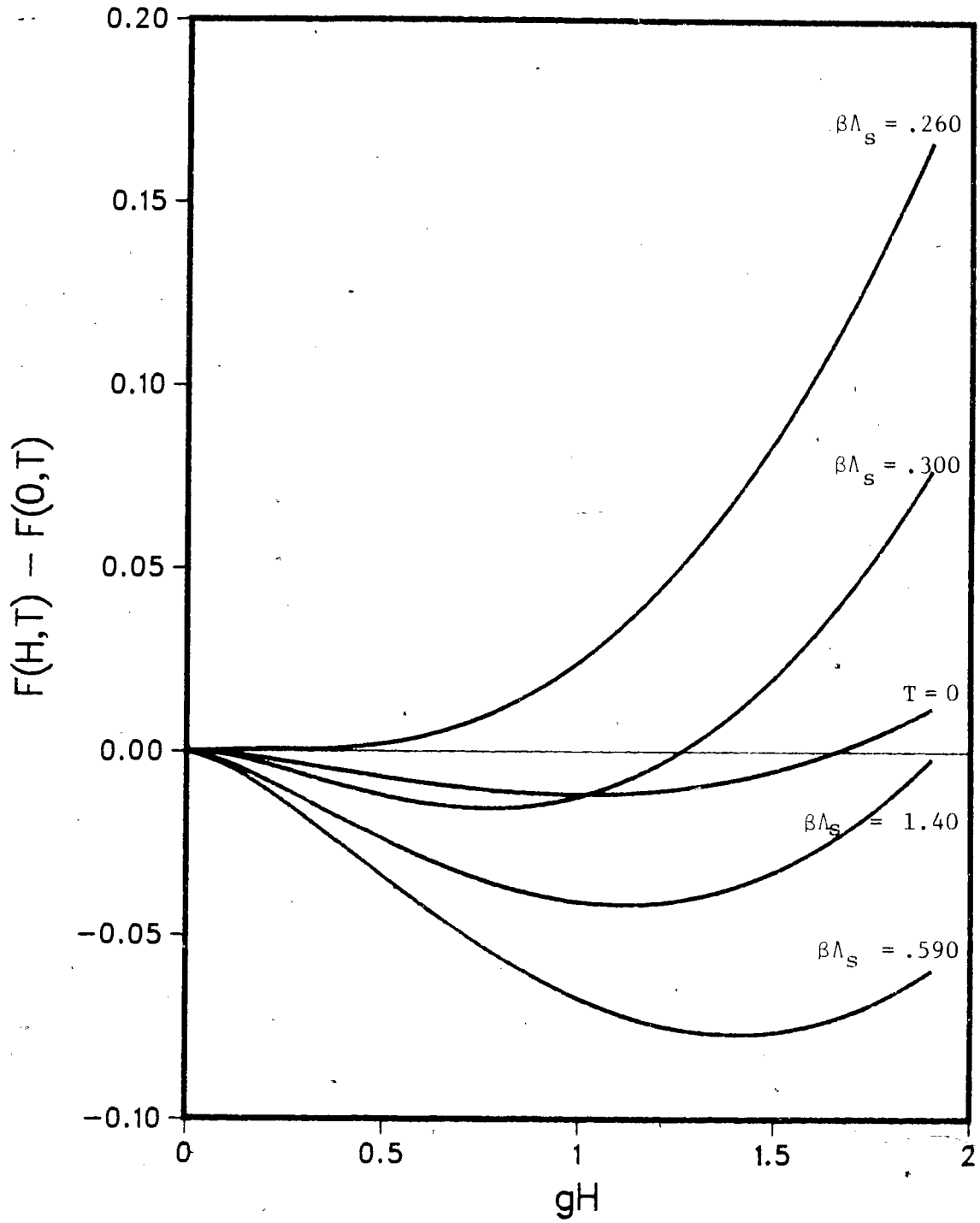

Fig. 5.1 Free-energy density curves for the Savvidy state, including an "all orders" contribution from the unstable modes.

strikingly with that in Figs. 4.1 and 4.2. First of all, as the temperature is increased from zero, the order parameter, gH, increases, and the free-energy density well gets deeper. The well reaches a maximum depth of about $.08\Lambda_s^4$ when $\beta\Lambda_s \approx .590$. As the temperature is increased further, the well gets shallower again. Finally, when $\beta\Lambda_s \approx .260$, the system undergoes a second order phase transition to a state of zero field.

The deepening of the free-energy density minimum is peculiar. We shall encounter this phenomenon again in Chapter 6, when we discuss fermions. In that case it is easier to understand what is happening. Here we can only assume that there is a very large entropy in these unstable modes which dominates the free-energy density at low temperatures. (We recall the definition of the free energy, F = U-TS, where U is the energy and S the entropy.) Mathematically, the source of this behavior lies with the fourth term in (5.22). This term is negative, and its magnitude increases both with increasing field and increasing temperature. The behavior of the third term depends on the choice for μ^2. For $\mu^2 = \Lambda_s^2$ or $\mu^2 = 1 + 1/\beta^2$, it is negative for small gH, but goes positive as H is increased. For choices such as $\mu^2 = 1 + gH$ or $\mu^2 = 1 + gH + 1/\beta^2$, it stays negative, approaching zero for large fields. This has the effect of further delaying the phase transition. These various choices have no effect on the order of the transition.

In conclusion, we have computed the free-energy density for the constant background color magnetic field by treating the unstable modes beyond one loop. We have seen that this treatment of the unstable modes leads to behavior which is qualitatively different from that exhibited by the stable modes alone, and from that caused by exciting the unstable modes as in Chapter 4. This treatment of the unstable modes leads to a more stable lowest-energy

state at low temperatures. Moreover, the system is seen to exhibit a second order phase transition to a state of zero field. Although there is an ambiguity in the epxression (5.22) due to the presence of an arbitrary scale μ^2, the qualitative behavior of the system is seen not to depend on reasonable choices for this parameter.

CHAPTER 6

Fermions at Finite Temperature

So far we have only considered the contribution of gluons to the free-energy density. However, QCD is a theory which contains quarks as well as gluons. In this chapter, we compute the effects of quarks on the free-energy density of the Savvidy state. This turns out to be very interesting. Intuitively one expects the additional degrees of freedom provided by the quarks to have a destabilizing effect on the ground state. However, this is not the case here. As the temperature is increased from zero, quark fluctuations initially increase the stability (the free-energy density well gets deeper). As the temperature is further increased this trend is reversed, and the system eventually undergoes a phase transition to a state of zero field.

For definiteness we consider two flavors of zero-mass quarks. Their one-loop finite-temperature contribution to the free-energy density is obtained directly from (C.4). Doing the sum over $S_i = \pm 1$, and rearranging terms in the remaining sum, we get

$$F_\beta^{(\text{quarks})} = \frac{-gH}{\beta^2}\left\{\frac{1}{6} + \frac{4}{\pi^2}\sum_{N=1}^{\infty}\int_0^\infty dx\, \ln\left[1 + \exp\left(-\sqrt{NgH\beta^2 + x^2}\right)\right]\right\}. \quad (6.1)$$

(The first term in (6.1) arises from the N = 0, S = -1 mode, the energy of which is independent of H.)

To illustrate the effects of quarks we shall ignore the question of the proper treatment of the unstable modes, and simply add (6.1) to the contribution from the stable gluon modes, (4.3). The total free-energy density is then

$$F = \frac{3(gH)^2}{16\pi^2}\left[\ln\frac{gH}{\Lambda_s^2} - \frac{1}{2}\right] + F_\beta^{(\text{stable})} + F_\beta^{(\text{quarks})}, \qquad (6.2)$$

where the first term is the zero temperature result from (3.31).

This expression can be evaluated numerically. (Some details are given in Appendix E.) In Figure 6.1 we have plotted $\Delta F = F(gH,T) - F(0,T)$ in units of Λ_s^4, versus gH in units of Λ_s^2, where

$$F(0,T) = \frac{-2\pi^2}{45\beta^4} - \frac{7\pi^2}{45\beta^4} = \frac{-\pi^2}{5\beta^4}$$

is the free-energy density of the charged gluon modes and the quark modes in zero field (see (C.5)). This should be compared with Fig. 4.2, ΔF vs. gH for the stable gluon modes only.

The contrast between the two figures is striking. When only the gluon stable modes are considered, the free-energy density shows the behavior one intuitively expects. As the temperature is increased from zero, the stable-mode fluctuations cause the order parameter, gH, to decrease and the free-energy density well to become shallower. At $\beta\Lambda_s = 1.07$ the minimum becomes degenerate, and the barrier height is about $.002\Lambda_s^4$. Above this temperature H = 0 is the global minimum.

Inclusion of quarks results in a rather different picture. As the temperature is increased from zero, two interesting things happen: the order parameter increases, and the free-energy density well gets deeper. (This is just the behavior we encountered in Chapter 5. Its source there is somewhat obscure, but is clearly associated with the unstable modes. As we shall see, the situation here is much easier to understand.) The well reaches a maximum depth of about $-.03\Lambda_s^4$ (compared with about $-0.1\Lambda_s^4$ at T = 0) at gH = 1.8. This happens at $\beta\Lambda_s = 1.35$. As the temperature is increased further, the

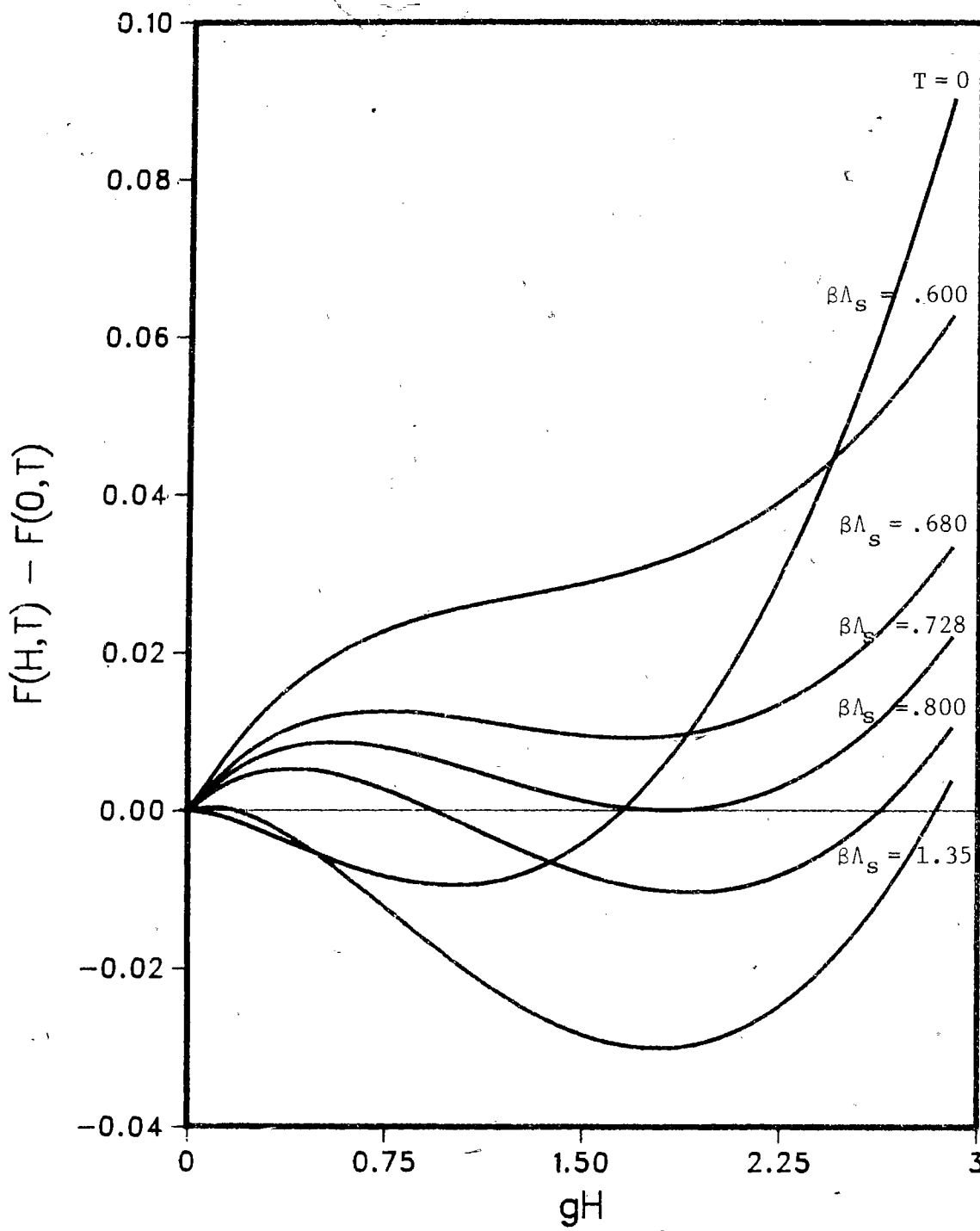

Fig. 6.1 Free-energy density curves for the Savvidy state, including 2 flavors of massless quarks, and excluding the finite-temperature unstable mode contribution.

well gets shallower (although the order parameter continues to increase slightly for a while, reaching a maximum value of 1.9). At $\beta\Lambda_s = .728$ the minimum becomes degenerate, and the barrier height is about $.009\Lambda_s^4$. Above this temperature H = 0 is the global minimum.

Before discussing these results, we point out that the two transition temperatures cannot be immediately compared since $\Lambda_s(N_f=0) \neq \Lambda_s(N_f=2)$. However, in Ref. 15 it is shown that the dependence of Λ_s on Λ_m (the scale parameter in the minimal-subtraction scheme) is only weakly dependent on N_f. Furthermore, Ref. 47 shows that Λ_m also depends only weakly on N_f. So, for the purposes of our discussion, we can ignore the dependence of Λ_s on N_f. We still have the qualitative result that inclusion of quarks increases the stability of the system, and this leads to an increase in the transition temperature (decrease in critical β).

(While writing this up, I discovered a recent paper by Page and Socolovsky [48] in which similar results are reported. They consider three massless quarks within SU(3), but the behavior is qualitatively the same. However, these authors seem to be unaware of the source of this unusual behavior.)

Now, what's going on? Let's look at Eqn. (6.1). The first term arises from the N=0, S=-1 mode. The energy of this mode is independent of H since the ground-state oscillator contribution 1/2 gH, is cancelled by the spin term (spin parallel to the field). (The only energy in this mode is in the free-particle momentum in the 3 direction.) However, this mode (in fact, all oscillator modes) has a density of states which goes linearly with the field, $gH/2\pi$; so it gives a contribution to the free-energy density which is also linear in the field. The contributions from all other quark modes, as well

as the stable gluon modes (see (4.3)), get damped out as gH is increased. So it is the quark zero modes which are responsible for the interesting behavior depicted in Fig. 6.1.

We can be more quantitative. If we make the approximation that only the quark zero modes are excited at low temperatures, we can write

$$F \approx \frac{3(gH)^2}{16\pi^2}\left[\ln\frac{gH}{\Lambda_s^{\,2}} - \frac{1}{2}\right] - \frac{gH}{6\beta^2} \quad . \tag{6.3}$$

Minimizing this with respect to the field gives

$$gH \ln\frac{gH}{\Lambda_s^{\,2}} = \frac{4\pi^2}{9\beta^2} \quad . \tag{6.4}$$

At $T = 0$ the minimum occurs, of course, for $gH = \Lambda_s^{\,2}$. As the temperature increases, the right hand side of (6.4) increases and so the value of gH minimizing (6.3) also increases. As the temperature increases further, the other quark and gluon modes get excited and come into play. A glance at (4.3) and (6.1) shows that taking derivatives of the terms coming from all other modes results in a sign change with respect to the zero-mode term. (This is due to the minus sign in the exponential.) This gives terms on the right hand side of (6.4) which are negative. These terms eventually dominate, and drive the magnitude of gH minimizing F back toward smaller values.

Although (6.4) is transcendental, we can get an approximate solution, valid for low temperatures, by setting

$$gH = \Lambda_s^{\,2} + \delta \quad , \tag{6.5}$$

where δ is small. Keeping only terms of order δ, we get

$$\delta \approx \frac{4\pi^2}{9\beta^2} \quad . \tag{6.6}$$

We then use (6.5) and (6.6) in (6.3) to get

$$F_{min} \approx \frac{-3}{32\pi^2}\Lambda_s^4 - \frac{(\Lambda_s^2+\delta)}{6\beta^2} + O(\delta^2) . \qquad (6.7)$$

At zero temperature, $F_{min} = -3\Lambda_s^4/32\pi^2$. Now the depth of the free-energy density well is the minimum of ΔF. However, $F(0,T)$ is proportional to T^4 and can be neglected at low temperatures compared to the T^2 term in (6.7). Therefore $\Delta F_{min} \approx F_{min}$, and (6.7) shows that the zero modes deepen the free-energy density minimum.

That's all mathematics. What about physics? It's not hard to see that we are observing an entropy effect. (We recall the definition of the free-energy, F = U-TS, where U is the energy and S the entropy.) At low temperatures the free-energy density is dominated by the entropy of the zero-energy modes. It is energetically favorable to produce these modes (since they cost nothing), and entropically favorable for the field to be large (to decrease F). At higher temperatures other modes become excited, and these eventually dominate the free-energy density, leading to the more usual behavior.

Finally, we mention that including the quark contribution in the cases studied in Chapters 4 and 5 produces nothing new. In either case, quarks, because of their zero modes, delay the phase transition. In addition, they have no effect on the order of the transition.

In conclusion, we have examined the effects of quark fluctuations on the free-energy density of the constant background color magnetic field. We have seen that the entropy of the zero modes dominates the free-energy density at low temperatures, and causes the phase transition to zero field to be delayed. In addition, we have seen that quark fluctuations do not change the qualitative nature of this transition (first or second order).

CHAPTER 7

Summary and Conclusions

I have made several studies of the vacuum structure of QCD. Using the background field technique, and the covariantly constant background color magnetic field Ansatz, I computed the effective action in several circumstances at finite, as well as zero, temperature. The emphasis has been on a proper treatment of the unstable modes associated with negative eigenvalues in the gauge field fluctuation spectrum.

I considered a supersymmetric QCD in Chapter 3, and computed its vacuum energy density. I have nothing to add to the discussion given there, but simply summarize the main conclusions: (1) SUSY QCD is less asymptotically free than ordinary QCD, due to fluctuations of the SUSY partners of the quarks and gluons (s-quarks and gluinos, respectively); (2) there are no negative eigenvalues in the s-quark or gluino spectra, so the SUSY state exhibits the same instability as the ordinary Savvidy state; (3) cancellation of quantum corrections found in some models (such as the self-dual background field described in Appendix F) is not a general feature of supersymmetry.

The work reported in Chapters 4 and 5 was concerned with a more rigorous treatment of the unstable gluon modes, in order to compute the free-energy density. Before discussing this, I want to return to something mentioned at the beginning of Chapter 5. It has been argued that the Savvidy state is not unstable (i.e. that the vacuum energy density is completely real) [44-46]. In these papers, it is the regularization procedure used in Appendix B which is attacked. Schanbacher [44] argued that the contour used to evaluate the negative eigenvalue contribution is wrong, implying an incorrect choice of

boundary conditions. However, his argument, which I am unable to follow completely, relies on an analogy with QED, and this may not be relevant for a non-Abelian theory. I can only say that I don't believe the result. Unfortunately, the work of Dittrich and Reuter [45] is short on details. They computed the one-loop vacuum energy density using the zeta-function regularization procedure, and found no imaginary part. However, in a more recent paper [23] these authors have retracted this conclusion. In addition, Elizalde [51] has also used the ζ-function procedure, and concluded that the one-loop vacuum energy does have an imaginary part.

In Chapters 4 and 5, I considered two different ways of treating the unstable modes: (1) exciting them à la Nielsen and Ninomiya; and (2) going beyond one loop à la Flory. I computed the free-energy density for both cases. The most important conclusion, in either case, is that there is a phase transition from a non-trivial lowest energy state to a trivial one. This is just what one expects of the color confinement-deconfinement transition. The semi-classical treatment of the excited unstable modes yields a first-order transition; the "all orders" treatment of these modes yields a second-order transition.

It is natural to wonder which of the two treatments of the unstable modes gives a better approximation to the QCD ground state. They each yield a completely real vacuum energy. Unfortunately, there seems to be no compelling reason for choosing one over the other. Of course, that doesn't prevent me from having an opinion. My hunch is that the flux-tube structure of the Copenhagen vacuum is closer to the truth. The constant magnetic field, even given gauge-invariant and Lorentz-invariant linear combinations, seems too trivial. It lacks structure. The complicated spaghetti structure of the

Copenhagen vacuum seems to me much more likely to provide the necessary confinement mechanism.

Finally, I considered the effects of quarks on the free-energy density in Chapter 6. The interesting result there is that quarks increase the stability of the lowest energy state at low temperatures. This is due to the presence of zero-energy modes in the quark fluctuation spectrum, the entropy of which dominates the free-energy density at low temperatures. This has the effect of increasing the transition temperature compared to the situation without quarks.

I'd like to conclude by suggesting some lines along which this work can be extended:

(1) It would be nice to refine the calculation for the Copenhagen vacuum. A possibility I briefly looked into, and which seemed promising, is to find a perturbative method for obtaining approximate eigenvalues for the fluctuations about the Nielsen-Nimomiya state.

(2) The presence of an arbitrary scale in the result of the "all orders" calculation in Chapter 5 is unsatisfying. Perhaps this could be resolved with a complete calculation, including all modes. This has recently been done at zero temperature [46], but adds nothing to Flory's result. At any rate, this needs more study.

(3) Another interesting problem would be to compute the free-energy density for the self-dual background field discussed in Appendix F. This would probably require use of the real-time formalism. It would be particularly interesting because of the zero modes. (This might take some thought. For the Savvidy state, the fermion zero modes are trivial. However, even at zero temperature, the zero modes for the self-dual background field must be treated separately.)

I am reminded of a discussion about progress in physics which I heard related by Frank Shu:

> With Newtonian physics we could solve the two-body problem, but nothing more complicated. In General Relativity we can't sove the two-body problem. In QED we can't solve the one-body problem. Now we have QCD, and we can't even solve the vacuum!

APPENDIX A

Eigenvalues of Fluctuation Operators

We want to solve the eigenvalue equations (3.15). Actually, we can get the eigenvalues without explicitly solving them. First, we write the scalar and vector equations in component form:

$$-\nabla_\mu^{ab}\nabla_\mu^{bc}\phi^c = \lambda\phi^a , \tag{A.1}$$

$$(-\nabla_\lambda^{ab}\nabla_\lambda^{bc}\delta_{\mu\nu} - 2gf_{abc}\tilde{F}_{\mu\nu}^{b})a_\mu^c = \lambda a_\nu^a . \tag{A.2}$$

Now we combine the two fermion equations in (3.15) to get

$$\sigma_\mu^E\bar{\sigma}_\nu^E\nabla_\mu\nabla_\nu\psi = \lambda^2\psi .$$

We use $\sigma_i^E\bar{\sigma}_j^E = -i\varepsilon_{ijk}\sigma^k$, $\sigma_o^E = i\sigma^o$, and $[\nabla_\mu,\nabla_\nu] = ig\tilde{F}_{\mu\nu}$ to write

$$[-\nabla_\mu\nabla_\mu + g(\sigma_i\tilde{F}_{oi} + \varepsilon_{ijk}\tilde{F}_{ij}\sigma_k)]\psi = \lambda^2\psi ,$$

where $i < j$. In components,

$$[-\nabla_\mu^{ab}\nabla_\mu^{bc} + g(\sigma_i\tilde{F}_{oi}^{ac} + \varepsilon_{ijk}\tilde{F}_{ij}^{ac}\sigma_k)]\psi^c = \lambda^2\psi^a . \tag{A.3}$$

The only group matrices appearing in (A.1), (A.2), and (A.3) are those which mutually commute. (We denote these by T_R^j, $j = 1,\ldots, r$, where r is the rank of the group. Note that, for the adjoint representation, $(T_b)_{ac} = if_{abc}$.) Therefore, the operators may be simultaneously diagonalized in color space. We write the diagonalized operators as

$$\nabla_\mu^{(\alpha)} \equiv \partial_\mu + ig\alpha_R^j A_\mu^j \equiv \partial_\mu + igA_\mu^{(\alpha)},$$

$$\tilde{F}_{\mu\nu}^{(\alpha)} \equiv \alpha_R^j\tilde{F}_{\mu\nu}^j , \tag{A.4}$$

where α_j^R is the j^{th} component of the weight of the basis state α in the representation R (i.e. $\alpha_R{}^j$ are the eigenvalues of the T_R^j). Then we write the operators of (A.1)-(A.3) as

$$-\nabla_\lambda^{(\alpha)}\nabla_\lambda^{(\alpha)},$$

$$-\nabla_\lambda^{(\alpha)}\nabla_\lambda^{(\alpha)}\delta_{\mu\nu} + 2ig\tilde{F}_{\mu\nu}^{(\alpha)}$$

$$-\nabla_\lambda^{(\alpha)}\nabla_\lambda^{(\alpha)} + g(\sigma_i\tilde{F}_{4i}^{(\alpha)} + \sum_{i<j}\varepsilon_{ijk}\tilde{F}_{ij}^{(\alpha)}\sigma_k). \qquad \text{(A.5)}$$

Since $\tilde{F}_{\mu\nu}^{(\alpha)}$ is constant, the operators may be simultaneously diagonalized in μ,ν (vectors) and spinor space (fermions). Recall that the background fields are $F_{12}{}^3 = F_{23}{}^8 = F_{31}{}^{15} = H$. Then, the eigenvalues of $F_{\mu\nu}^{(\alpha)}$ are $0,0,\pm i\alpha_R{}^j H$. The eigenvalues of σ_k are ± 1, so the second term of the fermion operator has eigenvalues $\pm\,\alpha_R{}^j H$.

The field strengths are perpendicular in real space (for SU(3) and SU(4)), so a state $^\alpha R$ is effectively coupled to a single field of magnitude $\sqrt{\sum_j (\alpha_R{}^j)^2 H} \equiv \chi_R H$. We choose the direction of the field to be along the third axis. Then we may write

$$-\nabla_\mu^{(\alpha)}\nabla_\mu^{(\alpha)} = -(\partial_1 + i\chi_R gHx_2) - \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2.$$

The operators ∂_1, ∂_3 and ∂_4 have plane-wave eigenstates:

$$-\nabla_\mu^{(\alpha)}\nabla_\mu^{(\alpha)} = -(-ik_1 + i\chi_R gHx_2) - \partial_2{}^2 + k_3{}^2 + k_4{}^2.$$

The change of variables $y = \dfrac{-k_1}{\sqrt{\chi_R gH}} + \sqrt{\chi_R gH}\, x_2$ gives

$$\begin{aligned} -\nabla_\mu^{(\alpha)}\nabla_\mu^{(\alpha)} &= (-\partial y^2 + y^2)\chi_R gH + k_3{}^2 + k_4{}^2 \\ &= (2N+1)\chi_R gH + k_3{}^2 + k_4{}^2, \end{aligned} \qquad \text{(A.6)}$$

where we have recognized the harmonic oscillator operator $-\partial y^2+y^2$.

Finally, with (A.6) and the results following (A.5), we may express all of the eigenvalues as

$$\lambda = [(2N+1+S)\chi_R gH + k_3^2 + k_4^2]^p , \qquad (A.7)$$

where $p = 1/2$ for fermions (recall (A.3)) and $p = 1$ otherwise. For scalars (and ghosts), $S = 0$; for spinors, $S = \pm 1$; and for vectors, $S = 0, \pm 2$.

APPENDIX B

Regularization of Determinants

To regularize (3.20) we shall adopt the method employed by Salam and Strathdee [41], which is closely related to Schwinger's proper time formalism [2]. (It is used extensively by the Danish group.) It uses two tricks. First, we write

$$\ln \lambda = \lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon} \lambda^{-\varepsilon} + \frac{1}{\varepsilon}\right) .$$

The term $1/\varepsilon$ may be dropped as an uninteresting (infinite) constant. Next, we use an integral representation of the gamma function:

$$\Gamma(\varepsilon) = \int_0^\infty dx\, x^{-1+\varepsilon} e^{-x} . \tag{B.1}$$

By analytic continuation we let $x = it\lambda$ to get

$$\lambda^{-\varepsilon} = \frac{i^\varepsilon}{\Gamma(\varepsilon)} \int_0^\infty dt\, t^{-1+\varepsilon} e^{-it\lambda} .$$

Then,

$$\ln \lambda = \lim_{\varepsilon \to 0} \left[\frac{-i^\varepsilon}{\varepsilon \Gamma(\varepsilon)}\right] \int_0^\infty dt\, t^{-1+\varepsilon} e^{-it\lambda} . \tag{B.2}$$

Hereafter, we suppress the limit notation.

In (3.20), we make the change of variable $k_3^2 + k_4^3 = (\chi_i gH)n^2$, use (B.2), and perform the momentum integration to get

$$\varepsilon^{(1)} = -\sum_i q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2} \left(\frac{\chi_i gH}{\mu^2}\right)^{-\varepsilon} \frac{i^{\varepsilon+1}}{\varepsilon \Gamma(\varepsilon)} \sum_{S_i} \sum_{N=0}^{\infty} \int dt\, t^{-2+\varepsilon} \times \exp[-it(2N+1+S_i)] . \tag{B.3}$$

Apart from the case N = 0, S = -2 (the vector field "unstable" mode), the sum over N and integration over t can be done after rotating the contour 90° clockwise. We set $t = -\frac{1}{2} i\tau$, and note the following relations:

$$\sum_{N=0}^{\infty} e^{-N\tau} = \frac{1}{1-e^{-\tau}} , \quad \sum_{N=1}^{\infty} e^{-N\tau} = \frac{e^{-\tau}}{1-e^{-\tau}} ,$$

$$\Gamma(\varepsilon) = (\varepsilon-1)\Gamma(\varepsilon-1) ,$$

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^{\infty} dt\, t^{-1+z} \frac{e^{-qt}}{1-e^{-t}} , \tag{B.4}$$

where $\zeta(z,q)$ is the generalized zeta function [42]. Then (B.3) becomes

$$\varepsilon^{(1)} = -2 \sum_i q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2} \left(\frac{2\chi_i gH}{\mu^2}\right)^{-\varepsilon} \left[\frac{-1}{\varepsilon(\varepsilon-1)}\right] \sum_{S_i} \zeta\left(\varepsilon-1, \frac{1+S_i}{2}\right) . \tag{B.5}$$

For S = -2, N ≠ 0 we get (B.5) with the zeta function changed to $\zeta(\varepsilon-1,1/2)$. For S = -2, N = 0 we rotate the contour in (B.3) 90° counter-clockwise. This gives (t = iτ)

$$\varepsilon^{(1)}_{\substack{S=-2\\N=0}} = -q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2} \left(\frac{-\chi_i gH}{\mu^2}\right)^{-\varepsilon} \frac{1}{\varepsilon(\varepsilon-1)} , \tag{B.6}$$

where we have also used (B.1).

Now we expand (B.5) and (B.6) in powers of ε:

$$\varepsilon^{(1)} = -2 \sum_i q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2} \sum_{S_i} \left\{ \frac{\zeta\left(-1, \frac{1+S_i}{2}\right)}{\varepsilon} + \left[\zeta\left(-1, \frac{1+S_i}{2}\right) + \right.\right.$$

$$\left.\left. + \partial_1 \zeta\left(-1, \frac{1+S_i}{2}\right)\right] - \zeta\left(-1, \frac{1+S_i}{2}\right) \ln \frac{2\chi_i gH}{\mu^2} + O(\varepsilon) \right\} , \tag{B.7}$$

$$\varepsilon^{(1)}_{\substack{S=-2\\N=0}} = -q_i p_i M_i \chi_i^2 \frac{(gH)^2}{8\pi^2} \left\{ -\frac{1}{\varepsilon} - 1 + \ln \frac{\chi_i gH}{\mu^2} - i\pi + O(\varepsilon) \right\} , \qquad (B.8)$$

where $\partial_1 \zeta$ means the derivative of the zeta function with respect to its first argument. Again, the contribution from $S = -2$, $N \neq 0$ is given by (B.7) with $\zeta(-1,1/2)$.

Finally, we note that the zeta functions in (B.7) may be expressed as Bernoulli polynomials [42]:

$$\begin{aligned} \zeta(-1,\alpha) &= -\frac{1}{2} B_2(\alpha) = -\frac{1}{2}(\alpha^2 - \alpha + \frac{1}{6}) , \\ \zeta(-1,\tfrac{1+S}{2}) &= \frac{1}{24} - \frac{S^2}{8} \\ &= \begin{cases} 1/24 , & S = 0 \\ -1/12 , & S = \pm 1 \\ -11/24 , & S = \pm 2 \end{cases} . \end{aligned} \qquad (B.9)$$

APPENDIX C

Finite Temperature Free Energies

As we saw in Chapter 2, (2.25), the finite-temperature free energy is obtained by the substitutions

$$k_4 \to \omega_n \quad , \quad \int \frac{dk_4}{2\pi} \to \frac{1}{\beta} \sum_n \quad ,$$

$$\omega_n = \begin{cases} \dfrac{2\pi n}{\beta} & , \text{ bosons} \\ \dfrac{(2n+1)\pi}{\beta} & , \text{ fermions} . \end{cases}$$

We must compute

$$\sum_{n=-\infty}^{\infty} \text{Tr } \ln(\omega_n^2+\omega^2) \ ,$$

where $\omega^2 = (2N+1+S_1)\chi gH+k_3^2$, and Tr means sum over N, integration over k_3, and sum over S_1.

Following Bernard [34] we write

$$\sum_n \ln(\omega_n^2+\omega^2) = \int_{1/\beta^2}^{\omega^2} da^2 \sum_n \frac{1}{\omega_n^2+a^2} + \sum_n \ln(\omega_n^2+1/\beta^2). \qquad (C.1)$$

The second term depends only on β and, as Bernard shows, cancels the β-dependent part of the functional integral's normalization constant. Now consider

$$\frac{1}{2\pi i} \oint_C \frac{f(\omega)}{\omega^2+a^2} d\omega \ ,$$

where

$$f(\omega) = \begin{cases} \frac{1}{2}\beta \cot \frac{1}{2}\beta\omega & , \text{ bosons} \\ -\frac{1}{2}\beta \tan \frac{1}{2}\beta\omega & , \text{ fermions} . \end{cases}$$

$f(\omega)$ has simple poles of residue 1 at $\omega = 2\pi n/\beta$ and $\omega = (2n+1)\pi/\beta$, respectively. The integrand also has simple poles at $\omega = \pm ia$. We choose the contour to be the circle at infinity, along which the integrand vanishes. Therefore, the sum we want in (C.1) is given by the negative of residues of the simple poles at $\omega = \pm ia$. We get

$$\sum_n \frac{1}{\omega_n^2+a^2} = \begin{cases} \frac{\beta}{2} \frac{\coth \frac{1}{2}\beta a}{a} & , \text{ bosons} \\ \frac{\beta}{2} \frac{\tanh \frac{1}{2}\beta a}{a} & , \text{ fermions} . \end{cases}$$

Putting this into the first term on the right in (C.1) yields

$$\sum_n \ln(\omega_n^2+\omega^2) = \begin{cases} 2[\frac{\beta\omega}{2} + \ln(1-e^{-\beta\omega}) - \ln 2], & \text{bosons} \\ 2[\frac{\beta\omega}{2} + \ln(1+e^{-\beta\omega}) - \ln 2] , & \text{fermions} . \end{cases} \qquad (C.2)$$

The first term in (C.2) is the zero-temperature result, and the last term is an irrelevant constant. So we write

$$\sum_n \mathrm{Tr} \ln(\omega_n^2+\omega^2) = (T=0) + 2\mathrm{Tr} \ln(1\pm e^{-\beta\omega}) . \qquad (C.3)$$

Finally, restricting ourselves to SU(2), we get the free energies:

$$F_\beta = \begin{Bmatrix} +1 \\ -N_f \end{Bmatrix} \frac{gH}{\pi^2\beta^2} \sum_{S_i} \sum_{N=0}^{\infty} \int_0^\infty dx \ln[1\mp\exp(-\sqrt{(2N+1+S_i)\chi g H\beta^2+x^2})], \qquad (C.4)$$

where the upper signs are for bosons, the lower ones for fermions, and we have set $x = \beta k_3$.

We also note the zero-field free energies:

$$F_\beta(H=0) = \begin{cases} -2\pi^2/45\beta^4 , & \text{gluons} \\ -7\pi^2 N_f/90\beta^4 , & \text{quarks} , \end{cases} \qquad \text{(C.5)}$$

where the gluon term includes only the contributions from the two charged states.

APPENDIX D

Eqn. 4.11

In this Appendix we provide some details in arriving at (4.11). (We proceed somewhat differently than the authors of Ref. 15.) We substitute (4.10) into (4.8) noting that $\partial_\mu \phi_{k_1} = 0$, and since nothing depends on x_3 or x_4, the integrals over these variables give "volume" terms ($L_3 L_4$):

$$
\begin{aligned}
S_1 &= (L_3L_4)gH\sqrt{\frac{\pi}{gH}}\,(2\pi|\Phi|)^2\int\frac{dk_1}{2\pi}\sum_m \delta(k_1-mc)\sum_{m'}\delta(k_1-m'c) \\
&- (L_3L_4)\,\frac{\pi}{gH}\sqrt{\frac{gH}{2\pi}}\,(2\pi|\Phi|)^4\int\frac{dk_1\,dk_1'\,dk_1''\,dk_1'''}{(2\pi)^4}\,2\pi\delta(k_1+k_1'-k_1''-k_1''') \\
&\times \exp\left\{\frac{-1}{2gH}\left[k_1^2+(k_1')^2+(k_1'')^2+(k_1''')^2-(k_1+k_1')^2\right]\right\} \\
&\times \sum_{m,m',m'',m'''}\delta(k_1-mc)\,\delta(k_1'-m'c)\,\delta(k_1''-m''c)\,\delta(k_1'''-m'''c) \\
&= (L_3L_4)\sqrt{\pi gH}\,\frac{2\pi}{c}\,|\Phi|^2\sum_{m,m'}\delta(m-m') \\
&- (L_3L_4)\,\frac{\sqrt{2\pi gH}}{2c}\,\frac{2\pi}{gH}\,|\Phi|^4\sum_{m,m',m'',m'''}\delta(m+m'-m''-m''') \\
&\times \exp\left\{\frac{-c^2}{2gH}\left[m^2+(m')^2+(m'')^2+(m''')^2-(m+m')^2\right]\right\}.
\end{aligned}
$$

In the last term we use the δ-function to set $m''' = m''-m-m'$, and rearrange terms in the exp to get

$$
\begin{aligned}
S_1 &= (L_3L_4)\sqrt{\pi gH}\,\frac{2\pi}{c}\,|\Phi|^2\sum_m\delta(0) \\
&- (L_3L_4)\,\frac{\sqrt{2\pi gH}}{2c}\,\frac{2\pi}{gH}\,|\Phi|^4\sum_m\delta(0)\sum_{m',m''}\exp\left\{\frac{-c^2}{2gH}\left[(m''-m)^2+(m''-m')^2\right]\right\}
\end{aligned}
$$

Now the area of a unit cell in the x_1-x_2 plane is $2\pi/gH$, so the total area is $\sum_m \delta(o) \frac{2\pi}{gH}$. The total volume is thus

$$\Omega = L_3 L_4 \sum_m \delta(o) \frac{2\pi}{gH} .$$

Finally, redefining the summation variables, we get

$$S_1 = \Omega \left\{ \frac{\sqrt{\pi}}{c} (gH)^{3/2} \left|\Phi\right|^2 - \frac{\sqrt{2\pi gH}}{2c} \left|\Phi\right|^4 \left(\sum_m e^{-\frac{c^2}{2gH} m^2} \right) \right\} .$$

APPENDIX E

Numerical Integrations

In this Appendix, we summarize a few technical details about the numerical evaluation of integrals in Chapters 4-6.

To evaluate a single integral of one variable, such as in the first and second terms of (4.3), it is easiest to use a standard integration package (such as from the NAG or IMSL libraries). However, this is not practical for a series of integrals, such as in the third term of (4.3) or the second term of (6.1). These series have the general form

$$\sum_{N=1}^{\infty} \int_0^{\infty} dx \, \ln f_N(x) \ .$$

Examination of the form of $f_N(x)$ in either case (they are very similar) shows that the series converge very slowly. To avoid a large number of numerical integrations, we may put the sum inside the logarithm to get

$$\int_0^{\infty} dx \, \ln \prod_{N=1}^{\infty} f_N(x) \ .$$

Although the product converges slowly, its numerical evaluation is quick. Because of the large number of products involved, it is best to use an integration routine which requires a fixed number of evaluations of the integrand. I used a 64-point Gauss-Laguerre quadrature, where the product was computed until the relative change was less than 10^{-8}.

The last two terms in (5.22) contain double integrals. The outcome of a double integration may be affected by the way in which the first integration is done. A numerical integration approximates the integrand in some way:

The safest procedure is to use a routine, for the first integration, which approximates the integrand with a smooth function. This suggests a Gaussian quadrature, which uses a polynomial approximation, rather than a standard adaptive quadrature, which uses a piecewise-linear approximation. So I used a Gauss-Laguerre quadrature for the integration over the amplitude $\hat{c}$, and an adaptive-quadrature routine from the NAG library for the integration over $\hat{k}$.

All calculations were done in double precision.

APPENDIX F

Self-Dual Background Field

In this Appendix, we compute the one-loop effective action for a self-dual background field. It is only slightly outside the line of development of this thesis, and I have also done some work on this case [50]. It's an important case since, as Leutwyler has shown [12], it is the only constant field configuration which is stable at one loop.

The dual field strength is defined by

$$ {}^*F^{\mu\nu a} \equiv \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma}^{\ a} . $$

In electrodynamics, the dual of $\vec{E}$ is $\vec{B}$, and the dual of $\vec{B}$ is $-\vec{E}$. A field strength is said to be self-dual (+) or anti-self-dual (-) if

$$ {}^*F^{\mu\nu a} = \pm F_{\mu\nu}^{\ a} , $$

or

$$ \vec{E}^a = \pm \vec{B}^a . $$

We shall just consider the SU(2) guage group, and choose as our Ansatz

$$ \tilde{A}_\mu^{\ 3} = (-Hx_2, 0, -Hx_4, 0), \qquad \tilde{A}_\mu^{\ 1,2} = 0 , $$

where $F_{12}^{\ 3} = F_{34}^{\ 3} = H$, $H > 0$. (The eigenvalues for the self-dual and anti-self-dual cases are the same, so we lose no generality here.) The eigenvalues are obtained as in Appendix A. First we have

$$ \begin{aligned} -\nabla_\mu^{(\alpha)}\nabla_\mu^{(\alpha)} &= -(\partial_1 + \chi_R gHx_2)^2 - \partial_2^{\ 2} - (\partial_3 + i\chi_R gHx_4)^2 - \partial_4^{\ 2} \\ &= (2N+2M+2S)\chi_R gH , \end{aligned} \qquad (F.1) $$

where, from the steps leading to (A.6), we recognize the sum of two harmonic oscillators. We introduce S for future convenience (S=1 here). The integers

The eigenvalues of the other operators in (A.5) follow directly. The operator $2ig\tilde{F}_{\mu\nu}^{(\alpha)}$ has eigenvalues $\pm 2\chi_R gH$, each appearing twice. (We recall that the Savvidy case has two zeros, the contribution from which is cancelled by that of the ghosts. Here we must explicitly compute the ghost contribution.) In the fermion operator, the term with $\tilde{F}_{4i}^{(\alpha)}$ now also gives $\pm\chi_R gH$. So the fermion eigenvalues are given by (F.1) $\pm\chi_R gH \pm \chi_R gH$. Therefore we can express all the eigenvalues as

$$\lambda = [(2N+2M+2S)\chi_R gH]^p , \qquad (F.2)$$

where $p = 1/2$ for fermions and $p = 1$ otherwise. For scalars (and ghosts), $S = 1$, for spinors, $S = 0,1,1,2$; and for vectors, $S = 0,2$ (twice each). We note that the eigenvalues are all non-negative.

The density of states for two harmonic oscillators is $(\chi_R gH/2\pi)^2$. So the expression for the one-loop correction to the energy density, analogous to (3.20), is

$$\varepsilon^{(1)} = -\sum_i q_i p_i \left(\frac{\chi_i gH}{2\pi}\right)^2 \sum_{S_i} M_i(S_i) \sum_{M,N=0}^{\infty} \ln \mu^2[(2M+2N+2S_i)\chi_i gH] , \qquad (F.3)$$

where $M_i(S_i)$ is the multiplicity of the eigenvalue. We note that the argument of the logarithm is non-negative, unlike in (3.20), so that $\varepsilon^{(1)}$ is real. We apply the regularization procedure of Appendix B. Using (B.2), and rotating the integration path 90° clockwise ($t = -\frac{1}{2} i \tau$), we get

$$\varepsilon^{(1)} = -\sum_i q_i p_i \left(\frac{\chi_i gH}{2\pi}\right)^2 \left(\frac{2\chi_i gH}{\mu^2}\right)^{-\varepsilon} \left[\frac{-1}{\varepsilon\Gamma(\varepsilon)}\right] \sum_{S_i} M_i(S_i) \sum_{M,N=0}$$

$$\times \int_0^\infty d\tau\, \tau^{-1+\varepsilon} e^{-(M+N+S)\tau} . \qquad (F.4)$$

The integral in (F.4) diverges for M = N = S = 0. These zero modes must be treated separately. We shall set them aside for the time being. We note the following relations:

$$\sum_{M,N=0}^{\infty} e^{-(M+N)\tau} = \frac{1}{(1-e^{-\tau})^2} = \frac{e^{2\tau}}{(e^{\tau}-1)^2} ,$$

$$\sum_{M\neq N\neq 0} e^{-(M+N)\tau} = \frac{e^{2\tau}}{(e^{\tau}-1)^2} - 1 = \frac{2e^{\tau}-1}{(e^{\tau}-1)^2} ,$$

$$\int_0^{\infty} \frac{x^{\nu-1}e^{-\mu x}}{(e^{x}-1)^2}\,dx = \Gamma(\nu)\left[\zeta(\nu-1,\mu+1) - (\mu+1)\zeta(\nu,\mu+1)\right] ,$$

$$\mathrm{Re}\mu > -2,\ \mathrm{Re}\nu > 2 \quad [42] ,$$

$$\Gamma(\varepsilon) = (\varepsilon-1)\Gamma(\varepsilon-1) . \qquad \text{(F.5)}$$

Now we use (F.5) in (F.4), expand in powers of ε, renormalize, and take the limit $\varepsilon \to 0$. However, we know from our previous calculations, that the interesting part is the logarithm term. This is given by

$$\varepsilon_+^{(1)} = \sum_i C_i \left(\frac{gH}{2\pi}\right)^{\varepsilon} \ln \frac{gH}{\mu^2} ,$$

where

$$C_i \equiv -q_i p_i \chi_i^2 \sum_{S_i} M_i(S_i)\zeta_i(S_i) ,$$

$$\zeta_i(0) \equiv 2\zeta(-1,0) - \zeta(-1,1) + \zeta(0,1),$$

$$\zeta_i(S_i\neq 0) = \zeta(-1,S-1) - (S-1)\zeta(0,S-1) . \qquad \text{(F.6)}$$

We recall that the zeta functions may be expressed as Bernoulli polynomials [42]:

$$\zeta(-1,a) = -\frac{1}{2}\left(a^2 - a + \frac{1}{6}\right) \quad \text{and} \quad \zeta(0,a) = \frac{1}{2} - a .$$

The C_i, and the various other parameters, are listed in Table III. The notation is the same as we used in Chapter 3, and we have included the data for the ghosts.

To obtain $\varepsilon^{(1)}$ for a SUSY vector multiplet we must sum C_i over the gluons, ghosts, and gluinos. From the last column in Table III we see that this is zero. This is the result of D'Adda and Di Vecchia [37]. Although they don't mention chiral multiplets, we see from the table that the quarks and s-quarks of these multiplets give cancelling contributions. So we have the result that the one-loop correction around a self-dual background field in SUSY QCD is given by the contribution from the zero modes only. In ordinary QCD we eliminate the s-quarks and gluinos to get

$$\varepsilon_+^{(1)} = \left(-1 + \frac{N_f}{2}\right) \frac{(gH)^2}{24\pi^2} \ln \frac{gH}{\mu^2} \quad . \tag{F.7}$$

We shall not go through a derivation of the contribution from the zero modes. We just state that they give a logarithmic correction like (F.7), the coefficient of which is just given by the density of zero modes [12,37]:

$$\pm\, M_i(S=0)\chi_i^{\,2} \left(\frac{gH}{2\pi}\right)^2 ,$$

where + is for bosons, - for fermions. We can read off the contributions from Table III. For SUSY we have contributions from gluons, gluinos, and quarks:

$$\text{SUSY:} \qquad \varepsilon_0^{(1)} = \left(1 - \frac{N_f}{2}\right) \frac{(gH)^2}{4\pi^2} \ln \frac{gH}{\mu^2} \quad . \tag{F.8}$$

This is the total correction.

TABLE III. Contributions of the various fields to $\varepsilon_+^{(1)}$ for a self-dual background field in SU(2).

Field	q	p	χ	S	ζ(S)	M(S)	C_i
ghosts	- 1	1	1	1	- 1/12	2	1/6
$a_\mu{}^a$ (gluons)	- 1/2	1	1	0	- 7/12	4	
				2	5/12	4	- 1/3
Λ (gluinos)	+ 1	1/2	1	0	- 7/12	1	
				1	- 1/12	2	
				2	5/12	1	1/6
ψ (quarks-2 comp.)	+ 1	1/2	1/2	0	- 7/12	$2N_f$	
				1	- 1/12	$4N_f$	
				2	5/12	$2N_f$	$N_f/12$
S (s-quarks)	- 1	1	1/2	1	- 1/12	$4N_f$	$-N_f/12$

In the non-SUSY case we eliminate the gluinos to get

$$\varepsilon_0^{(1)} = \left(2 - \frac{N_f}{2}\right) \frac{(gH)^2}{4\pi^2} \ln \frac{gH}{\mu^2} \quad .$$

We add this to (F.7) to get the complete one-loop correction:

$$\varepsilon^{(1)} = \frac{22-5N_f}{48\pi^2} (gH)^2 \ln \frac{gH}{\mu^2} \quad .$$

The complete renormalized vacuum energy density to one loop is then

$$\varepsilon = \frac{22-5N_f}{48\pi^2} (gH)^2 \left[\ln \frac{gH}{\Lambda_s^2} - \frac{1}{2}\right] . \qquad \text{(F.9)}$$

(This expression corrects an error in Ref. 50 where the factor $\chi = 1/2$ was overlooked in computing the fermion contribution.) We emphasize that the vacuum energy density is completely real.

(We conclude by mentioning that Ref. 50 also includes a calculation for massive quarks. More recently, Elizalde [51] has also treated massive quarks in both constant magnetic and constant self-dual background fields using the zeta-function regularization technique.)

REFERENCES

1. E.V. Shuryak, Phys. Rep. 115, 151 (1984).
2. J. Schwinger, Phys. Rev. 82, 664 (1951).
3. W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).
4. M.J. Duff and M. Ramon-Medrano, Phys. Rev. D12, 3357 (1975).
5. L.S. Brown and W.I. Weisberger, Nucl. Phys. B157, 285 (1979).
6. I.A. Batalin, S.G. Matinyan, and G.K. Savvidy, Sov. J. Nucl. Phys. 26, 214 (1977).
7. G.K. Savvidy, Phys. Lett. 71B, 133 (1977).
8. A. Yildiz and P. Cox, Phys. Rev. D21, 1095 (1980).
9. V.S. Vanyashin and M.V. Terentev, JETP (Sov. Phys.) 21, 375 (1965).
10. H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978).
11. N.K. Nielsen and P. Olesen, Nucl. Phys. B144, 376 (1978).
12. H. Leutwyler, Phys. Lett. 96B, 154 (1980); Nucl. Phys. B179, 129 (1981).
13. N.K. Nielsen and P. Olesen, Phys. Lett. 79B, 304 (1978).
14. J. Ambjørn, N.K. Nielsen, and P. Olesen, Nucl. Phys. B152, 75 (1979).
15. H.B. Nielsen and M. Ninomiya, Nucl. Phys. B156, 1 (1979).
16. H.B. Nielsen and P. Olesen, Nucl. Phys. B160, 380 (1979).
17. J. Ambjørn and P. Olesen, Nucl. Phys. B170 [FS1], 60 (1980).
18. J. Ambjørn and P. Olesen, Nucl. Phys. B170 [FS1], 265 (1980).
19. C.A. Flory, SLAC-PUB-3244 (1983).
20. J.I. Kapusta, Nucl. Phys. B190 [FS3], 425 (1981).
21. B. Müller and J. Rafelski, Phys. Lett. 101B, 111 (1981).
22. J. Chakrabarti, Phys. Rev. D24, 2232 (1981).
23. M. Reuter and W. Dittrich, Phys. Lett. 144B, 99 (1984).

24. W. Dittrich and V. Schanbacher, Phys. Lett. 100B, 415 (1981).

25. M. Ninomiya and N. Sakai, Nucl. Phys. B190 [FS3], 316 (1981).

26. J. Schwinger, Proc. Nat. Acad. Sci. 37, 452 (1951).

27. K. Symanzik, Comm. Math. Phys. 16, 48 (1970).

28. R. Jackiw and A. Kerman, Phys. Lett. 71A, 158 (1979).

29. S. Coleman, "Secret Symmetry," Erice 1973, ed. by A. Zichichi (Academic Press, New York, 1975).

30. S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

31. R. Jackiw, Phys. Rev. D9, 1686 (1974).

32. J. Iliopoulos, C. Itzykson, A. Martin, Rev. Mod. Phys. 47, 165 (1975).

33. K. Huang, Quarks, Leptons and Gauge Fields (World Scientific, Singapore, 1982), Chapters 7 and 10.

34. C.W. Bernard, Phys. Rev. D9 , 3312 (1974).

35. L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974).

36. D. Kay, Phys. Rev. D28, 1562 (1983).

37. A. D'Adda and P. Di Vecchia, Phys. Lett. 73B, 162 (1978).

38. J. Wess and J. Bagger, Supersymmetry and Supergravity (Princeton University Press, 1983).

39. L.F. Abbott, Nucl. Phys. B185, 189 (1981), and references therein.

40. H. Flyvbjerg, Nucl. Phys. B176, 379 (1980).

41. A. Salam and J. Strathdee, Nucl. Phys. B90, 203 (1975).

42. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 1965).

43. D. Kay, Phys. Rev. D29, 2409 (1984).

44. V. Schanbacher, Phys. Rev. D26, 489 (1982).

45. W. Dittrich and M. Reuter, Phys. Lett. 128B, 321 (1983).

46. J. Chakrabarti, Phys. Rev. D28, 2657 (1983).

47. F. Fucito, S. Solomon, and C. Rebbi, Phys. Rev. D31, 1460 (1985).

48. D.P. Page and M. Socolovsky, Phys. Rev. D31, 923 (1985).

49. E. Elizalde and J. Soto, Barcelona, preprints UBFT-FP-7-84 and UBFT-FP-1-85.

50. D. Kay, R. Parthasarathy, and K.S. Viswanathan, Phys. Rev. D28, 3116 (1983).

51. E. Elizalde, Nucl. Phys. B243, 398 (1984).