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SATURATED FORMATIONS, SCHUNCK CLASSES AND THE STRUCTURE OF FINITE GROUPS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics and Statistics

SIMON FRASER UNIVERSITY

April 1986

APPROVAL

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Degree: Master of Science

Title of Thesis: Saturated formations, Schunck classes and the structure of finite groups.

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Abstract

This is a study of the structure of finite groups from the standpoint of certain classes of groups. Examples of such classes are formations, saturated formations and Schunck classes. Generalizations of Hall subgroups are the dominating theme, and we begin with the theorems of Hall and Carter about the existence and conjugacy of Hall and Carter subgroups (respectively) in finite solvable groups.

A basic notion in recent work is that of covering subgroups of a finite group, where the main result is that if F is a saturated formation then every finite solvable group has F-covering subgroups and any two of them are conjugate.

A more general notion than that of covering subgroups is that of projectors; however, in the case of saturated formations the covering subgroups and projectors of any finite solvable group coincide and form a single conjugacy class. Moreover, the covering subgroups (projectors) for the formation of finite nilpotent groups are the Carter subgroups. In addition, pull-backs exist for the F-projectors associated with a saturated formation F of finite solvable groups, and yields the construction of certain formations.

A natural question is: which classes F give rise to F-projectors? The answer is that for Schunck class, which are more general than saturated formations, every solvable group has F-covering subgroups iff F is a Schunck class. In this case F-covering subgroups are conjugate.

This is now known to be true if "$\mathcal{F}$-covering subgroups" is replaced by "$\mathcal{F}$-projectors", and we prove this result using projectors.

In fact these theorems can be extended to finite π-solvable groups, and if $\mathcal{F}$ is a π-saturated formation (or a π-Schunck class) then every finite π-solvable group has $\mathcal{F}$-covering subgroups (projectors) and any two of them are conjugate.

The existence of projectors in any finite group is proved; however, they may not be conjugate and may not coincide with the covering subgroups, which might not at all exist, in this case. But if $\mathcal{F}$ is a Schunck class then $\mathcal{F}$-projectors and $\mathcal{F}$-covering subgroups do coincide in groups in $\mathcal{UF}$, although the $\mathcal{F}$-projectors need not be conjugate.

Acknowledgement

I wish to thank my senior supervisor Professor J.L. Berggren for suggesting the topic and for his sincere and friendly help and guidance. I would also like to thank Professor N.R. Reilly and Professor C. Godsil for their valuable remarks and comments. I wish also to thank all faculty and staff members of the Mathematics and Statistics Department. My thanks also to Cindy Lister for typing this manuscript.

NOTATION

$g \in G$		g is an element of G.
$x, y, h, \ldots$		elements of groups.
$G, H, \ldots$		sets, groups.
$F, Y, \ldots$		classes of groups.
$H \cong G$		H is isomorphic with G.
$H \leq G$		H is a subgroup of G.
$H < G$		H is a proper subgroup of G.
$H \trianglelefteq G$		H is a normal subgroup of G.
$\|G\|$		the order of the group G.
$\|G:H\|$		Index of the subgroup H in the group G.
$p \setminus q$		p divides q.
$\langle 1 \rangle$		the identity subgroup.
1		the identity element of a group.
$\mathrm{Aut}(G)$		the group of automorphisms of G.
$A \setminus B$	=	$\{x : x \in A$ and $X \notin B\}$.
$\pi(G)$	=	$\{p : P$ is a prime and $p \setminus \|G\|\}$.
h^g	=	$g^{-1}hg$ where $g, h \in G$ and G is a group.
$[x,y]$	=	$x^{-1}y^{-1}xy$.
$[H,K]$	=	$\langle [x,y] : x \in H$ and $y \in K \rangle$.
$Z(G)$	=	$\{g \in G : h^g = h$ for all $h \in G\}$ - the centre of G.
$C_G(H)$	=	$\{g \in G : h^g = h$ for all $h \in H\}$ - the centralizer of H in G.
$N_G(H)$	=	$\{g \in G : h^g \in H$ for all $h \in H\}$ - the normalizer of H in G.

p-group	a group with every element having order a power of the prime p.
π-group	a group with every element having order a power of p, where $p \in \pi$, and π is a fixed set of primes.
π'-group	a group with every element having order a power of p, where p is a prime not in π, π is a fixed set of primes.
$O_\pi(G)$	the largest normal π-subgroup of G.
$H \times K$	direct product of H and K.
Maximal subgroup :	proper subgroup, not contained in any greater proper subgroup.

Minimal normal subgroup of G : normal subgroup $M \neq \langle 1 \rangle$ of G which does not contain normal subgroups of G except $\langle 1 \rangle$ and M.

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Chapter 1

1.1 PRELIMINARIES:

All groups mentioned are assumed to be finite. The notation used is standard, but we have provided a list of symbols and their meanings. The rest of this section is a list of the basic results we will need most frequently. When proofs are not provided, they will be found in any basic textbook in group theory, e.g., Macdonald [4], Suzuki [7] or Robinson [5].

1.1.1 LEMMA: If A and B are subsets of G, then $|AB| = |BA| = |A||B|/|A \cap B|$.

1.1.2 LEMMA:

Let H,K be subgroups of a group G such that $(|G:H|,|G:K|) = 1$, then:

(a) $|G:H \cap K| = |G:H||G:K|$.

(b) $G = HK$.

Proof:

Set $|G| = g$, $|G:H| = m$, $|G:K| = n$ and $|H \cap K| = a$. Then $|H| = ah$, $|K| = ak$ for some integers h,k. Hence $g = ahm = akn$ which implies that $hm = kn$. Since $(m,n) = 1$, $h = nr$, $k = mr$ for some r. But then also $g = amnr$. On the other hand, $g = |G| \geq |HK| = |H||K|/|H \cap K| = (ah)(ak)/a = amnr^2$. So $amnr = amnr^2$ which implies that $r = 1$. Hence $|HK| = amn = |G|$. Therefore $G = HK$. Furthermore, $|G: H \cap K| = amn/a = mn = |G:H||G:K|$. □

<u>DEFINITION</u>: Two subgroups H and K of a group G are said to <u>permute</u> if $HK = KH$. This is in fact precisely the condition for HK to be a subgroup.

1.1.3 <u>LEMMA</u>:

If H and K are subgroups of a group G, then HK is a subgroup iff H and K permute. In this event $HK = \langle H,K \rangle = KH$.

<u>Proof</u>:

Suppose that $HK \leq G$. Then $H \leq HK$ and $K \leq HK$, so $KH \subseteq HK$. Taking inverses of each side we get $HK \subseteq KH$, whence $HK = KH$. Moreover, $\langle H,K \rangle \subseteq HK$, since HK is a subgroup of G, while $HK \subseteq \langle H,K \rangle$ is always true; thus $\langle H,K \rangle = HK$.

Conversely let $HK = KH$. If $h_i \in H$ and $k_i \in K$ $(i=1,2)$, then $h_1k_1(h_2k_2)^{-1} = h_1(k_1k_2^{-1})h_2^{-1}$. Now $(k_1k_2^{-1})h_2^{-1} = h_3k_3$ where $h_3 \in H$ and $k_3 \in K$. Hence $h_1k_1(h_2k_2)^{-1} = (h_1h_3)k_3 \in HK$ and so HK is a subgroup of G. □

1.1.4 <u>LEMMA</u> (Dedekind's Modular law):

Let H, K, L be subgroups of a group and assume $K \leq L$. Then $HK \cap L = (H \cap L)K$.

<u>Proof</u>:

$(H \cap L)K \leq HK$ and $(H \cap L)K \leq LK = L$; hence $(H \cap L)K \leq HK \cap L$. Conversely let $x \in HK \cap L$ and write $x = hk$ where $h \in H$, $k \in K$. Then $h = xk^{-1} \in LK = L$, so that $h \in H \cap L$. Hence $x \in (H \cap L)K$. □

1.1.5 <u>DEFINITIONS</u>: A chain of subgroups of a group

$$\langle 1 \rangle = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

will be called a <u>subnormal series</u> if $G_{i-1} \trianglelefteq G_i$, $i = 1,\ldots,n$, and a <u>normal series</u> if $G_i \trianglelefteq G$ $i = 0,.,\ldots,n$. If each inclusion is proper and the chain is maximal (i.e., no more terms may be inserted without causing some term to be repeated) then the series is called a <u>composition series</u> or a <u>chief series</u>. The corresponding quotient groups G_i/G_{i-1} are called <u>composition</u> and <u>chief factors</u>.

If H is a subgroup of G and H is a member of some subnormal series we say H is a <u>subnormal</u> subgroup of G.

1.1.6 <u>DEFINITION</u>: Let $H \leq G$. Then H is a <u>characteristic</u> subgroup of G if $\alpha(H) = H$ for all $\alpha \in \mathrm{Aut}(G)$. We write H char G. We have the following results:

1.1.7 <u>LEMMA</u>:

(a) Let H be a subgroup of a group G. If $\alpha(H) \subseteq H$ for all $\alpha \in \mathrm{Aut}(G)$ then H char G.

(b) $\langle 1 \rangle$ char G, G char G, $Z(G)$ char G and $G' = [G,G]$ char G. Also H char G implies that $C_G(H)$ char G and $N_G(H)$ char G.

(c) If a subgroup H of a finite group G is the unique subgroup of its order, then H is characteristic.

(d) If H char K char G, then H char G.

(e) If $K \trianglelefteq G$ and H char K then $H \trianglelefteq G$.

(f) If $H \trianglelefteq G$ and $(|H|,|G:H|) = 1$ then H char G.

Proof:

(a) By assumption, $\alpha(H) \subseteq H$ for all $\alpha \in \text{Aut}(G)$. Hence $\alpha^{-1}(H) \subseteq H$, and this implies $H \subseteq \alpha(H)$. So we have $H = \alpha(H)$ for all $\alpha \in \text{Aut}(G)$. Hence H char G.

(b) It is obvious that $\alpha(G) \subseteq G$ and $\alpha(<1>) = <1>$ so $<1>$ and G are characteristic subgroups of G. Choose an element g of $C_G(H)$. For any $\alpha \in \text{Aut}(G)$ and $x \in H$, we have $g\alpha^{-1}(x) = \alpha^{-1}(x)g$. Applying the automorphism α to both sides, we get $\alpha(g)x = x\alpha(g)$. Since this holds for any $x \in H$, we conclude that $\alpha(g) \in C_G(H)$. Hence $\alpha(C_G(H)) \subseteq C_G(H)$ for all $\alpha \in \text{Aut}(G)$; by (a), $C_G(H)$ char G. Similar argument shows that $N_G(H)$ char G.

(c) Since $|\alpha(H)| = |H|$, (c) follows easily from (a).

(d) Let $\alpha \in \text{Aut}(G)$. Since K char G, we have $\alpha(K) = K$. Thus the restriction $\beta = \alpha_k$ of α on K is an automorphism of K. By assumption H char K; so, we have $\beta(H) = H$. The function β is the restriction of α, so $\beta(H) = \alpha(H) = H$. The last equality holds for all $\alpha \in \text{Aut}(G)$; so, H char G.

(e) The preceding proof (d) is valid for $\alpha = i_g$, the inner automorphism of G, and shows that H is invariant by all inner automorphisms of G. Thus H is normal in G.

(f) Let $|H| = m$ and $|G:H| = n$, so that $(m,n) = 1$ and $|G| = mn$. If $\alpha \in \mathrm{Aut}(G)$ then $|\alpha(H)| = m$ and $\alpha(H)H$ is a subgroup of G. Setting $d = |\alpha(H) \cap H|$, we have that $d \setminus m$, $|\alpha(H)H| = m^2/d$, and $(m^2/d) \setminus mn$. Since $(m,n) = 1$, this forces $m = d$ and $H = \alpha(H)$. Thus H char G. ◫

The following lemma is elementary and we omit the proofs.

1.1.8 LEMMA:

(a) $Z(G)$ and $G' = [G,G]$ are normal subgroups of G.

(b) $G' = \langle 1\rangle$ iff G is abelian.

(c) If H is a subgroup of G then $H \trianglelefteq N_G(H)$.

(d) If $H \trianglelefteq G$ then $C_G(H) \trianglelefteq G$.

(e) If H is abelian then $H \trianglelefteq C_G(H)$.

(f) $H \trianglelefteq G$ iff $N_G(H) = G$.

We note the following properties of commutators.

1.1.9 LEMMA: Let $H, K \leq G$. Then

(a) $[H,K] \trianglelefteq \langle H,K\rangle$.

(b) $[H,K] = [K,H]$.

(c) $H \leq N_G(K)$ iff $[H,K] \leq K$.

(d) $H,K \trianglelefteq G$ implies that $[H,K] \trianglelefteq G$ and $[H,K] \leq H \cap K$.

(e) If $K \trianglelefteq G$ then G/K is abelian iff $[G,G] \leq K$.

(f) $H \leq C_G(K)$ iff $[H,K] = \langle 1 \rangle$.

(g) If $M \trianglelefteq G$ then $[HM/M, KM/M] = [H,K]M/M$.

(h) If $H, K \trianglelefteq G$ and $H \leq K$, then $K/H \leq Z(G/H)$ iff $[K,G] \leq H$.

1.1.10 DEFINITIONS: Let G be a finite group.

(a) A subgroup is called a p-subgroup if its order is a power of p.

(b) If $|G| = p^n m$, $(p,m) = 1$ then a subgroup of G is called a Sylow p-subgroup if its order is p^n. We note that 1.1.7(f) implies that all normal Sylow subgroups are characteristic.

1.1.11 THEOREM (Sylow): Let G be a finite group.

(i) G has a Sylow p-subgroup.

(ii) Any two Sylow p-subgroups of G are conjugate.

(iii) Any p-subgroup of G is contained in some Sylow p-subgroup.

1.1.12 LEMMA: Let $H \trianglelefteq G$ and let S be a Sylow p-subgroup of G. Then

(i) $S \cap H$ is a Sylow p-subgroup of H;

(ii) SH/H is a Sylow p-subgroup of G/H.

1.2 NILPOTENT, SOLVABLE AND SUPERSOLVABLE GROUPS:

This section contains most of the basic results on nilpotent, solvable and supersolvable groups that we will need.

1.2.1 DEFINITIONS:

(a) A central series in the group G is a normal series

$$\langle 1\rangle = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$$

such that:

(i) G_i is a normal subgroup of G, $0 \le i \le r$; and

(ii) $G_i/G_{i-1} \le Z(G/G_{i-1})$, for $1 \le i \le r$.

(b) A group is nilpotent if it has a central series.

We now define special central series called upper and lower central series.

DEFINITIONS:

(a) Let $Z^1(G) = G$, $Z^2(G) = [G,G]$, $Z^3(G) = [[G,G],G]$, ..., $Z^{n+1}(G) = [Z^n(G),G]$. The sequence of subgroups

$$Z^1(G) \ge Z^2(G) \ge \dots$$

is called the lower central series of G.

(b) Let $Z_0(G) = \langle 1\rangle$, and for $i > 0$, $Z_i(G)$ is the subgroup of G corresponding to $Z(G/Z_{i-1}(G))$; by the correspondence theorem $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$. The sequence of subgroups

$$Z_0(G) \le Z_1(G) \le Z_2(G) \le \dots$$

is called the upper central series of G.

1.2.2 <u>LEMMA</u>: Let

$$\langle 1\rangle = G_0 \le G_1 \le \ldots \le G_r = G$$

be a series of subgroups each normal in G. This is a central series iff $[G_i,G] \le G_{i-1}$, for $1 \le i \le r$.

1.2.3 <u>LEMMA</u>: If the nilpotent group G has a proper subgroup H then H is a proper subgroup of its normalizer.

<u>Proof</u>:

Take a central series for G:

$$\langle 1\rangle = G_0 \le G_1 \le \ldots \le G_r = G.$$

we have $[G_i,G] \le G_{i-1}$ for $1 \le i \le r$, by 1.2.2. Suppose that $G_{k-1} \le H$ while $G_k \not\le H$. Such a value of k exists because $\langle 1\rangle = G_0 \le H$ and $G = G_r \not\le H$, H being a proper subgroup of G. Then $[G_k,G] \le G_{k-1} \le H$, and so $[G_k,H] \le H$ and hence by 1.1.8(c), $G_k \le N_G(H)$. By the choice of k there is an element of G_k which does not lie in H, and it follows that $H \lneq N_G(H)$. ◻

The following lemma will be useful later when we study closure operations on classes of groups.

1.2.4 <u>LEMMA</u>:

(a) The class of nilpotent groups is closed under the formation of subgroups, quotients, and finite direct products.

(b) All p-groups are nilpotent.

(c) A group G is nilpotent iff all its maximal subgroups are normal.

(d) A group G is nilpotent iff all its Sylow subgroups are normal.

(e) A nilpotent group is the direct product of its Sylow subgroups.

1.2.5 LEMMA: If H is a non-trivial normal subgroup of the nilpotent group G, then $H \cap Z(G) \neq \langle 1 \rangle$.

Proof:

Let $\langle 1 \rangle = G_0 \leq G_1 \leq \ldots \leq G_r = G$ be a central series of G. There is a least integer k such that $H \cap G_k \neq \langle 1 \rangle$. Let x be a non-trivial element in $H \cap G_k$. Then $[x,G] \leq [H,G] \leq H$ as $H \trianglelefteq G$. Also $[x,G] \leq [G_k,G] \leq G_{k-1}$ by 1.2.2. Thus $[x,G] \leq H \cap G_{k-1} = \langle 1 \rangle$ by the choice of k. By 1.1.8(f), $x \in Z(G)$. □

1.2.6 LEMMA: The following are equivalent:

(i) G is nilpotent.

(ii) There exists an integer n such that $Z^{n+1}(G) = \langle 1 \rangle$.

(iii) There exists an integer n such that $Z_n(G) = G$.

It may be shown that the least n in (ii) is the same as the least n in (iii). This integer n is called the class of the nilpotent group.

Since a group G is abelian iff $G' = \langle 1 \rangle$, all abelian groups are nilpotent. On the other hand, A_4 is an example of a non-nilpotent group. This is because $Z_1(A_4) = \langle 1 \rangle$, as a direct check shows. Hence

$Z_1(A_4) = Z_2(A_4) = \ldots$, and thus $Z_n(A_4) \neq A_4$ for every n.

1.2.7 DEFINITION: A group G is solvable if it has a subnormal series

$$\langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_r = G$$

in which G_i/G_{i-1} is abelian for $1 \leq i \leq r$.

Comparison of definitions shows that every nilpotent group is solvable. On the other hand, there are solvable groups such as S_3 and A_4 that are not nilpotent.

Again we state some closure results for later use.

1.2.8 LEMMA:

(a) The class of solvable groups is closed with respect to the formation of subgroups, homomorphic images, and finite direct products.

(b) Let H be a normal subgroup of G. If H and G/H are both solvable then G is solvable.

(c) Put $D(G) = [G,G]$, define by induction $D_n(G) = [D_{n-1}(G), D_{n-1}(G)]$. Then G is solvable iff $D_n(G) = \langle 1 \rangle$ for some integer n.

Hence G is solvable implies that $G' < G$.

1.2.9 DEFINITION: Let p be a prime number.

An abelian group E is an elementary abelian p-group if every element of E has order p or 1.

1.2.10 LEMMA:

Chief factors of finite solvable groups are elementary abelian p-groups for some prime p.

1.2.11 <u>COROLLARY</u>: A minimal normal subgroup of a finite solvable group is an elementary abelian p-group for some prime p.

1.2.12 <u>DEFINITION</u>: A group is <u>supersolvable</u> if it has a normal series with cyclic factors.

Again we state some results on closure for future reference.

1.2.13 <u>LEMMA</u>:

(a) The class of supersolvable groups is closed with respect to the formation of subgroups, homomorphic images, and finite direct products.

(b) A group G is supersolvable iff its maximal subgroups have prime index.

(c) Nilpotent groups are supersolvable and supersolvable groups are solvable. In general, we have the following hierarchy of classes of groups:

Cyclic $\subset$ Abelian $\subset$ Nilpotent $\subset$ Supersolvable $\subset$ Solvable $\subset$ Group.

All inclusions are proper, and in particular A_4, which has no normal series

$$\langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = A_4$$

with each factor group G_i/G_{i-1} cyclic and with each $G_{i-1} \trianglelefteq G$, is the first example of a solvable group that is not supersolvable.

1.3 <u>EXTENSIONS</u>:

Let H and F be groups. The study of extensions involves finding all groups G (up to isomorphism) such that $H \trianglelefteq G$ and $G/H \cong F$.

1.3.1 <u>DEFINITIONS</u>: If K and F are groups, an <u>extension of K by F</u> is a group G such that:

(i) G contains K as a normal subgroup.

(ii) $G/K \cong F$.

If $H \leq G$ and there exists another proper subgroup K of G such that $H \cap K = \langle 1 \rangle$ and $G = HK$ we say H is <u>complemented</u> in G. If $H \cap K \neq \langle 1 \rangle$ then K is called a <u>partial complement</u> of H. In the following we point out some special cases of extensions.

If H and K are normal subgroups of G such that $G = HK$ and $H \cap K = \langle 1 \rangle$ then G is a <u>direct product</u> of H and K, written $G = H \times K$. We notice that if $G = H \times K$ then the elements of H commute with those in K, that is H and K commute elementwise. For if $h \in H$ and $k \in K$, $h \neq k$, then $h^{-1}k^{-1}hk = h^{-1}(k^{-1}hk) = (h^{-1}k^{-1}h)k \in H \cap K = \langle 1 \rangle$, hence $hk = kh$.

1.3.2 <u>LEMMA</u>: If $G = H \times K$ and $A \trianglelefteq H$ then $A \trianglelefteq G$.

<u>Proof</u>:

Let $a \in A$, $g \in G$, then $g = hk$ where $h \in H$, $k \in K$ and

$$g^{-1}ag = k^{-1}h^{-1}ahk = k^{-1}a'k \quad \text{where } h^{-1}ah = a' \in A \text{ as } A \trianglelefteq H$$
$$= a'k^{-1}k \quad \text{as } a' \in A \leq H,\ k^{-1} \in K$$
$$= a' \in A. \quad \square$$

From the definition of direct product we notice that the subgroups H and K are required to be normal. A natural generalization of direct products is the situation in which only one of the subgroups is required to be normal.

1.3.3 DEFINITION: A group G is a semidirect product of K by F in case G contains subgroups K and F such that:

(i) $K \trianglelefteq G$.

(ii) $KF = G$.

(iii) $K \cap F = \langle 1 \rangle$.

It follows from the second isomorphism theorem that a semidirect product of K by F is an extension of K by F.

1.3.4 LEMMA: If G is a semidirect product of K by F then there is a homomorphism $\theta : F \to \mathrm{Aut}(K)$ defined by $\theta_x(K) = xkx^{-1} = k^{x^{-1}}$, for all $k \in K$, $x \in F$. Moreover, $\theta_x(\theta_y(k)) = \theta_{xy}(k)$ and $\theta_1(k) = k$, $k \in K$, $x,y,1 \in F$.

Proof:

Straightforward, using the normality of K. ∎

DEFINITIONS:

(a) Given K, F, and $\theta : F \to \mathrm{Aut}(K)$, then a semidirect product G of K by F realizes θ in case $\theta_x(k) = k^{x^{-1}}$ for all $k \in K$.

(b) Let K, F, and $\theta : F \to \mathrm{Aut}(K)$ be given. Then $K \rtimes_\theta F$ is the set of all ordered pairs $(k,x) \in K \times F$ under the binary operation $(k,x)(k_1,y) = (kk_1^{x^{-1}},xy)$.

1.3.5 <u>THEOREM</u>: Let K, F, and $\theta : F \to \mathrm{Aut}(G)$ be given, then $G = K \rtimes_\theta F$ is a semidirect product of K by F that realizes θ.

<u>Proof</u>:

We first prove that $G = K \rtimes_\theta F$ is a group. Multiplication is associative:

$$[(k,x)(k_1,y)](k_2,z) \qquad\qquad (k,x)[(k_1,y)(k_2,z)]$$

$$= (kk_1^{x^{-1}},xy)(k_2,z) \qquad\qquad = (k,x)(k_1k_2^{y^{-1}},yz)$$

$$= (kk_1^{x^{-1}}k_2^{(xy)^{-1}},xyz) \qquad\qquad = (k(k_1k_2^{y^{-1}})^{x^{-1}},xyz)$$

$$= kk_1^{x^{-1}}k_2^{y^{-1}x^{-1}},xyz); \qquad\qquad = (kk_1^{x^{-1}}k_2^{y^{-1}x^{-1}},xyz).$$

It is easy to check that the identity element is (1,1) and $(k,x)^{-1} = ((k^{-1})^x,x^{-1})$.

Let us identify K with the subset of G consisting of all pairs of the form (k,1). Since the only "twist" occurs in the first coordinate, the map $\alpha : G \to F$ defined by $\alpha(k,x) = x$ is a homomorphism. It is easily checked that $\ker\alpha = k$, so that $K \trianglelefteq G$.

Identify F with all pairs (1,x). Then $F \leq G$ with $KF = G$ and $K \cap F = \langle(1,1)\rangle$. Therefore, G is a semidirect product of K by F.

To see that G realizes θ, compute:

$$(1,x)(k,1)(1,x)^{-1} = (k^{x^{-1}},x)(1,x^{-1}) = (k^{x^{-1}},1). \quad \square$$

1.3.6 LEMMA: If G is a semidirect product of K by F, then $G \cong K \rtimes_\theta F$ for some $\theta : F \to \mathrm{Aut}(G)$.

Proof:

Define $\theta_x(k) = k^{x^{-1}}$. Since $G = KF$, each $g \in G$ has the form $g = kx$, where $k \in K$, $x \in F$; this form is unique since $K \cap F = \langle 1 \rangle$. Multiplication in G satisfies $(kx)(k_1x_1) = k(xk_1x^{-1})xx_1 = kk_1^{x^{-1}}xx_1$ and it now is easy to see that the map $K \rtimes_\theta F \to G$ defined by $(k,x) \to kx$ is an isomorphism. □

1.4 THE FRATTINI AND FITTING SUBGROUPS:

1.4.1 DEFINITION: The intersection of all maximal subgroups of a group G is called the Frattini subgroup of G and is denoted by $\phi(G)$.

The Frattini subgroup has the remarkable property that it is the set of all nongenerators of the group, where an element g is nongenerator if $G = \langle g,X \rangle$ always implies that $G = \langle X \rangle$ when X is a subset of G. The following lemma states this formally and will be of constant use to us in the sequel.

1.4.2 LEMMA:

(a) For a subset X of G, $\langle X,\phi(G) \rangle = G$ iff $\langle X \rangle = G$. In particular, if $G = H\phi(G)$ for some subgroup H of G, then $G = H$.

(b) If $H \trianglelefteq G$, then H has a partial complement in G iff $H \not\leq \phi(G)$.

Proof:

(a) Suppose $G = \langle X, \phi(G)\rangle$. If $\langle X\rangle$ were a proper subgroup of G, then there would be a maximal subgroup M which would contain X. Then we would have $\langle X, \phi(G)\rangle \subseteq M$, contrary to the assumption. The converse implication is obvious. In particular, $G = H\phi(G) \subseteq \langle H, \phi(G)\rangle = \langle H\rangle = H$ as H is a subgroup. So $G = H$.

(b) Assume that $H \trianglelefteq G$ with $H \nleq \phi(G)$. Then there is a maximal subgroup M of G with $H \nleq M$. But HM is a group as $H \trianglelefteq G$ and $M < HM$ as $H \nleq M$. Maximality of M implies that $G = HM$ and M is a partial complement of H.

Conversely, suppose $H \leq \phi(G)$ and H has a partial complement K. Then $G = HK \leq \phi(G)K = K$, contrary to the definition of a partial complement. ◫

1.4.3 LEMMA: If H is a normal subgroup of the finite group G and P is a Sylow p-subgroup of H, then $G = N_G(P)H$.

Proof:

Let $g \in G$; then $P^g \leq H$ and P^g is Sylow p-subgroup of H. Hence $P^g = P^h$ for some $h \in H$ by Sylow's theorem. Consequently $gh^{-1} \in N_G(P)$ and $g \in N_G(P)H$. ◫

The proof of this enormously useful result is usually referred to as the Frattini argument and the technique in the proof is often used. One application is to show that the Frattini subgroup of a finite group is nilpotent, a fact first established by Frattini himself.

In the following lemma we collect some useful properties of the Frattini subgroup of a finite group. The proofs may be found in Gorenstein [1] or Suzuki [7] and we omit them.

1.4.4 LEMMA:

(a) $\phi(G)$ is a nilpotent, characteristic subgroup of G.

(b) If $H \trianglelefteq G$ then $\phi(H) \leq \phi(G)$.

(c) If $H \trianglelefteq G$ and $H \leq \phi(G)$ then $\phi(G)/H \cong \phi(G/H)$. In particular, $\phi(G/\phi(G)) = \langle 1 \rangle$.

(d) If $H \trianglelefteq G$, then H is nilpotent iff $[H,H] \leq \phi(G)$, In particular G is nilpotent iff $G' \leq \phi(G)$.

1.4.5 LEMMA: If L is abelian normal subgroup of G such that $L \cap \phi(G) = \langle 1 \rangle$, then L is complemented in G.

Proof:

Choose $H \leq G$ minimal subject to $G = HL$. (Such an H exists by 1.4.2(b)). Since L is an abelian normal subgroup of G, $H \cap L \trianglelefteq G$. If $H \cap L \leq \phi(H)$ then $H \cap L \leq \phi(H) \cap L \leq \phi(G) \cap L = \langle 1 \rangle$. So we may assume that $H \cap L \nleq \phi(H)$. Then $H \cap L \nleq M$ for some maximal subgroup M of H, and so $H = M(H \cap L)$. But then $ML = M(L \cap H)L = HL = G$, a contradiction to the minimality of H. □

Now to introduce another nilpotent subgroup of the finite group G we remind the reader that the product AB of two normal nilpotent subgroups A and B of a group G is again a nilpotent subgroup (Gorenstein [1], 6.1.1). We may now state:

1.4.6 <u>DEFINITION</u>: The subgroup generated by all the normal nilpotent subgroups of a group G is called the <u>Fitting subgroup</u> of G and is denoted by $F(G)$. It is evidently the unique largest normal nilpotent subgroup of G.

1.4.7 <u>LEMMA</u>: For a finite group G.

(i) $\phi(G) \leq F(G)$.

(ii) $F(G)/\phi(G)$ is abelian.

<u>Proof</u>:

(i) Obvious.

(ii) Let F denote $F(G)$. Since F is nilpotent, each maximal subgroup of F is normal and hence contains F'. Thus $F' \leq \phi(F) \leq \phi(G)$ from which it follows that $F/\phi(G)$ is abelian. □

1.4.8 <u>LEMMA</u>: If G is solvable and $\phi(G) = \langle 1 \rangle$ then $F(G)$ is the product of (abelian) minimal normal subgroups of G.

<u>Proof</u>:

Write $F(G) = L$. Since L is nilpotent, a maximal subgroup of L is normal and has prime index. Hence $L' \leq \phi(L) \leq \phi(G) = \langle 1 \rangle$ and so L is abelian.

Let N be the product of all the (abelian) minimal normal subgroups of G. N is the direct product because any two such subgroups intersect trivially. Then N is abelian and normal in G so by 1.4.5 there is a subgroup H of G such that $G = HN$ and $H \cap N = \langle 1 \rangle$. Now $H \cap L \trianglelefteq HL = G$. Since $(L \cap H) \cap N = L \cap (H \cap N) = \langle 1 \rangle$, the normal subgroup $H \cap L$ cannot contain a minimal normal subgroup of G; we conclude $H \cap L = \langle 1 \rangle$. Hence $L = L \cap HN = N(L \cap H) = N$. ◻

1.4.9 LEMMA: If G is a solvable group then $C_G(F(G)) \leq F(G)$.

Proof:

For brevity set $F(G) = F$ and $C_G(F) = C$. Suppose the lemma fails so that $CF \neq F$. Then CF/F is a non-trivial normal subgroup of G/F, hence contains a minimal normal subgroup Q/F of G/F. Solvability implies Q/F is abelian so we have $Q' \leq F < Q \leq CF$ and $(Q \cap C)F = Q$. On the other hand $Q \cap C \lhd G$ and $Q \cap C$ is nilpotent because $[Q \cap C, Q \cap C, Q \cap C] \leq [Q',C] \leq [F,C] = \langle 1 \rangle$. Hence $Q \cap C \leq F$ so that $(Q \cap C)F = F$. This and $(Q \cap C)F = Q$ yields the contradiction $F = Q$ completing the proof. ◻

1.5 HALL π-SUBGROUPS:

Let G be a finite group and let π be a non-empty set of primes. By a π-subgroup we mean a subgroup whose order is the product of primes in π. A Sylow π-subgroup of G is defined to be a maximal π-group. While Sylow π-subgroups always exist, they are usually not conjugate if π-contains more than one prime.

A more useful concept is that of a Hall π-subgroup. A π-subgroup H of G such that $|G:H|$ is not divisible by any prime in π is called a Hall π-subgroup of G. It is rather obvious that every Hall π-subgroup is a Sylow π-subgroup. In general, however, a group G need not contain any Hall π-subgroups as we will see from the following example.

1.5.1 <u>EXAMPLE</u>: Let $\pi = \{2,5\}$. A Hall π-subgroup of A_5 would have index 3, but A_5 has no such subgroups. For if there were a subgroup B of index 3, then B would have three conjugates; the intersect N of these would be a subgroup of index at most $3^3 = 27$. But then N would be a proper normal subgroup of A_5, whereas A_5 is simple. Therefore A_5 has no subgroup of order 20.

A_5 must have Sylow π-subgroups. Among these are the Sylow 2-subgroups, such as $V = \langle(12)(34), (13)(24)\rangle$, is one, for if it were contained in a larger π-subgroup then the latter would have order 20 by Lagrange's theorem. Another Sylow π-subgroup is $U = \langle(12345), (25)(34)\rangle$; for it will be found that this has order 10 and is therefore again a maximal π-subgroup. Here is a case in which Sylow π-subgroups are not conjugate.

We shall see now that in contrast to the situation in the non-solvable group A_5 in a finite solvable group Hall π-subgroups always exist and form a single conjugacy class.

1.5.2 <u>THEOREM</u> (P. Hall):

Let G be a finite solvable group and π a set of primes. Then

(i) G contains a Hall π-subgroup;

(ii) any two Hall π-subgroups of G are conjugate;

(iii) every π-subgroup of G lies in a suitable Hall π-subgroup.

Proof:

We proceed by induction on $|G|$. Let N be a minimal normal subgroup of G with $|N| = p^r$ where p is a prime. By induction G/N has a Hall π-subgroup H/N, moreover, any two Hall π-subgroups of G/N are conjugate in G/N. We consider separately the cases $p \in \pi$ and $p \notin \pi$.

Case 1: $p \in \pi$. In this case H is a π-subgroup of G and, since $|G/N : H/N| = |G : H|$, it is easily seen that H is in fact a Hall π-subgroup of G. If L is any other π-subgroup of G then by induction $LN/N = (H/N)^{xN}$ for some $x \in G$, and therefore $L \leq H^x$. If L is also a Hall π-subgroup of G, then $|L| = |H| = |H^x|$. This and $L \leq H^x$ give $L = H^x$.

Case 2: $p \notin \pi$. If $H < G$, then by induction, H has a Hall π-subgroup H_1 which must be a Hall π-subgroup of G. If L is any other π-subgroup of G, then, by induction LN/N lies in H^x/N for some x in G and hence $L \leq H^x$. Since H_1^x is a Hall π-subgroup of H^x, we may apply induction to conclude that L lies in H_1^{xy} for some $y \in H^x$. Moreover, if L is also a Hall subgroup of G then, as in case 1, L would coincide with H_1^{xy}.

We assume for the remainder of the proof that $G = H$ and, without loss of generality, that $G \neq N$. We also observe that under these conditions N is a Sylow p-subgroup of G. Let T/N be a minimal normal subgroup of G/N with $|T/N| = q^s$ for some prime $q \neq p$. Let Q be a Sylow q-subgroup of T, so that $T = NQ$. By the Frattini argument, $G = TN_G(Q) = NN_G(Q)$. If $G = N_G(Q)$, then we may apply the argument of case 1 using Q for N. We therefore assume that $G \neq N_G(Q)$. Since $N \cap N_G(Q) \trianglelefteq G$ as N is abelian, it follows that $N \cap N_G(Q) = \langle 1 \rangle$. Thus $|G:N| = |N_G(Q)|$ and $N_G(Q)$ is a Hall π-subgroup of G. Set $H_1 = N_G(Q)$ and let L be any π-subgroup of G. Since $G = H_1N$, $LN = LN \cap H_1N = N(H_1 \cap LN)$. Thus $H_1 \cap LN$ is a Hall π-subgroup of LN. If $LN \neq G$, then by induction $L \leq (H_1 \cap LN)^x \leq H_1^x$ for some x in LN. If $LN = G$, then $T = N(L \cap T)$ and in this situation $L \cap T = Q^x$ for some $x \in T$. Then $T \trianglelefteq G$ implies $L \cap T \trianglelefteq L$ so that $L \leq N_G(Q^x) = H_1^x$. Again, if L is also a Hall π-subgroup of G, then the argument of Case 1 can be used to show $L = H_1^x$. □

1.5.3 LEMMA: Assume G has a Hall π-subgroup H. Then:

(i) If $H \leq K \leq G$ then H is a Hall π-subgroup of K.

(ii) If $M \trianglelefteq G$ then $M \cap H$ and HM/M are Hall π-subgroups of M and G/M respectively.

Proof:

(i) Follows immediately from the definition of Hall π-subgroups.

(ii) By (i) H is a Hall π-subgroup of HM and since $|M : M \cap H| = |HM : H|$, $|M : M \cap H|$ is divisible only by primes in π', where π' is the complement of π in the set of all primes. Since H is a π-group and $|M \cap H|$ divides $|H|$, we have $H \cap M$ is a π-group. Therefore $H \cap M$ is a Hall π-subgroup of M. A similar argument shows that HM/M is a Hall π-subgroup of G/M. □

1.6 CARTER SUBGROUPS:

In 1961 R.W. Carter published a striking theorem about the nilpotent subgroups of a finite solvable group in his paper "Nilpotent self-normalizing subgroups of a solvable group".

1.6.1 THEOREM (Carter):

If G is a finite solvable group, then:

(a) G has a self-normalizing nilpotent subgroup.

(b) If H_1, H_2 are self-normalizing nilpotent subgroups of G, then $H_1 = H_2^g$ for some $g \in G$.

We shall prove a more general version of this theorem later.

1.6.2 DEFINITION

Self-normalizing nilpotent subgroups are now called Carter subgroups.

1.6.3 LEMMA:

Let E be a Carter subgroup of G. Then:

(i) If $E \leq F \leq G$, then E is a Carter subgroup of F.

(ii) If $M \trianglelefteq G$, then EM/M is a Carter subgroup of G/M.

Proof:

(i) As the definition of nilpotence takes no account of any group in which E may lie, we have E is a nilpotent subgroup of F.

Since E is self-normalizing in G, $E = N_G(E)$ and so if $E \leq F$, then $N_F(E) = N_G(E) \cap F = E \cap F = E$. Therefore E is a nilpotent self-normalizing subgroup of F. Thus E is a Carter subgroup of F.

(ii) Since $M \trianglelefteq G$, $E \cap M \trianglelefteq E$, and since E is nilpotent, $E/E \cap M$ is nilpotent. But $E/E \cap M \cong EM/M$, therefore EM/M is nilpotent.

To show EM/M is self-normalizing, suppose $EM/M \trianglelefteq F/M$. Then $EM \trianglelefteq F$. By (i) E is a Carter subgroup of EM and so by (1.6.1) all conjugates of E in EM under the action of F are conjugate in EM; that is for all $x \in F$, there is $g \in EM$ such that $E^{xg} = E$ which implies that $xg \in N_G(E) = E$ implies that $x \in Eg^{-1}$ implies that $F = E \cdot EM = EM$, so EM/M is self-normalizing in G/M and therefore EM/M is a Carter subgroup of G/M. □

Carter's discovery aroused considerable interest, although it was clear from the start that it could not be extended in an obvious way to arbitrary finite groups. The alternating group A_5 shows that Carter subgroups need not exist in insolvable groups.

Carter subgroups may be seen as analogues (for the class of nilpotent groups) of Sylow p-subgroups (for the class of p-groups) and Hall π-subgroups (for the class of π-groups). All are maximal subgroups of their class, are preserved under epimorphisms, and satisfy the existence and the conjugacy conditions, i.e. they exist and form a single conjugacy class. One important component of the theorems of Sylow and Hall that is missing from Carter's theorem, however, is the following theorem:

<u>THEOREM</u>:

Every p-subgroup of a group is contained in a Sylow p-subgroup; every π-subgroup of a solvable group is contained in a Hall π-subgroup.

It is not the case that every nilpotent subgroup of a solvable group is contained in a self-normalizing nilpotent subgroup.

EXAMPLE:

Consider S_3. The subgroup $H = \{I,(123),(132)\}$ is easily checked to be a normal in S_3. Since $H \cong C_3$, $S_3/H \cong C_2$, by 1.2.8(b), S_3 is solvable. $Z(S_3) = \langle 1 \rangle$ as a direct check shows. Hence $Z_1(S_3) = Z_2(S_3) = \ldots$, and thus $Z_n(S_3) \neq S_3$ for every n. Hence by 1.2.6(iii), S_3 is not nilpotent.

H is nilpotent and H itself is the largest nilpotent subgroup of S_3 that contains H. But H is not self-normalizing, since $H \trianglelefteq S_3$ which implies that $N_G(G) = S_3$.

Some years were to elapse before the discovery of new conjugacy classes of generalized Sylow subgroups satisfying Theorem 1.6.1.

Chapter 2

PROJECTORS and FORMATIONS

2.1 FORMATIONS:

By a class of groups F we mean a class - not a set - whose members are groups and which enjoys the following properties:

(i) F contains a group of order 1;

(ii) $G_1 \cong G \in F$ always implies that $G_1 \in F$.

2.1.1 DEFINITION:

Let F be a class of finite groups. F is called a homomorph if $G \in F$, $N \trianglelefteq G$ implies that $G/N \in F$.

Examples of homomorphs are readily found. The classes of finite groups, finite solvable groups, finite nilpotent groups, and finite super-solvable groups are homomorphs.

2.1.2 DEFINITION:

A homomorph F is called a formation if it satisfies the following condition:

(*) If $N_1,\ldots,N_k$ are normal subgroups of G such that $G/N_i \in F$ and $\bigcap_{i=1}^{k} N_i = \langle 1 \rangle$ $i = 1,\ldots,i$, then $G \in F$.

REMARK 1: Condition (*) of the definition 2.1.2 is equivalent to the following condition:

(**) If G/N_1, $G/N_2 \in F$, then $G/(N_1 \cap N_2) \in F$.

This is because: if $N_1, N_2 \trianglelefteq G$ with $G/N_1, G/N_2 \in F$, then

$$(G/N_1 \cap N_2)/(N_i/N_1 \cap N_2) \cong G/N_i \in F \quad i = 1,2,$$

and $(N_1/(N_1 \cap N_2)) \cap (N_2/N_1 \cap N_2) = \langle 1 \rangle$, and therefore $G/N_1 \cap N_2 \in F$. Thus we have shown (*) implies (**). On the other hand, if $N_1,\ldots,N_k$ are normal subgroups of G such that $G/N_i \in F$ and $\bigcap_{i=1}^{k} N_i = \langle 1 \rangle$, then assuming (**) we have $G/N_1, G/N_2 \in F$ implies $G/(N_1 \cap N_2) \in F$. Again $G/N_3 \in F$ and $G/N_1 \cap N_2 \in F$ implies $G/(N_1 \cap N_2 \cap N_3) \in F$. Continuing this we finally get $G/(N_1 \cap \ldots \cap N_{k-1})$, $G/N_k \in F$ implies $G/\bigcap_{i=1}^{k} N_i \in F$, but $\bigcap_{i=1}^{k} N_i = \langle 1 \rangle$, therefore we have $G \in F$.

REMARK 2:

Condition (**) can be replaced by the following apparently weaker condition: (***) If $N_1, N_2 \trianglelefteq G$ such that $G/N_1, G/N_2 \in F$ and $N_1 \cap N_2 = \langle 1 \rangle$, then $G \in F$.

REMARK 3:

If $G,H \in F$ then $G \times H \in F$.

We will provide examples of formations later. Now we have the following lemma.

2.1.3 LEMMA:

Let F be a formation, and G a group. Let $G_F = \cap\{H : H \trianglelefteq G, G/H \in F\}$. Then G_F is unique and minimal subject to the property that $G/G_F \in F$. Moreover, $G/H \in F$ iff $G_F \leq H$, $H \trianglelefteq G$. In particular, $G \in F$ iff $G_F = \langle 1 \rangle$.

Proof:

By (**) of the definition of formation we have $G/G_F \in F$. G_F is unique and minimal subject to $G/G_F \in F$ by its very definition.

Also if $G_F \leq H$, then $G/H \cong (G/G_F)/(H/G_F) \in F$, since $G/G_F \in F$ and F is closed under epimorphisms. □

G_F is often called the F-residual of G. It is characteristic in G by 1.1.8(c), and it may be characterized as the least normal subgroup with factor group in F.

EXAMPLE:

Let A be the class of finite abelian groups. We will see later that A is a formation. $G' = [G,G]$ is the A-residual, since for $N \trianglelefteq G$, $G/N \in A$ implies that $G/G' \in A$ and $G' \leq N$.

To provide examples, we denote by

- N: The class of all finite nilpotent groups.
- P: The class of all finite p-groups, where p is a fixed prime.
- T: The class of all finite supersolvable groups.
- $\underline{\pi}$: The class of all finite solvable π-groups (where π is a set of primes).
- U: The class of all finite solvable groups.
- A: The class of all finite abelian groups.

We have seen that formations are closed under taking factors and direct products. If on the other hand we require a class to be closed under both of these operations and taking subgroups it is, in fact, a formation.

2.1.4 THEOREM:

If $H, K \trianglelefteq G$, then there exists a monomorphism θ of $G/H \cap K$ into $G/H \times G/K$.

Proof:

Define $\theta : G/H \cap K \to G/H \times G/K$ by $\theta[g(H \cap K)] = (gH, gK)$. Then θ is well defined, since $g_1(H \cap K) = g_2(H \cap K)$ implies that $g_1 g_2^{-1} \in H \cap K$ implies that $g_1 g_2^{-1} \in H$ and $g_1 g_2^{-1} \in K$. Hence $g_1 H = g_2 H$ and $g_1 K = g_2 K$ and so $(g_1 H, g_1 K) = (g_2 H, g_2 K)$.

$$\begin{aligned} \text{Now,} \quad \theta[g_1(H \cap K) g_2(H \cap K)] = \theta(g_1 g_2(H \cap K)] &= (g_1 g_2 H,\ g_1 g_2 K) \\ &= (g_1 H g_2 H,\ g_1 K g_2 K) \\ &= (g_1 H,\ g_1 K)(g_2 H,\ g_2 K) \\ &= \theta[g_1(H \cap K)]\theta[g_2(H \cap K)] \end{aligned}$$

and therefore θ is a homomorphism. Now $\theta[g_1(H \cap K)] = \theta[g_2(H \cap K)$ implies that $(g_1 H,\ g_1 K) = (g_2 H,\ g_2 K)$ and so $g_1 H = g_2 H$ and $g_1 K = g_2 K$. Hence $g_1^{-1} g_2 \in H \cap K$ and therefore $g_1(H \cap K) = g_2(H \cap K)$. Thus θ is one-to-one homomorphism. □

The following corollary to Theorem 2.1.4 enables us to prove very simply that all the classes N, P, T, $\underline{\pi}$, U and A are formations.

2.1.5 COROLLARY:

Let F be a non-empty class of groups, closed under epimorphisms, subgroups, and direct products. Then F is a formation.

Proof:

Since F is a homomorph (i.e. closed under epimorphisms), we only need to show that if $G/H, G/K \in F$ then $G/H \cap K \in F$ where $H,K \trianglelefteq G$. Since $G/H \times G/K \in F$, and by 2.1.4, $G/H \cap K$ is isomorphic to a subgroup of $G/H \times G/K$, $G/H \cap K \in F$ as required. □

2.1.6 EXAMPLES:

(a) Since all the classes N, P, T, $\underline{\pi}$, U and A are clearly closed under epimorphisms, subgroups and direct products, then they are formations.

(b) In this example we show that formations are not necessarily closed under taking subgroups.

Let $F = \{G : G$ is solvable and $|M/N| \neq 2$ for all 2-chief factors M/N of $G\}$. We show that F is a formation. It is obvious that F is a homomorph, since all 2-chief factors of G/N (where $N \trianglelefteq G$) are isomorphic to a subset of those of G.

Now, let $G/H_1, G/H_2 \in F$ and by (***) of (2.1.2) we may assume that $H_1 \cap H_2 = \langle 1 \rangle$, and we want to show that $G \in F$.

If M/N is a chief factor of G, then, since $H_i \trianglelefteq G$ $(i = 1,2)$ we must have either H_i covers M/N or avoids it by (1.1.6). If H_i covers M/N, then $MH_i = NH_i$ so that $MH_i/NH_i = \langle 1 \rangle$. If H_i avoids M/N, then again by (1.1.6), $M/N \cong MH_i/NH_i \cong (MH_i/H_i)/(NH_i/H_i)$ which means that M/N is isomorphic to a chief factor of G/H_i. Therefore we have $G \in F$.

For instance, $A_4 \in F$ since its chief factors have order 4 and 3, but $V = \{(1),(12)(34),(13)(24),(14)(23)\}$ is a normal subgroup of A_4 and $V \notin F$ since C_2 is a 2-chief factor of V with order 2. Thus F is not closed under (normal) subgroups.

In order to generalize Carter's result (1.6.1), Gaschütz needed to consider a special kind of formations which he defined (in his paper "Zur theorie der endlichen auflösbaren Gruppen", Math Z. 80, 1963, 300-305) as follows:

2.1.7 DEFINITION:

Let F be a non-empty formation. Then F is said to be saturated provided that $G/\phi(G) \in F$ implies $G \in F$.

2.1.8 EXAMPLES:

(a) Consider N, the class of finite nilpotent groups. By 2.1.6(a), N is a formation. To show N is saturated, suppose $G/\phi(G) \in N$ and let P be a Sylow p-subgroup of G. Then $P\phi(G)/\phi(G)$ is a Sylow p-subgroup of $G/\phi(G)$. But $G/\phi(G) \in N$ so $P\phi(G)/\phi(G) \trianglelefteq G/\phi(G)$ which implies that $P\phi(G) \trianglelefteq G$.

If $N_G(P) < G$, then there exists a maximal subgroup M of G such that $N_G(P) \leq M$. Now P is a Sylow p-subgroup of $P\phi(G)$ so by the Frattini argument we have $G = (P\phi(G))N_G(P) \leq M$ (as $P \leq N_G(P) \leq M$ and $\phi(G) \leq M$) which is contradiction. Therefore $N_G(P) = G$ and hence $P \trianglelefteq G$. Thus G is nilpotent.

(b) By 2.1.6, T is a formation. Since G is finite, $G/\phi(G)$ is supersolvable implies that G is supersolvable (see Robinson 9.4.5). Thus, the class of finite supersolvable groups is a saturated formation.

(c) We know that A is a formation (Example 2.1.6), and if G is a non-abelian p-group, we know that $G/\phi(G) \in A$ but $G \notin A$ (e.g. take $G = Q_8$, the quaternion group). Thus A is not saturated.

The following characterization of saturated formations provides insight into their properties as well as a list of useful criteria.

2.1.9 THEOREM:

Let F be a formation and G a finite solvable group. Then, the following are equivalent:

(i) F is saturated;

(ii) if $G \notin F$ and M is a minimal normal subgroup of G such that $G/M \in F$, then M has a complement and all such complements are conjugate in G; and

(iii) if $G \notin F$, M is a minimal normal subgroup of G, and $G/M \in F$, then M has a complement in G.

Proof:

(i) $\Rightarrow$ (ii): Suppose F is saturated, $G \notin F$ and M is a minimal normal subgroup of G and $G/M \in F$. We need to show that M has a complement and any two complements of M are conjugate.

Since F is saturated and $G \notin F$, we conclude that $G/\phi(G) \notin F$.

If $\phi(G) \neq \langle 1 \rangle$, the result follows by induction. For, since $G/M \in F$ it follows that $M \not\leq \phi(G)$ and hence $M \cap \phi(G) = \langle 1 \rangle$. Then $M\phi(G)/\phi(G)$ is a minimal normal subgroup of $G/\phi(G)$ and so it has a complement $L/\phi(G)$. It follows that $G = ML$ and $M \cap L = \langle 1 \rangle$. Also, any two complements of M must be maximal subgroups and hence, again by induction, are conjugate in G.

So we can assume $\phi(G) = \langle 1 \rangle$. Hence, there exists a maximal subgroup H of G such that $G = MH$. Now $M \cap H \trianglelefteq MH = G$, as M is abelian, and so $M \cap H = \langle 1 \rangle$ by minimality of M and so H is a complement of M in G. It only remains to show that if K is another complement of M in G, then H and K are conjugate.

By induction we may assume that M is the unique minimal normal subgroup of G. For if N is another minimal normal subgroup of G then $M \cap N = \langle 1 \rangle$. It follows that $G/N \notin F$ (otherwise $G/N, G/M \in F$ implies $G \in F$) and that $N \leq H$ and $N \leq K$, for if $N \not\leq H$ then by maximality of H, $NH = G$ and hence $G/N \cong H/H \cap N \in F$, which was just seen to be false, so $N \leq H$. Similarly, $N \leq K$. Thus H/N and K/N are complements of MN/N in G/N. It follows (by induction) that H/N and K/N are conjugate in G/N and hence that H and K are conjugate in G.

So since M is the unique minimal normal subgroup of G and $\phi(G) = \langle 1 \rangle$, then by (1.4.8) we have $M = F(G)$, and by (1.4.9), $C_G(M) = M$. Suppose $|M| = p^k$ for some prime p. Then we conclude that $O_p(G/M) = \langle 1 \rangle$ (otherwise, the inverse image of $O_p(G/M)$ in G is a normal p-group strictly containing $M = F(G)$, a contradiction), and hence that $O_{p'}(G/M) \neq \langle 1 \rangle$.

Let R/M be a minimal normal subgroup of G/M contained in $O_{p'}(G/M)$. It follows that $(|R/M|, |M|) = 1$. Now R/M is an elementary abelian q-group, $q \neq p$, q a prime. Now $R = M(R \cap H) = M(R \cap K)$ so we conclude that $R \cap H$ and $R \cap K$ are Sylow q-subgroups of R. Hence there exists $x \in M$ so that $R \cap H = (R \cap K)^x = R \cap K^x$. As $R \trianglelefteq G$, we have $R \cap H \trianglelefteq H$ and $R \cap H = (R \cap K)^x \trianglelefteq K^x$. Hence $N_G(R \cap H) \geq H, K^x$.

Since M is the unique minimal normal subgroup of G, it follows that $R \cap H$ is not normal in G and that $\langle H, K^x \rangle$ is not all of G. As H and K^x are maximal subgroups of G, we conclude that $H = K^x$ as desired.

(ii) $\Rightarrow$ (iii): is trivial.

(iii) $\Rightarrow$ (i): Suppose (iii) is satisfied and $G/\phi(G) \in F$. Assume G is minimal with this property, subject to $G \notin F$. Then for any minimal normal subgroup M of G with $M \leq \phi(G)$ we have $G/\phi(G) \cong (G/M)/(\phi(G)/M) = (G/M)/\phi(G/M) \in F$ and so by minimality of G we have $G/M \in F$. (Thus we have a minimal normal subgroup M of G with $G/M \in F$ and $G \notin F$). Therefore by (iii) M has a complement in G, contracting $M \leq \phi(G)$. □

In 1963 Gaschütz published a far-reaching generalization of Hall and Carter subgroups.

2.1.10 DEFINITION:

Let F be a class of finite groups. A subgroup F of a group G is called an F-covering subgroup of G if:

(i) $F \in F$

(ii) If $F \leq H \leq G$ and $N \trianglelefteq H$ such that $H/N \in F$, then $H = FN$.

2.1.11 REMARK:

If F is a non-empty formation, then by definition of G_F, the F-residual of G, we have for any $N \trianglelefteq G$ with $G/N \in F$, $G_F \leq N$. It follows that a subgroup F of G so that $F \in F$ is an F-covering subgroup of G iff $F \leq H \leq G$ implies that $H = FH_{\mathcal{F}}$, where $H_{\mathcal{F}}$ is the $\mathcal{F}$-residual of H.

2.1.12 EXAMPLE:

Consider $\underline{\pi}$, the class of finite solvable π-groups, where π is a set of primes. Take F to be a Hall π-subgroup of the finite solvable group G. Then $F \in \underline{\pi}$. Moreover, if $F \leq H \leq G$ and $H_1 \trianglelefteq H$ such that $H/H_1 \in \underline{\pi}$, then clearly $(|H:H_1|, |H:F|) = 1$ and so by (1.1.2), $H = FH_1$. Thus Hall π-subgroups of G are $\underline{\pi}$-covering subgroups.

In particular, taking $\pi = \{p\}$ we see Sylow p-subgroups are the $\mathcal{P}$-covering subgroups, where $\mathcal{P}$, as before, is the class of finite p-groups.

2.1.13 LEMMA:

Let $\mathcal{F}$ be a homomorph and let F be an $\mathcal{F}$-covering subgroup of the finite group G. Then if $F \leq H \leq G$, we must have that F is an $\mathcal{F}$-covering subgroup of H.

Proof:

Clear from the definition. ◻

2.1.14 LEMMA:

Let $\mathcal{F}$ be a homomorph and F be an $\mathcal{F}$-covering subgroup of the finite group G, and suppose that $N \trianglelefteq G$. Then FN/N is an $\mathcal{F}$-covering subgroup of G/N.

Proof:

First we notice that $FN/N \in \mathcal{F}$ as $FN/N \cong F/F \cap N \in \mathcal{F}$, Since F is a homomorph. Now, suppose that $FN/N \leq H/N \leq G/N$ and $(H/N)/(M/N) \in \mathcal{F}$. Then $H/M \in \mathcal{F}$ and so as $F \leq H \leq G$ we get that $H = MF$ from the fact that F is an $\mathcal{F}$-covering subgroup of G. Hence it follows that $H/N = MF/N = (M/N)(FN/N)$ and consequently FN/N is an $\mathcal{F}$-covering subgroup of G/N. □

2.1.15 <u>REMARKS</u>:

(i) It follows trivially from (2.1.10) that if F is an $\mathcal{F}$-covering subgroup of $G, F \leq H \leq G$ and $H \in \mathcal{F}$, then $H = F$, which means that $\mathcal{F}$-covering subgroups are maximal $\mathcal{F}$-subgroups.

(ii) It also follows that if $G \in \mathcal{F}$, then G is its own $\mathcal{F}$-covering subgroup.

The next lemma allows us to pull an $\mathcal{F}$-covering subgroup of a factor group of G back to an $\mathcal{F}$-covering subgroup of G.

2.1.16 <u>LEMMA</u>:

Let $\mathcal{F}$ be a homomorph and N be a normal subgroup of the finite group G. Suppose $\overline{F}/N$ is an $\mathcal{F}$-covering subgroup of G/N. If F is an $\mathcal{F}$-covering subgroup of $\overline{F}$. Then F is an $\mathcal{F}$-covering subgroup of G.

<u>Proof</u>:

Clearly $F \in \mathcal{F}$. Suppose that $F \leq H \leq G$ and $H/M \in \mathcal{F}$. Since F is an $\mathcal{F}$-covering subgroup of $\overline{F}$ and $\overline{F}/N \in \mathcal{F}$, it follows that $\overline{F} = FN$.

Since $F \leq H$, $\overline{F} \leq HN$ and so $\overline{F}/N \leq HN/N$. Hence by 2.1.13, $\overline{F}/N$ is an $\mathcal{F}$-covering subgroup of HN/N. In view of the isomorphism

$H/H \cap N \cong HN/N$ we find that $(\bar{F} \cap H)/(H \cap N)$ is an F-covering subgroup of $H/H \cap N$; indeed: $(\bar{F} \cap H)/(H \cap N) = (FN \cap H)/(H \cap N) = F(H \cap N)/(H \cap N) \cong F/(F \cap H \cap N) = F/(F \cap N) \cong FN/N = \bar{F}/N$. By 2.1.13, F is an F-covering subgroup of $H \cap \bar{F}$. Now if $H < G$, it would follow by induction that F is an F-covering subgroup of H. As $H/M \in F$, this would give that $H = MF$ as required.

Thus we suppose that $H = G$ and hence $G/M \in F$. Now $\bar{F}/N$ is an F-covering subgroup of G/N and $(G/N)/(MN/N) \cong G/MN \cong (G/M)/(MN/M) = (H/M)/(MN/M) \in F$, and so $G/N = (\bar{F}/N)(MN/N)$. Thus $G = \bar{F}(MN) = \bar{F}M$.

Now $\bar{F}/\bar{F} \cap M \cong \bar{F}M/M = G/M \in F$ so $\bar{F} = F(\bar{F} \cap M)$. Thus $G = \bar{F}M = F(\bar{F} \cap M)M = FM$ as required. □

Now we come to the fundamental theorem on F-covering subgroups which yields numerous families of conjugate subgroups in a finite solvable group.

2.1.17 <u>THEOREM</u>:

Let F be a formation.

(i) If every finite group has an F-covering subgroup, then F is saturated.

(ii) If F is saturated, then every finite solvable group contains an F-covering subgroup and all of its F-covering subgroups are conjugate (in the group).

<u>Proof</u>:

(i) Let G be a finite group such that $G/\phi(G) \in F$. If F is an F-covering subgroup of G, then $G = F\phi(G)$, which implies that $G = F$ by by the non-generator property of $\phi(G)$. Thus $G \in F$ and so F is saturated.

(ii) This part will be established by induction on $|G|$. If $G \in F$, then G is evidently the only F-covering subgroup, so we shall exclude this case. So suppose $G \notin F$, and choose a minimal normal subgroup N of G. Then by induction G/N has an F-covering subgroup $\bar{F}/N$. We consider two cases.

Case 1: Suppose first that $G/N \notin F$, so that $\bar{F} \neq G$. By induction $\bar{F}$ has an F-covering subgroup F. We deduce directly from (2.1.16) that F is an F-covering subgroup of G. Now let F_1, F_2 be two F-covering subgroups of G. By 2.1.14, the subgroups F_1N/N and F_2N/N are F-covering subgroups of G/N, whence they are conjugate, say $F_1N = F_2^gN$ where $g \in G$. Now $F_1N \neq G$ because $G/N \notin F$. Hence F_1 and F_2^g, as F-covering subgroups of F_1N, are conjugate by induction, which implies that F_1 and F_2 are conjugate.

Case 2: Assume now that $G/N \in F$. Then 2.1.9 shows that there is a complement K of N in G. Moreover, since N is a minimal normal subgroup of G, K must be maximal in G. Since $G/N \in F$, we have $N = G_F$ and so $G = KG_F$. Therefore by Remark (2.1.11) K is an F-covering subgroup of G. (Notice that $K \in F$ as $K \cong G/N \in F$). If H is another such group, then $G = HN$, while $H \cap N = \langle 1 \rangle$, since N is abelian. Applying (2.1.9) again, we conclude that H and K are conjugate. ◫

The following lemma is simple but useful application of Theorem 2.1.17.

2.1.18 LEMMA:

Let F be a saturated formation and G a finite solvable group. If $N \trianglelefteq G$, then each F-covering subgroup of G/N has the form FN/N where F is an F-covering subgroup of G.

Proof:

Let $\overline{F}/N$ be an F-covering subgroup of G/N and let F_1 be an F-covering subgroup of G. Then by 2.1.14, F_1N/N is an F-covering subgroup of G/N. So by 2.1.17, F_1N/N is conjugate to $\overline{F}/N$. Hence $\overline{F} = (F_1N)^g$ for some $g \in G$. Define F to be F_1^g, then we have $\overline{F} = FN$. □

Before going any further we want to refer again to some examples of formations we looked at previously.

2.1.19 EXAMPLES:

(i) We saw in Example 2.1.8(c) that A, the class of finite abelian groups, does not form a saturated formation. It is easy to see that Q_8, the quaternion group of order 8, has no A-covering subgroups. In fact, no p-group in which $G' = \phi(G) \neq \langle 1\rangle$ can have an A-covering subgroup. For if F were such a covering subgroup then $\phi(G) = G'$ implies that $G/\phi(G)$ is abelian which implies that $F\phi(G) = G$ and hence $F = G$ by the non-generator property of $\phi(G)$. Thus we have $G \in A$, a contradiction. Q_8 is one such group.

On the other hand, S_3 is an example of a group which does have A-covering subgroups; these are the subgroups of order 2, and notice that they are all conjugate.

(ii) Let T, as before, be the class of finite supersolvable groups. We have seen in Example 2.1.8(b) that T is a saturated formation, so by Theorem 2.1.17(ii) if G is any finite solvable group then G must have T-covering subgroups. We will show that the T-covering subgroups of G are precisely the supersolvable subgroups F of G so that for every pair of subgroups H,K of G such that $F \leq H \leq K \leq G$, the index $|K:H|$ is not a prime.

First suppose that F is a T-covering subgroup of G and that $F \leq H \leq K \leq G$. Suppose that $|K:H| = p$, a prime. Then H is a maximal subgroup of K. Let $N = \mathrm{Core}_K(H)$ (i.e., N is the largest normal subgroup of K which is contained in H) and let A/N be a maximal abelian normal subgroup of K/N. Then $A \cap H \trianglelefteq K$ so $A \cap H \leq N$. Hence, as $N \leq A \cap H$, we have $N = A \cap H$. Now as H is a maximal subgroup of K, $K = HA$. Thus $|K| = |H||A|/|H \cap A| = |H|\cdot|A|/|N|$. It follows that $|A:N| = |K:H| = p$. It follows that AF/N is supersolvable $((AF/N)/(A/N) \cong AF/A \cong F/F \cap A$ is supersolvable, so AF/N is supersolvable by 1.2.14). Hence $AF = NF$, a contradiction. This establishes that all T-covering subgroups satisfy the above condition.

Now suppose F is a supersolvable subgroup of G satisfying the given condition. Let $F \leq H \leq G$ and suppose $H/N \in T$. Now if $NF \neq H$, then there would exist a maximal subgroup containing NF whose index in H would be a prime, contrary to assumption.

2.1.20 DEFINITIONS:

Let F be a class of finite groups and G any finite group.

(a) A subgroup H of G is called <u>F-maximal</u> if:

(i) $H \in F$.

(ii) $H \leq H_1 \leq G$, $H_1 \in F$ implies that $H = H_1$.

(b) A subgroup H of G is an F-projector of G if for any $N \trianglelefteq G$, HN/N is F-maximal in G/N.

2.1.21 LEMMA:

Let F be a homomorph and let G be a group.

(a) If H is an F-projector of G, then for any $x \in G$, H^x is an F-projector of G.

(b) If H is an F-projector of G and if $N \trianglelefteq G$, then HN/N is an F-projector of G/N.

(c) If $\overline{F}/N$ is an F-projector of G/N and if F is an F-projector of $\overline{F}$, then F is an F-projector of G.

Proof:

(a) and (b) are clear from the definition.

(c) Let M be any normal subgroup of G. Suppose $FM/M \leq H/M$ and $H/M \in F$. Since $\overline{F}MN/MN$ is an F-projector of G/MN and $HN/MN \cong (HN/M)/(MN/M) = ((H/M)(MN/M))/(MN/M) \cong (H/M)/((H/M) \cap (MN/M)) \in F$ (as F is a homomorph and $H/M \in F$), we have $\overline{F}MN/MN = HN/MN$ which implies that $HN = \overline{F}MN = \overline{F}M$. Thus $H \leq \overline{F}M$ and hence $H/M \leq \overline{F}M/M$.

Since F is an F-projector of $\overline{F}$, FM/M is F-maximal in $\overline{F}M/M$ by definition of F-projector. Thus $FM/M = H/M$ which means that F is an F-projector of G. □

The following lemma is useful for proofs by induction.

2.1.22 <u>LEMMA</u>:

Let F be a homomorph and G a group. A subgroup H of G is an F-projector of G iff H is F-maximal in G and HM/M is an F-projector of G/M for any minimal normal subgroup M of G.

<u>Proof</u>:

Suppose H is an F-projector of G. Since $\langle 1\rangle \trianglelefteq G$, H must be an F-maximal in G. Moreover, if M is a minimal normal subgroup of G, then by 2.1.21(b), HM/M is an F-projector of G/M.

On the other hand, suppose H is an F-maximal subgroup of G and HM/M is an F-projector of G/M for any minimal normal subgroup M of G. If $N \trianglelefteq G$ then there is a minimal normal subgroup M of G such that $M \leq N$. If $HN \leq H_1 \leq G$ with $H_1/N \in F$, then $(H_1/M)/(N/M) \in F$ and $(HM/M)(N/M)/(N/M) = (HN/M)/(N/M) \leq (H_1/M)/(N/M)$. But by definition of projectors, $(HM/M)(N/M)/(N/M)$ is F-maximal in $(G/M)/(N/M)$, so $HN = H_1$ and $HN/N = H_1/N \in F$. Thus H is an F-projector of G. □

2.1.23 <u>REMARKS AND EXAMPLES</u>:

(a) Let F be a homomorph and G a group. Then every F-covering subgroup of G is an F-projector of G. For if H is an F-covering subgroup of G and $H \leq H_1 \leq G$ such that $H_1/N \in F$. Then by definition of covering subgroups, $HN = H_1$ and so $HN/N = H_1/N \in F$. Therefore H is an F-projector of G.

For instance, Hall π-subgroups and Sylow p-subgroups are $\underline{\pi}$- and P-projectors respectively (see Example 2.1.12). Also from Example

2.1.19(ii), the supersolvable subgroups F of the finite solvable group G so that for every pair of subgroups H, K with $F \leq H \leq K \leq G$, the index $|K:H|$ is not a prime, are the $\mathcal{T}$-projectors.

(b) For a prime p, a non-abelian p-group G has no A-projectors. For if H were an A-projector, then by 2.1.22, H would be abelian and $HG'/G' = G/G'$, that is $HG' = G$. But since $G' \leq \phi(G)$ for any p-group G, it would then follow that $H = G$, in contradiction to the assumption that G is non-abelian.

(c) Consider the simple group A_5. A_5 has a subgroup $E \cong V$, the Klein four-group. In fact, $E \trianglelefteq A_4 \leq A_5$. $E \in \mathcal{N}$, since $V \in \mathcal{N}$, where $\mathcal{N}$ is the class of finite nilpotent groups. Also it is clear that EN/N is an $\mathcal{N}$-maximal in A_5/N for any $N \trianglelefteq A_5$. Thus E is an $\mathcal{N}$-projector of A_5.

On the other hand, E is not an $\mathcal{N}$-covering subgroup of A_5, since $E \trianglelefteq A_4 \leq A_5$ and $A_4/E \in \mathcal{N}$ but $E = EE \neq A_4$.

A_5 also has a subgroup $F \cong C_5$ and $F \trianglelefteq D \leq A_5$ where $D \cong D_{10}$. Similar argument shows that F is an $\mathcal{N}$-projector of A_5, and F is not an $\mathcal{N}$-covering subgroup of A_5.

Thus, the notion of projectors is more general than the notion of covering subgroups, therefore poorer in properties. However, the advantage of projectors over covering subgroups, apart from their greater generality, lies in the fact that they possess good duals (namely the injectors, which lie out of the scope of this research). Also there is a close connection between covering subgroups and projectors as we will see.

2.1.24 LEMMA:

If F is a homomorph and G a group, then the subgroup H of G is an F-covering subgroup of G iff H is an F-projector of K for all $H \leq K \leq G$.

Proof:

Let H be an F-covering subgroup of G and K a subgroup of G with $H \leq K$. By 2.1.13, H is an F-covering subgroup of K and by 2.1.23(a), H is an F-projector of K.

Conversely, let H be a subgroup of G which is an F-projector of any subgroup K of G with $H \leq K$. We have $H \in F$. Let given K and L, with $H \leq K \leq G$, $L \trianglelefteq K$, $K/L \in F$. H is an F-projector of K, so that HL/L is F-maximal in K/L. But $K/L \in F$. It follows that $HL/L = K/L$, hence $K = HL$. □

The following auxiliary lemma is due to Carter and Hawkes.

2.1.25 LEMMA:

Let F be a saturated formation and G a finite solvable group. If H is an F-subgroup of G such that $G = HF(G)$, then H is contained in an F-covering subgroup of G.

Proof:

We will argue by induction on $|G|$. We may assume $G \notin F$ (For otherwise G is evidently the F-covering subgroup which contains H). Let N be a minimal normal subgroup of G. Then HN/N inherits the hypothesis on H, so by induction it is contained in some F-covering subgroup K^*/N of

G/N. By 2.1.18, then K^*/N will be of the form KN/N where K is an $\mathcal{F}$-covering subgroup of G. Consequently $H \leq KN$. Now we consider two cases:

Case (i): $KN < G$. Then by the induction hypothesis, H is contained in some $\mathcal{F}$-covering subgroup M of KN. But as K is an $\mathcal{F}$-covering subgroup of G, it is $\mathcal{F}$-covering subgroup of KN and so it must be conjugate to M. This shows M to be an $\mathcal{F}$-covering subgroup of G.

Case (ii): $KN = G$. Let $F = F(G)$. Since N is a minimal normal subgroup of the solvable group G, N is abelian and so nilpotent and therefore contained in F. Indeed $N \leq Z(F)$ because $\langle 1\rangle \neq N \cap Z(F) \trianglelefteq G$ (see 1.2.5). Therefore $K \cap F \trianglelefteq KN = G$. Now, if $K \cap F \neq \langle 1\rangle$, we can apply the induction hypothesis to $G/K \cap F$, concluding that $H \leq T$ where $T/K \cap F$ is an $\mathcal{F}$-covering subgroup of $G/K \cap F$. Now $T < G$ (For otherwise $T = G$ would imply that $G/K \cap F = T/K \cap F \in \mathcal{F}$ and since K is an $\mathcal{F}$-covering subgroup of G, we must have $G = K(K \cap F) = K \in \mathcal{F}$, a contradiction). And so again by induction there is an $\mathcal{F}$-covering subgroup R of T containing H. But then by 2.1.16, R is an $\mathcal{F}$-covering subgroup of G.

Consequently we can assume that $K \cap F = \langle 1\rangle$. Hence $F = F \cap G = F \cap KN = N(F \cap K) = N$. So from the hypothesis of the lemma we have $G = HF = HN$. Also we notice that H is a maximal in G: for $H \neq G$ since $G \notin \mathcal{F}$. Finally $G_{\mathcal{F}} = N$, since $G/N \in \mathcal{F}$. Hence $G = HG_{\mathcal{F}}$. As G is the only subgroup of G that properly contains H (because of maximality of H), then by Remark 2.1.11, H itself is an $\mathcal{F}$-covering subgroup of G. ▥

2.1.26 THEOREM:

If F is a saturated formation and G a finite solvable group, then every F-projector of G is an F-covering subgroup of G.

Proof:

We shall argue by induction on $|G|$. Assume that H is an F-projector of G and let N be a minimal normal subgroup of G. Then HN/N is an F-projector of G/N. By induction hypothesis HN/N is an F-covering subgroup of G/N. Put $M = HN$. By 2.1.18 we can write $M = H^*N$ where H^* is an F-covering subgroup of G. Since N is abelian (so it is nilpotent), it contained in $F(M)$, and so $M = HN = HF(M) = H^*F(M)$. By 2.1.25, there is an F-covering subgroup $\overline{H}$ of M containing H. But H is F-maximal in G by 2.1.22, so $\overline{H} = H$, and therefore H is an F-covering subgroup of M. Also H^* is an F-covering subgroup of M since it is an F-covering subgroup of G. So it follows from 2.1.17(ii) that H and H^* are conjugate in M. Obviously this shows that H is an F-covering subgroup of G. □

The following theorem is analogous to 2.1.17, the fundamental theorem on covering subgroups.

2.1.27 THEOREM:

Let F be a formation.

(i) If every finite group has an F-projector, then F is saturated.

(ii) If F is saturated, then every finite solvable group possesses F-projectors and any two of these are conjugate (in the group).

Proof:

(i) Suppose that every finite group has an F-projector, and let G be a finite group such that $G/\phi(G) \in F$. If H is an F-projector of G, then $H\phi(G)/\phi(G)$ is an F-maximal in $G/\phi(G)$. Since $G/\phi(G) \in F$ and $H\phi(G)/\phi(G) \leq G/\phi(G)$, we must have $H\phi(G)/\phi(G) = G/\phi(G)$ which implies that $H\phi(G) = G$, which implies that $G = H \in F$ by the non-generator property of $\phi(G)$.

(ii) Follows from 2.1.17(ii) and 2.1.23(a) and 2.1.26. ◫

As we have seen in Theorem 2.1.26, there is a close connection between F-covering subgroups and F-projectors. The most important instance of this theory is when $F = N$, the class of finite nilpotent groups. Since F is saturated (see Example 2.1.8(b), the F-covering subgroups and F-projectors coincide and form a single conjugacy class of nilpotent self-normalizing (i.e. Carter) subgroups in any finite solvable group.

2.1.28 THEOREM (Carter):

Let G be a finite solvable group. Then the Carter subgroups of G are the covering subgroups (or projectors) for the formation of finite nilpotent groups.

Proof:

Let H be an N-covering subgroup of G, where N is the formation of finite nilpotent groups. Suppose $H < N_G(H)$. Then there is a subgroup K such that $H \trianglelefteq K$ and K/H has prime order. Now $K = HK_N$. However $K_N \leq H$ so $K = H$ a contradiction which shows that $H = N_G(H)$. Since $H \in N$, it is a Carter subgroup of G.

Conversely, suppose H is a Carter subgroup of G and $H \leq H_1 \leq G$ with $H_1/N \in \mathcal{N}$. Then by 1.6.3, HN/N is a Carter subgroup of H_1/N. But since H_1/N is nilpotent, its only self-normalizing subgroup is H_1/N itself (see 1.2.3). Thus $H_1 = HN$ as required. ◻

2.2 PULL-BACKS FOR COVERING SUBGROUPS (PROJECTORS):

In this section, we show that "Pull-backs" exist for the F-covering subgroups (or F-projectors) associated with a saturated formation F of finite solvable groups, and we will use this to construct several classes of formations. We first prove a number of preliminary results.

2.2.1 LEMMA:

If F is a formation, if F is an F-projector of a finite (not necessarily solvable) group G, and if M,N are normal subgroups of G, then $FM \cap FN = F(M \cap N)$.

Proof:

Let $L = FM \cap FN$. Then $F(M \cap N) \leq L$, $LM = FM$, $LN = FN$, $L/L \cap M \cong LM/M = FM/M \in F$ and $L/L \cap N \cong LN/N = FN/N \in F$. Hence $L/M \cap N = L/(L \cap M \cap N) \in F$. But $F(M \cap N)/(M \cap N)$ is F-maximal in $G/M \cap N$, so that $L = F(M \cap N)$. This completes the proof. □

2.2.2 LEMMA (Huppert):

Suppose that F is a saturated formation, and that F is an F-covering subgroup of G. Then for all normal subgroups N_1, N_2 of G, $F \cap N_1N_2 = (F \cap N_1)(F \cap N_2)$.

Proof:

We will argue by induction on $|G| > 1$. Put $M = N_1 \cap N_2 \trianglelefteq G$. Then by 2.1.14, FM/M is an F-covering subgroup of G/M. If $M \neq \langle 1 \rangle$, then by induction we have $(N_1N_2M/M) \cap (FM/M) = ((N_1M/M) \cap (FM/M))((N_2M/M) \cap (FM/M))$ which implies $N_1N_2M \cap FM = (N_1M \cap FM)(N_2M \cap FM)$ and so we have

$$N_1N_2 \cap F \leq (N_1N_2M \cap FM) \cap F = (N_1M \cap FM)(N_2M \cap FM) \cap F$$

$$= [M(N_1M \cap F)][M(N_2M \cap F)] \cap F \quad \text{by the modular law}$$

$$= (N_1M \cap F)(N_2M \cap F)M \cap F$$

$$= (N_1M \cap F)(N_2M \cap F)(M \cap F) \quad \text{by the modular law again}$$

since $(N_1M \cap F)(N_2M \cap F) \leq F$. Thus we have

$$N_1N_2 \cap F \leq (N_1M \cap F)(N_2M \cap F)(M \cap F) \quad ......(*).$$

Substituting for $M = N_1 \cap N_2$ in (*) we get

$$N_1N_2 \cap F \leq (N_1 \cap F)(N_2 \cap F)(N_1 \cap N_2 \cap F) = (N_1 \cap F)(N_2 \cap F).$$

But clearly $(N_1 \cap F)(N_2 \cap F) \leq N_1N_2 \cap F$ so the equality holds in this case.

So we now consider the case when $N_1 \cap N_2 = \langle 1 \rangle$: Since F is an $\mathcal{F}$-covering subgroup of G, F is an $\mathcal{F}$-covering subgroup of FN_1 by 2.1.13. So if $N_1F < G$ then by induction we have:

$$N_1N_2 \cap F = (N_1N_2 \cap N_1F) \cap F = N_1(N_2 \cap N_1F) \cap F =$$

$$= (N_1 \cap F)(N_2 \cap N_1F \cap F) = (N_1 \cap F)(N_2 \cap F).$$

Similarly if $N_2F < G$. So we may assume now that $N_1F = N_2F = G$. But then $G/N_i = FN_i/N_i \cong F/N_i \cap F \in \mathcal{F}$ for $i = 1,2$, and consequently $G/N_1 \cap N_2 \in \mathcal{F}$, since $\mathcal{F}$ is a formation. Since $N_1 \cap N_2 = \langle 1 \rangle$, $G \in \mathcal{F}$, and so $F = G$ by 2.1.17(ii). The conclusion then is trivial. □

2.2.3 LEMMA:

Denote by F_1F_2 the class of all groups which are extensions of groups in F_1 by groups in F_2, and let F_1, F_2 be saturated formations. Suppose that the order of each group in F_1 is coprime to the order of each group in F_2. Then F_1F_2 is a formation. If F_1 is an F_1-covering subgroup of G, and F_2 is an F_2-covering subgroup of $N_G(F_1)$, then F_1F_2 is an F_1F_2-covering subgroup of G.

Proof:

Let $G \in F_1F_2$ and $N \trianglelefteq G$. Since $G \in F_1F_2$, then for some normal subgroup M of G we have $M \in F_1$ and $G/M \in F_2$. Now $G/MN \cong (G/M)/(MN/M) \in F_2$, since F_2 is a formation; and $MN/N \cong M/M \cap N \in F_1$, since F_1 is a formation; and so $G/N \in F_1F_2$. Now suppose G/M, $G/N \in F_1F_2$ where $M,N \trianglelefteq G$. Then for some normal subgroups R, S of G we have G/R, $G/S \in F_2$ while R/M, $S/N \in F_1$. Let π be the set of primes dividing $|R/M|$ or $|S/N|$, and let H be a Hall π-subgroup of G. Then by the hypothesis on F_1 and F_2, $H \leq R \cap S$, and $HM = R$, $HN = S$. Thus $H/H \cap M \cong R/M \in F_1$ and $H/H \cap N \cong S/N \in F_1$, and so $H/(H \cap M \cap N) \in F_1$. Now $R \cap S = HM \cap HN = H(M \cap N)$ by 2.2.1, since H is $\underline{\pi}$-projector of G (see 2.1.23(a)). So $R \cap S/M \cap N \cong H/(H \cap M \cap N) \in F_1$. Since $G/R \cap S \in F_2$ we have $G/M \cap N \in F_1F_2$ as desired.

We consider the second conclusion. Certainly $F_1F_2 \in F_1F_2$. Now suppose that $F_1F_2 \leq V$, and $V/N \in F_1F_2$. We want to show that $F_1F_2N = V$. Since $V/N \in F_1F_2$, there is a normal subgroup K of V so that $V/K \in F_2$ and $K/N \in F_1$. Since $(|V/K|,|F_1|) = 1$, $F_1 \leq K$, and therefore F_1 is an F_1-covering subgroup of K. Thus $K = F_1N$. Since any two V-conjugates of

F_1 are necessarily K-conjugate, it follows that for any $v \in V$, $F_1^v = F_1^k$ for some $k \in K$, and so $F_1^{vk^{-1}} = F_1$ or $vk^{-1} \in N_V(F_1)$; that is $v \in N_V(F_1)K$ which means that $V = N_V(F_1) \cdot K$. Therefore $V/K \cong N_V(F_1)/N_V(F_1) \cap K \in \mathcal{F}_2$ as $V/K \in \mathcal{F}_2$. Since $F \leq V \cap N_G(F_1) = N_V(F_1)$ and F_2 is an $\mathcal{F}$-covering subgroup of $N_G(F_1)$, it follows that $N_V(F_1) = F_2(N_V(F_1) \cap K)$; hence $V = KN_V(F_1) = KF_2 = F_1F_2N$. □

Now we come to the main theorem in this section.

2.2.4 <u>THEOREM</u>:

Let $\mathcal{F}$ be a saturated formation. Let G be a finite solvable group with normal subgroups M and N, and suppose $M \cap N = \langle 1 \rangle$. If an $\mathcal{F}$-covering subgroup of G/M and an $\mathcal{F}$-covering subgroup of G/N have the same image in G/MN then they are both homomorphic images of some $\mathcal{F}$-covering subgroup of G.

<u>Proof</u>:

Let $\overline{F}_1/M$ and $\overline{F}_2/N$ be $\mathcal{F}$-covering subgroups of G/M and G/N respectively such that $\overline{F}_1/M$ and $\overline{F}_2/N$ have the same image in G/MN. By 2.1.18, $\overline{F}_1/M = F_1M/M$ and $\overline{F}_2/N = F_2N/N$ where F_1, F_2 are $\mathcal{F}$-covering subgroups of G. Thus we may suppose that there are $\mathcal{F}$-covering subgroups F_1, F_2 of G such that $F_1MN = F_2MN$. We must find an $\mathcal{F}$-covering subgroup F of G so that $FM = F_1M$ and $FN = F_2N$.

If G is a minimal counterexample to the theorem, then we claim that $G = F_1MN$ and M, N are minimal normal subgroups of G. To show this, suppose $F_1MN < G$. Then F_1, F_2 are $\mathcal{F}$-covering subgroups of $F_1MN = F_2MN$, and so by minimality of G there is an F-covering subgroup F of F_1MN such that $FM = F_1M$ and $FN = F_2N$. But since F is an F-covering subgroup of F_1MN and F_1MN/MN is an F-covering subgroup of G/MN, by 2.1.16, F must be an F-covering subgroup of G, a contradiction, since G has no such F-covering subgroups.

Now we show that M, N are minimal in G. Suppose not, and assume that M_1, N_1 are minimal normal subgroups of G such that $\langle 1\rangle < M_1 < M$ and $\langle 1\rangle < N_1 < N$. Since G is a counterexample to the theorem, then for any F-covering subgroup F of G we have $FM \neq F_1M$ or $FN \neq F_2N$. We may assume $FM \neq F_1M$. Now consider the smaller group G/M_1 with its normal subgroups M/M_1 and NM_1/M_1, see Fig. 1.

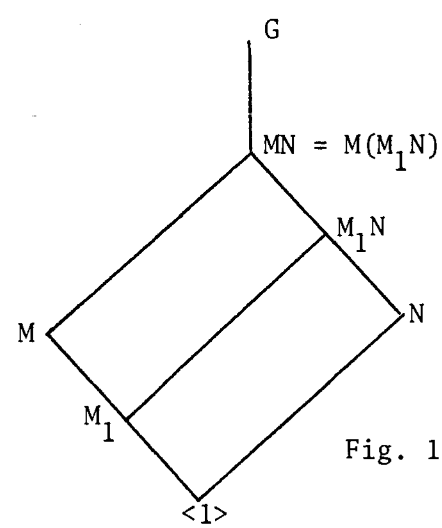

Fig. 1

Clearly $(M/M_1) \cap (NM_1/M_1) = \langle 1\rangle$. Now since F_1 is an F-covering subgroup of G, F_1M_1/M_1 is an F-covering subgroup of G/M_1 and therefore $(F_1M_1/M_1)(M/M_1)/(M/M_1) = (F_1M/M_1)/(M/M_1)$ is an F-covering subgroup of $(G/M_1)/(M/M_1)$. Similarly F_2 is an F-covering subgroup of G implies that

F_2M_1/M_1 is an $\mathcal{F}$-covering subgroup of G/M_1 and hence $(F_2M_1/M_1)(NM_1/M_1)/(NM_1/M_1) = (F_2M_1N/M_1)/(NM_1/M_1)$ is an $\mathcal{F}$-covering subgroup of $(G/M_1)/(NM_1/M_1)$. Since $(F_1M/M_1)/(M/M_1) \cong F_1M/M$ and $(F_2NM_1/M_1)/(NM_1/M_1) \cong F_2M_1N/M_1N$ and clearly F_1M/M, F_2MN/M_1N have the same image in G/MN, we conclude that $(F_1M/M_1)/(M/M_1)$ and $(F_2M_1N/M_1)/(NM_1/M_1)$ have the same image in $(G/M_1)/(MN/M_1)$, and therefore there is an $\mathcal{F}$-covering subgroup of $\overline{F}/M_1$ of G/M_1 such that $(\overline{F}/M_1)(M/M_1) = (F_1M_1/M_1)(M/M_1)$ and $(\overline{F}/M_1)(NM_1/M_1) = (F_2M_1/M_1)(NM_1/M_1)$. But $\overline{F}/M_1 = FM_1/M_1$ by 2.1.18 where F is an $\mathcal{F}$-covering subgroup of G. So we have $(FM_1/M_1)(M/M_1) = (\overline{F}/M_1)(M/M_1) = (F_1M_1/M_1)(M/M_1)$, that is $FM/M_1 = F_1M/M_1$ which implies that $FM = F_1M$, a contradiction. So if G is a minimal counterexample to the theorem, then we must have $G = F_1MN$ and M, N are minimal normal subgroups of G, and therefore abelian. Hence a non-trivial intersection $F_1 \cap M$ or $F_1 \cap N$ would be a normal in G (see 1.3.2), and therefore F_1 must either contain or have trivial intersection with each of M and N. Suppose that $M \leq F_1$. Then $G/N = F_1N/N \in \mathcal{F}$, so that $F_2N = G$. Thus we may take $F = F_1$ and we finished, in this case. By symmetry, we may suppose that $M \cap F_1 = N \cap F_1 = \langle 1 \rangle$. Now by 2.2.2, $F_1 \cap MN = \langle 1 \rangle$, and we may suppose that similar results hold for F_2. Now let $F = F_1M \cap F_2N$. Then $|F| = |F_1M||F_2N|/|G| = |F_1|$ (notice that $G = F_1MN = F_2MN = F_1MF_2N$ and $F_1 \cap M = F_2 \cap N = \langle 1 \rangle$). Now $MF = M(F_1M \cap F_2N) = F_1M \cap G = F\ M$. This shows that F and F_1 are both complements of M in MF_1, and are therefore conjugate. In particular, F is an $\mathcal{F}$-covering subgroup of G. Since we can show in identical fashion that $NF = NF_2$, our proof is complete. □

Now we see how can we construct new formations by comparing covering subgroups (or projectors).

2.2.5 LEMMA:

Let F_1 and F_2 be saturated formations and let Y be the class of groups for which F_1-covering subgroups belong to F_2. Then Y is a formation.

Proof:

Let $G \in Y$, $N \trianglelefteq G$. Let H_1/N be an F_1-covering subgroup of G/N. Then by 2.1.18, $H_1/N = HN/N$ where H is an F_1-covering subgroup of G. Now H is an F_1-covering subgroup of G implies that $H \in F_2$ and hence $HN/N \in F_2$, since F_2 is a formation. Thus $H_1/N \in F_2$ and so $G/N \in Y$.

Now let G/N, $G/M \in Y$. We want to show that $G/M \cap N \in Y$. Let $H_1/M \cap N$ be an F_1-covering subgroup of $G/M \cap N$. Then again by 2.1.18, $H_1/M \cap N = H(M \cap N)/(M \cap N)$ where H is an F_1-covering subgroup of G. But then HN/N, HM/M are F_1-covering subgroups of G/N and G/M respectively. Since G/N, $G/M \in Y$ it follows that HN/N, $HM/M \in F_2$. Now, since $H/H \cap N \cong HN/N \in F_2$ and $H/H \cap M \cong HM/M \in F_2$, we have $H/(H \cap M \cap N) \in F_2$ as F_2 is a formation. But $H/(H \cap M \cap N) \cong H(M \cap N)/(M \cap N) = H^*/M \cap N$. Thus $H^*/M \cap N \in F_2$ and therefore $G/M \cap N \in Y$. □

As an application of the theorem 2.2.4, we offer a more delicate construction.

2.2.6 THEOREM:

Suppose that F_1 and F_2 are saturated formations; then the class of groups in which F_1-covering subgroups are subgroups of F_2-covering subgroups is a formation. In particular, the class of groups for which F_1- and F_2-covering subgroups coincide is a formation.

Proof:

The second statement is a direct consequence of the first. Suppose G is a member in the class of groups in which F_1-covering subgroups are subgroups of F_2-covering subgroups, and $N \trianglelefteq G$. If $\overline{H}/N$ is an F_1-covering subgroup of G/N then by 2.1.18, $\overline{H}/N = H_1N/N$ where H_1 is an F_1-covering subgroup of G, but then $H_1 \leq H_2$ where H_2 is an F_2-covering subgroup of G; hence $\overline{H}/N = H_1N/N \leq H_2N/N$ and by 2.1.14, H_2N/N is an F_2-covering subgroup of G/N. Thus this class is a homomorph. Now suppose that M and N are normal subgroups of G, $M \cap N = \langle 1 \rangle$, and in both G/M and G/N, F_1-covering subgroups are subgroups of F_2-covering subgroups. Then for an F_1-covering subgroup of F_1 of G, F_1M/M and F_1N/N are F_1-covering subgroups of G/M and G/N respectively. Furthermore, by lifting back from G/MN into G/M and G/N, and then applying Theorem 2.2.4, we may assume that for some F_2-covering subgroup F_2 of G, we have $F_1M \leq F_2M$ and $F_1N \leq F_2N$. Applying 2.2.1 we see that

$$F_1 = F_1(M \cap N) = F_1M \cap F_1N \leq F_2M \cap F_2N = F_2(M \cap N) = F_2;$$

our proof is complete. □

Several important formations arise in the manner of Theorem 2.2.6. For instance, groups in which Carter subgroups (i.e. nilpotent covering subgroups (projectors)) are Hall subgroups (i.e. $\underline{\pi}$-covering subgroups ($\underline{\pi}$-projectors)).

Chapter 3

PROJECTORS and SCHUNCK CLASSES

In this chapter all objects called groups are supposed to belong to the class of finite solvable groups unless explicitly stated otherwise.

3.1 DEFINITION:

Let G be any finite group. G is called a primitive if there is a maximal subgroup S of G with $\mathrm{Core}_G(S) = \langle 1\rangle$, where

$$\mathrm{Core}_G(S) = \bigcap\{S^g : g \in G\}.$$

The subgroup S is called a primitivator of G.

3.2 REMARKS and EXAMPLES

(i) If G is a group, S is a maximal subgroup of G, then $G/\mathrm{Core}_G(S)$ is obviously primitive with $S/\mathrm{Core}_G(S)$ as primitivator, and all the primitive factor groups of G are obtained this way.

(ii) From (i) it follows immediately that

$$\phi(G) = \bigcap\{\mathrm{Core}_G(S) : G/\mathrm{Core}_G(S) \text{ is primitive}\}$$
$$= \bigcap\{\mathrm{Core}_G(S) : S \text{ is maximal in } G\}.$$

(iii) If $H \trianglelefteq G$ and $HN = G$ for all primitive factor groups G/N of G, then $H = G$. For otherwise there would exist a maximal subgroup S of G with $H \leq S$, and this would imply that $HN \leq SN = S$ for $N = \mathrm{Core}_G(S)$, a contradiction.

(iv) $\langle 1 \rangle$ is not primitive.

(v) If G is a group with a prime order, then G is primitive and $\langle 1 \rangle$ is a primitivator of G.

(vi) Since in a nilpotent group every maximal subgroup is normal, a nilpotent group is primitive iff it has prime order.

3.3 <u>LEMMA</u>:

Let G be a primitive group, S a primitivator of G, $\langle 1 \rangle \neq H \trianglelefteq G$ and H is nilpotent. Then S is a complement of H in G.

<u>Proof</u>:

$S \cap H < H$ since $\mathrm{Core}_G(S) = \langle 1 \rangle$; therefore $S \cap H < N_H(S \cap H)$ because of the nilpotence of H (see 1.2.3). Hence $S < SN_H(S \cap H)$ (for otherwise $N_H(S \cap H) \leq S \cap H$). Since S is a maximal subgroup of G, we must have $SN_H(S \cap H) = G$. But $H \cap S \trianglelefteq S$ and $H \cap S \trianglelefteq N_H(H \cap S)$, so $H \cap S \trianglelefteq SN_H(S \cap H) = G$. Since $\mathrm{Core}_G(S) = \langle 1 \rangle$, we have $H \cap S = \langle 1 \rangle$.

$SH = G$ now follows immediately from $H \not\leq M$ and the maximality of M. $\square$

3.4 <u>COROLLARY</u>:

Let G be a primitive group. Then G has a unique non-trivial nilpotent normal subgroup N.

In particular, N is the unique minimal normal subgroup of G.

Proof:

Let S be a primitivator of G, N a minimal normal subgroup of G, and H a nilpotent normal subgroup of G. Since N is abelian, HN is a non-trivial nilpotent normal subgroup of G. From this, by 3.3 $|HN| = |G:S| = |N|$ so $N = NH$ and $H = \langle 1\rangle$ or $H = N$. □

3.5 LEMMA:

Let G be a primitive group, $S < G$, and N minimal normal subgroup of G. Then S is a primitivator of G iff $SN = G$.

Proof:

Suppose S is a primitivator of G. Then S is a maximal subgroup of G and $N \not\leq S$, since $Core_G(S) = \langle 1\rangle$. Hence $G = SN$.

Conversely, suppose $G = SN$. Then by 3.4, S contains no minimal normal subgroup of G; hence $Core_G(S) = \langle 1\rangle$. Let $S \leq S_1 < G$. $NS = G$ implies that $N \not\leq S_1$. Since N is abelian and $NS_1 = G$, $N \cap S_1 \trianglelefteq G$. Since $N \not\leq S_1$, $N \cap S_1 = \langle 1\rangle$ so $N \cap S = \langle 1\rangle$ and $S = S_1$, and therefore S is a maximal subgroup of G. □

3.6 LEMMA:

Let G be a group, N minimal normal subgroup of G, and S_1, S_2 primitivators of G. Then S_1 and S_2 are conjugate under N.

Proof:

If $G = N$, then $S_1 = S_2 = \langle 1\rangle$ by 3.3. Otherwise let L/N be a chief factor of G, $|N| = p^{\alpha}$, and $|L/N| = q^{\beta}$ where p and q are primes. Then by 3.4, L is not nilpotent, so $p \neq q$. Since S_1, S_2 are complements of N in G by 3.3, it follows that $S_i \cap L$ is a Sylow q-subgroup of L $(i = 1,2)$. But then $S_1 \cap L = (S_2 \cap L)^x$ by Sylow's theorem, where $x \in L$. Then $S_1 \cap L \trianglelefteq S_1$ and $S_1 \cap L = (S_2 \cap L)^x \leq S_2^x$. If $S_1 \neq S_2^x$, then $\langle S_1, S_2^x \rangle = G$ and $S_1 \cap L$ would be a non-trivial nilpotent normal subgroup of G, in contradiction to 3.4. ◫

3.7 THEOREM:

A group G is primitive iff there exists a minimal normal subgroup N of G such that $C_G(N) = N$.

Proof:

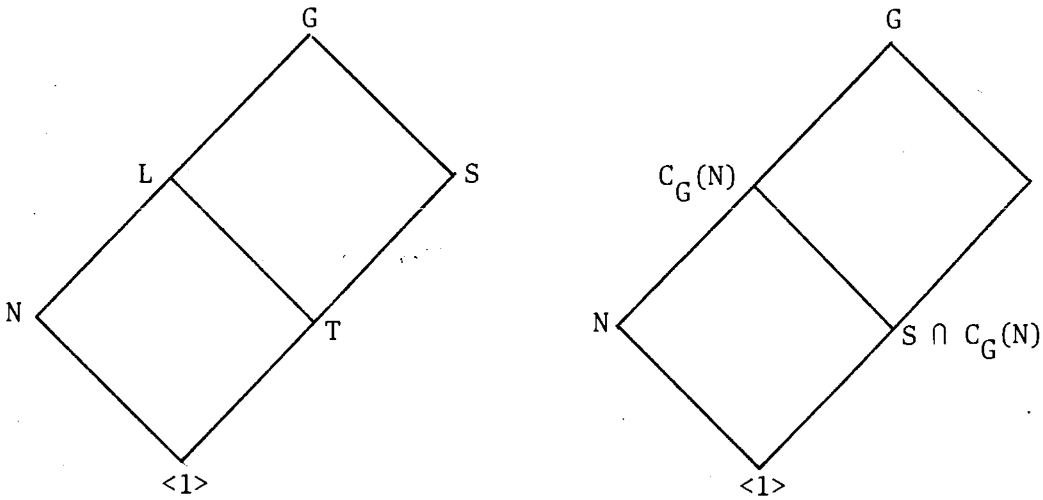

Fig. 2

Suppose G is primitive with primitivator S. Let N be the unique minimal normal nilpotent subgroup of G, whose existence is guaranteed by 3.4. Since N is abelian normal subgroup of G, $N \leq C_G(N) \trianglelefteq G$ by 1.1.9. By 3.3, $SN = G$ and $S \cap N = \langle 1 \rangle$. Hence $G = SC_G(N)$ and $S \cap C_G(N) \trianglelefteq G$. Therefore $S \cap C_G(N) = \langle 1 \rangle$ and so $C_G(N) = N$.

Conversely, suppose G has a minimal normal subgroup N such that $C_G(N) = N$. If $N = G$, then G has a prime order, and the theorem holds.

Otherwise let L/N be a chief factor of G, $|N| = p^{\alpha}$ and $|L/N| = q^{\beta}$ where p and q are primes. If $p = q$, then L would be a p-group and $\langle 1\rangle \neq N \cap Z(L) \trianglelefteq G$, hence $N \leq Z(L)$ in contradiction to $C_G(N) = N$ (Notice that $N \leq Z(L)$ implies that $L \leq C_G(N)$). Therefore $p \neq q$. Let T be a Sylow q-subgroup of L and $S = N_G(T)$. By the Frattini argument, $SL = G$. If $S = G$ then $T \trianglelefteq G$ so $T \leq C_G(N)$ since $T \leq N = \langle 1\rangle$ (see 1.1.10(f)), a contradiction. Therefore $S < G$ and $S \cap N = \langle 1\rangle$. Furthermore, $Core_G(S) \cap N = \langle 1\rangle$ and so by 1.1.10(f), $Core_G(S) \leq C_G(N) = N$, hence $Core_G(S) = \langle 1\rangle$ and therefore S is a primitivator of G. □

3.8 <u>DEFINITION:</u>

A homomorph F is called a <u>Schunck class</u> (or <u>saturated homomorph</u>) if it is primitively closed, i.e. if any finite (not necessarily solvable) group G, all of whose primitive factors are in F, is itself in F.

3.9 <u>EXAMPLES:</u>

(i) $\langle 1\rangle$ is a Schunck class and is contained in every Schunck class.

(ii) Consider N, the class of finite nilpotent groups, which is closed under epimorphisms. If for a group G all the primitive factor groups are nilpotent, then by 3.2(vi) and 3.2(i), this means that all the maximal subgroups of G are normal in G, so G is nilpotent.

(iii) Let $\underline{\pi}$ denote the class of solvable π-groups, where π is a fixed set of primes. Let $C(\pi)$ denote the class of groups G such that for all proper normal subgroups M of G, $G/M \notin \underline{\pi}$. $C(\pi)$ is called the class of π-perfect groups. $C(\pi)$ is closed under epimorphisms.

In 2.1.3 we showed that the group G has a unique normal subgroup N minimal subject to $G/N \in \underline{\pi}$. Now we show that if $N < G$, then G has a primitive quotient group in $\underline{\pi}$: For if N is a maximal in G, then G/N is cyclic of prime order and so primitive. Otherwise N is properly contained in a maximal subgroup S of G. Since $N \trianglelefteq G$, $N \leq \mathrm{Core}_G(S)$. By 3.2(i) $G/\mathrm{Core}_G(S)$ is primitive and as $N \leq \mathrm{Core}_G(S)$ we have $G/\mathrm{Core}_G(S) \cong (G/N)/(\mathrm{Core}_G(S)/N) \in \underline{\pi}$, as $G/N \in \underline{\pi}$. Thus if G has all its primitive quotient groups in $C(\pi)$, G itself must be in $C(\pi)$ and so $C(\pi)$ is a Schunck class.

There is a close connection between Schunck classes and saturated formations, as the next two Lemmas indicate.

3.10 LEMMA:

If F is a saturated formation then F is a Schunck class.

Proof:

Let G be a group and let $G/N \in F$ for all primitive factor groups G/N of G. Then G/N is of the form $G/\mathrm{Core}_G(S)$ where S is a maximal subgroup of G. By 3.2(ii), $G/\phi(G) = G/\cap\{\mathrm{Core}_G(S) : S \text{ is maximal in } G\} \in F$ as F is a formation. Since F is saturated, $G \in F$. □

The converse is not true in general.

3.11 EXAMPLE:

$C(\{2\})$ is a Schunck class, as has been established in Example 3.9(iii), but $C(\{2\})$ is not a formation. To see this, let $H = C_3Q_8$ be the semidirect product of Q_8 with a subgroup C_3 of $Aut(Q_8)$ It can easily be seen that $H \in C(\{2\})$. The group $G = C_2 \times H$ obviously does not belong to $C(\{2\})$. But in G there is another normal subgroup C_2^*, different from C_2, with $G = C_2^* \times H$. Therefore $G/C_2 \cong G/C_2^* \in C(\{2\})$ and $C_2 \cap C_2^* = <1>$.

3.12 LEMMA:

Let F be a formation. If F is a Schunck class then F is a saturated formation.

Proof:

Suppose $G/\phi(G) \in F$. If $G/Core_G(S)$ is a primitive factor group, then by 3.2(ii), $\phi(G) \leq Core_G(S)$ and F being a homomorph imply that $G/Core_G(S) \in F$. Since F is a Schunck class, $G \in F$, so F is a saturated formation. □

From Lemma 3.10 and Example 3.11 we conclude that Schunck classes are more general than saturated formations.

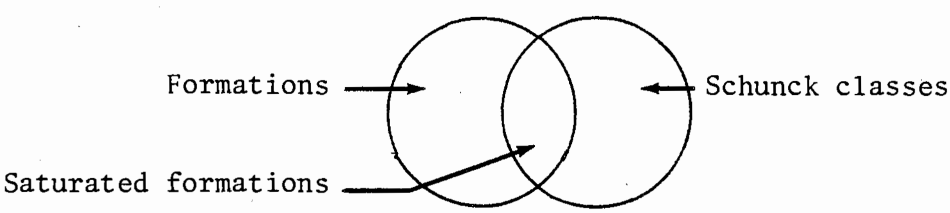

Fig. 3

The main object of this chapter is to extend the results of Gaschütz on the existence and conjugacy of covering subgroups (projectors) in saturated formations to the case of Schunck classes. First we start with the following definition.

3.13 DEFINITION:

Let F be a class of groups.

F is called projective if for any group G there exists an F-projector of G.

3.14 THEOREM:

If F is projective, then F is a Schunck class.

Proof:

Let $G \in F$. Then G is an F-projector of G and so G/N is an F-subgroup of G/N for any normal subgroup N of G, that is $G/N \in F$ for any $N \trianglelefteq G$ and therefore F is a homomorph.

Now let H be an F-projector of G and $G/N \in F$ for all primitive factor groups G/N of G. Then by the definition of F-projectors $HN/N = G/N$, that is $HN = G$ and $G = H \in F$ by 3.2(iii), as required. □

Now we show that a Schunck class F is projective and that in every group (i.e. finite solvable group) the F-projectors are conjugate.

3.15 MAIN LEMMA:

Let F be a Schunck class, H a nilpotent normal subgroup of G and $G/H \in F$. Then:

(i) There exists $M \leq G$, M $\mathcal{F}$-maximal in G, $MH = G$.

(ii) All such M are conjugate under H.

<u>Proof</u>:

(i) (see fig.4)

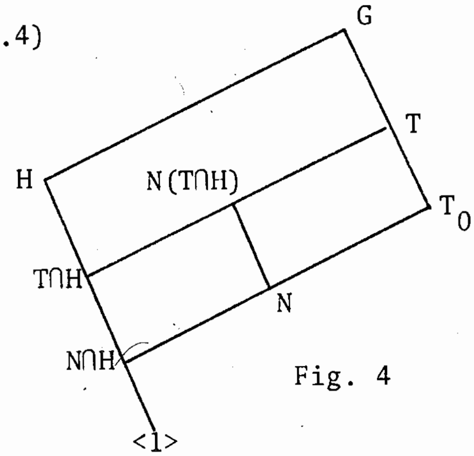

Fig. 4

It is clearly sufficient to construct $T \leq G$, $T \in \mathcal{F}$ with $TH = G$. For this purpose let T be minimal with $T \leq G$ and $TH = G$. If $T \notin \mathcal{F}$, then by definition of Schunck class there would exist a primitive factor group T/N of T with $T/N \notin \mathcal{F}$. Then $T \cap H \not\leq N$ because otherwise $G/H \cong T/T \cap H$ would have a factor group which does not belong to $\mathcal{F}$. However, by hypothesis $G/H \in \mathcal{F}$ and $\mathcal{F}$ is a homomorph so this is impossible. So $(T \cap H)N/N$ would be a non-trivial nilpotent normal subgroup, hence by 3.4 a minimal normal subgroup of T/N. Let T_o/N be a primitivator of T/N. Then $(T_o/N)(N(T \cap H)/N) = T/N$ by 3.5, that is $T_oN(T \cap H) = T$ and $G = TH = T_oN(T \cap H)H = T_oH$; since $T_o < T$, this would contradict the minimality of T. Hence $T \in \mathcal{F}$.

(ii) (see fig.5)

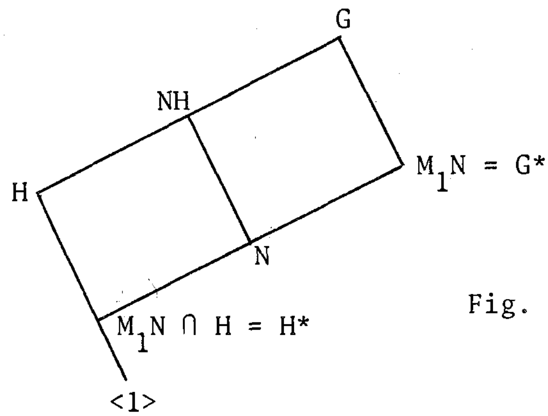

Fig. 5

Let M_1, M_2 be F-maximal in G and $M_1H = M_2H = G$.

If $G \in F$, then $M_1 = M_2 = G$ which proves the assertion.

If $G \notin F$, then by 3.8 there exists a primitive factor group G/N of G with $G/N \notin F$. Now $M_iN \neq G$ for $i = 1,2$, because otherwise $G/N \cong M_i/M_i \cap N \in F$. Furthermore $G/H \in F$ and, as F is a homomorph, $H \not\leq N$, so HN/N is a non-trivial nilpotent normal subgroup of G/N, hence by 3.4 HN/N is a minimal normal subgroup of G/N. Since $(M_iN/N)(HN/N) = M_iH/N = G/N$, M_iN/N are primitivators of G/N by 3.5 and conjugate under HN/N by 3.6. Therefore $M_1N = (M_2N)^h = M_2^hN$ for some $h \in H$. If we put $G^* = M_1N$ and $H^* = M_1N \cap H$. Then $G^*/H^* = M_1N/(M_1N \cap H) \cong G/H \in F$, and M_1, M_2^h are F-maximal in G^* with $M_1H^* = M_2^hH^* = G^*$. By induction it now follows that M_1 and M_2^h are conjugate under H^* and therefore M_1, M_2 are conjugate under H. □

3.16 <u>MAIN THEOREM:</u>

Let F be a Schunck class and G a finite solvable group. Then:

(i) There exists an F-projector of G.

(ii) All $\mathcal{F}$-projectors of G are conjugate under G.

Proof:

(see fig.6)

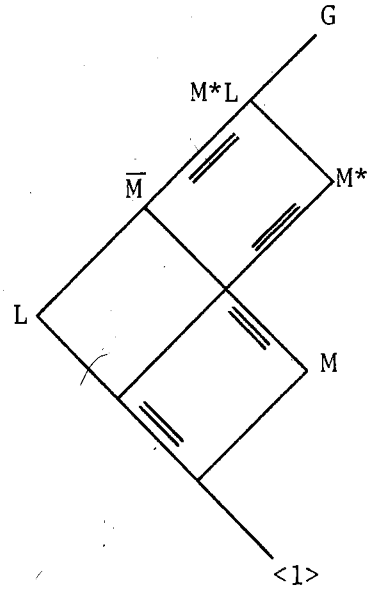

Fig. 6

The theorem obviously holds for $G = \langle 1 \rangle$. We carry out the proof by induction on $|G|$.

(i) Let $G \neq \langle 1 \rangle$, $\langle 1 \rangle \neq L \trianglelefteq G$, L nilpotent, and by induction $\overline{M}/L$ is an $\mathcal{F}$-projector of G/L. Then $\overline{M}/L \in \mathcal{F}$ and so by 3.15(i) there exists $M \leq \overline{M}$, M $\mathcal{F}$-maximal in $\overline{M}$ and $ML = \overline{M}$. We show that M is an $\mathcal{F}$-projector of G.

First we show M is $\mathcal{F}$-maximal in G: For if $M \leq M^* \leq G$, $M^* \in \mathcal{F}$ then $M^*L/L \cong M^*/M^* \cap L \in \mathcal{F}$ since $\mathcal{F}$ is closed under epimorphisms. Since $\overline{M}/L$ is $\mathcal{F}$-maximal in G/L, $M^*L/L \leq \overline{M}/L$ and $M^* \leq \overline{M}$. $M = M^*$ now follows from the $\mathcal{F}$-maximality of M in $\overline{M}$.

Now we show that MN/N is an $\mathcal{F}$-projector of G/N for any minimal normal subgroup N of G: By induction, let $\overline{T}/N$ be an $\mathcal{F}$-projector of G/N and by 3.15(i) let T be F-maximal in $\overline{T}$ with $TN = \overline{T}$. $H = NL$ is nilpotent. $\overline{M}H/H = MH/H$ and $\overline{T}H/H = TH/H$ are F-projectors of G/H, so by induction they are conjugate in G/H and therefore MH and TH are conjugate in G. So $MH = (TH)^{g_o} = T^{g_o}H$ for some $g_0 \in G$. Then T^{g_o} is also F-maximal in G, and by 3.15(ii) T^{g_o} and M are conjugate under H; that is $M = T^g$, $g \in G$. As $\overline{T}/N$ is an F-projector of G/N, so is $(\overline{T}/N)^{gN} = T^gN/N = MN/N$. Part (i) of the theorem now follows from 2.1.22.

(ii) If M_1 and M_2 are F-projectors of G and if $\langle 1 \rangle \neq H \trianglelefteq G$, H nilpotent, then M_1H/H and M_2H/H are F-projectors of G/H by 2.1.21(b) and by induction $M_1H/H = (M_2H/H)^{g_oH} = M_2^{g_o}H/H$ with $g_o \in G$. Then by 3.15(ii) $M_1 = M_2^g$ with $g \in G$; as required. ◫

Chapter 4

COVERING SUBGROUPS and PROJECTORS IN FINITE π-SOLVABLE GROUPS

In this chapter we study the existence and conjugacy of covering subgroups and projectors in finite π-solvable groups. We follow Brewster's and Covaci's proofs ([10], [15], [16] and [17]) that some results of Gaschütz and Schunck (Chapters 2, 3), originally proved only for solvable groups, can be extended to π-solvable groups. All groups considered here are finite.

We first start with some results that we shall use. These results, which are true for any finite group, are due to R. Baer.

4.1 <u>LEMMA</u>:

If M is a solvable minimal normal subgroup of a finite group G, then M is an abelian p-group, for some prime p.

If a maximal subgroup S of G does not contain M, then $G = MS$ and $M \cap S = \langle 1 \rangle$.

<u>Proof</u>:

Since M has no proper characteristic subgroups (because these would be normal subgroups of G), and since the commutator subgroup M' of the solvable group M is different from M, $M' = \langle 1 \rangle$ so that M is abelian. If the Sylow p-subgroup K (say) were a proper subgroup of M, then K would be a characteristic subgroup of M and so a normal subgroup of G, a contradiction. So M is a p-group.

If the maximal subgroup S of G does not contain M, then clearly $G = MS$. Since $M \trianglelefteq G$, $M \cap S \trianglelefteq S$ and since M is abelian, $M \cap S \trianglelefteq M$, so $M \cap S \trianglelefteq MS = G$, and $M \cap S = M$ or $M \cap S = \langle 1 \rangle$. But as $M \nleq S$, $M \cap S \neq M$ hence $M \cap S = \langle 1 \rangle$. ◫

4.2 LEMMA: Let G be a primitive group.

If S is a primitivator of G, if N is a non-trivial normal subgroup of G, and if $C = C_G(N)$, then $C \cap S = \langle 1 \rangle$ and C is either $\langle 1 \rangle$ or a minimal normal subgroup of G.

Proof: (see fig. 7)

Since $\mathrm{Core}_G(S) = \langle 1 \rangle$, $N \nleq S$ and so $G = NS$. Since $N \trianglelefteq G$, $C \trianglelefteq G$. Consequently $C \cap S \trianglelefteq S$ so that $S \leq N_G(C \cap S)$. Since N is contained in the centralizer of C, and hence $N \leq C_G(C \cap S) \leq N_G(C \cap S)$, $G = NS \leq N_G(C \cap S)$ and hence $C \cap S \trianglelefteq G$ so $S \cap C = \langle 1 \rangle$ because $\mathrm{Core}_G(S) = \langle 1 \rangle$.

Now suppose that C contains a non-trivial normal subgroup K of G. As before we see that $K \nleq S$ and so $G = KS$. Hence $K \leq C \leq KS$, and so by the modular law $C = K(C \cap S) = K$. Hence either $C = \langle 1 \rangle$ or else C is a minimal normal subgroup of G.

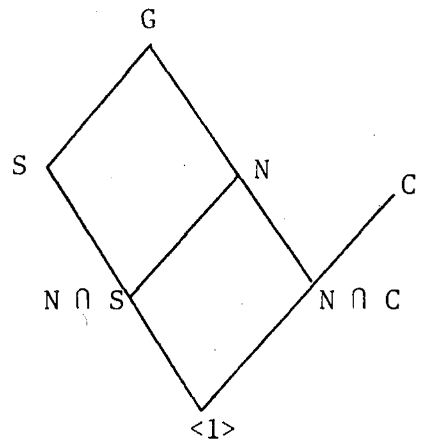

Fig. 7 □

4.3 <u>COROLLARY</u>:

If G is a primitive group and S is its primitivator. Then:

(i) There exists at most one non-trivial abelian normal subgroup of G.

(ii) There exists at most two different minimal normal subgroups of G.

<u>Proof</u>:

(i) If K is a non-trivial abelian normal subgroup of G, then $K \leq C_G(K)$. Hence $\langle 1\rangle < K \leq C_G(K)$; and by 4.2, $C_G(K)$ is a minimal normal subgroup of G. Consequently $K = C_G(K)$ is a minimal normal subgroup of G.

Assume now by way of contradiction the existence of abelian normal subgroups U and V of G such that $\langle 1\rangle \neq U \neq V \neq \langle 1\rangle$. By the preceding result U and V are both minimal normal subgroups of G so that in particular $U \cap V = \langle 1\rangle$. Consequently $U \leq C_G(V)$. But V has been shown to be its own centralizer so that $U \leq V$, a contradiction.

(ii) Assume by way of contradiction the existence of three different minimal normal subgroups P, Q and R of G. Then $P \cap R = P \cap Q = \langle 1\rangle$ so that R and Q are both contained in the centralizer of P. Since $R \cap Q = \langle 1 \rangle$, RQ is not a minimal normal subgroup of G so that the centralizer of P in G is neither $\langle 1 \rangle$ nor a minimal normal subgroup of G. This contradicts Lemma 4.2; and this contradiction proves (ii). □

4.4 <u>COROLLARY</u>:

If S is a primitivator of the finite group G and if A, B are two different minimal normal subgroups of G, then

(i) $G = AS = BS$, $\langle 1 \rangle = A \cap S = B \cap S$;

(ii) $A = C_G(B)$ (and $B = C_G(A)$);

(iii) A and B are non-abelian groups.

<u>Proof</u>:

Since A and B are two different minimal normal subgroups of G, $A \cap B = \langle 1 \rangle$ so that $B \leq C_G(A)$. We see from the proof of 4.3 that $B = C_G(A)$ and likewise that $A = C_G(B)$. This proves (ii).

Since $\mathrm{Core}_G(S) = \langle 1 \rangle$, neither A nor B is contained in S. Hence $SA = SB = G$. Since $A = C_G(B)$, $A \cap S = \langle 1 \rangle$ is a consequence of 4.2; and likewise we see that $B \cap S = \langle 1 \rangle$. This proves (i), and (iii) is a trivial consequence of (ii). □

4.5 <u>LEMMA</u>: Let G be a primitive group.

If the indices in G of all the primitivators of G are powers of one and the same prime p, then G has a unique minimal normal subgroup.

Proof:

Assume by the way of contradiction the existence of two minimal normal subgroups A and B of G. Because of the existence of maximal subgroups with Core $\langle 1\rangle$ we may deduce from Corollary 4.4 that A is non-abelian and that $G = AS$, $A \cap S = \langle 1\rangle$ for any maximal core-free subgroup S of G. Since $|A| = |G:S|$, and since the later is by the hypothesis is a power of p, A is a minimal normal subgroup of prime power order. Since such groups are solvable, Lemma 4.1 implies that A is abelian, and so we have arrived at a contradiction to 4.4(iii). □

4.6 LEMMA: Let G be a primitive group.

If G has a non-trivial normal solvable subgroup, then G has one and only one minimal normal subgroup.

Proof:

Let N be a solvable minimal normal subgroup of G. Then N is, by 4.1, an elementary abelian p-group. If S is a maximal subgroup of G with $\mathrm{Core}_G(S) = \langle 1\rangle$, then $N \not\leq S$; and $G = SN$, $N \cap S = \langle 1\rangle$ by 4.1. It follows that $|G:S| = |N|$ is a power of p. The result now follows by 4.5. □

4.7 THEOREM:

Assume that the group G possesses a non-trivial solvable normal subgroup, and that the core of the maximal subgroup S of G is $\langle 1\rangle$.

(a) The existence of a non-trivial solvable normal subgroup of S implies the existence of a non-trivial normal subgroup of S whose order is relatively prime to $|G:S|$.

(b) If there exists a non-trivial normal subgroup of S whose order is relatively prime to $|G:S|$, then S is conjugate to every maximal subgroup T of G whose core in G is $\langle 1 \rangle$.

Proof:

(a) From our hypothesis we deduce first the existence of a solvable minimal normal subgroup N of G. By 4.1, N is an abelian p-group for some

$\langle 1 \rangle$, then $N \not\leq X$; and 4.1 implies $G = NX$, $\langle 1 \rangle = N \cap X$. In particular $G/N \cong X$, and $|G:X| = |N|$ which is a power of p. (Note that this may be applied to $X = S$ too).

If there exists a non-trivial solvable normal subgroup of S, then the same is true for the isomorphic group G/N. Consequently there exists a solvable minimal normal subgroup M/N of G/N. By 4.1, M/N is an abelian q-group for some prime q. Assume by way of contradiction that $p = q$. Then M is a p-group. Since N is a non-trivial normal subgroup of the p-group M, N contains non-trivial central elements of M. But the center of M is a characteristic subgroup of a normal subgroup of G; and so $Z(M) \trianglelefteq G$. The minimality of N and $\langle 1 \rangle \neq N \cap Z(M)$ imply that $N \leq Z(M)$ and that therefore $M \leq C_G(N)$. Since G possesses maximal subgroups with Core $\langle 1 \rangle$, and since $\langle 1 \rangle < N \leq C_G(N)$ (as N is abelian), $C_G(N)$ is a minimal normal subgroup of G by 4.2. Hence $C_G(N) = N < M = C_G(N)$, a contradiction proving $p \neq q$. The isomorphic groups G/N and S contain therefore a non-trivial normal subgroups of order a power of q, whereas $|G:S| = |N|$, which is a power of the prime $p \neq q$.

(b) Assume the existence of a non-trivial normal subgroup of S whose order is prime to $|G:S|$. Then the group $G/N \cong S$ contains a non-trivial normal subgroup P/N whose order is prime to $|G:S|$. Since $|N| = |G:S|$, we see that $(|N|,|P/N|) = 1$. Consider now some maximal subgroup X of G such that $\mathrm{Core}_G(X) = \langle 1 \rangle$. Then $G = NX$ and $\langle 1 \rangle = N \cap X$. Because of $N \leq P \leq NX$ and the modular law, we have $P = N(P \cap X)$ so that $P \cap X$ is a complement of N in P. Since $P \cap X \trianglelefteq X$, $X \leq N_G(P \cap X)$. Since G possesses maximal subgroups with core $\langle 1 \rangle$ as well as the abelian minimal normal subgroup N, N is, by 4.3, 4.4 the unique minimal normal subgroup of G. Hence $P \cap X$ is not a normal subgroup of G so that $N_G(P \cap X)$ is exactly the maximal subgroup X of G.

Suppose now that T is a maximal subgroup of G with $\mathrm{Core}_G(T) = \langle 1 \rangle$. Applications of the results of the preceding paragraph of our proof show that $P \cap S$ and $P \cap T$ are both complements of N in P, that $S = N_G(P \cap S)$ and that $T = N_G(P \cap T)$. Since $(|N|,|P/N|) = 1$, and since N is abelian, any two complements of N in P are conjugate in P; (see KOCHENDÖRFFER [3], p.101, Theorem 6.2.3). Consequently there exists an element g in P conjugating $P \cap S$ to $P \cap T$, and this element g naturally conjugates the normalizer S of $P \cap S$ into the normalizer T of $P \cap T$. □

4.8 <u>DEFINITION:</u> Let π be a set of primes and π' the complement to π in the set of all primes.

A group is π-solvable if every chief factor is either a solvable π-group or a π'-group. Clearly solvable groups are π-solvable, and if π is the set of all primes, any π-solvable group is solvable.

It is also clear that subgroups of π-solvable groups are π-solvable. Also, if G is π-solvable group and $N \trianglelefteq G$ then G/N is π-solvable since chief factors of G/N are isomorphic to a subset of those of G.

4.9 DEFINITION: With π and π' as above.

A class F of groups is said to be π-closed if: $G/O_{\pi'}(G) \in F$ implies that $G \in F$, where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

We shall call a π-closed homomorph a π-homomorph, a π-closed Schunck class a π-Schunck class, and a π-closed saturated formation a π-saturated formation.

4.10 DEFINITION: If θ_1 and θ_2 are classes of groups, then by $\theta_1\theta_2$ we denote the class of groups G with a normal subgroup N such that $N \in \theta_1$ and $G/N \in \theta_2$.

It is not hard to show that if θ_1 and θ_2 are formations and θ_1 is closed under normal-subgroups, then $\theta_1\theta_2$ is a formation.

4.11 LEMMA: Let $M_{\pi'}$ denote the class of finite π'-groups and let F be a formation. Then F is π-saturated iff $F = M_{\pi'}F$.

Proof:

Suppose F is π-saturated formation. Clearly $F \subseteq M_{\pi'}F$. If $G \in M_{\pi'}F$, then there is $N \trianglelefteq G$ such that N is a π'-group and $G/N \in F$. But $N \leq O_{\pi'}(G)$ and so $G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N) \in F$ as F is a formation. Hence $G \in F$.

Conversely if $F = M_{\pi'}F$ suppose $G/O_{\pi'}(G) \in F$. Then $G \in M_{\pi'}F$ and this class is contained in F so $G \in F$. □

The following theorem, which is due to Brewster, is a generalization of Gaschütz theorem 2.1.17(ii).

4.12 THEOREM:

Let F be a π-saturated formation. If G is π-solvable, then G has F-covering subgroups and any two are conjugate.

Proof:

First the existence of F-covering subgroups is established by induction on $|G|$. If $G \in F$, there is nothing to show since G is its own F-covering subgroup. So suppose $G \notin F$ and let N be a minimal normal subgroup of G. Since G/N is π-solvable, there is an F-covering subgroup E/N of G/N. If E is a proper subgroup of G, then since E is π-solvable, by the induction, E has an F-covering subgroup E_o. By 2.1.16, E_o is an F-covering subgroup of G. Thus the result is proved unless $E = G$.

However, in this case $G/N \in F$ for each minimal normal subgroup N of G and so, since F is a formation and $G \notin F$, G has a unique minimal normal subgroup N. Because F is saturated, $\phi(G) = <1>$. Also because G is π-solvable N is either a solvable π-group or a π'-group. If N is a π'-group, then $N \leq O_{\pi'}(G)$ and so we have $G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N)$. But $G/N \in F$, hence $G/O_{\pi'}(G) \in F$, which implies, by the π-closure of F, the contradiction that $G \in F$. So N must be a solvable π-group, and so by 4.1, N is an abelian p-group for some $p \in \pi$.

Let S be a maximal subgroup of G such that $N \nleq S$. We shall show that S is an F-covering subgroup of G. First, by 4.1, $G = SN$ and $S \cap N = <1>$. Also $S = S/<1> = S/S \cap N \cong SN/N = G/N \in F$. Finally if $S \leq S^* \leq G$, $K \trianglelefteq S^*$ with $S^*/K \in F$ then either $S = S^*$ which implies that $SK = S^*$ or $S^* = G$ which implies that $N \leq K$ as N is the unique minimal normal subgroup of G and hence $SK = SN = G$. So in both cases $SK = S^*$ which means that S is an F-covering subgroup of G.

Similarly conjugacy of F-covering subgroups of G is shown by induction on $|G|$. Let E and E_o be any two F-covering subgroups of G. If $G \in F$, $E = G = E_o$. So suppose $G \notin F$. Let N be any minimal normal subgroup of G. By 2.1.14, EN/N and E_oN/N are F-covering subgroups of G/N and so for some $x \in G$, $EN = E_o^xN$. Thus if $EN \neq G$, then E, E_o^x are F-covering subgroups of EN and so are conjugate in EN.

So we may assume $EN = G = E_oN$ for each minimal normal subgroup N of G. Since F is a π-saturated formation and $G \notin F$, N is a unique minimal normal subgroup of G and is a solvable π-group. Hence N is an abelian p-group for some $p \in \pi$ and so E and E_o are maximal subgroups of G. Now E is π-solvable and so a minimal normal subgroup of E is either a π'-group or is solvable. By Theorem 4.7 E and E_o are conjugate. □

Some properties of these F-covering subgroups will now be established.

4.13 <u>COROLLARY</u>:

If F is a π-saturated formation and G is a π-solvable group, then an F-covering subgroup of G contains a Hall π'-subgroup of G.

<u>Proof</u>:

Using induction on $|G|$, let N be a minimal normal subgroup of G and suppose E is an F-covering subgroup of G. Then EN/N is an F-covering subgroup of G/N, and so by induction, EN/N contains a Hall π'-subgroup of G/N.

If N is a π-group, then $|G:E| = |G:EN||EN:N|$ is a π-number. If N is a π'-group, $EN \in M_{\pi'}F = F$ and so $E = EN$ by 2.1.15(i). In either case $|G:E|$ is a π-number and so a Hall π'-subgroup of E is a Hall π'-subgroup of G. □

<u>DEFINITION</u>: Let Γ be a set of primes. A group G is called <u>Γ-closed</u> if G has a normal Hall Γ-subgroup.

4.14 <u>LEMMA</u>: Let F be a π-saturated formation and let G be a π-solvable group in which the F-covering subgroup E is π'-closed. Denote the normal Hall π'-subgroup of E by $E_{\pi'}$. If H is a Hall π-subgroup of $N_G(E_{\pi'})$ such that $H \cap E$ is a Hall π-subgroup of E, then $H \cap E$ is an F-covering subgroup of H.

Proof:

We argue by induction on $|G|$. If $N_G(E_{\pi'}) \neq G$, then $E \leq N_G(E_{\pi'})$ and is an F-covering subgroup of $N_G(E_{\pi'})$. Since $N_G(E_{\pi'})$ is π-solvable, the induction implies $H \cap E$ is an F-covering subgroup of H.

So suppose $N_G(E_{\pi'}) = G$. If $E_{\pi'} = <1>$, then by 4.13, G is a π-group. Consequently $H = G$ and $H \cap E = E$ so that $H \cap E$ is an F-covering subgroup of H. Thus the case $E_{\pi'} \neq <1>$ is left for consideration. Then $|G/E_{\pi'}| < |G|$, $G/E_{\pi'}$ is π-solvable, $E/E_{\pi'}$ is a π'-closed F-covering subgroup of $G/E_{\pi'}$ and $HE_{\pi'}/E_{\pi'}$ is a Hall π-subgroup of $G/E_{\pi'} = N_{(G/E_{\pi'})}((E/E_{\pi'})_{\pi'})$. So by induction $(HE_{\pi'}/E_{\pi'}) \cap (E/E_{\pi'}) = (H \cap E)E_{\pi'}/E_{\pi'}$ is an F-covering subgroup of $HE_{\pi'}/E_{\pi'}$. But the natural isomorphism from $HE_{\pi'}/E_{\pi'}$ to H maps $(H \cap E)E_{\pi'}/E_{\pi'}$ onto $H \cap E$. Thus $H \cap E$ is an F-covering subgroup of H. ◫

The following two theorems, which are due to Covaci, give a generlization to Brewster's theorem (4.12).

4.15 THEOREM:

If F is a π-homomorph, then any two F-covering subgroups of a π-solvable group G are conjugate in G.

Proof:

By induction on $|G|$. Let E, E_o be two F-covering subgroups of G. If $G \in F$, then by definition of an F-covering subgroup, we obtain $E = E_o = G$. So suppose $G \notin F$. Let N be a minimal normal subgroup of G. By 2.1.14, EN/N and E_oN/N are F-covering subgroups of G/N. By the induction, EN/N and E_oN/N are conjugate in G/N and so for some $x \in G$, $EN = E_o^xN$. We distinguish two cases:

(i) There is a minimal normal subgroup M of G with $EM \neq G$. We put $N = M$. By 2.1.13, E and E_o^x are F-covering subgroups of EN; hence, by the induction, E and E_o^x are conjugate in EN and so E and E_o are conjugate in G.

(ii) For any minimal normal subgroup N of G, $EN = G = E_oN$. Then every minimal normal subgroup N of G is a solvable π-group. Indeed, since G is π-solvable, N is either a solvable π-group or a π'-group. Suppose that N is a π'-group. It follows that $N \leq O_{\pi'}(G)$ and we have $G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N)$. But $G/N = EN/N \cong E/E \cap N \in F$, hence $G/O_{\pi'}(G) \in F$, which implies, by the π-closure of F, the contradiction $G \in F$. So N is a solvable π-group. By 4.1, N is abelian. We shall prove that E and E_o are maximal subgroups of G. In the case of E, $E < G$ since $G \notin F$. Also $E \leq E^* < G$ implies $E = E^*$; for, if $E < E^*$, there is an element $e^* \in E^* \setminus E \subset G = EN$ so $e^* = eg$ with $e \in E$, $g \in N$. But, we see that $g \in N \cap E^* = \langle 1 \rangle$, which implies the contradiction $e^* = e \in E$. Let us notice that $\mathrm{core}_G(E) = \langle 1 \rangle = \mathrm{core}_G(E_o)$. If we suppose, for example, $\mathrm{core}_G(E) \neq \langle 1 \rangle$, putting $N \leq \mathrm{core}_G(E)$ we have $G = EN = E\,\mathrm{core}_G(E) = E$, a contradiction to $E < G$. Applying now Theorem 4.7, it follows that E and E_o are conjugate in G. ◫

4.16 <u>THEOREM</u>: A π-homomorph F is a Schunck class iff any π-solvable group has F-covering subgroups.

<u>Proof</u>:

In fact, for the proof in one direction, if F is a Schunck class (not necessarily π-closed), we prove by induction on $|G|$ that any π-solvable group G has F-covering subgroups. Two possibilities arise:

(i) There is a minimal normal subgroup M of G such that $G/M \notin F$. By induction G/M has an F-covering subgroup $\overline{H}/M$. Since $G/M \notin F$, $|\overline{H}| < |G|$ and so, by the induction, $\overline{H}$ has an F-covering subgroup H. Now, by 2.1.16, H is an F-covering subgroup of G.

(ii) Any minimal normal subgroup M of G satisfies $G/M \in F$. If G is not primitive, then by definition of primitivity, we conclude that for any maximal subgroup S of G, $\mathrm{Core}_G(S) \neq \langle 1 \rangle$ and so the core contains a minimal normal subgroup M of G and hence $G/\mathrm{Core}_G(S) \cong (G/M)/(\mathrm{Core}_G(S)/M) \in F$ as $G/M \in F$. Thus if G is not primitive then all primitive factor groups of G belong to F and hence $G \in F$ since F is a Schunck class and therefore G is its own F-covering subgroup. So we may assume that G is primitive and $G \notin F$. If S is a primitivator of G then we claim that S is an F-covering subgroup of G. First of all we notice that $S \in F$. Further, if $S \leq V \leq G$, $V_o \trianglelefteq V$, $V/V_o \in F$, we have, since S is a maximal subgroup of G, $V = S$ or $V = G$. If $V = S$ then $V = SV_o$, but if $V = G$, we choose a minimal normal subgroup M of G with $M \leq V_o$. By 3.5, $MS = G$ so $V = SV_o$.

Conversely, let F be a π-homomorph with the property that any π-solvable group has F-covering subgroups. We shall prove that F is a Schunck class. Suppose F is not a Schunck class and let G be a π-solvable group of minimal order with respect to the conditions: $G \notin F$ and any primitive factor group of G is in F. If M is a minimal normal subgroup of G then by the minimality of G, $G/M \in F$. Since G is a π-solvable group there is an F-covering subgroup H of G. Thus $H \leq G$,

$M \trianglelefteq G$ and $G/M \in F$ imply $G = MH$. By the π-closure of F and by the assumption $G \notin F$, we conclude, as in the proof of 4.15, that M is a solvable π-group, hence, by 4.1, M is abelian. So $M \cap H = \langle 1 \rangle$. As in 4.15, H is a maximal subgroup of G.

Now suppose G is not primitive. Then $Core_G(H) \neq \langle 1 \rangle$, so, by the minimality of G, $G/Core_G(H) \in F$. However, $H/Core_G(H)$ is an F-covering subgroup of $G/Core_G(H)$ and so $H = G$. But this is not possible because $H \in F$ and $G \notin F$. Thus G is primitive, contradicting the choice of G. ◻

Now after we proved the existence and conjugacy of F-covering subgroups in finite π-solvable groups, we extend our study one further step to F-projectors which, as we noticed before are more general than F-covering subgroups since every F-covering subgroup of the finite group G is an F-projector of G (see 2.1.23).

By the previous theorem 4.16 we have:

4.17 THEOREM:

If F is a π-Schunck class, then any finite π-solvable group has F-projectors. ◻

So it remains to prove that the projectors in a π-solvable group are conjugate. In preparation for this result we give the following theorem.

4.18 THEOREM:

Let F be a π-Schunck class, G a π-solvable group and A an abelian normal subgroup of G with $G/A \in F$. Then:

(i) There is a subgroup S of G with $S \in F$ and $AS = G$.

(ii) If S_1 and S_2 are F-maximal subgroups of G with $AS_1 = G = AS_2$, then S_1 and S_2 are conjugate in G.

Proof:

(i) Let $X = \{S^*: S \leq G, AS^* = G\}$. Since $G \in X$, $X \neq \phi$. Considering X ordered by inclusion, X has a minimal element S. We shall prove that $S \in F$.

Put $D = S \cap A$. Then $D \trianglelefteq G$. Let W be a maximal subgroup of S. We have $D \leq W$. Indeed, if we suppose that $D \not\leq W$, we obtain $DW = S$, hence $ADW = AS = G$, which means $W \in X$, in contradiction to the minimality of S in X. Put $N = \mathrm{Core}_G(W)$. Clearly $D \leq N$. Then $S/N \cong (S/D)/(N/D)$. Because $S/D = S/S \cap A \cong AS/A = G/A \in F$, we deduce, since F is a homomorph that $S/N \in F$.

For any primitive factor group S/N of S, we can find a maximal subgroup W of S such that $N = \mathrm{Core}_G(W)$. But this means by the above that any primitive factor group of S is in F which implies that $S \in F$, since F is a Schunck class.

(ii) We argue by induction on $|G|$. We distinguish two cases:

(a) If $G \in F$ then if S_1 and S_2 are F-maximal subgroups of G, $S_1 = S_2 = G$ and the theorem is proved.

(b) $G \notin F$. It means that there is a primitive factor group G/N with $G/N \notin F$. We have $NS_1 \neq G$ and $NS_2 \neq G$. Now we claim that AN/N is a minimal normal subgroup of G/N. Certainly $AN/N \trianglelefteq G/N$. Also AN/N is a non-trivial subgroup, for otherwise $AN = N$ which implies that $S_1N = S_1AN = S_1A = G$, a contradiction. Finally, if $H/N \trianglelefteq G/N$ with $\langle 1\rangle < H/N \leq AN/N$ then H/N is abelian since AN/N is abelian and so if S/N is a primitivator of G/N then $(H/N) \cap (S/N) = \langle 1\rangle = (AN/N) \cap (S/N)$ and $(S/N)\cdot(H/N) = G/N$ hence $|H/N| = |G/N:S/N| = |AN/N|$ and so $H/N = AN/N$ therefore AN/N is a minimal normal subgroup of G/N. Now put $M = AN$. Clearly $(NS_i)M = G$, $i = 1,2$.

M/N is a solvable π-group. Indeed, M/N being a minimal normal subgroup of G/N, M/N is a chief factor of the π-solvable group G/N, hence M/N is a solvable π-group or a π'-group. If M/N is a π'-group, $M/N < O_{\pi'}(G/N)$ and $(G/N)/O_{\pi'}(G/N) \cong ((G/N)/(M/N))/(O_{\pi'}(G/N)/(M/N))$; but $(G/N)/(M/N) \cong G/M = ANS_1/AN \cong S_1/S_1 \cap (AN) \in F$; it follows that $(G/N)/O_{\pi'}(G/N) \in F$, which implies, by the π-closure of F, the contradiction $G/N \in F$. Thus M/N is a solvable π-group.

By 4.6, M/N is the unique minimal normal subgroup of G/N. Hence S_iN/N contain no minimal normal subgroup of G/N $(i=1,2)$; therefore $\mathrm{Core}_{G/N}(S_iN/N) = \langle 1\rangle$. Also S_1N/N is a maximal in G/N. For otherwise $S_1N/N < S^*/N < G/N$ which implies that $M/N \not\leq S^*/N$. Since M/N is abelian and $(S^*/N)(M/N) = G/N$, $(S^*/N) \cap (M/N) \trianglelefteq G$; that is $(S/N) \cap (M/N) \leq (S^*/N) \cap (M/N) = \langle 1\rangle$, and $S^*/N = S/N$, a contradiction. Similarly S_2N/N is a maximal subgroup of G/N.

Now we prove that NS_1/N and NS_2/N are conjugate in G/N. If $S_1N/N = \langle 1\rangle$, we have $G/N = (S_1N/N)(M/N) = M/N$; but $G/N = (S_2N/N)(M/N)$; hence $(S_2N/N)(M/N) = M/N$, that is $NS_2/N \leq M/N$; it follows that $(S_2N/N) \cap (M/N) = S_2N/N$; but, on the other hand, 4.1 implies that $(S_2N/N) \cap (M/N) = \langle 1\rangle$; we conclude that $S_2N/N = \langle 1\rangle$. This shows that $S_1N/N = \langle 1\rangle$ implies $S_2N/N = \langle 1\rangle$ and so, S_1N/N and S_2N/N are conjugate in G/N in this case. Let us suppose now that $S_1N/N \neq \langle 1\rangle$. We shall use 4.7. We know that $M/N \neq \langle 1\rangle$ is a solvable normal subgroup of G/N and S_1N/N is a primitivator of G/N. Let us prove that S_1N/N has a normal subgroup $L/N \neq \langle 1\rangle$ with $(|L/N|,|G/N:S_1N/N|) = 1$. Indeed, S_1N/N being non-trivial, let K/N be a minimal normal subgroup of S_1N/N • K/N is either a solvable π-group or a π'-group. If K/N is a solvable π-group, then, by 4.7(a), there is a normal subgroup $L/N \neq \langle 1\rangle$ of S_1N/N with $(|L/N|,|G/N:S_1N/N| = 1$. If K/N is a π'-group, then even $K/N \neq \langle 1\rangle$ is a normal subgroup of S_1N/N with $(|K/N|,|G/N:S_1N/N| = 1$. Applying now 4.7(b), NS_1/N and NS_2/N are conjugate in G/N. Hence NS_1 and NS_2 are conjugate in G.

Put $G^* = NS_1 = (NS_2)^g = S_2^gN$, where $g \in G$, and $A^* = A \cap G^*$. We apply the induction to G^*. We notice that A^* is an abelian normal subgroup of G^*, with $G^*/A^* \in F$ and S_1, S_2^g are F-maximal subgroups of G^*, with $A^*S_1 = (A \cap G^*)S_1 = S_1(A \cap G^*) = S_1A \cap G^* = G \cap G^* = G^*$ and $A^*S_2^g = S_2^g(A \cap G^*) = S_2^gA \cap G^* = G \cap G^* = G^*$. By induction, S_1 and S_2^g are conjugate in G^*, hence S_1 and S_2 are conjugate in G. ◻

Now we come to a theorem which is one of the main results of this chapter.

4.19 THEOREM:

If F is a π-Schunck class then any two F-projectors of a π-solvable group G are conjugate in G.

Proof:

Induct on $|G|$ and let S_1, S_2 be two F-projectors of G and M a minimal normal subgroup of G. We put $\overline{S}_1 = MS_1$ and $\overline{S}_2 = MS_2$.

$\overline{S}_1$ and $\overline{S}_2$ are conjugate in G. Indeed, $\overline{S}_1/M$ and $\overline{S}_2/M$ are F-projectors of G/M and hence, by induction, they are conjugate in G/M. But this means that $\overline{S}_1$ and $\overline{S}_2$ are conjugate in G, i.e.
$MS_1 = \overline{S}_1 = \overline{S}_2^g = MS_2^g$, with $g \in G$.

In order to prove that S_1 and S_2 are conjugate in G, we notice that since M is a minimal normal subgroup of the π-solvable group G two cases can arise:

(i) M is a solvable π-group. By 4.1, M is abelian. Now we show that the hypothesis of theorem 4.18(ii) applies: F is a π-Schunck class, $\overline{S}_1$ is a π-solvable group, M is an abelian normal subgroup of $\overline{S}_1$ with $\overline{S}_1/M = S_1M/M \cong S_1/M \cap S_1 \in F$ and we have $\overline{S}_1 = S_1M = S_2^gM$ where S_1 and S_2^g are F-maximal subgroups of $\overline{S}_1$. It follows that S_1 and S_2^g are conjugate in $\overline{S}_1$, hence S_1 and S_2 are conjugate in G.

(ii) If M is a π'-group. Then $M \leq O_{\pi'}(\overline{S}_1)$ and so $\overline{S}_1/O_{\pi'}(\overline{S}_1) \cong (\overline{S}_1/M)/(O_{\pi'}(\overline{S}_1)/M)$. Since $\overline{S}_1/M \in F$ we deduce that $\overline{S}_1/O_{\pi'}(\overline{S}_1) \in F$, and hence, F being π-closed, $\overline{S}_1 \in F$. By the F-maximality of S_1 and S_2^g, $S_1 = \overline{S}_1 = S_2^g$ where $g \in G$. The theorem is completely proved. ◻

Combining Theorems 4.17 and 4.18 we get the following result:

4.20 THEOREM:

If F is a π-Schunck class, then any π-solvable group has

F-projectors and any two of them are conjugate. □

In order to make our study of this subject complete, we need to prove the converse of Theorem 4.20 to conclude that the only π-homomorphs for which the finite π-solvable groups have projectors are the π-Schunck classes. To do that we first need two lemmas.

4.21 LEMMA:

If F is a π-homomorph, G is a π-solvable group, H an F-maximal proper subgroup of G, and N a minimal normal subgroup of G with $HN = G$, then N is abelian.

Proof:

N is a chief factor of G so there are two possibilities:

(i) N is a solvable π-group, in which case by 4.1, N is abelian.

(ii) N is a π'-group. Then $N \le O_{\pi'}(G)$, hence $G/O_{\pi'}(G) \cong (G/N)/(O_{\pi'}(G)/N)$. But $G/N = HN/N \cong H/H \cap N \in F$, because $H \in F$ and F is a homomorph. It follows that $G/O_{\pi'}(G) \in F$, which implies, by the π-closure of F, that $G \in F$. This is a contradiction to the F-maximality of $H < G$. □

4.22 LEMMA: If F is a π-homomorph, G a π-solvable group, H an F-maximal proper subgroup of G, and if there is a minimal normal subgroup N of G with $HN = G$, then:

(a) H is maximal in G;

(b) $H \cap N = \langle 1 \rangle$.

Proof:

(a) Let H^* given with $H \leq H^* < G$. If $H < H^*$ then there is an element $h^* \in H^* \setminus H$. Because $G = HN$, $h^* = hx$, with $h \in H$ and $x \in N$. Suppose we can prove that $N \cap H^* = \langle 1 \rangle$. Then $x = h^{-1}h^*$ is in $N \cap H^*$, so $x = 1$. But this implies $h^* = h \in H$, contradicting the choice of h^*. It follows that $H = H^*$.

To prove that $N \cap H^* = \langle 1 \rangle$ observe that $N \cap H^* \trianglelefteq G$. Indeed, if $g \in G$ then $g = h^*x$, with $h^* \in H^*$, $x \in N$, because $G = HN = H^*N$, and if $y \in N \cap H^*$ then $g^{-1}yg = (h^*x)^{-1}y(h^*x) = x^{-1}(h^{*-1}yh^*)x$.

If $z = h^{*-1}yh^*$ then $z \in N \cap H^*$, since $N \cap H^* \trianglelefteq H^*$. So $z \in N$. But by Lemma 4.21, N is abelian. This implies that $g^{-1}yg = x^{-1}zx = x^{-1}xz = z \in N \cap H^*$, which proves that $N \cap H^* \trianglelefteq G$. Now, $N \cap H^* \neq N$, because $N \cap H^* = N$ implies $N \leq H^*$, hence the contradiction $G = HN = H^*N = H^*$. Since N is a minimal normal subgroup of G, $N \cap H^* \trianglelefteq G$ and $N \cap H^* \neq N$ imply that $N \cap H^* = \langle 1 \rangle$.

(b) Setting $H^* = H$ in the proof of (a), we obtain $H \cap N = \langle 1 \rangle$. □

Now we are ready to prove the theorem we promised.

4.23 THEOREM:

A homomorph F with the property that any finite π-solvable group has F-projectors is a Schunck class.

Proof:

To show F is a Schunck class, suppose the contrary and let G be a finite π-solvable group of minimal order with respect to the conditions: $G \notin F$ and any primitive factor group of G is in F. Let M be a minimal normal subgroup of G. Then $G/M \in F$ by definition of G. Let H be an F-projector of G. It follows that HM/M is F-maximal in G/M, so $G = HM$. Applying Lemma 4.22, we conclude that H is maximal in G. Suppose G is not primitive. We then have $Core_G(H) \neq \langle 1\rangle$. So that $G/Core_G(H) \in F$, by definition of G. But $H/Core_G(H)$ is an F-projector of $G/Core_G(H)$. Hence $H = G$, contradicting the hypothesis $G \notin F$ and $H \in F$. Thus G is primitive, in contradiction to the choice of G. ◫

Now we study some aspects of the connection between projectors and covering subgroups in finite π-solvable groups.

The following lemma, which we need here, is an immediate consequence of 2.1.24.

4.24 LEMMA:

If F is a homomorph, G a finite group and H an F-projector of G which is maximal in G, then H is an F-covering subgroup of G.

Proof:

Let K be a subgroup of G with $H \leq K$. We distinguish two cases:

(i) $K = G$. Then H is an F-projector of $G = K$.

(ii) $K < G$. Then $H = K$. But $H \in F$ is its own F-projector. The lemma is proved. □

4.25 COROLLARY:

If F is a homomorph and G a group then any subgroup H of G with the properties:

(i) H is an F-projector of G;

(ii) H is a primitivator of G

is an F-covering subgroup of G.

Lemma 4.22 has the following consequence:

4.26 LEMMA:

If F is a π-homomorph, G a π-solvable group and H an F-projector of G with the property that there is a minimal normal subgroup N of G such that $HN = G$, then H is an F-covering subgroup of G.

Proof:

Suppose without loss of generality that $H < G$. Then the hypothesis of Lemma 4.22 applies and it follows that H is maximal in G. Hence H is an F-covering subgroup of G by 4.24. □

4.27 LEMMA:

Let F be a π-homomorph, G a π-solvable group and $H < G$ with the property that H is F-maximal in G. Then the following are equivalent.

(i) For any minimal normal subgroup N of G, $HN = G$;

(ii) H is a primitivator of G.

Proof:

(i) $\Rightarrow$ (ii). H is maximal in G, by 4.22. Further, $\mathrm{Core}_G(H) = \langle 1 \rangle$, for, if we suppose that $\mathrm{Core}_G(H) \neq \langle 1 \rangle$, it follows that G has a minimal normal subgroup N with $N \leq \mathrm{Core}_G(H)$. But this means $G = HN \leq H.\mathrm{Core}_G(H) = H$, i.e., $H = G$, in contradiction with $H < G$.

(ii) $\Rightarrow$ (i). This follows from Lemma 4.1. ◫

Chapter 5

PROJECTORS OF FINITE GROUPS

The intent of this chapter is to investigate the properties of projectors in groups that are not necessarily solvable or π-solvable. All groups considered here are assumed to be finite. Our first aim is to prove the existence of $\mathcal{F}$-projectors in all finite groups, where $\mathcal{F}$ is a Schunck class, and we begin with a definition.

5.1 DEFINITION: Let $\mathcal{F}$ be a class of finite groups.

$\mathcal{F}$ is E_ϕ-closed if $G/\phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

DEFINITION:

Let $N \leq G$. The subgroup H of G is a supplement of N if $HN = G$.

5.2 LEMMA:

Let $\mathcal{F}$ be a homomorph and suppose $\mathcal{F}$ is E_ϕ-closed. If G is a group with normal subgroup N and if $G/N \in \mathcal{F}$, then every minimal supplement of N in G belongs to $\mathcal{F}$.

Proof:

Let H be a minimal member of the set of subgroups which supplement of N in G. Since any supplement of $N \cap H$ in H is also a supplement of N in G, H is a minimal supplement of $N \cap H$ in H and therefore $N \cap H \leq \phi(H)$. Hence $H/\phi(H) = (H/(H \cap N))/(\phi(H)/(H \cap N)) \in \mathcal{F}$, since

$H/H \cap N \cong HN/N = G/N \in F$ and F is a homomorph. Since F is E_ϕ-closed, $H \in F$. □

Now we are ready to our first main result of this chapter.

5.3 THEOREM:

Let F be a class of groups. If F is a Schunck class, then every finite group possesses an F-projector. Conversely, if Y is a Schunck class containing F and if every group in Y possesses an F-projector, then F is a Schunck class.

Proof:

Let F be a Schunck class and let G be a group of minimal order not possessing an F-projector. Then $G \notin F$ and $G > \langle 1 \rangle$. Let A be a minimal normal subgroup of G. Then G/A has an F-projector W/A. If $W < G$, then W has an F-projector E. But then 2.1.21(c) implies that E is an F-projector of G, a contradiction. Hence $W = G$ and $G/A \in F$ for each minimal normal subgroup A of G. Since F is a homomorph and $G/A \in F$ for any minimal normal subgroup A of G, every proper quotient of G belongs to F. Thus, if G is not primitive, every primitive quotient of G belongs to F and hence $G \in F$, a contradiction. Hence G is primitive.

Suppose first that G has a unique minimal normal subgroup A and let U be a minimal supplement of A in G. By Lemma 5.2, $U \in F$. Let E be an F-maximal subgroup of G containing U. If $N \trianglelefteq G$ and $N > \langle 1 \rangle$, then N contains A, so that $EN/N = G/N \in F$ and hence EN/N is F-maximal in G/N. Thus E is an F-projector of G, again a contradiction.

In view of Corollary 4.3 we conclude that G has exactly two minimal normal subgroups A and B, and by 4.4 both of them are complemented by a maximal subgroup E. Then $E \cong G/A \in F$ and an argument similar to that given in the preceding paragraph shows that E is an F-projector of G. This contradiction completes the proof of the first assertion of the theorem.

For the converse, assume that every finite group belonging to the Schunck class Y possesses an F-projector. Let $G \in F$ and $N \trianglelefteq G$. Then $G \in Y$ and hence G has an F-projector E. Then $E = G$ and $EN/N = G/N$ is F-maximal in G/N; in particular, $G/N \in F$. Hence F is a homomorph.

It remains to show that F is a Schunck class. Suppose this is false, and let G be a group of minimal order such that every primitive quotient of G belongs to F but G does not. Then G is not primitive. Since $F \subseteq Y$ and Y is a Schunck class, we have $G \in Y$. Let E be an F-projector of G, so $E < G$, and let A be a minimal normal subgroup of G. The minimal choice of G implies that $G/A \in F$. Hence $EA = G$. Let S be a maximal subgroup of G containing E. Then $SA = G$ and since A is an arbitrary minimal normal subgroup, S is a maximal subgroup of G with $\mathrm{Core}_G(S) = \langle 1 \rangle$ and hence G is primitive, a final contradiction. □

In Chapter 2 we have seen that if F is a saturated formation and G is a solvable group, then the F-projectors of G are conjugate and coincide with the F-covering subgroups. These results are also true when F is a Schunck class and G is a solvable group (see Erickson [19], 1.3.8). However, these results are not valid in arbitrary finite groups.

5.4 EXAMPLE:

As we have seen in Example 2.1.23(c), the subgroups E and F of the simple group A_5 are N-projectors but not N-covering subgroups of A_5, where N, as before, is the class of finite nilpotent groups, and it is clear that E and F are not conjugate since they have different orders. In fact A_5 has 3 conjugacy classes of N-projectors but no N-covering subgroups which means that the covering subgroups may not exist at all in the general case.

5.5 DEFINITION:

Let U denote the class of finite solvable groups. By $\underline{UF}$ we denote the class of groups that are extensions of solvable groups by F-groups.

In our next main result, we will show that the F-covering subgroups and F-projectors do coincide in groups in UF. To do this we need two preliminary lemmas.

5.6 LEMMA: Let F be a Schunck class. Let A be a minimal normal subgroup of the finite group G with A abelian, $G/A \in F$ and $G \notin F$. Then A is complemented in G and the complements are precisely the F-projectors of G. Moreover, every F-projectors of G is an F-covering subgroup of G.

Proof:

Since F is a Schunck class, F is E_ϕ-closed. Since $G/A \in F$ and $G \notin F$, $A \not\leq \phi(G)$, so there is a maximal subgroup S of G such that

$A \not\leq S$. By 4.1, S is a complement of A in G. Also it is easily seen that each F-projector of G is a complement of A and (since it is a maximal subgroup) also an F-covering subgroup of G. Thus the F-covering subgroups and F-projectors of G coincide, and we need only to show that every complement E of A is an F-covering subgroup of G.

Let G be a minimal counter-example. By Theorem 5.3, G has an F-projector L. First suppose that E is not corefree, that is $\mathrm{Core}_G(E) \neq \langle 1 \rangle$ and let N be a minimal normal subgroup of G with $N \leq E$. If N is also contained in L, that is $N \leq L \cap E$, then since G/N is not counter-example E/N is an F-covering subgroup of G/N. By 2.1.16, then, E is an F-covering subgroup of G, a contradiction.

So $N \leq E$ and $N \not\leq L$ so that $NL = G$. Since A is abelian, $AN \leq C_G(A)$. Thus $AN \cap L \trianglelefteq G$. Moreover, $AN \cap L \not\leq E$ for otherwise we would obtain the contradiction $A \leq AN = AN \cap NL = N(AN \cap L) \leq E$. Hence $E(AN \cap L) = G$ and $G/(AN \cap L) \cong E \in F$. Since L is an F-covering subgroup of G, we have $G = L(AN \cap L) = L$, a contradiction.

Thus $\mathrm{Core}_G(E) = \langle 1 \rangle$ and A is the unique minimal normal subgroup of G. It follows easily that E is an F-covering subgroup of G, a final contradiction. □

5.7 <u>LEMMA</u>: Let F be a Schunck class. Let G be a group with a nilpotent normal subgroup N, and let E be an F-subgroup of G that supplements N in G. Then E is contained in an F-covering subgroup of G. In particular, if E is F-maximal in G, then E is an F-covering subgroup of G.

Proof:

The lemma is trivial when $N = \langle 1 \rangle$. Suppose $N \neq \langle 1 \rangle$ and let A be a minimal normal subgroup of G with $A \leq N$. The hypotheses are satisfied in G/A, so by induction we conclude that EA/A is contained in F^*/A, an F-covering subgroup of G/A. Therefore, $E \leq F^*$. The hypotheses are satisfied by F^* and its subgroups $F^* \cap N$ and N. Thus, if $F^* < G$, then by induction we have $E \leq F$, an F-covering subgroup of F^*. Now by 2.1.16, F is an F-covering subgroup of G, and so the lemma is established unless $F^* = G$.

However, if $F^* = G$, then $G/A = F^*/A \in F$ for any minimal normal subgroup A of G contained in N. By Lemma 5.6, G has an F-covering subgroup W. Then W is maximal in G, so $W \cap N$ is maximal in N and since N is nilpotent we have $W \cap N \trianglelefteq G$. If $W \cap N > \langle 1 \rangle$, by choosing A to be contained in $W \cap N$, we have $G = WA = W(W \cap N) = W$, and the conclusion follows trivially. Thus we may assume that $W \cap N = \langle 1 \rangle$ and $G \notin F$. Then $G = EN \cap WA = E(N \cap WA) = EA(N \cap W) = EA$, and Lemma 5.6 implies that E is F-covering subgroup of G. □

5.8 THEOREM:

Let F be a Schunck class of π-groups and let G be an extension of a π-solvable group by an F-group. Then every F-projector of G is an F-covering subgroup of G.

Proof:

Let $N \trianglelefteq G$ with N π-solvable and $G/N \in F$, and let E be an F-projector of G. Let A be a minimal normal of G with $A \leq N$. By induction EA/A is an F-covering subgroup of G/A.

If A is abelian, then 5.7 implies that E is an F-covering subgroup of EA. This conclusion also holds if A is a π'-group. In either case, E is F-covering subgroup of G. □

Even in the setting of Theorem 5.8, the F-projectors need not be conjugate. To introduce the example given by SCHNACKENBERG, in his work ("On injectors, projectors and normalizers of finite groups", Ph.D. Dissertation, Univ. of Wisconsin, 1972) we first give the definition of holomorph.

DEFINITION:

The holomorph of the group K is: $K \rtimes_\theta \mathrm{Aut}(K)$, where $\theta\colon \mathrm{Aut}(K) \to \mathrm{Aut}(K)$ is the identity map.

Schnachenberg gives an example of a Schunck class F and a group $G \in \mathcal{U}F$ having non-conjugate F-covering subgroups. The group G is the holomorph of $V(3,2)$, a 3-dimensional vector space over $GF(2)$; thus $G = V(3,2) \rtimes_\theta \mathrm{Aut}(V(3,2))$. But $\mathrm{Aut}(V(3,2)) \cong GL(3,2)$, since $V(3,2)$ is an elementary abelian group of order 2^3. The class F is the smallest Schunck class containing $GL(3,2)$. By Lemma 5.6, $V(3,2)$ is complemented in G and all the complements are F-covering subgroups of G. But a remark on page 161 of B. Huppert, Endliche Gruppen I, shows that the complements are not all conjugate.

Projectors of Direct Products and Well-Placed Subgroups:

Gaschütz, in his work "Selected Topics in the Theory of Solvable Groups" 1963, proved some properties of projectors in the direct products. Gaschütz' proof, which uses the conjugacy of F-covering subgroups, does not apply to arbitrary finite groups, and to treat the general case, we first need a definition and preliminary lemma.

5.9 DEFINITION:

Let F be a class of groups. We say F is D_0-closed if it is closed under direct products.

5.10 LEMMA: Let F be a D_0-closed homomorph. Let $G = AB$ with $A \trianglelefteq G$, $B \trianglelefteq G$. If G/A has an F-projector V/A and G/B has an F-projector W/B, then $(V \cap W)/(A \cap B)$ is an F-projector of $G/A \cap B$.

Proof:

Since the hypotheses hold in group $G/A \cap B$, by induction we may assume that $A \cap B = \langle 1 \rangle$. Suppose $V < G$. In view of the isomorphism $V/V \cap B \cong VB/B = G/B$, we find that $(V \cap W)/(V \cap B)$ is an F-projector of $V/V \cap B$. Indeed, $(V \cap W)/(V \cap B) = (V \cap W)/(V \cap W \cap B) \cong B(V \cap W)/B = (BV \cap W)/B = (G \cap W)/B = W/B$.

Also, V/A is an F-projector of V/A, and $A(V \cap B) = V \cap AB = V \cap G = V$. So the hypotheses of the lemma are satisfied in the group V. By induction, $V \cap W$ is an F-projector of V. Since $V \cap W$ is an F-projector of V and V/A is an F-projector of G/A, $V \cap W$ is an F-projector of G.

So we may assume that $V = G$, and by symmetry, also that $W = G$. But then $W = V = G = A \times B$, and since $B \cong (A \times B)/A = V/A \in F$ and similarly $A \cong W/B \in F$ and since F is D_o-closed we conclude that $G \in F$ so $V \cap W = G$ is an F-projector of G. □

5.11 THEOREM: Let F be a D_o-closed homormorph. If A and B have F-projectors E and F, then $E \times F$ is an F-projector of $A \times B$.

Proof:

Apply Lemma 5.10 with $V = FA$ and $W = EB$. $V \cap W = FA \cap EB = F(A \cap EB) = FE(A \cap B)$ by the modular law. But this latter group is $FE = F \times E$, so $F \times E$ is an F-projector of $A \times B$. □

REMARKS:

(i) If we let $P(F)$ denote the class of groups possessing an F-projector, then $P(F)$ is a D_o-closed homomorph whenever F is. This is because, if $G \in P(F)$ then G has an F-projector E and so EN/N is an F-projector of G/N for any $N \trianglelefteq G$, so $G/N \in P(F)$. Also if F is D_o-closed and $G,H \in P(F)$ with F-projectors E and F respectively, then by 5.11, $E \times F$ is an F-projector of $G \times H$. So $G \times H \in P(F)$.

(ii) The analog of Theorem 5.11 for F-covering subgroups is also true.

(iii) The conclusion of Theorem 5.11 fails under the alternative hypothesis that F is a Schunck class, since there exists a Schunck class F that is not D_o-closed and groups A and B such that $A,B \in F$ but $A \times B \notin F$ (see Erickson [19], 2.4.10).

5.12 <u>DEFINITIONS</u>:

(a) A subgroup W of the group G is called <u>well-placed</u> in G, if there is a chain $W = M_o \leq M_1 \leq \ldots \leq M_n = G$ such that $M_i = M_{i-1}F(M_i)$ $(1 \leq i \leq n-1)$, where $F(M_i)$ is the fitting subgroup of M_i.

(b) For each class $\mathcal{F}$ let $S_w\mathcal{F}$ be the class of all groups which are well-placed in some $\mathcal{F}$-group. We say $\mathcal{F}$ is <u>S_w-closed</u> if $S_w\mathcal{F} = \mathcal{F}$.

Examples of well-placed subgroups are readily found. The subgroup $E = \langle(123)\rangle$ of A_4 is well-placed in A_4, since $F(A_4) = V = \{I, (14)(23), (12)(34), (13)(24)\}$ and $EV = A_4$.

5.13 <u>THEOREM</u>: Let $\mathcal{F}$ be an S_w-closed Schunck class. Let G be a group of the form UN, where $U \leq G$, $N \trianglelefteq G$ and N is nilpotent. Assume that G has an $\mathcal{F}$-projector F such that $F = (F \cap U)(F \cap N)$. Then $F \cap U$ is an $\mathcal{F}$-projector of U.

<u>Proof</u>:

Let G be a minimal counterexample. Then $U < G$ and $N > \langle 1\rangle$. Also, by hypothesis, $F \cap U$ is a well-placed subgroup of F, for $F \cap N \leq F(F)$ so that $F = (F \cap U)F(F)$, so $F \cap U \in S_w\mathcal{F} = \mathcal{F}$. It follows that $F < G$ and $G \notin \mathcal{F}$.

Suppose that $FN < G$. Now F is $\mathcal{F}$-maximal in FN and hence is an $\mathcal{F}$-projector of FN, by 5.7. Thus the hypotheses are satisfied by the group $FN = G \cap FN = UN \cap FN = (U \cap FN)N$ and its $\mathcal{F}$-projector F, and hence the minimality of G implies that $F \cap U$ is an $\mathcal{F}$-projector of $U \cap FN$. But, since FN/N is an $\mathcal{F}$-projector of G/N, $(U \cap FN)/(U \cap N)$ is an $\mathcal{F}$-projector of $U/U \cap N$; indeed, $U/U \cap N \cong UN/N = G/N$ and $(U \cap FN)/(U \cap N) = (U \cap FN)/(U \cap FN \cap N) \cong (U \cap FN)N/N = FN/N$. Thus we have $F \cap U$ is an $\mathcal{F}$-projector of $U \cap FN$ and $(FN \cap U)/(U \cap N)$ is an $\mathcal{F}$-projector of $U/(U \cap N)$, hence $F \cap U$ is an $\mathcal{F}$-projector of U, a contradiction.

So we may assume that $FN = G$. In the counterexample G we may also assume that U is chosen as large as possible. Suppose U is not maximal in G. Let U_1 be a maximal subgroup of G that properly contains U. Then $U_1N = G$, and $F = (F \cap U)(F \cap N) \leq (F \cap U_1)(F \cap N) \leq F$, and hence the equality holds. Now we show that $F \cap U_1$ is an $\mathcal{F}$-projector of U_1: clearly $F \cap U_1$ is a well-placed subgroup of F, so that $F \cap U_1 \in S_w\mathcal{F} = \mathcal{F}$, and since F is $\mathcal{F}$-maximal in G, $F \cap U_1$ is $\mathcal{F}$-maximal in U_1. Since $U_1 = FN \cap U_1 = (F \cap U_1)(F \cap N)N \cap U_1 = (F \cap U_1)N \cap U_1 = (F \cap U_1)(N \cap U_1)$, by 5.7, $F \cap U_1$ is an $\mathcal{F}$-projector of U_1. Now the hypotheses of the theorem are satisfied by the group $U_1 = U(N \cap U_1)$ and its $\mathcal{F}$-projector $F \cap U_1$, so that the minimality of G implies that the subgroup $(F \cap U_1) \cap U = F \cap U$ is an $\mathcal{F}$-projector of U, a contradiction.

Hence $FN = G$ and U is maximal in G. Since N is nilpotent normal subgroup of G, the subgroup $H = U \cap N$ is normal in G. Moreover, $U = FN \cap U = (F \cap U)N \cap U = (F \cap U)(N \cap U)$; that is $U = (F \cap U)H$. Now $U/H \cong G/N \cong F/F \cap N \in \mathcal{F}$, so we have two cases:

(i) either U/H is $\mathcal{F}$-maximal in G/H; or

(ii) $G/H \in \mathcal{F}$.

Suppose (i) holds. By 5.7, U/H is an $\mathcal{F}$-projector of G/H. But $U = (F \cap U)H \leq FH$, and FH/H is $\mathcal{F}$-maximal in G/H, so that $U = FH$

and $F \leq U$. But then Lemma 5.7 implies that F is an F-projector of U, a contradiction.

Thus (ii) holds: $G/H \in F$. Since $G \notin F$, we must have $H > \langle 1 \rangle$. Let A be a minimal normal subgroup of G with $A \leq H$. The hypotheses of the theorem are satisfied by the group G/A and its F-projector FA/A, so the minimality of G implies that $(U \cap FA)/A$ is an F-projector of U/A.

If $AF < G$, then the argument used earlier in the proof (with N replaced by A) shows that $F \cap U$ is an F-projector of $U \cap FA$. But then $F \cap U$ is an F-projector of U, a contradiction.

Thus $FA = G$. But then F is maximal in G, $F \cap N \trianglelefteq G$, and since $U = (F \cap U)H$ we have $U(F \cap N) = FH = G$. Now $F/F \cap N$ is an F-projector of $G/F \cap N$; in view of the isomorphism $U/F \cap H \cong G/F \cap N$ we conclude that $(F \cap U)/(F \cap H)$ is an F-projector of $U/F \cap H$. But $F \cap U \in F$, so that $F \cap U$ is an F-projector of U, a final contradiction. ◫

In the statement of the previous theorem (Theorem 5.13) we assume the existence of an F-projector F such that $F = (F \cap U)(F \cap N)$. Such an F-projector always exists, even without the assumption of S_w-closure. Indeed, a somewhat stronger conclusion holds.

5.14 <u>THEOREM</u>: Let F be a Schunck class, and G be a group. Let $U_o, U_1, \ldots, U_n, N_o, \ldots, N_n$ be subgroups of G such that:

(i) $U_o = G$, $N_o = \langle 1 \rangle$,

(ii) N_i is a nilpotent normal subgroup of U_{i-1} $(i=1,\ldots,n)$,

(iii) $U_{i-1} = U_iN_i$ $(i=1,\ldots,n)$; and

(iv) $U_i \cap N_i \leq N_{i+1}$ $(i=0,\ldots,n-1)$.

Then there is an F-projector F of G such that:

(v) $F \cap U_{i-1} = (F \cap U_i)(F \cap N_i)$ $(i=1,\ldots,n)$.

Proof:

By Theorem 5.3, G has an F-projector. If $N_i = \langle 1\rangle$ for each i, then (V) holds trivially (for an arbitrary F-projector F of G). So we may assume that $N_i > \langle 1\rangle$ for some i. Let j be minimal such that $N_j > \langle 1\rangle$. Then $N_j \trianglelefteq G$. We may assume that $j=1$.

Let A be a minimal normal subgroup of G with $A \leq N_1$. Then by (iv), $U_1 \cap A \leq U_1 \cap N_1 \leq N_2$, $U_2 \cap A \leq U_2 \cap U_1 \cap A \leq U_2 \cap N_2 \leq N_3$, and in general, $U_{i-1} \cap A \leq N_i$ $(i=1,2,\ldots,n)$. Hence for $i=1,2,\ldots,n-1$, we have:

$$(*) \quad U_iA \cap N_iA = A(U_i \cap N_iA) = A(U_i \cap U_{i-1} \cap N_iA)$$

$$= A(U_i \cap N_i(U_{i-1} \cap A)) = A(U_i \cap N_i),$$

which also holds when $i=0$. The subgroups:

$$U_oA/A,\ldots,U_nA/A,\ N_1/A,\ldots,N_nA/A$$

satisfy (i) - (iii) in the group G/A and in view of (*) they satisfy (iv) also. By induction, there exists an F-projector F^*/A of G/A such that

$$F^* \cap U_{i-1}A = (F^* \cap U_iA)(F^* \cap N_iA) = A(F^* \cap U_i)(F^* \cap N_iA) \text{ for } i=1,\ldots,n,$$

$$= (F^* \cap U_i)(F^* \cap N_iA), \text{ since } A \le F^* \cap N_iA.$$

Upon intersecting with U_{i-1}, we obtain, for $i=1,\ldots,n$,

$$F^* \cap U_{i-1} = (F^* \cap U_i)(F^* \cap N_iA) \cap U_{i-1}$$

$$= (F^* \cap U_i)(F^* \cap N_iA \cap U_{i-1})$$

$$= (F^* \cap U_i)(F^* \cap N_i(A \cap U_{i-1}))$$

$$= (F^* \cap U_i)(F^* \cap N_i), \text{ since } A \cap U_{i-1} \le N_i \ (i=1,\ldots,n).$$

Clearly, the subgroups $F^* \cap U_o,\ldots,F^* \cap U_n$, $F^* \cap N_1,\ldots,F^* \cap N_n$ satisfy (i) - (iv) in the group F^*.

Thus if $F^* < G$, by induction there is an $\mathcal{F}$-projector F of F^* such that for $i=1,2,\ldots,n$,

$$F \cap (F^* \cap U_{i-1}) = (F \cap F^* \cap U_i)(F \cap F^* \cap N_i); \text{ that is,}$$

$F \cap U_{i-1} = (F \cap U_i)(F \cap N_i)$. Since F is an $\mathcal{F}$-projector of F^* and F^*/A is an $\mathcal{F}$-projector of G/A, F is an $\mathcal{F}$-projector of G, and the proof is complete in this case.

Therefore, $F^* = G$ and $G/A \in \mathcal{F}$. We may assume that $G \notin \mathcal{F}$. If E is an arbitrary $\mathcal{F}$-projector of G, then $EA = G$ and $A \nleq E$. Since E is maximal in G and N_1 is nilpotent, we have $E \cap N_1 \trianglelefteq G$, so that $E \cap N_1 = \langle 1 \rangle$, for in the contrary case we could have chosen A to be contained in $E \cap N_1$. It follows that $N_1 = A$ a minimal normal subgroup of G. Hence U_1 is a maximal subgroup of G complementing N_1 in G.

By emma 5.6, U_1 is an $\mathcal{F}$-projector of G, and (V) holds with $F = U_1$, because $U_i \leq U_1$ and so $U_1 \cap U_{i-1} = U_1 \cap U_iN_i = U_i(U_1 \cap N_i) = (U_1 \cap U_i)(U_1 \cap N_i)$. □

5.15 <u>COROLLARY</u>: Let $\mathcal{F}$ be an S_w-closed Schunck class. Let U be a well-placed subgroup of a group G. Then there exists an $\mathcal{F}$-projector F of G such that $F \cap U$ is an $\mathcal{F}$-projector of U.

<u>Proof</u>:

Since U is a well-placed subgroup of G, there is a chain:

$$U = U_n \leq \ldots \leq U_o = G$$

such that $U_{i-1} = U_iN_i$ with $N_i = \mathcal{F}(U_i)$, for $i=1,\ldots,n$. By theorem 5.14, there is an $\mathcal{F}$-projector F of G such that $F \cap U_{i-1} = (F \cap U_i)(F \cap N_i)$, $(i=1,\ldots,n)$. By theorem 5.13, $F \cap U_1$ is an $\mathcal{F}$-projector of U_1. The hypothesis of Theorem 5.13 are then satisfied in the group $U_1 = U_2N_2$, so that $F \cap U_2 = (F \cap U_1)(F \cap N_2)$ is an $\mathcal{F}$-projector of U_2. Continuing in this fashion, we obtain the desired conclusion. □

<u>The Class of Groups Whose $\mathcal{F}$-Projectors are $\mathcal{F}$-Covering Groups:</u>

<u>DEFINITION</u>: For a homomorph $\mathcal{F}$, we define a class

$$W(\mathcal{F}) = \{G: \text{every } \mathcal{F}\text{-projector of } G \text{ is an } \mathcal{F}\text{-covering subgroup of } G\}.$$

Thus $W(\mathcal{F})$ consists of groups whose $\mathcal{F}$-projectors and $\mathcal{F}$-covering subgroups coincide. Clearly $\mathcal{F} \subseteq W(\mathcal{F})$, and if $\mathcal{F}$ is a Schunck class, then by theorem 5.8 we have $\mathcal{U}\mathcal{F} \subseteq W(\mathcal{F})$. In our next proposition we will see that

$W(F)$ inherits certain closure properties from F, but to do this we first need a lemma about F-covering groups, which is of considerable interest in its own right.

5.16 <u>LEMMA</u>: Let F be a formation and let $A, B \trianglelefteq G$. Assume that V/A is an F-covering subgroup of G/A and W/B is an F-covering subgroup of G/B, and that $VB = WA$. Then $(V \cap W)/(A \cap B)$ is an F-covering subgroup of $G/A \cap B$.

<u>Proof</u>:

Since $A \leq V$, by the modular law, we have $(V \cap W)A = V \cap WA = V \cap VB = V$. Similarly $(V \cap W)B = W$, so $(V \cap W)/(V \cap W \cap A) \cong (V \cap W)A/A = V/A \in F$, and similarly $(V \cap W)/(V \cap W \cap B) \in F$, and since F is a formation, $(V \cap W)/(A \cap B) \in F$.

By induction we may assume that $A \cap B = \langle 1 \rangle$. Let $L = VB = WA = VW$.

If $V = G$, then $W = W \cap V \in F$. Since W/B is an F-covering subgroup of G/B, $V \cap W = W$ is an F-covering subgroup of G as required.

So we may assume that $V < G$. In view of the isomorphism $V/V \cap B \cong VB/B = L/B$ we find that $(V \cap W)/(V \cap B)$ is an F-covering subgroup of $G/V \cap B$; indeed $(V \cap W)/(V \cap B) = (V \cap W)/(V \cap W \cap B) \cong (V \cap W)B/B = W/B$ which is F-covering subgroup of G/B, so W/B is an F-covering subgroup of L/B as $L \leq G$. Therefore $(V \cap W)/(V \cap B)$ is an F-covering subgroup of $V/V \cap B$.

Also V/A is an F-covering subgroup of V/A and $(V \cap W)A = V = V(V \cap B)$, so the hypotheses of the lemma are satisfied in the group V, By induction, $V \cap W$ is an F-covering subgroup of V and hence, since V/A is an F-covering subgroup of G/A, we have $V \cap W$ is an F-covering subgroup of G. □

5.17 REMARK: An examination of the proof of the previous lemma, shows that the statement of the lemma is true also for F-projectors in place of F-covering groups, assuming (in the notation of the lemma) that $VA=WB=G$.

In our consideration of the class $W(F)$, we restrict our attention to the case in which F is a Schunck class (so that F-projectors exist in every finite group).

5.18 PROPOSITION: Let F be a Schunck class. Then:

(a) $W(F)$ is an E_{ϕ}-closed homomorph.

(b) If F is a formation, then $W(F)$ is a (saturated) formation.

(c) If F is D_0-closed, so is $W(F)$.

(d) If F is S_W-closed, so is $W(F)$.

Proof:

For a group G, let $\mathrm{Proj}(G)$ be the set of F-projectors of G and $\mathrm{Cov}(G)$ the set of F-covering subgroups of G.

(a) Let $G \in W(F)$ and let $N \trianglelefteq G$. Let $F/N \in \mathrm{Proj}(G/N)$. Let $E \in \mathrm{Proj}(F)$. Then $E \in \mathrm{Proj}(G)$ which implies that $E \in \mathrm{Cov}(G)$ and so $F/N = EN/N \in \mathrm{Cov}(G/N)$. Hence $G/N \in W(F)$ and therefore $W(F)$ is a homomorph. homomorph.

To prove that $W(F)$ is E_{ϕ}-closed, let $N \trianglelefteq G$ with $N \leq \phi(G)$ and $G/N \in W(F)$. Let $E \in \mathrm{Proj}(G)$. Then $EN/N \in \mathrm{Proj}(G/N0 = \mathrm{Cov}(G/N)$. Since E is F-maximal in EN and N is nilpotent, by 5.7, we have $E \in \mathrm{Cov}(EN)$. Thus $E \in \mathrm{Cov}(G)$, so $G \in W(F)$.

(b) Let $G \in W(F)$ and let $A, B \trianglelefteq G$ with G/A, $G/B \in W(F)$; we show that $G/A \cap B \in W(F)$. We may assume that $A \cap B = \langle 1 \rangle$. Let $E \in \mathrm{Proj}(G)$. Then $EA/A \in \mathrm{Proj}(G/A) = \mathrm{Cov}(G/A)$, $EB/B \in \mathrm{Proj}(G/B) = \mathrm{Cov}(G/B)$ and $(EA)B = (EB)A$, so Lemma 5.16 implies that $EA \cap EB \in \mathrm{Cov}(G)$. But $E \leq EA \cap EB$ and E is F-maximal in G; so $E = EA \cap EB \in \mathrm{Cov}(G)$. Thus $G \in W(F)$. Since $W(F)$ is an E_ϕ-closed homomorph by (a), $W(F)$ is a saturated formation.

(c) Let $G = H \times K$ with $H, K \in W(F)$, and let $P \in \mathrm{Proj}(G)$. Then $PH/H \in \mathrm{Proj}(G/H)$ and $PK/K \in \mathrm{Proj}(G/K)$. In view of the isomorphism $G/K \cong H$ we have $PK \cap H \in \mathrm{Proj}(H)$. Indeed $PK \cap H = (PK \cap H)/(K \cap H \cap PK) \cong K(PK \cap H)/K = (PK \cap HK)/K = PK/K \in \mathrm{Proj}(G/K)$. A similar argument shows that $PH \cap K \in \mathrm{Proj}(K)$. Thus $PK \cap H \in \mathrm{Proj}(H) = \mathrm{Cov}(H)$ and $PH \cap K \in \mathrm{Proj}(K) = \mathrm{Cov}(K)$. By (ii) of the remark after the proof of theorem 5.11, we have $(PK \cap H)(PH \cap K) \in \mathrm{Cov}(G)$. But

$$\begin{aligned}(PK \cap H)(PH \cap K) &= PH \cap K(PK \cap H) && \text{by the modular law}\\ &= PH \cap PK \cap HK && \text{by the modular law again}\\ &= PH \cap PK.\end{aligned}$$

Thus $PH \cap PK \in \mathrm{Cov}(G)$, and by F-maximality of P, we get $P = PH \cap PK$, so that $P \in \mathrm{Cov}(G)$, and $G \in W(F)$.

(d) We want to show that if M is a well-placed subgroup of $W(F)$-group, then $M \in W(F)$. It suffices to show that if M is a maximal subgroup of G, if $MF(G) = G$, and if $G \in W(F)$ then $M \in W(F)$. Let $E \in \mathrm{Proj}(M)$ and let $N = F(G)$. Then $EN/N \in \mathrm{Proj}(G/N)$. Let C be F-maximal in EN then $E \leq C$, and by 5.7, $C \in \mathrm{Cov}(EN)$. Hence $C \in \mathrm{Cov}(G)$. Now $C = C \cap EN = E(C \cap N) \leq (C \cap M)(C \cap N) \leq C$ so the equality holds. An application of Theorem 5.13 shows that $C \cap M \in Cov(M)$. But E is $\mathcal{F}$-maximal in M, whence $E = C \cap M$ and $E \in Cov(M)$. Hence $M \in W(\mathcal{F})$. □

Somewhat more generally, if $\mathcal{F}$ is a Schunck class and $\mathcal{Y}$ is a homomorph, we can form the class $W(\mathcal{F},\mathcal{Y}) = \{G$: every $\mathcal{F}$-projector of G is $\mathcal{Y}$-covering subgroup of $G\}$. Then $W(\mathcal{F},\mathcal{Y})$ is a homomorph, is a formation when $\mathcal{F}$ and $\mathcal{Y}$ are formations, is D_o-closed when $\mathcal{F}$ and $\mathcal{Y}$ are D_o-closed, and is S_w-closed when $\mathcal{Y}$ is S_w-closed and $\mathcal{F} \subseteq \mathcal{Y}$. The proofs employ arguments similar to those above, together with Lemma 2.2.1.

The following theorem gives a method for constructing a formation by using the concept of $\mathcal{F}$-covering subgroups.

5.19 THEOREM:

Let $\mathcal{F}$ be a formation. Let $\mathcal{X} = \{G$: for all $H \leq G$, H has an $\mathcal{F}$-covering subgroup$\}$; that is $\mathcal{X}$ is the class of groups all of whose subgroups have an $\mathcal{F}$-covering subgroup. Then $\mathcal{X}$ is a formation.

Proof:

Let $G \in \mathcal{X}$ and $N \trianglelefteq G$. We want to show $G/N \in \mathcal{X}$. Let H/N be a subgroup of G/N. Then $H \leq G$ and so H has an $\mathcal{F}$-covering subgroup K (say). But then KN/N is an $\mathcal{F}$-covering subgroup of H/N. Thus every subgroup of G/N has an $\mathcal{F}$-covering subgroup and therefore $G/N \in \mathcal{X}$.

Now suppose $\mathcal{X}$ is not a formation and let G be a group of minimal order having normal subgroups A and B with $G/A \in \mathcal{X}$, $G/B \in \mathcal{X}$ but $G \notin \mathcal{X}$. Then $A \cap B = \langle 1 \rangle$ (For otherwise $G/A \cap B \in \mathcal{X}$ by minimality of G).

We first show that every proper subgroup of G belongs to X. If $U < G$ then UA/A, as a subgroup of G/A, belongs to X. But $UA/A \cong U/U \cap A$. Therefore $U/U \cap A \in X$. Similarly $U/U \cap B \in X$. By minimality of G we conclude that $U \in X$.

Now by definition of X, we conclude that G does not have an F-covering subgroup. Since $G/AB \cong (G/A)/(AB/A) \in X$ (as $G/A \in X$ and X is a homomorph by the first part), let F/AB be an F-covering subgroup of G/AB. If $F < G$, then $F \in X$ and hence F has an F-covering subgroup E, which is also F-covering subgroup of G, a contradiction.

Thus $F = G$ and $G/AB \in F$. Let V/A and W/B be F-covering subgroups of G/A and G/B respectively. Since $G/AB \in F$, $VB = G = WA$, so that Lemma 5.16 implies that $V \cap W$ is an F-covering subgroup of G, a final contradiction. □

By the Remark following Lemma 5.16, the proof of this theorem also holds for F-projectors, and so we obtain:

5.20 <u>THEOREM</u>:

Let F be a formation. The class of groups all of whose subgroups have an F-projector is a formation. □

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