EX-POST EXCHANGE ACTIVITY AND THE EXISTENCE OF RISK AFFINITY, AND LEARNING-BY-DOING AND THE STRUCTURE OF EMPLOYMENT CONTRACTS UNDER UNCERTAINTY

by

Brian T. Pratt

B.A. (Hons.), Simon Fraser University, 1973

A THESIS SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Economics

SIMON FRASER UNIVERSITY

April, 1985

APPROVAL

Name: Brian T. Pratt

Degree: Ph.D.

Title of Thesis: Two Papers

Ex Post Exchange Activity and the Existence of Risk Affinity, and

Learning by Doing and the Structure of Employment Contracts Under Uncertainty

Examining Committee:

Chairman: James W. Dean

Terence Heaps
Senior Supervisor

Pao Cheng

John Heaney

John Herzog

Robert A. Jones
External Examiner
Associate Professor
Business Administration
University of British Columbia

Date Approved: April 9, 1985

Title of Thesis/Project/Extended Essay

Ex Post Exchange Activity and the Existence of Risk Affinity, and

Learning by Doing and the Structure of Employment Contracts

Under Uncertainty.

Author: ______________________

(signature)

Brian T. Pratt

(name)

April 9, 1985

(date)

ABSTRACT

This thesis is composed of two papers on economic behavior under uncertainty. A major objective of both papers is to show that the existing theory of behavior under uncertainty can be extended by employing Duality Theory.

The first paper extends the recent work on risk bearing in the face of multiple sources of uncertainty by showing that the risk aversion criteria established in those studies focussing on uncertain consumption vectors do not imply that individuals are risk averse. The paper also shows that risk aversion in the Arrow-Pratt sense is not equivalent to risk aversion with respect to multivariate risks in commodity prices and nominal income.

The major conclusion of this paper is that risk affinity in the face of uncertainty about income and commodity prices cannot be ruled out by imposing restrictions on preferences if the compensated demand functions for commodities are downward sloping. This result also holds when random variations in the rate of pure monetary inflation and nominal income induce random fluctuations in real income.

Human capital formation creates an irreversible change in the stock of a durable capital asset. The second paper of this thesis, in contrast to previous studies of human capital formation under uncertainty, emphasizes this irreversibility by developing a **multiperiod** model of this investment process. A distinguishing characteristic of the model is that it shows that differences in the opportunities for learning-by-doing may explain why some employees are paid a salary, while others are paid by the hour. The model implies, in particular, that employees who are in occupations which offer relatively rich (poor) opportunities for learning-by-doing will be

paid a salary (by the hour).

The major reason why the preceding model yields some insight into the structure of labour contracts is that it assumes that learning-by-doing involves an immediate, but implicit, sacrifice of labour income. Given this assumption, the analysis shows that (individual specific) marginal valuations of human capital are proportional to the opportunities for learning-by-doing. This result suggests that salary-rated employees have a greater incentive to invest in human capital than hourly-rated employees - even if these investments involve 'off-the-job' training.

The analysis contained in this study also implies: (i) that economic welfare can never be increased by deferring investments in human capital, and (ii) that the returns on human capital are uncertain even when labour income is not.

To Bonnie, who stood by;

and to Dad, who failed to see

the end of this endeavor.

ACKNOWLEDGEMENTS

Chances are this thesis would not have been completed without the assistance of Terry Heaps. I wish to thank him for his patience, for his encouragement, and for his active involvement during the writing of this thesis.

The major result of this thesis springs from a comment made by John Heaney. I would like to thank him for that insight and for his critical examination of many wild ideas.

I would like to thank Pao Cheng for assistance over a long period of time. His criticisms of my work have always improved my understanding.

Thanks also to John Herzog and to Prof. R.A. Jones for reading and commenting on the final draft.

Finally, I would like to express my appreciation to Leslie Romines for a superb job of typing.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

EX-POST EXCHANGE ACTIVITY AND THE EXISTENCE OF RISK AFFINITY

IN THE FACE OF UNCERTAINTY ABOUT INCOME AND COMMODITY PRICES

INTRODUCTION

Previous investigations into the theory of behavior under uncertainty have shown that the Arrow-Pratt theorems on risk bearing fail to hold when either: (i) portfolio separation does not obtain; (ii) insurance contracts do not provide full coverage against all risks; or when (iii) there is no riskless asset (see: Cass and Stiglitz (1972), Hart (1975), Ross (1981)). This paper extends the theory of risk bearing by developing several theorems about the behavior of individuals in the face of uncertainty about commodity prices (relative or absolute) and income or wealth. The major conclusion of the paper is that there must exist at least one risky situation where an individual will voluntarily sacrifice resources in order to be exposed to contemporaneous (temporal) random variations in income (or wealth) and commodity prices, if his compensated demand curves for commodities are downward sloping. If the major result of ordinal demand theory is correct, in other words, then risk aversion in the Arrow-Pratt sense does not imply risk aversion with respect to multivariate price-income risks.

The preceding conclusion is derived by applying what might be called Hanoch's Principle; namely that ordinal demand theory can be used to refute propositions about behavior under risk but, alas, it can never be used to confirm them[1]. This principle is applied throughout this study in order to refute the proposition that individuals will never be risk lovers if their Arrow-Pratt coefficients of relative and absolute risk aversion are nonnegative.

Section 1 contains a summary of the major conclusions which are

---

[1] Hanoch (1977, p. 414) appears to be the first one to state this proposition.

contained in those studies which focus upon the violation of the Arrow-Pratt theorems. The major objective of this section is to show that the theorems contained in those papers which discuss risk aversion with respect to uncertainty in consumption vectors (see: Duncan (1977), Hanoch (1977), Keeney (1973), Kihlstrom and Mirman (1974), Paroush (1975)) are invalid if individuals can engage in exchange activity in the (ex-post) spot markets for commodities. Hanoch's claim[2], in particular, that risk aversion in the Arrow-Pratt sense implies, and is implied by, risk aversion with respect to multivariate price-income risks is shown to be false.

In his classic article on risk aversion Pratt (1964) proved that complete interpersonal rankings of attitudes towards risks involving random variations in nominal income (or wealth) can be constructed by comparing risk premia which are formally related to a local measure of absolute risk aversion. Karni (1979) has generalized this theorem to cover situations where individuals face uncertainty about relative commodity prices and nominal income. In Section 3 of his paper he proves a theorem which states that concavity of the indirect utility function with respect to both nominal income and commodity prices is a necessary and sufficient condition for ruling out the possibility that an individual will sacrifice resources in order to be exposed to uncertainty about nominal income and commodity prices.

After proving this theorem he states:

[2] Op. Cit. p. 418-419.

> "The interpretation of this result must be approached with some care. It is well known that the indirect utility function is quasi-convex in prices for a given nominal income. Hence decision makers are in general not averse to random variations in relative prices and will be willing to sacrifice some income in order to be exposed to such risks."[3].

The major objective of Section 2 of this study is to prove that these comments should have been:

> "This result implies that individuals must be willing to sacrifice income in at least one situation where prices and income are uncertain, because it is impossible to have an indirect utility function which is concave with respect to both income and commodity prices.".

Section 3 of this study extends the theoretical foundations of the welfare economics of commodity price stabilization by delineating the sufficient conditions for risk affinity with respect to commodity price uncertainty. This topic has a long history. By using consumers surplus as a measure of the welfare effects induced by changes in relative commodity prices, Waugh (1944) was able to claim that consumers will prefer to face random commodity prices over prices which are stabilized at their arithmetic means. This proposition is now called the 'Waugh Paradox'. Massell (1969) then demonstrated - by using the concept of economic surplus - that when both producers and consumers are considered in a closed model that a regime of commodity price instability is pareto inferior to a regime of price stability. Samuelson (1972) reached the same conclusion as Massell. He emphasized however that those who gain from price stability must actually compensate those who lose; i.e., in the absence of compensation price stabilization is only potentially pareto superior.

---

3 Karni (1979) p. 1395.

In Section 3, the key assumption of the preceding studies, namely that consumers surplus is an accurate measure of price induced changes in utility, is relaxed in order to focus upon the partial equilibrium question: will consumers sacrifice income in order to avoid commodity price instability, if their nominal income is nonstochastic? The answer to this question is shown to depend upon: (i) the coefficient of relative risk aversion; (ii) the income elasticities of demand; and (iii) the number of prices being stabilized. Waugh's Paradox, in particular, is shown to be equivalent to the following theorem:

> "If uncertainty prevails about all commodity prices, then a consumer will definitely sacrifice income in order to be exposed to any multivariate price risks if: (a) his preferences are homothetic; and (b) his coefficient of relative risk aversion is always less than two.".

Condition (a) of the preceding theorem implies, as demonstrated in Section 3, that consumers surplus is a unique monetary measure of price-induced changes in utility. This measure of changes in economic welfare is an ordinal measure; i.e., it is invariant with respect to all monotonic transformations of a given utility function. Condition (b), in contrast, imposes a restriction upon a cardinal property of a utility function. These properties, which by definition are invariant only with respect to linear (or affine) transformations, must be restricted in some way in any theorem which assumes that behavior under risk is consistent with Expected Utility Theory. Surprisingly, however, these restrictions do not appear in the previously cited studies by Waugh, Massell and Samuelson. This study corrects part of their analyses by adding condition (b) to their statement of the Waugh Paradox, but it does not address the issues which arise in a full general equilibrium analysis of the welfare effects of price instability.

Many of the issues discussed in Section 3 are covered in the study of Turnovsky, Shalit and Schmitz (1982). This study extends their analysis by linking their welfare criteria to risk premium or income sacrifice functions; and also by showing that their claims about the relationship between consumer's surplus and behavior under risk are invalid.

Section 4 extends the analysis of the previous sections by developing a risk premium function for those risks which involve random fluctuations in nominal income and the rate of pure monetary inflation; i.e., for those unforeseen changes in the cost of living which are not accompanied by changes in relative commodity prices. The major conclusion of this section is that the appropriate risk premium function must be negative for at least one of these bivariate risks. This result holds for all utility functions which have the properties required by ordinal demand theory. It suggests moreover that it would be difficult to prove that complete, or partial indexation of contracts against pure monetary inflation is a pareto superior reform of a contract structure which is not indexed.

1. LIMITATIONS OF THE ARROW-PRATT THEOREMS

In their seminal works on risk aversion, Arrow (1965; 1971) and Pratt (1964) prove that restrictions on von Neumann-Morgenstern utility functions are sufficient for obtaining unambiguous predictions about behavior when end of period wealth is a function of a single random variable. Arrow (1971, Chapter 3) couples the qualitative properties of the preference based risk aversion functions to the qualitative properties of portfolio choices within a setting where investment opportunities are confined to a single risky asset and a riskless asset. He proves, in particular, that the hypothesis of nonincreasing absolute (relative) risk aversion is equivalent to the proposition that the amount (proportion) of initial wealth invested in the risky asset is a nondecreasing function of initial wealth. Pratt, on the other hand, shows that one investor will always spend less money on the risky asset than another investor, who has the same expectations if, and only if, he is always more risk averse; i.e., if his coefficient of absolute risk aversion is always larger. He proves as well: (i) that the maximum premium which an individual will pay for an insurance policy which provides full coverage against a single source of risk will never increase, as initial wealth increases, if the absolute risk aversion function is a nonincreasing function of wealth, and (ii) that one individual's insurance premium for full coverage insurance will always be at least as large as the risk premium of another individual, who has the same expectations if, and only if, he is always at least as risk averse.

Although the preceding theorems are used extensively for analyzing problems in the microeconomics of uncertainty, recent research has shown that they do not hold when income (or wealth) is a function of more than one random variable. Cass and Stiglitz (1972) show, in particular, that Arrow's comparative statics theorem will hold for portfolios consisting of two or more risky assets, and a single riskless asset, <u>if</u> preferences have the portfolio separation property, but that it will not hold - for these portfolios - when preferences are not restricted to belong to the class which implies portfolio separation. Hart (1975) proves, moreover, that portfolio separation is a necessary condition; i.e., that preferences must have the separation property if Arrow's theorem holds when there is more than one risky asset and a riskless asset.

Ross (1981) extends the analysis of the two preceding studies by showing that none of the portfolio selection theorems developed by Arrow and Pratt hold when a riskless asset does not exist. He also proves that all of Pratt's theorems about the demand for insurance are violated when individuals cannot purchase insurance which provides full coverage against <u>all</u> risks.

The central reason why the Arrow-Pratt theorems are not valid in the cases discussed above, is that the choices of expected utility maximizers are extremely sensitive to the structure of expectations in these situations. Indeed, it is fairly obvious that restrictions must be imposed upon both preferences <u>and</u> expectations in order to obtain unambiguous predictions about behavior when there are multiple sources of risk.

Kihlstrom, Romer and Williams (1981) show, for example, that Pratt's theorem about the insurance premiums for different individuals will hold, when some risks are not insurable, if: (i) end of period wealth is an additive function of the multiple sources of uncertainty; (ii) if all existing insurance contracts provide full coverage; (iii) if at least one of the two individuals has a coefficient of absolute risk aversion which is a nonincreasing function of initial wealth; and (iv) if all random variables are independently distributed. Ross (1981), on the other hand, develops a more restrictive measure of risk aversion and then shows that his measure will yield unambiguous predictions about behavior in many of the cases where the Arrow-Pratt theorems fail, if all sources of risk are uncorrelated. Machina (1982) shows, however, that Ross's theorems about the effects of changes in the distribution of end of period wealth upon insurance premiums and upon the relative demands for two risky assets are too specialized; i.e., they hold only when changes in the distribution of wealth are restricted to changes in the expected value of wealth. Machina also argues that the expected utility hypothesis must be modified to allow for the possibility that expected utility is not "linear in the probabilities" in order to obtain determinate comparative static results when changes in wealth distributions are not confined to exact horizontal translations.

Although the studies discussed above provide ample evidence for the verdict that the existing theory of behavior under uncertainty is not very robust, they do not cover the complications which arise when uncertainty prevails about the composition of future consumption vectors. Keeney (1973), Kihlstrom and Mirman (1974), Paroush (1975) and Duncan (1977) have asserted, for example, that the Arrow-Pratt analysis does not apply when direct utility functions contain more than one argument. Unfortunately, however, these authors do not provide any justification for this assertion. In each of these papers, to be more specific, the analysis focuses upon the development of risk aversion measures for risks which involve random variations in the composition of a basket of perishable commodities. One of the striking characteristics of this analysis is that, for some unspecified reason, it is always based upon the assumption that individuals cannot engage in exchange activity in the spot markets for commodities. Consider the following diagram from H. Levy and A. Levy (1984, p. 46), for example. This diagram is used to explain one of the major conclusions of this literature, namely that it is impossible to compare the attitudes towards risk of two individuals, when there are many commodities, unless their indifference maps are identical.

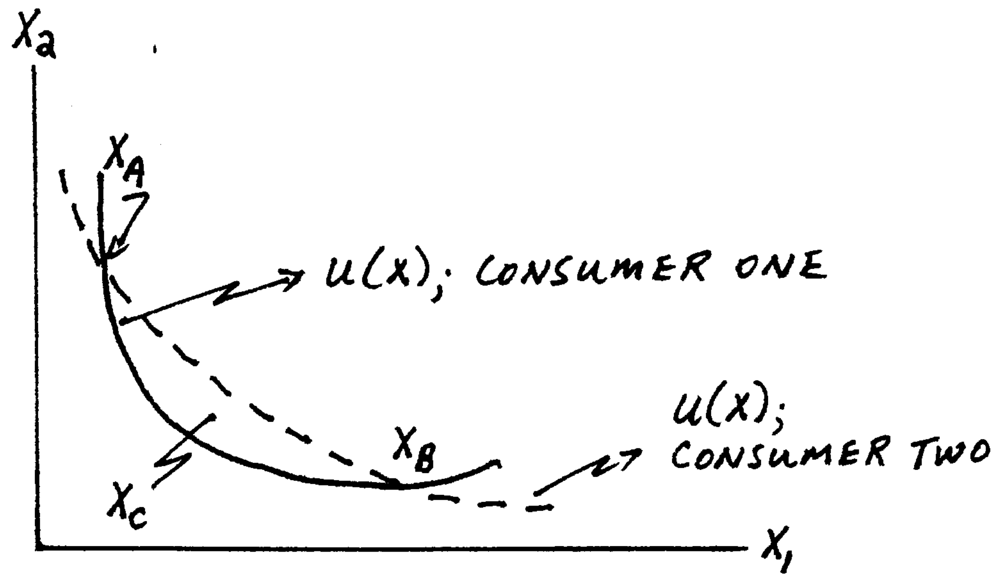

Figure 1

According to Levy and Levy this diagram implies that consumer 1 will always prefer the certain consumption vector X_C over a 50:50 gamble which yields either X_A or X_B; regardless of his attitudes towards risk, and also that consumer 2 will always prefer the gamble over X_C; regardless of his attitudes towards risk. Notice however that these assertions are based on the assumption that individuals cannot trade. Suppose, for example, that the "prizes" of the gamble could be sold for $Y_A = X_A'P$ and $Y_B = X_B'P$. Obviously, both individuals would prefer the gamble over X_C if $Y_A > X_C'P$ and $Y_B > X_C'P$. Consumer 1, in particular, will definitely prefer the gamble, given the price vector P, because it provides him with the opportunity to reach either the indifference surface $J(Y_A,P)$ or $J(Y_B,P)$, where $J(Y_A,P) > U(X_C)$ and $J(Y_B,P) > U(X_C)$.

Levy and Levy (1984), and the other authors who assert that the Arrow-Pratt framework necessarily breaks down when there is more than one consumption good, are concerned with the fact that the behavior of the two consumers in the situation described in Figure 1 does not depend on their attitudes towards risk; i.e., upon the cardinal properties of their direct utility functions. Unfortunately, however, they do not explain why they are interested in choice problems which assume that individuals cannot trade. The artificial nature of their analysis may be illustrated by Figure 2. This graph is taken directly from Kihlstrom and Mirman[4].

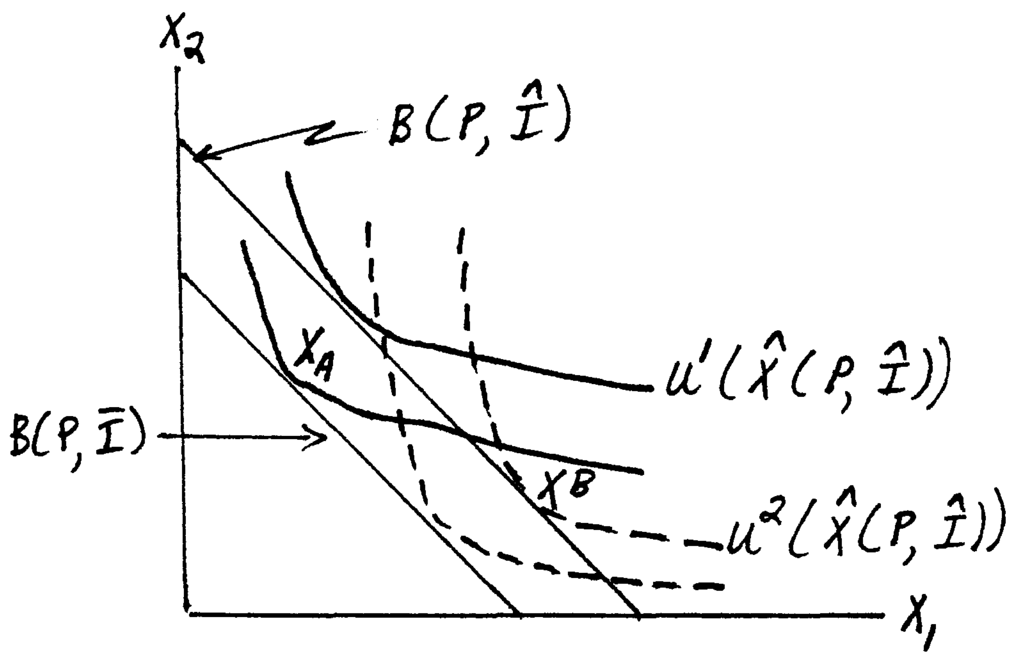

Figure 2

In their discussion of this graph, Kihlstrom and Mirman initially assume that both individuals are forced to consume the commodity basket X^A, when the budget constraint is equal to $B(P,\bar{I})$; and X^B when this constraint is equal to $B(P,\hat{I})$. This assumption implies that these

[4] Op. Cit., pp. 373-375.

individuals will disagree about a gamble which yields an outcome between $\bar{I}$ and $\hat{I}$. As Kihlstrom and Mirman point out, individual 1 will prefer the certain income $\bar{I}$ to the gamble, while individual 2 will prefer the gamble over the certain prospect $\bar{I}$.

In the second stage of their discussion, Kihlstrom and Mirman assume that both individuals can choose any consumption bundle which they can afford. Since all realizations of the random variable $\tilde{I}$ are always greater than or equal to $\bar{I}$ and since this immediately implies that both individuals will now prefer the gamble over the certain income $\bar{I}$, they acknowledge that risk aversion comparisons are now possible when they are based on the indirect utility functions. Surprisingly, however, they dismiss this conclusion with the assertion:

"The troublesome gambles ... (associated with forced consumption) ... are ruled out by this approach."[5].

Clearly, as their own analysis indicates, the Arrow-Pratt analysis was never intended to deal with these extremely artificial gambles.

In his study of the relationship between expected utility theory and ordinal demand theory, Hanoch (1977) claims that risk aversion in the face of uncertainty about both income and relative commodity prices implies, and is implied by, risk aversion in the face of uncertainty about nominal income. This claim is based on the following theorems.

<u>Theorem (H:1):</u> Hanoch (1977, p. 416)

If the commodity vector X and the Lagrange multiplier λ satisfy the optimium conditions for the utility maximizing problem:

---

[5] Loc. Cit.

$$\underset{X}{\text{MAX}}\ U(X) + \lambda(Y-P'X);\ P \gg O_{N-1},\ Y > 0;$$

then the Arrow-Pratt measure of relative risk aversion, say $\alpha(P,Y) \equiv -YJ_{YY}(Y,P)/J_Y(Y,P)$, is equivalent to:

$$\frac{U_X'X\,|U_{XX}|}{\begin{vmatrix} 0 & U_X' \\ U_X & U_{XX} \end{vmatrix}};$$

U_{XX} = the Hessian of U(X)

$U_X' = \left(U_1(X), \ldots, U_{N-1}(X)\right)$; $U_i(X)$ = marginal utility of X_i

After proving Theorem (H:1) Hanoch states that a consumer is risk averse to all (small) consumption risks of the form $\tilde{X} = (1+\tilde{\sigma})X$, $E(\tilde{\sigma}X) = O_{N-1}$; if, and only if, the direct utility function is concave at X, or equivalently if, and only if, $E\left(U(X+\tilde{\sigma}X)\right) < U(X)$. He then states that the following theorem holds when both income and relative prices are subject to uncertainty, if the deviations σX are not restricted to the expansion path.

Theorem (H:2): Hanoch (1977, p. 419)

U(X) is concave at X if, and only if, $\alpha(P,Y) \geq 0$; if $\alpha(P,Y) > 0$ then U(X) is strictly concave. That is, risk aversion with regard to income implies and is implied by risk aversion with regard to quantity bundles.

Hanoch's proof of Theorem(H:2) follows the following reasoning:

(i) $$E\left[U(X+\tilde{\sigma}X)\right] < U(X) \leftrightarrow \frac{U_X'X\,|U_{XX}|}{\begin{vmatrix} 0 & U_X' \\ U_X & U_{XX} \end{vmatrix}} > 0$$

(ii) Hence, from Theorem(H:1):

$$E\left[U(X+\tilde{\sigma}X)\right] < U(X) \leftrightarrow \alpha(P,Y) > 0$$

One of the missing elements in Hanoch's analysis is that he does not explain how Theorem (H:2) applies to situations where income and commodity prices are uncertain. In fact his interpretation of Theorem (H:2) appears to be incorrect.

Suppose, for example, that individuals could sell any realization, say X_1, of the random vector $\tilde{X} = (1+\tilde{\sigma})X$. Obviously, they will take advantage of this exchange opportunity if it allows them to attain the indifference surface $J(Y_1 = X_1'P_1;\ P_1)$; where P_1 is a realization of the random vector $\tilde{P}$, and $J(Y_1,P_1) > U(X_1)$. Indeed, given the fact that they can always consume the prize, X_1, it is obvious that $J(\ldots)$ must satisfy $J(Y_1,P_1) \geq U(X_1)$ for <u>any</u> vector (X_1,P_1).

The critical question then is whether or not $\alpha(P,Y) > 0$ is equivalent to $E\left(J(\tilde{Y},\tilde{P})\right) < U(X) = J(Y,P)$. This question, in turn, is equivalent to the question: can $J(Y,P)$ be concave in Y and P? The major result of Section 2 of this paper is: '<u>Since the compensated demand curves for commodities are downward sloping, J(...) cannot be concave in Y and P.</u>'. This result implies that Hanoch is incorrect; i.e., risk aversion with regard to uncertainty in nominal income does <u>not</u> imply risk aversion when income and commodity prices are uncertain, or conversely.

2. KARNI'S GENERALIZATION OF RISK AVERSION THEORY, HANOCH'S PRINCIPLE, AND THE EXISTENCE OF RISK AFFINITY

Consider a two-period setting where decisions about current consumption expenditures must be made prior to the resolution of uncertainty about nominal income and commodity spot prices in the second period. Suppose moreover that individuals respond to this temporal uncertainty by choosing that affordable (N-1) dimensional consumption vector, say $\hat{C}_0$, which maximizes the expected value of the conditional exchange plans $J(Y_1,P_1;C_0)$; where (Y_1,P_1) is some realization of the N dimensional random vector $(\tilde{Y},\tilde{P})$ and:

$$J(Y_1,P_1;C_0) \equiv \underset{C_1 \geq 0_{N-1}}{\text{MAX}} \quad U(C_1,C_0) + \lambda_1(Y_1-P_1'C_1); \quad \begin{matrix} \ldots P_1 \gg 0_{N-1} \\ \ldots Y_1 > 0 \end{matrix} \tag{2:1}$$

The maximum value function defined in (2:1) is a conditional or variable indirect utility function. As (2:1) indicates, it represents an individual's perception of his maximum attainable level of utility over the second period given: (i) some particular realization of the random vector $(\tilde{Y},\tilde{P})$, and (ii) <u>any</u> prior consumption decision $C_0 \geq 0_{N-1}$.

Let $f(\tilde{Z})$ represent the joint probability distribution of the N dimensional random vector $\tilde{Z} = (\tilde{Z}_Y,\tilde{Z}_P) = (\tilde{Y}-Y_0,\tilde{P}-P_0)$, where (Y_0,P_0) is an arbitrary vector and $Y_0 > 0$, $P_0 \gg 0_{N-1}$. Suppose also that an individual is offered a contract which guarantees that he will be able to choose a consumption vector at the beginning of period two which satisfies $Y_0 + E(\tilde{Z}_Y) - C_1'\left(P_0 + E(\tilde{Z}_P)\right) = 0$. Obviously if he accepts this contract his maximum attainable level of utility over the second period will be equal to $J(Y_0 + E(\tilde{Z}_Y),\ P_0 + E(\tilde{Z}_P);C_0)$; where this function is the attained maximum for:

$$\underset{C_1}{\text{MAX}}\ U(C_0,C_1) + \lambda\left[Y_0 + E(\tilde{Z}_Y) - C_1'\left(P_0 + E(\tilde{Z}_P)\right)\right] \tag{2:2}$$

The key idea of Karni's (1979) study is that interpersonal comparisons of risk aversion, in the face of contemporaneous random variations in Y and P, can be based upon a preference based measure of risk aversion: - if this measure ranks individuals according to their (maximum) demand prices for the contract described above. In the first stage of his analysis, these individual specific demand prices are measured by a risk premium function, $\pi(Y_0, P_0, f(\tilde{Z}); C_0)$; where $\pi(\ldots)$ is formally defined by:

$$J\left(Y_0 + E(\tilde{Z}_Y) - \pi,\ P_0 + E(\tilde{Z}_P); C_0\right) \equiv E\left(J(\tilde{Y}, \tilde{P}; C_0)\right) \equiv F(Y_0, P_0, f(\tilde{Z}); C_0) \quad (2:3)$$

Karni's definition of $\pi(\ldots)$ in (2:3) implies that the maximum demand price for the contract described earlier will be positive if, and only if, acceptance of the contract corresponds to an increase in economic welfare. To see this, consider the expenditure functions $Y_0 + E(\tilde{Z}_Y) \equiv e(P_0 + E(\tilde{Z}_P); J)$ and $Y_0 + E(\tilde{Z}_Y) - \pi \equiv e(P_0 + E(\tilde{Z}_P); F)$; where J is the attained level of utility in (2:2) and F is the conditional expected utility $E\left(J(\tilde{Y}, \tilde{P}; C_0)\right)$. Notice moreover that (2:3) implies:

$$\pi \equiv Y_0 + E(\tilde{Z}_Y) - e\left(P_0 + E(\tilde{Z}_P);\ F(..)\right);$$

or from (2:2) that

$$\pi \equiv e\left(P_0 + E(\tilde{Z}_P);\ J\right) - e(P_0 + E(\tilde{Z}_P);\ F) \quad (2:4)$$

Although the definition of π in (2:4) is equivalent to the definition in (2:3) it is more informative in the sense that it emphasizes that the sign of π depends on the difference between the conditional level of welfare offered by the risk avoiding contract, namely J, and the risk taking level of welfare, namely F. Given the usual assumption that preferences do not exhibit satiation; i.e., that e(..) is a monotonically increasing function of the level of utility, (2:4) implies, in particular, that $\pi \geq 0$ for all $f(\tilde{Z})$ is equivalent to $J \geq F$ for all $f(\tilde{Z})$.

In the second stage of his analysis, Karni assumes: (i) that the risks in Y and P are actuarially neutral; i.e., that $E(\tilde{Z}) = 0_N$; and (ii) that the price-income risks are small in the sense that $\text{PROB}\{(Z_1, \ldots, Z_N) \varepsilon B\} = 1$; where B is an N dimensional ball with center 0_N and radius $\delta \to 0$. Given these two assumptions he then derives the following approximate solution for his risk premium function:

$$\pi(Y_0, P_0, f(\tilde{Z}); C_0) = -1/2 \sum_{i=1,}^{N} \sum_{j=1}^{N} \sigma_{ij}(J_{ij}/J_Y)$$

$$= 1/2 \ \text{tr}(RV) \tag{2:5}$$

where: $\sigma_{ij} = \text{Cov}(\tilde{Z}_i, \tilde{Z}_j)$; $\tilde{Z}_1 = \tilde{Z}_Y$

V = the positive semidefinite covariance matrix $\{\sigma_{ij}\}$; i,j = 1,...,N;

R is a matrix measure of absolute risk aversion for small risks in Y and P, and;

$R = -[J_Y(Y_0, P_0; C_0)]^{-1} H_J(Y_0, P_0; C_0)$

H_J = the Hessian of J(...) evaluated at (Y_0, P_0, C_0)

Equation (2:5) is a key result in Karni's paper because he uses it to prove the following theorem on comparative risk aversion.

<u>Theorem (K:3):</u> Karni, pp. 1396-1397

Let $R_A(Y_0, P_0; C_0)$, $R_B(Y_0, P_0; C_0)$, $\pi_A(Y_0, P_0; C_0; f(\tilde{Z}))$ and $\pi_B(Y_0, P_0; C_0; f(\tilde{Z}))$ denote the matrix measures of local risk aversion and the risk premium functions corresponding to the (variable) indirect utility functions $J_A(Y_0,P_0;C_0)$ and $J_B(Y_0,P_0;C_0)$, respectively. Let $\tilde{Z}$ be an n dimensional random vector with a joint probability distribution, $f(\tilde{Z})$, $Y_0 + Z_Y \geq 0$, $P_0 + Z_P \gg O_{N-1}$ with $E(\tilde{Z}) = O_N$. Then the following conditions are equivalent representations of the proposition: 'individual B is at least as risk averse as individual A <u>regardless of any differences in their ordinal preferences</u>.':

(a) $R_B - R_A$ is positive semidefinite for all $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$

(b) $\pi_B \geq \pi_A$ for all $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$ and all N dimensional joint probability distributions $f(\tilde{Z})$ with mean $E(\tilde{Z}) = O_N$

(c) $\phi(t,P_0,C_0)$ is a concave function of t and P_0 for all $t \geq 0$, $P_0 \gg O_{N-1}$ for any given $C_0 \geq O_{N-1}$ where $t \equiv J_A(Y_0,P_0;C_0)$ and $\phi(t,P_0,C_0) \equiv J_B(e_A,P_0;C_0)$; e_A equal to the expenditure function $e_A\big(P_0,J_A(Y_0,P_0;C_0)\big)$.

Theorem (K:3) represents Karni's justification for considering R as a measure of risk aversion with respect to contemporaneous random variations in P and Y. The only essential difference between Theorem (K:3) and Theorem (1) in Pratt's study is that commodity prices are random. To see this, suppose that commodity prices are nonstochastic. In this case, Theorem (K:3) states that the following conditions are equivalent.

(a)
$$-\frac{B_{YY}(Y_0,P_0;C_0)}{B_Y(Y_0,P_0;C_0)} \geq -\frac{A_{YY}(Y_0,P_0;C_0)}{A_Y(Y_0,P_0;C_0)}$$

for all $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$

(b) $\pi_B \geq \pi_A$ for all $Y_0 \geq 0$, $P_0 > O_{N-1}$, $C_0 \geq O_{N-1}$ and all one dimensional probability distributions $f(\tilde{Z}_Y)$ with $E(\tilde{Z}_Y) = 0$

(c) There exists G: $J_B = G(J_A)$; $G' \geq 0$, $G'' \leq 0$

These three conditions are exactly the same as the conditions in Pratt's Theorem 1.

As indicated in the Introduction, the major problem with Karni's study is that he fails to recognize that his analysis implies that individuals cannot be risk averse for all multivariate risks in P and Y; i.e, for all V. Before he proves Theorem (K:3) he proves the following theorem:

Theorem (K:1): Karni, pp. 1394-1395

The following propositions are equivalent:

(a) R is positive semidefinite for all $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$

(b) $\pi\left(Y_0, P_0, C_0; f(\tilde{Z})\right) \geq 0$ for all $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$ and for all $f(\tilde{Z})$ with $E(\tilde{Z}) = O_N$

(c) $J(Y_0, P_0; C_0)$ is concave (and hence quasi-concave) with respect to all $Y_0 \geq 0$ and with respect to all $P_0 \gg O_{N-1}$; for any given $C_0 \geq O_{N-1}$

Karni's proof of Theorem (K:1) is based on the relationships depicted in (2:4) and (2:5)[6]. When satiation does not obtain or, equivalently, when $J(Y,P;C_0)$ is always a nondecreasing function of Y; (2:4) implies:

$$\pi(Y_0, P_0, f(\tilde{Z}); C_0) \geq 0 \quad \leftrightarrow \quad J\left(Y_0, P_0; C_0\right) - F\left(Y_0, P_0; C_0; f(\tilde{Z})\right) \geq 0 \quad (2:6)$$

Since the right-hand side of (2:6) is Jenson's inequality, (2:6) proves

6 Ibid., p. 1395.

(b) $\leftrightarrow$ (c).

(2:5), on the other hand, implies:

$$\pi \geq 0 \quad \leftrightarrow \quad 1/2 \ \mathrm{tr}(RV) \geq 0 \tag{2:7}$$

The trace on the right-hand side of (2:7), however, is non-negative for all V if, and only if, R is positive semidefinite; or, equivalently, if and only if $J(Y,P;C_0)$ is concave in Y and P[7]. This proves (a) $\leftrightarrow$ (b) and also, since (b) $\leftrightarrow$ (c), (a) $\leftrightarrow$ (c).

Although Theorem (K:1) is valid, it does not provide a theoretical definition of risk aversion with respect to multivariate price-income risks, as might be expected, because indirect utility functions cannot satisfy condition (c) of this theorem.

Consider the following theorem.

Theorem (A:3)

If $J(Y,P;C_0)$ is the maximum with respect to C_1 of $U(C_0,C_1)$ + $\lambda(Y-P'C_1)$, where $U(C_0,C_1)$ is monotonically increasing and quasi-concave in $C_0 \geq 0_{N-1}$, $C_1 \geq 0_{N-1}$, and, if there is more than one commodity, then J(...) cannot be jointly concave in both $P \gg 0_{N-1}$ and $Y \geq 0$, given any $C_0 \geq 0_{N-1}$.

There are at least two ways of proving Theorem (A:3). Appendix A contains a fairly abstract proof; i.e., it uses the properties of convex budget sets to generate the contradication: 'if J(...) is concave with respect to Y and P, and if J(...) is an indirect utility function, then it cannot be a maximum value function for a utility maximizing problem.'. The

7 The proof of this statement was developed by Terry Heaps; it can be supplied upon request.

following proof, which illustrates Hanoch's Principle, is more informative[8].

Proof of Theorem A:3

i) Suppose that the direct utility function, $U(C_0, C_1)$, is real valued, twice continuously differentiable, monotonically increasing, and quasi-concave with respect to all $C_0, C_1 \geq 0_{N-1}$. These conditions imply the existence of a maximum with respect to C_1 of $U(C_0, C_1) + \lambda(Y - P^1 C_1)$; $P \gg 0_{N-1}$, $Y > 0$. See Diewart (1973). Let $J(Y,P;C_0)$ denote this maximum.

ii) $J(\ldots)$ would be a concave function of Y and P if it was a linear function of Y and P. But can it be linear? Suppose that it was. Then, by Roy's Identity (see Appendix A), the demand functions $\hat{C}_{1i}$ would be equal to a_i/a_Y for all $i = 2,\ldots,N$, where a_Y and the a_i are constants; i.e., in this case there is no utility maximizing problem!

iii) Assume then, as required, that $J(\ldots)$ is nonlinear in Y and P but concave with respect to $P \gg 0_{N-1}$, $Y \geq 0$. Since any concave function is also quasi-concave (although not conversely), this assumption implies that $S = \{(Y,P)/J(Y,P;C_0) \geq \hat{J}\}$ is a convex set for $Y \geq 0$, $P \gg 0_{N-1}$. If S is a convex set, however, the level surfaces of J, or the expenditure functions $Y \equiv e(P;\hat{J})$ must be convex with respect to $P \gg 0_{N-1}$, given any parametric level of utility $\hat{J}$, when the conditions delineated in (i) above hold. These restrictions on $U(C_0, C_1)$ imply $J_Y > 0$ (or nonsatiation) and $\partial J/\partial P_i \leq 0$ for all $i = 2,\ldots,N$; $P \gg 0_{N-1}$; $Y \geq 0$. Hence,

8 This proof springs from an argument supplied to me by John Heaney.

in two dimensions:

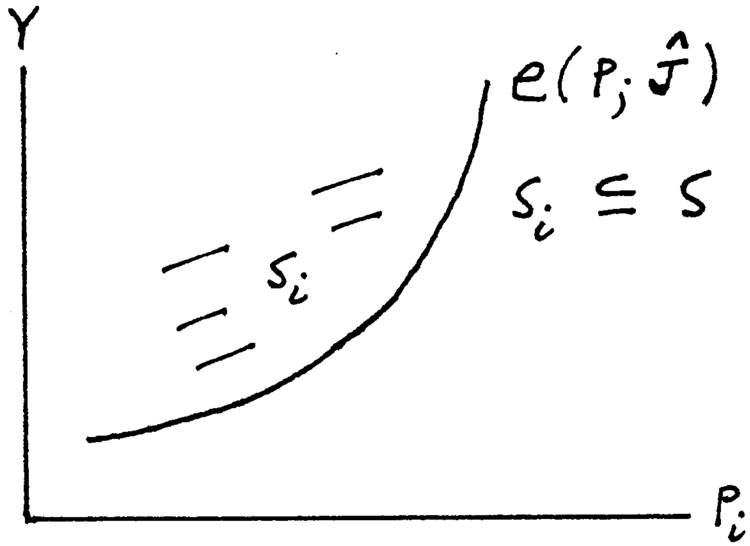

Figure 3

iv) Let $X_i^H(P,\hat{J})$ denote the compensated or 'Hicksian' demand function for commodity i. According to Shephard's lemma (see Appendix A), $X_i^H(P,\hat{J}) \equiv \partial e(P,\hat{J})/\partial P_i$. Hence, if $e(P,\hat{J})$ is convex in P, as indicated in (iii), all compensated demand functions must have a non-negative slope; i.e., $\partial X_i^H/\partial P_i \geq 0$ for all $i = 2,\ldots,N$.

v) Perhaps the most important result of ordinal demand theory is that $\partial X_i^H/\partial P_i \leq 0$ for all $i = 2,\ldots,N$ when the conditions delineated in (i) above hold. Since this result contradicts the implication of the assumption that J is concave in Y and P, this assumption must be rejected. This proves Theorem (A:3).

Theorem (K:1) and Theorem (A:3) jointly imply:

Theorem (2:1)

If behavior under uncertainty (in a two-period setting) is consistent with ordinal demand theory, then there must exist at least one 'small' price income risk for which $\pi\left(Y_0, P_0; C_0, f(\tilde{Z})\right) < 0$; i.e., risk affinity must obtain in at least one situation where commodity prices and nominal income are uncertain.

Notice, also, that (2:6) and Theorem (A:3) jointly imply:

Theorem (2:2)

Theorem (2:1) holds even if the multivariate price-income risks represented by $f(\tilde{Z})$ are large in the sense that $\text{PROB}\{(Z_1, \ldots Z_N) \varepsilon B\}$ is not equal to one where B is an N dimensional ball with center O_N and radius $\delta \to 0$.

The two preceding theorems, namely (2:1) and (2:2), are the most important theorems developed in this study. Although these theorems do not imply that economic behavior under uncertainty exhibits risk affinity more often than it exhibits risk aversion, they do suggest that risk affinity may be a fairly common phenomenon. They also suggest, for example, that the standard arguments about the gains from portfolio diversification are not very robust.

Before proceeding further, it is essential to explain why Ordinal Demand Theory can be used to refute conjectures about behavior under uncertainty and why the Arrow-Pratt analysis of risk aversion does not apply to price-income risks.

In order to highlight the essential difference between Ordinal Demand Theory (ODT) and Expected Utility Theory (EUT), consider a situation where two individuals, say A and B, have the same expectations about the risky prospect $B(\tilde{Y}, P) = \tilde{Y} - P'C_1$, where $\tilde{Y} = Y_L$ or Y_U and $E(\tilde{Y}) = Y_0$. Suppose, moreover, that these individuals have the same indifference maps and the same initial income (or wealth). See Figure 4 below.

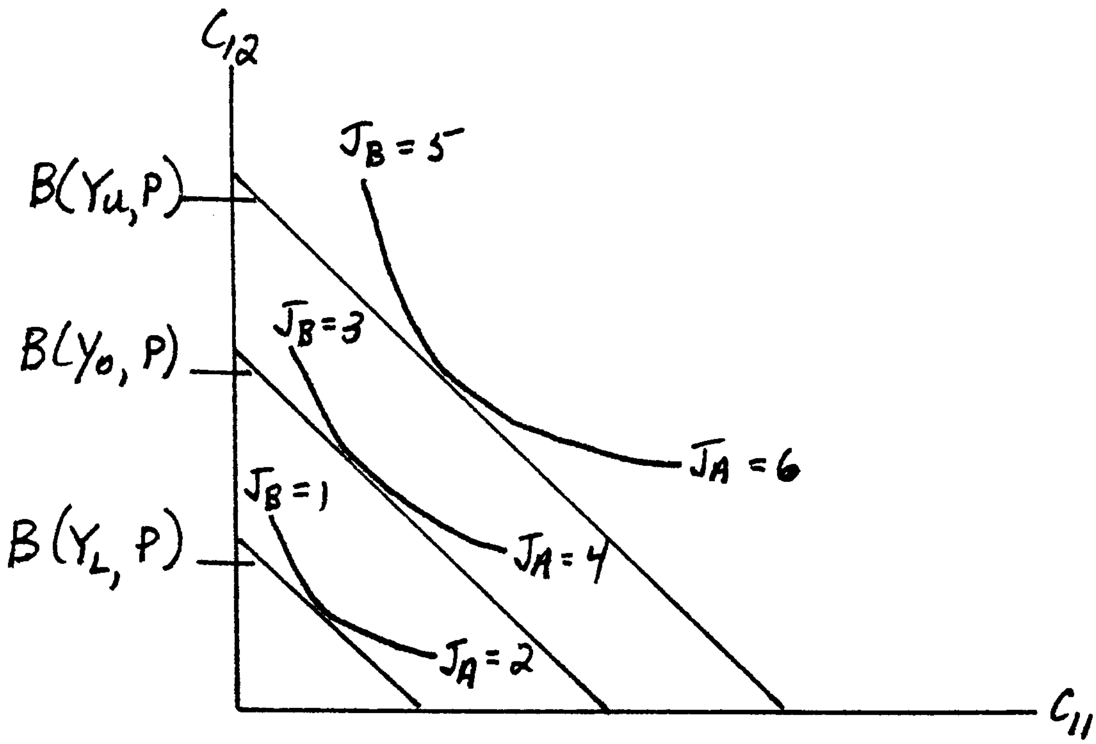

Figure 4

One of the basic characteristics of ODT is that utility functions (direct or indirect) are interpreted as mathematical devices for mapping preferences over commodity vectors onto the real line. Clearly, then, if this interpretation of utility functions was strictly adhered to in the development of EUT, this theory would imply that A and B would behave the same in the situation depicted in Figure 4.

The key idea of EUT, in contrast, is that the rule or utility function which is used to map commodity vectors onto the real line contains more information about behavior than the underlying indifference map. According to this theory, A and B will not behave the same because B's rule for assigning real numbers to commodity vectors yields different values for the differences between the numbers assigned to adjacent indifference surfaces (see Figure 4). In summary, then, the essential difference between EUT and ODT is that EUT uses utility functions for mapping preferences onto the real line and for developing additional conjectures about behavior. These additional conjectures are reflected in assumptions about the so-called cardinal properties of utility functions; i.e., they are logical implications of those properties of utility functions which are not invariant with respect to all monotonic transformations.

The key to understanding Hanoch's Principle is to recognize that the cardinal properties of utility functions yield information about their ordinal properties, but not conversely. Consider, for example, the hypothesis of constant relative risk aversion or, more specifically, $\alpha(P,Y;C_0) \neq f(P,Y;C_0)$ and $\alpha \neq 1$. As Hanoch has shown[9], this conjecture about economic behavior under uncertainty about Y alone is equivalent to assuming that preferences are described by:

$$J(Y,P) = (1-\alpha)^{-1}\big(Y/G(P)\big)^{1-\alpha} - H(P) \qquad (2:8)$$

where G(P) is homogeneous of degree 1, H(P) is homogeneous of degree zero, and $G(P) > 0$ for $P \gg 0_{N-1}$.

Since the indirect utility function has properties which are not invariant under transformations of the form $\phi(J)$, $\phi' > 0$, and properties which are, the common practice of calling it a 'cardinal utility function' is misleading. Notice, in particular, that the demand functions, $X_i^M(P,Y)$, implied by (2:8) and, hence, by 'α is equal to a constant,

9 Op. Cit. pp. 424-425.

and $\alpha \neq 1$', are ordinal properties of this function since, by Roy's Identity:

$$X_i^M(P,Y) \equiv - \frac{\phi'}{\phi'}\left(\frac{\partial J/\partial P_i}{\partial J/\partial Y}\right) \equiv - \frac{J_i(\ldots)}{J_Y(\ldots)} \; ; \; i = 2,\ldots,N \tag{2:9}$$

$$- \frac{J_i(\ldots)}{J_Y(\ldots)} \equiv \frac{G_i(P)}{G(P)} Y + H_i(P)G(P)^{1-\alpha} Y^{\alpha}; \; i = 2,\ldots,N \tag{2:10}$$

The demand functions in (2:10) are the necessary conditions of the preceding hypothesis about α. Clearly, then, this hypothesis can be rejected, at least in principle, by empirical investigation of the demands for commodities. Notice, however, that these empirical investigations can never be used to confirm this hypothesis because the demand functions in (2:10) are also the necessary conditions for the hypothesis:

$$\alpha = - \left\{\phi''(J_Y)^2 + \phi' J_{YY}\right\} / \phi' J_Y = \alpha(Y,P) \tag{2:11}$$

The central reason why the Arrow-Pratt analysis of risk aversion does not apply to multivariate price-income risks is that economic behavior under uncertainty depends, in general, upon both the cardinal and the ordinal properties of utility functions. The risk premium functions $\pi(\ldots f(\tilde{Z})\ldots)$, for example, can be written as $\pi^c + \pi^o$, where $\pi^c(\pi^o)$ is invariant under all affine (monotonic) transformations of $J(Y_0, P_0; C_0)$. To see this, consider the identity:

$$X_i^M(P,Y;C_0) \equiv X_i^H\big(P;J(Y,P;C_0)\big); \; i = 2,\ldots,N \tag{2:12}$$

Differentiating both sides of (2:12) with respect to P_j and then with respect to Y yields:

$$\partial X_i{}^M/\partial P_j \equiv \frac{\partial X_i{}^H}{\partial P_j} + \frac{\partial X_i{}^H}{\partial J}\frac{\partial J}{\partial P_j} \; ; \; i,j = 2,\ldots,N \qquad (2:13)$$

$$\partial X_i{}^M/\partial Y \equiv \frac{\partial X_i{}^H}{\partial J}\frac{\partial J}{\partial Y} \; ; \; i = 2,\ldots,N \qquad (2:14)$$

Substituting (2:14) into (2:13) yields the Slutsky equation:

$$\partial X_i{}^M/\partial P_j \equiv \frac{\partial X_i{}^H}{\partial P_j} + \frac{\partial X_i{}^M}{\partial Y}\left(\frac{\partial J/\partial P_j}{\partial J/\partial Y}\right)$$

$$\equiv K_{ij} - X_j{}^M \frac{\partial X_i{}^M}{\partial Y} \qquad (2:15)$$

where:

$$K_{ij} \equiv \partial X_i{}^H/\partial P_j$$

$$X_j{}^M \equiv -\frac{\partial J/\partial P_j}{\partial J/\partial Y} \text{ by Roy's Identity}$$

Recall, moreover, that $X_i{}^M(\ldots) \equiv -(\partial J/\partial P_j)/(\partial J/\partial Y)$ and, hence, that:

$$\partial X_i{}^M/\partial P_j \equiv R_{ij} + X_i{}^M R_{Yj} \qquad (2:16)$$

where:

$$R_{ij} \equiv -J_{ij}/J_Y; \; i,j = 2,\ldots,N$$

$$R_{Yj} \equiv -J_{Yj}/J_Y; \; j = 2,\ldots,N$$

(2:15) and (2:16) jointly imply:

$$R_{ij} = K_{ij} - X_j{}^M \frac{\partial X_i{}^M}{\partial Y} - X_i{}^M R_{Yj} \tag{2:17}$$

Notice, moreover, that Roy's Identity also implies:

$$\partial X_j{}^M/\partial Y \equiv R_{Yj} - X_j{}^M(J_{YY}/J_Y); \text{ or:}$$

$$R_{Yj} \equiv \partial X_j{}^M/\partial Y - X_j{}^M \alpha Y^{-1} \tag{2:18}$$

As indicated earlier, Karni has shown that:

$$\pi\left(Y_0, P_0; f(\tilde{Z}); C_0\right) = 1/2 \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} R_{ij};$$

$$\ldots R_{11} = R_{YY} = -J_{YY}/J_Y;$$

$$\ldots R_{Yj} = R_{1j} = -J_{Yj}/J_Y;$$

$$\ldots j = 2,\ldots,N \tag{2:5'}$$

When (2:17) and (2:18) are evaluated at (Y_0, P_0, C_0), therefore, these relationships imply (by substitution into (2:5')) that:

$$\pi = \pi^c + \pi^o \tag{2:19}$$

where π^c is a 'cardinal' risk premium function, π^o is an 'ordinal' risk premium function, and:

$$\pi^c(Y_0, P_0; f(\tilde{Z}); C_0) = \alpha Y_0{}^{-1}\left[1/2\ \sigma_Y^2 - X'V_{YP} + 1/2\ \mathrm{tr}(XX'V_P)\right] \tag{2:20}$$

$$\pi^o(Y_0, P_0; f(\tilde{Z}); C_0) = X_Y'V_{YP} + 1/2\ \mathrm{tr}\{(K-A)V_P\} \tag{2:21}$$

with:

$X' = \left(X_2{}^M(Y_0, P_0; C_0), \ldots, X_j{}^M(Y_0, P_0; C_0) \ldots \right)$

X'_Y = the N-1 row vector $\left\{\frac{\partial X_j{}^M}{\partial Y}(Y_0, P_0; C_0)\right\}$

K = the Slutsky matrix $\{K_{ij}\}$; i,j = 2,...,N

A = the matrix $\left\{X_i{}^M \frac{\partial X_j{}^M}{\partial Y} + X_j{}^M \frac{\partial X_i{}^M}{\partial Y}\right\}$; i,j = 2,...,N

V_{YP} = the N-1 row vector of covariances σ_{Yj}; j = 2,...,N

V_P = the commodity price covariance matrix $\{\sigma_{ij}\}$; i,j = 2,...,N

Equation (2:20) implies that Pratt's classic theorem on interpersonal comparisons of risk aversion, namely his Theorem (1), does not hold for (small) multivariate price-income risks. Suppose, for example, that individuals A and B have the same indifference map and the same expections but that $\alpha_A - \alpha_B > 0$, with α_A, $\alpha_B > 0$. In this case, $\pi_A{}^o - \pi_B{}^o = 0$ but, if $X'V_{YP} > 1/2\ \sigma_Y^2 + 1/2\ \mathrm{tr}(XX'V_P) > 0$, then:

$$\pi_A - \pi_B = \pi_A{}^c - \pi_B{}^c = (\alpha_A - \alpha_B)Y_0{}^{-1}\left(1/2\ \sigma_Y^2 - X'V_{YP} + 1/2\ \mathrm{tr}(XX'V_P)\right) < 0 \tag{2:22}$$

One of the basic characteristics of the Arrow-Pratt analysis is that restrictions on the cardinal properties of $J(Y,P;C_0)$ are sufficient for developing predictions about behavior. Equation (2:21) highlights the fact that these restrictions are not sufficient when uncertainty prevails about commodity prices. This ordinal risk premium function implies in particular that differences in the behavior of A and B will depend on whether or not they have the same indifference maps and, also, that risk affinity may obtain even if $\pi^c > 0$.

In closing this section, it should be pointed out that Markowitz's (1959) definition of a risk lover implicitly assumes that behavior under uncertainty about nominal income (or wealth) alone is not consistent with ordinal demand theory. Markowitz's definition of a risk lover is equivalent to the condition: $\pi \equiv e(P_0;J) - e(P_0,F) < 0$ for all univariate risks $\tilde{Z}_Y$, where $J = J\big(Y_0+E(\tilde{Z}_Y);P_0\big)$; $F = E\big(J(\tilde{Y};P_0)\big) = F(P_0;f(\tilde{Z}_Y);Y_0)$. Notice, however, that this condition violates the following proposition.

<u>Proposition (2:1)</u>

A minimum condition of ordinal demand theory is that $U(C_0,C_1)$ be quasi-concave in $C_0,C_1 \geq 0_{N-1}$. According to duality theory (see Appendix A), this condition is equivalent to the statement '$J(Y,P;C_0)$ is quasi-concave <u>or not convex</u>, with respect to all $Y \geq 0$, for any given $P \gg 0_{N-1}$; $C_0 \geq 0_{N-1}$'. Hence, if behavior under uncertainty is consistent with ordinal demand theory, then risk affinity, in the Markowitz sense, cannot obtain in all situations where Y is uncertain but P is certain.

3. <u>THE WELFARE ECONOMICS OF COMMODITY PRICE STABILIZATION: A PARTIAL EQUILIBRIUM ANALYSIS</u>

The two major objectives of the previously cited study of commodity price stabilization by Turnovsky <u>et al</u>. are: (i) to delineate the effects of commodity price stabilization upon the economic welfare of individuals when consumer's surplus does not provide a unique (monetary) measure of changes in utility; and (ii) to ascertain whether or not those arguments which use consumer's surplus as a welfare criterion (cf. Waugh, Massell and Samuelson, Op. Cit.) are correct. One of the limitations of their analysis is that they use the curvature properties of J(...) with respect to P = $(P_2,\ldots,P_T)$, $2 \leq T \leq N$, as welfare criteria without explaining how these properties are related to any monetary equivalent of price-induced changes in utility. This study closes this gap in their analysis by relating these properties to the sign of the following risk premium function.

$$\pi\left(Y_0, P_0, f(\tilde{Z}_P); C_0\right) \equiv e(P_0+E(\tilde{Z}_P);J) - e(P_0+E(\tilde{Z}_P);F);$$

$$\ldots\; J = J(Y_0, P_0+E(\tilde{Z}_P); C_0);$$

$$\ldots\; F = E\left(J(Y_0, \tilde{P}; C_0)\right) = F\left(Y_0, P_0, f(\tilde{Z}_P); C_0\right);$$

$$\ldots\; \tilde{Z}_P = (\tilde{Z}_2, \ldots \tilde{Z}_M),\; 2 \leq M \leq N \qquad (3:1)$$

The major limitation of the study of Turnovsky et al., however, is that some of their conclusions are not consistent with the major axiom of EUT; namely, that behavior under risk depends upon the cardinal properties of utility functions. Many of their theorems, to be more specific, contain statements of the form: 'if consumer's surplus is a unique measure of price-induced changes in utility, then their measure will also (correctly) indicate a welfare loss from price stabilization in this case'[10]. These statements are incorrect for one basic reason: consumer's surplus is an ordinal property of any given utility function, and these properties alone cannot be sufficient conditions in any theorem about behavior under uncertainty, if this behavior is consistent with EUT!

---

[10] Cf. Turnovsky et al., Op. Cit.; Propositions 1, 5 and 6.

Surprisingly, perhaps, those conclusions in Turnovsky et al., which are inconsistent with EUT, are developed from an analysis of consumer's surplus which is not correct. In the initial stages of their formal analysis, they claim that the coefficient of relative risk aversion must be equal to all of the income elasticities of demand of those goods whose prices are changing, if consumer's surplus is a unique measure of changes in utility[11]. This proposition, which violates Hanoch's Principle, is derived from a prior statement which asserts that consumer's surplus will be unique if, and only if, the marginal utility of income does not vary with any of those prices which are undergoing discrete changes[12]. This claim, and the implied predictions about behavior which are based on it, is incorrect. Indeed, as shown below, consumer's surplus can be unique even if λ or $J_Y(\ldots)$ is a function of all commodity prices.

The references cited by Turnovsky et al. indicate that their discussion of consumer's surplus was based upon Silberberg's (1972) paper. Since the analysis in the latter paper is not completely correct however (see also Varian (1978) and Burns (1973)), Section (3:1) below derives the necessary and sufficient conditions under which consumer's surplus is unique. This analysis implies, contrary to Turnovsky et al., that the original statement of Waugh Paradox is incorrect.

(3:1) Consumer's Surplus[13]

Given any indirect utility function, J(Y,P), the monetary equivalent

---

11 Ibid., p. 141.

12 Ibid., p. 140.

13 Given the previously cited errors in the literature on consumer's surplus, the analysis in this section was developed independently. Since all of the analysis is a straightforward application of Microeconomic Theory, however, it should be considered to be in the public domain.

of the total change in the maximum obtainable level of utility arising from the vector of price changes $dP = (dP_2, \ldots, dP_N)$ is given by:

$$dJ/J_Y = - \sum_{i=2}^{N} X_i^M dP_i ; \; dY = 0 \qquad (3:2)$$

In order to derive the monetary equivalent of a discrete price change, $\Delta P = (\Delta P_2, \ldots, \Delta P_N)$, it is necessary to integrate (3:2) over the closed interval $[P_H P_L]$, where $P_{Hi} > P_{Li}$ for at least one P_i. When more than one price changes, this integral is a line integral; i.e., its value depends upon the sequence of price changes or upon the path of integration. The critical question, as far as the uniqueness of consumer's surplus is concerned, is: What are the conditions under which the integral

$$\int_{P_L}^{P_H} X' dP = - \int_{P_L}^{P_H} dJ/J_Y$$

is path independent?

In the case where T commodity prices are changing, $1 \leq T \leq N-1$, the necessary and sufficient conditions for path independence are[14]:

$$\partial X_i^M / \partial P_j = \partial X_j^M / \partial P_i ; \; i,j = 1, \ldots, T \qquad (3:3)$$

The conditions in (3:3) must hold for all P_H, P_L, Y. They are equivalent, moreover, to the condition that all goods whose prices are subject to variation have the same income elasticities of demand.

[14] These conditions are well known; for example, see Silberberg (1972) or any advanced calculus text.

Proof

According to the Slutsky equation, the conditions in (3:3) are equivalent to:

$$\frac{\partial X_i^H}{\partial P_j} - X_j^M \frac{\partial X_i^M}{\partial Y} = \frac{\partial X_j^H}{\partial P_i} - X_i^M \frac{\partial X_j^M}{\partial Y};\ i,j = 1,\ldots,T \qquad (3:3)$$

Since the Slutsky matrix is symmetric (and negative semidefinite), however;

$$\partial X_i^H/\partial P_j = \partial X_j^H/\partial P_i \text{ for all } i,j = 2,\ldots,N$$

Hence, (3:3) is equivalent to:

$$\eta_i(Y,P) = (Y/X_i^M)\frac{\partial X_i^M}{\partial Y} = \eta_j(Y,P);\ i,j = 1,\ldots,T \qquad (3:4)$$

Q.E.D.

In the limiting case, where all commodity prices are undergoing discrete changes, (3:4) will hold if, and only if, the underlying direct utility function is homothetic or, equivalently, when J(Y,P) has the form $\phi(Yf(P))$.

Proof

Differentiating the budget constraint $Y - P'X = 0$, with respect to Y, and then using $\partial X_i^M/\partial Y = \eta_i X_i^M Y^{-1}$ yields the relationship $\sum_i B_i\eta_i = 1$, where B_i is the fraction of money income expended on commodity i. When (3:3) or, equivalently, (3:4) holds, then:

$$\eta \sum_{i=1}^{T} B_i + \sum_{i=T+1}^{N-1} B_i\eta_i = 1 \qquad (3:5)$$

Hence, if T = N-1 and (3:3) or (3:4) holds, then:

$$\eta \sum_{i=2}^{N-1} B_i = \eta(1) = 1 \tag{3:6}$$

(3:6) holds, however, if and only if $J = \phi\big(Yf(P)\big)$; that is, if and only if U(X) is homothetic[15].

Q.E.D.

Silberberg and, hence, Turnovsky et al. claim that the integrability conditions in (3:3) will hold if and only if the marginal utility of money income is independent of all P_i and P_j, where $i,j = 1,\ldots,T$ and $1 \leq T \leq N-1$. This independence condition is a sufficient conditon but it is not a necessary condition.

Proof

Consider the equality:

$$\lambda\left(\frac{\partial X_i^M}{\partial P_j} - \frac{\partial X_j^M}{\partial P_i}\right) = X_j^M \frac{\partial \lambda}{\partial P_i} - X_i^M \frac{\partial \lambda}{\partial P_j} \tag{3:7}$$

(3:7) is derived from Roy's Identity, $J_i = -X_i^M\lambda$, and the equality $J_{ij} = J_{ji}$; $i,j = 2,\ldots,N$. Clearly, if the right-hand side of (3:7) equals zero even when $\partial\lambda/\partial P_i$ and $\partial\lambda/\partial P_j$ are not equal to 0, then the integrability conditions in (3:3) will hold for $\lambda > 0$. Conversely, if $\lambda > 0$ and (3:3) holds, then:

$$X_j^M \frac{\partial \lambda}{\partial P_i} = X_i^M \frac{\partial \lambda}{\partial P_j} \tag{3:8}$$

---
15 See Varian (1978).

(3:8) holds of course when $\partial\lambda/\partial P_i = \partial\lambda/\partial P_j = 0$; clearly, however, this is not required.

Q.E.D.

Consider any homothetic indirect utility function which has the form:

$$J(Y,P) = \phi\left((1-\alpha)^{-1}\left(Y/G(P)\right)^{1-\alpha}-H\right);\ \alpha \neq 1,\ \phi' > 0 \tag{3:9}$$

Any member of (3:9) is a counterexample to the claim that λ must be independent of those prices which are changing if consumer's surplus is unique. To see this, notice first that:

$$X_i^M(P,Y) = G_i(P)G(P)^{-1}Y;\ i = 2,\ldots,N \tag{3:10}$$

$$J_Y \equiv \lambda(Y,P) = Y^{-\alpha}G(P)^{\alpha-1} \cdot \phi' \tag{3:11}$$

(3:10) satisfies the path independence conditions; i.e.,

$$\frac{\partial X_i^M}{\partial P_j} = G_{ij}(P)G(P)^{-1}Y - G_i(P)G_j(P)G(P)^{-2}Y = \frac{\partial X_j^M}{\partial P_i};\ i,j = 2,\ldots,N \tag{3:12}$$

One of the characteristics of the path independence conditions in (3:12) is that they hold for any utility function described by (3:9) and, hence, for an extremely large number of values for the coefficient of relative risk aversion. Clearly, then, Turnovsky et al. must be incorrect in stating that restrictions on the coefficient of relative risk aversion are necessary conditions for the uniqueness of consumer's surplus.

The following propositions summarize the main results of the preceding discussion.

Proposition (3:1)

If all commodity prices are undergoing discrete changes, then consumer's surplus is a unique measure of changes in welfare if, and only if, preferences are homothetic.

Proposition (3:2)

If more than one but not all prices are changing, then consumer's surplus is a unique welfare measure if, and only if, $\eta_i = \eta_j$ for all of those commodities whose prices are changing.

Proposition (3:3)

The conditions in Propositions (3:1) and (3:2) can hold for any value of the coefficient of relative risk aversion. Hence, consumer's surplus can be a unique welfare measure for any value of $\alpha(Y,P)$.

(3:2) Sufficient Conditions for Risk Affinity When Commodity Prices are Uncertain

The risk premium function in (3:1) and Theorem (A:3) of Section 2 jointly rule out the possibility that risk aversion will obtain for all multivariate commodity price risks when future nominal income is non-stochastic. Hence, the following analysis concentrates upon identifying the conditions which imply that individuals will sacrifice resources in order to be exposed to uncertainty about commodity prices.

Consider the following theorem:

Theorem (3:1)

If a consumer is uncertain about all of the second period spot prices for commodities and if his nominal income over the second period is nonstochastic, then he will always sacrifice current income in order to be exposed to these (large or small) multivariate price risks if: (a) his preferences, $U(C_0, C_1)$, are homothetic with respect to C_1, and (b) his coefficient of relative risk aversion, $\alpha(Y_0, P_0; C_0)$ is always less than 2.

Proof

By using the identity in (3:1), Jenson's inequality and the definition of convexity, it is easy to show that the following statements are equivalent.

(i) $\pi\left(Y_0, P_0, f(\tilde{Z}p); C_0\right) < 0$ for all $f(\tilde{Z}p)$ with mean $E(\tilde{Z}p)$, $Y_0 \geq 0$, $P_0 \gg O_{N-1}$, $C_0 \geq O_{N-1}$

(ii) $J(Y_0, P_0; C_0)$ is strictly convex with respect to all $P_0 \gg O_{N-1}$; for any given $Y_0 > 0$, $C_0 \geq O_{N-1}$.

(iii) The matrix $Jpp = (J_{ij})$; $i,j = 2,\ldots,N$ is positive semi-definite so that $t'Jppt \geq 0$ for all vectors $t \neq O_{N-1}$.

From (2:17) and (2:18), however:

$$J_{ij} = -J_Y\left(K_{ij} - Y^{-1}(\eta_i + \eta_j - \alpha)X_i^M X_j^M\right)$$

If preferences are homothetic with respect to C_1 or, equivalently, if $\eta_i \equiv 1$ for all $i = 2,\ldots,N$, then

$$J_{ij} = -J_Y\left(K_{ij} - X_i{}^M X_j{}^M Y^{-1}(2-\alpha)\right)$$

$$t'J_{pp}t = -J_Y t'Kt + J_Y Y^{-1}(2-\alpha)t'XX't$$

K is the symmetric, negative semidefinite Slutsky substitution matrix; hence, $-J_Y t'Kt \geq 0$, given nonsatiation, or $J_Y > 0$. Since $X_i{}^M > 0$ for all $i = 2,\ldots,N$ however (by nonsatiation and quasi-concavity of $U(C_0, C_1)$), $t'XX't > 0$. Clearly, then, if α is always less than 2, then $|J_{PP}|$ is always greater than 0 or, equivalently, $\pi(\ldots f(\tilde{Z}_P))$ is always less than 0.

Theorem (3:1) is an exact statement of the Waugh Paradox. It completes Waugh's original argument, which was based on consumer's surplus alone, by adding the condition that $\alpha < 2$. The essential difference between Theorem (3:1) and Proposition 6 in Turnovsky et al. is that the latter proposition contains the statement: "For consumer's surplus to be an exact measure of utility change in this case requires $\alpha \equiv 1$, in which case this measure (alone) will also indicate a (welfare) loss from (price) stabilization.". This statement, as indicated previously, is not consistent with their maintained assumption that behavior under uncertainty is in accordance with E.U.T.

When uncertainty prevails about a subset of commodity prices, say $S = \{P_i > 0 \mid i = 1,\ldots,T;\ 1 < T \leq N\}$, then risk affinity or $\pi(\ldots f(\tilde{Z}_P)\ldots) < 0$ will obtain for any multivariate risk described by $f(\tilde{Z}_P)$ if: (a) $\eta_i = \eta_S$ for all $X_i{}^M$ associated with S; and (b) α is always less than $2\eta_S$. In this case, (2:17) and (2:18) imply:

$$J_{ij} = -J_Y\left(K_{ij} - X_i^M X_j^M Y^{-1}(2\eta_S - \alpha)\right)$$

$$t'J_{pp}t = -J_Y t'Kt + J_Y Y^{-1}(2\eta_S - \alpha)t'XX't \qquad (3{:}13)$$

(3:13) implies that J(...) is strictly convex with respect to the prices in S or, equivalently, that $\pi(\ldots f(\tilde{Z}p))$ is always less than zero <u>if</u> $2\eta_S - \alpha > 0$ always holds. This proves:

<u>Theorem (3:2)</u>

When uncertainty prevails about more than one but not all commodity prices, an individual will always sacrifice income in order to be exposed to price uncertainty if: (a) his preference structure implies that consumer's surplus is a unique welfare measure; and (b) his coefficient of relative risk aversion is always less than $2\eta_S$, where η_S is the common income elasticity of demand for those commodities whose prices are uncertain.

Finally, for completeness, consider:

<u>Theorem (3:3)</u>

A consumer will always sacrifice current income in order to gamble on uncertainty about a single commodity price, say P_i, if his α always satisfies $\alpha < 2\eta_i$.

Theorem (3:3) follows immediately from the definition of $\pi(\ldots f(\tilde{Z}_i))$ implied by Equation (3:1) and the fact that $J_{ii} = -J_Y\left(K_{ii} - (2\eta_i - \alpha)(X_i^M)^2\right)Y^{-1}$ is greater than 0 when $(2\eta_i - \alpha) > 0$.

In closing this section, it should be noted that the sufficient conditions in all of the three preceding theorems always include restrictions upon the cardinal properties of $J(Y,P;C_0)$. This property of these theorems is essential, given that behavior under uncertainty is consistent with Expected Utility Theory.

4. RISK AFFINITY IN THE FACE OF UNCERTAINTY ABOUT NOMINAL INCOME AND THE RATE OF PURE (MONETARY) INFLATION

This section extends the analysis of the previous sections by proving two theorems about the effect of unpredictable fluctuations in real income upon the economic welfare of individuals. The analysis assumes: (a) that all variations in the cost of living are not accompanied by changes in the relative spot prices of commodities; and (b) that there is no unique relationship between nominal income and the rate of pure (monetary) inflation. Within this setting, the critical question is: Will individuals sacrifice current income in order to avoid uncertainty in their real income?

Consider:

Theorem (4:1)

An individual must be willing to sacrifice current income in order to be exposed to random variations in his real income for at least one risk which involves contemporaneous random variations in the cost of living and in his nominal income if:

i) relative commodity prices are not changing; and

ii) there is no unique relationship between his future nominal income and the rate of pure (monetary) inflation.

Theorem (4:1) is implied by one of the classic propositions of economic theory; i.e., it must hold if economic behavior does not exhibit 'money illusion'. It holds, in other words, because $J(Y,P;C_0)$ is homogeneous of degree zero in Y and P for any given $C_0 \geq O_{N-1}$.

To see this, suppose that an individual's expectations about the rate of pure inflation over the second period, say $Z_I = (P/P_0)-1$, and about his future nominal income are described by a bivariate probability distribution, $f(\tilde{Z}_{YI})$, where $\tilde{Z}_{YI}$ is the random vector $(\tilde{Z}_Y,\tilde{Z}_I) = \left(\tilde{Y}-Y_0,(\tilde{P}/P_0)-1\right)$. Assume, moreover, that this individual is offered a contract which guarantees attainment of $J^* = J\left(Y_0+E(\tilde{Z}_Y),E(\tilde{P});C_0\right)$ over the second period. His maximum demand price for this contract, say $\pi\left(Y_0,P_0,F(\tilde{Z}_{YI});C_0\right)$, is defined by:

$$\pi\left(\ldots f(\tilde{Z}_{YI})\ldots\right) = e\left(E(\tilde{P});J^*\right) - e\left(E(\tilde{P});F\right);$$
$$\ldots F = E\left(J(\tilde{Y},P_0(1+\tilde{Z}_I);C_0)\right)$$

One of the basic propositions of ordinal demand theory, as indicated previously, is that $J(Y,P;C_0)$ is homogeneous of degree zero with respect to Y and P. This condition implies:

$$\sum_{i=2}^{N} J_i P_i + J_Y Y = P'J_P + J_Y Y = 0 \qquad (4:2)$$

and also:

$$\sum_{i=2}^{N} J_{Yi} P_i + J_{YY} Y = P'J_{YP} + J_{YY} Y = -J_Y \qquad (4:3)$$

$$\sum_{i=2}^{N}\sum_{j=2}^{N} J_{ij} P_i P_j + Y \sum_{i=2}^{N} P_i J_{iY} = P'J_{PP}P + YP'J_{YP} \qquad (4:4)$$
$$= -P'J_P$$

According to Equation (4:3):

$$\alpha(Y,P;C_0) = 1 + \frac{P'J_{YP}}{J_Y} \qquad (4:5)$$

Notice, moreover, that (4:2), (4:4) and (4:5) yield:

$$\alpha(Y,P;C_0) = 2 - \frac{P'J_{PP}P}{J_Y Y} \qquad (4:6)$$

Consider the Hessian of $J\big(Y,P_0(1+Z_I);C_0\big)$ with respect to Y and Z_I; namely:

$$H = \begin{pmatrix} J_{YY} & J_{YI} \\ J_{IY} & J_{II} \end{pmatrix} = \begin{pmatrix} -J_Y Y^{-1}\alpha & P'_0 J_{YP} \\ P_0{}'J_{YP} & P_0{}'J_{PP}P_0 \end{pmatrix} \qquad (4:7)$$

When $P = P_0(1+Z_I)$ is substituted into (4:5) and (4:6) and when the resulting expressions are substituted into (4:7), we obtain:

$$H = J_Y \begin{pmatrix} -Y^{-1}\alpha & (\alpha-1)(1+Z_I)^{-1} \\ (\alpha-1)(1+Z_I)^{-1} & (2-\alpha)Y(1+Z_I)^{-2} \end{pmatrix} \qquad (4:8)$$

The major implication of (4:8) is that it rules out the possibility of having $\pi(\ldots f(\tilde{Z}_{YI})\ldots) \geq 0$ for <u>all</u> bivariate risks in Y and Z_I. This condition requires both $J_{YY} \leq 0$ (or $-J_Y Y^{-1}\alpha \leq 0$) and $|H| \geq 0$. These two inequalities cannot hold simultaneously however since, according to (4:8), $|H| = -J_Y^2(1+Z_I)^{-2} < 0$. In summary, then, if $\pi\big(\ldots f(\tilde{Z}_{YI})\big)$ cannot always be non-negative, there must exist at least one risk, $f(\tilde{Z}_{YI})$, where $\pi\big(\ldots f(\tilde{Z}_{YI})\big) < 0$, as Theorem (4:1) states.

As indicated earlier in Proposition (2:1), ordinal demand theory implies that $J(Y,P;C_0)$ must be at least quasi-concave, or not convex, in Y. This characteristic of these indirect utility functions implies that $\pi(\ldots f(\tilde{Z}_{YI}))<0$ cannot hold for all risks in Y and Z_I. In summary:

Theorem (4:2)

Risk affinity, or $\pi(\ldots f(\tilde{Z}_{YI}\ldots)<0$, cannot hold for all bivariate risks in Y and Z_I when conditions (i) and (ii) of Theorem (4:1) hold.

Intuition suggests that investors will prefer portfolios whose nominal returns have a relatively large covariance with the rate of inflation and that they would also demand higher expected nominal rates of return on risky assets with increases in the variance in the cost of living. Sercu (1981) has shown that these suppositions are not valid in general. His analysis is based on the assumption that expected utility is a function of the mean and variance of real portfolio returns. He demonstrates:
(a) that required nominal rates of return will increase with increases in the variance of $\tilde{Z}_I$ only if $\alpha > 2$; and (b) that investors will prefer portfolios with relatively high covariances with $\tilde{Z}_I$ only if $\alpha > 1$. The analysis presented in this study suggests that these conditions are also sufficient. It also suggests, moreover, that they are valid even when individuals choose portfolios which are not mean-variance efficient.

Suppose, in particular, that $E(\tilde{Z}_Y) = E(\tilde{Z}_I) = 0$ and that the risks in Y and Z_I are small in the sense defined earlier in this paper. In this case, the usual second order Taylor series expansion for $\pi(\ldots f(\tilde{Z}_{YI})\ldots)$ yields:

$$\pi(\ldots f(\tilde{Z}\quad)\ldots) = -1/2\ J_Y^{-1}\ \mathrm{tr}(H\Omega) \tag{4:9}$$

where all derivatives are evaluated at (Y_0, P_0, C_0) and Ω is the covariance matrix corresponding to $f(\tilde{Z}_{YI})$.

(4:9) and (4:8) jointly imply:

$$\frac{\partial \pi(\ldots f(\tilde{Z}_{YI}))}{\partial \, Var(\tilde{Z}_I)} = 1/2 \; Y_0 (\alpha - 2) \tag{4:10}$$

$$\frac{\partial \pi(\ldots f(\tilde{Z}_{YI}))}{\partial \, Cov(\tilde{Y}, \tilde{Z}_I)} = 1 - \alpha \tag{4:11}$$

According to (4:10), $\pi(\ldots)$ increases with $Var(\tilde{Z}_I)$ if, and only if, $\alpha > 2$. If individuals prefer to be exposed to increases in the covariance between Y and Z_I however, then from (4:11), $\partial\pi/\partial \, Cov(\tilde{Y},\tilde{Z}_I) < 0$ or $\alpha > 1$. Conversely, if $\alpha > 1$, then $\partial\pi/\partial \, Cov(\tilde{Y},\tilde{Z}_I) < 0$.

The underlying economic reason why individuals may prefer to gamble on uncertainty in their real income is that their expectations may be dominated by the fact that changes in the cost of living have an asymmetric effect upon real income. When the cost of living, say $I = 1+Z_I$, decreases by 50%, for example, real income, or YI^{-1}, doubles. When I increases by 50%, on the other hand, real income decreases by only one-third. Clearly, then, an individual may prefer to gamble because his marginal utility of income does not decline so rapidly that a decrease in real income of 33% reduces his welfare more than a doubling of his real income enhances it.

One of the puzzling aspects of the contract formation process is that contracts are not indexed against inflation as frequently as might be expected. Although Theorem (4:1) does not imply that individuals prefer to hold securities which are not indexed against inflation, it does suggest that it would be difficult to show that indexed contracts are always pareto superior.

5. CONCLUDING REMARKS

Perhaps the major contribution of this study is that it uses microeconomic theory to show that any argument which maintains that risk affinity is a prominent characteristic of behavior under uncertainty should not be automatically rejected. The major limitation of the analysis presented in this paper, however, is that it fails to cast any insight into attitudes towards risk when time horizons extend beyond two exchange periods. This characteristic of the paper raises the question: can risk affinity obtain when consumption decisions in the second period and in subsequent periods are made in the face of uncertainty about future budget constraints? Given the fact that the curvature properties of the maximum value functions which are used in dynamic programming have not been thoroughly explored, this question should be the topic of future research.

APPENDIX A

This appendix summarizes the major results of duality theory and develops a proof for Theorem (A:3). For a more rigorous and complete presentation of duality theory see: Diewart (1973, 1974), Epstein (1975), Hanoch (1977), Lau (1969), Roy (1947), Samuelson (1965, 1972), Shephard (1953) and Varian (1978).

Duality Relationships Between Direct and Variable Indirect Utility Functions

Let $U(C_0, C_1)$ represent a direct utility function which is defined over the first and second period consumption vectors C_0 and C_1. Suppose that $U(C_0, C_1)$ is (i) real valued and twice continuously differentiable with respect to $C_0 \geq O_{N-1}$, $C_1 \geq O_{N-1}$; and (ii) monotonically increasing and quasi-concave for all $C_0, C_1 \geq O_{N-1}$.

Consider a situation where decisions with respect to C_0 must be made prior to decisions about C_1. In this setting, the maximum of $U(C_0, C_1)$ with respect to C_1 can be represented by a variable indirect utility function $J(Y,P;C_0)$, where:

$$J(Y,P;C_0) \equiv \max_{C_1 \geq O_{N-1}} U(C_0,C_1)/P'C_1 \leq Y;\ P \gg O_{N-1};\ Y \geq 0 \qquad \text{(A:1)}$$

Y is nominal income over the second period and P is a vector of spot prices for commodities over the second period. The restriction $P \gg O_{N-1}$ implies that the budget set, say $B = \{C_1/P'C_1 \leq Y;\ Y \geq 0\}$ is compact. When $U(\ldots)$ has the properties listed above this ensures the existence of $J(Y,P;C_0)$.

Diewart (1973; Theorems (2:4) and (2:5)) has proven the following theorem.

Theorem (D:1)

When $U(C_0, C_1)$ has the properties enumerated above, then $J(Y,P;C_0)$ will be: (i) real valued and twice continuously differentiable for $Y \geq 0$, $P \gg 0_{N-1}$, $C \geq 0_{N-1}$; (ii) nonincreasing and quasi-convex in $P \gg 0_{N-1}$ for any $C_0 \geq 0_{N-1}$, $Y \geq 0$; (iii) homogeneous of degree zero in $Y \geq 0$, $P \geq 0_{N-1}$ for any $C_0 \geq 0$; and (iv) nondecreasing and quasi-concave in $Y \geq 0$, $C_0 \geq 0_{N-1}$ for any $P \gg 0_{N-1}$.

Epstein (1975, Appendix A) has proven the converse of Theorem (D:1).

Theorem (E:1)

Suppose that $J(Y,P;C_0)$ has all of the properties listed in Theorem (D:1). Consider the reconstructed direct utility function $f(C_0, C_1)$, where:

$$f(C_0, C_1) \equiv \min_{Y \geq 0,\ P \geq 0_{N-1}} J(Y,P;C_0) / P'C_1 \leq Y;\ C_1 \gg 0_{N-1}$$

When $f(C_0, C_1)$ is attained then $f(C_0, C_1) \equiv U(C_0, C_1)$ for $C_0 \geq 0_{N-1}$, $C_1 \gg 0_{N-1}$.

Theorem (D:1) and Theorem (E:1) jointly imply that $U(C_0, C_1)$ and $J(Y,P;C_0)$ are equivalent representations of preferences with respect to C_0 and C_1.

Roy's Identity

In his 1947 paper, Roy showed that the uncompensated demand function for any commodity C_{1i} always satisfies:

$$\hat{C}_{1i}(Y,P;C_0) \equiv - \frac{\partial J(\ldots)/\partial P_i}{\partial J(\ldots)/\partial Y} \tag{A:2}$$

This identity follows immediately from the envelope theorem. When the maximum in (A:1) is attained; i.e., when $J(Y,P;C_0)$ exists, then:

$$J(Y,P;C_0) \equiv U\big(C_0,\hat{C}_1(Y,P;C_0)\big) \tag{A:3}$$

$$\partial U/\partial C_{1i} - \hat{\lambda}P_i \equiv 0;\ i = 2,\ldots,N \tag{A:4}$$

$$Y - P'\hat{C}(Y,P;C_0) \equiv 0 \tag{A:5}$$

From (A:3) and (A:5):

$$\frac{\partial J(\ldots)}{\partial P_i} \equiv \sum_j \frac{\partial U}{\partial C_{1j}} \frac{\partial \hat{C}_{1j}}{\partial P_i} \tag{A:6}$$

$$- \hat{\lambda}\Big(\sum_j P_j \frac{\partial C_{1j}}{\partial P_i} + \hat{C}_{1i}\Big) \equiv 0 \tag{A:7}$$

Adding (A:6) to (A:7) and then using (A:4) yields:

$$\frac{\partial J(\ldots)}{\partial P_i} \equiv - \hat{\lambda}\ \hat{C}_{1i}\ (Y,P;C_0) \tag{A:8}$$

From (A:3), (A:5) and (A:4) however:

$$\partial J(\ldots)/\partial Y \equiv \hat{\lambda}(Y,P;C_0) \tag{A:9}$$

(A:8) and (A:9) jointly imply, as required, that

$$\hat{C}_{1i} \equiv - \frac{\partial J/\partial P_i}{\partial J/\partial Y}$$

The Expenditure Function and Shephard's Lemma

The inverse of $J(Y,P;C_0)$, where $J(\ldots)$ has the properties enumerated in Theorem (D:1), is called the expenditure function. This function can also be defined as follows, for any given parameter J^*:

$$e(P,J^*) \equiv \min_{C_1 \geq 0_{N-1}} \quad P'C_1/U(C_0,C_1) \geq J^*$$

When $U(C_0,C_1)$ has the properties enumerated at the beginning of this appendix, $e(P,J^*)$ is: (i) real valued and twice continuously differentiable with respect to $P \gg 0_{N-1}$; (ii) nondecreasing and concave with respect to $P \gg 0_{N-1}$; and (iii) homogeneous of degree 1 in P. (See Varian (1978).)

Shephard's Lemma (Shephard (1953)) states that the 'Hicksian' or compensated demand function for any commodity C_{1i}, say $C_{1i}(P,J^*)$, is identically equal to $\partial e(P,J^*)/\partial P_i$. This identity follows immediately from the envelope theorem. Since $e(P,J^*)$ is concave with respect to $P \gg 0_{N-1}$, for any given $J^* > 0$, this identity implies $\partial C_{1i}^H(P,J^*)/\partial P_i \leq 0$ or that own substitution effects are never positive.

Theorem (A:3)

It is impossible for the variable indirect utility function defined by (A:1) to be concave with respect to both $Y \geq 0$ and $P \gg 0_{N-1}$, for any given $C_0 \geq 0_{N-1}$, when the consumption vector C_1 has more than one element.

Proof

(i) $J(Y,P,C_0)$ would be concave in both Y and P if it was a linear (or an affine) function of both Y and P. This is a degenerate case, however, since

$$\hat{C}_{1i} \equiv - \frac{\partial J/\partial P_i}{\partial J/\partial Y} \neq f(P,Y)$$

(ii) If $J(Y,P;C_0)$ was nonlinear and concave with respect to both Y and P, then:

$$J(Y_a,P_a;C_0) \geq tJ(Y_b,P_b;C_0) + (1-t)J(Y_c,P_c;C_0);$$

where $0 \leq t \leq 1$, $Y_a = tY_b + (1-t)Y_c$; and $P_a = tP_b + (1-t)P_c$. Since the preceding inequality must hold for all (Y_b,Y_c) and (P_b,P_c) choose these vectors so that $J(Y_b,P_b;C_0) = J(Y_c,P_c;C_0)$. In this case: $J(Y_a,P_a;C_0) \geq J(Y_b,P_b;C_0) = J(Y_c,P_c;C_0)$.

(iii) Consider the following budget sets: $B_a = \{C_{1a}/P_a'C_{1a} \leq Y_a\}$; $B_b = \{C_{1b}/P_b'C_{1b} \leq Y_b\}$; $B_c = \{C_{1c}/P_c'C_{1c} \leq Y_c\}$. The last inequality in (ii) above entails both: $C_{1a} \geq C_{1b}$ and $C_{1a} \geq C_{1c}$; or, equivalently: $P_b'c_{1a} \geq Y_b$ and $P_c'C_{1a} \geq Y_c$. These last two inequalities imply: $(tP_b' + (1-t)P_c')C_{1a} \geq tY_b + (1-t)Y_c$ or $P_a'C_{1a} \geq Y_a$. But this contradicts

$$J(Y_a,P_a;C_0) = \max_{C_1 \geq 0_{N-1}} : U(C_0,C_1) \text{ s.t. } P_a'C_1 \leq Y_a.$$

In summary, then, if $J(Y,P;C_0)$ was concave with respect to both Y and P, it could not be a (variable) indirect utility function.

BIBLIOGRAPHY

Arrow, K.J. (1965), Aspects of the Theory of Risk-Bearing, (Yrjo Jahnsson Lectures), Helsinki, Yrjo Jahnssonin Saatio.

Arrow, K.J. (1971), Essays in the Theory of Risk Bearing, Markham, Chicago.

Biswas, T. (1983), "A Note on the Generalised Measures of Risk Aversion", Journal of Economic Theory, 29, 347-352.

Burns, M.E. (1973), "A Note on the Concept and Measure of Consumer's Surplus", American Economic Review, Vol. 63, No. 3, 335-344.

Cass, D. and Stiglitz, J.E. (1972), "Risk Aversion and Wealth Effects on Portfolios with Many Assets", Review of Economic Studies, 39, 331-354.

Deschamps, R. (1973), "Risk Aversion and Demand Functions", Econometrica, Vol. 41, No. 3, 455-465.

Diewart, W.E. (1973), "Hicks Aggregation Theorem and the Existence of a Real Value Added Production Function", Department of Manpower and Immigration, Ottawa.

Diewart, W.E. (1974), "Applications of Duality Theory", Frontiers of Quantitative Economics, Vol. 2, ed. M.D. Intrilligator and D.A. Kendrick, North Holland, Amsterdam.

Duncan, G.T. (1977), "A Matrix Measure of Multivariate Local Risk Aversion", Econometrica, Vol. 45, No. 4, 895-903.

Epstein, L. (1975), "A Disaggregated Analysis of Consumer Choice Under Uncertainty", Econometrica, Vol. 43, 877-892.

Fama, E.F. (1972), "Ordinal and Measurable Utility", Studies in the Theory of Capital Markets, ed. M.C. Jensen, Praeger, New York, 125-145.

Hanoch, G. (1977), "Risk Aversion and Consumer Preferences", Econometrica, Vol. 45, No. 2, 413-426.

Hart, O.D. (1975), "Some Negative Results on the Existence of Comparative Statics Results in Portfolio Theory", Review of Economic Studies, 42, 615-622.

Karni, E. (1979), "On Multivariate Risk Aversion", Econometrica, Vol. 47, No. 6, 1391-1401.

Keeney, R.L. (1973), "Risk Independence and Multiattributed Utility Functions", Econometrica, Vol. 41, 27-34.

Kihlstrom, R.E. and Mirman, L.J. (1974), "Risk Aversion with Many Commodities", Journal of Economic Theory, 8, 361-388.

Kihlstrom, R.E., Romer, D. and Williams, S. (1981), "Risk Aversion with Random Initial Wealth", Econometrica, Vol. 49, No. 4, 911-920.

Lau, R.J. (1969), "Duality and the Structure of Utility Functions", Journal of Economic Theory, 1, 374-396.

Levy, H. and Levy, A. (1984), "Multivariate Decision Making", Journal of Economic Theory, 32, 36-51.

Machina, M.J. (1982), "A Stronger Characterization of Declining Risk Aversion", Econometrica, Vol. 50, No. 4, 1069-1077.

Massell, B.F. (1969), "Price Stabilization and Welfare", Quarterly Journal of Economics, 83, 284-298.

Oi, W.Y. (1961), "The Desirability of Price Instability under Perfect Competition", Econometrica, 29, 58-64.

Paroush, J. (1975), "Risk Premium with Many Commodities", Journal of Economic Theory, 11, 283-286.

Pratt, J.W. (1964), "Risk Aversion in the Small and in the Large", Econometrica, Vol. 32, Nos. 1-2, 122-136.

Ross, S.A. (1981), "Some Stronger Measures of Risk Aversion in the Small and the Large with Applications", Econometrica, Vol. 49, No. 3, 621-638.

Roy, R. (1947), "La Distribution de Revenu entre les Divers Biens", Econometrica, Vol. 15, 205-225.

Samuelson, P.A. (1972), "The Consumer Does Benefit from Feasible Price Stability", Quarterly Journal of Economics, 86, 476-493.

Samuelson, P.A. (1965), "Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Price Elasticity of Demand", Econometrica, 33, 781-796.

Sercu, P. (1981), "A Note on Real and Nominal Efficient Sets", The Journal of Finance, Vol. 36, No. 3, 721-737.

Shephard, R.H. (1953), Cost and Production Functions, Princeton University Press, Princeton.

Silberberg, E. (1972), "Duality and the Many Consumers' Surpluses", American Economic Review, 62, 942-952.

Stiglitz, J.E. (1969), "Behavior Towards Risk with Many Commodities", Econometrica, 37, 660-667.

Turnovsky, S.J., Shalit, H. and Schmitz, A. (1980), "Consumers' Surplus Price Instability and Consumers' Welfare", Econometrica, Vol. 48, No. 1, 135-152.

Varian, H.R. (1978), Microeconomic Analysis, W.W. Norton & Company, Inc., New York.

Waugh, F.V. (1944), "Does the Consumer Benefit from Price Instability", Quarterly Journal of Economics, 58, 602-614.

CHAPTER 2

HUMAN CAPITAL FORMATION AND THE STRUCTURE OF EMPLOYMENT CONTRACTS UNDER UNCERTAINTY

1. INTRODUCTION

In his seminal work on the portfolio selection problem, Markowitz (1959, pp. 298-299) emphasized that his analysis was incomplete because human capital was not included as an investment opportunity. Although this caveat has not been completely ignored, the major elements of human capital formation have yet to be incorporated into the modern extensions and generalizations of Markowitz's original work. The major limitation of the two-period models developed by Mayers (1972, 1973), Levhari and Weiss (1974) and Williams (1978), to be more specific, is that they fail to reflect the fact that investments in human capital create irreversible changes in the stock of a durable capital good. In fact, the major problem with any two-period investment model is that human capital is implicitly assumed to be completely perishable.

Clearly, the common practice of excluding human capital formation from both theoretical and empirical work in Finance may be justified if theoretically-predicted effects are not empirically significant. Fama and Schwert (1977) and Liberman (1980) present some evidence which suggests that this may be the case. In these studies, changes in per capita labour income are used as measures of the return to human capital in order to test whether the covariance effects predicted in Mayers' extended versions of the Sharpe-Lintner-Black (SLB) capital asset pricing model are significant[1]. Fama and Schwert use a measure of the return to aggregate human capital for the entire U.S. labour force to estimate the differences between the Mayers and SLB measures of systematic risk for various classes of financial securities. Since these differences are found to be extremely

[1] Op. Cit.

small, they conclude that it is unnecessary to take account of human capital in asset pricing theory[2]. Liberman, on the other hand, uses data on labour income for various industries and occupational groups in order to test the Mayers model. He not only corroborates the results of Fama and Schwert, he also finds that the composition of individual security portfolios is not sensitive to labour income[3].

As Liberman acknowledges, the basic reason why both studies fail to find any significant evidence for the effects predicted by Mayers is that there is very little fluctuation in the rate of change of labour income over time - at least in the U.S.A. Strictly speaking, however, this characteristic of the data does not imply that asset pricing theory can safely ignore human capital or that human capital is virtually riskless. Before drawing these conclusions, it is necessary to show that changes in labour income correspond to the changes in welfare associated with changes in human capital. The analysis presented in this paper implies: (a) that period specific changes in labour income are not measures of the returns to human capital; and (b) that the correct measure of returns on human capital is always uncertain even when labour income is not.

The major objective of this study is to extend both portfolio selection theory and human capital theory to cover the most important aspects of investment in human capital under uncertainty. In Section 2, the main ideas of human capital theory are summarized and then integrated into a model of human capital formation. Although this model assumes that investments in human capital are completely irreversible, it does allow for the

---

[2] Op. Cit., p. 120.

[3] This result is contrary to the predictions about portfolio composition which are contained in Mayers' (1972, 1973) papers.

possibility that an individual's stock of human capital may become obsolete. The model is also more general than the highly specialized models contained in the literature; i.e., the investment process is generalized to encompass both "learning-by-doing" and education. Perhaps the major contribution of the model, however, is that it shows that differences in the opportunities for learning-by-doing across occupations may explain why some employees are paid a salary; while others are paid by the hour. The model implies that those employees who are in occupations which offer relatively rich (poor) opportunities for learning-by-doing will be paid a salary (by the hour). It also suggests that the opportunity to earn 'participation payments', such as overtime pay, is a form of compensation for restricted opportunities to accumulate human capital through learning-by-doing.

In Section 3, the model of human capital formation developed in Section 2 is integrated into a discrete time, multiperiod model of consumption and portfolio selection decisions. Restrictions on preferences and expectations are kept to a minimum; i.e., the distributions describing expectations are assumed to possess finite moments and direct utility functions are assumed to be quasi-concave, monotonically increasing and non-separable with respect to time. Individuals are assumed to have the opportunity to invest in marketable securities, default-free debt instruments of various maturities and, of course, human capital. With the exception of those debt instruments which mature at the end of one period, the returns on all assets are assumed to be uncertain at the time of investment. Decisions to supply default-free debt instruments, on the other hand, are assumed to be constrained by a borrowing capacity function. This credit rationing constraint is built into the model in order to reflect the fact that the ability to borrow against future labour income is partially restricted by contract formation constraints.

In Section 4, the maintained assumption that investments in human capital are irreversible is shown to imply that economic welfare can never be increased by deferring investments in human capital. This result implies that individuals have a powerful incentive to accumulate human capital as rapidly as possible in their youth. Although it would be difficult to corroborate it empirically, the behavior of many individuals is probably consistent with this theorem.

One of the interesting aspects of the analysis presented in Section 4 is that it suggests (but only suggests) that the marginal value of human capital at any point in time depends on the nature of the employment contract. The valuation equations of this section imply that the marginal value of human capital is proportional to the sum of the marginal value of leisure and the value of the marginal changes in future labour income. These marginal changes in future labour income can be interpreted as effective wage rates. If these effective wage rates are a decreasing (increasing) function of the number of hours of labour supplied, the marginal value of human capital will increase (decrease) in the absence of offsetting movements in the (endogenous) marginal value of leisure[4]. One hypothesis which deserves further exploration, then, is that individuals in salaried occupations have a greater incentive to invest in human capital than individuals who are paid by the hour.

One of the basic results of portfolio selection theory is that individual specific risk premiums on any risky asset are proportional to the

---

4 The behavior of these wage rates is a function of the costs of learning-by-doing and the participation effects discussed in Section 2.

covariance between its rate of return and the marginal utility of wealth[5]. In Section 5, this result is shown to hold for human capital. The covariance expression for this asset is unique, however, because it decomposes into a linear combination of the variance in the marginal utility of wealth and its covariance with the marginal utility of human capital. This decomposition implies that the variance of non-human wealth, the variance of labour income and the covariance between security returns and labour income belong to the set of factors which underlie the individual specific risk premiums on human capital. It also implies, as indicated previously, that investments in human capital are risky even when the variance in labour income is equal to zero.

Although the required analysis is not presented in this study, it is possible to show that the risk premium on any capital asset will have the same structure as the risk premium on human capital if optimizing decisions are based on a provisional plan to hold a particular asset for several exchange periods. Suppose, for example, that an individual has a provisional holding period for housing which extends several exchange periods into the future. In this case, it is possible to show that his risk premium on housing will depend on his expectations about the variance in his marketable wealth. Clearly, since these holding period horizons exist in reality, it might be interesting to develop their implications for asset pricing in another study.

---

5 For reference, see Rubinstein (1974).

2. A MODEL OF HUMAN CAPITAL FORMATION

Human Capital Theory is based on three main ideas: (i) that wage rates are dividends on individual specific holdings of a scarce and intangible stock of knowledge or "human capital"; (ii) that the magnitude of these "dividends" is a function of both the current stock of human capital and its rate of change; and (iii) that investments in human capital are irreversible.

The human capital models contained in the literature can be segregated into two groups. One group consists of the so-called 'pure training' models, the other consists of 'pure experience' or 'learning-by-doing' models[6]. In the pure training models[7], the maximum dividend associated with the accumulated stock of human capital is called the potential wage rate. This wage rate exceeds the realized wage rate when an individual is accumulating human capital by spending time on those off-the-job training activities which induce systematic increases in his future potential wage rate. When the stock of human capital is not changing, however, the potential wage rate is assumed to be equal to the realized wage rate. A common feature of the pure training models then is that current earnings must be sacrificed in order to enhance future earning opportunities.

One of the limitations of the pure training models is that they do not provide an explanation for those empirical observations which indicate that wage rates are a function of accumulated experience in the labour force[8].

---

6 For a concise review of human capital theory under certainty, see Killingsworth (1982).

7 See, for example, Ghez and Becker (1975), Heckman (1976) and Blinder and Weiss (1976).

8 See Mincer and Polachek (1978), Sandell and Shapiro (1978) and Corcoran and Duncan (1979).

The 'learning-by-doing' models[9] provide an explanation for these observations by making the accumulation of human capital a by-product of work. According to these models, increases in the rate of accumulation of 'experience' are accompanied by increases in both current wage rates <u>and</u> future wage rates. The costs of investing in human capital, in other words, are restricted to the foregone benefits of leisure.

Since human capital can be accumulated by a combination of learning-by-doing and off-the-job training, both of these phenomena are built into the model presented below. Although this model does not allow for those 'technological' risks arising from uncertainty about learning abilities or the quality of training, it does incorporate the uncertainty associated with random variations in output prices, factor mix ratios and production technologies, etc. One of the distinguishing characteristics of the model is that the costs of 'learning-by-doing' are not restricted to the foregone value of leisure.

Consider a setting, then, where individuals cannot change their allocation of time between work and training activities as rapidly as they can change their investments in nonhuman capital and their consumption decisions. Suppose, moreover, that the change in the stock of human capital during period t+1, say K(t+1) - K(t), is a monotonically increasing function of N(t), S(t) and K(t); where N(t) and S(t) are period t decision variables. N(t) denotes the number of hours that an individual plans to spend in the labour force during period t+1, while S(t) denotes the number of hours that he plans to spend in 'off-the-job' training during period t+1. K(t), on the other hand, represents the accumulated stock of human

---

[9] See Arrow (1962, p. 155), Eckhaus (1963) and Rosen (1972).

capital at the _end_ of period t.

Let G represent the production function for human capital so that, for any t:

$$K(t+1) - K(t) = G\big(N(t),S(t);K(t)\big) = G(t+1) \tag{2:1}$$

Assume, moreover:

(i) that investments in human capital are irreversible, or $G(t+1) \geq 0$ for all $N(t)$, $S(t) \geq 0$;

(ii) that N(t) and S(t) are the only factors of production, or $G\big(0,0;K(t)\big) \equiv 0$;

(iii) that the marginal productivities of N(t) and S(t) are always non-negative:

$$G_N(t+1) \equiv \frac{\partial G(t+1)}{\partial N(t)} \geq 0; \; N(t) \geq 0$$

$$G_S(t+1) \equiv \frac{\partial G(t+1)}{\partial S(t)} \geq 0; \; S(t) \geq 0$$

(iv) that the net output of human capital during any period t+1 is a non-decreasing function of the accumulated capital stock K(t), or:

$$G_K(t+1) \equiv \frac{\partial G(t+1)}{\partial K(t)} \geq 0; \; K(t) \geq 0$$

An interesting feature of the preceding technology is that the _rate of growth_ in human capital, say $g(t+1) = G(t+1)/K(t)$, may increase or decrease as K(t) increases. Notice, in particular, that (i) and (iv) are consistent with the notion that 'too much' knowledge decreases the rate at which individuals are willing (able?) to learn new ideas and new skills, or with:

$$g_K(t+1) = \frac{\partial g(t+1)}{\partial K(t)} = K(t)^{-2}\left[K(t)G_K(t+1) - G(t+1)\right] < 0$$

The major reason for assuming that (iv) holds, instead of $G_K(t+1) \leq 0$, is that $G_K(t+1) \leq 0$ and assumption (i) jointly rule out $g_K(t+1) > 0$. Loosely speaking, this is essentially the same as ruling out the possibility that increases in knowledge cause individuals to become more receptive to learning new ideas and new skills.

One of the characteristics of human capital formation is that the relationship between investments in this asset and the average realized wage rate during any period t+1, say $R(t+1) \geq 0$, are seldom written down in any explicit contract between an employee and his employer. Although the absence of these contracts implies the existence of a 'first order' level of uncertainty about the returns on human capital and the costs of accumulating it, this study abstracts from this phenomenon. It assumes, in particular: (i) that the functions which relate realized wage rates to the formation of human capital are non-stochastic; and (ii) that all individuals have complete knowledge of the function which pertains to them.

Suppose, then, that the effects of human capital formation upon R(t+1) depend upon the rental rate on K(t); where this rental rate, namely $\rho(t+1) \geq 0$, is a function of those variables which affect the derived demand for labour. Assume, moreover, that R(t+1) is described by the following (individual specific) function:

$$R(t+1) = \rho(t+1)K(t) - \rho(t+1)\phi(t+1) \qquad (2{:}2)$$

where:

$\rho(t+1)K(t)$ = the potential wage rate during period t+1;

$\phi(t+1)$ = a wage change function, $\phi\big(G(t+1),N(t);K(t)\big)$.

When restrictions are imposed upon $\phi(t+1)$, (2:2) yields most of the major predictions of Human Capital Theory. Suppose, to be more specific, that $\phi(t+1)$ has the following properties for all t:

$$\phi_1: \quad \phi\big(G(t+1),N(t);K(t)\big) \geq 0;\ G(t+1) \geq 0,\ N(t) \geq 0$$

$$\phi_2: \quad \partial\phi(t+1)/\partial G(t+1) \equiv \phi_G(t+1) \geq 0$$

$$\phi_3: \quad \frac{\partial\phi(t+1)}{\partial G(t+1)}\frac{\partial G(t+1)}{\partial K(t)} + \frac{\partial\phi(t+1)}{\partial K(t)} \equiv \phi_K(t+1) \leq 0$$

ϕ_2 and the second inequality in the third assumption about the properties of the production function G(t+1), namely $G_S(t+1) \geq 0$, jointly imply:

$$\phi_4: \quad \frac{\partial\phi(t+1)}{\partial G(t+1)}\frac{\partial G(t+1)}{\partial S(t)} \equiv \phi_S(t+1) \geq 0$$

Let $\phi_N(t+1) \equiv \dfrac{\partial\phi(t+1)}{\partial G(t+1)}\dfrac{\partial G(t+1)}{\partial N(t)} + \dfrac{\partial\phi(t+1)}{\partial N(t)}$.

Assume as well that:

$$\phi_5: \quad \frac{\partial\phi(t+1)}{\partial N(t)} \leq 0$$

The first property of $\phi(t+1)$, namely ϕ_1, implies that the average realized wage rate, R(t+1), is bounded from above by the potential wage rate $\rho(t+1)K(t)$ or, from (2:2), that:

$$\psi(t+1) \equiv \rho(t+1)K(t) - R(t+1) = \rho(t+1)\phi(t+1) \geq 0 \qquad (2:3)$$

ϕ_2 implies that the marginal (wage) costs of accumulating human capital are non-negative or, from (2:3), that:

$$\frac{\partial\psi(t+1)}{\partial G(t+1)} = \rho(t+1)\phi_G(t+1) \geq 0 \qquad (2:4)$$

ϕ_4, on the other hand, implies that the gap between the potential and the average realized wage rate is a nondecreasing function of the number of hours spent in off-the-job training or, from (2:3), that:

$$\frac{\partial\psi(t+1)}{\partial S(t)} = \rho(t+1)\phi_S(t+1) \geq 0 \qquad (2:5)$$

Given the maintained assumption that the ability to produce human capital is never diminished by previous knowledge, or $\partial G(t+1)/\partial K(t) \geq 0$, ϕ_3 and (2:2) jointly imply:

$$\frac{\partial R(t+2)}{\partial K(t)} = \rho(t+2)[1-\phi_K(t+2)][1+G_K(t+1)] \geq 0;$$

or more generally that:

$$\frac{\partial R(t+j)}{\partial K(t)} = \rho(t+j)[1-\phi_K(t+j)] \prod_{i=1}^{j-1} \left(1+G_K(t+i)\right) \geq 0 \qquad (2:6)$$

(2:6) states that human capital is a capital asset which yields a stream of non-negative 'dividends' over time. When j=1, this inequality implies:

$$\frac{\partial R(t+1)}{\partial K(t)} = \rho(t+1)[1-\phi_K(t+1)] \geq 0 \qquad (2:7)$$

(2:7) is a specialized version of the major prediction of Human

Capital Theory; namely, that realized wage rates are never diminished by historical accumulations of knowledge.

One of the distinguishing characteristics of the model developed above is that it suggests that differences in the opportunities for learning-by-doing may explain why employees in some occupations are paid by the hour, while employees in other occupations are paid a salary.

To see this, let $Y(t+1) = N(t)R(t+1)$ represent the realized level of labour income during period t+1. Assume, moreover, that the effective marginal wage rate, say $Y_N(t+1) \equiv \partial Y(t+1)/\partial N(t)$, is equal to zero for all $N(t) \geq N^* > 0$, where N^* is a parameter. According to (2:2), this implies:

$$Y_N(t+1) = R(t+1) - N(t)\rho(t+1)\phi_N(t+1) = 0 \qquad (2:8)$$

By assumption, however: $R(t+1) \geq 0$, $\rho(t+1) \geq 0$ and $N(t) > 0$. Hence, if $Y_N(t+1) = 0$, then:

$$N(t)\rho(t+1)\phi_N(t+1) = R(t+1) \geq 0;$$

or:

$$\phi_N(t+1) = \frac{\partial\phi(t+1)}{\partial G(t+1)}\frac{\partial G(t+1)}{\partial N(t)} + \frac{\partial\phi(t+1)}{\partial N(t)} \geq 0 \qquad (2:9)$$

The second term on the right-hand side of (2:9) is restricted, by ϕ_5, to satisfy $\partial\phi(t+1)/\partial N(t) \leq 0$. This restriction is imposed upon the model to capture the effects of overtime pay and movements from part-time to full-time employment upon labour income. When these participation effects have a positive effect upon labour income, i.e. when $\partial\phi(t+1)/\partial N(t) < 0$, they tend to drive the effective marginal wage rate above the average realized wage rate. Suppose, for example, that there was no learning-by-doing at the margin. In this case, the first term on the right-hand side of (2:9) would be equal to zero and $Y_N(t+1)$ would satisfy:

$$Y_N(t+1) - R(t+1) = -N(t)\rho(t+1)\,\frac{\partial\phi(t+1)}{\partial N(t)} \geq 0$$

The first term on the right-hand side of (2:9), however, tends to drive the effective marginal wage rate below the average realized wage rate. This term captures the effects of learning-by-doing upon Y(t+1). Given the maintained assumption that the marginal costs of accumulating human capital are non-negative (see (2:4)) and the assumption that labour is a productive factor of production, or $\partial G(t+1)/\partial N(t) \geq 0$, this term satisfies:

$$\frac{\partial\phi(t+1)}{\partial G(t+1)}\,\frac{\partial G(t+1)}{\partial N(t)} \geq 0.$$

The immediately preceding inequality implies that the marginal (wage) costs of learning-by-doing are non-negative; i.e., from (2:3) that:

$$\frac{\partial\psi(t+1)}{\partial N(t)} = \rho(t+1)\,\frac{\partial\phi(t+1)}{\partial G(t+1)}\,\frac{\partial G(t+1)}{\partial N(t)} \geq 0$$

One of the characteristics of many salary-rated jobs is that effective marginal wage rates are equal to zero for a fairly wide range of values for N(t). Since (2:9) is implied by $Y_N(t+1) = 0$ and since it states that the effects of learning-by-doing on $Y_N(t+1)$ are not dominated, in an absolute value sense, by those participation effects which might be induced by increases in the supply of labour, it suggests that salary-rated employees will be in those occupations which offer extensive opportunities for learning-by-doing. This conjecture is supported, moreover, by an argument which proceeds directly from (2:9).

Suppose, in particular, that the learning-by-doing effect dominates the participation effect for all $N(t) > 0$. In this case, (2:9) will hold and $Y_N(t+1)$ will satisfy:

$$R(t+1) - Y_N(t+1) = \rho(t+1)N(t)\phi_N(t+1) \geq 0 \qquad (2:10)$$

or:

$$Y_N(t+1) \leq R(t+1) \qquad (2:11)$$

Since (2:11) holds for many salary-rated occupations, it also suggests that learning-by-doing is a prominent phenomenon in these occupations.

One of the characteristics of hourly-rated jobs, in contrast, is that (2:11) seldom holds. Indeed, it is reasonable to assume that $Y_N(t+1) \geq R(t+1)$ always holds for these occupations. When this inequality holds, however, it implies, along with (2:2), that:

$$Y_N(t+1) - R(t+1) = -N(t)\rho(t+1)\phi_N(t+1) \geq 0 \qquad (2:12)$$

or, from ϕ_2, ϕ_5 and $\partial G(t+1)/\partial N(t) \geq 0$, that:

$$0 \leq \frac{\partial\phi(t+1)}{\partial G(t+1)}\frac{\partial G(t+1)}{\partial N(t)} \leq -\frac{\partial\phi(t+1)}{\partial N(t)} \qquad (2:13)$$

(2:13) states that the absolute value of the participation effect is never dominated by the influence of learning-by-doing on $Y(t+1)$. Since this condition implies, and is implied by, $Y_N(t+1) \geq R(t+1)$, it suggests that employees will be paid by the hour if they are in a job which does not provide many opportunities for accumulating human capital through learning-by-doing[10].

In their previously-cited models of human capital formation, Rosen and Killingsworth assume: (a) that there is no lag between investments in human capital and increases in realized wage rates; (b) that there are no participation effects; and (c) that learning-by-doing is not accompanied by any reduction in labour income. The major reason why these assumptions are relaxed in this study is that they jointly imply: (i) that the total and marginal (wage) costs of accumulating human capital are less than or equal to zero; and (ii) that the (wage) costs of learning-by-doing are equal to zero. If assumption (c) was imposed upon the model presented above, moreover, it would imply (see (2:8) and (2:9)):

$$Y_N(t+1) - R(t+1) = -N(t)\rho(t+1)\frac{\partial\phi(t+1)}{\partial N(t)} \qquad (2:14)$$

Given the sensible assumption that an increase in the supply of labour is not associated with a decrease in labour income, or $-\partial\phi(t+1)/\partial N(t) \geq 0$, (2:14) rules out the situation which frequently obtains in many salary-related jobs; namely,

$$Y_N(t+1) \leq R(t+1)$$

Perhaps the major distinguishing characteristic of decisions to invest in human capital is that they are frequently made in the face of uncertainty about the form of the labour income function. This 'first order'

---

[10] (2:13) is equivalent to $\phi_N(t+1) \leq 0$, by definition; hence, it implies (2:12).

level of uncertainty may be induced by uncertainty about: (i) the technology of human capital formation; (ii) the compensation relationship which connects K(t) and K(t+1) - K(t) to Y(t+1); and (iii) the demand relationships which link $\rho(t+1)$ to Y(t+1). Although the model developed in this section abstracts from the first two sources of uncertainty about labour income, the stochastic implications of the model are broadly consistent with some of the stylized facts about human capital formation. The marginal cost functions in (2:4) and (2:5), for example, imply:

$$E_t\left\{\frac{\partial\psi(t+1)}{\partial G(t+1)}\right\} = E_t\{\tilde{\rho}(t+1)\}\phi_G(t+1) \qquad (2:15)$$

$$E_t\left\{\frac{\partial\psi(t+1)}{\partial S(t+1)}\right\} = E_t\{\tilde{\rho}(t+1)\}\phi_S(t+1) \qquad (2:16)$$

Equation (2:15) states that the expected marginal (wage) costs of accumulating human capital are proportional to the expected rental rate on this asset. (2:16) asserts that the expected marginal (wage) costs of going to school are proportional to the expected rental rate. These two equations suggest that the incentives to accumulate human capital and to engage in off-the-job training activities will be weak (strong) when expectations about the derived demand for labour are optimistic (pessimistic).

Equation (2:2), on the other hand, implies:

$$\bar{R}(t+j,t) = \bar{\rho}(t+j,t)\left[K(t+j-1) - \phi(t+j)\right] \qquad (2:17)$$

$$\sigma_N(t+j,t) = \frac{\sigma\rho(t+j,t)}{\bar{\rho}(t+j,t)}\,\bar{R}(t+j,t) \qquad (2:18)$$

where:

$\bar{R}(t+j,t) = E_t\{\tilde{R}(t+j)\}$

$\sigma_N(t+j,t) = \left(Var_t\{\tilde{R}(t+j)\}\right)^{1/2}$

$\bar{\rho}(\ldots), \sigma\rho(\ldots)$ = the corresponding moments for $\tilde{\rho}(t+j)$.

Differentiating (2:17) and (2:18) with respect to K(t) and then using (2:6) yields:

$$\frac{\partial\sigma_N(t+j;t)}{\partial K(t)} = \left(\frac{\sigma\rho(t+j;t)}{\bar{\rho}(t+j;t)}\right) \frac{\partial \bar{R}(t+j;t)}{\partial K(t)} \geq 0 \qquad (2:19)$$

(2:19) states that increases in K(t) are accompanied by increases in the (conditional) mean and standard deviation of the average realized wage rates in all future periods. Obviously, this is somewhat arbitrary; indeed, it must be admitted that there is no compelling reason to assume that it holds in reality. Levhari and Weiss[11] cite several empirical studies, however, which suggest that the mean and variance of labour income tends to increase with attained levels of education. Perhaps, then, (2:19) is a decent approximation.

3. CONSUMPTION, INVESTMENT AND FACTOR SUPPLY DECISIONS UNDER TEMPORAL UNCERTAINTY

The choice theoretic model developed in this section assumes:

(a) that all current consumption, investment and factor supply decisions must be made prior to the resolution of uncertainty about: (i) the wage rates R(t+j), j > 0; (ii) the (marketable) capital asset return vectors Z_{t+j}, j > 0; and (iii) the commodity and asset price vectors P_{t+j}, j > 0;

[11] Op Cit.

(b) that individuals respond to temporal uncertainty by engaging in the planning process which is built into dynamic programming;

(c) that time horizons are finite; i.e., that t+j = t,...,t+T, where T is the end of an individual's lifetime;

(d) that nominal wealth at the beginning of the terminal period, namely $W(t+T) = Y(t+T) + (\alpha_{t+T-1})'Z_{t+T}$, always satisfies $W(t+t) > 0$ for all realizations of the random vector $(\tilde{Y}(t+T), \tilde{Z}_{t+T})$, and for all portfolio selection decisions α_{t+T-1};

(e) that all elements of the choice vectors $X_{t+j} = (C_{t+j}, \alpha_{t+j}, N(t), S(t))$, including all elements of the consumption vector C_{t+j}, are nonnegative except for the elements of the portfolio selection vector α_{t+j};

(f) that all short, intermediate and long term debt instruments are default-free;

(g) that borrowing is restricted by a credit rationing constraint of the form $B(t+j) + P(^B_{t+j})'\alpha_{t+j} \geq 0$, where $-(P^B_{t+j})'\alpha^B_{t+j} \geq 0$ is the total amount of money borrowed at the beginning of period t+j and $B(t+j) \geq 0$;

(h) that the underlying direct utility function is a nonstochastic, monotonically increasing, quasi-concave and nonseparable function of the vector $(C_t, \ldots C_{t+T}, \ell_t, \ldots \ell_{t+T})$, so that:

$$U(t) = U(C_t, \ldots C_{t+T}, \ell_t, \ldots \ell_{t+T}) \qquad (3:1)$$

where $\ell(t+j) = \ell(N(t+j-1), S(t+j-1))$ is the amount of time available for leisure in period t+j[12].

[12] U(t) is also a function of C_{t-1}, ..., C_{t-j}, ℓ_{t-j}, etc., but these have been suppressed.

The credit rationing constaint outlined in assumption (g) is introduced in order to allow for the possibility that borrowing opportunities may be affected by incomplete capitalizations of future labour income. Although these 'capitalization failures' may affect borrowing capacity in a fairly complex way, suppose: (i) that borrowing capacity in any period t+j is an increasing function of W(t+j); (ii) that increases in the supply of any debt instrument reduces borrowing capacity; and (iii) that increases in the 'bond prices' $P_i{}^B(t+j)$ enhance borrowing capacity. Given these assumptions, the period specific borrowing capacity functions can be written as:

$$B(t+j) = B\left(W(t+j)\right) + (P^B_{t+j})'\alpha^B_{t+j})\Big);$$

where:

$$B(0) \equiv 0; \; B' \geq 0$$

$$W(t+j) + (P^B_{t+j})'\alpha^B_{t+j} \geq 0$$

$$B_W(t+j) \equiv \frac{\partial B(t+j)}{\partial W(t+j)} \geq 0$$

$$B_i(t+j) \equiv \frac{\partial B(t+j)}{\partial \alpha_i{}^B(t+j)} = B'P_i{}^B(t+j) \leq 0$$

$$\frac{\partial B(t+j)}{\partial P_i{}^B(t+j)} = B'\alpha_i{}^B(t+j) \geq 0$$

Let J(t+T) represent the conditional or variable indirect utility function for the terminal period t+T. Although the definition of this maximum value function should reflect the existence of bequest functions, suppose, with some loss of generality, that it is defined as follows:

$$J(t+T) \equiv \underset{C_{t+T}}{\text{Max}} \; U(C_{t+T},\ldots) + \lambda(t+T)\left[W(t+T) - (P^{C}_{t+T})'C_{t+T}\right] \qquad (3:2)$$

where $(P^{C}_{t+T})'$ is a vector of commodity prices, and:

$$W(t+T) = N(t+T-1)R(t+T) + (\alpha_{t+T-1})'Z_{t+T} \qquad (3:3)$$

$$= Y(t+T) + (\alpha_{t+T-1})'Z_{t+T} > 0$$

The preceding definition of J(t+T) is consistent with the model of human capital formation developed in Section 2; i.e., it assumes that decisions about the human capital inputs N(t+T-1) and S(t+T-1) were made in the previous period. It also implicitly assumes, in accordance with assumption (d), that an individual has enough wealth to retire all of his outstanding debt liabilities.

A defining characteristic of decision making under temporal uncertainty is that individuals cannot attain a 'once and for all' solution to the problem of maximizing the expected utility of lifetime consumption. This characteristic of temporal uncertainty is incorporated into the analysis presented below by assuming that decisions are based upon a set of provisional exchange plans. J(t+T), to be more specific, will be interpreted as a provisional exchange plan because, given any previous set of decisions X_{t+j}, $t+j = t,\ldots,t+T-1$, and any realization of the random vector $(\tilde{W}_{t+T}, \tilde{P}^{C}_{t+T})$, it represents an individual's perception of his maximum attainable level of utility at the beginning of period t+T. The provisional exchange plans for period t+T-1, on the other hand, are defined as follows:

$$J(t+T-1) \equiv \max_{X_{t+T-1}} \; E_{t+T-1}\{J(t+T)\} \qquad (3:4)$$

$$+ \lambda(t+T-1)\left[W(t+T-1) - (P_{t+T-1})'X_{t+T-1}\right]$$

$$+ \lambda^B(t+T-1)\left[B(t+T-1) - (P_{t+T-1})'\gamma X_{t+T-1}\right]$$

where X_{t+T-1} is the M dimensional choice vector $\{X_i(t+T-1)\}$ and:

$$\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \gamma_i & \\ 0 & & \gamma_m \end{pmatrix}$$

$$\gamma_i = \begin{cases} -1 & \text{for } X_i(t+j) = \alpha_i^B(t+j) \\ 0 & \text{otherwise} \end{cases}$$

$$B(t+T-1) = B\left(W(t+T-1) - (P_{t+T-1})'\gamma X_{t+T-1}\right)$$

$$P_i(t+j)X_i(t+j) \equiv 0 \text{ when } X_i(t+j) \text{ refers to } N(t+j) \text{ or } S(t+j)$$

The provisional exchange plans for all preceding periods are defined, in a recursive manner, from (3:4). J(t+1), in particular, is defined as follows:

$$J(t+1) \equiv \max_{X_{t+1}} \; E_{t+1}\{J(t+2)\} \qquad (3:5)$$

$$+ \lambda(t+1)\left[W(t+1) - (P_{t+1})'X_{t+1}\right]$$

$$+ \lambda^B(t+1)\left[B(t+1) - (P_{t+1})'\gamma X_{t+1}\right]$$

Suppose that expectations about the random vector $\left(\tilde{Z}_{t+2}, \tilde{P}_{t+2}, \tilde{R}(t+2)\right)$ are described by the vector Ω_{t+1}, where each element of Ω_{t+1} is a parameter. Assume, as well, that the maximum in (3:5) is attained. In this case, J(t+1) can be written as:

$$J(t+1) = J\big(W(t+1),\ P_{t+1},\ \Omega_{t+1};\ X_t;\ K(t+1)\big) \qquad (3:6)$$

where K(t+1) is the accumulated stock of human capital at the end of period t+1, and:

$$W(t+1) = N(t)R(t+1) + \alpha_t{}'Z_{t+1} \qquad (3:7)$$

$$R(t+1) = \rho(t+1)K(t) - \rho(t+1)\phi(t+1) \qquad (2:2)'$$

K(t+1) appears as a parameter in (3:6) because human capital formation creates an irreversible change in the stock of a durable, and valuable, capital asset. This characteristic of human capital formation is reflected in the following relationships:

$$J_K(t+T) \equiv \frac{\partial J(t+T)}{\partial K(t+T)} \equiv 0 \qquad (3:8)$$

$$J_K(t+j) \equiv \frac{\partial J(t+j)}{\partial K(t+j)}$$

$$\equiv E_{t+j}\left\{\frac{\partial J(t+j+1)}{\partial Y(t+j+1)}\,\frac{\partial Y(t+j+1)}{\partial K(t+j)} + J_K(t+j+1)\,\frac{\partial K(t+j+1)}{\partial K(t+j)}\right\} \geq 0 \qquad (3:9)$$

$$\ldots j = 0,\ldots,T-1$$

$$J_K(t+1) \equiv \frac{\partial J(t+1)}{\partial K(t+1)}$$

$$\equiv E_{t+1}\left\{\frac{\partial J(t+2)}{\partial Y(t+2)}\,\frac{\partial Y(t+2)}{\partial K(t+1)} + J_K(t+2)\,\frac{\partial K(t+2)}{\partial K(t+1)}\right\} \geq 0 \qquad (3:10)$$

These three identifies are derived by using the envelope theorem (see Appendix A). (3:8) states, as might be expected, that economic welfare during the terminal period is neither enhanced nor diminished by an increase in the 'terminal' capital stock K(t+T). Each member of the set of identities in (3:9), including (3:10), states that economic welfare is never diminished by an increase in the end of period stock of human capital. These identities can be rewritten in the form:

$$J_K(t+j) = E_{t+j}\left\{\frac{\partial J(t+j+1)}{\partial Y(t+j+1)}\frac{\partial R(t+j+1)}{\partial K(t+j)}\right\}N(t+j) + E_{t+j}\left\{J_K(t+j+1)\right\}\left[1 + G_K(t+j+1)\right] \quad (3:11)$$

where:

$$1 + G_K(t+j+1) \equiv \frac{\partial K(t+j+1)}{\partial K(t+j)} \geq 1 \quad (3:12)$$

and:

$$\frac{\partial R(t+j+1)}{\partial K(t+j)} = \rho(t+j+1)\left[1 - \phi_K(t+j+1)\right] \geq 0 \quad (2:7)'$$

(3:12) and (2:7)' jointly imply $J_K(t+j) \geq 0$, given nonsatiation, or $\partial J(\ldots)/\partial Y(\ldots) \geq 0$. (3:12) follows from property (iv) of Equation (2:1); i.e., it states that the ability to produce human capital is a nondecreasing function of the accumulated stock. It also reflects the maintained assumption that human capital is a (perfectly) durable capital asset. (2:7)' states, as indicated in Section 2, that realized wage rates are never diminished by historical accumulations of knowledge.

The first term on the right-hand side of (3:11) obviously captures the effects of increases in K(t+j) upon labour income in period t+j+1. This term, or a variant thereof, will appear in any two-period model of human capital formation. The second term on the right-hand side of (3:11), however, does not appear in a two-period model of human capital formation because these models (implicitly) assume that human capital completely decays, or perishes, at the end of the second period.

The Marginal Cost of a Reduction in Debt Capacity

Before proceeding to a discussion of the valuation relationships for human capital, it is necessary to clarify the economic interpretation of the shadow prices $\lambda(t+j)$ and $\lambda^B(t+j)$. The analysis presented below shows that the marginal (dollar) value of additional borrowing capacity may be positive or negative. It also shows: (i) that $\lambda^B(t+j)$ must be negative if the marginal cost of a reduction in borrowing capacity is positive; and (ii) that $[\partial J(t+j)/\partial W(t+j)]^{-1}$ is not the relevant factor for converting marginal utilities into marginal (dollar) valuations when the credit rationing constraint is binding.

Consider:

$$J(t) \equiv \underset{X_t}{\text{Max:}} \quad F(t) + \lambda(t)[W(t) - P_t{}'X_t] \qquad (3:13)$$

$$+ \lambda^B(t)[B(t) - P_t{}'\gamma X_t]$$

where $F(t) \equiv E_t\{J(t+1)\}$; and $B(t) = B(W(t) - P_t{}'\gamma X_t)$.

When the maximum in (3:13) is attained, the vector $(\hat{X}_t, \hat{\lambda}(t), \hat{\lambda}^B(t))$ must satisfy the constraints and the M first order conditions:

$$F_i(t) \equiv \frac{\partial F(t)}{\partial X_i(t)} \leq P_i(t)[\hat{\lambda}(t) + \hat{\lambda}^B(t)\ \gamma_i(B'+1)] \qquad (3:14)$$

where, according to the envelope theorem:

$$J_W(t) \equiv \frac{\partial J(t)}{\partial W(t)} \equiv \hat{\lambda}(t) + \hat{\lambda}_B(t)B'(t) > 0 \qquad (3:15)$$

Suppose that $\hat{X}_i(t)$ represents a perishable consumption good, $C_i(t)$, so that $\gamma_i = 0$. In this case, (3:14) implies:

$$\frac{F_i(t)}{\hat{\lambda}(t)} \leq P_i(t) \qquad (3:16)$$

(3:16) highlights the importance of providing a correct economic interpretation of the Lagrange multipliers. Although this inequality does suggest that $[\lambda(t)]^{-1}$, rather than $[J_W(t)]^{-1}$, is the factor which converts any expected marginal utility $F_i(t)$ into a marginal (dollar) valuation of the corresponding choice variable, it certainly does not imply that this is the case. Before jumping to this conclusion, it is necessary to show that $[\lambda(t)]^{-1}$ is equal to the marginal cost of expected utility at the optimum position J(t).

Perhaps the easiest way to clarify the economic interpretation of $\lambda(t)$ and $\lambda^B(t)$ is to use the expenditure function $e(t) = e(P_t, B(t), K(t), \Omega_t; J^*)$. This function is defined by the dual to (3:13); namely:

$$e(t) \equiv \underset{X_t}{\text{Min}}\ P_t'X_t + \mu(t)[J^* - F(t)] + U^B(t)[P_t'\gamma X_t - B(t)] \qquad (3:17)$$

where J^* is any parametric level of expected utility.

When (3:17) is attained and when $J^* = J(t)$, the vector of compensated or 'Hicksian' demands, say X_t^*, must satisfy the following identity:

$$e(\ldots; J(t)) \equiv P_t'X_t^* \equiv W(t) \equiv P_t'\hat{X}_t \qquad (3:18)$$

(3:18) implies:

$$\hat{X}_i(t) \equiv X_i^*\left(P_t, B^*(t), K(t), \Omega_t; J(t)\right) \tag{3:19}$$

where: $B^*(t) \equiv \hat{B}(t) = B\left(W(t), P_t, K(t), \Omega_t\right)$.

(3:18) and (3:19) state that the compensated demands $X_i^*(t)$ must be equal to the uncompensated demands $\hat{X}_i(t)$ when the objective of maximizing expected utility has been attained. When (3:18) is differentiated with respect to $P_i(t) \neq 0$, we obtain:

$$e_i(t) + \left(\frac{\partial e(t)}{\partial J(t)}\right) J_i(t) \equiv \bar{X}_i(t) \tag{3:20}$$

where:

$$e_i(t) \equiv \partial e/\partial P_i + \frac{\partial e}{\partial B}\frac{\partial B}{\partial P_i}$$

$$J_i(t) \equiv \partial J/\partial P_i + \frac{\partial J}{\partial B}\frac{\partial B}{\partial P_i} + \frac{\partial J}{\partial W}\bar{X}_i(t)$$

$\bar{X}_i(t)$ is the endowment of $X_i(t)$

Notice, however, that when the envelope theorem is applied to (3:17), that:

$$e_i(t) \equiv X_i^*(t)\left[1 + \mu^B(t)\, \gamma_i (1+B') - \mu^B(t)\, B'\bar{X}_i(t)\right] \tag{3:21}$$

and also that, when it is applied to (3:13), that:

$$J_i(t) \equiv -\hat{X}_i(t)\left[\hat{\lambda}(t) + \hat{\lambda}_i^B(t)\, \gamma\, (1+B')\right] + \bar{X}_i(t)\left[\hat{\lambda}(t) + \lambda^B(t)B'\right] \tag{3:22}$$

(3:21) is a generalized version of Shephard's lemma, while (3:22) is a generalized version of Roy's identity. When these identities are substituted into (3:20) and when (3:20) is the relevant equation for a perishable consumption good, so that $\bar{X}_i(t) = \gamma_i = 0$, we have:

$$X_i^*(t) - \left(\frac{\partial e(t)}{\partial J(t)}\right) \hat{X}_i(t)\, \hat{\lambda}(t) \equiv 0 \qquad (3:23)$$

or since $X_i^*(t) \equiv \hat{X}_i(t)$:

$$\left(1 - \frac{\partial e(t)}{\partial J(t)}\, \hat{\lambda}(t)\right) \equiv 0 \qquad (3:23)'$$

According to the envelope theorem, however, $\partial e(t)/\partial J(t) \equiv \mu^*(t)$. Hence, from (3:23)':

$$\mu^*(t)\, \hat{\lambda}(t) \equiv 1 \qquad (3:24)$$

$\mu^*(t)$ is the marginal (dollar) cost of obtaining an additional 'unit' of expected utility at the optimum position J(t). (3:24) implies, therefore, that $[\hat{\lambda}(t)]^{-1}$ is the variable which converts expected marginal utilities into marginal (dollar) valuations of the choice variables $X_i(t)$.

In order to develop the economic interpretation of $\lambda^B(t)$ and some of its properties, notice first that (3:18) implies:

$$\frac{\partial e(t)}{\partial W(t)} + \frac{\partial e(t)}{\partial J(t)} \left(\frac{\partial J(t)}{\partial W(t)} + \frac{\partial J(t)}{\partial B(t)} B'\right) \equiv 1 \qquad (3:25)$$

By using the envelope theorem and (3:24), it is easy to show that (3:25) implies[13]:

$$\frac{\partial e(t)}{\partial W(t)} \equiv -\mu^B(t)B' \equiv -\hat{\lambda}(t)^{-1}\, \hat{\lambda}^B(t)B' \qquad (3:26)$$

---

[13] See Appendix A.

or, since $\frac{\partial e(t)}{\partial W(t)} = \frac{\partial e(t)}{\partial B(t)} B'$:

$$\frac{\partial e(t)}{\partial B(t)} \equiv -\mu^B(t) \equiv -\hat{\lambda}(t)^{-1}\lambda^B(t) \qquad (3:26)'$$

(3:26)' states that the marginal value of an increase in borrowing capacity is equal to $-\hat{\lambda}(t)^{-1}\hat{\lambda}^B(t)$ at the optimum position J(t). Although intuition may suggest that this shadow price is nonnegative, especially given the assumption that preferences do not exhibit nonsatiation, its sign is indeterminate. The essential reason appears to be that increases in borrowing capacity have two potentially offsetting effects: the aggregate amount of money borrowed increases when expected utility is held constant, but the expenditures on the remaining choice variables may increase or decrease when individuals are compensated for changes in J(t).

Consider the following comparative static system:

$$(A)\begin{pmatrix} \partial X/\partial\theta \\ \partial\mu/\partial\theta \\ \partial\mu^B/\partial\theta \end{pmatrix} = \begin{pmatrix} h_X \\ h_J \\ h_B \end{pmatrix} \qquad (3:27)$$

θ can be any parameter underlying the dual optimizing problem in (3:17).

(A) is the bordered Hessian:

$$\begin{pmatrix} \{a_{ij}\} & -F_X & (1+B')\{P_i\gamma_i\} \\ -F_X & 0 & 0 \\ (1+B')\{P_j\gamma_j\} & 0 & 0 \end{pmatrix}$$

where:

$a_{ij} = -(\mu F_{ij} + B''\mu^B P_i P_j \gamma_i \gamma_j)$ ... $i,j = 1,\ldots,M$

F_X is the M vector $(F_1,\ldots F_i,\ldots,F_M)$

h_X' is the M vector $(h_1, \ldots, h_i, \ldots, h_M)$

$$h_i = -\frac{\partial P_i}{\partial \theta}\left(1 + \mu^B \gamma_i (1+B')\right) + \mu\, F_{i\theta}$$

$$- \mu^B P_i \gamma_i B'' \left(\frac{\partial W}{\partial \theta} - \sum_{j=1}^{M} \gamma_j X_j^* \frac{\partial P_i}{\partial \theta}\right)$$

$$h_J = F_\theta - J_\theta;\ F_\theta \equiv \frac{\partial F(t)}{\partial \theta},\ J_\theta \equiv \frac{\partial J(t)}{\partial \theta}$$

$$h_B = B' \frac{\partial W}{\partial \theta} - (1+B\ ) \sum_{j}^{M} \gamma_j\ X_j^* \frac{\partial P_j}{\partial \theta}$$

Solving (3:27) for $\partial X_j^*/\partial\theta$ yields:

$$\frac{\partial X_j^*}{\partial \theta} = |A|^{-1} \left[\sum_{i=1}^{M} |A_{ij}| h_i + |A_{J_j}| h_J + |A_{B_j}| h_B\right] \qquad (3:28)$$

where $|A_{ij}|$, $|A_{J_j}|$ and $|A_{B_j}|$ are the cofactors of the elements a_{ij}, $-F_j$ and $(1+B)' P_j \gamma_j$. If the second order conditions for (3:17) hold, then $|A| < 0$ and $|A_{ii}| < 0$ for all $i = 1, \ldots, M$.

Suppose, then, that $\theta = W(t)$. In this case:

$$h_i = -\mu^B B'' P_i \gamma_i,\ i = 1, \ldots, M$$

$$h_j = 0$$

$$h_B = B'$$

According to (3:28) therefore:

$$\frac{\partial X_j^*}{\partial W} = -\frac{\mu^B B''}{|A|} \sum_{i=1}^{M} |A_{ij}| P_i \gamma_i + \frac{B'}{|A|} |A_{B_j}| \qquad (3:29)$$

$$\ldots\ j = 1, \ldots, M$$

Notice, however, that a Laplace expansion around the last column of (A) yields:

$$(1+B')\sum_{i=1}^{M}\left|A_{iB}\right|P_i\gamma_i = \left|A\right| \tag{3:30}$$

Notice, moreover, that:

$$(1+B)'\sum_{i=1}^{M}\left|A_{ij}\right|P_i\gamma_i = 0 \tag{3:31}$$

(3:31) holds because it is an alien cofactor expansion.

Substituting (3:31) into (3:29) yields:

$$\frac{\partial X_j^*}{\partial W} = \frac{B'}{\left|A\right|}\left|A_{B_j}\right|;\ j = 1,\ldots,M;\ B' \geq 0 \tag{3:32}$$

Unfortunately, the sign of (3:32) is indeterminate. As indicated previously, however, the effect of an increase in W upon the aggregate amount of money borrowed is determinate when expected utility is held constant. (3:32) implies:

$$-\sum_{j\varepsilon Q} P_j \frac{\partial X_j^*}{\partial W} = -\frac{B'}{\left|A\right|}\sum_{j\varepsilon Q}\left|A_{B_j}\right|P_j \tag{3:33}$$

where Q is the set of those $X_i(t)$ which represent either short or long term borrowing.

Since $\gamma_i = -1$ when $X_i(t)$ refers to a decision to supply a debt instrument, substituting (3:30) into (3:33) yields:

$$- \sum_{j \varepsilon Q} P_j \frac{\partial X_j^*}{\partial W} = \frac{B'}{(1+B)'} > 0;\ B' > 0 \tag{3:34}$$

The left-hand side of (3:34) is equal to the rate of increase in the aggregate amount of money borrowed with respect to a wealth-induced increase in borrowing capacity when J(t) is held constant ($P_j < 0$ for loans).

Let $q = - \sum_{j \varepsilon Q} P_j X_j^*$. Since $dB = B'dW$, (3:34) implies:

$$dq = \frac{\partial q}{\partial W} dW = (1+B')^{-1}\ dB;\ B' > 0 \tag{3:35}$$

(3:35) states that an individual will increase his debt liabilities when his borrowing capacity is increased by $1; and, also, that his increase will be less than $1 when $B' > 0$ if he is compensated for the changes in J(t).

Unfortunately, the determinant response of the aggregate (compensated) supply of debt instruments to wealth induced changes in borrowing capacity does not yield any information about the sign of the marginal value of additional borrowing capacity since $\partial X_j^*/\partial W$ has an indeterminate sign for the other choice variables. Suppose, however, that $J_W(t)/\lambda(t) > 0$ and that $0<B'<1$. These two assumptions and (3:14) jointly imply:

$$\frac{J_W(t)}{\lambda(t)} = 1 + \frac{\lambda^B(t)}{\lambda(t)} B'(t) > 0, \text{ or:}$$

$$- \frac{\lambda^B(t)}{\lambda(t)} < \frac{1}{B'} \tag{3:36}$$

(3:36) does not rule out the possibility that the marginal value of additional borrowing capacity, namely $-\lambda^{B}(t)/\lambda(t)$, is negative. This possibility is disturbing because a negative value for $-\lambda^{B}(t)/\lambda(t)$ implies that an individual cannot be simultaneously changing his debt liabilities and his holdings of the debt instruments supplied by others when borrowing rates exceed lending rates. To see this, suppose that the interest rate for a single period loan, say $r_B(t)$, exceeds the single period lending rate, $r_L(t)$. Assume, as well, that an individual is changing his single period debt obligations and his investments in the single period debt instruments supplied by other individuals. In this case, (3:14) requires:

$$1 - \frac{\lambda^{B}(t)(B'+1)}{\lambda(t)} = \frac{1+r_B(t)}{1+r_L(t)} > 1 \qquad (3:37)$$

Clearly, this necessary condition cannot hold when $-\lambda^{B}(t)/\lambda(t)$ is negative.

In reality, of course, individuals frequently change their debt liabilities and their investments in default free assets at the same time. The analysis in the following sections allows for this by assuming, as (3:37) requires, that $-\lambda^{B}(t)/\lambda(t) \geq 0$ for all t. This is equivalent to assuming that the marginal cost of a reduction in debt capacity is nonnegative.

4. MARGINAL VALUATIONS OF HUMAN CAPITAL

The analysis contained in this section shows: (i) that equilibrium marginal valuations of human capital are positively (negatively) related to the learning-by-doing (participation) effects discussed in Section 2; and

(ii) that current marginal valuations of the capital stock K(t) are never less than the current marginal valuations of K(t+1), K(t+2), ..., K(t+T). These two results suggest:

(a) that salary-rated employees have a greater incentive to invest in human capital than hourly-rated employees; and

(b) that individuals have a strong incentive to build up their stock of human capital as rapidly as possible.

For expositional purposes recall that:

$$J(t) \equiv \max_{X_t} F(t) + \lambda(t)\left[W(t) - P_t{}'X_t\right] + \lambda^B(t)\left[B(t) - P_t{}'\gamma X_t\right] \qquad (3:13)'$$

When this maximum is attained, the vector $\left(\hat{X}_t, \hat{\lambda}(t), \hat{\lambda}^B(t)\right)$ must satisfy the constraints and the M first order conditions:

$$F_i(t) \leq P_i(t)\left[\hat{\lambda}(t) + \hat{\lambda}^B(t)\gamma_i(B'+1)\right] \qquad (3:14)'$$
$$i = 1,\ldots,M$$

Let $\hat{N}(t)$ and $\hat{S}(t)$ denote the optimal choices for N(t) and S(t). The first order conditions for these two (unconstrained) choice variables can be written as:

$$\frac{E_t\{J_K(t+1)\}G_N(t+1)}{\hat{\lambda}(t)} \leq \frac{E_t\{J_L(t+1) - J_Y(t+1)Y_N(t+1)\}}{\hat{\lambda}(t)} \qquad (4:1)$$

$$\frac{E_t\{J_K(t+1)\}G_S(t+1)}{\hat{\lambda}(t)} \leq \frac{E_t\{J_L(t+1) - J_Y(t+1)Y_S(t+1)\}}{\hat{\lambda}(t)} \qquad (4:2)$$

(4:1) and (4:2) are derived from (2:1), (3:1), (3:6), (3:7), (3:10) and (3:14)'. These two cost minimizing conditions simply state that the expected marginal revenue products of the two factor inputs must not exceed their expected marginal costs when (3:13)' is attained. They also imply that the expected marginal costs of obtaining the capital stock K(t+1) must be at least as large as the expected marginal value of K(t+1), or that:

$$V_K\big(t;K(t+1)\big) \leq P_K\big(t;K(t+1)\big) \tag{4:3}$$

where:

$$V_K\big(t;K(t+1)\big) \equiv \hat{\lambda}(t)^{-1}E_t\{J_K(t+1)\}$$

$$P_K\big(t;K(t+1)\big) \equiv \text{Min}\begin{cases} G_N(t+1)^{-1}P_N\big(t;K(t+1)\big) \\ G_S(t+1)^{-1}P_S\big(t;K(t+1)\big) \end{cases}$$

$$P_N\big(t;K(t+1)\big) \equiv \hat{\lambda}(t)^{-1}E_t\{J_L(t+1) - J_Y(t+1)Y_N(t+1)\}$$

$$P_S\big(t;K(t+1)\big) \equiv \hat{\lambda}(t)^{-1}E_t\{J_L(t+1) - J_Y(t+1)Y_S(t+1)\}$$

One of the characteristics of (4:3) is that it always satisfies:

$$P_K\big(t;K(t+1)\big) \geq V_K\big(\big(t;K(t+1)\big) \geq 0 \tag{4:4}$$

The reason (4:4) always holds, for any t, is that (3:12) and (2:7)' jointly imply (as shown in Section 3) that $J_K(t+j) \geq 0$; i.e., if the ability to produce human capital in a nondecreasing function of the accumulated capital stocks and if realized wage rates are never diminished by increases in human capital, then:

$$V_K\big(t;K(t+1)\big) = E_t\{J_K(t+1)\}\hat{\lambda}(t)^{-1} \geq 0 \tag{4:5}$$

$$\ldots\ \hat{\lambda}(t) > 0$$

Suppose that an interior solution obtains for N(t). In this case, (4:3) becomes:

$$V_K\big(t;K(t+1)\big) = \frac{E_t\{J_L(t+1) - J_Y(t+1)Y_N(t+1)\}}{\hat{\lambda}(t)G_N(t+1)} \qquad (4:6)$$

Assume, as well, that the effects of learning-by-doing upon Y(t+1) are dominated by the participation effects. In this case, (2:12) and (2:13) imply:

$$\tilde{Y}_N(t+1) = \tilde{R}(t+1) - \hat{N}(t)\tilde{\rho}(t+1)\phi_N(t+1) \geq 0 \qquad (4:7)$$

$$-\hat{N}(t)\tilde{\rho}(t+1)\phi_N(t+1) \geq 0 \qquad (4:8)$$

These latter two inequalities hold for hourly-rated employees. Substituting them into (4:6) yields:

$$V_K\big(t;K(t+1)\big) - \frac{E_t\{J_L(t+1) - J_Y(t+1)R(t+1)\}}{\hat{\lambda}(t)G_N(t+1)} \leq 0 \qquad (4:9)$$

Suppose, in contrast, that the participation effects are dominated by the learning-by-doing effects. In this case, (4:7) holds, but:

$$-\hat{N}(t)\tilde{\rho}(t+1)\phi_N(t+1) \leq 0 \qquad (4:10)$$

(4:10) is one of the characteristics of salary-rated jobs: see (2:10) and (2:11). It implies, along with (4:7) and (4:6) that:

$$V_K\big(t;K(t+1)\big) - \frac{E_t\{J_L(t+1) - J_Y(t+1)R(t+1)\}}{\hat{\lambda}(t)G_N(t+1)} \geq 0 \qquad (4:11)$$

Strictly speaking, (4:9) and (4:11) imply only that $V_K\big(t;K(t+1)\big)$ is positively (negatively) related to the learning-by-doing (participation) effects upon Y(t+1). Although these two results do not imply that the marginal value of human capital for a salary-rated employee is always larger than the corresponding value for an hourly-rated employee, they do show that the marginal value of human capital is connected, in a plausible way, to the structure of employment contracts.

Clearly, any model of human capital formation should provide some insight into the life cycle characteristics of investments in this asset. The model developed in this paper passes this test in the sense that it implies that it is never advantageous to defer these investment decisions. It indicates, in other words, that individuals have a very strong incentive to build up their stock of human capital as rapidly as they can.

To see this, notice that repeated application of the envelope theorem to any problem which has the same form as (3:5) yields:

$$\frac{\partial J(t+j)}{\partial K(t+j-1)} \equiv \frac{\partial F(t+j)}{\partial K(t+j-1)} + J_W(t+j)Y_K(t+j) \qquad (4:12)$$

$$\frac{\partial J(t+j)}{\partial K(t+j)} \equiv \frac{\partial F(t+j)}{\partial K(t+j)} \qquad (4:13)$$

Recall, moreover, that:

$$\frac{\partial K(t+j)}{\partial K(t+j-1)} = 1 + G_K(t+j) \geq 1 \qquad (3:12)'$$

Substituting (3:12)' and (4:13) into (4:12) yields:

$$\frac{\partial J(t+j)}{\partial K(t+j-1)} \equiv \frac{\partial J(t+j)}{\partial K(t+j)} \left[1 + G_K(t+j)\right] + J_W(t+j)Y_K(t+j) \qquad (4:14)$$

(3:12)' and (4:14) jointly imply that changes in K(t+j-1) will always produce larger changes in the attained level of expected utility in period t+j than changes in K(t+j). This dominance of earlier investments in human capital is reflected in the valuation expressions. When j = 0, (4:14) implies:

$$V_K\big(t;K(t-1)\big) \geq V_K\big(t;K(t)\big) \tag{4:15}$$

(4:15) holds even though $K(t) \geq K(t-1)$. Notice, moreover, that:

$$V_K\big(t;K(t)\big) = \hat{\lambda}(t)^{-1}E_t\Big\{J_Y(t+1)\,\frac{\partial Y(t+1)}{\partial K(t)}\Big\} + \lambda(t)^{-1}E_t\big\{J_K(t+1)\big\}\big(1 + G_K(t+1)\big) \tag{4:16}$$

The first term on the right-hand side of (4:16) is always nonnegative, given nonsatiation because, from (2:7):

$$\frac{\partial Y(t+1)}{\partial K(t)} = N(t)\big[1 - \phi_K(t+1)\big] \geq 0 \tag{2:7$'$}$$

The second term, on the other hand, is equal to the nonnegative expression $\big(1 + G_K(t+1)\big)V_K\big(t;K(t+1)\big)$: see (4:5).

(4:15) and (4:16) jointly imply:

$$V_K\big(t;K(t-1)\big) \geq V_K\big(t;K(t)\big) \geq V_K\big(t;K(t+1)\big) \tag{4:17}$$

even though $K(t+1) \geq K(t) \geq K(t-1)$.

(4:17) will hold as an equality only when previous experience in acquiring knowledge has a zero marginal product, so that $G_K(t+j) \equiv 0$ <u>and</u> when $\partial Y(t+j)/\partial K(t+j-1) \equiv 0$.

According to (2:7)', however, this situation can hold only when $\hat{N}(t+j) = 0$ for all t+j. In summary, then, if $\hat{N}(t+j) \geq 0$ for all exchange periods, we must have strong dominance, or:

$$V_K(t;K(t-1)) > V_K(t;K(t)) > V_K(t;K(t+1)) \qquad (4:18)$$

even though $K(t+1) > K(t) > K(t-1)$.

5. RISK-RETURN CONDITIONS FOR HUMAN CAPITAL

Human capital theory is plagued by an unsolved problem: it appears to be impossible to develop measures for costs and returns which are both observable and theoretically sound. Although the measures of per capita labour income which are used in empirical work (see, for example, Fama and Schwert (1977) and Liberman (1980)) may be the best return proxies available, they have three major limitations: (i) they implicitly assume that the costs of accumulating human capital are equal to zero; (ii) they fail to incorporate any valuation of the stream of benefits accruing to human capital; and (iii) they do not correspond to the changes in economic welfare associated with investments in human capital.

The return measures developed below satisfy all of the preceding three conditions but, alas, they are not observable. The main reason for developing them is to gain some insight into the economic structure of the risk premiums on human capital.

Suppose that $\hat{X}_i(t) \neq 0$, where $X_i(t)$ is either a marketable capital asset or a single period debt instrument. In this case, (3:14) implies:

$$E_t\{J_W(t+1)Z_i(t+1)\} = P_i(t)(\hat{\lambda}(t) + \hat{\lambda}_B(t)\gamma_i(B'+1)) \qquad (5:1)$$

Let $\bar{r}_i(t+1,t)$ or $(\bar{Z}_i(t+1,t)/P_i(t))-1$ denote the expected rate of return on asset i over period t. Using this definition, (5:1) can be written as:

$$1 + \bar{r}_i(t+1,t) = \frac{\hat{\lambda}(t) + \hat{\lambda}^B(t)\gamma_i(B'+1)}{\bar{J}_W(t+1;t)} - \frac{\mathrm{Cov}_t\left(J_W(t+1), r_i(t+1)\right)}{\bar{J}_W(t+1;t)} \qquad (5:2)$$

where $\bar{J}_W(t+1;t) \equiv E_t\{\tilde{J}_W(t+1)\}$.

Suppose that an individual has chosen both to issue short term debt and to hold the short term debt instruments supplied by others. If these instruments are default free, then, from (5:2):

$$\bar{J}_W(t+1,t) = \hat{\lambda}(t)/1 + r_L(t) = \frac{\hat{\lambda}(t) - \hat{\lambda}^B(t)(B'+1)}{1 + r_B(t)} \qquad (5:3)$$

Substituting (5:3) into (5:2) and subtracting $1+r_L(t)$ from both sides yields:

$$\bar{r}_i(t+1,t) - r_L(t) = \left(1+r_L(t)\right)\left(\frac{\hat{\lambda}^B(t)\gamma_i(B'+1)}{\hat{\lambda}(t)}\right) + \left(1+r_L(t)\right)\pi_i(t) \qquad (5:4)$$

with:

$$\pi_i(t) \equiv - \frac{\mathrm{Cov}_t\left(J_W(t+1), \tilde{r}_i(t+1)\right)}{\hat{\lambda}(t)}$$

When $\bar{r}_i(t+1,t)$ in (5:4) equals $r_B(t)$, then:

$$\left(1+r_L(t)\right) \frac{\hat{\lambda}^B(t)(B'+1)}{\hat{\lambda}(t)} = r_L(t) - r_B(t) < 0 \qquad (5:5)$$

(5:5) and (5:4) jointly imply:

$$\bar{r}_i(t+1,t) - r_L(t) = \gamma_i\left(r_L(t) - r_B(t)\right) + \left(1+r_L(t)\right)\pi_i(t) \qquad (5:6)$$

The first term in (5:6) is equal to zero for all those securities which are not debt liabilities. If, for example, an individual issues a callable debt instrument which matures at M = t+j, then $\gamma_i = -1$ and:

$$\bar{r}_B(t+1,t;M) = r_B(t) + (1+r_L)\pi_i(t) \qquad (5:7)$$

where $1+\tilde{r}_B(t+1,M)$ is equal to the (perhaps uncertain) redemption value at t+1 divided by the issue price at time t.

Clearly, if the redemption value, say P(t+1,M) is nonstochastic, then from (5:7):

$$r_B(t+1,M) = r_B(t) \qquad (5:8)$$

(5:8) holds under credit rationing when the rationing coefficients, namely γ_i, are not sensitive to the terms of maturity on debt liabilities. Short and long term debt will not be perfect substituents, given nonstochastic redemption values, however, if the credit rationing parameters depend on the term to maturity. Assume, for example, that $\gamma_i < -1$ when the term to maturity is more than one period. In this case, (5:6) implies:

$$r_B(t+1,M) - r_B(t) = \left(1 + \gamma_i(M)\right)\left(r_L(t) - r_B(t)\right) > 0 \qquad (5:9)$$

(5:9) states that long term debt is more expensive than short term debt when borrowing capacity is a decreasing function of the terms to maturity.

One of the essential differences between human capital and other capital assets is that the returns on this asset are implicit returns. The implicit marginal return on K(t+1) in particular is equal to $J_K(t+1)$. If an interior solution obtains for either N(t) or S(t), moreover, then from (4:3):

$$E_t\{J_K(t+1)\} = \hat{\lambda}(t)\big(P_K(t;K(t+1))\big) \tag{5:10}$$

An equivalent form of (5:10), however, is:

$$E_t\{\lambda(t+1)V_K\big(t+1;K(t+1)\big)\} = \hat{\lambda}(t)P_K\big(t;K(t+1)\big) \tag{5:10)'$$

where $V_K\big(t+1;K(t+1)\big)$ is the marginal dollar value of $K(t+1)$ at the beginning of period $t+1$; i.e.,

$$V_K\big(t+1;K(t+1)\big) \equiv \lambda(t+1)^{-1}J_K(t+1)$$

(5:10)' suggests that we define the random rate of return on human capital, say $\tilde{r}_K(t+1)$, by:

$$1 + \tilde{r}_K(t+1) \equiv \frac{V_K\big(t+1;K(t+1)\big)}{P_K\big(t;K(t+1)\big)} \tag{5:11}$$

Substituting (5:11) into (5:10)' yields:

$$1 + \bar{r}_K(t+1,t) = \frac{\hat{\lambda}(t)}{\bar{\lambda}(t+1,t)} - \frac{\mathrm{Cov}_t\big(\tilde{r}_K(t+1),\tilde{\lambda}(t+1)\big)}{\bar{\lambda}(t+1,t)} \tag{5:12}$$

or:

$$1 + \bar{r}_K(t+1,t) = \frac{\hat{\lambda}(t)}{\bar{\lambda}(t+1,t)}\big(1 + \pi_K(t)\big) \tag{5:12)'$$

where:

$$\pi_K(t) \equiv -\hat{\lambda}(t)^{-1}\mathrm{Cov}_t\big(\tilde{\lambda}(t+1),\tilde{r}_K(t+1)\big)$$

From (5:3), however, $\hat{\lambda}(t) = \bar{J}_W(t+1,t)(1+r_L)$; hence, by substitution:

$$\bar{r}_K(t+1,t) - r_L(t) = \theta_K(t) + \big(1+r_L(t)\big)\pi_K(t) \qquad (5:13)$$

with:

$$\theta_K(t) \equiv \big(1+r_L(t)\big)\big(1+\pi_K(t)\big)\left[\frac{\bar{J}_W(t+1,t)}{\bar{\lambda}(t+1,t)} - 1\right]$$

(5:13) is one form of the risk-return condition on human capital; $\theta_K(t)$ is a credit rationing premium. This term appears in this equilibrium condition because borrowing capacity in period t+1 is influenced by an investment in a default free credit instrument but not by a marginal increase in the capital stock K(t+1). (5:13) indicates, then, that the expected rate of return on human capital will be influenced by those contract formation constraints which affect the ability of individuals to borrow against the returns on this asset.

In order to highlight what are perhaps the major differences between the risk premiums on human capital and those on marketable assets, suppose that credit rationing constraints do not exist. In this case, (3:29) requires that $r_B(t) = r_L(t) = r_F(t)$, where $r_F(t)$ is the interest rate on all short term debt instruments. The risk-return conditions for all assets, moreover, will have the same form; namely,

$$\bar{r}_i(t+1,t) - r_F(t) = \big(1+r_F(t)\big)\cdot\pi_i(t) \qquad (5:6)'$$

$$\bar{r}_K(t+1,t) - r_F(t) = \big(1+r_F(t)\big)\cdot\pi_K(t) \qquad (5:13)'$$

Although these equilibrium conditions have the same form, there are some significant differences between $\pi_i(t)$ and $\pi_K(t)$. Notice, in particular, that, by using the definition for $r_K(t+1)$ and a second order approximation for $E_t\{J_K/\lambda\}$, that $\pi_K(t)$ can be decomposed into the linear combination:

$$\left(\hat{\lambda}(t)\cdot P_K\big(t;K(t+1)\big)\right)^{-1}\left[\frac{\bar{J}_K(t+1,t)}{\bar{\lambda}(t+1,t)^2}\,\mathrm{Var}_t\big(\tilde{\lambda}(t+1)\big) - \frac{\mathrm{Cov}_t\big(\tilde{\lambda}(t+1),\tilde{J}_K(t+1)\big)}{\bar{\lambda}(t+1,t)}\right] \tag{5:14}$$

Substituting (5:14) into (5:13)' and using the first order condition (5:10) yields:

$$\bar{r}_K(t+1,t) - r_F(t) = \left[\frac{1+r_F(t)}{\bar{\lambda}(t+1,t)}\right]\left[\frac{\mathrm{Var}_t\big(\tilde{\lambda}(t+1)\big)}{\bar{\lambda}(t+1,t)} - \frac{\mathrm{Cov}_t\big(\tilde{\lambda}(t+1),\tilde{J}_K(t+1)\big)}{\bar{J}_K(t+1,t)}\right] \tag{5:15}$$

(5:15) highlights the fact that human capital is a risky capital asset even when there is no uncertainty about labour income. The variance in the marginal utility of wealth, in particular, is a function of the variance in labour income and the variance in nonhuman wealth. Hence, given uncertainty about the returns on marketable assets, it is always positive, even when labour income is nonstochastic. The covariance term in (5:15), on the other hand, is a complicated serial covariance: see the expression for $J_K(t+1)$ in (3:10). It appears to be a complicated function of the serial covariances $\mathrm{Cov}_t\{\lambda(t+1),J_W(t+j)\}$, where $j > 1$. Again, given uncertainty about the returns on nonhuman capital, it is non-zero.

APPENDIX A

Consider

$$J(t+j) \equiv \underset{X_{t+j}}{\text{Max}} \;\; E_{t+j}\{J(t+j+1)\}$$
$$+ \lambda(t+j)[W(t+j) - (P_{t+j})'X_{t+j}]$$
$$+ \lambda^B(t+j)[B(t+j) - (P_{t+j})'\gamma X_{t+j}] \qquad (3:5)'$$

The first order conditions associated with (3:5)' can be written as:

$$F_i(t+j) - P_i(t+j)[\hat{\lambda}(t+j) + \hat{\lambda}^B(t+j)\gamma_i(B'+1)] = 0 \qquad (A:1)$$
$$\ldots\; i = 1, \ldots, m$$

$$W(t+j) - (P_{t+j})'\hat{X}_{t+j} = 0 \qquad (A:2)$$

$$B(t+j) - (P_{t+j})'\gamma\hat{X}_{t+j} = 0 \qquad (A:3)$$

where

$$F_i(t+j) \equiv E_{t+j}\left\{\frac{\partial J(t+j+1)}{\partial X_i(t+j)}\right\}$$

(A:1), (A:2) and (A:3) have exactly the same form as the first order conditions in (3:14).

The envelope theorem states that the rate of change in the optimal value of the objective function with respect to the change in any parameter is equal to the rate of change in the Lagrangian with respect to that parameter when the latter derivative is evaluated at the optimal values of the decision variables [see Varian (1978)]. Applying this theorem to (3:5)' yields the results presented in (3:9). To see this, notice that:

$$\frac{\partial J(t+j)}{\partial K(t+j)} = E_{t+j}\left\{\frac{\partial J(t+j+1)}{\partial Y(t+j+1)}\,\frac{\partial Y(t+j+1)}{\partial K(t+j)} + J_K(t+j+1)\,\frac{\partial K(t+j+1)}{\partial K(t+j)}\right\}$$

$$+ \sum_i \frac{\partial X_i(t+j)}{\partial K(t+j)}\left(F_i(t+j) - P_i(t+j)[\hat{\lambda}(t+j) + \hat{\lambda}^B(t+j)\gamma_i(B'+1)]\right)$$

$$+ \frac{\partial \lambda(t+j)}{\partial K(t+j)}\left[W(t+j) - (P_{t+j})'X_{t+j}\right]$$

$$+ \frac{\partial \lambda^B(t+j)}{\partial K(t+j)}\left[B(t+j) - (P_{t+j})'\gamma X_{t+j}\right] \qquad (A:4)$$

When (A:4) is evaluated at the optimum choice vector $\hat{X}_{t+j}$, i.e. when (A:1), (A:2) and (A:3) hold, then

$$\frac{\partial J(t+j)}{\partial K(t+j)} \equiv E_{t+j}\left\{\frac{\partial J(t+j+1)}{\partial Y(t+j+1)}\,\frac{\partial Y(t+j+1)}{\partial K(t+j)} + J_K(t+j+1)\,\frac{\partial K(t+j+1)}{\partial K(t+j)}\right\} \qquad (3:9)$$

where:

$$J_K(t+j+1) \equiv E_{t+j+1}\left\{\frac{\partial J(t+j+2)}{\partial K(t+j+1)}\right\}$$

The envelope theorem result presented in (3:26) follows immediately from the first order conditions for the minimization problem in (3:7); i.e., differentiating both sides of (3:17) with respect to W(t) and then evaluating the resulting expression at the optimal choice X_t^* yields:

$$\frac{\partial e(t)}{\partial W(t)} \equiv -\mu^B(t)B' \qquad (3:26)'$$

BIBLIOGRAPHY

Arrow, K.J. (1962), "The Economic Implications of Learning by Doing", Review of Economic Studies, Vol. 24, No. 3, 155-173.

Blinder, A.S. and Weiss, Y. (1976), "Human Capital and Labor Supply: A Synthesis", Journal of Political Economy, Vol. 84, No. 3, 449-472.

Corcoran, M. and Duncan, G.J. (1979), "Work History, Labor Force Attachment and Earnings Differences Between Races and Sexes", Journal of Human Resources, Vol. 14, No. 1, 3-20.

Dixit, A.K. (1978), Optimization in Economic Theory, Oxford University Press, Toronto.

Eckaus, R.S. (1963), "Investment in Human Capital: A Comment", Journal of Political Economy, Vol. 71, No. 5, 501-504.

Fama, E.F. (1970), "Multiperiod Consumption-Investment Decisions", American Economic Review, Vol. 60, No. 1, 163-174.

Fama, E.F. and Schwert, G.W. (1977), "Human Capital and Capital Market Equilibrium", Journal of Financial Economics, Vol. 4, 95-125.

Ghez, G. and Becker, G.S. (1975), "The Allocation of Time and Goods Over the Life Cycle", Natural Bureau of Economic Research, New York.

Heckman, J.J. (1976), "A Life-Cycle Model of Earnings, Learning and Consumption", Journal of Political Economy, Vol. 84, No. 4, (Part 2), 511-544.

Killingsworth, M.R. (1982), "Learning by Doing and Investment in Training: A Synthesis of Two Rival Models of the Life Cycle", Review of Economic Studies, Vol. 49, 263-271.

Levhari, D. and Weiss, Y. (1974), "The Effect of Risk on the Investment in Human Capital", The American Economic Review, Vol. 64, No. 6, 950-962.

Mayers, D. (1972), "Non-marketable Assets and Capital Market Equilibrium Under Uncertainty", Studies in the Theory of Capital Markets, ed. M. Jensen, Praeger, New York.

Mayers, D. (1973), "Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset", Journal of Business, Vol. 46, No. 2, 258-267.

Mincer, J. and Polachek, S. (1978), "The Theory of Human Capital and the Earnings of Women: Women's Earnings Reexamined", Journal of Human Resources, Vol. 13, No. 1, 118-134.

Rosen, S. (1972), "Learning by Experience as Joint Production", Quarterly Journal of Economics, Vol. 86, No. 3, 366-382.

Rubinstein, M. (1974), "An Aggregation Theorem for Securities Markets", Journal of Financial Economics, Vol. 1, No. 3, 225-244.

Sandell, S.H. and Shapiro, D. (1978), "The Theory of Human Capital and the Earnings of Women: A Reexamination of the Evidence", Journal of Human Resources, Vol. 13, No. 1, 103-117.

Varian, H.R. (1978), Microeconomic Analysis, W.W. Norton & Company, Inc., New York.

Williams, J.T. (1978), "Risk, Human Capital and the Investor's Portfolio", Journal of Business, Vol. 51, No. 1, 67-90.