

**PRE-SERVICE ELEMENTARY SCHOOL TEACHERS'
EXPERIENCES WITH INTERPRETING AND CREATING
PROOFS**

by

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THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

In the
Faculty of Education

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SIMON FRASER UNIVERSITY

Fall 2006

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ABSTRACT

Having an ability to appreciate, understand, and create proofs is crucial in being able to evaluate students' mathematical arguments and reasoning. As such, the development of this ability in pre-service teachers is imperative. Research, however, has repeatedly shown that the ability to understand and create proofs is difficult for students in general and for pre-service elementary school teachers in particular.

This study aimed at extending the views and insights about the difficulties that pre-service elementary school teachers experience in dealing with the notion of mathematical proof. For this purpose I analysed students' discourses when they attempted to interpret or create proofs for some propositions related to elementary number theory.

The communicational approach to learning is the theoretical perspective that I adopted to investigate the difficulties students experience in generating proofs. According to communicational approach to cognition, thinking is a special case of the activity of communication, and learning mathematics is an initiation in a certain type of discourse, which is called literate mathematical discourse.

In this study, I have introduced the notion of dialogue as a tool for involving students in the process of creating a proof. Based on the idea that thinking can be considered as an act of communication that one has with oneself, I introduced dialogue as a self-dialogue or a conversation that a person has with oneself while she/he is thinking. I encouraged students to write a dialogue while they were thinking to interpret or create a proof. For this purpose, I designed six tasks. The results revealed that the main difficulty that students experienced in creating a proof is that they do not know how to communicate their idea mathematically.

There are several contributions of this study to the field of mathematics education focusing on pedagogy, methodology and theory.

Keywords: mathematics education, proof, pre-service elementary school teachers, elementary number theory, dialogue.

To the memory of my parents

ACKNOWLEDGEMENTS

My sincere thanks goes to my senior supervisor Professor Rina Zazkis who patiently and generously supported and guided me throughout this program with her knowledge and understanding. I am thankful to my committee member Dr. Peter Liljedahl for all his helpful comments. I would like to thank my first teachers in mathematics education Dr. Zahra Gooya and Dr. Bijan Zangeneh for inspiring and encouraging me to pursue my study. I also would like to thank my colleagues and friends Dr. Marianna Bogomolny and Shoaleh Bigdeli for being there for me when I needed it the most, and thanks to all my friends in SFU who were my family in Canada.

I would like to extend my thanks to my dearest sister and brothers who nurtured and supported me with their love throughout my life. And, my most special thanks to my uncle without his support I could not initiate this journey.

Thank you all for believing in me.

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CHAPTER 1: INTRODUCTION

“There is a traditional story about Newton: As a young student, he began the study of geometry, as was usual in his time, with the reading of the Elements of Euclid. He read the theorems, saw that they were true, and omitted the proofs. He wondered why anybody should take pains to prove things so evident. Many years later, however, he changed his opinion and praised Euclid.”

(Polya, 1957, p.215)

Many mathematicians, or those who are involved in mathematics as teachers or students, have had similar thoughts as Newton. Why should we prove a statement that seems obvious? What is the role of proof and its importance in mathematics and in mathematical thinking? The answers to these questions vary considerably from person to person, and are informed mainly through doing mathematics.

As a mathematics student, I learned to see the proof of a mathematical theorem as a sequence of steps that leads to a conclusion. As I studied more mathematics, I learned more about the essence of a mathematical proof and how it helps distinguish mathematics from other sciences. However, for many people, especially non-mathematicians, a verification of an argument, by considering several possible cases, can be the most convincing way of proving.

This raises an important question: what is in mathematical proof that is not in verification? Rota (1997) proposes that even though verification in its ideal form, i.e. consideration of all the possible cases, is a proof, it may not give the reason for why the

statement is true. Many good proofs provide the reason and explanation. The other value of a proof is that a given proof can be turned into a proof technique, suited to proving other theorems. Furthermore, on an advanced level, a proof may open up other mathematical possibilities, as seen in the proof of Fermat's last theorem by Andrew Wiles. One may gain and appreciate these types of experiences, in my opinion, only through doing mathematics.

With this perspective on a proof I started my work with pre-service elementary school teachers. In a short time, I learned that for majority of these students the notion of a proof does not have the same meaning as the shared understanding of proof in a mathematical context. In fact, a proof for them was mainly a verification based on some confirming examples. This group of students had studied the notion of proof in the context of elementary number theory. Considering the elementary level of number theory in their program, many of the propositions seemed too trivial to students so it was a challenge for them to get the main idea and importance of proofs in that context.

Reflecting on my own experience, I believe the best way to help students is by meaningfully involving them in the practice of doing proof. In my opinion, understanding the essence of proof is very personal, and it may not happen unless one is engaged in the process of proving. Research has repeatedly shown that proof is a difficult notion for students to understand. As a researcher, I believe that we can help our students only when we acquire a better understanding of the difficulties that they experience while doing mathematics and, this is especially the case in constructing a mathematical proof. The question then becomes: what can be an effective way to access student difficulties in writing proofs in a natural context?

In the past decades, a growing body of research on students' understanding of the notion of proof as well as writing a proof has accumulated (e.g. Dreyfus, 1999; Harel & Sowder, 1998; Moore, 1994). However, there has not been a significant amount of research on pre-service elementary teachers' understanding of, and their difficulties with, the notion of proof. My study attempts to fill this gap.

In my search, I encountered the work of Anna Sfard (2001), who introduced the communicational approach to cognition for the study of students' difficulties with mathematics in general. According to the communicational approach to cognition, thinking is a special case of activity of communication that a person has with oneself and learning mathematics is an initiation into a certain type of discourse.

As I became familiar with this theoretical framework, I was led to the idea that if we have access to the students' thinking process we may recognize students' difficulties better. In several studies, proof has been viewed as means for communicating ideas between mathematicians (Balacheff, 1991; Hanna, 1983; Hersh, 1993; Knuth, 2002; Schoenfeld, 1994). Indeed, a proof from this perspective can be considered as a self-sufficient discourse that includes the answers to all possible '*whys*' related to the argument. Furthermore, we can regard the process of constructing a proof as engaging in self-dialogue to satisfy the above requirement.

With this view, I attempted to create a learning environment that would implement the communicational nature of proof and make explicit students' thinking processes while they are creating a proof. I encouraged students to write a dialogue that they had with themselves, while they were trying to understand or create a proof. The

written dialogues could provide a partial picture of students' thinking process as well as a rich source of students' discourses. I propose to examine whether organizing and analysing students' discourses through a communicational framework provides a better understanding of the difficulties that students experience in creating a proof.

Overall, this study addresses the following research questions:

1. What difficulties do pre-service elementary school teachers experience in writing and interpreting proofs for propositions related to elementary number theory?
2. What are the outcomes of students' activity of creating a dialogue?
 - (a) Does it facilitate students' participation in the process of proving?
 - (b) Does it reveal their difficulties in this process?
3. Can a communicational approach to cognition serve as a tool for researchers in recognizing and identifying factors that impede pre-service elementary school teachers' participation in the process of creating and interpreting proofs?

Dissertation organization

I present my investigation and argument through the following chapter organization. First, in Chapter 2, I review the chronological pathway of emergence and evolution of mathematical proof. I begin with a brief review on the history of ancient Greek mathematics and a brief review on the history of mathematics and mathematics education in the last two centuries mainly in western societies. I then review the literature that examines the difficulties that students experience in learning proof, and the different approaches towards teaching proof. At the end of this chapter, I discuss the necessity of including proof in the curriculum for the preparation of elementary school teachers.

In chapter 3, I describe the communication approach to learning as a theoretical perspective that is based on the participation metaphor. In this theoretical perspective, learning mathematics is tantamount to becoming a participant in the mathematical discourse. I make a case for mathematical proof as a discourse for communicating mathematical ideas.

Having laid the foundation with a framework, I then move into presenting my study. Chapter 4 is a review of the background of this study. I describe the three studies through which my main research evolved over the last three years: ‘one line proof’, ‘what counts as a proof’, and ‘proof as a discourse’. The focus of chapter 5 is the setting of the study. First, I describe how the idea of proof as dialogue emerged from a communicational approach to cognition. Then I introduce the tasks designed for the purpose of the study and describe the rationale for this choice.

In chapter 6, I describe the participants’ performance on each task, with a quantitative summary of the responses, followed by samples of students’ work and their analysis. The participants’ discourses are next analysed through the lens of the communicational framework, in chapter 7. In particular, I examine the four features of the literate mathematical discourses across the tasks: (1) the mathematical uses of words (2) the use of mediating tools (3) the routines, and (4) endorsed narratives (Sfard, 2002; Ben-Yehuda, et. al., 2005). Moreover, I propose a further refinement of these features as applicable to proofs in the elementary number theory discourse. Finally, in chapter 8 the outcomes of this study are discussed as well as its contributions to the field of mathematics education.

CHAPTER 2: THE NOTION OF PROOF IN MATHEMATICS AND IN MATHEMATICS EDUCATION

In this chapter I consider the history of mathematics education, mainly in the western society. I analyse why proof became a part of the mathematics curriculum, how mathematicians defined proof, and how their description of proof and its role in mathematics education evolved during the last decades. Then, based on the literature, I examine the difficulties that students have in learning proof.

The teaching of proof has been one of the controversial issues during the last decades. As such, I review the different approaches that mathematicians and mathematics educators had toward teaching proof in classrooms. Finally, I discuss the necessity of including proof in the curriculum for the preparation of elementary school teachers.

Introduction

Historically the discovery of the process of proof is attributed to the Greek mathematicians and philosophers. The primary purpose of proof was providing validation and certification for claims (Davis & Hersh, 1980). As Schoenfeld (1994) mentions, “One of the glorious things about proof is that it yields certainty: when you have a proof of something you know it has to be true, and why” (p. 74). In fact, “proof, in its best instance, increases understanding by revealing the heart of the matter” (Davis & Hersh, 1980, p.151).

Traditionally, in schools proof was introduced and taught in Euclidian geometry. Euclidian geometry is the first example of a formalized deductive system and has become the model for all such systems. Through “new math” movement, however, in the mid 1950s, proof received a broader place in school mathematics. In this era the notion of proof not only taught in geometry but also in other high school courses such as algebra. In addition, to introduce the axiomatic method and proof, set theory was proposed as another subject matter in school curriculum (Davis & Hersh, 1980).

However, studies of high school students (and also university students) have repeatedly shown that only a limited number of them acquire a respectable degree of proof understanding and proof-writing ability during their high school mathematics (Harel & Sowder, 1998). “Proof is one of the most misunderstood notions of the mathematics curriculum” (Schoenfeld, 1994, p. 75) and therefore one of the greatest challenges for researchers and mathematics educators. Considering the important role of proof in teaching and learning mathematics, much research has been carried out in the last few decades to diagnose students’ difficulties and provide opportunities for them to get better understanding of proof. Prior to going through the details of the literature of proof it would be helpful to review the historical background of proof in mathematics education.

Historical background

A glance at the history of mathematics education in western society shows that mathematics taught as a part of formal education has reformed dramatically during the last two centuries. The reforms were due to the influence of social and philosophical conditions of the eras and also the influence of the mathematicians’ point of view. In this

section I review the history to see how the notion of proof emerged in the mathematics curriculum, and how it developed and changed. To situate the notion of proof in mathematics education I integrate a synopsis of the development of this notion in mathematics, starting with the ancient Greeks.

The ancient Greeks

Thales (600 B.C.) was the first mathematician who saw the necessity of proving a general geometric proposition (Davis & Hersh, 1980). “The angles of the base of an isosceles triangle are equal”, is one of the propositions, the proof of which is ascribed to Thales. Nobody, however, knows how he proved it (DeLong, 1970). The same is true for the most famous theorem in the history of mathematics: the Pythagorean Theorem. Nevertheless, since then the idea of the proof, as deduction of facts from (apparently) simpler facts, established itself as the characteristic aspect of mathematics.

Citizenship in ancient Greece required one to speak in public forums (Reed & Johnson, 2000). The sophists were the first professional Greek teachers who would travel and for a fee would teach their students how to speak persuasively to win arguments. In order to refute the arguments of sophists whose conclusions were either false or paradoxical Aristotle was the one who tried to devise a set of principles by which one could determine whether any given argument was a good one. Aristotle, with some help of his predecessors, formulated a rather extensive theory of logic.

DeLong (1970) summarizes Aristotle’s motivation in inventing logic as follows:

First, there is a desire to know the truth about the nature of argument, an intellectual curiosity, which needs no further account or justification. Second, there is the desire to know the condition under which something is proved.... Third, there is the desire to refute opponents (p. 13).

Certainly one of the greatest achievements of the early Greek mathematicians was the creation of the axiomatic form of thinking through Aristotle's and Euclid's works. Euclid's most famous work is his treatise on mathematics the *Elements*. For more than two millennia this work has dominated teaching of geometry. The text began with definitions, postulates, and common notions, then proceeded to obtain results by proof. Contrary to the common ideas that Euclid's *Elements* is on geometry alone, these books contain a large body of knowledge about number theory and elementary algebra as well. Probably few results in the *Elements* were first proved by Euclid, but "the skilful selection of propositions and their arrangement into a logical sequence are certainly due to him" (Eves, 1966, p. 115). In fact, Euclid's *Elements* has become the prototype of modern mathematical form.

For centuries it was thought the Euclid's *Elements* covered the whole study of elementary synthetic geometry of the triangle and the circle. However, the findings of mathematicians in the nineteenth century initiated a new approach to the study of geometry (Eves, 1966).

19th century

The beginning of the nineteenth century marked the new emphasis on mathematics as a foundation for rational thinking. That was a reaction toward the eighteenth-century mathematics texts that were mainly based on memory work at the expense of logic. In 1818 Samuel Goodrich, author of *The Child's Arithmetic*, believed that for children to understand arithmetic, rote learning and doing arithmetic just by following the rules do not suffice. He also suggested that children should find the rules through manipulating concrete objects (Cohen, 1982).

Warren Colburn, who was a Harvard-trained mathematics major, took Goodrich's idea and developed it to a new system, "mental arithmetic". In 1821, he published *First Lessons, Or Intellectual Arithmetic On The Plan Of Pestalozzi*. This book that contained no rules and no memory work, was the most popular arithmetic text ever published (Hanna, 1983).

In the early years of the nineteenth century the accepted method of teaching was "to state a rule, give examples, and provide problems" (Jones & Coxford, 1970, p. 21, cited in Hanna, 1983). The focus was thus on the application of rules. But, Colburn did suggest an alternate procedure in which general principles were built up from examples. At the heart of his method was inductive reasoning. This method made arithmetic closer to rational thinking where students learn to make conclusion based on the given facts (Cohen, 1982).

The enhanced position of mathematics in the school curriculum, during the first half of the nineteenth century, was parallel with the important developments in mathematics itself. There were many great mathematicians like Gauss, Galois, Riemann, Lobachevsky, and Cayley who published during this period. In addition, several events in the nineteenth century such as the industrial revolution, the development of the physical science and the expansion of the universities lead to the growth of mathematical science and increased attention to structure and methods of mathematics (Hanna, 1983).

Late in the nineteenth century stronger arguments for a closer relationship between professional mathematics and school curricula began to be formulated and heard in educational communities (Hanna, 1983). In Italy under impression of Peano's idea an axiomatic approach to mathematics became popular in schools. In 1893, Felix Klein in

Germany started a reform in content and pedagogy of school mathematics. He believed intensifying rigor in mathematics textbooks and programs will help students to have an understanding similar to that of practicing mathematicians. He also proposed the use of function to bring algebra and geometry together. However, in Britain and North America, under impression of John Perry's work, the main concern was the application of education. As a result of this view, education must introduce practical skills. Therefore, the selection of mathematical contents of high school was based on their utility (Hanna, 1983).

Until the nineteenth century the axiomatic method, far from being a general device in mathematics, was limited to Euclidean geometry. The development of non-Euclidean geometries in the nineteenth century laid the foundation for the development of new axiomatic structures in other areas of mathematics and coincided with growing attention to rigorous definitions and proofs.

Early 20th century

During the early years of the twentieth century the secondary school mathematics curriculum changed considerably in response to the changing demands of society.

Until 1892, the purpose of mathematics in the secondary school curriculum was the education of mathematicians; by 1920, its purpose was the education of well-informed citizens. With the discrediting of "faculty psychology" and its replacement by stimulus response psychology and Thorndike's theory of bonds, it was no longer necessary for educators to see mathematics primarily as a discipline for the mind, or to believe that it had to be hard to be valuable; they were free to treat it as a means of imparting useful knowledge (Hanna, 1983, p. 9).

In 1923 the National Committee on Mathematical Requirement of the International Commission on the Teaching of mathematics, issued a report entitled

Reorganization of Mathematics in Secondary Education. This committee recognized the pedagogical theories, based on which the secondary curriculum has been designed, as the reason for the diverse problems of the curriculum. At the time, the curriculum had over emphasis on practical mathematics inspired by Perry movement. However, due to the work of Peano, Klein, Hilbert, and Poincare the necessity of greater rigor in mathematics textbooks was come out. As the result, the committee suggested less stress on arithmetic processes and formal presentation of geometry in junior high school and more emphasis on the business application of mathematics in that level (Hanna, 1983).

During 1930-1950, despite the World War II many of the suggested reforms were successfully implemented in schools. The domination of arithmetic, besides manipulation and memorization decreased in junior high school curriculum. Instead usefulness and application of mathematics received more attention. However, none of these increased students' motivation for the study of mathematics.

In the first half of the twentieth century mathematics itself vastly developed. During this period new fields such as modern mathematical statistics, the theory of games, queuing theory, and graph theory emerged (Hanna, 1991).

Three schools of thought

The growth of mathematics was accompanied by investigation of the foundation of mathematics from which three main schools of thought emerged: the logicist school in England, with Russell and Whitehead; the formalist school in Germany, with Hilbert; and intuitionist school in Holland, with Brouwer and Heyting. Although, they differ widely in their approach to the definition of numbers, in their view of mathematical infinity, and

the role they assign to logic, they did share an emphasis on the importance of formal proof (Hanna, 1991).

The Bourbaki approach

By the growth of mathematics in the first half of the twentieth century, many mathematicians turn their attention to the axiomatic method. At that time, axiomatic method was considered as a means for unifying the disciplines that were unrelated formerly into some mathematical structures. This approach mainly emerged through the work of influential French mathematicians, such as Dieudonne. They wrote and promulgated their view under the name of Bourbaki. The Bourbaki group had a great influence on mathematical research at international level. Besides focusing on new topics in mathematics, the group introduced the “Bourbaki approach: formal, abstract, and rigorous approach, emphasizing precise definitions and formal proof” (Hanna, 1991, p. 54).

The “New Math” era

In the early 1950s, under the influence of the views of certain mathematicians and social and philosophical conditions, a new movement began. That movement, known as “New Math”, was characterized primarily by the promotion of a more abstract approach to mathematics in the schools. The main reason for initiating the movement was the existing gap between high school mathematics and university mathematics (Hanna, 1983).

The first new math project began in 1951 by three faculty members of mathematics and education at the university of Illinois. The project denoted as UICSM

[the University of Illinois Committee on School Mathematics] (Usiskin, 1999). Six years later, in 1957, the new math received its biggest push when the Soviet Union launched Sputnik, the first artificial satellite. This achievement of the Soviet Union intensified the arguments for reform. As a result, the US federal government proposed to change the school curriculum. For this purpose some curriculum development projects designed to advance the school mathematics and science programs. Among different projects that performed for this purpose SMSG [the School Mathematics Study Group] was one of the earliest and may be the most famous of these projects (Stanic & Kilpatrick, 1992).

The central focus of the studies was in making school mathematics similar in structure, content, and manner of presentation to what was perceived to be the theory and practice of mathematics as an academic discipline. In the elementary school curriculum, this idea appeared as stress on conceptualization. In high school mathematics, along with the introduction of some of the more abstract areas of modern mathematics into the curriculum, there was a new emphasis on the notion of mathematics as a unified axiomatic structure. Also, there was a new emphasis on logic and proof (Hanna, 1983; Usiskin, 1999). By the beginning of the new math era the notion of proof not only taught in geometry but also in other high school courses such as algebra. In addition, to introduce the axiomatic method and proof, set theory was proposed as another subject matter in school curriculum (Davis & Hersh, 1980).

After being introduced, the new math movement was very well accepted by mathematics educators and general public. However, in 1960s the gradual sign of disagreement began to grow. One of the main reasons for the objections was that the new abstract mathematics curriculum could not serve all the students, particularly slower ones

(Usiskin, 1999). Mathematicians such as Morris Kline also criticized the new math reform. Kline along with other mathematicians (Ahlfors¹ *et al.*, 1962) believed that the new math was too abstract, impractical, and confusing.

The “Back to Basics” approach

As a response to the new math approach, in the mid 1970s, a series of new textbooks for grades K-12 published. The new textbooks followed a new approach, called “Back to Basics”. This approach encouraged proficiency on skills without caring about properties or applications. In fact, the books did not contain any or much explanation. As a result of this approach teachers were encouraged to teach skills, such as algebraic skills without understanding, and paying much less attention to proof in geometry (Usiskin, 1999).

The 1970s was challenging for mathematics education community. Throughout these years there was a struggle between those who were supporter of mathematics reforms and those who preferred to put stress on the basics (O’Shea, 1998). To resolve the existing confusion, the National Council of Teachers of Mathematics (NCTM) decided to take the leadership of the teaching mathematics, mathematics curriculum designers and policy makers. The result was the publication of the *Agenda for Action* (NCTM 1980), which included eight recommendations. These recommendations form the foundation for *Curriculum and Evaluation Standards for School Mathematics*, produced by the NCTM in 1989.

¹ The comment signed by sixty-five mathematicians from various geographical locations in the United States and Canada.

The NCTM standards

By the time the *Curriculum and Evaluation Standards for School Mathematics* (1989) was published, the concept of proof had almost disappeared from the curriculum (Greeno, 1994, cited in Hanna, 2000) or shrunk to a meaningless ritual (Wu, 1996, cited in Hanna 2000). The NCTM did not see any necessity for changing the situation of proof in the mathematics curriculum. It even suggested less emphasis on proof in geometry, especially in the form of two column proof.

On the other hand, the *Standards* (1989) did propose greater emphasis on the testing of conjectures, the formulation of counter examples and the construction and examination of valid argument, as well as on the ability to use these techniques in the context of non-routine problem solving (Hanna & Jahnke, 1996). In the *Standards* (1989) there were even two topics, among the seven recommended for greater attention, which had a distinct essence of proof: (1) short sequence of theorems, and (2) deductive arguments expressed orally and in sentence form (pp. 126-127).

The standards approach (1989) was to increase students' motivation and involvement in heuristic argument. Therefore implicitly proof lost its role as a teaching tool. This document highlighted the importance of heuristic argument for the sake of discovery and understanding mathematics, but did not link this approach to mathematical proof (Hanna, 2000).

The new version of the NCTM *Principles and Standards* (2000) has remedied this situation by recommending that reasoning and proof be a part of mathematics curriculum at all levels. One of the process standards of this document called "Reasoning and Proof" states that students should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

But even after inclusion of proofs in the *Principles and Standards* (2000), what it mostly emphasized is not proofs, but reasoning (except in geometry).

Next, with reference to the literature I discuss what a proof really is. And, what roles proof could have in mathematics and mathematics education. Then I examine what difficulties students might have in learning the concept of proof, and what the different approaches toward teaching proof are.

What is proof?

For many years the only answer to the question “what is proof?” was: “a formal proof of a given sentence is a finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentence by applying rules of inference and the last sentence is the one to be proved” (Hanna, 1990, p. 6). And, establishing the mathematical certainty of a theorem was the main purpose of its proving (Weber, 2003).

In the last three decades both mathematicians and mathematics educators have begun to reassess the role of axiomatic structures and formal proof. They agreed that proofs might have different degree of formal validity and still gain the same degree of

acceptance. In this regards, Hanna (1990) makes distinction among different perceptions of proof in mathematics education. She considers three aspects:

- Formal proof: proof as a theoretical concept in formal logic (or metalogic), which may be thought of as the ideal which actual mathematical practice only approximates.
- Acceptable proof: proof as a normative concept that defines what is acceptable to qualified mathematicians.
- The teaching of proof: proof as an activity in mathematics education, which serves to elucidate ideas worth conveying to the student (p. 6).

In the latter case she introduces an explicit distinction between proofs that prove and proofs that explain, as two legitimate proofs. She mentions a very important difference between these two kinds of proof. She believes a proof that proves shows only that a theorem is true. While, a proof that explains, also shows why a theorem is true and may cause a better understanding of mathematics.

Hersh (1993), after Hanna, distinguishes between proofs that convince and proofs that explain:

Mathematical proof can convince, and it can explain. In mathematical research, its primary role is convincing. At the high school or undergraduate level, its primary role is explaining (p. 398).

Weber (2002) makes this distinction finer. He describes four types of proof. According to Weber, two types of proof – proofs that convince and proofs that explain, provide knowledge about mathematical truth. Then he describes, two other types of proof – proofs that justify the use of definition or axiomatic structure and proofs that illustrate

technique. As an example for the proof that justifies the structure, Weber mentions the Peano's proof for "two plus two equals four". He argues that the purpose of this proof is not persuasion of mathematicians or explanation for its truth, but to show that the Peano's system of arithmetic is a reasonable one. As an example for proofs that illustrate technique, Weber considers " $f(x)=x^2$ is a continuous function". This proof can be used to show students how to prove this kind of mathematical statement. Students' awareness of different uses of proof may promote their appreciation of the essential roles that proof plays in mathematical science (Weber, 2003).

Knuth (2002, p. 63) summarizes the various roles that mathematics educators suggested for proof in mathematics as follows:

- to verify that a statement is true,
- to explain why a statement is true,
- to communicate mathematical knowledge,
- to discover or create new mathematics,
- to systematize a statement into an axiomatic system.

Further, NCTM's (2000) document gives a central role to proof for all students: "reasoning and proof should be a consistent part of students' mathematical experiences in pre-kindergarten through grade 12" (p. 56). Nevertheless, Knuth's (2002) research on 17 experienced secondary school mathematics teachers shows, "teachers still tend to view proof as an appropriate goal for the mathematics education of only a minority of students (p. 83)". The reality of mathematics classrooms, also, shows proof is a difficult mathematical concept for students (Wheeler, 1990). The question that arises from this is:

what difficulty might students have with the concept of proof, and how then could this concept be taught more effectively?

What are students' difficulties in understanding the concept of proof?

The notion of proof has an especial interpretation in mathematics, which is not the same as its common application in everyday language. This variation may lead mathematics students to have a different interpretation of proof (Tall, 1989). They usually consider non-deductive arguments as a proof (Weber, 2003). This claim agrees with Schoenfeld's (1985) observations of the empirical nature of students' beliefs about mathematics and their failure to use deductive reasoning as a mathematical tool. Harel and Sowder (1998) have given an inclusive classification of such beliefs. Some of these beliefs, denoted by Harel and Sowder as "proof schemes", are as follows:

- Authoritative: an argument is a proof if it is appeared in a textbook, presented by or approved by an established authority, such as a teacher or famous mathematician.
- Ritual: an argument is a proof if its appearance is in accordance with common formalism of mathematical convention.
- Inductive: a general statement is true if it holds for a number of examples
- Perceptual: by way of a basic mental image, such as an appropriate diagram, one can visually show that a certain property holds. For instance, by looking at an isosceles triangle ABC , one might perceptually observe two equalities: $AB = AC$ and $\angle C = \angle B$, without seeing the causality relationship between these two equalities.

In recent decades research has focused on why students may have these beliefs about proof. The results of Dreyfus' (1999) research on students' conceptions of proof show they mostly do not have a clear idea about proof and the purpose of proving. According to Harel (1998), a most important reason that students are not interested to involve in proof process is that they do not see the necessity to prove some results that seem obvious to them. This attitude leads students to see proof as a bunch of formal rules and unimportant point of learning.

Historically, proof has nominally been a major ingredient of high school mathematics through the medium of Euclidian geometry. Dina and Pierre van Hiele, two Dutch researchers, proposed five levels through which students may progress as they learn Euclidian geometry: visualization, analysis, informal deduction, deduction, and rigor. They proposed that students need to pass through all the levels in sequence, and their instruction should be in accordance with their learning level. In fact, according to Van Hiele's idea students in a lower level can never have required understanding for the higher level. Senk (1989) in her study, conducted in United States, mentioned the reason that high school students have difficulty in geometry is that they enter high school with a lower required level of understanding of Van Hiele model.

In the process of solving a problem, it happens very often that students show they have a good understanding but they are not capable to present it mathematically (Dreyfus, 1999). In other words, they are not capable to use mathematical notations and language to present their idea. Many college students begin higher level courses which require writing proof without receiving any instruction for it. In fact, for most of them high

school geometry is the only experience of writing proof. So they usually are not familiar with different possible methods of proving and writing it (Moore 1994).

Even in the cases that students decently present familiar proofs, there is no guarantee that they can present any thing beyond that in a new situation (Weber, 2003). Moore (1994) in a study observed five undergraduate students as they progressed through an introductory proof course. He found that many of the students' difficulties were cognitive. In this regard he established some major sources of the student difficulty in doing proofs as follows:

- The students did not know the definitions, that is, they were unable to state the definitions.
- The students had little intuitive understanding of the concepts.
- The students' concept images were inadequate for doing the proofs.
- The students were unable, or unwilling, to generate and use their own examples.
- The students did not know how to use definitions to obtain the overall structure of proofs.
- The students were unable to understand and use mathematical language and notation.
- The students did not know how to begin proofs (p. 251-252).

Altogether, the literature (cited in Moore, 1994, p. 250) suggests the following areas of potential difficulty that students encounter in learning to do proofs: perception of

the nature of proof (Balacheff, 1988; Bell, 1976; Galbraith, 1981; Lewis, 1987; Schoenfeld, 1985), logic and methods of proof (Bittinger, 1969; Duval, 1991; Morgan, 1972; Solow, 1990), problem solving skills (Goldberg, 1975; Schoenfeld, 1985), mathematical language (Laborde, 1990, Leron, 1985; Rin, 1983), and concept understanding (Dubinsky and Lewin, 1986; Hart, 1987; Tall, and Vinner, 1981; Vinner and Dreyfus, 1989).

These studies show that the combination of beliefs, knowledge, and cognitive skills is required for the study of abstract mathematics and creating proofs (Moore, 1994).

More research is needed to recognize what combination of these factors could be more useful for students to have a better understanding of the notion of proof.

Teaching proof

Improving the understanding of proof among all students requires effective mathematics teaching. Teaching mathematics well is a complex endeavour, and there is not any special method that works for all students (NCTM, 2000). Considering the different contributions that proof might have in mathematics during the last decades, mathematicians and mathematics educators have introduced different approaches toward teaching proof. In what follows I provide an overview of these approaches.

Explanation

According to Hanna (2000) the primary responsibility of proof in the classrooms is answering the “why” questions. In other words, in the educational domain the main role of proof is to explain why a claim is true. Hence it would be a great advantage for teachers who used explanatory proofs. However, it is not possible to choose explanatory

proofs for all the theorems, because “some theorems need to be proved using contradiction, mathematical induction, or other non explanatory methods” (p. 9).

Conviction

One of the common mathematical meanings of proof is: an argument that convinces qualified judges (Hersh, 1993). To help the students focus on the various stages of putting up a convincing argument, Mason, Burton, and Stacey (1982) suggest three stages: convince yourself, convince your friend, and convince your enemy.

Convincing oneself involves having an idea of why some statement might be true, but convincing a friend requires that the arguments be organized in a more coherent way.

Convincing an enemy means that the argument must now be analysed and refined so that it will stand the test of criticism.

Rigor

Tall (1989) believes that in Mason et al’s approach the formal notion of mathematical thinking is absent. He acknowledges that it is not because the authors do not believe in formal mathematical proof, but because “the nature of formal mathematical proof is very difficult for students to comprehend” (p. 30). Tall considers clearly formulated definitions and statements, and rules of deduction as two important ideas that mathematical proof must be based on them.

However, Hanna (1983) in *Rigorous Proof in Mathematics Education*, without rejecting well formulated and well presented proof in mathematics, criticizes the view of formal proof in education. This view, which emphasized rigorous proof has been adopted by the new math movement of the 1950s and 1960s. By “rigorous proof” she means “a

finite sequence of formulae of some given system, where each formulae of the sequence is either an axiom of the system or a formulae derived by a rule of the system from some of its preceding formulae” (p. 66). In her study Hanna argues that a stress on rigorous proof does not necessarily bring a mathematics curriculum closer to mathematical practice. She came to the conclusion that in educational settings understanding has priority over rigorous proof.

Heuristics

Considering these critiques and including the important role of proof in mathematical practice, the 1990s saw substantial changes in both school mathematics curricula and teachers’ instructional practices based on the NCTM *standards* (1989). This book did not focused on proof, but; it suggested the use of conjecturing, reasoning, validating claims to discuss and question thinking of their own and others. This point of view encourages students to use heuristic and inductive approach to support their mathematical perception (Hanna & Jahnke, 1996).

Mathematicians have no doubt that intuition, speculation and heuristics are very useful for initiating a mathematical proof. However, they emphasize the distinction between proof and heuristic argument as well. Indeed, there is a consensus among mathematicians that the only way to validate a mathematical result is to use proof (Hanna, 2000).

Hanna (1991) discusses naïve mathematical ideas emerge from routine daily experiences. These experiences must be refined and developed to become explicit through including an amount of formalism. In other words, in teaching mathematics in

general and promoting reasoning skills in particular a degree of formalism is required. For this purpose, formalism would be considered as a crucial tool for better clarifying, understanding and validating a mathematical result. Applying an appropriate amount of rigor to justify any theorem could immensely improve learning.

Generic example

In mathematics the purposes of proof may be one or more. These purposes include assurance of truth, explanation of observed regularities, and clarification of claims (Hersh, 1993). The generic example is a mode of explanation. In this mode students generate examples from their empirical mathematical experiences. Also, teachers may use these examples as didactic tools to clarify some proofs that students would otherwise find complicated (Rowland, 2000).

According to Mason and Pimm (1984, p. 287), “a generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general”. In fact, a generic example inductively deduces that the result is true in general.

Rowland (2002) argues that in mathematical community, generic proof is not accepted as a proof. In its best case, generic proof can be considered as an intermediate stage between naïve empiricism and perception, and a general argument. Nevertheless, without rejecting this view he states that mathematics teachers and mathematical texts could assist all mathematics learners to recognize and value the generics in their insights, explanations, and arguments.

Visualization

Traditionally, diagrams and other visual representations have been an essential component of mathematics curriculum to facilitate insight and understanding of mathematical knowledge. Considering the existence of misleading diagrams the question arises as to what extent we should use them as evidence or even justification of mathematical statements.

In the last decade a number of mathematicians, logicians and researchers (Borwein & Jorgenson, 1997; Brown, 1999; Francis, 1996; Nelsen, 1993; Palais, 1999; cited in Hanna, 2000) have been investigating the use of visual representations, and their potential contribution to mathematical proofs. Among them, Borwein and Jorgenson (1997) would consider a great role for visualization in reasoning. They believe an image can act as a form of visual proof provided it meets certain qualifications, such as reliability, consistency, and repeatability.

As an example of visual proofs we can consider proofs without words, which began to appear in modern mathematical texts about 1975 (Nelsen, 2000). “Proofs without words are pictures or diagrams that help the reader see *why* a particular mathematical statement may be true, and also to see *how* one might begin to go about proving it true” (Nelsen, 2000, p. ix). For many researchers, however, visual representations are really no more than heuristic devices. According to them visual representations are psychologically suggestive and pedagogically important, but they prove nothing.

Exploration

One of the educational innovations in the last decades is introducing dynamic geometry software. Geometer Sketchpad and Cabri Geometry are two dynamic geometry softwares that are designed for teaching geometry. These softwares by providing the opportunity for constructing geometric shapes with high degree of accuracy facilitate students understanding. In fact, the flexible environments of these softwares help students to make conjectures and explore propositions based on the accurate construction. However, these also lead students to make conclusions as a general result based on their exploration (Hanna, 2000).

Despite mathematics educators' agreement on the usefulness of teaching students how to explore, formulate and test conjectures, they just consider it as a step towards constructing a proof. Exploration alone does not satisfy the required generality of a proof. In this regard Hanna (2000, p. 14) says: "what we really need to do, of course, is not to replace proof by exploration, but to make use of both."

The NCTM's approach to teaching proof

By reviewing the NCTM's (2000) document it seems that it tries to make use of all the former attempts of the educational communities to give suggestions for teaching and learning proof more effectively.

- "At all levels, students should reason inductively from patterns and specific cases. Increasingly over the grades, they should also learn to make effective deductive arguments based on the mathematical truths they are establishing in class (p. 59)."
- "By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from

hypotheses—and should appreciate the value of such arguments (p.56).”

- “Students at all grade levels should learn to investigate their conjectures using concrete materials, calculators and other tools, and increasingly through the grades, mathematical representations and symbols (p. 57).”
- “Along with making and investigating conjectures, students should learn to answer the question, Why does this work? (p. 58)”
- “High school students using dynamic geometry software could be asked to make observations about the figure ...and attempt to prove them (p. 57).”
- “High school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians (p. 58).”

In my opinion the main challenge for meeting these demands in the K-12 curriculum, is having teachers who have good understanding of the notion of proof, and teachers who know what kind of proof for what purpose and for what level is appropriate.

The fact that mathematical proof, in its common meaning, is not a separate topic in the mathematics curriculum of elementary school, brings up the question: Is it really important for elementary school teachers to have deep understanding of and have ability to generate mathematical proof? If yes, what would be the characteristic of the proofs that they need to know and how could this understanding serve them to be more efficient teachers?

The role of mathematical proof in elementary school teacher education

Traditionally, mathematics education for pre-service elementary school teachers aims at providing a certain level of understanding of mathematics and mathematical

methods. Most of the students in this group will often not continue their studies of mathematics at more advanced levels, but almost all of them will have to apply their knowledge of mathematics in their future profession as a teacher.

According to Polya (1957), “the first rule of teaching is to know what you are supposed to teach. The second rule of teaching is to know a little more than what you are supposed to teach” (p. 173). The authors of the standards believed that “knowing” mathematics is “doing” mathematics and “what” students learn, highly depends on “how” they learn it (Smith, Smith, & Romberg, 1993). If we consider the classroom as a place in which students should develop their mathematical beliefs and values and consequently their intellectual autonomy in mathematics (Yackel & Cobb, 1996), then the critical and central role of mathematics teachers as conductors and facilitators of establishing the mathematical thinking becomes clearer. In this regard, mathematical proof, in its common essence can be held to be a topic that has much to offer in the promotion of mathematical thinking. For this reason alone, the discussion of having mathematical proofs as a part of pre-service elementary teachers curriculum should be intensified.

Research, however, has repeatedly shown that proofs and the ability to understand and generate proofs is difficult for students in general (Hoyles, 1997) and for pre-service elementary school teachers in particular (Gholamazad, Liljedahl, & Zazkis, 2003, 2004; Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Ma 1999; Martin & Harel, 1989; Simon & Blume, 1996). The evidence from these studies suggests that pre-service elementary teachers tend to accept inductive evidence, such as a series of empirical examples or a pattern as being sufficient to establish the validity of a claim. Considering the high tendency of pre-service elementary teachers toward inductive reasoning, Martin and

Harel (1989) make an argument on the importance of the teachers understanding of what constitutes mathematical proof. They argue that, since the primary source of children's experience with verification and proof is their teacher, therefore if elementary teachers lead their students to accept a few examples as a proof, it is natural that the students have difficulty with the idea of proof at the secondary level. In other words, the inductive proof frame, which is constructed at an earlier stage, is not deleted from their memory, and the requirement of the deductive proof frame presents an obstacle.

Developing the role of proof in the classroom requires a great amount of teachers' understanding of the nature and role of proof. The mathematics teacher education and professional development programs can play a key role for meeting this demand. In a regular university mathematics course, mathematical proof is presented according to modern standards of rigor. But, pre-service elementary school teachers usually have neither strong background in mathematics nor enough interest to struggle through long proofs or to appreciate subtleties. Considering this fact, we need to know what kind of proof might better serve pre-service elementary school teachers.

The idea of improving pre-service elementary school teachers understanding and improving their active involvement in the process of creating proofs is the initial driving force for this study. The coming chapters explore and discuss one possible approach to this issue.

Conclusion

A review of the evolution and state of the status of proof in school mathematics still raises the fundamental question: Do we need proof in school mathematics?

Schoenfeld's response to this question is "Absolutely". This would be the response of most people who have been involved in professional mathematical practices, because for this group mathematics without proof does not make sense. But we should make it clear what we mean by proof, because as Hanna (1983, p. 29) mentions, "there is no consensus today among mathematicians as to what constitutes an acceptable proof and there never has been".

Considering the literature, I place the different understandings of proof on a vast spectrum: on one end, there is rigorous proof, on the other end, visual proof, and between them different kinds of arguments, justifications, verifications, and explanations. The important point is that all the views of proof are acceptable, but how we use them depends on the purposes of their usage. As Manin (1977) says, "a proof becomes a proof after the social act of accepting it as a proof" (p. 48). So it seems to me that mathematicians and mathematics educators based on different scientific, philosophic, and social requirements have introduced a view of proof, which could satisfy their needs in the specific paradigm. For example we can consider the proof of the four-colour theorem, which is a computer-assisted proof. Even for the educational purposes we observed radical changes in expectations about an acceptable proof during the last half of the 20th century.

In the 1950s, as a response to the inadequate preparation of high school students for the mathematics courses offered by universities, the emphasis was placed on rigorous proof. That approach could not last for long because mathematics educators very soon reached the result that "premature formalization may lead to sterility" (Ahlfors et al, 1962, p. 192). As result of social objection against rigorous proof in mathematics

curriculum, the role of proof has been changed to explanation. Proof in its new form – explanation – produces reasons more than examine the strength of these reasons, something that was needed in formal proof. Many of the mathematicians who support this approach share “the view that a proof is most valuable when it leads to understanding, helping (learners) think more clearly and effectively about mathematics” (Rav, 1999; Manin, 1992, 1998; Thurston, 1994; cited in Hanna, 2000, p. 7).

The explanation role of proof opened up new approaches to the teaching of proof: heuristic, exploration, and visualization. Each of these approaches, in my view, has a relationship with empirical verification. While empirical verification is very useful in clarifying a problem, it is only a preliminary stage toward a proof. However, it is seen for many students who were presented by a deductive proof still further empirical verification is required to be convincing (Fischbein & Kedem, 1982). This suggests that the activation of both the inductive and the deductive proof frames may be required for students to reach a particular conclusion. Here the main question is: what could be an appropriate teaching approach, which helps students to move from inductive to deductive reasoning?

We can see an effort towards the remedying of this problem in the latest version of NCTM standards (2000) with the recommendation that reasoning and proof be a part of the mathematics curriculum at all levels. But, in reality, there is always a gap between intention and implementation. Professional teachers, in my opinion, can play the main role in reducing this gap.

I conclude this chapter with a quote from Schoenfeld (1994), because I strongly agree with his words.

I think that if students grew up in a mathematical culture where discourse, thinking things through, and convincing were important parts of their engagement with mathematics, then proofs would be seen as a natural part of their mathematics (why is this true? It's because ...) rather than as an artificial imposition (p. 76).

Therefore, I focus my study on pre-service elementary school teacher education, which is the starting point of the "mathematical culture" that Schoenfeld refers to.

CHAPTER 3: THEORETICAL PERSPECTIVES

In this chapter I describe how the study emerged from the theoretical perspective. I begin by giving a very brief overview of the background of two approaches to the notion of ‘learning’ in mathematics education: learning as ‘acquiring mathematical knowledge’, and learning as ‘becoming a participant in mathematical discourse’. I then describe how the idea of learning as becoming a participant in mathematical discourse emerged from a communicational approach to learning.

Background of the study

Mathematics and psychology are two disciplines that have had a seminal influence on research in mathematics education (Kilpatrick, 1992). Research in this field has developed as mathematicians and educators have turned their attention to how and what kind of mathematics might be taught and learned. The history of mathematics education shows that research in this field, influenced by psychological theories, has been subject to a number of major shifts during the last century – from behaviourism to constructivism. Behaviourism is mostly based on the repetition of stimulus and response among different creatures (eg. human beings or animals) and it does not focus on the functions of the mind. Conversely, constructivism relies on the shaping of ideas in the mind (Tall, 1991).

Introducing the idea of learning-with understanding in cognitive psychology brought the behaviourist era to an end. Cognitive psychology through equating “understanding with perfecting mental representations” and defining “learning-with-understanding as one that effectively relates new knowledge to the knowledge already possessed” (Sfard, 2001, p. 21) opened up a new trends of study on human cognition.

The ways in which most educational researchers have been looking at learning during the last half century may vary from gradual reception to an acquisition by development or construction. Currently these ways were unified by the metaphor of learning as acquisition of knowledge, which justifies the individuality of efforts. In this framework by acquisition of information individuals become enriched, and what they acquire or learn becomes their own possession (Sfard, 1998). Acquisition of knowledge happens actively or passively and through which individuals make their own concepts and procedures. However, sometimes the acquired knowledge shapes from misconceptions rather than formal accepted conceptions (Sfard, 2001).

Recently different publications paved the way for the emergence of another learning metaphor, which is the “participation metaphor”. An example of those publications is *Situated learning* (Lave & Wenger, 1991), which refers to learning as a legitimate peripheral participation. Basically, participationist approach to learning grows from the sociocultural tradition. In the participationist framework, unlike the acquisitionist approach, learning is becoming a participant in certain activities and its goal is community building. In this metaphor knowledge is considered as an aspect of discourse (Sfard 1998).

There is no doubt that the emergence of the idea of learning-with-understanding had a beneficial impact on the study of mathematics education in the last decades. Nevertheless, some researchers believe the notions grounded in the acquisition metaphor are not sufficient for some of their more advanced needs. Anna Sfard (2001) is one of those who believe that acquisition-based theories serve only a restricted part of the learning processes.

Overall, it is important to emphasise that substituting the words “acquiring mathematical knowledge” with “becoming a participant in mathematical discourse” implies a different way of looking and researching (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005). Without rejecting the long-standing acquisition metaphor, Sfard (2001), through a communication approach to cognition, supplements it with socio-culturally grounded metaphor of participation. The following section is an overview of the communicational approach to cognition and the essential role of the participation metaphor in this approach.

The communicational approach to cognition

Currently the attention of researchers in the areas of human and mathematical thinking is attracted to communicational approach grounded on Vygotsky’s theory. In this approach priority is given to communicative public speech rather than inner private speech (Vygotsky, 1987). Human cognitive processes are based on the need for communication, which is “the primary driving force behind human cognitive processes ..., understanding thinking requires understanding the ways people communicate with one another” (Sfard, 2000a, p. 320). Therefore, better understanding of public discourse deepens our insight into a dialogue that one leads with oneself.

Let us pause to examine what communication is, and what researchers mean by this notion. Generally speaking, communication is “a process by which information is exchanged between individuals through a common system of symbols, signs, or behaviour” (*Merriam-Webster Dictionary*, 2006). In this definition information may be considered as an objective entity, which could be exchanged among individuals while its identity remains stable. In other words, the nature of these experiences does not change among individuals. Due to this perspective individual experiences would be as “comparable and measurable as material objects are, and they may therefore be used as explanatory devices in the study of human communication” (Sfard, 2000a, p. 299). Rather, this kind of process is more interpretative than explanatory.

According to Sfard (2000a) communication is an activity through which one tries to make their interlocutor act or feel in a certain way. It is effective if it achieves its goal of evoking reactions in tune with the interlocutor’s expectations. Hence, paying attention, thinking, and attempting to remember can be consider as different forms of communication.

In the study of human cognition, communicational approach considers thinking as a kind of communication between one and itself. Regarding ideas of Bakhtin (1986) and Vygotsky (1962, 1978, 1987), Sfard (2000a, 2001) argues that thinking is similar to conversation between two people and the same as any other conversation it involves turn taking, asking questions, giving answers, and building a new interconnected audible or silent utterance, in words or in other symbols.

The principal assumption of considering thinking as a special case of the activity of communication is that what happens in a public conversation is indicative of what

might be taking place in the individual's head as well (Sfard, 2000a). Accepting this assumption gives an essential role to language and discourse in the genesis, acquisition, communication, formulation and justification of all knowledge in general, and mathematical knowledge in particular. Therefore, a close analysis of the public discourse may reveal much about learning.

Within a communicational framework the focus of a study would be on discourse. Indeed, in this framework learning mathematics is defined as an initiation to mathematical discourse, that is, initiation to a special form of communication known as mathematical. In the following section I examine the notion of mathematical discourse to see what must be learned if a person is to become a skilful participant in a given mathematical discourse.

Learning mathematics in terms of discourse

According to current scholarship, rather than considering learning as “acquisition of knowledge”, it is possible to view learning as “becoming a participant in a certain discourse” (Sfard, 2000b). The word discourse has a very vast meaning. Indeed, discourse includes all the communicative activities that may be practiced by a given community. As Gee (1997, cited in Sfard 2000b) says:

Discourses are sociohistorical coordinations of people, objects (props), ways of talking, acting, interacting, thinking, valuing, and (sometimes) writing and reading that allow for the display and recognition of socially significant identities, like being (certain sort of) African American, boardroom executive, feminist, lawyer, street-gang member, theoretical physicist, 18th-century midwife, 19th-century modernist, Soviet or Russian, schoolchild, teacher, and so on through innumerable possibilities. If you destroy a discourse (and they do die), you also destroy its cultural models, situated meanings, and its concomitant identities (pp. 255-256).

It seems that any community may be characterized by the distinctive discourses they create. The mathematical community is one of them.

Mathematics as a form of knowledge is considered as a kind of discourse in the communicational framework (Sfard & Cole, 2002). In mathematical discourse individual learning emerges from communication with others and adjusting one's discursive ways to those of theirs (Sfard, 2002). A mathematical discourse deals with mathematical objects such as quantities and shapes. Depending on who is communicating about mathematical objects there are two types of discourses: colloquial and literate mathematical discourses (Sfard & Cole, 2002).

The colloquial discourses are everyday natural discourses that shape as a result of repetitive actions. These discourses take place in different situations, in other words, they are situation specific with limited applicability. Indeed, the colloquial discourses are suitable for the situations in which they develop. On the other hand, literate mathematical discourses are purposeful goals of teaching and schooling (Sfard & Cole, 2002; Ernest, 2003; Ben-Yehuda et al, 2005), and include generality and more applicability.

Literate mathematical discourse as the objective of school learning are distinguished from other types of communication through four criteria: (1) their special vocabulary, (2) their special mediating tools, (3) their discursive routines, and (4) their particular endorsed narratives (Sfard, 2002; Ben-Yehuda et al, 2005).

The special mathematical **vocabulary** is what renders mathematical discourse its distinctive identity (Sfard, 2002; Ben-Yehuda et al, 2005). While becoming a participant in a mathematical discourse, the student may learn terms that she/he has never used before. Expressions such as odd or even number that are unique to mathematics are good

examples for these words. Another word-related type of change that is often necessary in the course of learning is the use of words that are already known to the children from other discourses. Once these words become a part of mathematics they must be applied quite differently. For example, the word proof, familiar to children from spontaneously learned everyday discourse, will have to be applied in a somewhat different manner once the child begins learning mathematics at school.

“Every discourse is about something, and if the discourse is to go on, this something must be either actually visible or imagined” (Sfard & Cole, 2002, p. 4).

Mediators as vehicles for ‘somethings’ are mainly visible means with which people help themselves while communicating. In more concrete discourses, independent material objects could generate images that can visually support them. Unlike, mathematical discourse is mediated by signs and symbols designed for the purpose of mathematical communication. Within the communicational framework the designed symbolic artefacts are not used just as auxiliary means. It means that one considers them as an inseparable part of mathematical communication and cognitive processes (Sfard, 2001).

Discursive routines refer to patterns that can be noticed in discursive activities. This kind of repetitive patterns can be seen in any form of mathematical discourse. In fact, participants of mathematical discourse use these routine patterns to response a well-defined familiar type of request, question, task or problem, which happens in similar situations (Sfard & Cole, 2002). As examples of discursive routines that a person may perform in typical mathematical tasks we can refer to calculation, estimation, explanation, justification, and exemplification. These routines, which are used by discourse participants, may vary significantly in different mathematical discourse. In the

literate mathematical discourse these routines may vary considerably, but they are all “particularly strict and rigorous” (Ben-Yehuda et al, 2005, p.182).

Discursive patterns could be recognized only by professional observers. These rules are called meta-discursive rules. Being formulated, they take a form of propositions about the discourse (Ben-Yehuda et al, 2005). For instance, the rules such as “If you are to solve the equation $2x - 5 = 9$, the actual physical shape of the letter used is unimportant” or “If it is true that statement A entails B and statement B entails statement C, then statement A entails C” are meta-discursive since their objects are mathematical statements (Sfard, 2000b). Indeed, meta-discursive rules are mostly invisible rules that guide the general course of communicational activities. Discursive routine, as opposed to an actual discursive action, is a set of meta-discursive rules that specify the when and how of such action (Ben-Yehuda et al, 2005).

Endorsed narratives are the product of typical mathematical routines. These are narratives that are accepted by mathematical communities and are labelled as true. Endorsed narratives include discursive constructs (e.g. definitions, proofs, and theorems), and some of which are in the form of formulae and identities. (Ben-Yehuda et al, 2005).

Regarding communicational approach, learning mathematics means creating changes in students’ discourse. Such a change may be expressed within the above-mentioned conceptual framework. Indeed, features of mathematical discourse must be learned if a person is to become a skilful participant of a given mathematical discourse. Now, the question might arise as to whether all the mathematical notions could be analysed under the lens of this conceptual framework. In what follows I explain the

applicability of this framework for analysing proof as a form of a literate mathematical discourse, which facilitates the mathematical communication activities.

Proof as discourse

For years the only role of proof was showing the correctness of results and providing certainty. During the last decades, however, many researchers turn their attention to the social aspect of proof as well (e.g. Davis, 1986; Hanna, 1983; Hersh, 1993; Richards, 1996). They basically consider proof as a social construct and the product of mathematical discourse. From this perspective proof is also considered as a means to communicate mathematical knowledge. Nevertheless, this aspect of proof has not have a great role in the practicing proof in school mathematics yet (Knuth, 2002). Balacheff (1991) also noted the limited attention given to the social nature of proof: “What does not appear in the school context is that a mathematical proof is a tool for mathematicians for both establishing the validity of some statement, as well as a tool for communication with other mathematicians” (p. 178).

According to Schoenfeld (1994), “proof is not a thing separable from mathematics, ...; it is an essential component of doing, communicating, and recording mathematics” (p. 76). Indeed, proof is what distinguishes mathematics from natural science. A well-structured deductive proof offers humans the purest form of reasoning to establish certainty. Rorty (1979, cited in Ernest, 2003) relates the persuasion aspect of proof, as its very origin, to the conversational nature of that.

If, however, we think of “rational certainty” as a matter of victory in argument rather than of relation to an object known, we shall look toward our interlocutors rather than to our faculties for the explanation of the phenomenon. If we think of our certainty about the Pythagorean Theorem

as our confidence, based on experience with arguments on such matter, that nobody will find an objection to the premises from which we infer it, then we shall not seek to explain it by the relation of reason to triangularity. Our certainty will be a matter of conversation between persons, rather than an interaction with nonhuman reality (pp. 156-157).

Ernest (2003), also, considers dialectics and conversation as the origins of mathematical proof and logic. According to him,

Mathematical proof is a special form of text, which since the time of the ancient Greek, has been presented in monological form. This reflects the absolutist idea that total precision, rigour and perfection are attainable in mathematics. Thus the monologicality of the concealed voice uttering a proof itself belies and denies the presence of the silent listener. But as it is an argument intended to convince, a listener is presupposed. The monologicality of proof tries to forestall the listener by anticipating all of her possible objections. So the dialectical response is condensed into the ideal perfection of a monologic argument, in which no sign of speaker or listener remain (p. 5).

The ideas of these two philosophers provide a basis for accepting mathematical proof as a form of discourse, and pave the way for analysing proof under the lens of communicational framework. Seeing proof from this perspective could open a new window toward the essence and the nature of proof, which could reveal the factors that impede students' active participation in creating a proof, and communication through proof. These assumptions based so far on theory need to be validated through research.

To sum up, let us return to a definition of mathematical proof. "A mathematical proof of a given sentence is a finite sequence of sentences such that the first sentence is an axiom or has been derived from preceding sentence by applying rules of inference and the last sentence is the one to be proved" (Hanna, 1990, p. 6). Regarding the precise use of well-defined axioms, sentences, and rules of inference in the construction of a proof, it would be considered as a most literate form of a text in mathematics. On the other hand,

the persuasive nature of it as an argument that compels the mind to accept an assertion as true, would give it a role as an objective tool for communication in a mathematical community. Hence, examining the above mentioned features of literate mathematical discourse – mathematical vocabulary, mediators, routines, and endorsed narratives – in students arguments could provide researcher with a better understanding of the nature of their arguments and the possible factors comprising their notion of proof. These features are explored in the following chapters.

CHAPTER 4: HOW THE RESEARCH EMERGED

In this chapter I review the studies through which my main research evolved over the years. The preliminary investigations of ‘one line proof’ and ‘what counts as a proof’ by pre-service elementary school teachers were followed by the study of ‘proof as a discourse’. The extension and expansion of the latter study culminated in the current work. In what follows I describe the three studies: ‘one line proof’, ‘what counts as a proof’, and ‘proof as a discourse’, to describe how they paved the way for the current research.

One line proof

In this study we (Gholamazad, Liljedahl, & Zazkis, 2003) narrowed down the wide area of mathematical proof to very specific proofs in number theory that we denote “One Line Proof”, metaphorically referring to very short proofs. For those we provided a framework that allowed us to carry out a fine grain analysis of participants’ work, and gave insight into the complex coordination of competencies that is required for generating such short proofs.

Participants of this study were 116 prospective elementary teachers, enrolled in the course ‘Principles of Mathematics for Teachers’ (the detailed information about this course is provided in chapter 5). During the course the students were exposed to the concept of closure as part of the discussion of number systems. The formal definition was

provided: “a set is said to be closed under an operation if and only if for any two elements in the set the result of the operation is in the set”. Further, a variety of examples of sets closed or not closed with respect to certain operations were provided and students were engaged in a variety of problems in which they had to prove or disprove claims.

In this study we analysed the participants written response for the following two questions.

(Q1) The set of perfect squares is closed under multiplication. Prove the statement or provide a counterexample.

(Q2) The set of odd numbers is closed under multiplication. Prove the statement or provide a counterexample.

For our purpose, we considered the “ideal” solutions (that is, proofs) of these statements to be:

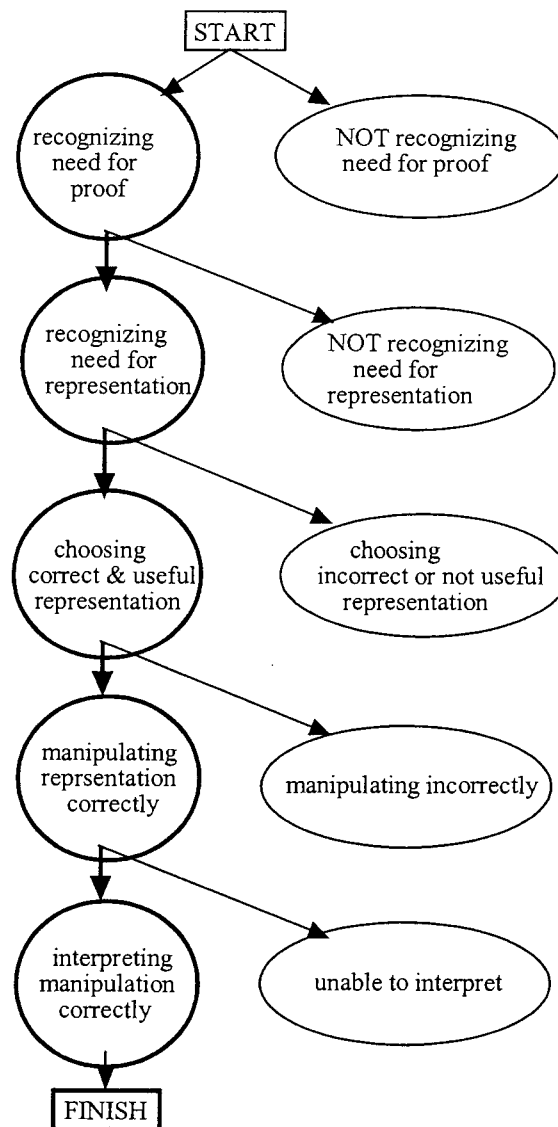
(Q1) Let a^2 and b^2 be any two square numbers. Then, $a^2 \times b^2 = (ab)^2$ which is itself a square number.

(Q2) Let $(2m + 1)$ and $(2n + 1)$ be two odd numbers. Then $(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$ which is itself odd.

However, the generation of such seemingly simple and short proofs is deceptively intricate, requiring an appreciation of the need for, and the coordination of many skills (see Figure 1). First and foremost is the recognition that a proof is indeed required for the purposes of establishing the truth of a statement. From a mathematical perspective, such a requirement is obvious. The establishment of the validity of a statement requires the treatment of the statement in general, as opposed to the examination of a few particular cases. Once a need for a proof has been established, the students then need to be sensitive to the fact that treatment of the general case requires the selection of some form of

representation. Representations play a crucial role in mathematics; they are considered as tools for communication, as tools for symbolic manipulation, and as tools that promote and support thinking (e.g. Skemp, 1986; Kaput, 1991). Furthermore, the choice of representation is often linked to students' understanding of the content (Lamon, 2001).

Figure 1: Pathway towards (and digression from) a one line proof



However, the recognition that a representation is needed is not enough. The students must select one that is both correct and useful for the purposes of generating a proof. For Q1 (above), for example, choosing to represent the two square numbers as X and Y is in itself not incorrect, but for the purposes of generating a proof, it is completely useless. A much more effective (and natural) representation of two square numbers is a^2 and b^2 . Once such a representation is established, the students must then be able to work with it. That is, they must be able to perform correctly any manipulations necessary to transform the expression into the form that clearly represents the nature of the number. In the example of Q1 such a manipulation is not onerous. Q2, however, requires much greater adeptness with algebraic manipulation in order to mould the expression into one that clearly expresses its inherent ‘oddness’. There is an assumption in this last sentence, though. The phrase *clearly expresses* assumes that the students are able to interpret the result of their manipulation as representative of what they are aiming to show. This is the last step in the proof process. The students must be able to constantly interpret their manipulations in order to know what they have found, and when they have found it. By analysing the students’ work, we realized that the lack of understanding of each of these steps could be a potential obstacle in a pathway towards presenting an authentic proof.

A complete and correct proof was provided by 19% of the participants for Q1 and by 37% of the participants for Q2. We organized students’ incorrect responses according to the framework provided above. The potential obstacles at every step include:

- (Not) recognizing the need for a proof
- (Not) recognizing the need for representation
- (Not) providing a useful representation

- (Not) manipulating representation correctly
- (Not) interpreting the manipulation

In general, the study demonstrated that the concept of closure was generally well grasped. That is to say, the majority of students understood that they were expected to show that the product of two perfect squares is a perfect square, and the product of two odd numbers results in an odd number.

However, it is troublesome that what prevented some students from completing the proof was not their understanding of closure, or appreciation of the need for a proof, but a poor ability to choose an appropriate representation or inability to manipulate the chosen representation. The latter draws the focus from undergraduate teacher education and invites regression to skills of simple algebraic manipulation. Lack of competence in these skills presents an obstacle not only for correct manipulation, but also for interpreting the meaning of manipulation, that is, the ability to represent the manipulated expression in a desired form.

Overall, the result of this study showed that students have different attitudes towards and understanding about proof. Hence, this study encouraged us to examine what really counts as a proof for pre-service elementary school teachers.

What counts as a proof?

Everyone who is involved in doing serious mathematics has asked himself or herself at least once in their lifetime, the basic question: what counts as *proof*? As simple as the question is, the answer to it is rather complicated. As already discussed in chapter 2, the main reason for this is that, although there is an expectation that every individual

mathematician should have an operational understanding of what a *proof* is, there seems to exist no succinct definition of *proof*. In fact, the varying interpretations of what constitutes a proof revolve largely around the notion of rigor. The question here is: how much rigor is required for various groups of students?

In this study we (Gholamazad, Liljedahl, & Zazkis, 2004) investigated students' ability to evaluate the correctness of a given 'proof', that is, the ability to judge whether a given argument, or sequence of arguments, proves a given statement. We saw this ability as an important precursor to the ability to generate correct proofs. Furthermore, we believe that this ability is an extremely important one for teachers to have or acquire because they are to be the facilitators of mathematical understanding of their students.

As such, we chose to focus our investigation on prospective elementary school teachers' ability to judge the validity of presented arguments as proofs. More specifically, we focused on the tendencies and trends in the accepting or rejecting of arguments as proofs and examined the role that numerical examples played in participants' reasoning.

In this study we used the framework presented-above, referred to as "one line proofs" (Gholamazad, Liljedahl, & Zazkis, 2003). As already mentioned, the framework describes five competencies necessary for the generation of a complete and correct proof. Itemized competencies of this framework not only detail what is needed in generating short proofs of number properties, but also provide a tool for the diagnosis of possible obstacles in generating such proofs. As such, we used this framework as a guiding tool in the design of the instrument for this study.

The seventy-five participants in this study were prospective elementary school teachers enrolled in the course "Principles of Mathematics for Teachers" (the detailed

information about this course is provided in chapter 5). The participants responded to a written questionnaire in which they were asked to consider the validity of arguments purporting to ‘prove’ five different statements related to set closure (See the statements along with their ‘proofs’ that were used in the study in appendix 1). They were asked to examine the arguments and decide, in each case, whether the argument was acceptable as a proof for the given statement or not. In the case that an argument was not acceptable, the participants were asked to provide an acceptable proof either by editing or by augmenting the presented argument as necessary. In particular, they were invited to delete parts of the presented arguments that they perceived as unnecessary.

As mentioned, the ‘proofs’ were constructed from plausible errors as indicated by the framework. We examined the participants’ awareness of situations in which a proof can rely on exhaustive consideration of all possible cases (‘proof’ #1) and those where one example is sufficient (‘proof’ #2). Furthermore, we examined the participants’ awareness of the need for representation when a multitude of examples does not constitute a proof (‘proof’ #3) and the existence of valid argument not involving algebraic symbolism (‘proof’ #4). The final item (‘proof’ #5) addressed the participants’ attentiveness to the correctness of symbolic manipulation.

The results of the study indicated that:

- The majority of participants accepted, as valid, proofs that consider all possible cases in a finite set.
- The majority of participants were not satisfied with the use of a single counterexample to disprove a claim. There was a tendency toward having more than one counterexample.

- The majority of participants accepted the confirmation of examples as a valid method of proof.
- The conventional form of presenting the proof seemed to play no role in the decision of a proof's validity.
- The majority of participants did not detect the error in algebraic manipulation of a given 'proof'.

In general, the participants' feedback demonstrated that, although the mathematical concepts of the statements were generally well grasped, the concept of proof was not. As such, they often believed that non-deductive arguments constitute a proof. For the majority of participants it seemed so clear that the sum of two multiples of five would be a multiple of five, or the product of two odd numbers would be an odd number, that they were unable to see the need for anything more than a few confirming examples as support. In the cases where the truth value of the statements was not as 'obvious', the results still showed the tendency of the prospective elementary school teachers to acknowledge empirical verification as an acceptable proof. Together, these results confirm the findings of prior research (Harel & Sowder, 1998; Martin & Harel, 1989; Fischbein & Kedem, 1982) that suggests a strong reliance on empirical proof schemes. However, an interesting contribution of our study dealt with a question of how many examples constitute a 'proof', as perceived by our participants. For the majority, two or three examples seemed to be sufficient, as evidenced by the way in which the participants deleted or added numerical examples to the provided arguments.

Although, some research (Vinner, 1983; Selden & Selden, 2003) suggests that students tend to judge a mathematical argument on its appearance, we did not find high

reliance on this 'ritualistic' aspect of proof. Instead, the arguments in items four and five were either augmented or verified with numerical examples before they were accepted as 'proofs'. This finding, however, can be explained by the participants' mathematical background. Prospective elementary school teachers experience only minor exposure to the rituals of the proof, as compared to the participants (mathematics majors) in the aforementioned studies. As such, mathematical proof does not become a part of the prospective teachers' mathematical culture and beliefs in the same way that it does for mathematics majors.

The results of this study drew my attention to the way that prospective elementary school teachers usually think while they are proving a mathematical statement. Considering the close relationship between language and knowing (Vygotsky, 1987) I decided to study the kind of language or discourse that the students use in their mathematical arguments. It seemed to me that considering students' discourse as a tool, would provide a valuable insight into the way that they think. In a further study (Gholamazed, 2005) I compared the discourses that are used by students in mathematical arguments, with literate mathematical discourses that are available in the literature.

Proof as discourse

Considering the idea that the language of proof can also be used to communicate and to debate, I examined students' generated proofs to see how successful they are in communicating their mathematical knowledge through it. I considered several theorems or claims and examined different kinds of proofs for these claims. The proofs were drawn from the work of Euclid, from work of a typical contemporary mathematics instructor, and from proofs provided by university students.

In this study I adopted the communicational framework (Sfard, 2002; Ben-Yehuda et al, 2005). (See chapter 3 for the description of this framework). In my opinion this theoretical framework of mathematical discourse offers a new approach to the analysis of mathematical proof and students' learning of proof. Since itemized components of literate mathematical discourse not only detail what should be considered in generating proof and communicating through that, but also provide a tool for the diagnosis of possible obstacles in generating such proof. Furthermore, it offers a framework for the examination of proof as an evolving discourse, for in the work of Euclid, although crude by modern standards, there is available the first illustration of what present-day mathematicians would call a mathematical discourse (Newsom, 1964).

For this study I adopted three propositions from *Euclid's Elements* (2002), which can be also found in most number theory textbooks.

Proposition 24, from book seven:

If two numbers be prime to any number, their product also will be prime to the same.

Proposition 30, from book seven:

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

Proposition 29, from book nine:

If an odd number by multiplying an odd number, make some number, the product will be odd.

I adjusted the language of these propositions and presented them to 110 prospective elementary school teachers enrolled in the course "Principles of Mathematics for

Teachers” (the detailed information about this course is provided in chapter 5). They were invited to determine whether the statement was true or false, and then to prove it or provide a counterexample respectively. Students’ proofs for these propositions, in addition to Euclid’s proofs (See appendix 2) and proofs presented by a typical contemporary math instructor, provided a good variety of discourses on mathematics.

I had a close look at the role and the form of the characteristics of the literate mathematical discourses, according to the aforementioned framework, in each of the proofs. Although in different forms, the observations verified the inevitable presence of the components of the literate mathematical discourses in Euclid’s and contemporary proofs. Non-attendance or wrong attendance of those components in students’ works, however, can be seen as the factors that impede students’ active participation in creating a proof, and communication through proof.

Mathematical vocabulary: The hallmark of Euclid’s proof is the precise use of well-defined words for concepts, numbers, operations, and the rules of inference. However, the same precision can be seen in a contemporary proof, where there can also be found an equivalent sign or symbol for most of the technical words. This is the distinguishing characteristic of contemporary mathematical discourse.

Analysis of students’ works revealed that familiarity with the colloquial uses of words might have given them means for an ad hoc interpretation. Indeed, their lack of understanding of mathematical concepts usually led them to use technical words in an inappropriate manner or using inappropriate words for a technical purpose.

Mediators: The strong use of language in presenting a mathematical idea is the salient aspect of Euclid’s proofs. The only non-lingual mediator in his proof is the line

segments that he used as an icon or pictorial means for representing numbers. In a contemporary proof, algebraic symbols are the main mediator tools created specially for the sake of literate mathematical discourse. Surprisingly, in students' works there was no high tendency to use algebraic symbols. The majority of students used numerical examples, as a visual mediator for understanding and showing the validity of the proposition.

Routines: In Euclid's work, Aristotelian logic rules are the set of meta-rules or the routine that specify the 'how' of the proof. Considering the major role of the algebraic notations and symbols in contemporary proofs, the routine (beside the logic rules) is the rules for manipulating the algebraic notations and symbols. However, generalization based on a limited number of numerical examples was the dominating routine that was used by the majority of the students. Results also revealed that the mostly invisible rules that guide the general course of students' communicational activities are influenced by their everyday discourse.

Endorsed Narratives: Endorsed narratives, in general, are produced throughout the discursive activities. Indeed, as a result of mathematical routines, according to a set of well-defined rules, the new endorsed narratives are constructed from previous endorsed narratives. This chain of constructing endorsed narratives from previous ones is the main characteristic of Euclid's proofs.

In the contemporary proof of the first and second proposition, for example, we can see the key role of the Fundamental Theorem of Arithmetic, as an endorsed narrative that supports and guides the whole process of the proof. The students' work, however, was mainly based on their intuition. Some of students' work demonstrated that they had a

good understanding of the proposition but they did not know how to present it in the form of a mathematical proof.

Examining the components of literate mathematical discourses in different discourses in past and present provides an opportunity to see their importance. However, an interesting contribution of my study deals with the gaps that the lack of each of these communicational means can cause. Indeed, these gaps may impede students' active participation in creating a proof, and communication through proof.

In the work of Euclid's and modern texts however, based on totally different discourses, the persuasion aspect of proofs is salient. This aspect was the weakness of students' arguments. The results showed that the students' proofs were very subjective and based on their intuition, which is not enough to satisfy the social nature of proof. It seems that students did not consider the social aspect of the proof, that a proof should be convincing for a third person who reads it. Indeed, they were satisfied with an argument that was convincing enough for them.

Overall, results showed that the students' arguments were significantly influenced by their colloquial discourse. Indeed, the results might remind us that we cannot teach or evaluate students in isolation and the fact that we cannot disconnect them from their everyday experiences.

Conclusion

All the above-mentioned studies led me to the question: how can we help students acquire a better understanding of a given proof and also how can we help them participate in a process of creating a mathematical proof? Since communication is a

dialogic process, the meanings that are made by speakers and listeners or writers and readers with respect to individual utterances are strongly influenced by the discourse context in which they occur. This is the approach I will be using in my search for an effective method for teaching proof. In the following chapters I examine the students' engagement in creating a dialogue as a means towards understanding and creating proofs.

CHAPTER 5: RESEARCH SETTING

In this chapter I explain the research setting. First, I focus on how I extracted the idea of proof as a dialogue from the communicational approach to cognition. Next, I describe the educational environment in which this study took place, including the course setting, and the participants of the study. Then I introduce the tasks designed for the study and describe a rationale for this choice.

Proof as a dialogue

In this study I adopted the communicational approach to cognition (introduced in chapter 3), based on the learning-as-participation metaphor, and conceptualisation of thinking as an instance of communication. Considering the idea that thinking is a kind of communication that one has with oneself (Sfard, 2002), I encouraged students to write down the dialogue that they have with/or among themselves while they were thinking to understand or create a proof. The main purpose of this kind of task was to satisfy the convincing aspect of a student-generated proof, not only for the writer but also for the third person that might read it. Therefore, such dialogue should answer all the possible questions related to the mathematical properties or arguments used in a proof.

Harel and Sowder (1998) consider two sub processes for the process of proving: ascertaining, the process an individual employs to remove her or his own doubts about the truth of an observation, and persuading, the process an individual employs to remove

others' doubts about the truth of an observation. Writing down a dialogue provides students an opportunity to reflect on their ascertaining and thinking process and to organize it in a convincing or persuading way. From this perspective the dialogue can be considered as an intermediate stage between having an overview of a proof and writing a mathematical proof. It also offers an opportunity for researchers to examine students' arguments, presented in their discourse, and to investigate the possible obstacles that impede their understanding and creating a proof.

To introduce the idea of writing a proof through a dialogue the participants received a sample of a dialogue (see figure 2). The sample dialogue is between two imaginary persona, EXPLORER, the one who tries to prove the proposition, and WHYer, the one who asks all the possible questions related to the process of the proof. The main idea of designing these two personas was to consider two aspects of the character of an individual who is proving a mathematical statement. The dialogue is supposed to illustrate a conversation that one might have with oneself while one is thinking.

Figure 2: Exploring proof via dialogue

1. **EXPLORER:** If from an odd number an even number be subtracted, the result will be odd.
2. **WHYer:** What is this?
3. **EXPLORER:** This is a proposition.
4. **WHYer:** What is a proposition?
5. **EXPLORER:** A statement in terms of a truth to be demonstrated.
6. **WHYer:** What do you mean by demonstrating?
7. **EXPLORER:** To prove or make clear by reasoning or evidence.
8. **WHYer:** What is proof?
9. **EXPLORER:** Something that induces certainty or establishes validity.

10. **WHYer:** Are you going to prove this proposition?
11. **EXPLORER:** Yes! But I need to read it again to have better understanding of it.....
12. **WHYer:** What are even and odd numbers?
13. **EXPLORER:** An Even number is a whole number which is divisible by 2, such as 2,4,6,8,.... In other words they are multiples of 2.
14. **WHYer:** What about odd numbers?
15. **EXPLORER:** An Odd number is a whole number which leaves a remainder of 1 when divided by 2, such as 1,3,5,7,... and so on.
16. **WHYer:** How many of these numbers you might have?
17. **EXPLORER:** Infinitely many.
18. **WHYer:** So how are you going to prove the proposition?
19. **EXPLORER:** Let me start with an example.
20. **WHYer:** Why example?
21. **EXPLORER:** Numerical example helps to set a sense of what the proposition is talking about.
22. **WHYer:** Well, what is your example?
23. **EXPLORER:** Let us take 7 as an odd number and 4 as an even. Then $7 - 4 = 3$, which is an odd number.
24. **WHYer:** What about other possible choices?
25. **EXPLORER:** We can test it for other numbers as well.
 $5 - 2 = 3$, $11 - 6 = 5$, $21 - 12 = 9$, $9 - 4 = 5$, $15 - 8 = 7$
 All the examples confirm the proposition.
26. **WHYer:** But you test it for small numbers. How can you be sure that the result will be valid for big numbers as well?
27. **EXPLORER:** You are right. Let us test it for some bigger numbers:
 $573 - 198 = 375$, $48105 - 7964 = 40141$, $12649729 - 9478920 = 3170809$
 These examples confirm the proposition as well.
28. **WHYer:** Yes, but as you said there are infinitely many even and odd numbers that you have not tried them yet. How can you be sure that the proposition is valid for the cases that you didn't test?
29. **EXPLORER:** Your doubt is reasonable, but it is impossible to test the proposition for all the numbers. We will never finish.
30. **WHYer:** Is this why you need a proof?

31. **EXPLORER:** Exactly. The proof will show that the proposition is true in general, that it is true for all the possible choices of numbers.
32. **WHYer:** How can you do that?
33. **EXPLORER:** We first need to use a form of representation for even and odd numbers.
34. **WHYer:** What kind of representation?
35. **EXPLORER:** A representation for an even and an odd number can be an algebraic expression, which shows the structure and properties of the number.
36. **WHYer:** Like what?
37. **EXPLORER:** Since an even number is a multiple of 2, we can represent it in the form of “ $2n$ ”.
38. **WHYer:** What is n ?
39. **EXPLORER:** n can be any whole number. What ever we chose for n , $2n$ is always even.
40. **WHYer:** And, what about odd numbers?
41. **EXPLORER:** Similarly, since an odd number has remainder 1 after dividing by 2, we can represent it by “ $2m+1$ ” where “ m ” can be any whole number. We also assure then $m \geq n$.
42. **WHYer:** Why did you say $m \geq n$.
43. **EXPLORER:** We want to stay with whole numbers, and if this condition doesn’t hold, the subtraction may take us outside of whole numbers.
44. **WHYer:** And, why did you use n and m ?
45. **EXPLORER:** Actually it doesn’t matter what letter you use, m, n, r, k, \dots .
It is just a symbol that can be substituted with any whole number.
46. **WHYer:** So, can we use $2n$ for an even and $2n+1$ for an odd?
47. **EXPLORER:** This would mean that we have chosen consecutive numbers. To emphasis that the proposition is about any even and any odd we chose different letters.
48. **WHYer:** I see, you are saying that $2n$ is a representation for any even number and $2m+1$ is a representation for any odd number. Now, I’m wondering in what way are these useful?
49. **EXPLORER:** Subtract $2n$ from $2m+1$ and have: $(2m + 1) - 2n$.
50. **WHYer:** How can you say if the result is an odd number or not?
51. **EXPLORER:** For being an odd number, $(2m + 1) - 2n$ should be in the form of “2 times some whole number, plus 1”.
52. **WHYer:** Yes. But is it?
53. **EXPLORER:** Not apparently. But we can manipulate it based on properties of numbers.

- 54. WHYer:** What manipulation? It is not clear for me!
- 55. EXPLORER:** By using the properties of numbers we can rearrange the terms of our expression, $(2m + 1) - 2n$, and obtain its equivalent expression.
- 56. WHYer:** could you show it?
- 57. EXPLORER:** Yes, based on commutative property and associative property of whole numbers we can write
- $$(2m + 1) - 2n = (2m - 2n) + 1$$
- 58. WHYer:** Are you saying that $(2m - 2n) + 1$ is an odd number?
- 59. EXPLORER:** Yes, but to see this clearly we can pull out 2 from the first two terms, as their common factor, as follows:
- $$(2m - 2n) + 1 = 2(m - n) + 1$$
- In fact we have applied the distributive property, $ab - ac = a(b - c)$.
- 60. WHYer:** Okay. What is next?
- 61. EXPLORER:** Since m and n are two whole numbers, and m is bigger or equal to n , the result $(m - n)$ would be also a whole number.
- 62. WHYer:** Is this what you wanted?
- 63. EXPLORER:** Indeed, we had to show that the result is of the form $2 \times (\text{some whole number}) + 1$. To see this more clear we can say that $m - n = k$. Then $2(m - n) + 1$ can be written $2k + 1$, which has the structure of an odd number.
- 64. WHYer:** Sorry to say that but I lost the point of your last interpretation. Can you summarize your argument?
- 65. EXPLORER:** We wanted to show if we subtract an even number from an odd number the result would be an odd number. We considered $(2m + 1)$ as an odd number and $2n$ as an even number, where m and n are whole numbers and $m \geq n$.
- Therefore: $(2m + 1) - 2n = (2m - 2n) + 1$
 $= 2(m - n) + 1$ ($m - n$ is a whole number, k)
 $= 2k + 1$ which is an odd number.
- Now we can say that we proved the validity of the proposition.
- 66. WHYer:** Do you always make it so complicated?
- 67. EXPLORER:** In fact, the three lines above are sufficient for a proof. What we did here was to explain every symbol and every step in this derivation.

My claim is that such a self-dialogue happens in the mind of whoever is proving a mathematical statement. However, depending on the familiarity of a person with a mathematical discourse, the dialogue might take varying amounts of time and may express diverse qualities or rationality. For instance, for a mathematician it might take half a second and the answers to some of the '*whys*' are implicit in his or her understanding. However, for a naïve student it might take much longer and it might not include the answer to all the possible '*whys*'. In my experience, the main problem with the naïve students' self-dialogue is that many of the reasonable questions in the process of proving are not usually '*a question for them*'.

According to Sfard (2002) the process of thoughtful imitation is the most natural, or rather, the only imaginable way to enter new discourse. Therefore observing the more experienced players in mathematics may enable students to make sense of the new rules of the mathematical games and may enable them to participate in the activity. Hence, in this study, I brought the participants into a discourse through imitation. I did this by giving the students the sample dialogue to encourage them to pose questions in the environment of dialogue. I believe that writing a dialogue could cultivate the art of question posing, and after a while would become a part of the culture of students' mathematical thinking.

For a teacher, the dialogues offer an opportunity to examine students' arguments, and through posing more appropriate questions, lead the students to refine and strengthen their arguments. On the other hand, for researchers, written dialogues provide a rich source of students' discourses. Analysis of the discourses may reveal the factors that block or mislead the learners' understanding and production of proof.

To remind the reader, this study sought to address the following research questions:

1. What difficulties do pre-service elementary school teachers experience in writing and interpreting proofs for propositions related to elementary number theory?
2. What are the outcomes of students' activity of creating a dialogue?
 - (a) Does it facilitate students' participation in the process of proving?
 - (b) Does it reveal their difficulties in this process?
3. Can communicational approach to cognition serve as a tool for researchers in recognizing and identifying factors that impede pre-service elementary school teachers' participation in the process of creating and interpreting proofs?

For the purpose of answering the questions, I explored students' engagement in creating proofs in the environment of a dialogue.

Participants

Participants in this study were 93 pre-service elementary school teachers enrolled in the course "Principles of Mathematics for Teachers" described in the next section. A survey on the background of the participants showed that none of them had mathematics or science as her/his major or minor. The students mostly majored or intended to complete their degree in History, Geography, Psychology, English, Interactive Arts, and Sociology. Their background in mathematics was not strong. 70% of the participants mentioned that the last mathematics course that they had taken was mathematics 11 in high school, and the rest of them had some college courses such as statistics, accounting,

and business math. The main reason for students' taking the course was to fulfil the requirements and get into the Professional Development Program (PDP), which is a program for teacher certification. In general, most of the students enrolled in the course did not show much enthusiasm for mathematics. In other words, they did not consider a great role for mathematics as a part of their academic identity.

The course setting

One of the prerequisite courses for entry into PDP at Simon Fraser University is 'Principles of Mathematics for Teachers', or Math190. The prerequisite for this course is Grade 11 Mathematics (or equivalent) with a grade of at least 'C' or permission of the Mathematics department. The goal of this course is to promote the understanding of mathematical concepts and relationships. It concentrates on investigating *why* we do something in mathematical activities rather than *how* we do it. It looks at the language of mathematics, patterns, and problem solving.

The course has been designed to cover mathematical ideas involved in number systems and geometry in the elementary school curriculum. Whole number, fractional number, and rational number systems, plane geometry, solid geometry, and motion geometry constitute the content of the course. One of the issues that wove itself through all the topics of the course is the need for support of mathematical claims. Extensive discussion and exercises are usually aimed at helping students understand when and where a general argument, or a proof, is needed and when an example is sufficient.

The course runs for one semester, and has two, two-hour sessions each week. The course mark is determined from performance on weekly assignments, a project, two

midterms, and a final exam. The students in this course are provided with support through an open tutorial lab. The lab is a place where students can go to seek help with their homework problems as well as a place to meet each other, exchange their experiences, and work on assignments. The lab is open to students from 20 to 30 hours per week and is staffed at all times with one to four teaching assistants.

Tasks

Number Theory is one of the chapters taught in Math 190. The topics include:

- Prime numbers as the building blocks for counting numbers,
- Divisibility of the counting numbers and its properties,
- The greatest common factor and the lowest common multiple of the counting numbers and the application of these concepts in solving different problems,
- The Fundamental Theorem of Arithmetic and its application in solving problems or proving other propositions,
- Euclid's Theorem about infinitely many prime numbers.

In this study, the students were presented with several tasks related to number theory as a part of the course assignments, the course project, and the exams. Details of each task are summarized in Table 1. Further detail is provided in the next section.

Based on the purpose of the tasks, the participants performed them in small groups or individually. Task 1 was designed as a diagnostic tool to assess students understanding of what is considered a proof. Given the developmental priority of

communicative public speech over inner private speech (Vygotsky, 1987) discussed in chapter 3, the students were asked to work on Task 2 and Task 3 in a group of two to four. The main purpose of the group work was to encourage the students to have a discussion and to provide a reason to convince each other about their ideas and claims.

For the remaining tasks (4, 5, and 6) students were asked to work individually. They were expected to incorporate the experiences from the group discussions on prior tasks into their individual performance on tasks 4, 5, and 6. The students received detailed written comments on their performance.

Table 1: Synopsis of tasks

Task	Time of the task	Form of the task	Description of the task
1	Week 5	-Individual -Midterm question	The sum of any 3 odd numbers is odd. Is it true or false? Provide a convincing argument to justify your decision.
2	Week 6	-Group work -Homework assignment	The set of odd numbers is closed under multiplication. Prove the statement by filling the blanks in the given dialogue. Make it convincing for yourself and any other reader.
3	Week 8	-Group work -Homework assignment	Consider the following statement and its given proof. For any two whole numbers a and b, if a and b are relatively prime then a^2 and b are also relatively prime. Write a dialogue between two characters that will convince yourself and everybody else about the validity of the given proof.
4	Week 10	-Individual -Midterm question	a) Prove the following statement by filling the blanks in the given dialogue. For any three whole numbers a, m, and n, if $a m$ and $a n$, then $a m+n$. b) Consider the following statement for natural numbers a , b and c . If $a b$ and $b c$ then $a c$. Circle one TRUE/FALSE. Prove it or provide a counterexample.

Task	Time of the task	Form of the task	Description of the task
5	Week 9 (Students were allowed to work on the project for three weeks.)	-Individual -Course project	<p>a) Consider the following statement and its given proof. Write a dialogue that you have with yourself while you are convincing yourself about the validity of the given proof.</p> <p>If p is a prime number and $p ab$ then $p a$ or $p b$.</p> <p>b) Write a dialogue that you have with yourself for proving the following proposition.</p> <p>Let a, b, and c be whole numbers. If a and c are relatively prime, and b and c are relatively prime then ab and c are relatively prime.</p>
6	Week 14	-Individual -Final exam	<p>a) Consider the following statement</p> <p>If a and c are relatively prime and b and c are relatively prime, then $a \times b$ and c are relatively prime.</p> <p>Is this statement true or false? Circle TRUE/FALSE. If true-justify. If false-provide a counterexample.</p> <p>b) Consider the following statement:</p> <p>The sum of any 3 multiple of 7 is a multiple of 7.</p> <p>Is this statement true or false? Circle TRUE/FALSE. If true-justify. If false-provide a counterexample.</p>

Task analysis

The selection of the tasks was guided by the progression of thinking about proof.

In what follows I discuss the purpose of including the tasks for data collection. The pedagogical value of the tasks will be offered in the last chapter.

Task 1

The sum of any 3 odd numbers is odd.

Is it true or false?

Provide a convincing argument to justify your decision.

Task 1 was a part of a question in the first midterm exam. A typical mathematical proof for the statement is:

Consider $2n+1$, $2m+1$, $2k+1$ as three odd numbers where n , m , and k are whole numbers. Then $(2n+1)+(2m+1)+(2k+1) = 2(n+m+k+1)+1$, which is an odd number.

The purpose of asking this question was to investigate whether students pay attention to the fact that a convincing argument in mathematics, even for such simple and obvious statements, required a deductive reasoning that satisfied the generality of the claim.

Task 2

Prove the given statement by filling the blanks in the following dialogue. Make it convincing for yourself and any other reader.

1. **EXPLORER:** I am going to prove that “the set of odd numbers is closed under multiplication.”

2. **WHYer:** Could you tell me what is an odd number?

3. **EXPLORER:** Sure, an Odd number is _____

4. **WHYer:** And what is the set of odd numbers? Can you show it?

5. **EXPLORER:** Yes, we can show it by _____

6. **WHYer:** Fine. But, what does it mean that the set “is closed under multiplication”?

7. **EXPLORER:** It means _____

8. **WHYer:** Can you give me an example of it?

9. **EXPLORER:** For example _____

10. **WHYer:** I see. Did you prove the statement by these examples?

11. **EXPLORER:** These numerical examples _____

12. **WHYer:** Why?

13. **EXPLORER:** Because _____

14. **WHYer:** How can you show that the statement is true in general?

15. **EXPLORER:** First we need to select a notation that represents odd numbers in general.

16. **WHYer:** Can you explain it more?

17. **EXPLORER:** Sure. We can consider $2m+1$ as an odd number, where m is a whole number, and _____

18. **WHYer:** How can you multiply them?

19. **EXPLORER:** By using the properties of multiplication and addition:
 $(2m + 1) \times (\quad) =$ _____

20. **WHYer:** But how can you say that the result is an odd number?

21. **EXPLORER:** To make it more clear we can manipulate the result and simplify it in a form of an odd number _____

22. **WHYer:** Could you please repeat and summarize what you have done?

23. **EXPLORER:** Sure. I considered ____ and ____ as two odd numbers in general and _____

24. **WHYer:** Can we consider it now as a proof for the given statement?

25. **EXPLORER:** _____

The participants received Task 2 attached to a copy of the sample dialogue (see p. 61) The sample dialogue addresses the proposition: *'The difference of an odd number and an even number is an odd number'*. The similarity of this statement with the one in the first task provided the participants with an opportunity to have a reflection on their previous performance in Task 1. Indeed, the main purpose of giving the sample dialogue was to show the students how many reasonable questions the proof of such simple

statement might bring up. Also, I wanted to encourage the students to have such a self-dialogue when they were making a deductive argument.

In Task 2 participants were asked to complete the given dialogue (see p. 71) to prove that *'the set of odd numbers is closed under multiplication'*. The reason for choosing "incomplete dialogue" was to make students more familiar with the possible questions in a process of proving a statement. The incomplete dialogue provided them with an opportunity to face the questions that they should answer to prove the statement. The design of the questions was based mainly on the most plausible errors as indicated by the "one line proof" framework (Gholamazad, Liljedahl, & Zazkis, 2003). As was mentioned in chapter 4, the framework describes five competencies necessary for the generation of a complete and correct proof. Answering the questions of WHYer in Task 2 leads the students to a proof for the statement.

Task 3

Consider the proposition and its proof.

Write a dialogue that you have with yourself while you are convincing yourself about the validity of the given proof.

Proposition: For any two whole numbers a and b , if a and b are relatively prime then a^2 and b are also relatively prime.

Proof: By the Fundamental Theorem of Arithmetic each whole number can be expressed as the product of primes in exactly one way.

Let a and b be expressed as the product of primes as follows:

$$a = p_1 p_2 \dots p_m \text{ and } b = q_1 q_2 \dots q_n$$

$$\text{Then } a^2 = p_1^2 p_2^2 \dots p_m^2.$$

Since a and b do not have any common prime factor therefore a^2 and b do not have either.

Therefore a^2 and b are relatively prime.

Having an ability to understand proofs is crucial for being able to evaluate and generate mathematical arguments and reasoning. Task 3 provides a researcher an opportunity to explore how students go through the different steps in the process of interpreting. In the task students were asked to write a dialogue to extend the given proof for a number theory proposition.

The proposition was related to the course material, but it was unfamiliar to the participants. They saw the proposition and its proof in this task for the first time. This task was also designed to be completed in a group. It was expected that students would discuss and question all the steps of the proof in their group, and through posing and answering the questions make the given proof clear and convincing for themselves and for the reader.

Task 4

- a) Prove the given statement by filling the blanks in the following dialogue.
1. **EXPLORER:** I am going to prove that “for any three whole numbers a , m , and n , if $a|m$ and $a|n$, then $a|(m + n)$.”
 2. **WHYer:** How do you read $a|m$?
 3. **EXPLORER:** We say a divides m .
 4. **WHYer:** What does “ a divides m mean”?
 5. **EXPLORER:** “ a divides m ” means for some whole number x we have $ax = m$.
 6. **WHYer:** Can you give me an example of it?
 7. **EXPLORER:** Sure. For example _____ divides _____ because _____
 8. **WHYer:** Can you explain what you are going to prove?
 9. **EXPLORER:** Well, let’s get started with an example of it. If $a =$ _____, $m =$ _____, and $n =$ _____, you see _____
Also, _____
 10. **WHYer:** So does the example prove the statement?
 11. **EXPLORER:** _____, limited number of examples _____ a proof for a general statement.

12. WHYer: Then what are you going to do?

13. EXPLORER: I will use the definition of divisibility to prove the statement.

If $a|m$ then _____ = _____ for some whole number _____, and

If $a|n$ then _____ = _____ for some whole number _____.

Adding the respective sides of the two equations we have

And that implies $a|(m + n)$.

14. WHYer: Do you know if the converse of the statement is also a true statement or not?

15. EXPLORER: What do you mean?

16. WHYer: I mean, shall we say for any three whole numbers a , m , and n ,

if $a|(m + n)$, then $a|m$ and $a|n$?

17. EXPLORER: I don't know. Let's see _____

b) Consider the following statement:

For natural numbers a , b and c ,

if $a|b$ and $b|c$, then $a|c$.

Circle one TRUE/FALSE. Prove it or provide a counterexample.

Task 4 was a part of the second midterm exam, and therefore, performed individually. This task consisted of two parts. In the first part students were asked to fill the blanks of the given incomplete dialogue, and in the second part they were asked to prove the given proposition.

In the beginning of the given incomplete dialogue students were reminded of the algebraic definition of the divisibility of one whole number by another (see line 5). The main purpose of the task was to see whether and how students interpret the given

definitions and then implement them in the process of proving the statement. The students also received the converse of the given statement (see line 16), and they were asked to examine whether it was valid or not. The purpose of including this part was to see whether students could recognize that for rejecting a general statement one counterexample is enough.

The purpose of the second part of the task was to examine participants' awareness of their work on the first part. In other words, the main purpose was to see if the participants could implement the definition of divisibility in a new situation and if they were capable of creating and presenting a reasonable argument without further hint.

Task 5

a) Consider the following proposition and its proof.

Write a dialogue that you have with yourself while you are convincing yourself about the validity of the given proof.

Proposition: If p is a prime number and $p|ab$ then $p|a$ or $p|b$.

Proof: Since $p|ab$, then $ab=px$ for some whole number x .

Let a and b be expressed as the product of primes as follows:

$$a = p_1 p_2 \dots p_m \text{ and } b = q_1 q_2 \dots q_n$$

$$\text{Therefore } ab = p_1 p_2 \dots p_m q_1 q_2 \dots q_n$$

$$\text{Or } px = p_1 p_2 \dots p_m q_1 q_2 \dots q_n$$

By the Fundamental Theorem of Arithmetic each composite number can be expressed as the product of primes in exactly one way,

and so $p = p_i$ or q_j for some i, j ($1 \leq i \leq m$, $1 \leq j \leq n$).

If $p = p_i$ then $p|a$,

and if $p = q_j$ then $p|b$.

b) Write a dialogue that you have with yourself for proving the following proposition.

Proposition: Let a , b , and c be whole numbers.

If a and c are relatively prime, and b and c are relatively prime then ab and c are relatively prime.

Task 5 was a part of the course project. The participants were asked to perform this Task individually, and they had three weeks to work on it. As pointed out earlier, the main purpose of asking the participants to extend a proof in the form of a dialogue was to provide a researcher and an instructor with an access to the possible difficulties that participants may experience. Indeed, investigating the learners' questions and answers, as well as the important points that are not included in their questions, may reveal the quality of the participants' engagement in the process of understanding a given proof.

The purpose of the second part of the task was to encourage students to prove the given statement through writing down their self-dialogue. The given statement in this part was unfamiliar for the students; however, it could be considered as a kind of generalization for the statement presented in Task 3. Hence, performing this task would be an evaluation for students' progress in communicating mathematically with themselves and creating a proof through their self-dialogue.

Task 6

a) Consider the following statement

If a and c are relatively prime and b and c are relatively prime, then $a \times b$ and c are relatively prime.

Is this statement true or false? Circle TRUE/FALSE.

If true-justify. If false-provide a counterexample.

b) Consider the following statement:

The sum of any 3 multiple of 7 is a multiple of 7.

Is this statement true or false? Circle TRUE/FALSE.

If true-justify. If false-provide a counterexample.

Task 6 was a part of the final exam, and had two parts. In the first part participants were exposed to the proposition that they had already written a dialogue for in the project:

Let a , b , and c be whole numbers. If a and c are relatively prime, and b and c are relatively prime then $a \times b$ and c are relatively prime.

And, in the second part of the task they were asked to provide a proof for the unfamiliar statement:

The sum of any 3 multiple of 7 is a multiple of 7.

The main purpose of Task 6 was to evaluate the participants' improvement in presenting mathematical argument, particularly, to evaluate students' performance outside of the environment of dialogue. Task 6 also investigates participants' understanding and engagement in creating a proof in a form of a literate mathematical discourse.

Summary

In this chapter I introduced the setting for the research and the tasks designed for this study. In particular, I discussed the purpose and the rationale for including the tasks. The selection of the tasks was guided by two criteria: mathematical content and engagement of students in the process of proving. The tasks addressed several key

concepts of elementary number theory: whole numbers, prime numbers, relatively prime numbers, divisibility and Fundamental Theorem of Arithmetic. The progression of the tasks followed the gradual involvement of students in the process of proving, from public to private, moving from imitation of sample dialogue by completing and explaining a proof in a group to creating a proof individually.

Task 1 reveals the students' initial mathematical discourse for presenting a convincing argument. Tasks 2-5 require students to actively participate in the construction of a proof in a form of a dialogue through completing the given dialogue and writing their own dialogue. Task 6 evaluates the students' writing of a proof for the familiar and unfamiliar propositions. Tasks 6(a) and 5(b) introduce the same proposition but require a different approach to presenting an argument: as a formal proof and as a dialogue, respectively. Considering the idea that learning means a change in the manner of communication (Sfard, 2002), the similarity of Tasks 1 and 6(b) reveals a possible change in the students' discourse for presenting a mathematical proof. Chapters 6 and 7 discuss and analyse the results of the study.

CHAPTER 6: RESULTS AND ANALYSIS, BY TASKS

In this chapter I describe the participants' performance on each task. It was not the purpose of the study to quantify participants' responses. However, when presenting the results I will give a quantitative summary of the responses if it is applicable, followed by the samples of students' work and their analysis.

Task 1

The sum of any 3 odd numbers is odd.
Is it true or false?
Provide a convincing argument to justify your decision.

In Task 1, the participants were asked to provide a convincing² argument for the mathematical statement. 93 pre-service elementary school teachers provided arguments for the given statement. All the participants acknowledged that the statement is true. None of the students presented a correct algebraic proof for the statement. Different types of arguments and mediators used by participants for communicating their ideas are presented in Table 2.

² By "convincing" I mean, "convincing according to mathematical convention".

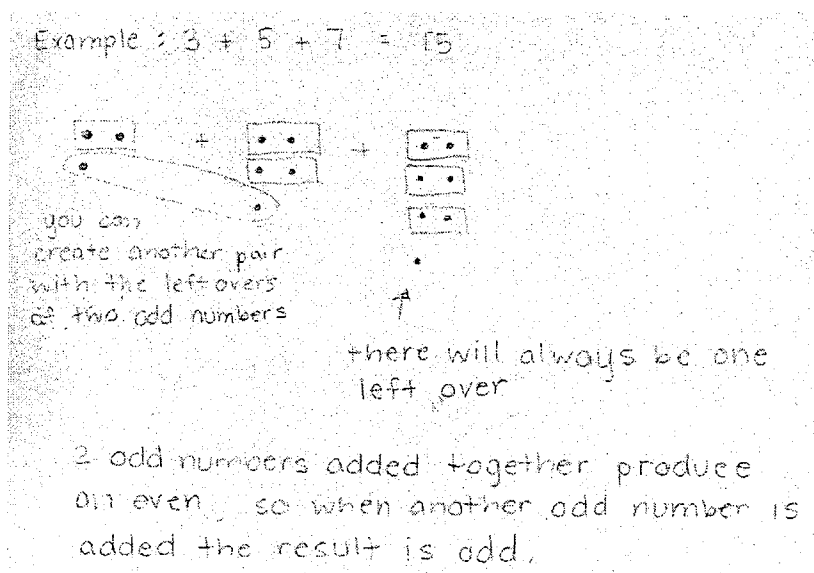
Table 2: Summary of students' responses to Task 1

Type of argument	The applied mediators	Number of students (n = 93)
Potentially reasonable arguments	Pictorial mediators and numbers	5
Empirical reasoning	Verbal explanation and numbers	54
Intuitive reasoning	Verbal explanation	11
Superficial algebraic reasoning	Algebraic symbols and numbers	11
Wrong reasoning	Verbal explanation	12

Potentially reasonable arguments

Five of the participants created reasonable arguments based on their common sense. These students used all the tools available to them to create a convincing argument. Four of them presented a generic proof by use of pictorial mediators. They applied the basic property and structure of odd numbers without using algebraic symbols. Indeed, their work illustrate a very good use of non-algebraic symbols in presenting a convincing argument (see Fig. 3)

Figure 3: Use of non-algebraic symbols as a mediator for Task 1



Another form of a reasonable argument was provided with a student who tried to test all the possible cases. In other words, she tried to present a proof by exhaustion. She wrote:

$$1+3+5=9 \quad 7+9+11=27 \quad (1+3+5)+7+9=25=\text{ODD}$$

$$n + m = \text{Odd}$$

The last # of any odd # will be either 1,3,5,7 or 9

When any combination of these are added together the outcome is an odd # so therefore the sum of 3 odd #'s will be odd

$$1+3+5=9 \quad 1+5+7=13 \quad 3+5+7=15 \quad 5+7+9=21$$

$$1+3+7=11 \quad 1+5+9=15 \quad 3+5+9=17$$

$$1+3+9=13 \quad 1+7+9=17 \quad 3+7+9=19$$

It is obvious that in writing “the last # of any odd #” the student meant to say “the last digit of an odd number”. Her argument, if it were complete, could be a very natural and reasonable approach for showing the validity of the statement.

Empirical reasoning

The results of this task show the high reliance of the participants on numerical evidence and rhetorical tools. As shown in Table 2 the majority of the participants (54 out of 93) established their argument considering numerical examples. Based on the numerical evidence the statement sounded too obvious to these students. The following example is a sample of this approach:

An example of how the statement is true is:

ie $3+5+7=15$

ie. $25+97+3=125$

ie. $2003+5197+27=7227$

you can keep providing examples and discover that the sum of three odd numbers will always end up an odd number.

Intuitive reasoning

The common explanation used by the participants in this category shows their intuitive understanding of the statement. They explained that since the sum of two odd numbers is an even number and the sum of an odd and an even number is an odd number, therefore the sum of three odd numbers is an odd number. Indeed, based on their intuition and despite their deficiency in reasonable representation, the students presented a correct overview of the proof without going through its deductive details, such as why the sum of two odd numbers is an even number. An example of these arguments is as follows:

The statement is true since the sum of 2 odd numbers is even (odd + odd = even) and the sum of an odd and even number is odd (odd + even = odd), the sum of the first two odd numbers equals an even number to which an odd number is added, thus giving us an odd total, or odd + odd = even + odd = odd.

Superficial algebraic reasoning

None of the participants who tried to use the algebraic notations in their arguments were successful. Indeed, the superficial use of notations in the arguments revealed the participants' weakness in use of algebraic tools. For instance, some of them used consecutive notations, X , $X + 2$, $X + 4$, without even referring to the structure of X

as an odd number. Others, took the same representation for the three odd numbers, $X + 1$, $X + 1$, $X + 1$, again without referring to the structure of X as an even number. The following argument exemplifies the superficial use of symbols in some of the participants work.

Two odd numbers added together will always have an even sum.

D {odd numbers} F {even numbers}

$a \in D$ $b \in D$ $c \in F$

$a + b = c$

$3a + 3a = 6a$

An even number and an odd number added together will always have an odd sum

$x \in D$

$2 + 3 = 5$

$2a + a = a$

$a + c = x$

Wrong reasoning

Twelve of the students presented different kinds of incorrect reasoning. The root of most of the students' problems was in misunderstanding the mathematical rules. For example one of the students wrote:

The sum of any 3 odd numbers is odd because each of the addends cannot be divided by an even number. Therefore the sum of those numbers would have the same property as well.

As it can be seen the student generalized the property of each addend to their sum, which is not true in general. These kinds of reasoning, however incorrect, are a rich source of the discourses and logical inferences used by the students. For example, an incorrect wording in students' discourse can be seen in the following argument.

Any two number added together can be divided by 2. That is, $2 + 2$ can be split into its original 2 parts. $3 + 3$, $4 + 4$, $5 + 5$, every two numbers added together can be divided back into its 2 original numbers.

$$2 + 2 = 4 \quad 4 \div 2 = 2$$

$$3 + 3 = 6 \quad 6 \div 2 = 3$$

3 whole numbers added together cannot be split into 2 numbers if the result of the split must stay a whole number.

$$3 + 5 + 7 = 15 \quad 15 \div 2 = 7.5$$

In this argument the student expressed her belief that “every two numbers added together can be divided back into its 2 original numbers” However, the presented examples show that she meant the statement for “every two equal numbers”, not every two numbers in general. She also, without expressing it explicitly, considered an odd number as a number which is not divisible by 2, and without any more explanation, seeming only to base it on the example, evaluated the given statement as true.

In general, the results of this task showed the majority of students had a personal understanding of the given statement. Indeed, the statement was too obvious for the majority of them. Therefore, they skipped the answers of the basic ‘whys’ that could have made their argument self-sufficient and convincing from a mathematical point of view.

Task 2

Prove the given statement by filling the blanks in the following dialogue (see p. 71). Make it convincing for yourself and any other reader.

The set of odd numbers is closed under multiplication.

In this task students were supposed to fill in the blanks of the given incomplete dialogue to prove that ‘the set of odd numbers is closed under multiplication’. They completed this dialogue in groups of 2 to 4. The number of completed dialogues was 26. The questions in the incomplete dialogue addressed the key points, definitions, and properties that were required for proving the proposition. The questions and analysis were designed based on the possible obstacles that were explained in the ‘one line proof’ framework in chapter 4. The results showed four groups of the participants could complete the argument properly. A summary of the results is presented in the Table 3.

Table 3: Summary of students’ responses to Task 2

Number of groups (n = 26)	Description of the responses
6	Acknowledged numerical examples as a proof
4	Could not choose appropriate representation
8	Could not manipulate the chosen representations correctly
4	Could not interpret the results correctly
4	Could complete the proof reasonably well

All of the groups presented correct definitions for an odd number (line 3), however, some of them had a weak performance for showing the set of odd numbers (line 5). All the groups, also, presented a correct definition for the closure property of a set, and described it further by presenting some related numerical examples.

To answer the question “did you prove the statement by these examples?” (line 10), more than half of the groups, in different phrases, acknowledged that some numerical example could not be a proof for the statement. However, some of them accepted numerical examples as a proof. For example, one of the groups wrote:

These numerical examples prove this as the product is always a whole number try it yourself and see if you can come up with one answer to disprove this statement. That's all you need. One example to disprove³.

And to answer the “why” (line 12) they explained:

Because the number system we use base 10 makes it so.

This rhetorical answer, although not wrong, cannot be considered as an acceptable reason from a mathematical point of view. In another dialogue, students gave more examples to support their idea:

These numerical examples show that the set is closed because if you multiply any of the two numbers together the answer will also be in the set. Ex. $1 \times 3 = 3$, $3 \times 5 = 15$, and so on.

And to answer “why” they explained:

Because the set contains all of the odd whole numbers and the product of two odd numbers is always an odd number.

³ The italics are added to represent students' writings.

As it can be seen, this group used the statement itself in its proof. This was a common problem in students' arguments. Several groups, to answer the above-mentioned questions (lines 10 and 12), gave some explanations based on their empirical results and intuitions. For example one of the groups wrote:

These numerical examples help set an example for what a number closed under multiplication looks like.

And to answer "why" they explained:

Because our examples show that an odd number multiplied by an odd number equals an odd number.

Choosing an appropriate representation for odd numbers and manipulating them correctly is an important part of writing a proof for the statement. In the dialogue, a representation for an odd number was given. The point was to see if the participants could choose an appropriate representation for the second odd number (line 19) that satisfied the generality of the argument, because keeping the generality of the proof was not possible without choosing appropriate representations.

Results showed that in several dialogues the chosen algebraic notations were incorrect or inappropriate. For example, in the following excerpt the chosen notations are representing two consecutive odd numbers, which does not satisfy the generality of the argument.

$$(2m + 1) \times (2m + 3) = 4m^2 + 6m + 2m + 3 = 4m^2 + 8m + 3$$

Manipulating the algebraic notations is very challenging for pre-service elementary teachers. It can be seen in some dialogues that their sound reasoning was established on the incorrect results. For example one of the groups wrote:

$$(2m + 1) \times (2n + 1) = 2(m + n) + 1$$

And for simplification (line21) they wrote

We say that $m + n = k$, when $k \in W$ so then $2(m + n) + 1$ can be written as $2k + 1$ which has the structure of an odd number.

There were also some groups that wrote a correct product but they could not interpret the result. One example of it is as follows:

$$(2m + 1) \times (2k + 1) = 2(2mk + m + k) + 1$$

For simplification (line 21) they wrote,

$$\text{if } m = 3 \text{ and } k = 5 \text{ then } 2(2(3)(5) + (3) + (5)) + 1 = 77$$

This shows the high reliance of this group on numbers as the main mediator and the superficial use of algebraic notation. As it is observed, even for substituting m and k the students used odd numbers, which may show that the structure of $2m+1$ has not had that much meaning for them.

For presenting a summary of their argument, most of the groups used the algebraic notations that they had already introduced in their argument as the main tool for

communicating their idea. However, some groups, despite working with algebraic notations, summarized their argument in a narrative style. For example:

I considered $2m+1$ and $2n+1$ as two odd numbers in general and multiplied them together. With properties of multiplication and addition we recognize the product into a simple form of an even number plus one, making an odd number.

There were also two groups that summarized their argument by using more numerical examples:

I considered 3 and 7 as two odd numbers in general and when we multiply 3×7 it equal 21 which is odd. When using algebra equation, any number can be put into the equation to create an odd answer.

In general, the results of this task revealed yet again the participants' high reliance on the numerical examples and their poor skill in working with the algebraic notations.

Task 3

Consider the proposition and its proof (see p. 73).

Write a dialogue that you have with yourself while you are convincing yourself about the validity of the given proof.

Proposition: For any two whole numbers a and b , if a and b are relatively prime then a^2 and b are also relatively prime.

In Task 3 participants were given the statement and its proof (see chapter 5), and were asked to write a dialogue they would have while convincing themselves about the validity of the given proof. They performed the task in groups of 2 to 4. Altogether 25

dialogues were created. In this task participants wrote a complete dialogue for the first time. Results showed many of the sentences that they used in their dialogues were an exact duplication of the sentences from the sample dialogue. The artificial use of those sentences revealed students' lack of understanding in different places. However, such duplication can be considered as the first step in imitation, and may be a necessary step in the progress of students' evolution towards creating a proof. Almost none of the groups created a dialogue that answered all of the implicit key '*whys*' of the proof. In what follows I explain some of the characteristics and problems in students' discourses that the environment of dialogue made them known for the researcher.

All of the dialogues began by referring to the definition of 'relatively prime numbers'. In addition to the main concept in the statement – 'relatively prime numbers' – the majority of participants recalled some other concepts as well. They defined the concepts such as whole numbers, prime numbers, composite number, greatest common factor, factor, divisor, and prime factorisation. It seems that for these students the concept of relatively prime number has not been objectified yet. So they situated its definition in relation to the other connected concepts.

11 out of 25 groups began their dialogue with this question: "what is a whole number?" To answer this question, 4 groups referred to the counting numbers:

A whole number is a counting number together with zero. i.e. 0,1,2,3,...

While the remaining 7 groups introduced whole numbers by using their intuitive understanding. An example of these descriptions is as follows:

All numbers greater than or equal to zero with no fractional parts.

This description, although it can be considered as an intuitively based explanation for whole numbers, is not mathematically correct because based on this definition, for example, $\sqrt{2}$ can be also considered as a whole number.

As mentioned in chapter 5, one of the main purposes of using a dialogue was to encourage students to pose appropriate questions related to the proposition or the process of its proof. The results showed that the environment of a dialogue was very appropriate for this purpose. The following three excerpts illustrate some of these questions that were presented in the group dialogues:

Explorer: We can show a and b as the product of prime by:

$$a = p_1 p_2 \dots p_m \text{ and } b = q_1 q_2 \dots q_n$$

Whyer: Fine. But why do you only look at the prime factors of the number?

What about the other factors?

Explorer: Because the prime factors are factors of all the larger composite factors of the number.

The above excerpt reveals that the group of the students who created this dialogue made a distinction between prime and composite factors of a number, which is a principal point for working on number theory statements. The following excerpt illustrates another good question referring to the prime factors of a number.

Whyer: What is prime factorisation?

Explorer: Prime factorisation means that we can express the number only using its factors which are prime. For example: $24=2^3 \times 3$.

Wyer: But how do we know we have the right prime factors?

Explorer: By the Fundamental Theorem of Arithmetic each whole number can be expressed as the product of prime numbers in exactly one way. For example

...

The first step in finding the prime factorisation of a number, usually using a factor tree as a mediator, is writing it as a product of two of its factors. For most of the numbers, especially large numbers, this step can be performed in several different ways, depending on the chosen factors. Hence, it usually causes a doubt for many of the students, where, depending on the first two factors they may end up with different prime factorisation. Addressing this issue in the above-mentioned excerpt may show that the students overcame this common misunderstanding. Often times, students do not differentiate between the use of mathematical terminology in different contexts. However, the following excerpt illustrates the students' attentiveness to this issue.

Wyer: Are relatively prime numbers always prime?

Explorer: Not necessarily. For instance, 4 and 9 are relatively prime numbers because they do not have a common factor other than 1.

As can be seen, this excerpt adequately addresses a potential misunderstanding that may be caused by using the word 'prime' in different concepts: prime number, and relatively prime numbers.

Recognizing the hidden aspects of students' difficulties in the dialogues

The results showed the environment of a dialogue revealed some aspects of learners' difficulties that may not be recognizable in their regular presentation of a proof. Therefore, these dialogues provided me with an access to the misunderstandings that might weaken learners' performance. For instance, from the Fundamental Theorem of Arithmetic students know that "each composite number can be expressed as the product of primes in exactly one way, except for the order of the factors" (Musser, Burger, & Peterson, 2003, p. 186). In the following excerpt we can see that the students ignored the phrase "... except for the order of the factors" of the theorem. Indeed, these students included the order of the prime factors in the unique representation of it.

Bert⁴: How do we find a numbers' prime factors?

Ernie: You perform prime decomposition. It looks like this: (a tree diagram)

The prime factors of 8 are 2,2,2 this is written as 2^3 .

Bert: Can I write the prime factors of 8 like this: $2^2 \times 2$?

Ernie: No, each whole number, can be expressed as the product of its prime in EXACTLY ONE way.

Another problem that students usually have in mathematical discourse is related to the use of quantifiers. Through the course material the students might have learned that a mathematical argument should satisfy the generality of a statement. It was observed that some students try to satisfy the generality of their argument by including quantifiers in their discourse. However, most of the times they used the quantifiers in an inappropriate

⁴ Notice the choice of personas. Participants' choices will be discussed under "Personas in students' dialogues" at the end of this chapter.

place. In the following excerpt we can see the students' effort to show the generality of the argument. But, the rather weak use of the word 'any' reveals their poor skill in working with algebraic notations.

W: How do you prove it?

E: We can show the proposition is true in general, for **any** whole #'s. we can start by expressing a and b as the product of **any** primes, such as:

$$a = p_1 p_2 \dots p_m \text{ and } b = q_1 q_2 \dots q_n.$$

W: What are p and q ?

E: These are just different symbols to present **any** whole #, in this case the individual factors of a and b .

One of the main issues that implicitly impede the students' participation in mathematical discourse is their own attitude toward mathematics. This attitude can be seen in some of their dialogues.

Clark Kent: I am going to prove to you that for any two whole numbers a and b , if a and b are relatively prime, then a^2 and b are also relatively prime.

Bruce Wayne: Sorry Clark, was that English? Lets start at the beginning. I know that whole numbers are a set of numbers $\{1,2,3,4,\dots\}$ but what does it mean to be relatively prime?

.....

B. W. : Fundamental Theorem? **Is that like a law?**

C.K. : Yes Bruce, a law of arithmetic. So let's

Considering a theorem as a “law”, in the above-mentioned excerpt, may show that the students did not perceive the persuading aspect of a mathematical theorem. It seems they see theorems as some rules or laws to obey without questioning, which is far from the real nature of a mathematical theorem.

As already mentioned, there is a strong belief among students that numerical examples can guarantee the validity of a claim. The following excerpt illustrates this tendency,

Whyer: I see. Did you prove the statements by these examples?

Explorer: Yes. The examples induced certainty and established validity.

Whyer: You have not tried every integer. How can you be sure that the proposition is valid for the case that you didn't test?

Explorer: Your doubt is reasonable, but it is impossible to test the proposition for all the numbers. We will never finish.

As can be seen, the above dialogue includes duplication of some lines from the sample dialogue (see Figure 2, p. 61), however with a totally different intention. We can learn from this excerpt that what the students perceived from sample dialogue is not necessarily the same as what they read. It sounds as if the students' belief and insight about acceptability of numerical examples creates a robust obstacle that does not let them acknowledge a new approach or extend their perspectives easily.

Overall, the results showed 12 out of 25 group-generated dialogues explained the given proof just by using some numerical examples. In fact, what they did was translating the given proof to the language of numbers, which apparently is more understandable for

the students. The rest of the dialogues (13 out of 25) provided interpretation for the given algebraic proof. In 6 of these dialogues students could not expand the proof more; indeed, they just reiterated what they received in the task. The remaining 7 dialogues could not even re-present the given proof correctly. In the following excerpt we can observe a sample of the weak performances with the algebraic notations and the use of words:

WHYER: How can you show that the statement is true in general?

EXPLORER: First we need to select a notation that represents **a whole counting number and its products in general**. We will use: $a = p_1, p_2, \dots, p_m$.

We will use b for the 2nd number.

WHYER: So how do you show that a^2 & b are also relatively prime?

EXPLORER: Well, $a^2 = p_1^2, p_2^2, \dots, p_m^2$. $b = q_1, q_2, \dots, q_n$. These numbers still don't have any common prime factors, therefore they are relatively prime.

In the above excerpt, it appears that the students interpreted the prime decomposition of a whole number as "... a whole counting number and its products in general", which conveys a totally different idea. Despite having the algebraic representation for prime decomposition of numbers, a and b , in the given proof, it can be seen that the students for re-presenting the prime decompositions in their dialogue used commas between prime factors. This misrepresentation may show students' poor understanding of the notations, which had an effect on their imitation. Also, it may reveal that they are not completely aware of what the given argument is presenting.

Overall, students' work on this task showed the usefulness of their written dialogues for a researcher in order to have access to the hidden aspects of students' difficulties.

Task 4

a) Prove the given statement by filling the blanks in the following dialogue (see p. 74).

For any three whole numbers a , m , and n , if $a|m$ and $a|n$, then $a|(m + n)$.

b) Consider the following statement:

For natural numbers a , b and c ,

if $a|b$ and $b|c$, then $a|c$.

Circle one TRUE/FALSE. Prove it or provide a counterexample.

93 students performed the task. More than half of the students completed the given dialogue (Task 4 part a) correctly. The result of the first part of the task is summarized in the Table 4.

Table 4: Summary of students' responses to Task 4(a)

Number of the students (n=93)	Description of the responses
8	Could not provide a numerical example for the statement (line 9)
25	Could not write the definition of divisibility by algebraic notations (line 13)
11	Could not manipulate and interpret the algebraic notations to get the desired results (line 13)
49	Could complete the proof reasonably well

In the first part of task 4 participants were reminded of the definition of divisibility (line 5). Nevertheless, the results show that several students had a problem with providing a numerical example for the definition or the statement (line 7 and 9).

Almost one fourth of participants could not repeat the definition with different letters (line 13). Also, despite the given hints for operations, some of the participants could not manipulate the representation and interpret the results to get the desired conclusion. In general, results showed that working with the algebraic symbols and signs was the most challenging part for the participants. This is both surprising and frustrating, given that the algebraic operations involved are very basic and part of curriculum in introductory school Algebra.

However, to a much lesser degree than in previous tasks the reliance on the numerical examples was seen in the results of this task. In answer to the line 10 – “So does the example prove the statement?” – 80% of the participants (74 out of 93) acknowledged that numerical examples cannot be considered as a proof for the statement.

To show whether the converse of the statement is true or not (line 16-17), one third of the participants correctly provided a counterexample. The rest of them tried to prove the converse of the statement by applying the definition of divisibility or by providing a few confirming examples. For instance in the following excerpt we see that a student by using the definition, showed ‘ $a|(m+n)$ ’ means ‘ $\frac{(m+n)}{a}$ is a whole number’, which is correct. But, in the next step of his reasoning he tacitly considered, every whole number is only the sum of two whole numbers, which is not necessarily correct in general.

If $a|(m+n)$ then $ax = (m+n)$

$$x = \frac{(m+n)}{a}$$

$$x = \frac{m}{a} + \frac{n}{a} \quad \text{property of distributivity of addition}$$

\therefore If $a|(m + n)$, then $a|m$ and $a|n$

In the second part of Task 4 participants were asked to prove, or disprove with a counterexample, that for any three whole numbers a , b , and c if $a|b$ and $b|c$ then $a|c$.

About 10% of the participants (9 out of 93), by misusing the logical rules of inference or by misusing the definition of divisibility claimed that the statement was false. For example one of the students wrote:

If $a|b$ and $b|c$ then $a|c$ if and only if c is a multiple of both a and b .

Ex. $a = 4$, $b = 2$, $c = 12$ then $4|2$, $2|12$, and $4|12$

Counterexample: $a = 4$, $b = 12$, $c = 3$ then $a|b = 4|12$ and $b|c = 12|3$ but 4 doesn't divide 3.

In this argument, the student interpreted "If $a|b$ and $b|c$ then $a|c$ " as " c is a multiple of both a and b ", which is true. But by misusing the phrase "if and only if" she also considered if " c is a multiple of both a and b " then " $a|b$ " which is not necessarily true. The presented example and the counterexample reveal the possible root of this mistake, which can be the wrong perception of the writer about the divisibility of a number by another.

The rest of the students (84 out of 93) acknowledged that the statement was true. However, only 34% (32 out of 93) of the students could provide a deductive argument for the statement. The description of the students' performance is summarized in the Table 5.

Table 5: Summary of students' responses to Task 4(b)

Number of students (n = 93)	Description of the presented arguments
9	Acknowledged the statement as false
18	Only provided some confirming examples
27	Repeated the definition of divisibility and provided some confirming examples
4	Only reiterated the statement
3	Provided some verbal explanation
32	Provided deductive argument

The results showed that working with algebraic notations is still the main challenge for the students. Nevertheless, we see an improvement in a dialogue but still a major deficiency in writing a proof. The comparison between the results of two parts of Task 4 shows the importance of a dialogue as a step towards a proof.

Task 5

<p>a) Consider the following proposition and its proof (see p. 76). Write a dialogue that you have with yourself while you are convincing yourself about the validity of the given proof.</p> <p>Proposition: If p is a prime number and $p ab$ then $p a$ or $p b$.</p>
<p>b) Write a dialogue that you have with yourself for proving the following proposition.</p> <p>Proposition: Let a, b, and c are whole numbers. If a and c are relatively prime, and b and c are relatively prime then $a \times b$ and c are relatively prime.</p>

Task 5 of the study was part of the course project to be completed within three weeks. As such, any sloppiness due to time constraints would not be an issue for this

task. Altogether 83 students⁵ performed this task. In the first part of the task the participants were required to write a dialogue to extend the given proof of the proposition. In this part, almost all of the participants began their dialogues defining the prime numbers followed by the definition of divisibility. Providing numerical examples for the definitions and for the proposition was the other common aspect of all the dialogues. Indeed, for the majority of the participants, empirical verification was the first step towards understanding the proposition. This corresponds to ‘specializing’ as a type of mathematical thinking discussed in Mason et al (1982).

21 (out of 83) of the participants presented a reasonable dialogue by going through all the steps of the proof and addressing the key points of the given argument. The following dialogue illustrates the performance of this category of students.

Explorer: Hello Whyer, I have a proposition for you, if p is a prime number and $p|ab$ then $p|a$ or $p|b$.

Wyer: That is nice but what is a prime number, and what does this symbol ($|$) represent?

Explorer: I’ll start by explaining what a prime number is. A prime number belong to the set of whole numbers and has only two divisors, 1 and the number itself (p). As for your second question, ($|$) that symbol tells us that the number on the right (a , b , or ab) is divisible by the number on the left (p). Another way of putting $p|a$ for instance is that $p \times n = a$, where n is an element in the whole number set.

Wyer: When you said “ $p|a$ or $p|b$ ” in your original proposition what exactly did you mean?

⁵ 10 students did not submit this task.

Explorer: The word “or” implies that if either or both of the criteria are met then the statement is true. Both $p|a$ and $p|b$ must be untrue in at least one case to disprove my original statement.

Whyer: So have you proved your proposition yet?

Explorer: Not in the slightest. If I want to prove the proposition I will have to show you it is true in all cases.

Whyer: I don’t even understand one case let alone all of them, can you give me an example?

Explorer: Certainly, $5|50$ and $5|25$, although $5|2$ is untrue the proposition is still true because $5|25$. In this example $p = 5$, $a = 25$, $b = 2$, $ab = 50$.

Whyer: Ok, that makes sense, the prime number 5 divides 50 and also divides 25 so your proposition is true in this case, but what about all cases like you mentioned earlier?

Explorer: well, for the general case we have to leave the proposition in it’s variable form with all of the p , a , b and ab , this way all of the possible situations are represented. So, if $p|ab$ then $p|a$ or $p|b$.

First, I will begin by writing the problem in another way.

If $p \times w = ab$, for some whole number w

Then $p \times y = a$ or $p \times z = b$, for some whole numbers y, z .

If a is written as a product of primes:

$a = p_1 \times p_2 \times p_3 \dots p_m$ and $b = q_1 \times q_2 \times q_3 \dots q_n$, where q is also prime.

$ab = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$.

Whyer: Hold on a minute, what does p_m represent?

Explorer: Numbers may have different numbers of prime factors, for example 60 can be decomposed into the primes $5 \times 3 \times 2^2$. p_m is just a way of implying

that there may be different numbers of primes and includes the entire prime decomposition of the number.

Whyer: I understand. You can continue with what you were saying.

Explorer: Substituting from the above equation $p \times w = ab$,

$$p \times w = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$$

The fundamental theorem of arithmetic sequence confirms that a composite number may be expressed as the product of primes in exactly one way, so our prime number p must be equal to one of the primes that compose the prime decomposition of ab . This can be demonstrated with an example:

$p|210$ the prime decomposition of 210 is $2 \times 3 \times 5 \times 7$, this also means that because 210 can only be expressed in one way the prime p must be a prime from the group 2, 3, 5 or 7 because they are the only primes that divide 210.

In the general case p must be equal to one of the primes between $p_1, p_2 \dots p_m$ or $q_1, q_2 \dots q_n$. Because p is equal to one of the primes that compose ab it also must help compose a or b or both.

Whyer: Does that now prove your proposition?

Explorer: Yes.

Whyer: Can you sum it briefly one more time?

Explorer: Sure,

If $p|ab$ then $p|a$ or $p|b$ for some prime p .

$p \times w = ab$, for some whole number w ,

$a = p_1 \times p_2 \times p_3 \dots p_m$ and $b = q_1 \times q_2 \times q_3 \dots q_n$, where q is also prime.

$$ab = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$$

$$p \times w = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$$

By fundamental theorem of arithmetic sequence,

$P = \text{one of the primes } p_1, p_2 \dots p_m \text{ or } q_1, q_2 \dots q_n$

So, $p|a$ or $p|b$.

In this dialogue the student began with clarification of the statement through describing related vocabulary and symbols, and providing a numerical example. She, then by choosing appropriate representations, manipulating them correctly, interpreting and supporting the results by the Fundamental Theorem of Arithmetic drew a conclusion and created a convincing argument. The reasonable flow of this dialogue indicates the expected change in the writer's mathematical discourse and her improvement in presenting a mathematical argument.

19 (out of 83) participants just reiterated the given proof in the form of the question and answer. The dialogues created by these students showed that they just mimicked the proof that they received. Therefore, it is hard to assess whether they had any difficulty or true understanding of the proof.

The rest of the dialogues (43 out of 83) revealed the different kinds of challenges or misunderstandings that the students had. Indeed, these dialogues illustrated how different kinds of a weak or wrong use of the mathematical concepts, routines, or language may mislead the process of understanding or creating a proof. The following excerpt of one of the students' dialogue exemplified the challenges that students may experience in working with the algebraic symbols:

1. **Me2:** What is a and b ?
2. **Me1:** a and b are any whole numbers in this case we will use a and b to be expressed as the product of primes.

3. **Me2:** How do you express these products of primes if you don't use numerical symbols?
4. **Me1:** We can allow $a=p_1p_2\dots p_m$ and $b=q_1q_2\dots q_n$
5. **Me2:** What is this showing us?
6. **Me1:** $p_1p_2\dots p_m$ show that we continue to double, triple so on until m number of times the prime number so that no matter what our prime number can go into the m th number evenly and this also applies to $q_1q_2\dots q_n$. Therefore $ab=p_1p_2\dots p_m q_1q_2\dots q_n$.
7. **Me2:** Is that all?
8. **Me1:** No, if $ab=p_1p_2\dots p_m q_1q_2\dots q_n$ then $px=p_1p_2\dots p_m q_1q_2\dots q_n$.
9. **Me2:** What does this confirm in our proposition?
10. **Me1:** By the Fundamental Theorem of Arithmetic each composite number can be expressed as a product of primes in exactly one way.
11. **Me2:** What is that mean?
12. **Me1:** Let take p_1, p_7 we will use 2 as our prime number $2^7=128$. As we can see (in prime factor trees) no matter how we write the composite number we still end up with the same product of primes, (same way every time)
13. **Me2:** What are composite numbers?
14. **Me1:** Composite numbers are counting numbers with more than two factors, such as 128, 9, 12, 1065, 32.
15. **Me2:** Okay now knowing all this how do we express the theorem in our proof?
16. **Me1:** So we use $p=p_i$, or q for some i, j .
17. **Me2:** What is i and j ?
18. **Me1:** It is any whole number between 1 and m th and n th number. $(1 \leq i \leq m) (1 \leq j \leq n)$.

19. **Me2:** Why did you say $1 \leq i \leq m$ and $1 \leq j \leq n$?
20. **Me1:** We need to keep our conditions where it holds so i and j must be greater than 1 and less than the m th number and n th number in the sequence of the products of primes.
21. **Me2:** So can you summarize what you have done?
22. **Me1:** If $p=p_i$ the $p|a$ and if $p=q_j$ the $p|b$.

Lines 6 and 12 of the above dialogue reveal that the writer assumed the prime decomposition of a is showing the number of times that prime factor p exists in a , or as if every composite number is represented as a power of one prime. This misunderstanding led her whole argument towards a wrong direction.

Overall, the results showed, because of the algebraic structure of the given proof the main emphasis of the dialogues were on description and interpretation of the symbols and representations. Further discussion on details of students' work related to representations is presented in chapter 7.

In Task 5(b), participants were asked to write a dialogue and through that prove the given proposition. The results showed that all the participants began their dialogues by recalling the definition of relatively prime numbers. Similarly to Task 3 and Task 5(a), many of the students recalled some other related definitions as well. An example of this presentation can be seen in the following excerpt from one of the dialogues:

A. The proposition begins with a , b and c being whole numbers.

Q. What are whole numbers?

A. Whole numbers are counting numbers and zero. For example $0, 1, 2, 3, \dots$ are whole numbers. Next the proposition states that a and c are relatively prime.

Q. What does relatively prime mean?

A. Relatively prime is when there are two counting numbers whose greatest common factor is 1. For example, the greatest common factor of 2 and 3 is 1.

Q. What are factors?

A. They are the numbers whose product are other numbers. For example, the factors of 10 are 2 and 5. In this case these are the prime factors.

Q. What are prime factors?

A. Prime factors are numbers that only have factors of 1 and itself. For example, the only prime factors of 3 are 1 and 3. b and c are also relatively prime.

Q. So a and c have the greatest factor of 1, the same with b and c .

A. That's correct. What is being proved is that the product a and b , which will be referred to as x will still be relatively prime with c .

Q. What does product mean?

A. Product is the answer when two numbers are multiplied together.

For presenting their arguments, participants had recourse to different types of the mediating tools that served as a communication means such as numbers, verbal explanation, algebraic representation, and set theory symbols and diagrams. An outline of the participants' tendency to use each of these tools is summarized in the Table 6.

Table 6: Summary of mediating tools used in students' responses to Task 5(b)

Number of students (n = 83)	Mediating tool
57	Algebraic representation
14	Verbal explanation
7	Numerical examples
5	Set theory symbols and diagrams

Yet again, the main common aspect of all the dialogues was an empirical verification of the proposition. Almost all the participants provided some numerical examples and used them to explain the main idea of the proposition. The results, however, showed that only 8% of the participants used only this form of mediator and finished their argument in this step, which is a promising result. The following excerpt illustrates this approach. In this dialogue the writer after recalling the related definitions continued:

Whyer: can you explain what are you going to prove?

Explorer: Sure. Let's start with an example of it. If $a = 5$, $b = 9$, $c = 7$ then the GCF of a & $c = 1$ and the GCF of c & $b = 1$ then $a.b$ and c are relatively prime. E.g. $5.9 = 45$, and 45 is relatively prime with 7 GCF = 1

Whyer: So does the example prove the statement?

Explorer: Yes.

Whyer: Can you summarize what you have said?

Explorer: Yes, I considered for any three whole numbers a , b , c if a and c are relatively prime and b and c are relatively prime then together $a.b$ and c are relatively prime.

Another example of a logical derivation that does not rely on a mathematical formalism can be seen in the responses of some participants who used the verbal explanation as a mediator for communicating their ideas. The following excerpt illustrates this approach. The verbal explanation is a good example of the student's logical reasoning.

Whyer: How can we generalize the statement?

Explorer: Let's look at what we already know. a and c have to be relatively prime so writing a and c in prime factorisation we will notice they have no common primes. We also know that b and c are relatively prime. So, writing b and c in prime factorisation we will notice they have no common primes. We can see then that a and b can have the same prime factors therefore multiplying a and b together will give you the same prime factors of a and b , when we compare them to c 's prime factors they will still have nothing in common and therefore be relatively prime. This concludes the proof of our proposition.

The majority of the participants, following an empirical verification, tried to use a representation in their argument such as algebraic notations, set theory symbols, or diagrams. Around two thirds of the participants, used the algebraic representation for prime decomposition of a whole number. However, a poor competence in using algebraic notations and algebraic routine procedures led almost half of the students to a superficial presentation. For example, in the following dialogue the student, after an empirical verification, continues:

Aye: So, are we done?

Myself: No, because we have not proven the proposition yet, we have just seen that one example works. To prove the example, we have to go back to using letters, which represent all possibilities.

Aye: sounds good.

Myself: To start, we will express a , b , and c as a product of their primes such that $a=p_1,p_2\dots p_m$, $b=q_1,q_2\dots q_n$ and $c=r_1,r_2\dots r_q$. Therefore $ab=p_1q_1,p_2,q_2\dots pmqn$. Since the Fundamental Theorem of Arithmetic states that each composite number can be expressed as the product of primes in exactly one way and we know that $ab=pmqn$ and $c=rq$, we can state that ab and c are relatively prime.

Aye: And that was all we needed for proof?

Myself: Yes!

A few of the participants used set theory symbols or diagrams to present their argument. These dialogues revealed the writers' understanding of the proposition and its proof; however the poor access to the conventional symbolism and operations weakened their presentation. For example in the following excerpt (see Fig. 4) by using Venn diagrams the student distinguished between all the possible different cases, which shows the writer's understanding. However the written discussion does not satisfy the required precision of a mathematical discourse.

Figure 4: Use of set theory symbols and diagrams as a mediator for Task 5(b)

Student: OK, is there any way of showing this using drawings to make it more clear?

Teacher: Sure, lets use Venn Diagrams to show the relationships between these sets. This way we can explore the options of relationships between A and B as well.

Option 1: $A \cap B$, so $A \times B = A \cup B = A$
We can see that $A \cup B$ still does not intersect C.

Option 2: $A \cap B$, so $A \times B = A \cup B =$
Again the union of A and B does not intersect C.

Option 3: $A \subset B$ (or $B \subset A$)
Where $A \times B = A$ or $B \times A =$

Option 4: $A = B$, so $A \times B = A \cup B$
Still, $A \cap B \cap C$

* A never intersects with C b/c they are relatively prime and B never intersects with C b/c they are relatively prime.

The diagrams show three circles labeled A, B, and C. In Option 1, A and B are overlapping circles, and C is a separate circle below them. In Option 2, A and B are overlapping circles, and C is a separate circle below them. In Option 3, A is inside B, and C is a separate circle below them. In Option 4, A and B are overlapping circles, and C is a separate circle below them.

Overall, 43 (out of 83) participants through implementing different types of mediators were able to create a reasonable argument to justify the given statement in

Task 5(b). The performance of this group of students is a promising result of the study. The dialogues created by these students indicate their improvement in presenting an argument in the form of a mathematical discourse, by using appropriate mathematical words, mediators, routines, and endorsed narratives. A more detailed discussion of students' work as it relates to the use of different features of mathematical discourse is presented in chapter 7.

In general, the results of this task showed that the flexible environment of dialogue could be very helpful for involving students in the process of proving.

Task 6

a) Consider the following statement
If a and c are relatively prime and b and c are relatively prime. Then $a \times b$ and c are relatively prime.
Is this statement true or false? Circle TRUE/FALSE.
If true – justify. If false – provide a counterexample.

b) Consider the following statement:
The sum of any 3 multiple of 7 is a multiple of 7.
Is this statement true or false? Circle TRUE/FALSE.
If true – justify. If false – provide a counterexample.

Task 6 was part of the final exam. 92 students⁶ performed the task. In the first part of the task the participants were required to provide a justification or a counterexample for the familiar statement. Students had already written a dialogue for this statement in Task 5(b). All the students, except 3, acknowledged the given statement as a true one,

⁶ One of the students missed the final exam

and supported their decision with some sort of argument. 31 (out of 92) students, mainly by using algebraic representations, provided acceptable justification for the statement.

The rest of the results are summarized in the Table 7.

Table 7: Summary of students' responses to Task 6(a)

Number of students (n = 92)	Description of the presented argument
31	Correct proof of the statement
14	Empirical verification
11	Incorrect use of representations
23	Weak explanation
13	Wrong explanation

Despite high reliance of students on empirical verification at the beginning of the study, here, the results showed that only 15% of the students established their argument based on the numerical examples. A sample of these arguments follows:

We know that two numbers that have GCF of 1 is called relative prime.

First use example to prove: $a = 6$ $b = 3$ $c = 17$

Sub statement: If 2 and 17 are relatively prime and 3 and 17 are relatively prime then $(3 \times 2) = 6$ and 17 are relative prime.

\therefore therefore the statement is true based on the # example.

In this task, also, the incorrect numerical example misled 3 of the students. The following argument is one of those:

False, because if $a = 3$, $b = 2$, and $c = 9$ then $a \times b$ and c are relatively prime.

Well, $3 \times 2 = 6$ and $c = 9$.

6 and 9 are not relatively prime numbers therefore this statement is false.

Among those students who tried to use a type of algebraic representation in their argument, 11 students were not successful. The following argument exemplified the performance of this category of students:

$$a = 2 \quad c = 11 \quad b = 4$$

$$\text{GCF}(2, 11) = 1 \quad (2, 11) \text{ are relatively prime}$$

$$\text{GCF}(4, 11) = 1 \quad (4, 11) \text{ are relatively prime}$$

$$a \times b = 8 \quad \text{GCF}(8, 11) = 1 \quad (8, 11) \text{ are relatively prime}$$

Let the prime decomposition of $a = a_1 a_2 a_3$

the prime decomposition of $b = b_1 b_2 b_3$

and, the prime decomposition of $c = c_1 c_2 c_3$

The GCF of a, c is 1, the GCF of b, c is 1

$\therefore a$ and c are relatively prime and b and c are relatively prime.

Prime decomposition of $a \times b = ab_1, ab_2, ab_3$

$\text{GCF}(a \times b, c)$ is 1, $\therefore a \times b$ and c are relatively prime.

Using (ab_1, ab_2, ab_3) as prime factorisation of $a \times b$, reveals the student's weak understanding about algebraic representation of a prime factorisation..

Several participants tried to communicate their idea through verbal explanation. Here, I distinguish between arguments that presented a weak explanation for the statement and those that presented a wrong explanation. In weak explanation I categorized those arguments that do not include any wrong claim but do not answer the entire key "whys" either. The following argument illustrates this kind of argument:

If the prime factors of a and c are different, and the prime factors of b and c are different, the $a \times b$ will always be relatively prime with c , because the prime factors will be different.

As can be seen the above argument is mainly the reiteration of the given statement, and hence cannot be acceptable as a mathematical proof for the statement.

By wrong explanation I mean those arguments that included incorrect information. The following argument is an example of this category, which includes an incorrect definition of relatively prime numbers.

If a & c are relatively prime, that means a & c have only 2 factors, 1 & itself. If b & c are relatively prime then b & c only have 2 factors, one & itself. Therefore, it follows a & b are relatively prime sharing the same common factors 1 & itself. Therefore, when multiplying them together, you still get a larger relatively prime number. It is like the GCF. When every composite # can be broken down into a product of primes, if 2 numbers share the same primes, they are factors of each other. They can all be broken down into their common elements.

In Task 6(b) students were required to provide a justification or a counterexample for an unfamiliar statement. All the students acknowledged that the given statement was true. 57 (out of 92) students provided a mathematical proof for the given statement. The following argument illustrates the proofs provided by a student using algebraic notations.

A multiple of 7 can be expressed as 7 times some whole number.

Let 3 multiples of 7 be represented as $7x$, $7y$, and $7z$ where x , y , and $z \in W$

Their sum is $7x + 7y + 7z$, using distributivity of multiplication over addition,

$$7x + 7y + 7z = 7(x + y + z)$$

We know $x + y + z$ is a whole # because the set of whole numbers is closed under addition, so since $x, y, z \in W$, $x + y + z \in W$

Therefore we have expressed the sum of 3 multiples of 7 as a multiple of 7, namely

$$7(x + y + z).$$

The students' arguments are summarized in the Table 8.

Table 8: Summary of students' responses to Task 6(b)

Number of students (n = 92)	Description of the arguments
57	Provided a mathematical proof
7	Empirical verification
7	Using incorrect representation
11	Incorrect manipulation
10	Weak explanation

The result of this task showed only 7.5% of the participants established their argument based on numerical example. A sample of this type of arguments is as follows;

Any numbers that are multiples of 7 when they are added together are still multiples of 7. For example:

$$14 + 21 + 28 = 98 \quad 7 \mid 98$$

$$35 + 63 + 49 = 147 \quad 7 \mid 147$$

$$70 + 77 + 84 = 231 \quad 7 \mid 231$$

An algebraic proof for this task requires a selection of a correct and useful representation. For some of the students who wanted to present an algebraic proof, lack of access to appropriate representation was an obstacle. For instance, the chosen

representations in the following argument, even if they appear correct, do not satisfy the required generality of a proof.

If x is a whole number we say that $7x + 7x + 7x = 21x$

Knowing this we have any variable representing a number it is a multiple of 7.

x can be any whole number.

The other required competence for creating a proof is the ability to manipulate the chosen representation correctly. The result of this task showed almost 12% of the participants could not complete their arguments correctly because of their weak skill in manipulating the algebraic symbols. The following argument illustrates one of the most common mistakes that students had.

$$21 + 28 + 7 = 56 \div 7 = 8$$

$$7m + 7n + 7r = 21m + n + r \text{ which is divisible by 7 because } 7 \mid 21$$

In addition to the misuse of the equality sign, the writer considered $7m + 7n + 7r$ as $(7 + 7 + 7) \times m + n + r$, which is totally wrong.

The rest of the participants provided weak explanation for the statement. As I mentioned above, by weak explanation I mean a verbal explanation that does not include any wrong point but does not answer the entire key “*why*’s” related to the statement either. An example of this kind of argument is as follows:

The multiples of 7 means that 7 is their common factor. This indicates when 3 multiples of 7 add together, the sum must still have 7 as its common factor

because the sum is created by the multiples that have 7 in them. So, when the sum has the common factor of 7, then it is the multiple of 7.

In general the results of this task showed the students' progress in providing a deductive argument for a mathematical statement. Also, in comparison with the results of Task 1, the results of this task showed a dramatic decrease in students' tendency towards accepting an empirical verification as a proof.

Personas in students' dialogues

One of the noteworthy points about the students' dialogues is the selection of the personas. Even though many of the students use the same personas that were introduced in the sample dialogue: EXPLORER and WHYer, still some of the students created their own personas. A list of some of them is as follows:

EXPLORER	WHYer
Teacher	Student
Me1	Me2
Me	My head
Questionee	Questioner
Plato	Aristotle
Simon says	Curious George asks
Einstein	Curious George
Me	Myself
Sara1	Sara2
Tweedle Dee	Tweedle dumb

A	B
Batman	Robin
Text	Travis
Clark Kent	Bruce Wayne
Eros	Ethos
Dumbledore	Harry
Hee	Haw

As can be seen, the selection of the names carries a message about students' interpretation of and the attitude towards the idea of a dialogue, a self-dialogue. For example, Sara1 and Sara2 can represent two aspects of one persona having a dialogue with oneself. The case of Student and Teacher may show the authority that this student places on a teacher as a source of knowledge. In an informal conversation with a student I inquired about the reasons for choosing his personas as Travis and Text. He said: "I consider Travis as myself. Because for answering my questions I usually refer to the textbook, I chose the Text as the other one who is a source of knowledge."

The selection of the names may also reveal some of students' perception of the idea of a dialogue and also sometimes indicates their beliefs about the source of mathematical knowledge, their learning styles, and their attitude towards mathematics.

Summary

In this chapter I summarized students' work on each task, considering the purpose of designing the tasks. The tasks were designed first, to involve the students in the process of writing a proof; second, to see the applicability of using dialogue for this

process; and third, to see the usefulness of a written dialogue for researcher as a tool to investigate students' understanding.

For each task I provided a numerical summary of students' work, exemplifying the key features. Further, I outlined the mediators (such as numbers and symbolic representations) and possible source of obstacles. The repeating features in all the tasks were over-reliance on numerical verifications and inability to perform basic algebraic manipulations. However, comparing students' work in Tasks 1 and 6 (beginning and end of the course), significant improvement is noticeable. In the next chapter I analyse the results through the lens of a communicational framework.

CHAPTER 7: RESULTS AND ANALYSIS ACROSS THE TASKS

In this chapter, I analyse the participants' discourses through the lens of the communicational framework (proposed in Chapter 3). This approach will, I propose, make visible aspects relating to the process of creating proof that might otherwise remain invisible. In particular, I examine the four features of the literate mathematical discourses across the tasks. As mentioned in chapter 3, the four features that distinguish literate mathematical discourses from colloquial mathematical discourses are, (1) the mathematical uses of words (2) the use of mediating tools (3) the routines, and (4) endorsed narratives (Sfard, 2002; Ben-Yehuda et al, 2005). Moreover, I propose a further refinement of these features as applicable to proofs in the elementary number theory discourse.

Mathematical vocabulary

Having a clear understanding of mathematical vocabulary and using it appropriately is very important for participating in a mathematical discourse. One of the features that distinguish a literate mathematical discourse from a colloquial or everyday discourse is its special vocabulary. However, when introducing students to mathematical words, it is important to note that the work never starts from nothing, because students' familiarity with the colloquial use of some of the words might have given them the means for an ad hoc interpretation. Indeed, it happens very often that students who are not

familiar with mathematical meaning of a given word assign to it meaning from their everyday experiences. Consider, for example, the term proof. Students live in a world in which the term proof may mean different things in different contexts. For example, a person can prove that she/he was not at a crime scene because she/he has a proof, such as a witness saying that she/he was somewhere else. As such, we see students often do not recognize the necessity of having a mathematical proof for some statements. As the results showed, at the beginning of the study a majority of participants did not see any need for providing a deductive argument for Task 1. In fact, the presented arguments for Task 1 were mainly subjective, based on the students' intuition and empirical evidences.

Within the communicational framework, according to Vygotsky's theory, there is no split between a word and its meaning. Considering Vygotsky's (1962, cited in Sfard, 2000c, p. 45) idea that "Thought is not merely translated in words; it comes into existence through them", there is no split between what a person says and what she/he thinks. Hence, for analysing a number theory proof from the perspective of a communicational framework it is important to distinguish between different kinds of words that are used in this type of discourse.

In the spirit of the categorization of word use in the arithmetic discourse (Ben-Yehuda, et al, 2005), I propose the following refinement for the keywords in a proof for an elementary number theory proposition: (1) **descriptive words**, (2) **quantifiers**, and (3) **operation words**. This refinement emerged through analysing students' arguments. In what follows, by analysing some typical misuse of these mathematical terminologies in students' discourses, I examine how the students think when they are proving a statement.

Descriptive words

Descriptive words, in this work, refer to nouns and adjectives that are assigned for a number such as a *whole* number or a *prime* number. The data revealed that students had different approaches to introducing and implementing the descriptive words in different tasks depending on the nature of the task. For example, in Tasks 1 or 4(b), when the students were asked to present a proof, they rarely provided a definition for the words related to the statement or its proof. So, we could simply judge students' understanding of the words in terms of their performance because the meaning is embodied in use. However, the environment of a dialogue in Tasks 3 and 5 provided students the opportunity to recall all the meaning, and provided a researcher the opportunity to have access to students' thinking process in this regard.

As mentioned in chapter 6, all the dialogues began with recalling the definitions of the descriptive words that were present in the propositions, such as 'relatively prime numbers' in Tasks 3 and 5(b). The majority of students, however, saw the necessity to situate the concept of relatively prime numbers in relation with others such as whole number, prime number, factor, and the greatest common factor. The following excerpt is part of a dialogue for Task 5(b).

WHYer: Could you tell me what is a **whole number**?

EXPLORER: Sure, a whole number is a member of the set of positive integers and zero (0, 1, 2, 3, 4, and go on). **Integers** are defined as the set of numbers consisting of the **counting numbers** (that is, 1, 2, 3, 4, 5, ...), their opposites (that is **negative numbers**, -1, -2, -3, ...), and zero.

WHYer: And what is **prime number**? Can you show some examples of prime numbers?

EXPLORER: Some examples of **prime numbers** are **1, 3, 5, 7, and 11**. A prime number is a number that cannot be divided evenly by any other number except itself and the number one.

WHYer: How many of these numbers you might have?

EXPLORER: Infinitely many.

WHYer: Fine. But what is a **relatively prime number**?

EXPLORER: Two integers are relatively prime if there is no integer greater than one that divides them both (which means that their greatest common divisor is one).

WHYer: OK. What is **greatest common divisor**?

EXPLORER: The greatest common divisor is the largest factor two numbers have in common.

Recalling all the definitions in the above excerpt shows that the writer did not take any one of the mathematical words for granted. In this situation, a written dialogue provides the students with a big picture of related vocabulary for further use in their argument. It also offers the researcher access to students' possible misuse or misunderstanding of the concepts. As can be seen in this dialogue the student incorrectly considered 1 as a prime number, and excluded 2 from the set of prime numbers.

Below is an excerpt from a dialogue for Task 3. In this excerpt, the word 'prime' acts as a noun rather than an adjective for a number.

A: What are Prime factors?

B: Prime factors are the factors of a number, which are prime. **Prime** is when the number's only factors are 1 and itself.

A: So how do we look at the **primes** of a and b ?

B: a 's primes are p_1, p_2, \dots, p_m and b 's primes are q_1, q_2, \dots, q_n . So.....

In the phrase: ‘**Prime** is when the number’s only factors are 1 and itself’, it appears that for the writers of the dialogue, ‘primes’ are well-known objects, not a description of a number that may have some other characteristics as well.

One of the common problems in the students’ argument is that they have difficulty distinguishing between two words, *digit* and *number*. We saw an example of this confusion in chapter 6, considering Task 1. The following excerpt is another example of the confusion between *digit* and *number*. To answer, “what is the set of odd numbers? Can you show it?” (Task 2, line 4), a group of students wrote:

We can show it by *multiplying any digit by 2 in order to yield an even product and then adding 1 to create an odd sum: $(2m+1)$.*

The other noteworthy point in this excerpt is that the students introduced an even as a product, and an odd as a sum, rather than a *number* with some properties. In other words, we can say that, for this group of students, odd and even are the words used to announce the results of the operations rather than an adjective for a number. Also, in this response, the students did not show the set of odd numbers but the structure of an odd number.

Quantifiers

One of the important characteristics of a proof is its generality. This aspect of a proof is hardly perceived by students in general and by pre-service elementary teachers in

particular. Students' high reliance on the limited number of numerical examples as a proof confirms this claim. The results of this study, especially students' responses to Tasks 1, 2, and 3 acknowledged this tendency. In my opinion, one of the reasons why students do not pay attention to the 'generality' aspect of a proof is that they may not have a good understanding of the meaning of quantifiers in a mathematical context, such as in the structure of propositions. They usually implement the colloquial usage of these words in their argument. For example in the statements of Tasks 1, 3, 4, 5, and 6, the word '*any*' is equivalent to '*every*' that necessitates the consideration of the statement in general. However, some students did not recognize this necessity. A typical misuse of the word '*any*' can be illustrated in the following sample of argument for Task 1.

$$1+3+5=9, 3+5+9=15, 5+7+9=21, 7+9+11=27$$

In the following 4 attempts, 3 odd numbers were added and resulted with an odd number. Therefore, the sum of **any** 3 odd numbers is odd.

On the other hand, in some students' work we can see the superficial overemphasis resulting from inappropriate use of quantifiers, such as misusing the word '*any*' in a dialogue for Task 5(a).

Since $p|ab$ then there was some whole number x such that $px=ab$. If a and b can be expressed as the product of primes in exactly one way

$$a = p_1 p_2 \dots p_m \text{ for any prime number}$$

$$b = q_1 q_2 \dots q_n \text{ for any prime number}$$

In this argument there is a contradiction between two claims that: “ $a \dots$ can be expressed as the product of primes in exactly one way” and “ $a = p_1 p_2 \dots p_m$ for any prime number”.

The misuse of ‘any’ in this argument indicates that the student either did not know the application of the word ‘any’ in a mathematical context or did not have understanding of the unique factorisation of a whole number.

Operation words

Examining the students’ arguments revealed the effect of their colloquial discourse on their writing. In daily life, students deal with the basic mathematical operations: addition, subtraction, multiplication, and division. Hence, they use many of the required words implicitly and ignore them in their utterance. The following example is a typical utterance made by students for Task 1.

Two odd numbers equals an even number. Two even numbers equals an even number. But adding an odd and an even number which is what you get after the first two are added results in an odd number. $1+3+5=9$, $11+9+13=33$

In the first two sentences of the argument, we can see that the student is talking about the sum of two odd numbers without using any word that indicates the operation. The following excerpt, from a dialogue for Task 3, shows another misuse of an operation word: ‘product’.

WHYER: How can you show that the statement is true in general?

EXPLORER: First we need to select a notation that represents a **whole counting number and its products** in general. We will use: $a = p_1 \cdot p_2 \cdot \dots \cdot p_m$. We will use b for the second number.

Here, the use of ‘a whole number and it’s product’ for ‘ $a = p_1 p_2 \dots p_m$ ’ shows that the writer either did not have understanding of a product of numbers or did not have understanding of the prime factorisation of a number. Another misuse of the word ‘product’ is illustrated in the following excerpt from a dialogue for Task 5(b).

Whyer: How does this show that a and c and b and c are relatively prime?

Explorer: a and c are relatively prime because their **product** of primes do not have a common element. The same is true for b and c .

In this excerpt, it seems the student used ‘their product of primes’ for ‘the prime decomposition’ of a and c . This, again, shows the student’s problem with seeing a ‘product’ as a number rather than a list of numbers.

Presenting a proof in the form of mathematical discourse, as it is discussed in chapters 3 and 4, requires disciplined use of words. According to the communicational framework (Sfard, 2002; Ben-Yehuda, 2005), students’ use of words could reveal their thinking process. Having a close look on students’ discourses and examining students’ word use is helpful in recognizing the possible misunderstanding of the words that misleads and weakens students’ arguments.

Mediator use

According to Sfard (2001, p. 28) “Communication either inter-personal or self-oriented (thinking) would not be possible without symbolic tools, with language being

the most prominent among them”. In the case of building a deductive argument, we usually have recourse to certain means that serve as communication mediators.

The results of the study showed the emergence of different types of mediators, which were used by students to communicate their ideas. The mediators used by students were: (1) **numbers**, (2) **verbal explanation**, (3) **algebraic representation**, and (4) **set theory symbols and diagrams**. In many of the cases, students implemented the combination of these mediators in their arguments. Examining the different combinations of mediators in students’ arguments provides us with an insight into their thinking process and the depth of their understanding. In what follows, I describe the use of mediators in students’ arguments across the different tasks.

Numbers

The results of the study showed that numerical examples are the most common type of mediators for pre-service elementary school teachers for communicating their ideas. Indeed, students’ use of numerical examples shows they are very comfortable with using numbers for verification, and the results are convincing and working as a proof for them.

Although the results showed that a significant number of participants in each task established their argument based on the empirical verification of a proposition, they also showed that this tendency decreased through the semester, from 60% for Task 1 to 7.5% for Task 6(b). The reason for this promising result may lie in the flexible environment of a dialogue, which encourages students to use all mediators available to them to create a reasonable argument.

Here I would like to distinguish between two different types of numerical examples that students presented in their arguments: a numerical example that just verifies the proposition and a generic example. To reiterate, by generic example I mean an example that tacitly expresses a process of a proof. Indeed, the use of a generic proof may compensate students' lack of access to an appropriate mediator through recourse to numbers. The following excerpt is part of a dialogue for Task 5(b). In this dialogue, the writer, after recalling the related definitions continued:

Whyer: Can you give me an example?

Explorer: Sure, let's use the numbers 15, 16, and 7. a being 15, b being 16, and c being 7. 15 is thus relatively prime with 7 and 16 is relatively prime to 7.

Whyer: How is 15 and 16 relatively prime to 7?

Explorer: When breaking down 15 the factors are 3 and 5, and when breaking down 16 they are 2^4 . These numbers are not equal to the breakdown of 7 which is 7.

Whyer: I understand! But when a and b multiplied to be ab then how does this work?

Explorer: Well when 15 and 16 are multiplied together it equals 240 but the prime factorisation still remains $2^4 \times 3$ and 5. Thus the prime factors will never equal to 7.

Whyer: Why does this always happen again?

Explorer: This is the fundamental theory of arithmetic.

Whyer: Does this prove your statement?

Explorer: Indeed, it does! You'll remember that I stated that a and c do not have any common prime factors and b and c do not have any common prime factors. Therefore when a and b are multiplied and become ab their prime factors will not be the same as c .

As can be seen, the student drew a conclusion by choosing arbitrary relatively prime numbers and applying the Fundamental Theorem of Arithmetic to their prime decompositions. This argument, even though it is not a mathematical proof, illustrates the logic that the student has in her reasoning.

This kind of an argument, a proof by using a generic example, was mostly presented in Tasks 3 and 5(a), where students were supposed to extend the given proof through writing a dialogue. In these cases students mainly went through the details of the different steps of the proof by using numbers. However, the argument nowhere relied on the properties of the chosen numbers. In such cases we can consider the presented argument as a generic proof, as described by Rowland (2002). For example in the following excerpt from a dialogue for Task 5(a), Sue went through all the details of the proof by using arbitrary numbers that satisfy the assumption of the proposition.

Explorer: “ p divides ab ” means for some whole number x we have $px = ab$.

Whyer: Can you give me an example of it?

Explorer: Sure. For example 5 being a prime number divides 40 or $(4)(10)$ because 5 multiplied with 8 equals 40.

Whyer: Can you explain what you are going to prove?

Explorer: Well lets give an example of ab . If $ab = 40$ and $a = 10$ and $b = 4$, then p can divide either 4 or 10.

Whyer: How do you know this?

Explorer: If 5 is the prime number and it multiplied with x being 8 then the prime factors px will equal the prime factors of ab .

Whyer: What do you mean by prime factors?

Explorer: Well prime factors are the lowest factorisation of a number. For example the number 10, its prime factors are 2, 5 because 2 and 5 cannot be broken down any more than 2.5.

Whyer: How does this then relates to your statement?

Explorer: Well when ab is broken down into its prime factors it will look like this $a = t_1t_2$ and $b = q_1q_2$ then ab equals $t_1t_2q_1q_2$.

Whyer: Ok I understand. But can I have an example?

Explorer: Of course you can. Now lets go back to the old example where a is 10 and b is 4. The prime factorisation of a equals 2.5, and the prime factorisation of b equals 2.2 which can be changed to 2^2 . Now when a is multiplied with b , the product equal 40, yet the prime factors remain the same.

Whyer: 40 can be divided in other ways too, like 8 multiplied with 5. So how can the prime factors stay the same?

Explorer: Lets use your example 5 and 8. 5 is a prime number but 8 needs to be broken down. 8 broken down can equal 2.2.2, where this equals 2^3 . Thus the prime factorisation still remains the same.

Whyer: WOW! Does this always happen?

Explorer: Yes! According to the Fundamental Theorem of Arithmetic, each whole number can be expressed as the product of primes in exactly one way.

The key question in the proof of the proposition is how prime number p emerged in the prime factorisation of a or b . In Sue's dialogue, we see that she first showed this with numbers and then supported her claim with the Fundamental Theorem of Arithmetic.

Not all students who used numbers as a mediator employed the numbers as a generic example. Indeed, several students used numbers just to illustrate the statements in the given proof, as can be seen in Travis' dialogue for Task 5(a).

Text: If $p|ab$ then $ab = px$, this means that because p divides ab then there must be x that multiplies p . $ab = px$

Travis: What?

Text: Lets use some numbers. $P = 2, a = 4, b = 10$. So $p|ab = 2|4 \times 10 = 2|40 = 20$ such that $2 \times 20 = 40$.

Travis: Great! **Why use letters at all, numbers seem so user friendly.**

Indeed, one of the noteworthy findings in the analysis of numerical examples for Task 5(a) was that only a few students considered ab as one number. The majority of students first chose a prime number for p , and then two numbers for a and b such that at least one of them was a multiple of p . In fact, they used the converse of the proposition for choosing a numerical example. The following excerpt from Anna's dialogue for Task 5(a), illustrates this approach.

Aristotle: I'm sorry, but can you please give me an example?

Plato: Okay, Let's say that p is 3, a is 15 and b is 20. ab equals 300. We know that 300 is divisible by 3, therefore 300 is divisible by p . This means that ab is divisible by p . because 3 divides into 300, we know that 3 divides into either 15 or 20 (a or b).

Aristotle: Hang on a second. 3 does not divide **into** 20.

Plato: That does not matter because it divides **into** 15. When 15 is multiplied with 20, 3 will divide into the product. We needed to show that either a is divisible by p or b is divisible by p .

The order of selecting the numbers: " p is 3, a is 15 and b is 20. ab equals 300", does not reflect the original statement of the proposition: " $\text{if } p|ab \text{ then } p|a \text{ or } p|b$ ".

However the selection follows the converse of the statement: 'if $p|a$ or $p|b$ then $p|ab$ '. On the other hand, Sue's choice of numbers indicates the correct interpretation of the proposition: 'if $ab = 40$, $a = 10$ and $b = 4$ then $p (= 5)$ can divide either 4 or 10.'

In Anna's dialogue we can also observe one of the common colloquial uses of the words for divisibility: 'divide into'. As discussed in Zazkis (2000, 2002), this type of colloquial word use is very common in students' discourses; however, it is not a part of formal mathematical terminology.

Verbal explanation

Verbal explanation is a very common type of mediator in the colloquial forms of reasoning. Hence, it could be considered as the basic mediator for students to communicate their ideas. The results showed in most cases that the students implemented a verbal explanation to support and interpret other mediators used in their arguments, such as numbers or different kinds of representation. The results of each task also showed there was a group of students that established their arguments mainly based on verbal explanation.

Most of the students that just used verbal explanation to present their argument, reiterated a given statement, or relied upon other statements that also required a proof. The following example illustrates this kind of presentation for Task 1.

Because there are an odd number of elements the sum will always be odd. If there were an even number of elements of odd numbers the sum would be an even number.

In this argument the student made a conclusion by referring to a more general statement without providing a proof: 'The sum of an odd number of odd numbers is an odd number'. The other common type of argument for Task 1 is as follows:

Because anytime you add 2 odd numbers together you receive an even #.

Anytime you add an even # to an odd # you receive an odd number. The sum of any 3 odd #'s is odd.

This kind of explanation reveals the students' personal understanding; however, it cannot be considered as a complete mathematical proof, because there are still some "why's" that the argument did not answer yet, such as: why is the sum of two odd numbers even? And, why is the sum of an odd number and an even number is an odd number? We, also, can see this kind of arguments in students' responses for Task 4(b) as well:

Yes, if a is a factor of b , then a will also be a factor of any multiple of b .

This argument, which is a verbal interpretation of the statement, may show the writer's insight into the statement, and also that the statement may have been too trivial for her to present any more reasoning for that. In fact, what she wrote is exactly what she had to show.

A combination of a verbal explanation with other mediators was very common in the dialogues, especially in Tasks 3 and 5(a) where students required to expand the given proof. In these cases, the students were asked to expand the given algebraic proofs. For this purpose, students mostly used verbal explanations to interpret the algebraic

presentation of the argument. The following excerpt, which is a part of a dialogue that Kate wrote for Task 5(a), illustrates this use of verbal explanation in the dialogues. In this dialogue, Kate began with recalling the definition of a prime number, factor, composite number, divisibility and some numerical example for each of them. Then she continued:

1. **EXPLORER:** Let a and b be expressed as the product of prime as follows: $a = p_1 \cdot p_2 \cdots p_m$ and $b = q_1 \cdot q_2 \cdots q_n$. **Where a is a whole number and p is a prime number and b is a whole number and q is a prime number.** Therefore, $ab = p_1 \cdot p_2 \cdots p_m \cdot q_1 \cdot q_2 \cdots q_n$ or $px = p_1 \cdot p_2 \cdots p_m \cdot q_1 \cdot q_2 \cdots q_n$.
2. **WHYER:** what is n and m ?
3. **EXPLORER:** It does not matter what letters you use m, n, r, k, \dots n and m are random letters that have be chosen to represent the total number of primes in the set.
4. **WHYER:** Ok, so why does ab equal px ?
5. **EXPLORER:** According to the rule of division both ab and px are equal equations. As $p|ab = x$ and x is some whole number. $p \times x = ab$.
6. **WHYER:** Alright, I understand the connection now. Please continue.
7. **EXPLORER:** **By the Fundamental Theorem of Arithmetic each composite number can be expressed as the product of two primes in exactly one way**, and so $p = p_i$ or q_j for some i, j ($1 \leq i \leq m, 1 \leq j \leq n$).
The symbol means less than or equal to. If $p = p_i$ then $p|a$ and if $p = q_j$ then $p|b$.
8. **EXPLORER:** this statements show that, a composite number is a positive integer, which has factors other than 1 and itself. **Composite numbers are also the result of multiplying two prime numbers.** Also, p_i or q_j has to be less than or equal to m or j in order for the rules of

divisibility to work. Therefore, if $p = p_i$ then $p|a$ and if $p = q_j$ then $p|b$.

9. **WHYER:** Sorry to say but I lost the point of your last interpretation. Can you summarize your argument?
10. **EXPLORER:** Sure we wanted to show that a and b be expressed as the product of primes as follows: $a = p_1 \cdot p_2 \dots p_m$ and $b = q_1 \cdot q_2 \dots q_n$.
Where a is a whole number and p is a prime number and b is a whole number and q is a prime number. In my examples, **I have illustrated that $p|ab$, $p|a$, and $p|b$ when p is a prime number and a and b are products of two prime numbers.** It was also seen that $p \times x$ is equal to $a \times b$. Now we can say that we proved the validity of the proposition.

Kate's verbal explanations, rather than explaining the provided algebraic proof, revealed her misunderstandings of it. In line 1, Kate's verbal explanation of prime decomposition for a and b indicates that she did not distinguish between different prime factors of a by referring to only p as a factor: "where a is a whole number and p is a prime number". Kate's later incorrect interpretation of Fundamental Theorem of Arithmetic (line 7) is another evidence of her poor understanding of algebraic representation of prime decomposition of a number.

Verbal explanations are not necessarily indication of weakness, as the above examples may suggest, but of a preferred form of communication. The results showed that some students established their argument for Task 5(b) mainly based on verbal explanation. In these cases students implemented their common sense to administer the argument. For example:

Sara 2: What is a factor again and how do they differ from numbers?

Sara 1: The factors of a number are all the numbers that multiply together to yield that number.

Sara 2: So none of the factors that multiply together to form a are the same as any of the numbers that multiply together to form c .

Sara 1: that's correct!

Sara 2: and none of the factors that multiply to yield b are the same as the factors that multiply together to create c .

Sara 1: exactly!

Sara 2: So how does that prove that ab is relatively prime to c ?

Sara 1: ab are combined factors of a and b because the factors of the two numbers are combined when they multiply. $10 \times 5 = 50 = 5 \times 2 \times 5 \times 1$.

Sara 2: What does that tell me?

Sara 1: Since factors of ' a ' and the factors of ' b ' have no overlap with the factors of c , this is true for the factors of ab . Therefore ab is relatively prime to c .

Sara presented a correct explanation of the proof without using any symbolic representations. Her argument can be considered as an outline of a mathematical proof of the proposition.

Algebraic representation

Algebraic symbols are visual mediators created specially for the sake of literate mathematical discourse. However, as mentioned in chapter 6, working with algebraic representation was the most challenging part of the tasks for pre-service elementary school teachers. The results showed that none of the students successfully used algebraic symbols as a mediator to communicate their idea in Task 1. Indeed, they could not

provide a correct algebraic representation for an odd number. For example one of the students wrote:

$n+n+n=\text{odd}$, $3+3+3=9$ Same odd number added 3 times together gets an odd number.

$3+5+7=15$, if n is 3 then, $n+(n+2)+(n+4)=\text{odd}$

This response showed that the student extracted the algebraic representation of three odd numbers from the numerical examples, which does not satisfy the required generality of an algebraic representation. In the first case she incorrectly used the same symbol to represent the same three odd numbers and in the second case she again incorrectly used consecutive odd numbers to illustrate the statement. The formal representation of an odd number utilizes the structure of odd numbers, namely, an odd number is an even number plus 1. Some students used this interpretation of the structure of odd numbers in their dialogues; however, the algebraic representation the students provided was not useful for the proof. In particular, student's representations do not reveal the 'evenness' of an even number part of the structure. The following example illustrates a typical argument based on this representation.

$$1+3+5=9, 21+5+35=35$$

An odd number is basically an even number +1. If even numbers are represented by X , the odds would be $X+1$.

$$(X+1) + (X+1) + (X+1) = \text{odd number}$$

because 2 odd equal an even, $(X+1) + (X+1) = 2X+2$

adding another odd would make it odd, $2X+2 + (X+1) = 3X+3$

The chosen representation for odd numbers does not satisfy the generality of the argument. The writer took it for granted that the sum of three even numbers, $3X$, is always an even number; however, this claim has the same complexity as why the sum of any three odd numbers is an odd number. Although, this argument is not mathematically correct, it shows the writer's effort to combine different types of mediators – numbers, verbal explanation, and representations – to support her argument.

In Task 2 the algebraic representation for an odd number was given to students. Nevertheless, the results showed several students had a problem with representing another odd number (line 10) and performing the required manipulation.

Further, the results of Task 4 also revealed that, despite having the definition of divisibility in the dialogue, students had problems stating the definition using the algebraic symbols: “ $a|m$ means for some whole number x we have $ax = m$ ”. In what follows, we can see some of the students' incorrect presentations:

- If $a|m$ then $x \in w=w$ for some whole number x
- If $a|m$ then $a|m = a.x$ for some whole number m
- If $a|m$ then $a.x = ax$ for some whole number m

The inappropriate use of symbols such as \in in the definitions reveals the students' lack of understanding.

In Tasks 3 and 5(a), students were exposed to a proof for the given propositions through the use of algebraic symbols, and were asked to expand the given proof by writing a dialogue. Some of the written dialogues were mainly explanations for the

symbols presented in the proof. This approach shows students did not feel comfortable using the symbols and each of them needed an explanation. Also, the provided explanations revealed the misunderstandings that students have in regards to algebraic symbols and the concepts.

The following excerpt depicts the writer's difficulty in implementing algebraic notations into the definition of a prime number and prime decomposition.

Curious George: So what are $i, j, m,$ and n anyways?

Einstein: Well, remember the prime factors of a and b ? they were as follows:

$$a = p_1 p_2 \dots p_m \text{ and } b = q_1 q_2 \dots q_n$$

m is the greatest prime factor of a , and n is the greatest prime factor of b .

p_i is a prime factor of p that is also a prime factor of a .

q_j is a prime factor of p that is also a prime factor of b .

Curious George: Oh, Okay. So if p has the same prime factor as a ($p = p_i$) then

$p|a$, but if p has the same prime factor as b ($p = q_j$) then $p|b$?

Einstein: right.

Curious George: I understand a lot better now, thanks!

Einstein: no problem ... me too!

It is interesting to note that the writer began her dialogue with the correct verbal definition of a prime number. However, the subsequent conversation shows the student's difficulty with interpreting the meaning of symbols.

The result of Task 5(b) shows that the majority of the participants used the algebraic notations following an empirical verification. These students used an algebraic representation for prime decomposition of a whole number. However, at the time, their poor background in using algebraic notations led almost half of them to a superficial presentation. For example, in the following dialogue the student, after an empirical verification, continues:

Aye: So, are we done?

Myself: No, because we have not proven the proposition yet, we have just seen that one example works. To prove the example, we have to go back to using letters, which represent all possibilities.

Aye: sounds good.

Myself: To start, we will express a , b , and c as a product of their primes such that $a=p_1p_2\dots p_m$, $b=q_1q_2\dots q_n$ and $c=r_1r_2\dots r_q$. Therefore $ab=p_1q_1p_2q_2\dots p_mq_n$. Since the Fundamental Theorem of Arithmetic states that each composite number can be expressed as the product of primes in exactly one way and we know that $ab=pmqn$ and $c=rq$, we can state that ab and c are relatively prime.

Aye: And that was all we needed for proof?

Myself: Yes!

The use of comma in the prime decomposition of a and b may indicate that the writer perceived " $p_1p_2\dots p_m$ " as a set of prime numbers rather than a product. And for making a conclusion she did not even present all the factors but just compared " $pmqn$ " with " rq ". (The use of indices, in a form of a coefficient, such as ' pm ', is reproduced as was shown in the student's work).

Although the results confirmed the general weakness in students' use of algebraic representation, it also revealed good progress in some students' performance. As the results showed (in chapter 6) more than half of the participants were able to create a reasonable argument for Task 5(b) by using the environment of dialogue. The following excerpt from a dialogue for Task 5(b) illustrates these presentations.

WHYer: Could you please repeat and summarize what you have done?

EXPLORERE: sure, we set out to prove that for the whole numbers, a , b , and c , if a and c are relatively prime, and b and c are relatively prime then ab and c are relatively prime.

We considered that:

$a = p_1 \times p_2 \times p_3 \dots p_m$, where $p_1, p_2, p_3 \dots p_m$ are prime factors of a

$b = q_1 \times q_2 \times q_3 \dots q_n$, Where $q_1, q_2, q_3 \dots q_n$, are prime factors of b

$c = r_1 \times r_2 \times r_3 \dots r_t$, where $r_1, r_2, r_3 \dots r_t$, are prime factors of c

and noted that it was not necessary that a and b were relatively prime for the purpose of our proposition.

We determined that a and c were relatively prime because they shared no common prime factors and b and c were relatively prime because they shared no common prime factors either.

We also determined that if $a = p_1 \times p_2 \times p_3 \dots p_m$ and $b = q_1 \times q_2 \times q_3 \dots q_n$, then $ab = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$.

We then showed that if $ab = p_1 \times p_2 \times p_3 \dots p_m \times q_1 \times q_2 \times q_3 \dots q_n$ and $c = r_1 \times r_2 \times r_3 \dots r_t$ then ab and c were also relatively prime because they shared no common prime factors.

The Fundamental Theorem of Arithmetic states that a composite number can be expressed as the product of prime factors in only one way (except for the order of

the factors), which means there is no other way that a , b , c , or ab can be expressed other than what we have stated and since we have clearly illustrated that neither a and c nor b and c nor ab and c have any prime factors in common, we can conclude that if a and c are relatively prime and b and c are relatively prime, then ab and c are also relatively prime.

WHYer: Can we now consider this as proof for the given statement?

EXPLORER: Yes, we can now say that we have proven the validity of the statement that for any three whole numbers a , b , and c if a and c are relatively prime, b and c are relatively prime then ab and c are relatively prime.

Set theory symbols and diagrams

The analysis of the dialogues showed that some of the participants selected set theory language, symbols, and diagrams as mediators in their arguments. This kind of mediators were mainly seen in the dialogues for Tasks 1, 3 and 5, where students were dealing with the set of prime factors of a whole number. A sample of this approach can be seen in the following excerpts from a dialogue for Task 3. In this dialogue the writers refers to “set a ”, which is not commonly used in mathematical language.

Explorer: Prime #'s are those that have exactly 2 factors: 1 and itself. Relatively prime means that none of the numbers in set a are found in set b .

Whyer: What is set a and set b ?

Explorer: Set a is all the prime numbers that represent a . set b is all the prime numbers that represent b .

Whyer: can you give me an example?

Explorer: Set a includes p_1, p_2, \dots, p_m and set b includes q_1, q_2, \dots, q_n . Since a & b are relatively prime, they don't have any common prime factors.

Here, the writer defined ‘set a ’ as “all the prime numbers that represent a ”, but again the “prime numbers that represent a ” is also not commonly used in mathematical language. This discourse, although not mathematically conventional, depicts the writer’s correct perception of the relatively prime numbers, through indicating that ‘set a ’ and ‘set b ’ do not have any common prime factors. The following excerpt is part of a dialogue that Sue provided for Task 5.

Figure 5: Use of set theory symbols as a mediator for Task 5(b)

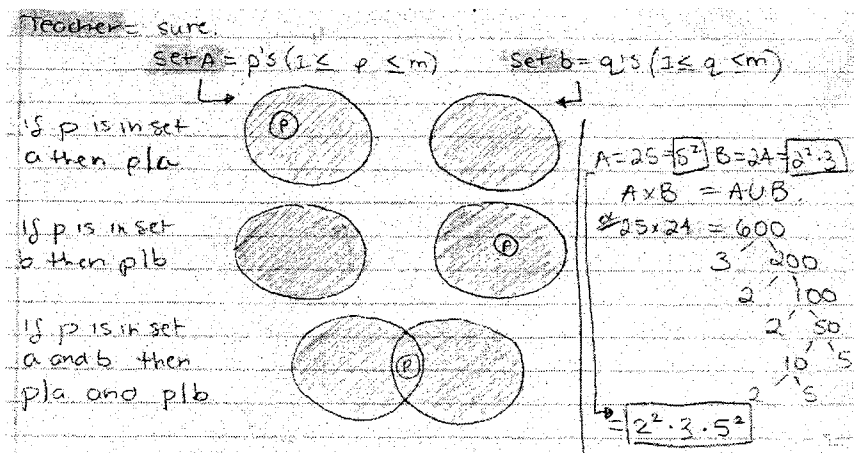
Student: ok. so a and c , and b and c are relatively prime.
 How can we show this?
 Teacher: This can be shown using variables. where:
 $A = \{p_1, p_2, p_3, \dots, p_x\}$ $C = \{q_1, q_2, q_3, \dots, q_n\}$
 All the prime factors of A are distinct from the factors of C .
 In other words, does not intersect c . [$a \nabla c$] or $p \neq q$
 $B = \{r_1, r_2, r_3, \dots, r_z\}$ Where all the factors of B are distinct
 from C . [$B \nabla c$] or, $r \neq q$

Figure 5 shows Sue’s correct understanding of the concept; however, the use of symbols is not mathematically conventional. As we can see, due to lack of access to correct symbols to represent two disjoint sets, she invented a ∇ symbol – “does not intersect” – incorporating the available ones. It seems that the words were not sufficient for this student to express her idea, and she needed symbols to further support it. Another possible interpretation may stem from students’ tendency to satisfy the ritualistic aspect of a proof by using more symbols.

Diagrams are a very effective means for communication. Several students included diagrams to give a clear picture of their argument. Figure 6, which is a part of a dialogue that Pari provided for Task 5(a), illustrates the use of a diagram in students' arguments.

In this excerpt, Pari used Venn diagrams to represent the sets of prime factors of a and b . She considered very well, the three possible conditions that the prime factor, p , may have in relation with these two sets. This visual presentation may illustrate the insight that Pari had into the proof. Even though she did not present it in a conventional mathematical form, it can be a good indication of her understanding and logical reasoning.

Figure 6: Use of diagram as a mediator for Task 5(a)



Overall, we observed that students used different types of mediators to communicate their ideas. Even though the algebraic representations were the most problematic for them, the flexible environment of dialogue invited the students to implement the other accessible mediators to communicate their ideas.

Routines

In a communicational framework the term ‘routine’ refers to repetitive patterns that can be considered in discursive activities. However, the routines with which we react to prove a proposition may vary from one mathematical discourse to another. For example, an arithmetic discourse varies from a geometrical discourse. In this study, I examined the possible repetitive patterns in a proof for elementary number theory propositions. Based on the analysis of students’ work, I categorized the emerged rules in three groups: (1) **operation routines**, (2) **clarification routines**, and (3) **semantic routines**. In what follows I explain each of these routines in students’ arguments to see how these routines appear in their work and how the misuse of each of them may weaken students’ arguments.

Operational routines

By operational routines I mean all the repetitive patterns that students use for manipulating numbers or different types of representations. The results of the study showed that the most challenging part of the Tasks for students was working with the algebraic representations, especially their manipulations. Some of these problems point to the general carelessness that we usually see in students’ performance. A sample of this kind of problem can be seen in students’ performance for Task 2, where they multiplied two binomials incorrectly:

$$(2m + 1) \times (2b + 1) = 2mb + 2m + 2b + 1$$

There were, also, some deeper problems in students’ mathematical discourses that were caused by unconscious use of some meta-rules. One of the most common problems

observed in students' arguments was in presenting the prime decomposition of a whole number. Despite having the prime decomposition of a whole number in the given proof in Tasks 3 and 5(a), many of the students used commas between the factors when they rewrote the prime decomposition in their dialogue. This problem might have originated from presenting the prime factors of a whole number in a form of a list of numbers. As a result, some students may unconsciously transfer this representation to the prime decomposition of a whole number as a list of prime numbers, separated by commas. The other common problem, related to misuse of symbols for operation can be seen in the definition of divisibility. The following excerpt from Betty's dialogue for Task 5(a), illustrates these and some other problems.

1. **WHYer:** Where did you get x from? What does it represent?
2. **EXPLORER:** From my definition of multiplication and division which I previously discussed, $a|b=c$ and this equation can be seen as $c \times a=b$ and $c \times b=a$. In relation to $p|ab=x$, and x is some whole number.
 $p \times x=ab$.
3. **WHYer:** Further explain a and b .
4. **EXPLORERE:** Moreover, let a and b be expressed as the product of primes as follows: $a=p_1, p_2 \dots p_m$ and $b=q_1, q_2 \dots q_n$. In this equation, a and b are any whole number and it equals the products of primes. And m and n stand for the amount of prime numbers. For example, if $a=84$. $84 = 4 \times 21 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$. I have just expressed 84 as the product of primes. $P_1=2^2$, $p_2=3$, $p_3=7$. In the same way as the proposition, a and b is also a whole number which is also expressed as the product of primes. Different letters are used in the equation to symbolize different numbers.
5. **WHYer:** What happens when you multiply a with b ?

6. **EXPLORER:** keeping in mind what I just explained, $ab = p_1 p_2 \dots p_m$, $q_1 q_2 \dots q_n$ or $px = p_1 p_2 \dots p_m q_1 q_2 \dots q_n$. When you multiply a with b or p with x , the product of primes is represented by multiplying $p_1 p_2 \dots p_m$ and $q_1 q_2 \dots q_n$. And m and n stand for the amount of prime numbers. For example if $a = 84 = 2^2 \times 3 \times 7$ and $b = 12 = 2 \times 2 \times 3 = 2^2 \times 3$. $ab = 2^4 \times 3^2 \times 7$. Remember that $a_n \times a_k = a_{n+k}$ which is a rule for exponents.
7. **WHYer:** Alright, keep explaining please. Why does ab equals px ?
8. **EXPLORER:** Both ab and px are equal equations as a result of a fundamental rule of division. Since $p|ab = x$ and x is some whole number. $p \times x = ab$.
9. **WHYer:** Ok, I can see why now. Go on.
10. **EXPLORER:** I think it is important to illustrate the Fundamental Theorem of Arithmetic at this point. **By the Fundamental Theorem of Arithmetic, each composite number can be expressed as the product of two prime in exactly one way**, and so $p = p_i$ or $p = q_j$ for some i and j ($1 \leq i \leq m$, $1 \leq j \leq n$). i and j stand for the product of primes. The sign \leq means equal and or less than. Moreover, if $p = p_i$ then $p|a$ and if $p = q_j$ then $p|b$.

First, Betty incorrectly used the divisibility notation ' $a|b$ ' as ' $a|b = c$ ' (line 2).

The source of this confusion might stem from the use of a common division symbol as in

' $b/a = c$ ' or ' $\frac{b}{a} = c$ '. The more substantial mistake comes from Betty's interpretation of

the operation: ' $a|b = c$ which means $c \times a = b$ and $c \times b = a$ '. Further, the idea of

considering two primes (line 10) might be a superficial inference from using p_i 's and q_j 's

as a representation for the prime decompositions of a and b .

Another example of an incorrect reliance on the appearance of the symbolic representations can be seen in the following excerpt from Roshe's dialogue for Task 5(a).

A: So $p = p_i$ or q_j for some i, j ($1 \leq i \leq m, 1 \leq j \leq n$).

B: Can you explain this more?

A: Sure, so basically you can look at it this way. We know that $ab = px$, and $a = p_1 p_2 \dots p_m$ and $b = q_1 q_2 \dots q_n$; if we know that $ab = px$ then we also know that $p = p_1 p_2 \dots p_m$ and $x = q_1 q_2 \dots q_n$.

It seems that for Roshe the symbols do not carry any meanings. She simply assigned a to p and b to x from the operation $ab = px$ without any attention to what the symbols stood for. Her assignment of $p = p_1 p_2 \dots p_m$ shows Roshe's negligence of the primeness of p , that is the key point of the proposition.

A few of the students used set theory symbols as mediators for presenting their arguments for Task 5(b). At the same time, these arguments illustrate the role of meta-rules, discussed in chapter 3, that led students to misuse some operational rules or symbols. The following excerpt from Sahar's dialogue depicts this phenomenon.

1. **A:** So we need to find the relationship between ab and c .
2. **A:** So we know that $a = p_1 p_2 \dots p_m, b = q_1 q_2 \dots q_n, c = x_1 x_2 \dots x_k$
3. when $\{p_1 p_2 \dots p_m\} \cap \{x_1 x_2 \dots x_k\} = 1$
4. and $\{q_1 q_2 \dots q_n\} \cap \{x_1 x_2 \dots x_k\} = 1$
5. we know that a and c, b and c are relatively prime
6. $ab = p_1 p_2 \dots p_m q_1 q_2 \dots q_n, c = x_1 x_2 \dots x_k$

7. so $ab = a \cup b$

8. $a \cup b = a + b - a \cap b$

9. $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$

10. since $a \cap c = \{p_1 p_2 \dots p_m\} \cap \{x_1 x_2 \dots x_k\} = 1$

11. and $b \cap c = \{q_1 q_2 \dots q_n\} \cap \{x_1 x_2 \dots x_k\} = 1$

12. then $(a \cap c) \cup (b \cap c) = \{1\} \cup \{1\} = 1$

13. meaning

$$(a \cup b) \cap c = ab \cap c = \{p_1 p_2 \dots p_m q_1 q_2 \dots q_n\} \cap \{x_1 x_2 \dots x_k\} = 1$$

14. Thus ab and c are relatively prime!

Sahar incorrectly implemented the rules of algebraic operations through the set theory symbols. This misunderstanding may have originated from the way the basic arithmetic operations were introduced to students from set theory perspective⁷ (Musser, et al, 2003). In lines 3 and 4, Sahar unconventionally matched the set theory representation of two disjoint sets to the concept of relatively prime numbers by using ‘1’ as the only common factor of two numbers instead of the empty set, ‘ ϕ ’. In addition, in line 7, she related the multiplication of two whole numbers to the union of the sets of their prime factors, which is not correct in general. Also, in line 8, Sahar misused the rule ‘ $n(a \cup b) = n(a) + n(b) - n(a \cap b)$ ’ for ‘ $a \cup b = a + b - a \cap b$ ’ without any application of this rule in her argument. In line 9, she again combined her interpretations of the union and the intersection of sets, as a multiplication of two whole numbers and a tool for showing two

⁷ For example the addition of whole numbers defined as follows:
 “Let a and b be any two whole numbers. If A and B are disjoint set with $a=n(A)$ and $b=n(B)$, then $a+b=n(A \cup B)$ ” (Musser, et al, 2003, p.96).

relatively prime numbers, respectively, with the correct set theory rule ' $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$ '. Following this rule allowed Sahar to make her desired conclusion that was consistent with her invented assumptions. Despite all the flaws, this argument reflects the student's sound thinking process.

Clarification routines

By clarification routines I mean all the meta-rules that a person may consciously or unconsciously apply to make a mathematical statement or a proof clear for oneself. These meta-rules may include recalling or searching for the meaning and definition of words or concepts, and also all the activity that someone may do to make an idea clear for oneself, such as making a conjecture or finding a more sensible evidence for that.

In a process of creating a proof, through clarification routines, students recognize definitions and statements to implement into their arguments. Indeed, when students give an overview of a proof based on their intuition they are clarifying the idea of a proof without providing all the required steps, and most of the time they consider what they obtained through clarification as a proof. In such cases, even though the provided argument cannot be considered as a mathematical proof, it depicts the writer's perception. The following argument is the proof that one of the participants provided for Task 4(b).

If $a|b$, then b is a multiple of a . Therefore, any number for which b is a factor also has a for a factor. Thus, would be divisible by a . For example, let $a = 13$, $b = 26$ and $c = 156$. $13|26$ because $26 \div 2 = 13$ and $26|156$ because $156 \div 6 = 26$ therefore $156 \div 13 = 12$, as such, $13|156$.

This argument shows that the student has clear idea of the proposition; however, in her argument she did not provide any answer for some possible ‘whys’ such as, why “any number for which b is a factor also has a for a factor”?

When students provide a numerical example for a given statement, we can also say that they are using numbers to clarify the meaning of the statement. The following dialogue exemplifies such meta-rules that lead the student in her clarification procedure for Task 5(a).

1. **Me:** So the proposition is that if p is a prime number and $p|ab$ then $p|a$ or $p|b$.
2. **Myself:** What’s a prime number?
3. **Me:** A counting number with exactly two different factors: itself and one.
4. **Myself:** And factors are ...
5. **Me:** A number’s factors are all the other numbers that evenly divide that number. For example, 3 is a factor of 9 because $3 + 3 + 3 = 9$.
6. **Myself:** Right.
7. **Me:** OK, let’s try this proposition assign actual numbers. Let’s say $p = 3$ (because it’s prime number) $a = 6$, $b = 4$. $6 \times 4 = 24$ and 3 divides 24, which means so far we are OK. Now 3 does not divide 4, but 3 divides 6, and the proposition states that p does not have to divide both a and b , it just has to divide one of them, so this example appears to work.
8. **Myself:** Yay! Now what?
9. **Me:** Now, let’s try to make some sense out of this written proof. The first step says, “since $p|ab$ then $ab = px$ for some whole number x .”
10. **Myself:** Zah?

11. **Me:** It's not as confusing as it sounds. Remember in our example $ab = 24$, all this statement is saying that p (which in our example equals 3) times some number (which they distinguish as x) is going to equal ab (24). In our example $x = 8$ because $3 \times 8 = 24$.
12. **Myself:** Gotcha.
13. **Me:** Alright, next step. "Let a and b be expressed as the product of primes as follow: $a = p_1, p_2 \dots p_m$ and $b = q_1, q_2 \dots q_n$ or $px = p_1, p_2 \dots p_m q_1, q_2 \dots q_m$."
14. **Myself:** I think I get that. It is basically states that we should think of a and b as their prime components, and add up the primes until you reach the number. $a = 6$, so listed in its prime components, it would look like 3, 6, 9, 12, 15, 18, 24. $b = 4$, so listed in its prime components, it would look like 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24.
15. **Me:** Did you notice that the primes for a and the number we are dividing by are both represented by the letter p ?
16. **Myself:** No I didn't, but that would make sense because both represent the same number, which in our example is 3.

Since meta-rules are usually invisible, the written dialogue may provide us with a partial picture of it. As can be seen, in the above dialogue the student tried to explain the relations and notations by substituting numbers. However, her explanation in line 14 reveals her incorrect perception about the given representation for prime decomposition of numbers.

Semantic routines

Semantic routines refer to all the meta-rules (explained in chapter 3) that administer the structure of mathematical discourses. Logic rules can be considered as a

part of semantic routines. The results of this study show that the prevailing rules that govern the students' arguments was their common sense.

One of the common logical issues in students' arguments is that they usually consider $(P \Rightarrow Q)$, and $(Q \Rightarrow P)$ equivalent. This logical confusion was observed mainly in students' performance in Task 4, where they were asked about the validity of the converse of the statement (line 14, p. 78). 60 out of 93 students gave an incorrect answer because they could not see the statement conversely. The following excerpt shows a typical answer provided by students (line 17)

$$a=5, m=20, n=5 \quad 5|20, 5|5, 5|(20+5)=5|25$$

Yes because if m and n are each divisible by a then you add them together their sum will also be divisible by 5. as long as $a, m, n \in W$.

The other common logical confusion is when students take $(P \Rightarrow Q)$ equivalent to $(\neg P \Rightarrow \neg Q)$. For example in the following argument (for Task 4(b)), the student provided an example in which a does not divide b and b does not divide c ($\neg P$) but a divide c (Q). Through implicit acceptance of the equivalency of $(P \Rightarrow Q)$ and $(\neg P \Rightarrow \neg Q)$ the students considered her example as a counterexample for the statement and rejected the validity of the statement on that understanding.

The statement is not necessarily true, because if $a=2, b=3, c=4$ then a doesn't divide b , and b doesn't divide c but a divides c .

The other observed issue relates to colloquial use of 'or' in students' mathematical discourse. As we know 'or' is always used inclusively in mathematics,

while 'or' in colloquial discourse is usually used exclusively. Under the influence of their colloquial discourse, the over emphasis on an exclusive aspect of 'or' was observed in some students' arguments. The following excerpt from one of the student's dialogue (for Task 5(a)) illustrates this misunderstanding.

Explorer: To sum up, we have shown that if p divides ab ($5|10 \times 4$) then p divides a or b , but not both. In the case of our examples $p|a$ ($5|10$) not b ($5|4$). So although when a & b are multiplied, their sum is divisible by p , when they are on their own, p will divide one or the other, not both.

As it is mentioned in chapter 3, the characteristic of the literate mathematical discourse is that its routines are particularly strict and rigorous (Ben-Yehuda et al, 2005). The results of the study showed how the lack of this characteristic in students discourses weaken or mislead their whole process of proving. The analysis of students' arguments revealed their poor skill in discursive routines related to manipulation of symbolic representations. The detailed presentation of students' argument in the form of a dialogue disclosed how the rules of their colloquial discourses administer their arguments. In other words, the main problem with some students' arguments was that they implicitly (or unconsciously) implemented the rules of their colloquial discourse in creating or interpreting a mathematical proof.

Endorsed narratives

The endorsed narratives are the production of discursive activities and mathematical routines (Ben-Yehuda et al, 2005). In other words, the endorsed narratives are narratives that are accepted by mathematical communities and are labelled as true.

Therefore, we can say that producing endorsed narratives is the main aim of a mathematical proof. On the other hand, the process of proving can be also considered as producing a chain of endorsed narratives that leads the argument to the desired conclusion. Indeed, all the definitions and axioms, and pre-proven propositions can be also considered as endorsed narratives that are employed for creating a formal mathematical proof.

The inquiring nature of a dialogue implicitly motivated students to create endorsed narratives in each step of their argument. The presented endorsed narratives in the dialogues were mainly the results of discursive activities such as (1) *manipulation* of numbers or different form of representations, (2) *memorization* of definitions, rules, or pre-proved proposition, and (3) *composition* of an argument in the form of a proof.

The results of this study indicate that this aspect of proving, using endorsed narratives for supporting each step of a proof, has been missing in some students' work. In several dialogues, for example, it was observed that students made a claim about the prime decomposition of whole numbers without referring to the Fundamental Theorem of Arithmetic. The following excerpt is part of a dialogue that one of the students created for Task 5(b).

Whyer: I get it. So now we can express a , b , and c as their product of their primes right?

Explorer: Right. Let a , b , and c be expressed as their products of primes as follows: $a = p_1 p_2 p_3 \dots p_l$, $b = q_1 q_2 q_3 \dots q_f$, and $c = r_1 r_2 r_3 \dots r_y$

Whyer: What does this tell us?

Explorer: This shows that a and c are relatively prime because their products of primes do not have any common primes. It also shows that b and c are relatively prime because they too do not have any common primes.

Whyer: But, then how do we know that ab and c are relatively prime?

Explorer: We know this because $ab = (p_1 p_2 p_3 \dots p_t)(q_1 q_2 q_3 \dots q_r)$ and these do not have any common primes with $c, (r_1 r_2 r_3 \dots r_y)$. Therefore ab and c are also relatively prime because their greatest common factor is 1.

Whyer: So can we now say that we proved the statement?

Explorer: Yes.

This student did not see the necessity to support her argument explicitly by presenting required endorsed narratives. In this dialogue the student made the conclusion without any reference to the Fundamental Theorem of Arithmetic, and simply based on the appearance of the selected letters for prime factorisation of a , b , and c . However, in mathematical argument we know that different letters necessarily do not refer to different numbers unless we indicate it in our argument.

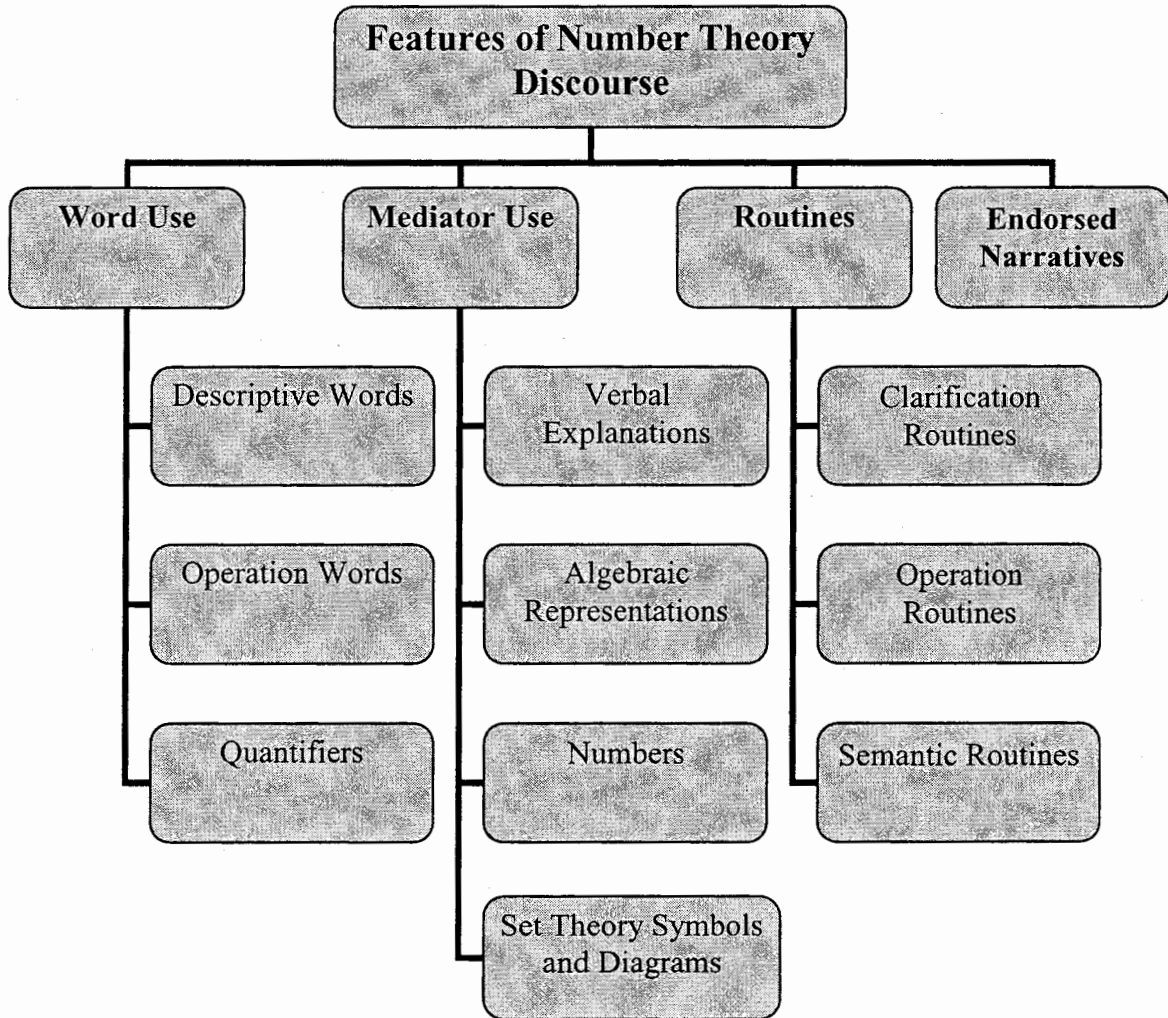
Summary

In this chapter the communicational framework provided us with a tool for analysing students' mathematical discourses. Through the analysis of students' dialogues a further refinement of the communicational framework emerged: the word use has been broken into (1) descriptive words, (2) quantifiers, and (3) operation words; the mediators – (1) numbers, (2) verbal explanation, (3) algebraic representation, and (4) set theory symbols and diagrams; routines – (1) operation routines, (2) clarification routines, and (3) semantic routines; and endorsed narratives resulted from discursive activities such as (1) *manipulation* of numbers or different form of representations, (2) *memorization* of

definitions, rules, or pre-proved proposition, and (3) *composition* of an argument in the form of a proof. The profile of this refinement is presented in Figure 7.

The examination of the four features of a literate mathematical discourse in students' work gave the researcher an insight into the possible obstacles to students' endeavour of creating a proof. The results showed that the students' colloquial use of words might weaken or mislead their arguments. Even though students had poor access to formal mathematical representations for expressing their argument, students' ability to communicate their reasoning and logic through the alternative mediators should be acknowledged. It was also observed that the manipulation of different types of representation was the main challenge for students. The construction of the written dialogues within this classroom context can be considered as a series of endorsed narratives. However, inadequate understanding of mathematical words, mediators, and routines prevented the majority of students from producing mathematically valid endorsed narratives.

Figure 7: Profile of Number Theory Discourse



CHAPTER 8: DISCUSSION AND CONCLUSION

“The birth of mathematical proof is essentially the result of the willingness of some philosophers to reject mere observation and pragmatism, to break off perception (le monde sensible), to base knowledge and truth on reason.”

(Balacheff, 1991, p. 187)

My research was aimed at extending the views and insights about the difficulties that pre-service elementary school teachers experience in dealing with the notion of mathematical proof. I have done so by analysing students' discourses when they attempted to interpret or create proofs for some propositions related to elementary number theory.

The communicational approach to learning is the theoretical perspective that I adopted to investigate the difficulties students experience in generating proofs. According to the communicational approach to cognition, thinking is a special case of the activity of communication, and learning mathematics is an initiation into a certain type of discourse, which is called literate mathematical discourse (Sfard, 2001; Sfard & Cole, 2002). From this theoretical perspective, the four features that render literate mathematical discourse its distinctive identity are its special vocabulary, mediating tools, routines, and endorsed narratives (Sfard, 2002; Ben-Yehuda, et. al., 2005).

In this study, I have introduced the notion of dialogue as a tool for involving pre-service elementary school teachers in the process of creating a proof. Based on the idea

that thinking can be considered as an act of communication that one has with oneself, I introduced dialogue as a self-dialogue or a conversation that a person has with oneself while she/he is thinking. I encouraged students to write a dialogue while they were thinking in order to interpret or create a proof. For this purpose, I designed six tasks. The progression of the tasks followed the gradual involvement of students in the process of proving, from public to private, and moving from imitation to creating a proof individually. I examined the collected data to answer the following research questions:

1. What difficulties do pre-service elementary school teachers experience in writing and interpreting proofs for propositions related to elementary number theory?
2. What are the outcomes of students' activity of creating a dialogue?
 - (a) Does it facilitate students' participation in the process of proving?
 - (b) Does it reveal their difficulties in this process?
3. Can communicational approach to cognition serve as a tool for researchers in recognizing and identifying factors that impede pre-service elementary school teachers' participation in the process of creating and interpreting proofs?

The results revealed that the main difficulty that students experienced in creating a proof is that they did not know how to communicate mathematically. In other words, they did not have the required competences to be an active participant in mathematical discourse. The students' communication was highly influenced by their colloquial discourse, and their arguments were mainly subjective.

In mathematics we say that “logically establishing the truth” can be considered as a proof. This definition might raise the question what we mean by logic. Many pre-service elementary school teachers use the informal logic of their colloquial discourse to prove a mathematics statement. This is a point where problems might originate. In a colloquial discourse, we may establish a truth based on limited observation. However, in mathematics the universality of truth is needed. This study aimed at moving students from colloquial to literate mathematical discourse.

At the beginning of the study (based on the results of tasks 1, 2, and 3), some confirming examples were sufficient for the majority of students to be convinced about the truth of the statement. For these students, their presented argument, based on empirical verification, was reasonable because they evaluated it in terms of their own logic. And, we cannot ignore the fact that the visual or inductive argument provides students with an insight into the meaning of a proposition. Nonetheless, it cannot provide a mathematical persuasion.

According to Balacheff (1991), an epistemological obstacle is “a genuine piece of knowledge which resists to the construction of the new one, but such that the overcoming of this resistance is part of a full understanding of the new knowledge” (p. 191). From this point of view, a high reliance on numerical example or empirical verification is an epistemological obstacle for pre-service elementary school teachers. In fact, this is one of the main obstacles that prevent students from understanding the essence of the notion of proof.

The lack of understanding and proficiency in using and manipulating algebraic symbols and representations are also the root of many difficulties that pre-service

elementary school teachers experienced in writing and interpreting proof for propositions related to elementary number theory. This problem returns to their poor background in mathematics. This poor background formed the mathematical world view or belief system of pre-service elementary school teachers. The belief system, formed through this poor background, may have weakened students' self-confidence and motivation for involvement in the process of creating a mathematical proof (Schoenfeld, 1985).

The results of the study showed that writing a dialogue was a useful tool for involving students in the process of proving and changing their attitude about what a proof is, as well as their own capability of producing a proof. The environment of dialogue involved students in the process of questioning about all the details related to a given statement, which is the best support for writing a mathematical proof. The atmosphere of dialogue provided students not only with a space to practice questioning, but also with a space for reflection, which improves mathematical thinking.

I found that the common aspects of all the students' written dialogues were recalling the related definitions, and making sense of the idea of a given proposition by presenting some confirming examples. Even though none of these two are part of a proof, both of them are very useful preliminary steps for initiating a proof. The result of the study showed the decrease of mere reliance on empirical verification. One of the common questions in all the dialogues for Task 5(b) was (in different phrases): "does a numerical example prove the proposition?" Almost all of the participants answered 'no', which is a promising result of the study. After this question, 92% of the participants continued their argument in its general form by using different types of mediators. The results showed that using dialogue was useful for changing students' attitudes toward empirical proof

that was a kind of epistemological obstacle in their understanding of proof. In fact, it provided the students with an opportunity to face the conflict of whether some limited number of examples can guarantee the validity of a statement in general, and mediate colloquial understanding of proof.

The atmosphere of dialogue encouraged students to create an honest conversation, that is, to answer the most basic and natural questions that came to their mind without any concern about judgment, and with ample space and time. Hence, created dialogues revealed most of the difficulties that students may experience in the process of creating or understanding a proof. In other words, the written dialogues revealed the details of misunderstandings that misled students' arguments and therefore, provided me with rich source of students' discourses. From this perspective, the written dialogue can also be considered as a research tool for collecting data related to students' discourses and students thinking process.

As I mentioned earlier, poor access to appropriate mediators, especially in the form of an algebraic representation and poor skills in discursive routines, are still the main challenges for pre-service elementary school teachers to present their argument mathematically. Nevertheless, I would like to acknowledge students' ability to implement, and in some cases even invent different kinds of mediators to present and communicate their idea. Indeed, dialogues provided the students with a flexible environment where they could cultivate their reasoning in the form of a literate mathematical discourse.

According to the communicational approach to cognition, "we can define learning as the process of changing one's discursive way in a certain well-defined manner" (Sfard,

2002, p. 26). The new discourse may facilitate communication and participation in mathematical processes including proving a proposition. On the other hand, we know that for making any change we need to have a clear idea about the current condition.

The communicational framework with its explanatory power helped me to organize the interrelated phenomena that incorporate an understanding and creating a proof. Indeed, examining the four features of literate mathematical discourse in students' arguments revealed some roots of many difficulties that pre-service elementary school teachers may experience.

Having a clear understanding of the mathematical vocabulary and using it appropriately is the very important requirement for being a legitimate active participant in a mathematical discourse. The study demonstrated that explaining mathematical words is a common part of the dialogues. The majority of students posed questions regarding the explanation and application of words or symbols and they answered them. That is to say, writing dialogues led them to make all the related definitions available and engaged them in the process of proof. However, the explanations very often revealed students' misunderstandings or misuse of terms, more than describing the words or concepts. Following this, this misuse of words would be helpful to identify a possible root of students' failure in thinking mathematically and understanding or creating a proof.

According to the communicational approach to cognition, mediating tools and routines are two things that must be learned if a person is to become a skilful participant of a given discourse. The results of the study showed the implemented mediators and routines in students' arguments were strongly influenced by their colloquial discourse. Having a close look at the pre-service elementary school teachers' argument through the

lens of the communicational framework provided the researcher with indicators for recognizing the factors that impeded students in the process of creating and interpreting a proof.

Specific contributions of the study

There are several contributions of this study to the field of mathematics education, particularly to the research on pre-service elementary school teachers' understanding and creating proofs for propositions related to elementary number theory. The contributions can be viewed from different perspectives: methodological, pedagogical, and theoretical.

As a methodological contribution, this study introduced the idea of writing dialogue as a data collection tool to investigate not only students' discourses but also students' thinking processes while they try to prove a mathematical statement. The results of the designed tasks based on the idea of writing dialogue revealed the difficulties that students experienced in writing and interpreting proofs for propositions related to elementary number theory. The created dialogues provided the researcher an opportunity to examine students' arguments and, by posing more appropriate questions, led the students to refine and strengthen their arguments. From this perspective it provided a researcher with an opportunity to trace the students' progress.

As a pedagogical contribution, this study introduced the activity of writing dialogue as a heuristic tool to involve students in the process of creating a proof. The benefit of writing dialogue is that it encourages students to 'explain *why* and *how* to do' instead of just doing. This is what Schoenfeld (1994) calls mathematical culture, where discourse, thinking things through, and convincing are important parts of students'

engagement with mathematics. The students' dialogues revealed their improvement in producing a reasonable mathematical argument, and also revealed their appreciation of the space of dialogue that gave them a chance to present their ideas in a flexible, explorative environment. From this perspective we can consider the activity of writing proof as an intermediate stage between having an overview of a proof and writing a mathematical proof.

This study was guided by the communicational approach to cognition. As described in chapter 2, the literate mathematical discourse is characterized with its four features: mathematical vocabulary, mediators, routines and endorsed narratives (Sfard, 2002; Ben-Yehuda et. al., 2005). The theoretical contribution of the study is the further refinement of these features, applicable to elementary number theory discourse, that emerged through analysing students' arguments (see Figure 7, p. 160). The proposed finer categorization enables a researcher to analyse students' discourse in more detail.

Limitations of the study and suggestions for further exploration

The main limitation of this study was the time frame of the course. Considering the number of participants, the time frame of the course did not allow me to follow the individual's progress. However, each student received a written feedback on her/his work; it could have been much more effective if it were supplemented by some discussion on each dialogue as well.

As noted above, the results reported in this dissertation are from involvement of pre-service elementary school teachers in the activity of creating dialogues for proving

propositions related to elementary number theory. We can extend and explore the idea of the study in other directions focusing on methodology, teaching, or content.

The claim of the study is that writing dialogue provides students with an opportunity to revise their ways of thinking about mathematical thinking and practice. Therefore, it should not be limited to interpreting and creating proof. The method of writing dialogue could be applicable for students' involvement in the process of mathematical problem solving, where students need to engage their knowledge sources to understand a problem, make a conjecture, and by reflecting on their own ideas find the solution. From this perspective the method of writing dialogue not only works as a heuristic tool to activate students participation in a mathematical practice, but may also provide a research tool for collecting data related to students' thinking process while they are solving a problem.

The exploratory nature of activity of writing dialogue aimed at cultivating a mathematics culture, including posing questions and providing reasonable explanation for 'why' and 'how' to do what to do. Considering the formation of mathematical thinking in early schooling, this study can be extended to investigate the applicability of writing dialogue for introducing and teaching proof to high school students.

This study examined students' discourses in their attempts for creating a proof for propositions related to elementary number theory through the communicational framework. Another extension of this study might focus on using communicational framework for analysing students' discourse for different topics such as geometry or algebra.

APPENDICES

Appendix 1

- Consider each of the following statements along with its justification (proof).
- Examine each proof carefully and decide whether it is satisfactory for proving/validating the given statements.
- If you believe it is mathematically acceptable write OK.
- If not, either add lines to provide an acceptable justification (valid proof) or delete lines to avoid statements that are unnecessary for the proof. You may rewrite the entire proof if you wish.

- (1) Statement: The finite set $B = \{0,1\}$ is closed under multiplication.
Proof: $0 \times 1 = 0$
 $0 \times 0 = 0$
 $1 \times 1 = 1$
 $1 \times 0 = 0$
 \therefore the set B is closed under multiplication
- (2) Statement: The set of prime numbers is closed under addition.
Proof: The set of prime numbers = $\{2,3,5,7,11,13,17,19,23,29,31,37,\dots\}$
 $2 + 3 = 5$ is a prime number
 $2 + 5 = 7$ is a prime number
 $17 + 2 = 19$ is a prime number
but $3 + 5 = 8$ is not a prime number
and $19 + 13 = 32$ is not a prime number
So, the set of prime numbers is not closed under addition.
- (3) Statement: The set of multiples of thirteen is closed under addition.
Proof: The set of multiples of thirteen = $\{0, 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, \dots\}$
 $13 + 26 = 39$ is a multiple of thirteen
 $39 + 52 = 91$ is a multiple of thirteen
 $65 + 78 = 143$ is a multiple of thirteen
 $91 + 104 = 195, 195 = 15 \times 13$ so it is a multiple of 13
 $117 + 156 = 273, 273 = 21 \times 13$
 $130 + 169 = 299, 299 = 23 \times 13$
 $195 + 143 = 338, 338 = 26 \times 13$
 $1300 + 2613 = 3913, 3913 = 301 \times 13$
We have seen that the sum of two multiples of thirteen is another multiple of thirteen so we can say this set is closed under addition.

(4) Statement: The set of multiples of five is closed under addition.
Proof: True, because for a multiple of five the last digit is 0 or 5. When we add up two numbers, which are multiples of five, then the last digit could be 0+0, 0+5, 5+5, which would be again a number with the last digit 0 or 5. Therefore, the set of multiples of five is closed under addition

(5) Statement: The set of odd numbers is closed under multiplication.
Proof: $O = \{1, 3, 5, 7, 9, 11, 13, 15, \dots\}$ = set of odd numbers
For any $n, m \in \mathbb{W}$, $(2n + 1) \in O$ and $(2m + 1) \in O$
[$2n+1$ and $2m+1$ are two odd numbers]
 $(2n + 1)(2m + 1) = 4nm + 1 = 2(2nm) + 1 = 2k + 1 \in O$
[$2nm$ is a whole number like k]
So, the set of odd numbers is closed under multiplication

Appendix 2

Proposition 24, from book seven

If two numbers be prime to any number, their product also will be prime to the same.

Proof⁸:

For let the two numbers A, B be prime to any number C , and

let A by multiplying B make D ;

A _____

I say that C, D are prime to one another.

B _____

For, if C, D are not prime to one another, some number will measure C, D .

C _____

D _____

Let a number measure them, and let it be E .

E _____

Now since C, A are prime to one another,

F _____

and a certain number E measures C ,

therefore A, E are prime to one another

[VII. 23]⁹

As many times, then, as E measures D , so many units let there be in F ;

therefore F also measures D according to the units in E .

[VII. 16]¹⁰

Therefore E by multiplying F has made D .

[VII. Def. 15]¹¹

But, further, A by multiplying B has also made D ;

Therefore the product of E, F is equal to the product of A, B .

But, if the product of the extremes be equal to that of the means, the four numbers are proportional;

[VII. 19]¹²

Therefore, as E is to A , so is B to F .

But A, E are prime to one another,

numbers which are prime to one another are also the least of those which have the same ratio,

[VII. 21]¹³

⁸ The proof is from *Euclid's Elements* (Euclid, 2002, p. 173).

⁹ [VII. 23] If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

¹⁰ [VII. 16] If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

¹¹ [VII. Def. 15] A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

¹² [VII. 19] If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

and the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is the antecedent the antecedent and the consequent the consequent; [VII. 20]¹⁴

therefore E measures B .

But it also measures C ;

therefore E measures B, C which are prime to one another:

which is impossible.

[VII. Def. 12]¹⁵

Therefore no number will measure the numbers C, D .

Therefore C, D are prime to one another.

Q.E.D.

¹³ [VII. 21] Numbers prime to one another are the least of those which have the same ratio with them.

¹⁴ [VII. 20] The least number of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

¹⁵ [VII. Def. 12] Numbers prime to one another are those which are measured by an unit alone as a common measure.

Proposition 30, from book seven

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

Proof¹⁶:

For let two numbers A, B by multiplying one another make C , and let any prime

A _____

number D measure C ;

B _____

I say that D measures one of the numbers A, B .

C _____

For let it not measure A .

D _____

E _____

Now D is prime;

therefore A, D are prime to one another.

[VII. 29]¹⁷

And, as many times as D measures C , so many units let there be in E .

Since then D measures C according to the unit in E ,

therefore D by multiplying E has made C .

[VII. Def. 15]¹⁸

Further, A by multiplying B has also made C ;

therefore the product of D, E is equal to the product of A, B .

Therefore, as D is to A , so is B to E .

[VII. 19]¹⁹

But D, A are prime to one another,

Prime are also least,

[VII. 21]²⁰

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent;

[VII. 20]²¹

therefore D measures B .

Similarly we can also show that, if D do not measure B , it will measure A .

Therefore D measure one of the numbers A, B .

Q.E.D.

¹⁶ The proof is from *Euclid's Elements* (Euclid, 2002, p. 177).

¹⁷ [VII. 29] Any prime number is prime to any number which it does not measure.

¹⁸ [VII. Def. 15] A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

¹⁹ [VII. 19] If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

²⁰ [VII.21] Numbers prime to one another are the least of those which have the same ratio with them.

²¹ [VII. 20] The least number of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

Proposition 29, from book nine

If an odd number by multiplying an odd number make some number, the product will be odd.

Proof²²:

For let the odd number A by multiplying the odd number B make C ;

I say that C is odd.

A _____
 B _____
 C _____

For, since A by multiplying B has made C , therefore C is made up of as many numbers equal to B as there are units in A . [VII. Def. 15]²³

And each of the numbers A, B is odd;

therefore C is made up of numbers the multitude of which is odd.

Thus C is odd.

[IX. 23]²⁴

Q.E.D.

²² The proof is from *Euclid's Elements* (Euclid, 2002, p. 230).

²³ [VII. Def. 15] A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

²⁴ [IX.23] If as many odd number as we please be added together, and their multitude be odd, the whole will also be odd.

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