# THE USE OF PARAMETRIC REPRESENTATIONS TO SIMULATE MOTION ON A GRAPHICS CALCULATOR

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN THE FACULTY

OF EDUCATION



SIMON FRASER UNIVERSITY

NOVEMBER 1993

ISBN 0-612-01133-X

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Degree: Master of Science

Title of Thesis: The Use of Parametric Representations to Simulate Motion on a Graphics Calculator

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Title of Thesis/Project/Extended Essay

**The Use of Parametric Representations to Simulate Motion on a Graphics Calculator**

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## ABSTRACT

The study involved the creation and field-testing of a curriculum package on the use of parametric representations of functions leading to the simulation of problems involving motion. The curriculum was written for use on the TI-81 graphics calculator and can easily be adapted to the TI-82 or the TI-85. It would also be useful as a resource for doing work on parametrics regardless of the technology used.

The study documents the growing interest in the topic manifested in the literature and at mathematics conferences over a three year period from the spring of1991 to the fall of 1993.

The curriculum package consists of an introduction to the graphics capabilities of the calculator, a detailed overview of graphing using parametrics, a section describing how parametrics can be used to simulate the motion of objects in two-dimensional space, and a set of solved and unsolved problems.

The curriculum package was piloted with a small group of first-year technology students at a junior college and then field-tested with an honours Mathematics 11 class in a senior secondary school. The field-test involved student presentations of problems they solved from the included problem set.

The results of the questionnaires indicated that the use of parametric representations to simulate problems involving motion helped the students visualize and understand the problems, thereby acting as an aid in the process of problem-solving.

# ACKNOWLEDGEMENTS

I would like to thank Dr. Tom O'Shea and Dr. Harvey Gerber for setting up a program at SFU that would attract mathematics teachers genuinely interested in improving their understanding of the subject. Their leadership allowed for a collegial atmosphere in which the students could pursue their own diverse interests.

I would also like to thank Dr. Chuck Vonder Embse at Central Michigan University for responding to my request for materials to get me started on the curriculum package.

The students in the pilot and the field test were very helpful and I appreciated their input.

My final thanks go to my wife, Kathleen, whose patience and technical skills helped polish up the curriculum package.

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## Background

Most teachers of mathematics would probably agree that the two most feared words in the teaching and learning of mathematics would be: *word problems.* Many teachers are able to assuage these fears by the clever use of algorithms specific to the type of problem being studied. Most of these, although leading to successful solutions, do not address *understanding* but rather an abstract analytical reinforcement of the initial source of the confusion: the *words.* This is especially true for problems involving **motion**. Here more than ever is the need for an effective mode of **visualization** for the type of problems that at once raise the levels of anxiety and lower the levels of confidence of the students, and at the same time, discourage them from careers in mathematics or related sciences. After all, a picture is worth a thousand words, especially one that moves.

Another factor affecting the difficulty in getting started on word problems is that of a lack of sufficient motivation on the part of the student. This lack of involvement may stem from an incomplete understanding of the problem. When faced with an algebraic algorithm for its solution, based on two-dimensional tables, the student, although perhaps cabable of generating this solution on demand, probably does so without really connecting the abstract solution to the real *live* problem.

The use of parametric representations of functions can bring problems involving motion to life on the screens of graphic calculators and computers. This *video* of the problem, that can be analyzed at any instant of time and replayed at slower or faster tempos, could complete the understanding of the problem, and provide the link between the problem and its solution.

Some of the strongest recommendations to improve the connections between representations have been made recently in a set of curriculum standards published by the National Council of Teachers of Mathematics (NCTM, 1989). To say that all students should be able to make connections between different representations as well as different disciplines and model real-world phenomena using a variety of functions is a considerable challenge. Strengthening these statements further is the recommendation to make full use of the graphing technology available in the belief that this use will not act as a crutch, but will instead strengthen existing concepts and even aid in further developmental learning. Seeing is not just believing, it is much more than that.

Although graphs have been recommended and used as a problem solving strategy their effectiveness in problems involving motion is limited by two factors. First, a single representation can involve only two variables at a time. In problems consisting of motion in a two-dimensional space, it is impossible to represent both dimensions as well as representing time. Secondly, even if there are only two dimensions, such as in a problem involving linear motion, the graphs drawn can only be static and never a dynamic representation of the actual problem. In both cases the graphs become another abstraction, marginally more representative than the algebraic one. By using the parametric capabilities of recent technology, both displacement coordinates can be shown, and the object can be seen to move in its two-dimensional space in real time.

## Purpose of the Study

The purpose was to design, test and revise a curriculum package on the use of parametric representations of functions leading to the simulation of problems involving motion using the parametric capability of the TI-81 graphics calculator. This technology was used fully during the pilot and the field test of the curriculum. The curriculum was written for use on the TI-81 but can be easily adapted to the TI-82 or the TI-85.

The pilot was carried out with a group of students who had completed their first year of a civil engineering technology program at a junior college. For them the advantage would be the realization that problems involving motion on a line or in space, as well as related rate problems can be simulated using parametrics. This visualization of the problem was intended to provide them with a stronger conceptual foundation, so that their formulations of these problems using calculus would make more sense to them.

The field test was conducted on a group of grade 11 students who were familiar with functions and graphing but who had little or no experience with the use of graphics calculators and certainly not with parametric representations. For them the advantages would be to familiarize themselves with the technology and to participate in the simulation of real-world application-type problems.

For both trials, pre-study and post-study questionnaires were used to determine whether this use of curriculum and technology could reduce the level of anxiety related to problems involving motion, lead to a better understanding of problems of this type, and be an aid in the solution of these problems.

## Justification

Problems involving motion where *time* is a variable can be made to come to life on the screens of graphics calculators and computers with parametric capabilities. The simplest of these problems would consist of only linear motion, as in the example of two trains approaching each other at different speeds on parallel tracks. Projectile problems where gravity is a factor and two dimensional problems where horizontal and vertical rates are different, such as a ladder sliding down a wall, can also be simulated using parametric representations.

To be shown, or better yet, to be able to create and adapt a *video* of a motion problem should be a confidence builder, and provide a strong motivational link between the different representations of the problem. These are:

1) the algebraic equations used to construct the graph

2) the simulated motion of the object, which is the graph

3) the values of the variables, the position coordinates (**x**,y) and the time **T**, which can be read off the graph using the **TRACE** key.

Using the **TRACE** option, the problem can be studied at any of its coordinate points and so the simulation can be used as an aid to the solution of the problem itself, or at least as a check of the analytical calculations.

For example, Figure 1 shows a problem from the curriculum of an object moving on a parabolic path. If a **TRACE** is done and stopped at T = 5 the display on the screen would contain the following information:

**T = 5, X = 125, Y = 27.5**

A table of values for various values of T, X, and Y, has been included to help the reader get a sense of time passing.

---

- A pair of parametric equations of the form:

  $X_{2T} = 25T$ & $Y_{2T} = 50 + 20T - 4.9T^2$

  describe the path of an object from an initial position ( 0, 50 )

- **25T** is the horizontal displacement (meters) at time **T**
- $50 + 20T - 4.9T^2$ gives the vertical displacement

  **50** = the initial vertical position ( T = 0 )

  **+20T** represents an initial vertical velocity of 20m/sec upward

  $- 4.9T^2$ represents the vertical effect of the gravitational pull on an object causing the vertical velocity to continually decrease at 9.8 m/sec each second. ( $at^2/2 = -9.8t^2/2 = -4.9t^2$ )

| T | X | Y |
|---|---|---|
| 0 | 0 | 50 |
| 1 | 25 | 65.1 |
| 2 | 50 | 80.2 |
| 5 | 125 | 27.5 |
| 10 | 250 | -240 |

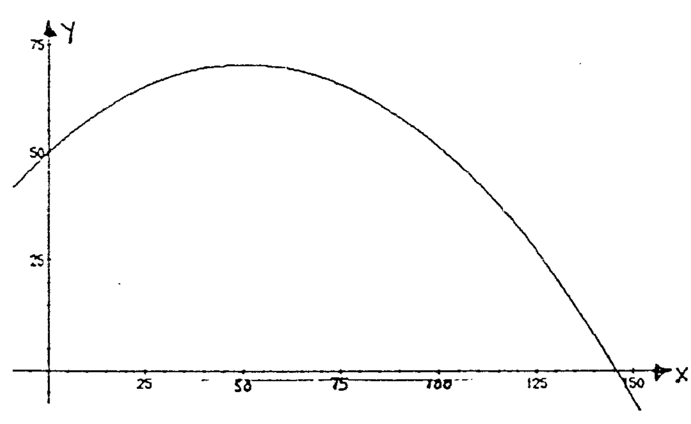


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**Figure 1.** Parabolic motion (quadratic function)

The fact that a problem can not only be displayed graphically, but can actually be simulated using time as a variable, is more than just a breakthrough in providing a sense of the dynamics of a problem. It also holds the exciting possibility of allowing students to discover greater understanding of the meaning and uses of mathematics. Graphing in secondary school mathematics is generally a static phenomenon. Even the so-called motion problems, when solved graphically, are not done in a way that gives a sense of action. Using parametric representations and simultaneously graphing the two or more functions, motion can be simulated on the screen of the TI-81 calculator. This is done by switching the **MODE** of graphing from **function** to **parameter** The third variable, time, can be slowed down or sped up to suit the problem. This is done by adjusting the **Tstep** in the **RANGE**. The problem can be studied at any point and all three coordinates--the two displacement coordinates and time--can be displayed by using the **TRACE** key and moving the cursor along the graph.

Parametric representations of functions is a fairly obscure topic, visited briefly by calculus students doing related rates problems. To subject students and teachers to yet another branch of mathematics, which itself is already too abstract for most students seems unreasonable, especially since a common complaint is that the curriculum is already overloaded. The most significant strength of this proposal is that algebraic, parametric representations of verbal problems, will be translated into graphical simulations on the calculator screen. The student can then solve the problem using the **TRACE** option, or revise the algebraic representation, thus producing a newer, and possibly a better fitting visual representation, and so on until the problem is completed. This process can be illustrated by the golfing problem shown in Figure 2. The algebraic solution is at the end of the problem.

---

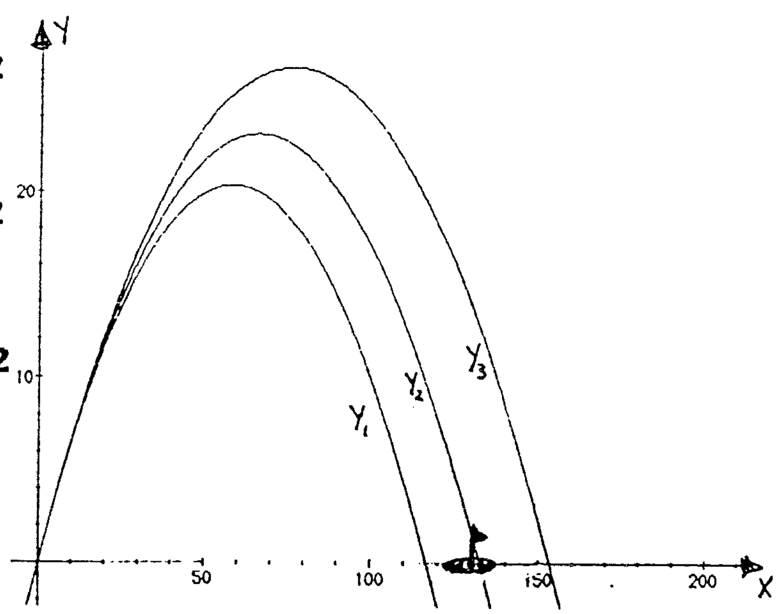
A golfer is contemplating a 7-iron shot towards a circular green with the flag located at the center. The hole is 130 meters away and the radius of the green is 8 meters. His 7-iron hits the ball at a 35° angle. Assuming the ball will be hit straight and that there is no wind, what velocity will it be necessary to impart to the ball so that it hits the green?

- The equations for the path of the ball become:
  **$X(T) = VTCos\ 35^\circ$ & $Y(T) = VTSin\ 35^\circ - 4.9T^2$**
- Choose different values of **V** and **TRACE** the graphs drawn to find the **X**-value when **Y = 0**. This is the distance the ball travels in the air. Y - 0 is ground. The desired X-value is 130. The algebraic solution of V is shown below.

- **$X1T = 35TCos\ 35^\circ$**
  **$Y1T = 35TSin\ 35^\circ - 4.9T^2$**

  **$X2T = 37TCos\ 35^\circ$**
  **$Y2T = 37TSin\ 35^\circ - 4.9T^2$**

  **$X3T = 40TCos\ 35^\circ$**
  **$Y3T = 40TSin\ 35^\circ - 4.9T^2$**
- Set **Range**:
  **T**: 0 -> 6 (TStep = .02)
  **X**: 0 -> 200 (XScl = 50)
  **Y**: 0 -> 40 (YScl - 10)




**Algebraic solution:**

The solution is a 130 meter shot ( X = 130 & Y = 0 ).

(A) $130 - VTCos35^\circ$

(B) $0 = VT\text{Sin}35^\circ - 4.9T^2$ -> $T(V\text{Sin}35^\circ - 4.9T) = 0$

-> $T = 0$ or $\mathbf{T = V\text{Sin}35^\circ/4.9}$

Substituting this back in (A) -> $130 = V^2\text{Sin}35^\circ\text{Cos}35^\circ/4.9$

-> $\mathbf{V = 36.82\ m/sec}$

---

**Figure 2.** The golf problem: Part D - problem 3

Another strength is that this approach is function based. Every parametric function is represented by two functions **X(t)**, the horizontal component, and **Y(t)**, the vertical. This should help reinforce the concept of functions, a major concern theme in the standards recommended in the recent NCTM report (NCTM,1989).

## Chapter Organization

The thesis consists of five chapters. The experimental curriculum is added as an appendix.

Chapter 1 consists of the background, including a description of some of the limitations of present day mathematics teachings, the purpose of the study and the justification of it as well as a brief introduction to the conceptual basis of parametric representations, how they can be used to strengthen the problem-solving process, and some examples to illustrate the visualization aspect of this approach.

Chapter 2 contains a brief review of the literature that recommends change and the specific use of parametric representations.

Chapter 3 describes the development of the curriculum, how it was piloted, and how the technology was used.

Chapter 4 consists of a description of the six-day field test, the student comments, and the results of the questionnaires.

Chapter 5 contains an analysis of the results as well as a discussion of some of the implications involving the use of parametric representations.

Appendix A contains the curriculum package itself. Revisions made to the curriculum, as a result of the field test, are described in Chapter 4.

# Chapter 2. Literature Review

The literature reviewed in this section was chosen by focusing on a particular theme: that the learning of mathematics will be enhanced by representing or visualizing problems in a variety of ways. The emphasis on alternative strategies for problem solving was reinforced by the advent of graphic calculators. The literature that followed suggested parametric representations of functions as a means of the simulation of problems involving motion in 1989. Since then there has been a growing interest in this topic. The remainder of the review focuses on literature specifically on parametric representations.

## Recent Recommendations

A key problem area in the teaching of mathematics has been its perceived lack of relevance to real-world situations. The different disciplines of mathematics were not seen as connected to each other, much less to other subject areas. Efforts were made in the late 1970s to correct this perception by providing teachers with the skills and tools to present mathematics as an interconnected and application-oriented subject. The first step was to establish problem solving as the focus of teaching.

### Expanding the Instructional Strategy Base

Problem solving was recommended to be the number one basic skill as early as 1978 by the National Council of Supervisors of Mathematics (NCTM, 1978), and adopted as the focus of school mathematics for the

1980's in the NCTM's Agenda for Action (NCTM, 1980). The recommendation below from this Agenda makes a clear case to include the use of technology in the problem-solving process.

> Mathematics programs of the 1980s must be designed to equip students with the mathematical methods that support the full range of problem solving, including -
> - the traditional concepts and techniques of computation and applications of mathematics to solve real-world problems
> - the use of the problem-solving capacities of computers to extend traditional problem-solving approaches and to implement new strategies of interaction and simulation
> - the use of imagery, visualization, and spatial concepts (NCTM, 1980, p. 13)

Before the advent of graphics calculators, the only tools available to provide any kind of visualization of graphic events were computers, with software devoted mainly to diagnostic work. Since then there has been an enormous effort on the part of all those active in the field to come to grips with a variety of instructional strategies for problem solving, and to devise appropriate approaches to what most people agree is the essence of mathematics, the "doing" of mathematics. The work done by those in developing thorough problem solving plans and lists of solution strategies (notably Charles & Lester, 1982, 1987) suggest that teachers model the use of different strategies, that students be encouraged to use a variety of strategies even after the successful completion of a problem.

## The Use of Multiple Linked Representations

The discussion of different strategies was subsequently focused in the direction of different representations of the same concept or problem. It was agreed that the learning environment could be enriched by a presentation that involved more than one way of visualizing or interpreting the solution of a problem.

> Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept, will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics. ... Students' understanding of the connections among mathematical ideas facilitates their ability to formulate and deductively verify conjectures across topics. (NCTM, 1989, p. 147)

The above is a quote from the NCTM's Standards document and is part of the description of Standard 4 titled Mathematical Connections, which reads:

> In grades 9-12, the mathematics curriculum should include investigation of the connections and interplay among various mathematical topics and their applications so that all students can:
> * recognize equivalent representations of the same concept;
> * relate procedures in one representation to procedures in an equivalent representation;
> * use and value the connections among mathematical topics;
> * use and value the connections between mathematics and other disciplines (NCTM, 1989, p. 146)

New technologies can help to make these connections by providing the link between the algebraic (syntactic) and the graphical or visual (semantic). In a study that recommends an algebra learning environment that supports the *linking* of different algebraic representations, Kaput (1989) argues that the problem of student alienation is

> the result of teaching algebra syntax instead of semantics ... dealing with the syntax of formal algebraic symbols and the lack of linkages to other representations that might provide informative feedback on the appropriateness of actions taken. (p.168)

Syntactic actions are described as opaque, whereas semantic ones are transparent in which the user sees through the notation. On the use of new technologies, Kaput (1989) says

> the bird of mathematical competence cannot fly on one wing ... neither the syntactic nor the semantic suffices alone. By providing the means to link actions on a notation with their consequences in a reference field, new technologies may help redress the semantic/conceptual balance without giving up syntactic/procedural power. (p.176)

**Assumptions Underlying the NCTM Standards**

In March 1989, the Working Groups of the Commission on Standards for School Mathematics of the National Council of Teachers of Mathematics issued a report recommending sweeping, definitive changes in the way mathematics is taught in the classroom. A common thread throughout the assumptions governing the selection of the K - 12 standards was that the presentation of mathematics needs to be broader, more flexible, and more applicable to real-world situations:

> Programs that provide limited developmental work, that emphasize symbol manipulation and computational rules, and that rely heavily on pencil and paper worksheets do not fit the natural learning patterns of children and do not contribute to important aspects of children's mathematical development. ... Extensive and thoughtful use must be made of physical materials to foster the learning of abstract ideas. ... Children also need to understand that mathematics is an

> integral part of real-world situations and activities in other curricular areas. (NCTM, 1989, pp. 16,17,18)
>
> The vision articulated in the standards is of a broad, concept driven curriculum, one that reflects the full breadth of relevant mathematics and its interrelationships with technology. (NCTM, 1989, p. 86)

The broadened view of mathematics described in the introduction to this document under the rubric *mathematical power*, together with the capabilities of available and emerging technology, suggests a need for changes in instructional patterns and in the roles of both teachers and students:

> A variety of instructional methods should be used in classrooms in order to cultivate students' abilities to investigate, to make sense of, and to construct meanings of new situations; to make and provide arguments for conjectures; and to use a flexible set of strategies to solve problems. (NCTM, 1989, p. 125)

The possibility of a more varied instructional approach that centers on the investigative powers of the students themselves is strengthened by emergence of the kind of technology that allows students to visualize and even simulate mathematical processes. This kind of exploration is inevitable, and if directed in an effective way, cannot help but add to the confidence and skill levels of the students:

> an environment that encourages students to explore, formulate and test conjectures, prove generalizations and discuss and apply the results of their investigations. Such an instructional setting enables students to approach the learning of mathematics both creatively and independently and thereby strengthen their confidence and skill in doing mathematics. ..... Technology .... can be used effectively for class demonstrations and independently by students to explore additional examples, perform independent investigations, generate

> and summarize ... Technology ... transforms the mathematics classroom into a laboratory much like the environment in many science classes, where students use technology to investigate, conjecture and verify their findings. (NCTM, 1989, p.128)

**NCTM Standards**

The first recommended standard repeats the call of ten years earlier to focus on mathematics as problem solving. The problem-solving process, spread across the curriculum, should entail appropriate use of calculator and computer technology:

> * apply integrated mathematical problem-solving strategies to solve problems from within and outside mathematics
> * apply the process of mathematical modelling to real-world problem situations (NCTM, 1989, p. 137)

The pervasiveness of functions and the ease by which functions can be displayed on computer and calculator screens once again encourages the expansion of this topic as a major recommendation:

> In grades 9-12, the mathematics curriculum should include the continued study of functions so that all students can -
> * model real world phenomena with a variety of functions
> * translate among tabular, symbolic, and graphical representations of functions
> * recognize that a variety of problem situations can be modeled by the same type of function (NCTM, 1989, p. 154)

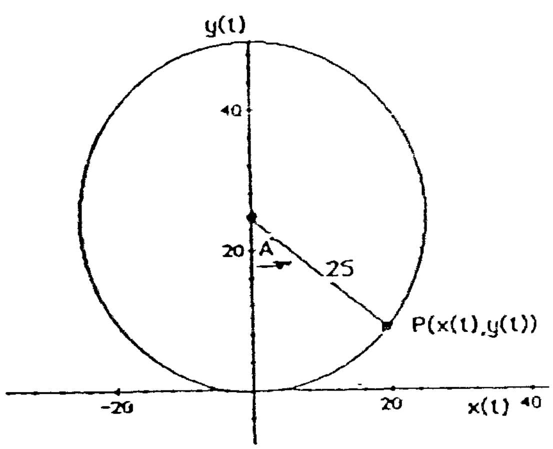
A separate recommendation is reserved for trigonometry and emphasizes the importance of trigonometric functions:

> * explore periodic real-world phenomena using the sine and cosine

functions
* use circular functions to model periodic real-world phenomena (NCTM, 1989, p.163)

Probably the first indication of a recommended use of parametric representations for secondary students occurs in the text of the trigonometric section of the report (NCTM, 1989, p. 165):

> Problem: Suppose a Ferris wheel with a radius of 25 feet makes a complete revolution in 12 seconds. Develop a mathematical model that describes the relationship between the height of a rider above the bottom of the Ferris wheel and time.
>
> At a certain level students would use right triangle trigonometry and simple proportions to derive the parametric representation of a point P = [ x(t), y(t) ] on the rotating Ferris wheel as a function of time, thereby establishing that the height is a sinusoidal function of 't'. They could then use a parametric graphing utility to simulate the motion of a point moving on the Ferris wheel.




$\sin A = x(t)/25$
$\longrightarrow x(t) = 25\sin A$

$\cos A = (25 - y(t))/25$
$\longrightarrow y(t) = -25\cos A + 25$

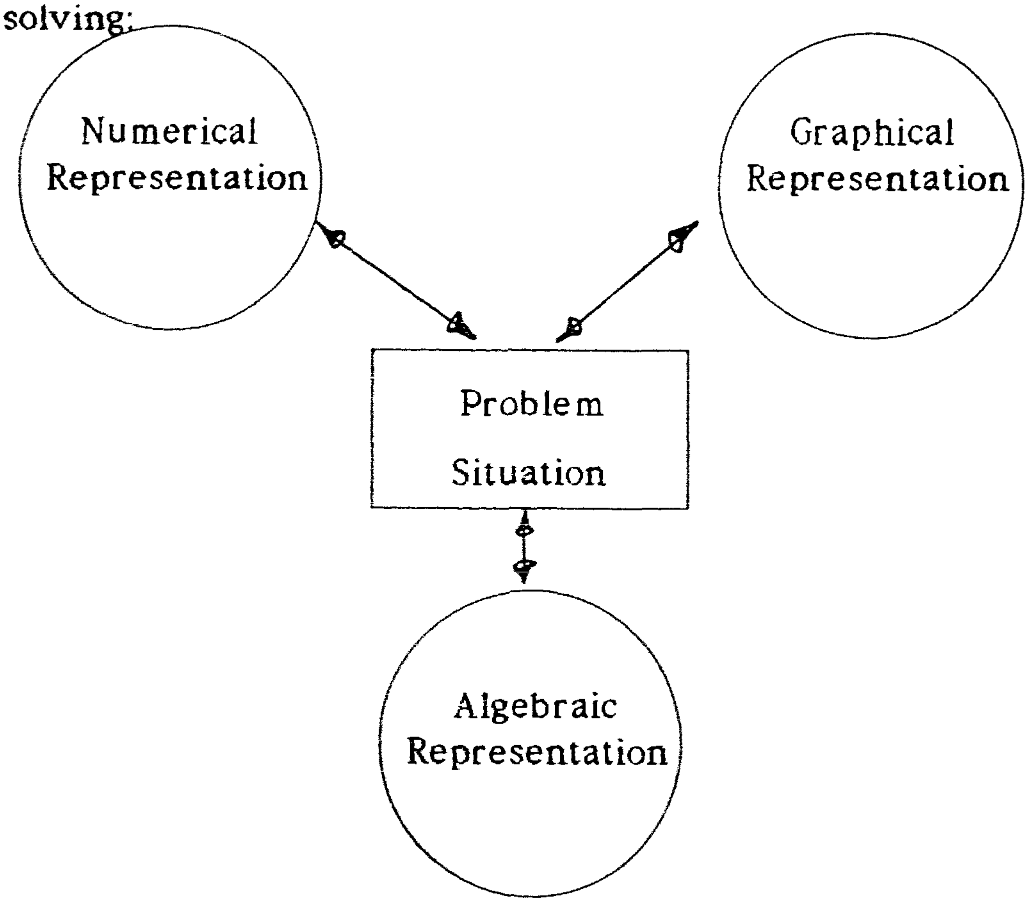
$2\pi/12 = A/t \longrightarrow A = (\pi/6)t$

## On the Use of Parametric Representations

The literature in this section deals exclusively with recommendations and applications on the use of parametric representations to simulate motion. Without a graphics calculator or computer that has a parametric graphing utility, the reader would have no sense of the motion in the problem. The graph is the simulation of the event. The literature for the most part, focuses on specific simulations and does not attempt to teach a conceptual or experiential development of the topic itself. The curriculum package was designed as an introduction and a step by step development of use of parametric representations.

**Existing literature**

Various educators at the forefront of the drive for increased use of technology in the classroom are making a strong case to include parametric representations as a component of mathematics instruction. At a recent presentation at the Third International Conference on Technology in Collegiate Mathematics, Vonder Embse (1991) used the following model to show the role that linking different representations plays in problem solving:



> By using parametric representations of relationships, we can graph any function or relatonship done in two dimensions and we have the ability to add yet a third variable to the set. This third variable becomes the independent variable. It is not represented visually on the graph, but is implicit to the graph in the same way that time is a fourth dimension implicit to the three dimensional world in which we live. The TI-81 graphing calculator has a built-in parametric graphing utility which allows students to interactively explore problems and concepts in ways not possible without a third variable. (Vonder Embse, 1991, pp. 3,4)

He provides a wide-ranging exploration of the use of parametric representations from junior high level graphing to the construction of exotic curves using polar equations. He classifies motion simulation problems into two categories. The first is motion on the earth's surface and he uses a "two-train problem" as an example. He then borrows a problem from a pre-calculus textbook to illustrate his motion in space category:

> A baseball is hit when the ball is 3 feet above the ground and leaves the bat with initial velocity of 150 feet per second and at an angle of elevation of 20°. A 6-mph wind is blowing in the horizontal direction against the batter. A 20-foot high fence is 400 feet from home plate. Will the hit go over the fence and be a home run?(Demana & Waits, 1993, p.497)

Horizontal and vertical components involving trigonometry may seem a bit intimidating. However, since every parametric function is represented by two functions X(t), the horizontal component, and Y(t), the vertical, students will have to deal with this type of thinking initially. By the time fly balls drift across the calculator screen towards the fence in center field student anxiety should be ameliorated. In recent issues of the

TI-81 newsletters two mathematics specialists have picked up the ball, so to speak, and come up with golfing and basketball variations of the above baseball theme (Texas Instruments, 1992). Foley (1992) argues that:

> Parametric representations can illuminate the relationships between rotation and angle measure, between vectors and their components, and between the geometry of curves and the motion of objects. In the past, constructing the graph of a pair of parametric equations was a laborious task. Now hand-held graphing calculators (e.g. the Texas Instruments TI-81) have built-in parametric graphing utilities that automate the curve construction process. They can simultaneously plot related curves and provide a user-controlled Trace that displays a numerical readout of the parameter value and coordinates associated with each plotted point. Graphing calculators reveal the dynamic nature of parametric representations. They can simulate the flight of a projectile or the path swept out by a point on a rolling wheel. The plotting speed can be adjusted to give the effect of a slow motion instant replay. Students and teachers can use this technology to gain relatively easy access to the interesting and useful mathematics of parametric equations and their graphs.(p. 138)

The power of visualization is again reinforced in an article (Demana & Waits, 1992) on the motion of a particle on a horizontal line. A particle moving according to the position function:

$$\mathbf{X(t) = 2t^3 - 13t^2 + 22t - 5}$$

can be analyzed as to position and velocity by simply doing a trace on the motion of the particle. By setting **Y = T** , the motion is easier to follow. Since the particle is simulated to rise, it doesn't double back on itself and the acceleration is evident:

> By pressing either the up or down arrow key you can look at the same motion on the vertically expanded view. It's not exactly the path of the particle, but makes the left to right motion of the particle

*visual*. (p.3)

**Workshops on Parametric Representations at NCTM 93**

At the April,1993 NCTM Conference in Seattle there were four standing-room-only workshops on simulating motion using parametrics. At the 1991 conference in New Orleans, there were none. I attended two in Seattle and acquired the handout from a third. The first highlighted a project entered into by the presenter with his precalculus class to stir up the interest of the punter on the varsity football team. The parametric equations:

$$X(T) = VTCos\varnothing \quad \& \quad Y(T) = VTSin\varnothing - 16T^2$$

were used to analyze his kicks to come up with the ideal angle:

> Pick some values for Ø and V and the graph shows the path of the ball over time. Let us assume that we know how far our punters' best kicks travel in the air. Assuming a constant initial velocity, what happens as we vary the angle? What angle produces the best kicks? (Barnard, 1993, p. 3)

He also posed an interesting question. Given the coordinates of a triangle, how do you come up with the parametric equations to graph the triangle?

The second presentation focused on the "monkey in the tree" problem which is usually a popular feature at science fairs. If a projectile is aimed directly, at any angle, at a target (the monkey), and if at the same instant that the projectile is fired, at any initial velocity, the target falls (out of the tree), then the projectile will hit the target every time. This was demonstrated using the parametric capabilities of the TI-82 graphics calculator. He had also prepared a video in conjunction with the physics department with ball bearings and tin cans. The tin can was connected electromagnetically to the trigger so that it would fall at the instant the ball bearing was fired. The handout included some interesting investigations, such as:

> Experiment with different bullet velocities. Does the bullet always hit the falling target? Why? What would happen if this experiment was performed on a different planet? ..... What angle would be necessary to hit the target if it was stationary? (Vonder Embse, Engebretsen, Carlson, 1993, pp. 11, 14)

The presenter of a workshop running concurrently with the one I was in arrived as we were filing out. He was demonstrating a program simulating the shooting of a free throw of a basketball. What I saw on the screen of his calculator was a basketball hoop attached to a backboard. He entered the parametric equations for the path of a basketball, pressed GRAPH and the ball arced through the air bounced off the front of the rim, off the backboard, then fell through the hoop hit the floor bounced a few times and rolled to a stop. I later acquired the handout from a friend who had attended the workshop.

> In this simulation the concepts are intended to be learned through trial and error. The graphics calculator is used as a tool to model the path of the basketball. The user "shoots" the basketball by inputting values for the height that the basketball is released above the floor, the initial angle the released ball makes with the horizontal, and the initial velocity of the basketball upon release. The path of the ball is graphed and the user can adjust the values until a 'successful' free throw is made (Engebretsen & Vonder Embse,1993, p.14).

The workshops were extremely well attended, and the participants were mostly in awe that finally mathematics could come alive on a calculator screen, and that students could experiment with different representations of problems much like science students in laboratories. There was a heady mix of fun with a sense of being on the edge of breakthroughs in some significant areas of the teaching of mathematics.

**Summary**

In the first workshop I attended in April 1991 one of the presenters suggested to the participants that they use parametric representations to simulate motion. After mostly blank stares and chaotic fumbling for an hour or so, the groups handed in their work and I overheard the presenters conferring in husheds tones to the effect that they were really on to something. Even though we were promised some form of summary of the ideas generated at the workshop, it never materialized and I left feeling somewhat like a guinea pig but with the distinct impression that, yes, something was going on that was bordering on revolutionary. I presented a workshop on the topic in Oct 1991 at the Northwest Conference in Richmond, B.C. Those in attendance seemed interested, especially a physics teacher who appreciated the connections between the two disciplines. I decided at that point that a comprehensive curriculum package on the use of parametrics was necessary so that students can translate a problem involving motion into parametric equations that will effectively simulate that motion on a calculator screen. Not only will this enable students to visualize, trace or freeze the actual motion of an object on a screen, but it will allow students to have in their hands, for the first time, the technology to understand the mathematics of motion.

# Chapter 3. Method

My first exposure to simulations, using parametric representations, of problems involving motion was a challenging experience. It was necessary for me to work backwards from the solution given in the problem, to try to understand the conceptual basis involved. The result of that struggle was my decision to develop a curriculum on parametric representations that started at a level of mathematics that included right triangle trigonometry and a knowledge of functions and graphing. The examples I'd seen, mostly from sports, were the most straightforward examples since they represented the visualizations of definite projectiles. In the curriculum package I planned to include problems involving the motion of trains, rectangles, or ladders sliding down a wall which are more like simulations of events rather than actual visualizations.

## Design of the Curriculum Package

The curriculum is designed in five parts which are labelled Parts A through E. Part A consists of an introduction to the function graphing capabilities of graphics calculators. Part B is an introduction to the use of parametric representations. Part C describes the use of parametrics to represent motion. Part D consists of specific problems and their solutions. Part E is a problem set with no solutions given. After the field test I created an addendum to improve on a particular topic. The curriculum package is Appendix A, and the addendum, (see Figure 17), is described in Chapter 4.

**Use of Technology and Graphing Capabilities - Part A**

My first thoughts in designing a curriculum that relied heavily on the use of calculators was to keep it untechnical and to include as many diagrams as possible. This meant teaching largely by example, and choosing the examples in such a way as to provide the conceptual development of the topic. In this way I hoped to avoid the dulling effects of some textbooks and most technical manuals. One constraint I felt obliged to uphold was that the package should be designed to fit a 6 to 10 hour time frame and that the students should have completed the grade 11 graphing and trigonometry units.

The first hour would have to be an introduction to the technology and its graphing capabilities. In Part A of the package I included a few technical terms, and referred students to the appendix for elaborations for features such as the **ZOOM** function. I chose as an example the solution of the equation:

$$0.5X + 1 = 2X^2 - 5X - 2$$

by graphing the linear and quadratic functions represented by the opposite sides of the equation, and finding the x-values of the points of intersection of the two graphs.

**Parametric Representations of Functions - Part B**

In the introduction to parametric representations I wanted to convey two major concepts. The first was that motion occurs according to a controllable parameter. Setting the **RANGE** for **T** is analagous to setting a stopwatch for a specific interval of time (**Tmin -> Tmax**), and deciding on how fast time should travel. A small **TStep** would slow the graph down.

The second concept was that two equations, each one a function of **T**, are necessary to describe the motion of a particle. A good example to illustrate this is shown in Figure 3. I had previously set TStep = 0.05 and had mentioned that doubling the TStep would double the speed of the graph:

---

**1. b) Drawing a line joining two given points:**
**(-5, 8) & (7,-2)**

- Press **Range** and set **Tmin** = 0.
  If T represents TIME then starting at T = 0 would be natural.

| T | X | Y |
|---|---|---|
| 0 | -5 | 8 |
| ? | 7 | -2 |

***One way to draw this line:***

- To increase X & T at the same rate enter
  **X1T** = **-5 + T** and vary T from 0 -> 12.
  **Note**: **X** changes from -5 to 7 as **T** changes at the same rate. **Y**, however, changes at a different rate, as it decreases from 8 to -2.

- Calculate the function **Y1T**, by setting — Y1T = 8 + **k**T
- Substitute T & Y (values from table) — -2 = 8 + 12**k**
  -10 = 12**k**
- Solve for k — -5/6 = **k**
- Enter **Y1T** = **8 + (-5/6)T.**

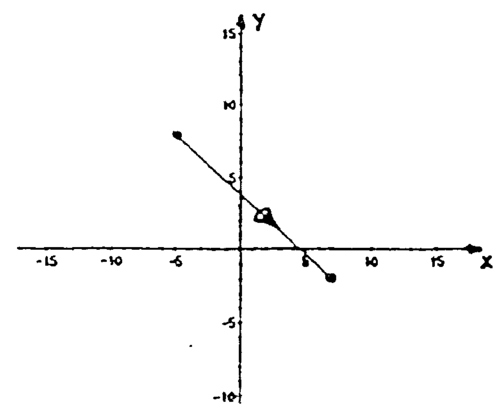



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**Figure 3.** Drawing a line joining two given points

For quadratic functions I used the example shown in Figure 4 to represent the motion of particles in two-dimensional space. My intent was to design three parabolas that would simultaneously leave the point (0,20) and hit the spot (20,0) at the same time. The graphs would be seen as having a projectile quality.

**b) Designing a Quadratic Function to connect (0,20) & (20,0)**

**Note:** This can be done any number of ways. In the three cases below I've kept **X** = **T** so the graphs will proceed left to right. In each of the three cases a value of k can be determined by substituting from the table:

| T | X | Y |
|---|---|---|
| 0 | 0 | 20 |
| 20 | 20 | 0 |

**i)** Enter $\mathbf{X_{1T} = T}$ & Set $Y_{1T} = 20 + \mathbf{k}T^2$

Substitute $0 = 20 + \mathbf{k}(20^2)$

Solve for **k** $\mathbf{k} = -0.05$

Enter $\mathbf{Y_{1T} = 20 - 0.05T^2}$

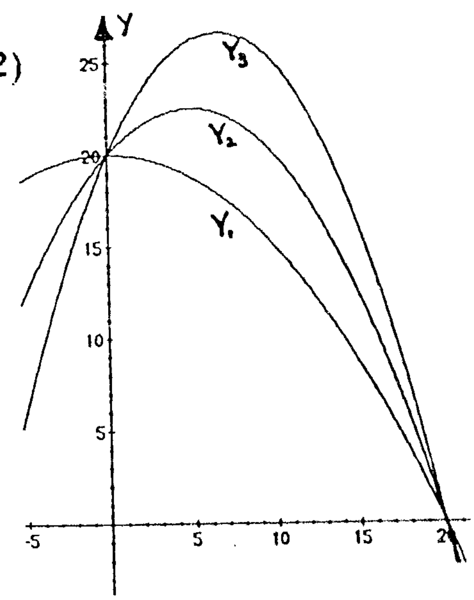
**ii)** Enter $\mathbf{X_{2T} - T}$ & $\mathbf{Y_{2T} = 20 + T - 0.1T^2}$

**iii)** Enter $\mathbf{X_{3T} = T}$

Set $Y_{3T} = 20 + 2T + \mathbf{k}T^2$

$0 = 20 + 40 + \mathbf{k}(20^2)$

$\mathbf{k} = -0.15$




- Enter $\mathbf{Y_{3T} = 20 + 2T - 0.15T^2}$
- Experiment with **TStep** changes
- **GRAPH** in **Sequence** or **Simul**taneously ( a **MODE** menu option )

**Figure 4.** Three quadratic functions using parametrics

I wanted to make the connection between linear functions and trigonometry. In Figure 5, I described how a linear graph with a slope of 30° could be drawn using trigonometry. I avoided using the concept of motion at this point, because that would involve talking about velocity. I followed this with another graph starting at the point (0,3) and rising at

30° but pulled down by a gravitational effect. Notice that the graph is the line Y = (Tan30°) X in regular function notation.

---

**e) Graphing a linear function using trigonometry.**

- If a straight line has an angle of elevation of 30° then the line can be described parametrically as follows:
  **$X_{1T}$ = TCos 30° & $Y_{1T}$ = TSin 30°**

**f) Graphing a quadratic function using trigonometry:**

- Enter **$X_{2T}$ = TCos 30°** & **$Y_{2T}$ = TSin 30° - 0.02$T^2$ + 3**

**Note**: This graph will also have an initial slope of 30°. It will start at + 3 on the Y-axis and will be pulled down by the gravitational effect of - 0.02$T^2$.

| T | $X_1$ | $Y_1$ | $X_2$ | $Y_2$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 3 |
| 1 | .87 | .5 | .87 | 3.48 |
| 2 | 1.73 | 2 | 1.73 | 4.92 |
| 10 | 8.7 | 5 | 8.7 | 6.0 |
| 20 | 17.3 | 10 | 17.3 | 5.0 |
| 30 | 26.0 | 15 | 26.0 | 0 |

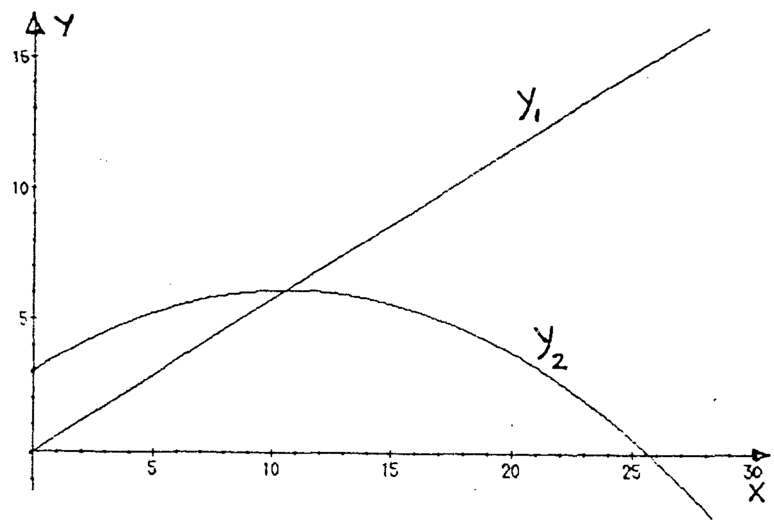



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**Figure 5.** Graphing functions using trigonometry

## Motion Simulation from a Given Point - Part C

This is the section where graphing becomes the simulation of the motion of a particle. I started this section with motion on a horizontal line. I had three particles moving on the lines Y = 4, Y = 8, and Y = 12. The bottom one travels at a constant rate of 30 units/sec to the right, the middle one has no initial velocity but accelerates at 4 units/sec$^2$ to the right, and the top one has an initial velocity to the left then accelerates to the right to try to catch the other two (see Figure 6). All three particles started at X = 50:

---

**1. Horizontal Motion**

- Set **Range** T: 0 -> 20; X: 0 -> 1000; Y: 0 -> 20

**a) Objects travelling at a constant speed**: bicycles, trains ....

- An object starts from ( 50,4 ) and travels at a constant Horizontal rate of 30 units/sec. This can be represented by:

$$X_{1T} = 50 + 30T \qquad \& \qquad Y_{1T} = 4$$

**b) Objects accelerating:**

- An object starts at (50,8) with acceleration **a** = 4 units/sec$^2$. Distance travelled is given by $D(t) = D_0 + V_0t + at^2/2$
- This can be represented parametrically by:

$$X_{2T} = 50 + 2T^2 \qquad \& \qquad Y_{2T} = 8$$

- If there is an initial velocity of $\mathbf{V_0}$ = - 20 units/sec, then using the same acceleration and starting from ( 50, 12 ) we get:

$$X_{3T} = 50 - 20T + 2T^2 \qquad \& \qquad Y_{3T} = 12$$

**Note**: Do a **TRACE** on this one and notice the cursor start towards the left (negative velocity) but it changes direction eventually because the acceleration is positive.

**Note:** A constant headwind of 20m/sec. would have the same effect as the initial velocity.

---

**Figure 6.** Horizontal motion of three particles

The first contact I had with parametric representations that would simulate the path of a projectile in the air was the baseball problem from a precalculus textbook (Demana & Waits, 1993):

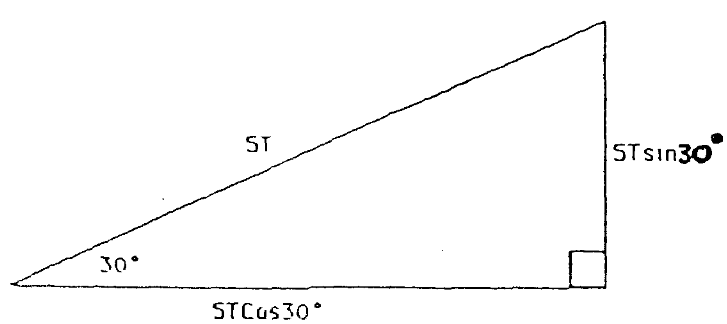
> A baseball is hit when the ball is 3 feet above the ground and leaves the bat with initial velocity of 150 feet per second and at an angle of elevation of 20°. A 6-mph wind is blowing in the horizontal direction against the batter. A 20-foot high fence is 400 feet from home plate. Will the hit go over the fence and be a home run?( p. 497 )

The 6-mph, (8.8 ft/sec), wind acts as a negative horizontal velocity. The parametric equations for this problem were given as:

$$X = 150T\cos 20^\circ - 8.8T \quad \& \quad Y = 3 + 150T\sin 20^\circ - 16T^2$$

Even though I knew these equations worked I didn't really understand that **150TCos20°** and **150TSin20°** represented the initial horizontal and vertical displacement components of the motion of the baseball. I knew the **T** was necessary for the graph but I wasn't comfortable with this until I designed an addendum to the curriculum that involved using the **TRACE** to calculate the velocity of such an object, as shown in Figure 7.

---




- Enter **X(T) = 5TCos 30°** & **Y(T) = 5TSin 30°** as the horizontal and vertical components of the displacement of an object travelling along a 30° path
- Set **RANGE** as T: 0 → 5, X: -5 → 25, Y: -5 → 15 ( Note: 3:2 ratio)
- Press **GRAPH**

--- **What is the velocity of the object along the path?** ---

Velocity - distance/ time and the hypotenuse, which is equal to 5T, represents the distance the object travels along the 30° path:

V = distance/time = 5T/T = 5

We can check this by doing a **TRACE** on our object

- Press **TRACE** and proceed until T = 3.8
- Calculate Distance = $\sqrt{(X^2 + Y^2)} = \sqrt{(16.4545^2 + 9.5^2)} = 19$
- Calculate velocity along the path V = D/T = 19/3.8 = 5

---

**Figure 7.** Calculation of velocity

## The Simulation of Specific Motion Problems - Part D

In this section I wanted specific problems and their solutions, representing a variety of motion situations. Problem #3, is a variation of a golfing problem from a TI-81 newsletter (Texas Instruments, 1992). The rest I designed myself. The first two problems are standard motion problems. Problem #4 involving horizontal motion, is useful in studying velocity. Problem #5, (see Figure 8), involves two cars approaching an intersection. When the equations and the range are set, the graph will show a visualization of the two cars approaching each other. To answer the questions about the distance between the cars a third graph needs to be drawn showing the shrinking distance between the cars. This graph is not so much a visualization as an abstract simulation of the distance. In a typical related rate problem in first year calculus, a question might be "How fast is the distance between the cars changing at a particular value of T?" A more difficult question would be to find the minimum distance between the cars. To determine this from the graph, without calculus, is perhaps the most powerful, and controversial aspect of this technique. I attended a workshop on calculus at the college level given by Bert Waits at the Northwest Mathematics Conference in Richmond, B.C. in Oct.1991. He used a more difficult variation of this problem using two people on two different sized ferris wheels to demonstrate the ease at which new technology can make traditional calculus teachers nervous.

---

Two cars are approaching an intersection at location ( 50, 50 ). Car A is coming from the north, 300 km away, at a speed of 40 km/h. Car B is approaching from the east, 400 km from the intersection, at a speed of 60 km/h.

a) At what time will the two cars be 100 km apart?
b) What is the minimum distance between them?

**Note:** The ( 50, 50 ) location for the intersection, off the X & Y axes, is chosen to allow for better viewing on the calculator.

- Set the path of car A as: **$X_{1T} = 50$ & $Y_{1T} = 350 - 40T$**
- Set the path of car B as: **$X_{2T} = 450 - 60T$ & $Y_{2T} = 50$**

**Note**: Car A starts at Y = 350 and moves downwards on the vertical line X = 50 at 40 km/h. Car B starts at X = 450 and moves to the left on the horizontal line Y = 50 at 60 km/h.

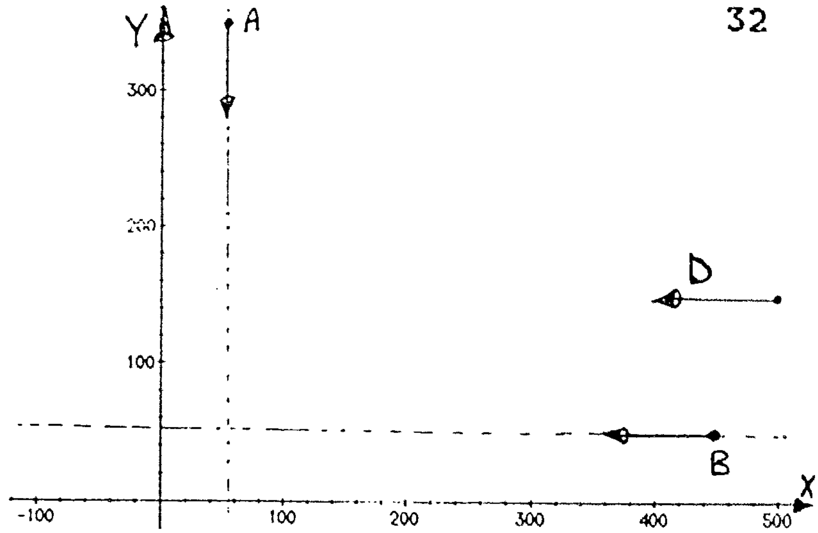
- To calculate the distance between the two cars, which is always the hypotenuse of a right triangle, use Pythagoras:

$$\text{Distance} = X_{3T} = \sqrt{(Y_{1T} - 50)^2 + (X_{2T} - 50)^2}$$

- Set **$Y_{3T}$ = 150**. The distance will be seen to be shrinking along this horizontal path towards a minimum then will increase once the cars pass each other ( you'll need a **TRACE** to see this ).

- Set **Range:**
  T: 0 → 10 (TStep = 0.05)
  X: -200 → 500 (XScl = 100)
  Y: -200 → 400 (YScl = 100)

| T | Y1 | X2 | D (X3) |
|---|---|---|---|
| 0 | 350 | 450 | 500 |
| 2 | 270 | 330 | 356.1 |
| 5 | 150 | 150 | 141.4 |
| 6 | 110 | 90 | 72.1 |
| 7 | 70 | 30 | 28.3 |
| 8 | 30 | -30 | 82.5 |
| 10 | -50 | -150 | 223.6 |




**Answers**
**a)** T = 8.25 h & T = 5.59 h. **b)** 27.74 m. (at T = 6.92 h.)

---

**Figure 8.** The two-car problem Part D - problem 5

At this point I decided to reverse the given information in a typical problem. Instead of looking for a point of intersection of two given functions, a problem could be devised to calculate the function necessary to intercept a given function at a given point. If two trains can be simulated to travel along tracks and the point of intersection can be calculated then a projectile could be launched to intercept a target at a pre-determined location. In target-shooting, two things need to be calculated - the time until contact and the parametric equations of the projectile. I demonstrated this in the first target shooting problem (see Figure 9).

---

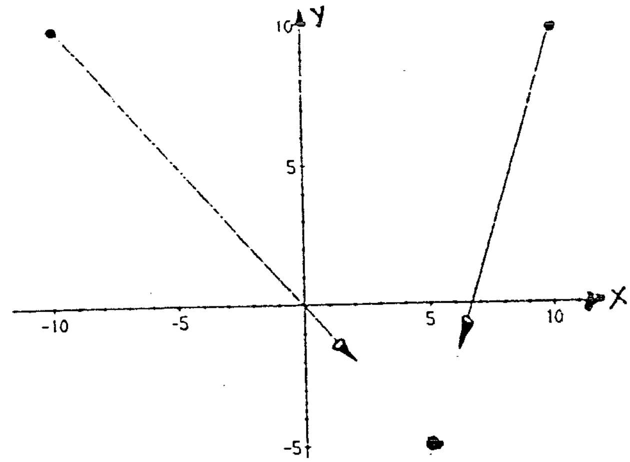
<u>**Target shooting#1:**</u>

A target is moving according to the function rules:

$$X_{1T} = -10 + T \quad \& \quad Y_{1T} = 10 - T$$

This target starts at the point (-10,10) and moves diagonally through the origin. The goal is to shoot the target at one of its locus points, say (5,-5) from a specified point say (10,10). This means
to mathematically devise the necessary parametric functions of a projectile that will intercept the target at the precise specified point at the ***exact***

***same time*.**




| T | X1 | Y1 | X2 | Y2 |
|---|---|---|---|---|
| 0 | -10 | 10 | 10 | 10 |
| 5 | -5 | 5 | | |
| 10 | 0 | 0 | | |
| ? | 5 | -5 | ? | ? |
| 20 | 10 | -10 | | |

**Step 1:**

- Find **T** by setting the target equations equal to the coordinates of the impact point (bold print in the table).

  $X_{1T} = -10 + \mathbf{T} = 5$

  $\mathbf{T} = 15$

**Step 2:**

- Find $X_{2T}$ & $Y_{2T}$
- $X_{2T} = 10 + \mathbf{h}T$ (at $T = 0$, $X_2 = 10$)

  $5 = 10 + \mathbf{h}(15)$ (at impact point, $T = 15$ and $X_2 = 5$)

  $-5 = 15\mathbf{h}$

  $\mathbf{h} = -1/3$
- $Y_{2T} = 10 + \mathbf{k}T$ (at $T = 0$, $Y_2 = 10$)

  $-5 = 10 + \mathbf{k}(15)$ (at impact point, $T = 15$ and $Y_2 = -5$)

  $\mathbf{k} = -1$

**Step 3:**

- Enter $\mathbf{X_{2T} = 10 - 1/3T}$ & $\mathbf{Y_{2T} = 10 - T}$
- Press **MODE**, choose the **SIMUL**taneous option then **GRAPH**.

---

**Figure 9.** Part D - Target shooting#1

## Design of the Problem Set - Part E

I set up the problem set ( part E ) as an assignment for the students in the field test. In designing the problem set I wanted to set up problems that could be solved by using algebra or by parametric representations. Problem #1 is a motion problem suitable for Grade 10 students:

1. Two planes leave airports 1000 km. apart, flying towards each other. Plane A leaves airport A at 1400h and travels at a constant speed of 200 km/h. Plane B leaves airport B at 1600h and travels at 300km/h. At what time and at what location will the two planes pass each other?

Problem #2 is a variation of the baseball problem previously described (Demana & Waits, 1993, p.497), where the velocity at impact, the angle and the initial height are known and the ball travels towards a 5m wall 150m out into center field. In this problem the student enters the equations and checks the trajectory and answers the questions.

Problem #3, (see Figure 10), is a metricised variation of the golf problem in the TI-81 Newsletter (Texas Instruments, 1992). Students are expected to experiment with different clubs, different angles and impact velocities, and choose the best club (greatest possible loft) for the shot.

---

3. A golfer is faced with a 140 m shot directly into a 5 m/sec wind. The problem in golf is to choose the best club for the shot. The best club is the highest numbered club to provide the greatest possible loft, thus minimizing the distance travelled after it hits the green. Each club imparts a different angle and a different maximum velocity to the ball at impact.

| **Club** | **Angle** | **Max. speed of ball** |
|---|---|---|
| 4-iron | 28° | 50 m/sec. |
| 5-iron | 32° | 48 m/sec |
| 6-iron | 36° | 45 m/sec |
| 7-iron | 39° | 40 m/sec |
| 8-iron | 43° | 36 m/sec |

Determine the best club and the velocity needed to drop the ball as close as possible to the hole.

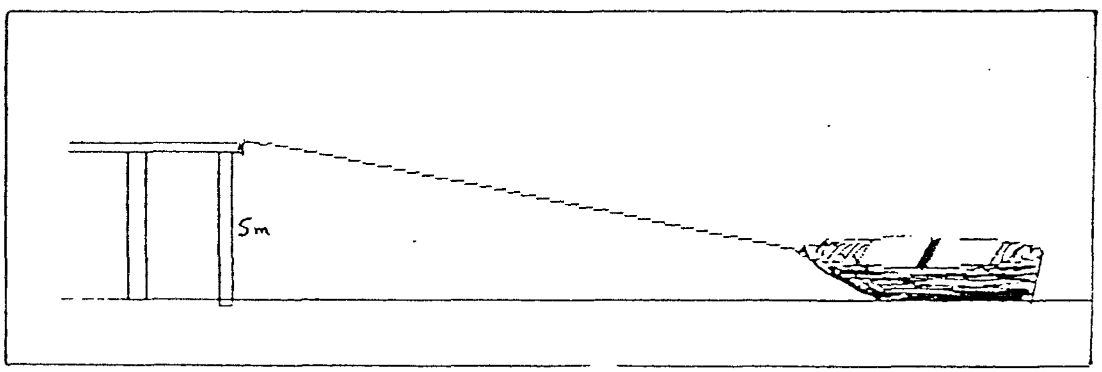
---

**Figure 10.** Golf problem: Part E - problem 3

For the benefit of the college students in the pilot I revised a related rate problem from their calculus text (Washington, 1990, p.710). A question specific to their curriculum would be "How fast is the boat moving at a particular value of T?" In this problem, (see Figure 11), the rope is being pulled in at a constant rate but the boat is accelerating. My intention with the table of values was for the students to see that the boat is accelerating since the distance travelled in each second increases. This simulation becomes in a sense an experiment, math-lab style:

---

**4.** A 25 m rope has one end attached to a boat and the other to a pulley on a dock. The rope is being pulled in at a constant rate of 2.50 m/sec. The water is 5 m below the pulley.




**a)** Set up two functions that will decrease, $X_{1T}$ showing the length of the rope, and $X_{2T}$ representing the distance of the boat from the dock. Graph both these functions on parallel horizontal lines.
**b)** Is the speed of the boat constant? Why or why not?
**c)** Complete the table below calculating the distance the boat travels during the 6th, 7th, and 8th seconds.

| T | $X_{2T}$ (Dist. from dock) | Dist. travelled that second |
|---|---|---|
| 5 | | XXXXXXXXXXXXXXXXXXXXX |
| 6 | | |
| 7 | | |
| 8 | | |

---

**Figure 11.** The boat problem Part E - problem 4

In designing the target shooting problems I wanted to set up a variety of paths that a projectile (target) could take. The target in problem #7, (see Figure 12), is travelling in a parabolic path. The problem seems difficult and I wasn't sure it would be successfully solved by the students in the field-test.

---

7. A ball is tossed from a height of 2 m. at an initial velocity of 25 m/sec. at an angle of 60°. Its path is described by:

$$X_{1T} = 25T \cos 60° \quad \& \quad Y_{1T} = 2 + 25T \sin 60° - 4.9T^2$$

From the point ( 100,0 ) a bullet is to be fired that will travel in a straight line at 100m/sec. The problem is to fire the gun so that it makes contact with the ball exactly 3 seconds after the ball is tossed.

**a)** How long after the ball is tossed should the gun be fired?
**b)** At what angle will the bullet travel?
**c)** What are the parametric equations simulating the path of the projectile?

---

**Figure 12.** A projectile problem Part E - problem 7

In all of the problems discussed so far the parameter, **T**, has stood for time. If motion is circular or elliptical then it becomes convenient to set up T as an angle although it can still be thought of as representing time. In a target shooting problem, (see Figure 13), the equations of the target are given. The target is moving in a circular path starting on the Y-axis and rotating counter-clockwise. The challenge is to hit the target at the exact time that it reaches the X-axis.

A particle is moving in a circular path starting at the point (0,3) and moving in a counter-clockwise direction according to the functions:

$$X_{1T} = -3\text{Sin}T \quad \& \quad Y_{1T} = 3\text{Cos}T$$

From a specified point, ( 10,10 ) launch a projectile that will intercept this particle as it reaches the point (3,0).

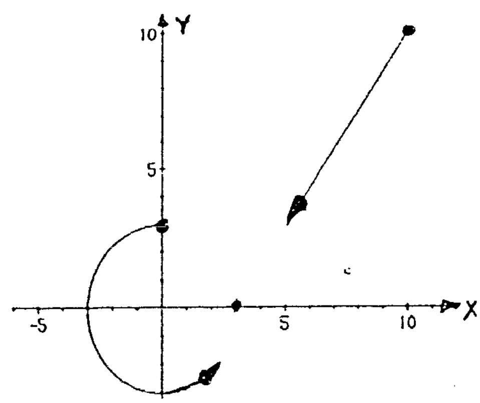
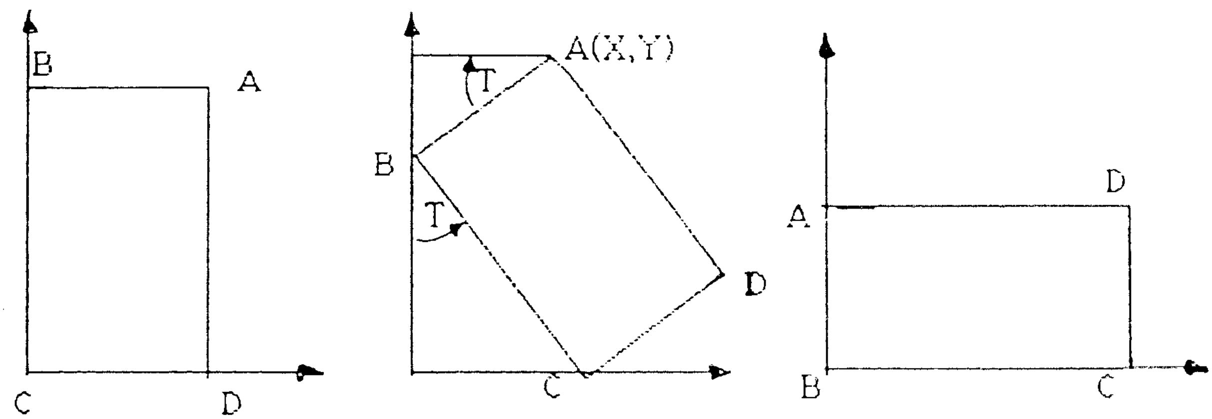



**Figure 13.** Part D - Target shooting #2

In a problem involving a rectangle that is touching both axes as it rotates, (see Figure 14), the student is asked to set up parametric functions representing the path of the tip of a rotating rectangle. **T**, in this case, is an angle and is further removed from being a representation of time. I didn't really expect anyone to solve this particular problem for a variety of reasons. The problem is a revision of the final question on a Euclid contest and as such is considered very difficult. The Euclid contest is written by the top grade 12 students in each school across Canada and the last question is often the one that determines the winners. Even with the hint of setting up the angle as a parameter the problem still requires a conceptual leap in the understanding of the role of a parameter:

6. A rectangle **ABCD** is moved so that vertex **B** always touches the Y-axis and vertex **C** always touches the X-axis. **AB** = 3 and **AD** = 5. Let ( X, Y ) be the coordinates of point **A**, and **T** be the angle shown in the diagram.




**a**) Find a pair of parametric representations **X(T)** and **Y(T)**.
**b**) Graph the path of point **A**.
**c**) Find the maximum value of **Y**.

**Figure 14.** Problem 6 - Part E

**Summary**

The curriculum package contains the range of features and applications of parametric representations. It asks students to:

1) graph functions starting at any point and going in either direction,

2) graph non-functions such as circles,

3) graph and trace the motion of an object that is moving on a horizontal or vertical line according to some function rule, and

4) simulate the motion of an object in two-dimensional space when the horizontal and vertical component functions are known, and the

parameter, T, represents time or an angle of rotation.

The crucial difference between this type of graphing and traditional graphing is that the student can observe the motion of an object as it moves through time, play it back at whatever speed is desired, and trace the graph to obtain the values of the three coordinates. In traditional graphing the graphs are static because they can only represent two out of the three coordinates.

## Pilot Study

The pilot of the curriculum package was with a group of technology-literate first year students at Camosun College. They were about half way through first year calculus having just finished a unit on applications of derivatives. They were excited at the technological capacity of simulating projectile problems and visualizing related rate problems.

I bypassed about half of the introductory material and tried to get to the applications as quickly as possible. For the first two hours with them, during their regular class time, I introduced the technology and the concept of graphing functions parametrically. We used a borrowed set of TI-81s in conjunction with the overhead version of the TI-85. To try to get them to come to the 3-hour Saturday morning session, I emphasized how much it would help them with understanding their calculus problems. They attended the final Saturday morning session with the promise from me that I would concentrate on applications that would be helpful for their upcoming calculus test. So after simulating the problems on related rates and projectile motion using parametric representations, we analyzed and solved them using calculus.

On Saturday morning five students showed up, and my son, who had just completed a Mathematics 12 course, operated the video camera and offered some suggestions as well. We spent most of an hour on the car problem (see Figure 15). Using calculus we were able to calculate at what time the distance between the two cars was a minimum. The simulation of the distance function allowed them to understand the problem in greater depth, and to check the accuracy of the answers that were obtained using calculus.

We then did the ladder problem, which was fortunate for them since coincidentally they were working on a similar project in physics where they had to find out what would be happening to someone standing on the center of the ladder. We also did the boat and the helicopter problems ( #4 & #5 from the problem set ), analyzing both of these problems using calculus at different values of time.

There were a few interesting observations that the students made. The most positive comment was that the simulation on the screen acted like a bridge between the problem and the calculus necessary to solve it. Certain problems, especially those involving related rates, are simplified by the use of parametric representations, because the problems can be visualized or simulated. The students felt more confident of the use of calculus once the problem was understood. The response was also quite strong to the whole concept of providing not just a graph but a "video of the event". One student, having just acquired an HP-48, regretted having sold her more user-friendly TI-81.

Two cars are approaching an intersection at location ( 50, 50 ). Car A is coming from the north, 300 km away, at a speed of 40 km/h. Car B is approaching from the east, 400 km from the intersection, at a speed of 60 km/h.

a) At what time will the two cars be 100 km apart?
b) What is the minimum distance between them?

- Set the path of car A as: **$X_{1T}$ = 50 & $Y_{1T}$ = 350 - 40T**
- Set the path of car B as: **$X_{2T}$ = 450 - 60T & $Y_{2T}$ = 50**

**Note**: Car A starts at Y = 350 and moves downwards on the vertical line X = 50 at 40 km/h. Car B starts at X = 450 and moves to the left on the horizontal line Y = 50 at 60km/h.

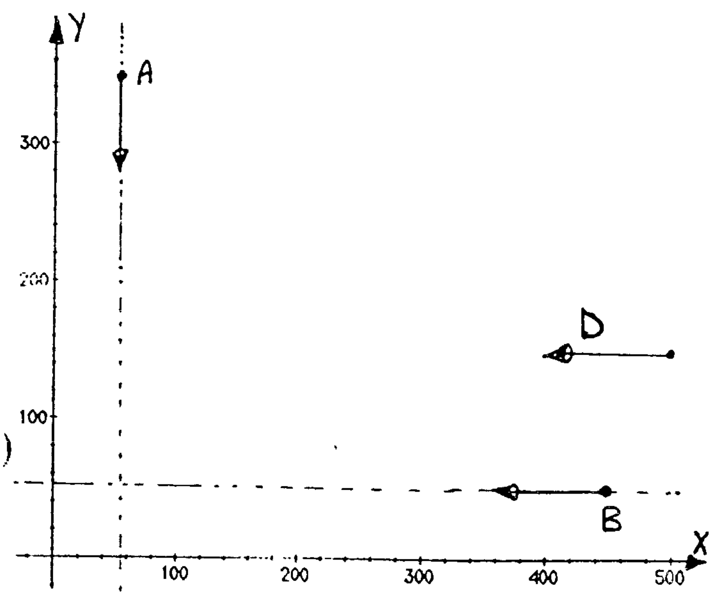
- To calculate the distance between the two cars, which is always the hypoteneuse of a right triangle, use Pythagoras:

$$\text{Distance} = X_{3T} = \sqrt{(Y_{1T} - 50)^2 + (X_{2T} - 50)^2}$$

- Set **$Y_{3T}$ = 150.** The distance will be seen to be shrinking along this horizontal path towards a minimum then will increase once the cars pass each other ( you'll need a **TRACE** to see this ).

| T | Y1 | X2 | D (X3) |
|---|---|---|---|
| 0 | 350 | 450 | 500 |
| 2 | 270 | 330 | 356.1 |
| 5 | 150 | 150 | 141.4 |
| 6 | 110 | 90 | 72.1 |
| 7 | 70 | 30 | 28.3 |
| 8 | 30 | -30 | 82.5 |
| 10 | -50 | -150 | 223.6 |

- Set **Range:**
  T: 0 → 10 (TStep = 0.05)
  X: -200 → 500 (XScl = 100)
  Y: -200 → 400 (YScl = 100)




**Answers**
**a)** T = 8.25 h & T = 5.59 h. **b)** 27.74 m. (at T = 6.92 h.)

**Figure 15.** Problem #5 - Part D

**Results of Questionnaires**

The five students who participated in the Saturday morning session of the pilot completed a pre-activity (see Table 1), and post-activity questionnaire (see Table 2). The choices for the questions were:

SD - Strongly Disagree, D - Disagree, N - No Opinion,

A - Agree, SA - Strongly Agree

The number of students are noted in bold underneath their choice.

**Table 1**

Pilot Pre-Activity Questionnnaire Results

| **Questions** | **Answers** | | | | |
|---|---|---|---|---|---|
| | SD | D | N | A | SA |
| 1. Problems involving motion like these are usually difficult for me to visualize. | | | **1** | **2** | **2** |
| 2. I often don't understand these problems even when I'm able to do the calculations. | | **2** | **1** | **2** | |
| 3. I have a hard time getting started on these problems. | | **1** | | **2** | **2** |
| 4. I usually get confused by these problems. | | | **2** | **2** | **1** |
| 5. I often experience a fair degree of anxiety when faced with problems like these. | | **1** | | **3** | **1** |

**Table 2**
Pilot Post-Activity Questionnaire Results

| Questions | SD | D | N | A | SA |
|---|---|---|---|---|---|
| 1. The simulation of these problems on a calculator screen, using the parametric representation of functions, helps me to visualize them. | | | | 2 | 3 |
| 2. A visual simulation of these problems helps me to understand what's happening in them. | | | | 2 | 3 |
| 3. When I see these problems simulated on a screen I feel more confident that I will be able to come up with a correct solution. | | | 1 | 2 | 2 |
| 4. My level of anxiety towards these problems became less when I saw them come to life on the screen of a graphics calculator. | | | | 3 | 2 |
| 5 I feel more confident in my ability to understand and solve these types of problems. | | 1 | 2 | 2 | |
| 6. The simulation of motion problems by parametric representations is an aid to problem solving. | | | | 2 | 3 |

(Answers: SD, D, N, A, SA)

**Summary**

The results of the questions #1, #2, and #6 (see Table 2) are an indication that the students agreed that this use of technology helped them to visualize and understand the problems, and as such become an aid in the solution of problems involving motion. It was clear to me from the experience of the pilot that the students felt that the use of parametric representations would be a worthwhile addition to their curriculum. For me, I was able to get a feel for the time necessary to teach the concepts and demonstrate the examples.

The inconclusive results of question #5 could be reflected in the ambiguity of the question itself. If a student disagrees that his or her level of confidence rose, it's because either the confidence was already high, or it was at an unshakeable low level. So I don't think the results of question #5 should be seen as a contradiction to the results of the other questions, which indicated that the simulation of these problems is an aid in understanding them and in their solution.

# Chapter 4. Results

This chapter contains four sections all pertaining to the field-testing of the curriculum package. The first section is a day-to-day description of the activities in the classroom. The second section contains tables of the results of the questionnaires. The third is a list of the written comments of the students. The fourth describes the revisions made to the curriculum.

## Field-test - Mt. Douglas Secondary Students

The field-test was with a group of Math 11 Enriched students from Mt.Douglas Secondary. They had little or no experience with graphics calculators much less with motion simulations with time as a parameter. This section includes the curriculum topics covered, the questions and comments of the students, and descriptions of their presentations during the final class. There were six hours of class time available. To fit the necessary components in to complete the main sections of the curriculum I omitted Part A - sections 3 and 4, Part B - section 1b) p.7, and Part B - section 3a), b), and c). In Part D, I demonstrated problems 1, 3, 5, and target shooting #1. Some of these sections involved understanding graphs of trigonometric functions, which they had not yet been taught at that point.

### Structure of Lessons - use of Technology

Each class with the high school students at Mt. Douglas Secondary was conducted using their class set of TI-81 calculators and the overhead version of the TI-85 belonging to Camosun College. The differences between the two technologies was minimal and probably worried me more than it did the students. When it came time for them to make their presentations on the sixth and final class they easily adapted to the TI-85 in front of the class. There were 23 students on the first day and 19 students on the final day.

The first four classes were taken up with my teaching the material and demonstrating some examples and solving some problems with the TI-85 on the screen. The students followed along on their calculators working loosely in groups which were forming by the students themselves so they could help each other and work together. It wasn't possible to assign any homework since they had no access to the calculators. The groups tightened up on the fifth day as the students worked on their group assignments which was to solve at least two of the problems from *Part E: The Problem Set* one of which had to be presented by the group in front of the class on the sixth day. The groups had three to four members in each and during the last two classes they worked with their desks pushed together. The students were told that the assignment was to be evaluated by the home teacher although this was purely to motivate them, since she thought they would work harder if there were marks involved.

The first five classes were recorded on audio tape and the presentations in the final class were video-taped.

**Day 1**

The first day was taken up mostly to introduce the graphing capacity of the calculators. The key point was to demonstrate to the students that a distance-time graph, such as one describing a ball thrown straight up, was not a simulation of the motion of the ball. They were asked to consider how to graph the motion of a ball thrown at an angle since there would then be three variables: the horizontal co-ordinate, X, the vertical co -ordinate, Y, and the time, T. We discussed how to graph circles under the regime of a calculator driven by a function mechanism.

The students then worked on the pre-assignment, which consisted of problems 1, 3 and 5 from Part D. They were asked to try to solve the problems and answer the pre-activity questionnaire at the end. My intention was to familiarize the students with the kinds of problems suitable for parametrics, before handing out the curriculum package. These three problems are solved using parametrics in the package.

**Day 2**

The second class was spent introducing the parametric representation of functions and the role of the variable *T*. Each student was given a copy of the curriculum package and the students experimented with this new way of graphing by following the instructions from me or from the package. Many of them were quite responsive and could tell that when a graph stopped it had run out of 'time'. We concentrated solely on linear functions and linear motion, including motion with a given angle, but with no acceleration component in either case. On this day we completed Part B - sections 1a), 1b) and 3e). It was in this class that I recognized the need for an addendum to make a better link between analytical geometry and the physics of motion. We finished the class by doing the *Girl on the bike* problem, (Part D - problem 1), using parametrics. When I asked which of the graphs represented the girl on the bike and which one the father in the car, the answer came quickly: " The top one's the father's because it's moving faster."

**Day 3**

The third day was spent on quadratic functions, starting with Part B - sections 2a). We started with **X** = **T** and Y as a function of T and the parabolas were familiar to the students except for the fact that they would stop because they ran out of time or they would go backwards, from right to left, if we changed to **X** = **5** - **T**. On a simultaneous graph of three parabolic graphs, ( Part B - section 2b), arching out from the same spot at different angles and meeting at the ground at the same spot and at the same time, the spirit of what they were seeing was evidenced by the sound effects of the whistling sounds of projectiles. We then extended the linear motion problem done the day before, (Part B - section 3e), given the initial angle and velocity by including gravitational effects which in effect makes it quadratic.

We spent about twenty minutes on Part C - section 1, which involved the horizontal motion of three objects according to the functions:

$$X_{1T} = 50 + 30T \quad \& \quad Y_{1T} = 4$$

$$X_{2T} = 50 + 2T^2 \quad \& \quad Y_{2T} = 8$$

$$X_{3T} = 50 - 20T + 2T^2 \quad \& \quad Y_{3T} = 12$$

When we were simulating motion on a straight line according to the third function one student cleverly pointed out that this was like a horizontal (one dimensional) parabola. This was at the end of the third hour with them and they were helping each other w .h tracing the object using the cursor buttons, wondering why the object moved to the left at the beginning then slowed, stopped and accelerated to the right. One student who had gone backwards in time got stuck at the beginning and wondered why but was quickly rescued by another student. The students experimented a lot with this section by trying to set up a race between the objects by changing the initial conditions or by entering a cubic function. There was some discussion on the velocity and acceleration of each object, and relating these "horizontal parabolas" to previous ones they had done. For the third graph we related the point at which the object stopped to the "vertex of the parabola".

**Day 4**

On the fourth day we started with the simulation of the motion of three objects dropping under three different gravitational environments (Part C - section 2). We also demonstrated this using DOT Mode with TStep = 1 so the students could see how far the objects would fall in each second. We then calculated the speed of the objects along linear paths as is done in Part C - section 3. We also did this calculation using trigonometry where the initial angle is given. This is explained in the addendum which is included at the end of this chapter (see Figure 17). Perhaps the most positive comments came when we were measuring the velocity of an object along a 30° slanted path. We had entered:

**X = 5TCos30 & Y = 5TSin30**

When we did a trace, we used the X & Y coordinates to calculate the distance travelled, divided the distance by the time coordinate and verified that in fact the object was travelling with a velocity of 5 along a 30° path. The comments indicated that we were really onto a real-life simulation and not just doing parlour tricks. This section was not in my curriculum package, but I considered it a break-through, which is why I created the addendum. One student cleverly suggested setting up DOT Mode with TStep = 1 and working out the distance between the dots, which of course is correct and probably easier. To end the class we played with the golf problem, Problem #3 in Part D.

**Day 5**

On the fifth day we solved three problems using parametric representations, each one from Part D: the car problem, Problem #5, and the target shooting problem. The rest of the time was group work on specific problems chosen by the students from the set of unsolved problems in Part E. The regular teacher at this point seemed a bit concerned that some of the students would not come up with much for presentation day. At this point the students probably realized that marks were not at issue and a few had become spectators for whatever reason.

**Day 6**

On the sixth day I set up the video while the students worked on the problems. I suggested that everyone choose the plane problem ( #1 ) just to get started and at least one other, and that it would be all right if two groups presented the same problem.

The first group presented the helicopter problem (Part E #5). One student, who had already contributed some interesting comments during Day 4, suggested that they graph not just the falling steel ball but also the helicopter overhead. It was clear immediately to him that the horizontal component of the helicopter would be the same as that of the ball. When his solution came up on the screen the helicopter was always directly above the ball as it arched towards the ground.

The second group presented the plane problem, (Part E #1), and the next twenty minutes at least were spent working on other problems. The third presentation was the baseball problem (Part E #2 ). The student graphed the 5m wall at 150m from home plate and showed the fly ball hit the wall at about 4m from the ground. There was some discussion about whether that ball could be caught. The student built in a wind factor of 1.5 m/sec and on this attempt the ball cleared the wall. He had started with 5 m/sec then worked down to 1.5 m/sec.

The fourth and final presentation, ( Part E #7 ), was the projectile (not under gravity's influence) shooting a ball that had been tossed (gravity included). It was a challenge but was quite popular and was successfully solved by two of the groups that attempted it. The problem was to calculate when the shot should be fired so that the ball would be hit exactly three seconds after it was tossed. This problem involved a lot of calculations and another group solved it correctly, but for some reason couldn't simulate it properly. The problem was successfully completed by the bell and the students by this time had finished their post-activity questionnaire. The applause for the final student group presentation, which was the successful completion of problem 7, the shooting of the tossed ball with a bullet, was quite enthusiastic. It was a dramatic finish, made more so by the timing, a few seconds before the noon announcements. One student who thanked me afterwards was honest in his belief that this six-hour session would help him a lot with his next year's math and physics courses.

## Results of Questionnaires - Field-Test

Twenty-four students participated in the pre-activity questionnaire, (see Table 3), and nineteen were present for the post-activity questionnaire (see Table 4). The choices for each question were:

SD - Strongly Disagree, D - Disagree, N - No Opinion,

A - Agree, SA - Strongly Agree

The number of students are noted in bold underneath their choice.

**Table 3**
Field Test Pre-Activity Questionnaire Results

| **Questions** | **Answers** | | | | |
|---|---|---|---|---|---|
| | SD | D | N | A | SA |
| 1. I've encountered problems similar to these before | **0** | **5** | **1** | **15** | **3** |
| 2. Problems involving motion like these are usually difficult for me to visualize. | **1** | **14** | **4** | **5** | **0** |
| 3. I often don't understand these problems even when I'm able to do the calculations. | **5** | **10** | **2** | **4** | **2** |
| 4. I have a hard time getting started on these problems. | **1** | **5** | **5** | **6** | **6** |
| 5. I usually get confused by these problems. | **0** | **6** | **7** | **10** | **0** |
| 6. I often experience a fair degree of anxiety when faced with problems like these. | **1** | **10** | **3** | **5** | **4** |

**Table 4**
Field Test Post-Activity Questionnaire Results

| Questions | Answers SD | D | N | A | SA |
|---|---|---|---|---|---|
| 1. The simulation of these problems on a calculator screen, using the parametric representation of functions, helps me to visualize them. | 2 | 1 | 2 | 3 | 11 |
| 2. A visual simulation of these problems helps me to understand what's happening in them. | 3 | 1 | 0 | 9 | 6 |
| 3. When I see these problems simulated on a screen I feel more confident that I will be able to come up with a correct solution. | 3 | 4 | 2 | 6 | 4 |
| 4. My level of anxiety towards these problems became less when I saw them come to life on the screen of a graphics calculator. | 4 | 3 | 3 | 8 | 0 |
| 5 I feel more confident in my ability to understand and solve these types of problems. | 6 | 3 | 3 | 7 | 0 |
| 6. The simulation of motion problems by parametric representations is an aid to problem solving. | 3 | 0 | 1 | 12 | 4 |
| 7. This work we did with calculators will help me deal with motion problems in the future. | 4 | 2 | 5 | 7 | 1 |

## Written comments

The two questions below were at the end of the post-activity questionnaire and allowed for open comments about the students' experiences.

**How do you think this work will help you deal with these kinds of motion problems?**

- would help me visualize the motion of objects
- it may be easier to write on a calculator but it doesn't seem to make much of a difference
- makes horizontal & vertical components of problem more clear
- when you have something real to work with you can better visualize the relationship ( relative or not ! )
- it will take less time because you don't have to do guesswork
- I'll be able to picture what's happening and the speed, height etc..
- it might not unless I have a graphics calculator, but I think I have a better understanding
- just to be able to able to visualize the problem, if the graph looks like what you think it should look like in real life you know you're on the right track
- need more time with the calculators to understand

**Are there any other comments, critical, helpful, suggestions etc.... that you would like to add**

- I needed more reasons for why we did what we did.
- interesting but confusing
- more relation to the math we are presently learning would be helpfull,

although more interaction would have been appreciated, both student & teacher

- I think sometimes the instructor assumed that we were clear on things before we totally understood them
- more time to work on group (or individual) problems

## Revision of Curriculum Package - Addendum

The field-test involved more work than the pilot on Part B and Part C where the concentration was mainly on the demonstration of problems by using parametric representations and calculus. It was still difficult to get through the material, largely because of the students' unfamiliarity with the calculators and their lack of access to the calculators outside of class time. We concentrated only on motion in a straight line, or in parabolic arches such as projectiles. It was during the introduction of the parabolic motion of projectiles using trigonometry, where the initial angle is known, that I came up with a link that was missing in the curriculum package.

When I was writing the curriculum I remember being uncomfortable with the transition I was making between analytical geometry and the physics of motion. I remember not being convinced, or at least uncertain that:

$$X = 25TCos35 \text{ and } Y = 25TSin35$$

would in fact propel an object along a 35° path at a velocity of 25. Part C - section 3 has been changed as shown in the revised table of contents (see Figure 16). The new section 3b) is the addendum (see Figure 17). The original section 3b) would be split into two, section 3c) and 3d).

**Original**

**Part C : Motion Simulation From a Given Point**



**Revised**

**Part C : Motion Simulation From a Given Point**

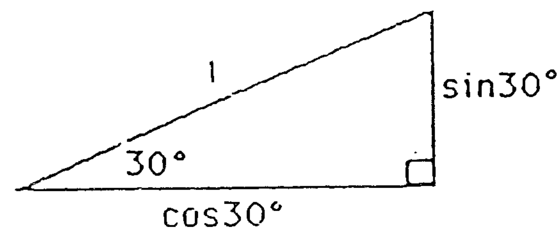


**Figure 16.** Revised table of contents

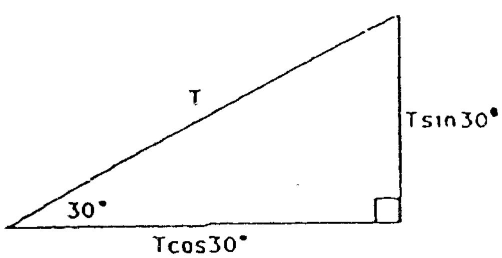
## 3 b) Linear Motion - given the velocity and angle

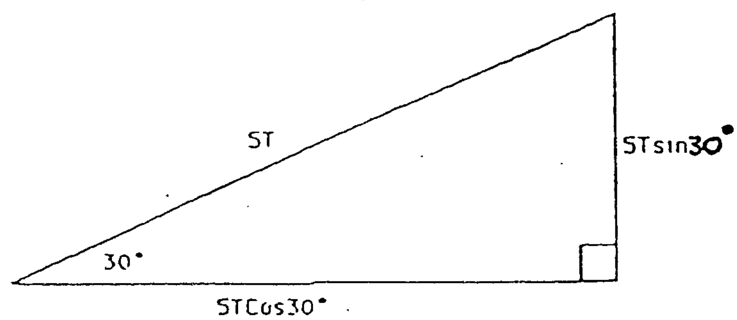
In a right triangle with a base angle of 30° and a hypoteneuse of one unit in length, the adjacent and opposite sides ( X & Y ), can be calculated by:

Sin 30° = opposite/hypoteneuse
= Y/1
---> Y = Sin 30°
Cos 30° = adjacent/hypoteneuse
= X/1
----> X = Cos 30°




If the Hypoteneuse is changed to T or 5T, and the base angle is kept at 30° the triangles will be similar, so the new X & Y values will just be multiples of Cos 30° and Sin 30° as shown in the diagrams below.




- Enter **X(T) = 5TCos 30° & Y(T) = 5TSin 30°** as the horizontal and vertical components of the displacement of an object travelling along a 30° path
- Set **RANGE** as T: 0 → 5, X: -5 → 25, Y: -5 → 15 ( Note: 3:2 ratio)
- Press **GRAPH**

--- **What is the velocity of the object along the path?** ---

Velocity - distance/ time and the hypoteneuse, which is equal to 5T, represents the distance the object travels along the 30° path:

V = distance/time = 5T/T = 5

We can check this by doing a **TRACE** on our object

- Press **TRACE** and proceed until T = 3.8
- Calculate Distance = $\sqrt{(X^2 + Y^2)} = \sqrt{(16.4545^2 + 9.5^2)} = 19$
- Calculate velocity along the path V = D/T = 19/3.8 = 5

**Figure 17**. Addendum: Revised Part 3b

# Chapter 5. Discussion and Conclusions

## Conclusions from Questionnaires

A total of 29 students ( 5 from pilot, 24 from field-test) participated in the pre-activity questionnaire and 24 answered the post-activity questionnaire. Five high school students in the field-test were not present on the final day. The first questionnaire focused on difficulties students sometimes experience when faced with problems involving motion. These can take the form of high levels of anxiety or difficulties in getting started on or even visualizing the problem. The college students, whose high school mathematics experiences were of mixed success, were in strong agreement that these difficulties were very real for problems of this type, and unanimous in agreeing that representing these problems on a screen using parametric representations, both alleviated these difficulties and aided in their solution. The high school students were somewhat more confident of their abilities, had low levels of anxiety, but admitted to some confusion and considerable difficulty in getting started on problems of this type. The post-activity questionnaire results were much more conclusive. Over 70% of the students agreed that using parametric representations helped them visualize the problems and aided them in understanding and solving the problems.

The results of question #5 seemed to contradict those of questions #2 and #6 (see Table 4). The responses to questions #2 and #6 indicate that the visualization of the problems on the screen is an aid in the understanding and the solution of the problems. The responses to question #5 indicate that the confidence levels of the students in approaching problems of this type remained unchanged. As previously discussed with the results of the pilot questionnaire, the responses to this question are hard to interpret, since for the better students their confidence levels may have been high to begin with.

## Conclusions from Comments

Once the students overcame the mysteries of the technology and the strange way of representing functions their participation and sense of experimentation increased. This was evidenced during the classes by sound effects accompanying the projectiles and comments such as "this is like a video" or "this is useful (meaning relevant) stuff". The generally high level of activity, curiosity, and to some extent excitement, especially during the final student presentation was captured on the video, but the most significant comment came from one student on the open-ended question at the end of the questionnaire.

- *just to be able to able to visualize the problem, if the graph looks like what you think it should look like in real life you know you're on the right track*

Often teachers and other mathematics practitioners worry about giving away too much, that the extended use of graphics calculators will reduce the students' thinking capacity and involvement in the problem. This student's comment should alleviate such concerns.

## Changing Roles of Students and Teachers

The use of this curriculum implies a change in teaching style. The students will have in their hands a problem solving tool that they will be experimenting with, coming up with ideas or methods of solution in unpredictable ways. Teachers may be reluctant to involve themselves with the use of technology in this way for a variety of reasons. Even if their level of confidence, knowledge and organizational skills are high they may not be prepared to surrender their control and switch to an environment that is more experimental and as a consequence more chaotic. Students naturally fall into 'What if?' situations in this kind of environment. One student, not happy with my solution of the bicycle problem pushed the problem by fiddling with the equations to see what was really going on. On another occasion when we had three objects moving on horizontal paths questions arose as to how we could make one of the particles speed up to catch up with another or what would happen if we changed the equation to a cubic one. With the machines in their hands it's to be expected that students are going to play, and as a result come up with some discoveries. Another example of this occurred after I showed them briefly how to draw a circle parametrically using trigonometric functions. One student, by varying the TStep and the equations as well, came up with a star with about 100 points and wanted to know why.

## The Effect of the Increase of Real-World Applications

In an already overloaded mathematics curriculum at the Grade 11 and 12 level, there may not be a place for experimenting with the simulation of real-world problems involving motion. Teachers are far too busy with a tightly packed curriculum consisting for the most part of basic skills. Mathematics teachers may feel uncomfortable straying outside the borders of their discipline. Comments that students made during the trials of the curriculum dealing with the relevance of what we were doing, and the likening of what we were doing to a "video of the problem", and the enthusiasm of the participants of the workshop participants indicate that change is inevitable. And that change must fall to the classroom teacher to concentrate on the kind of applications that he or she never had time to do before, and to the central authorities to create the time to do it.

## The Effect of an Increase in Experimentation

Perhaps the most subtle change will come about as a result of the increase in experimentation in the mathematics classroom. Algebraic skills and processes will still be necessary for formal and generalized solutions of problems, but never before have students had the power to *learn* by trial and error and feedback. Instead of learning algorithms for specific problems, students will be trying different approaches and coming up with new ideas on their own, with feedback that is close to being instantaneous. When I was calculating the speed of an object along a linear path by dividing the distance travelled by the time, one student suggested setting the TStep = 1 making the distance between the dots equivalent to the speed. Instead of a substitution for thought which is what many educators are afraid of, my experience is that calculators encourage thought, and with the speed of the technology, can reinforce a good idea instantly.

## Implications and Summary

In designing this package I needed to consider the difficulties involved in field-testing the curriculum. Given the pressured state of most senior mathematics classrooms in British Columbia, I felt the only possibility would be a Math 11 Enriched class since the better students often complete the grade 11 curriculum sooner than a regular class. Even so, I felt that the maximum time that I could hope to acquire would be six hours. If more time, and calculators, were available the package could be used more independently by the students. As it was the students in the field-test felt as I did that they were being pushed along. With less teacher input, and a greater variety of problem assignments, the package could be used more as a reference manual by the students. The focus would then be not just on the use of parametric representations as an aid to problem-solving, but on the development of the individual problem-solving skills of the students themselves. The experimental nature of the topic combined with the use of graphics calculators, imply that the best environment would be something like a math lab where the students could proceed at their own pace and design their own assignments.

When graphic calculators first appeared in mathematics classrooms about six years ago there was some concern and still is, about their role, and whether students would use them as a substitute for thought. Students must spend their time, not relying just on their calculators, but analyzing the problems algebraically, and later using calculus, so that they can appreciate that calculators and computers do not solve problems, but more and more are becoming an integrated and essential part of the process of problem-solving.

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APPENDIX

# The Simulation of Motion Problems by the Parametric Representation of Functions using the TI-81 Graphics Calculator

by
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January 1993

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**Part D: The Simulation of Specific Motion Problems**



**Part E: Problem Set**

Nine challenges:



**Appendix**

# Part A: Graphing Functions of the Form y = F(x)

**How to activate options:**

- Options can be selected by using the arrow keys to move a flashing cursor. When the cursor is over a desired option press **ENTER** to activate the option. An option is active if shaded.

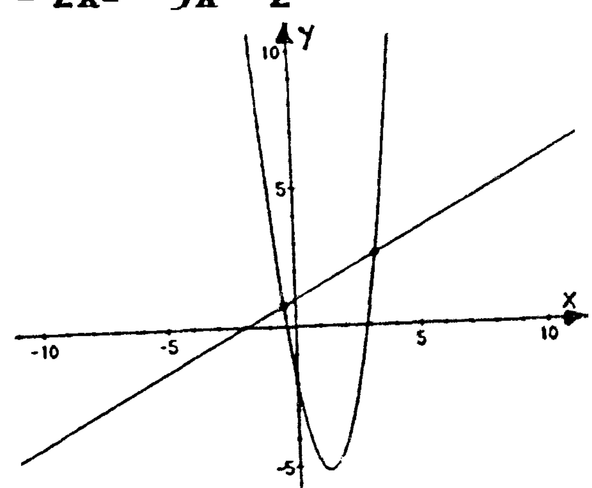
**How to graph functions of the form y - F(x):**

**1. Graphing linear functions.**

- Press **MODE** to check that the **Function** option is activated. It will be shaded when activated.
- Press the **Y -** key. There are four choices for entering functions: **$Y_1$=, $Y_2$=, $Y_3$=, or $Y_4$=.**
- At **$Y_1$=**, where the cursor is flashing, type **0.5X + 1**. (Find X on the X | T key). Press **ZOOM.** Select menu item **6** to see the plot of the graph using the standard range.
- Experiment with the **TRACE** and **ZOOM** functions. Refer to the appendix for instructions on using these two functions.

**2. Graphing quadratic functions.**

- Press the **Y -** key. Enter **$Y_2 - 2X^2 - 5X - 2$.** The **$Y_1$** and **$Y_2$** functions can be graphed in sequence or simultaneously Press the **MODE** key to select the **SEQUENCE** or **SIMUL** option. The **TRACE** can be used to get the points of intersection of these two functions. The X-values of these two points of intersection are the solutions of the quadratic equation: **$0.5X + 1 = 2X^2 - 5X - 2$**

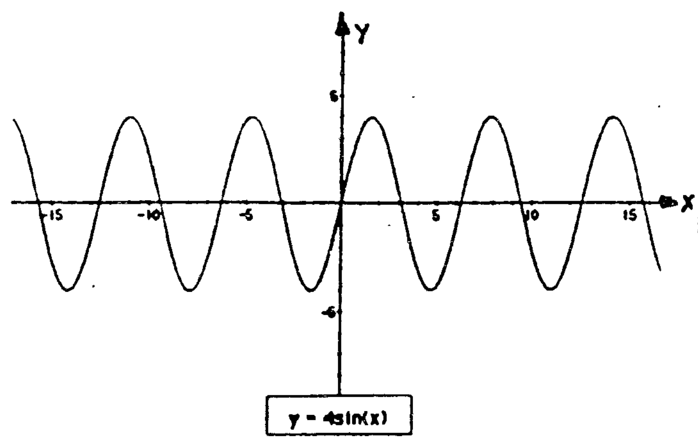



- Even without zooming, a **TRACE** on the graphs shown at the right will produce X-coordinates:
  **{ 3.3, -0.47 }**

- The solution of this equation using the quadratic formula is:
  **{ 3.216 , -0.466 }**

### 3. Graphing trigonometric functions.

- Press **Y** = key. Enter **Y3** = **4 Sin X**. Press the **MODE** key and select the **Radian** option. (To deactivate the earlier entered functions **Y1** & **Y2**, position the cursor over each equal sign, in turn, and press the **ENTER** key.) Now press the **GRAPH** key. The trigonometric function entered at **Y3**- will be graphed as shown below.




### 4. Combining graphs.

- Press **Y** = and perform the following steps to set up the function **Y4** = **Y1** + **Y2**. First press the blue **2nd** key then the **Y-Vars** key. Select menu item **1** and press **ENTER**. Press the following keys in sequence: **+**, **2nd**, **Y-Vars**, **2**. Deactivate **Y2**. Press **GRAPH** and function **Y4** will be displayed.

- To graph the **Y1**,**Y2**, & **Y4** functions simultaneously press **Y**=. Re-activate **Y1** & **Y2**. Press the **MODE** key. Select the **SIMUL** option. Press **GRAPH**. Notice that **Y4** is a quadratic function with function values equal to the sum of the function values of the other two functions.

- Change **Y4** to **Y1** * **Y2** (Multiplication), deactivate **Y1** & **Y2**. Graph the new function **Y4**. Notice that the new function is cubic because of the multiplication Of the **X** in **Y1** by the **$X^2$** in **Y2**.

- Press **Y** = and enter **Y4** = **Y1** + **Y3**. Ensure **Y2** is deactivated. Can you predict what the graph of the new **Y4** will look like? Now graph **Y4**.

# Part B: Introduction to Parametric Representations of Functions.

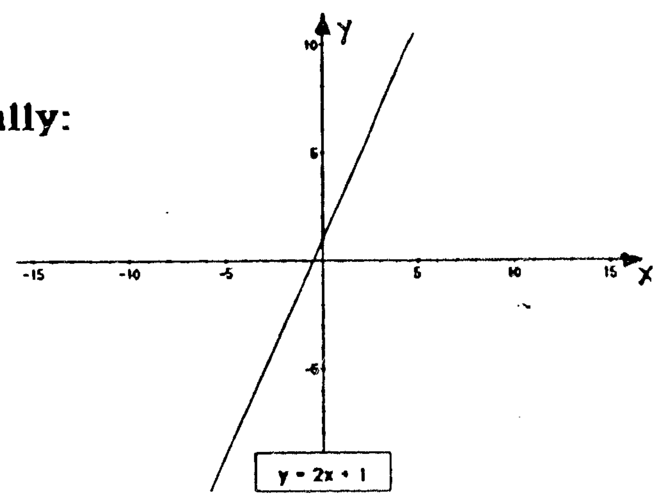
## Using the PARAMeter option:

- Pressing **MODE** and selecting the **PARAM** option activates the parametric mode. In this mode the **X | T** key will display a **T** instead of an **X**.

- Pressing the **Y –** key displays three pairs of functions. To graph a single function parametrically, a pair of functions, **$X_{1T}$ & $Y_{1T}$**, or **$X_{2T}$ & $Y_{2T}$**, or **$X_{3T}$ & $Y_{3T}$** is entered. Each member of each pair is a function of the parameter **T** which can be thought of as representing **TIME**. [ $X_{2T}$ - $X_2(T)$ means $X_2$ is a function of T.]

- The speed of the graph can be changed by varying the **TStep** which is found in the **RANGE.**

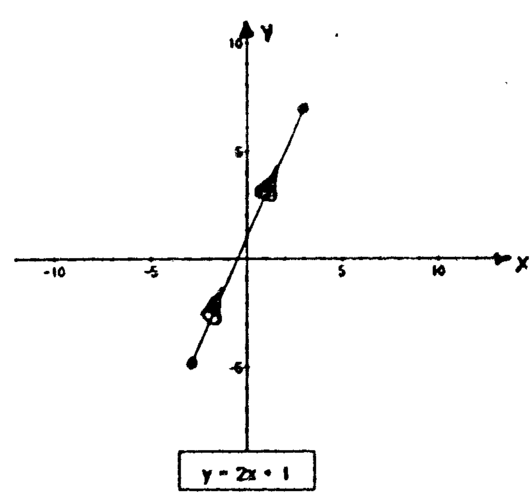
## How to graph functions parametrically:




### 1. a) Given a linear function.

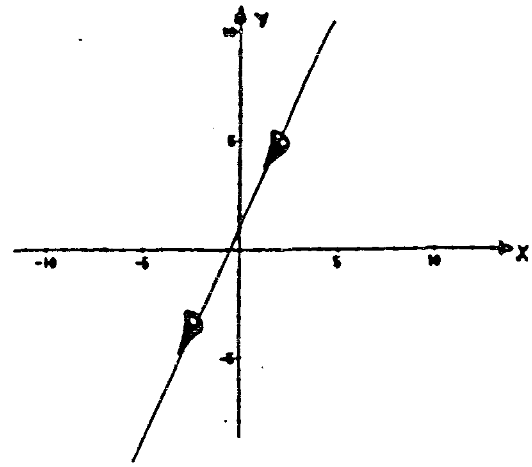
- The linear function **Y = 2X + 1** whose graph is shown at the right can be represented parametrically by introducing a parameter **T**. This can be done in many ways.

***One way:***




- Press **Y =** and enter **$X_{1T}$ – T & $Y_{1T}$ = 2T + 1**
- Press **Range** and vary T from -3 -> 3.
- Set **TStep** - 0.05. Press **GRAPH.** Note the direction of the graph.
- Change the **TStep**. Press **TRACE.** The instantaneous values of **T**, **X**, & **Y** are displayed.

*Another way:*




- Press **Y** = and enter
  **X2T = 2 - 5T.**
  (**Note**: If **2 - 5T** is algebraically substituted for X in Y = 2X + 1, the result is Y = 5 - 10T.)
- Enter **Y2T = 5 - 10T.**
- Press **GRAPH.** Using the T-Range from -3 -> 3, as set in the first way, the same line will be graphed but in a significantly different way as shown.
- A table of values, showing the X and Y values as T changes, demonstrates the mechanics of the graphing process.

| T | X | Y |
|---|---|---|
| -3 | 17 | 35 |
| 0 | 2 | 5 |
| 3 | -13 | -25 |

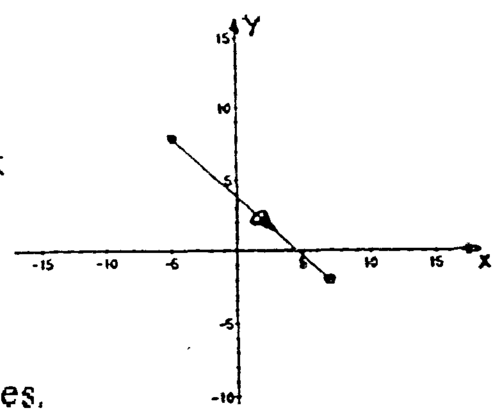
**1. b) Drawing a line joining two given points: (-5, 8) & (7,-2)**

- Press **Range** and set **Tmin** = 0.
  If T represents TIME then starting at T = 0 would be natural.

| T | X | Y |
|---|---|---|
| 0 | -5 | 8 |
| ? | 7 | -2 |

*One way to draw this line:*

- To increase X & T at the same rate enter
  **X1T** = -5 + **T** and vary T from 0 -> 12.
  **Note**: **X** changes from -5 to 7 as **T** changes at the same rate. **Y**, however, changes at a different rate, as it decreases from 8 to -2.




- Calculate the function **Y1T**, by setting  Y1T = 8 + kT
- Substitute T & Y (values from table)  -2 = 8 + 12k
  -10 = 12k
- Solve for k  -5/6 = k
- Enter **Y1T = 8 + (-5/6)T.**

- Set the **RANGE** using the table values.
- Press **GRAPH.**
  **Note**: The slope of the line is -5/6 which represents how Y is changing compared to T ( and in this case X).

***Another way to draw this line:***

- To increase X at twice the rate of T, enter
  **X2T = -5 + 2T**

- To calculate the final value of T, substitute the final value of X from the table into the equation

  X2T = -5 + 2T
  7 = -5 + 2T

- Solve for T  6 = T

**Note:** X changes at twice the rate of T, reaching its final value of 7 when T = 6. (In half the *time*.)

| T | X | Y |
|---|---|---|
| 0 | -5 | 8 |
| 6 | 7 | -2 |

- Calculate the function **Y2T,** by setting

  Y2T = 8 + kT

- Substitute Y = -2 & T = 6  -2 = 8 + 6k

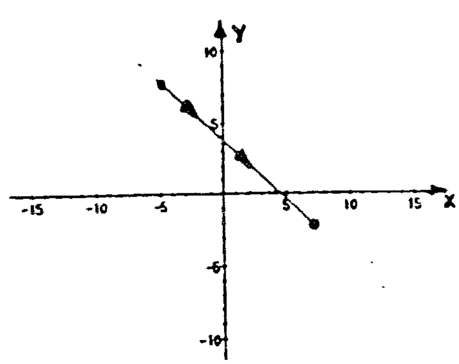
  -10 = 6k

- Solve for k  -5/3 = k

- Enter **Y2T = 8 + (-5/3)T.**

- Set the **RANGE** using the final table values of X and Y and Tmax = 6.

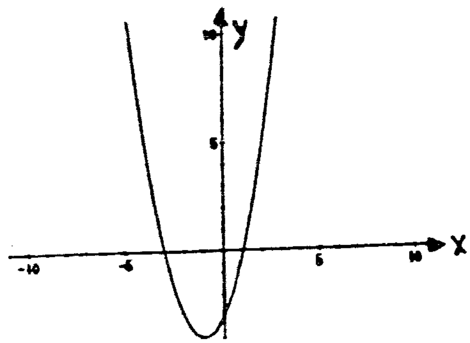
- Press **GRAPH.**




**Note:** The slope of the line graphed is the same with both approaches, but with this second approach, X & Y are each increasing at twice the previous rate - in half the *time*.

**2. a) Given a quadratic function:**

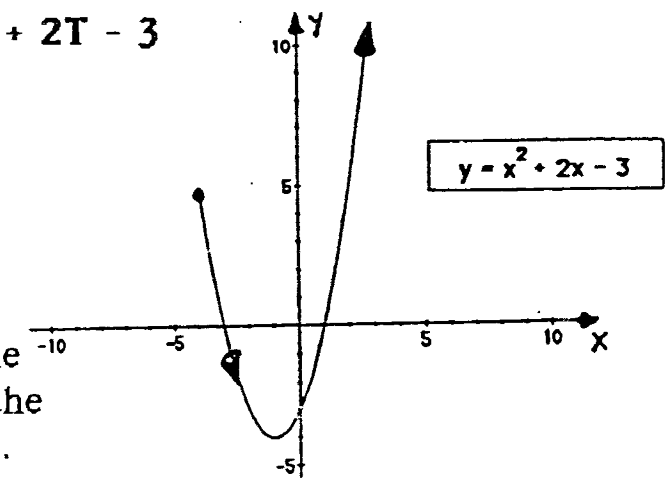
- The graph of $Y = X^2 + 2X - 3 = (X + 3)(X - 1)$ is shown:



- This function can be drawn parametrically by setting:

$$X_{1T} = T \quad \& \quad Y_{1T} = T^2 + 2T - 3$$

- Set **Range** as below
  T: -4 -> 4
  X: -6 -> 6
  Y: -5 -> 25
  TStep = 0.05




- Press **Graph**. The graph on the right shows which section of the original function was graphed.

- By varying the starting point ( initial value of T ) this graph can be viewed in sections.

- Also by varying the **TStep** and using the **TRACE** the behavior of this function can be investigated for different values of T. For example as T increases from T = -1 the cursor can be seen to steadily accelerate upward.

**Note:** In this case since the choice was made to let **X = T** the graph was drawn from left to right with X increasing at the same rate as T. As in the case of the linear function we could graph the same function in a different way. For example if we choose: $X_{2T} = 2 - T$
and substitute this for X in $Y = X^2 + 2X - 3$ we get:

$$Y_{2T} = T^2 - 6T + 5$$

- Enter this pair of functions
- Press **GRAPH** and notice that the end result is the same

## b) Designing a Quadratic Function to connect (0,20) & (20,0)

**Note:** This can be done any number of ways. In the three cases below I've kept **X** = **T** so the graphs will proceed left to right. In each of the three cases a value of k can be determined by substituting from the table:

| T | X | Y |
|---|---|---|
| 0 | 0 | 20 |
| 20 | 20 | 0 |

**i)** Enter $X_{1T} = T$ & Set $Y_{1T} = 20 + \mathbf{k}T^2$

Substitute $0 = 20 + \mathbf{k}(20^2)$

Solve for **k** $\mathbf{k} = -0.05$

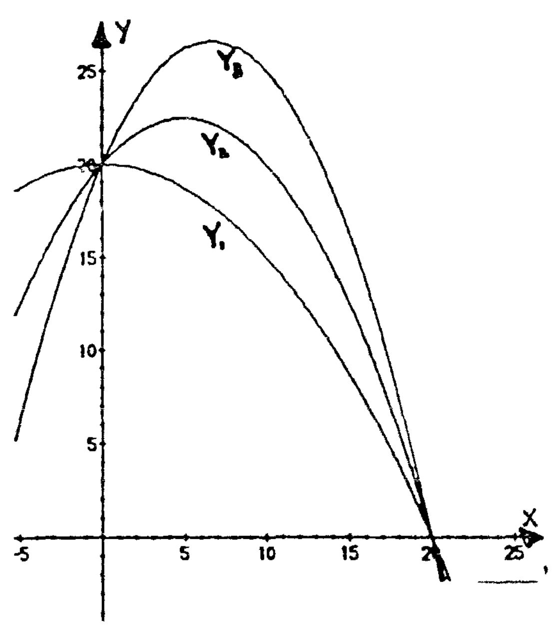
- Enter $Y_{1T} = 20 - 0.05T^2$
- Set appropriate ranges and **GRAPH** the first function
  **Note**: Y decreases steadily and if you do a **TRACE** you'll notice the cursor accelerate downwards.

**ii)** Enter $X_{2T} = T$ & Set $Y_{2T} = 20 + T + \mathbf{k}T^2$

Substitute $0 = 20 + 20 + \mathbf{k}(20^2)$

Solve for k $\mathbf{k} = -0.1$

- Enter $Y_{2T} = 20 + T - 0.1T^2$
- For the second function notice how the Y-values increase at first then decrease down to zero.




**iii)** Enter $X_{3T} = T$

Set $Y_{3T} = 20 + 2T + \mathbf{k}T^2$

$0 = 20 + 40 + \mathbf{k}(20^2)$

$\mathbf{k} = -0.15$

- Enter $Y_{3T} = 20 + 2T - 0.15T^2$
- Experiment with **TStep** changes
- **GRAPH** in **Sequence** or **Simultaneously** ( a **MODE** menu option )
- For **DOT** graphs ( a **MODE** menu option ), set **TStep** = **1 or 2**.

## 3. Trigonometric Functions

**Note**: The variable **T** on this page represents an angle and we'll vary **T** from 0° -> 360°

- Press **MODE** and choose **Degrees** instead of **Radians**

**a)** Enter $X_{1T} = T$ & $Y_{1T} = \sin T$

- Press **Range** and set: T : 0 -> 360 TStep = 10
  X : 0 -> 360 XScl = 90
  Y : -2 -> 2 YScl = 1
- Press **Graph** and notice how **Y** oscillates as **T** increases

**b)** Switch the role of the X & Y Variables by entering
$X_{1T} = \sin T$ & $Y_{1T} = T$

- Reverse the X & Y Range values and notice how the **X**-values now oscillate as **T** increases

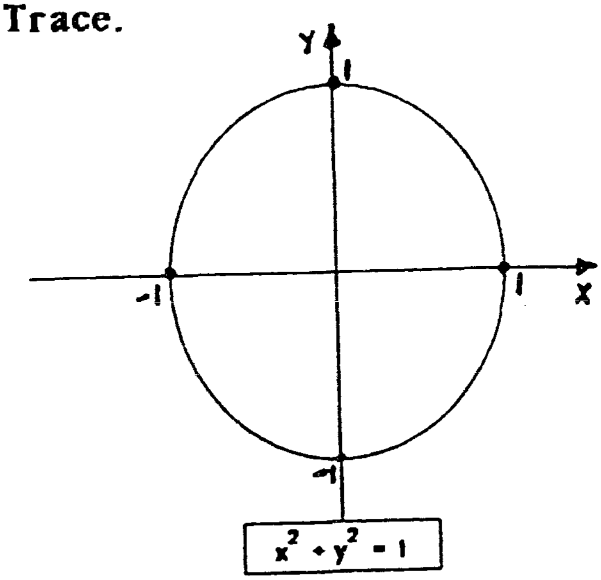
**c)** Try changing $X_{1T} = \sin T$ & $Y_{1T} = \sin T$

- Press **RANGE** and set: X : -3 -> 3
  Y : -2 -> 2
- Press **GRAPH** and do a **Trace** if you were surprised

**Note**: The Trace moves on the line **Y = X**
The angle will look like 45° because of the **3:2** Range ratio

**d)** Finally, set $X_{1T} = \cos T$ & $Y_{1T} = \sin T$ and using the same Range values draw the **Graph** and do a **Trace.**

| T | X | Y |
|---|---|---|
| 0 | 1 | 0 |
| 30 | 0.866 | 0.5 |
| 40 | 0.766 | 0.643 |
| 90 | 0 | 1 |
| 260 | -0.1736 | -0.9848 |




- The graph drawn is called the unit circle, since the radius, r = 1
- The non-parametric equation is:
  $X^2 + Y^2 = 1$ which shows why $\cos^2 T + \sin^2 T = 1$ for all values of **T**. You can check this using the values in the table.

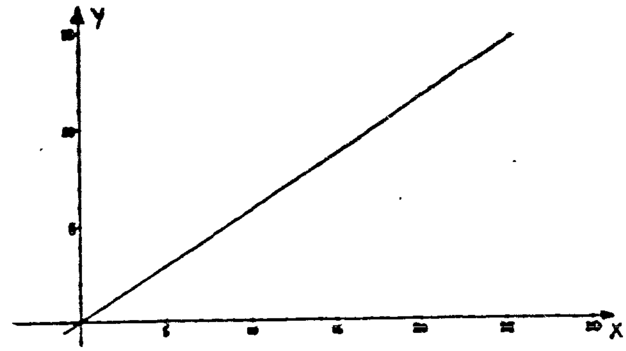
**Note:** The variable T on this page can be thought of as representing **TIME.**

**e) Graphing a linear function using trigonometry.**

- If a straight line has an angle of elevation of 30° then the line can be described parametrically as follows:

  **$X_{1T}$ - TCos 30°**

  **$Y_{1T}$ - TSin 30°**




- Set **Range** values:

  T: 0 -> 30 (TStep - 0.5)

  X: -5 -> 30 (XScl - 5)

  Y: -5 -> 20 (YScl - 5)

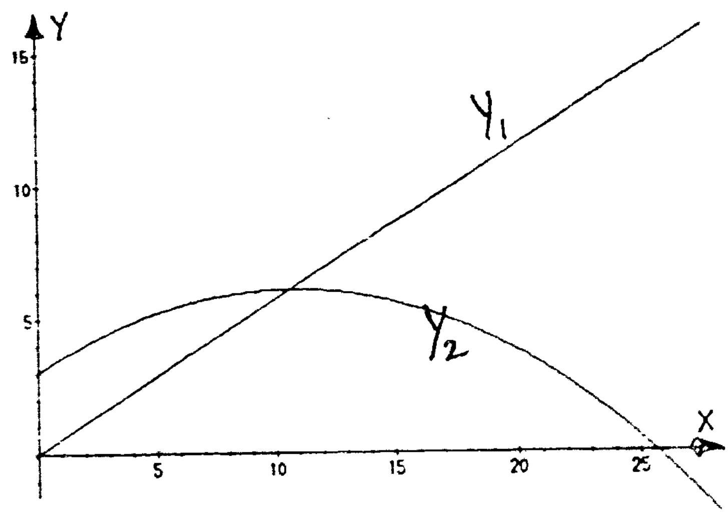
- Press **GRAPH.**

**f) Graphing a quadratic function using trigonometry:**

- Enter **$X_{2T}$ = TCos 30° &** **$Y_{2T}$ = TSin 30° - $0.02T^2$ + 3**

**Note**: This graph will also have an initial slope of 30°. It will start at + 3 on the Y-axis and will be pulled down by the gravitational effect of the term - **$0.02T^2$.**

| T | $X_1$ | $Y_1$ | $X_2$ | $Y_2$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 3 |
| 1 | .87 | .5 | .87 | 3.48 |
| 2 | 1.73 | 2 | 1.73 | 4.92 |
| 10 | 8.7 | 5 | 8.7 | 6.0 |
| 20 | 17.3 | 10 | 17.3 | 5.0 |
| 30 | 26.0 | 15 | 26.0 | 0 |




- **GRAPH** the two functions above in **Sequence** or **Simultaneously** ( a **MODE** option ). Now use the **TRACE** to answer the following questions. Check using Algebra.

- What is the maximum value of **$Y_2$** as **T** goes from 0 -> 30?
- For what value of **T** will it be true that **$Y_1$ = $Y_2$** ?

## Part C: Motion Simulation From a Given Point

### 1. Horizontal Motion

- Set **Range** T: 0 -> 20; X: 0 -> 1000; Y: 0 -> 20

**a) Objects travelling at a constant speed**: bicycles, trains ....

- An object starts from ( 50,4 ) and travels at a constant Horizontal rate of 30 units/sec. This can be represented by:

$$X_{1T} = 50 + 30T \quad \& \quad Y_{1T} = 4$$

**b) Objects accelerating:**

- An object starts at (50,8) with acceleration **a** = 4 units/sec$^2$. Distance travelled is given by $D(t) = D_0 + V_0t + at^2/2$
- This can be represented parametrically by:

$$X_{2T} = 50 + 2T^2 \qquad \& \qquad Y_{2T} = 8$$

- If there is an initial velocity of $\mathbf{V_0} = -20$ units/sec, then using the same acceleration and starting from ( 50, 12 ) we get:

$$X_{3T} = 50 - 20T + 2T^2 \qquad \& \qquad Y_{3T} = 12$$

**Note**: Do a **TRACE** on this one and notice the cursor start towards the left (negative velocity) but it changes direction eventually because the acceleration is positive.

**Note:** A constant headwind of 20m/sec. would have the same effect as the initial velocity.

### 2. Vertical Motion

- Set **Range:** T: 0 -> 10; X: 0 -> 20; Y: 0 -> 500
- Suppose an object is dropped from a height of 400m. We can represent this event in the gravitational environments of the Moon, Mars, and Earth by the following:

| **Moon** | **Mars** | **Earth** |
|---|---|---|
| $X_{1T} = 4$ | $X_{2T} = 8$ | $X_{3T} = 12$ |
| $Y_{1T} = 400 - 0.8T^2$ | $Y_{2T} = 400 - 1.9T^2$ | $Y_{3T} = 400 - 4.9T^2$ |

- For either of the simulations on this page a strobe-light effect can be generated by moving from **Connected** to **Dot MODE.** A suitable **TStep** would be 1 or 2. (TStep = 2 means one light flash every 2sec)

**3. Projectile Motion:** motion in two-dimensional space

**a) Linear Motion:**

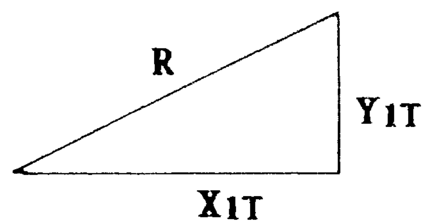
A pair of parametric equations of the form:

$X_{1T} = 25T$ ( 25 cm/sec for T sec. )

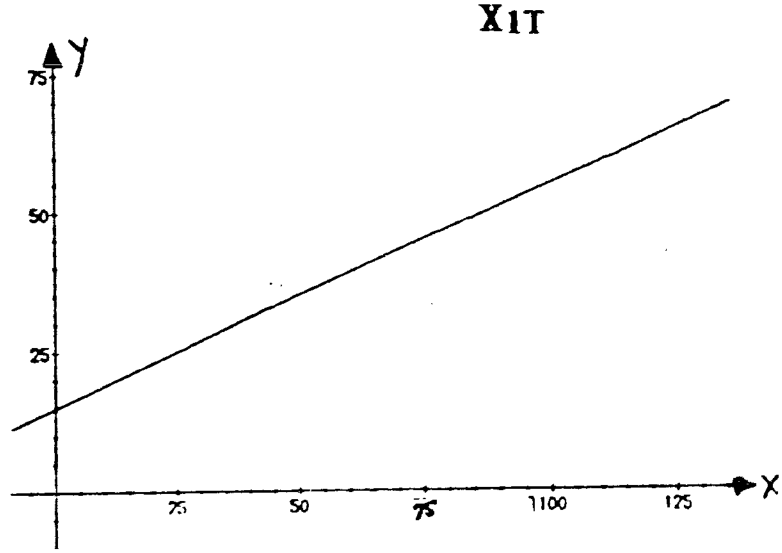
$Y_{1T} = 15 + 10T$ ( 10 cm/sec for T sec.)

describes the motion of an object in two-dimensional space, that has an initial position of ( 0, 15 ).

- $X_{1T}$ is the **Horizontal Component** of the position (displacement).
- $Y_{1T}$ is the **Vertical Component.**
- The **Resultant** ( **R** ) is the path taken by the object.




| T | X | Y |
|---|---|---|
| 0 | 0 | 15 |
| 1 | 25 | 25 |
| 2 | 50 | 35 |
| 10 | 250 | 115 |




**Notes:**

- In this example the path is linear with a slope of $10/25 = 2/5$ since $Y_{1T}$ is changing at a rate of 2/5 that of $X_{1T}$

- The non-parametric equation would be $Y = 2/5\ X + 15$ which can be arrived at algebraicly by solving for $T = X_{1T}/25$ and substituting this for T in the expression $Y_{1T} = 15 + 10T$.

- The Horizontal component of the velocity is 25 cm/sec
- The Vertical component of the velocity is 10 cm/sec
- The actual velocity is $\sqrt{10^2 + 25^2} = 26.9$ cm/sec. This is the velocity along the path of motion.

- This type of motion would be on a frictionless surface or in the absence of a gravitational pull.

## b) Parabolic Motion (Quadratic function)

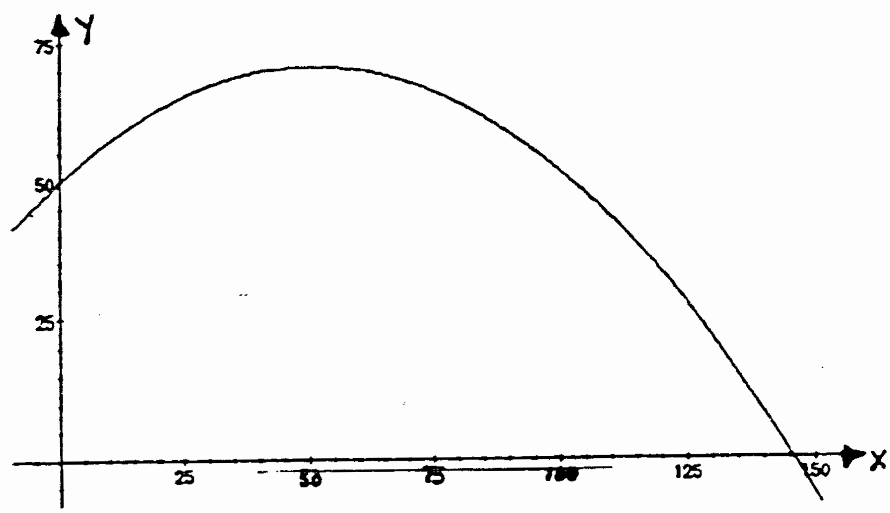
- A pair of parametric equations of the form:

  $$X_{2T} = 25T \qquad \& \qquad Y_{2T} = 50 + 20T - 4.9T^2$$

  describe the path of an object from an initial position ( 0, 50 )

- **25T** is the horizontal displacement (meters) at time **T**
- $\mathbf{50 + 20T - 4.9T^2}$ gives the vertical displacement
  **50** = the initial vertical position ( T = 0 )
  **+20T** represents an initial vertical velocity of 20m/sec upward
  $-\mathbf{4.9T^2}$ represents the vertical effect of the gravitatonal pull on an object causing the vertical velocity to continually decrease at 9.8 m/sec each second. ( $at^2/2 = -9.8t^2/2 = -4.9t^2$ )

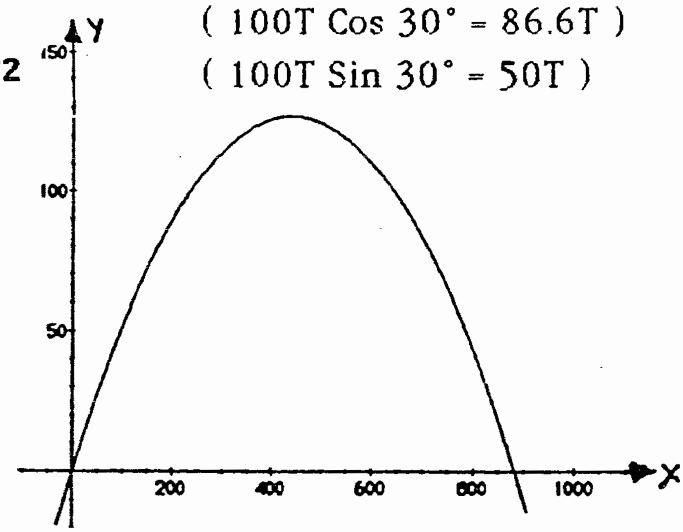
| T | X | Y |
|---|---|---|
| 0 | 0 | 50 |
| 1 | 25 | 65.1 |
| 2 | 50 | 80.2 |
| 5 | 125 | 27.5 |
| 10 | 250 | -240 |




- If the actual initial velocity ( 100m/sec ) and the angle of the path is known ( 30° ) then **Trigonometry** can be used as follows:

$$X_{3T} = 100T \cos 30^\circ$$
$$Y_{3T} = 100T \sin 30^\circ - 4.9T^2$$

( $100T \cos 30^\circ = 86.6T$ )
( $100T \sin 30^\circ = 50T$ )




- Set **Range** values:

  **T**: 0 → 10 ( TStep = 0.1 )
  **X**: 0 → 1000 ( XScl = 100 )
  **Y**: 0 → 200 (YScl = 100)

## Part D: The Simulation of Specific Motion Problems

**Problem #1:**

A girl leaves on a camping trip on a bicycle travels at a constant speed of 20 km/h. Two hours later, her father, realizing she has left her glasses at home, pursues her in his car, averaging 60 km/h. How long will it take him to overtake his daughter?

- The girl's horizontal displacement can be represented by:

  **X1T = 20T**

- Her father's horizontal displacement then becomes:

  **X2T = 60(T - 2)**

**Note**: Her father's positive displacement starts at **T > 2**

- To set up parallel horizontal paths for both parties, enter:

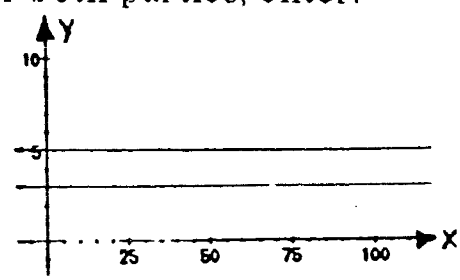
  **Y1T = 3 & Y2T = 5**

- Set **Range** then press **Graph**:

  **T**: 0 → 5 (TStep = 0.02)

  **X**: 0 → 100 (XScl = 50)

  Y: 0 → 8 (YScl = 5)

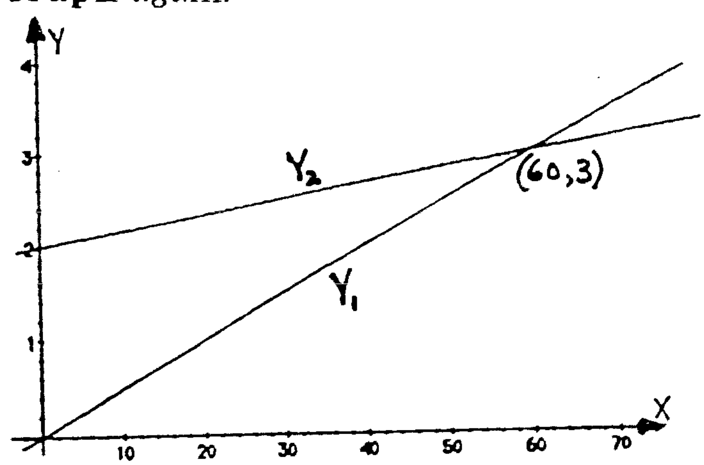



- Since there is no intersection of the two paths do a **TRACE** until an approximation can be made of the position and time of their meeting.

- To get a point of intersection, the Y-coordinates can arbitrarily be made to vary as T does. To do this enter:

  **Y1T = T & Y2T = T** then press **Graph** again:

**Note**: The girl is travelling at a slower horizontal rate than her father and therefore covers less distance from left to right in the same period of time. So her graph,using this simulation, has a greater vertical slope because the horizontal component of her motion is less than her fathers.




- The intersection point ( 60, 3 ) tells us that 60 km. from home, 3 hours after she leaves her father catches her.

**Problem#2:**

Two trains leave at the same time and approach each other on parallel tracks from stations 280 km apart. One has an average speed of 60 km/h, and the other 80 km/h. How long until and at what point will they pass each other?

- Set the parametric functions for the first train to be:

  $X_{1T} = 60T$ & $Y_{1T} = T$

- The second train, starting 280 km. down the track, becomes:

  $X_{2T} = 280 - 80T$ & $Y_{2T} = T$

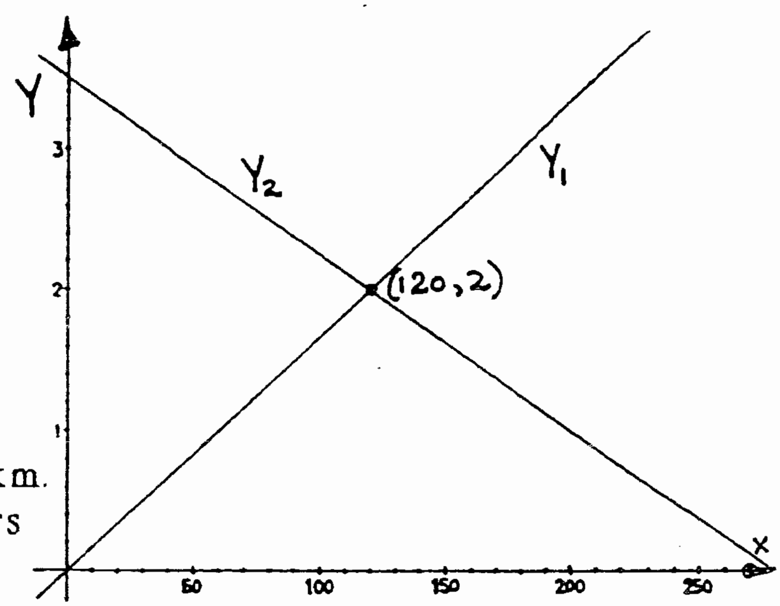
**Note**: The starting position of the second train is ( 280, 0 ). The negative velocity ( -80 ) signifies that the second train is travelling in the opposite direction from that of the first train.

- Set **Range** then press **GRAPH**:

  **T**: 0 → 5 (TStep - 0.02)
  **X**: 0 → 300 (XScl - 50)
  **Y**: 0 → 5 (YScl = 1)

- Do a **TRACE** to get the time and location where the two trains pass each other.

- The intersection point ( 120, 2 ) tells us that the trains pass each other 120 km. from the first station 2 hours after they each leave.



## Problem # 3

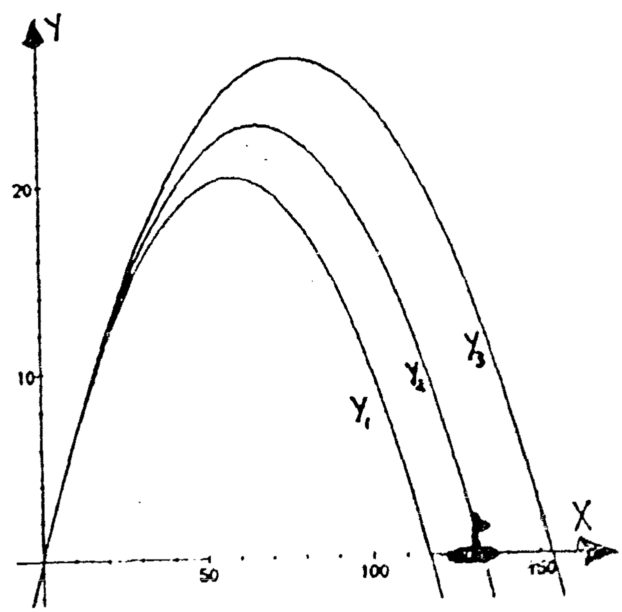
A golfer is contemplating a 7-iron shot towards a circular green with the flag located at the center. The hole is 130 meters away and the radius of the green is 8 meters. His 7-iron hits the ball at a 35° angle. Assuming the ball will be hit straight and that there is no wind, what velocity will it be necessary to impart to the ball so that it hits the green?

- If **V** is the velocity of the ball along the 35° path then:
  **VCos 35°** = the horizontal component of V
  **VSin 35°** = the vertical component of V

- The equations for the path of the ball become:
  **X(T) = VTCos 35°** & **Y(T) = VTSin 35° - 4.9T²**
- Choose different values of **V** and **TRACE** the graphs drawn to find the **X**-value when **Y = 0**. This is the distance the ball travels in the air. Y = 0 is ground. The desired X-value is 130. The algebraic solution of V is shown below.
- **$X_1$T = 35TCos 35°**
  **$Y_1$T = 35TSin 35° - 4.9T²**

  **$X_2$T = 37TCos 35°**
  **$Y_2$T = 37TSin 35° - 4.9T²**

  **$X_3$T = 40TCos 35°**
  **$Y_3$T = 40TSin 35° - 4.9T²**
- Set **Range:**
  **T**: 0 → 6 (TStep = .02)
  **X**: 0 → 200 (XScl = 50)
  **Y**: 0 → 40 (YScl = 10)




**Algebraic solution:**

The solution is a 130 meter shot ( X = 130 & Y = 0 ).

(A) 130 = VTCos35°

(B) 0 = VTSin35° - 4.9T² → T( VSin35° - 4.9T) = 0
→ T = 0 or **T = VSin35°/4.9**

Substituting this back in (A) → 130 = V²Sin35°Cos35°/4.9
→ **V = 36.82 m/sec**

**Problem#4:**

A particle moves on a horizontal line and its displacement is given by the formula: $\mathbf{X(T) = -T^3 + 4T^2 - 3T}$
where T is measured in seconds, and $\mathbf{0 \le T \le 4}$.

a) Which direction is the particle moving at T - 1s, 2s, 3s, 4s?
b) At approximately what times is the particle stopped?
c) At what times is the displacement zero?
d) At what time is the speed the greatest?

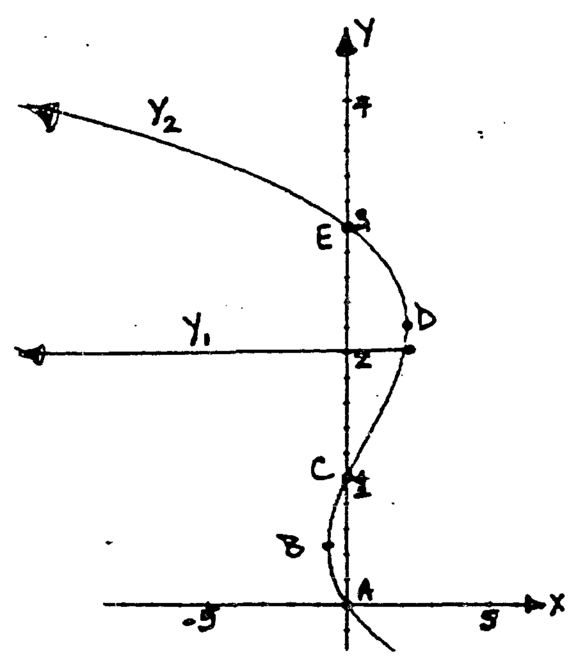
**Note:** These are questions that can be answered without calculus using parametrics on the TI-81.

- The motion can be simulated on any horizontal line:
- Set $\mathbf{X_{1T} - -T^3 + 4T^2 - 3T}$ & $\mathbf{Y_{1T} - 2}$.
- Press **TRACE** and follow the motion of the particle.

- Set $\mathbf{Y_{2T} - T}$ & $\mathbf{X_{1T} - X_{2T}}$ The cubic nature of the function can be seen in the second graph. Again use the **TRACE**, graph the functions **SIMUL**taneously and answer questions a) -> d).

- Set **Range** as follows:

| | |
|---|---|
| **T**: 0 → 5 | (TStep - .02) |
| **X**: -20 → 10 | (XScl - 10) |
| **Y**: -5 → 5 | (YScl - 2) |




**Answers**

**a)** motion is to the right at 1 sec. & 2 sec. and to the left at 3 sec. & 4 sec.

**b)** From the graph of $Y_2$ or by doing a **TRACE** on $Y_1$ we can see that the particle is stopped at T - 0.5 (point B) & T - 2.2 (point D).

**c)** The displacement ( X-value ) is zero at T = 0 (point A), T = 1 (point C), & T - 3 (point E)

**d)** The particle is moving the fastest, although with negative velocity, at T - 4

## Problem # 5

Two cars are approaching an intersecion at location ( 50, 50 ). Car A is coming from the north, 300 km. away, at a speed of 40 km/h. Car B is approaching from the east, 400 km. from the intersection, at a speed of 60 km/h.

a) At what time will the two cars be 100 km. apart?

b) What is the minimum distance between them?

**Note:** The ( 50, 50 ) location for the intersection, off the X & Y axes, is chosen to allow for better viewing on the calculator.

- Set the path of car A as: $\mathbf{X_{1T} = 50}$ & $\mathbf{Y_{1T} = 350 - 40T}$
- Set the path of car B as: $\mathbf{X_{2T} - 450 - 60T}$ & $\mathbf{Y_{2T} - 50}$

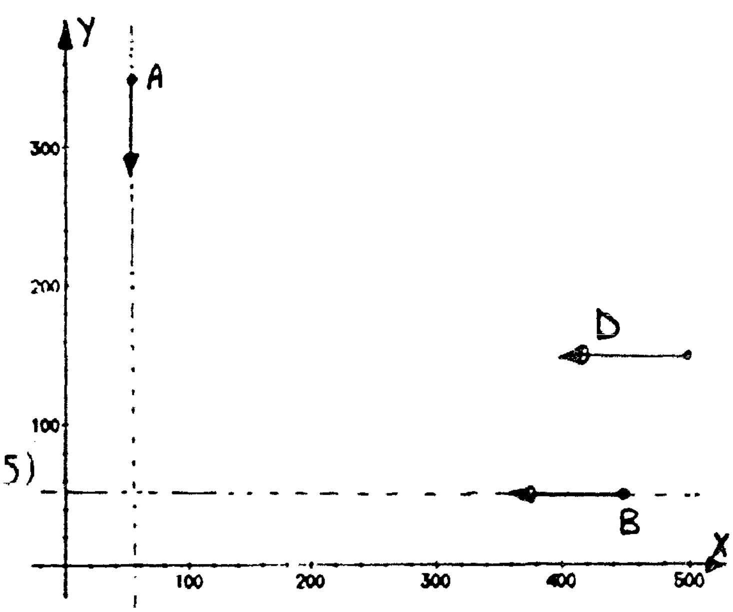
**Note:** Car A starts at Y - 350 and moves downwards on the vertical line X - 50 at 40 km/h. Car B starts at X - 450 and moves to the left on the horizontal line Y - 50 at 60km/h.

- To calculate the distance between the two cars, which is always the hypoteneuse of a right triangle, use Pythagoras:

$$\text{Distance} - \mathbf{X_{3T} - \sqrt{(Y_{1T} - 50)^2 + (X_{2T} - 50)^2}}$$

- Set $\mathbf{Y_{3T} - 150}$. The distance will be seen to be shrinking along this horizontal path towards a minimum then will increase once the cars pass each other ( you'll need a **TRACE** to see this ).

| T | $Y_1$ | $X_2$ | D ($X_3$) |
|---|---|---|---|
| 0 | 350 | 450 | 500 |
| 2 | 270 | 330 | 356.1 |
| 5 | 150 | 150 | 141.4 |
| 6 | 110 | 90 | 72.1 |
| 7 | 70 | 30 | 28.3 |
| 8 | 30 | -30 | 82.5 |
| 10 | -50 | -150 | 223.6 |




- Set **Range:**

T: 0 → 10 (TStep - 0.05)

X: -200 → 500 (XScl - 100)

Y: -200 → 400 (YScl - 100)

**Answers**

**a)** T - 8.25 h & T - 5.59 h **b)** 27.74 m. (at T - 6.92 h)

**Problem#6:**

A ten metre ladder is resting against a wall one metre from the base of the wall. The ladder starts to slide down the wall in such a fashion that the foot of the ladder is moving away from the wall at the rate of 2 m/s.

a) Is the speed of fall of the top of the ladder constant?

b) What is the path described by the center point on the ladder?

- For convenience of viewing choose the point where the ground meets the wall to be ( 5,5 ). Three parametric representations can be set up. ( Refer to the diagram below).

**a)** The foot of the ladder, moving at a constant speed:

$$X_{1T} = 5 + 2T \qquad \& \qquad Y_{1T} = 5$$

**b)** The Y-coordinate ($Y_{2T}$) of the top of the ladder can be calculated by using Pythagoras:

$$(X_1 - 5)^2 + (Y_2 - 5)^2 = 10^2 \text{ \& solving for } Y_2 \text{ we get:}$$

$$Y_{2T} = 5 + \sqrt{(100 - (X_{1T} - 5)^2)} \qquad \& \qquad X_{2T} = 5$$

**c)** The X-coordinate of the center of the ladder has a constant horizontal velocity equal to half that of the foot of the ladder: $X_{3T} = 5 + T$

The center of the ladder is always 5 meters from the point (5,5) so again using Pythagoras or the distance formula:

$$(X_{3T} - 5)^2 + (Y_{3T} - 5)^2 = 5^2 \text{ \& solving for } Y_{3T} \text{ we get:}$$

$$Y_{3T} = 5 + \sqrt{(25 - (X_{3T} - 5)^2)}$$

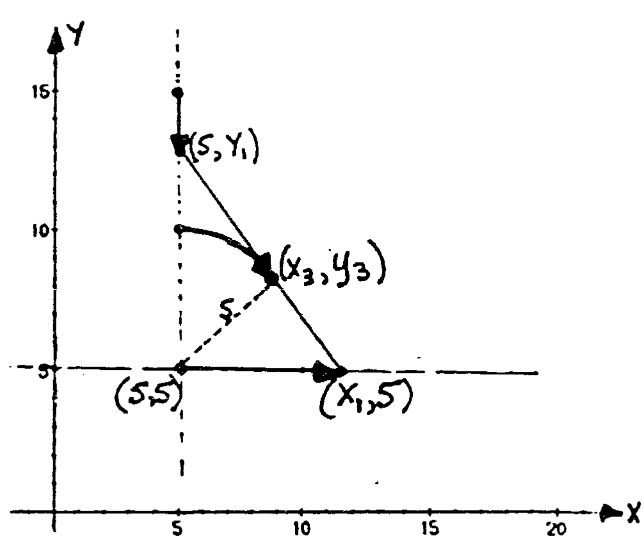
- These three functions when graphed simultaneously trace the fall of the ladder. If you sight the cursor of each function it can be seen that they are concurrent and represent the ladder itself as it falls.

- Set **Range:**

**T**: 0 → 5 (TStep = 0.02)

**X**: 0 → 15 (XScl = 5)

**Y**: 0 → 15 (YScl = 5)




**Answers**

**a)** No. If a **TRACE** is done the top of the ladder can be seen to accelerate.

**b)** A quarter-circle

## Target shooting# 1:

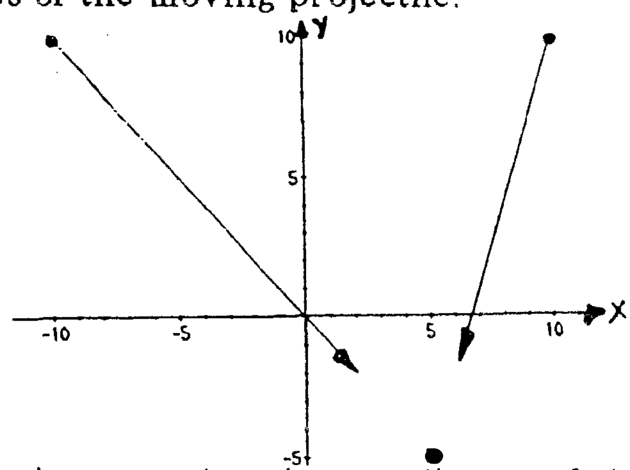
A target is moving according to the function rules:

$$X_{1T} = -10 + T \quad \& \quad Y_{1T} = 10 - T$$

This target starts at the point (-10,10) and moves diagonally through the origin. The goal is to shoot the target at one of its locus points, say (5,-5) from a specified point say (10,10). This means to mathematically devise the necessary parametric functions of a projectile that will intercept the target at the precise specified point at the ***exact same time***.

**Note:** ( $X_1$,$Y_1$ ) are the coordinates of the moving target.
( $X_2$,$Y_2$ ) are the coordinates of the moving projectile.

| T | $X_1$ | $Y_1$ | $X_2$ | $Y_2$ |
|---|---|---|---|---|
| 0 | -10 | 10 | 10 | 10 |
| 5 | -5 | 5 | | |
| 10 | 0 | 0 | | |
| ? | **5** | **-5** | ? | ? |
| 20 | 10 | -10 | | |




**Step 1:**

- Find **T** by setting the target equations equal to the coordinates of the impact point (bold print in the table).

$$X_{1T} = -10 + \mathbf{T} = 5$$
$$\mathbf{T} = 15$$

**Step 2:**

- Find $X_{2T}$ & $Y_{2T}$
- $X_{2T} = 10 + \mathbf{h}T$ (at T = 0, $X_2$ = 10)
  $5 = 10 + \mathbf{h}(15)$ (at impact point, T = 15 and $X_2$ = 5)
  $-5 = 15\mathbf{h}$
  $\mathbf{h} = -1/3$
- $Y_{2T} = 10 + \mathbf{k}T$ (at T = 0, $Y_2$ = 10)
  $-5 = 10 + \mathbf{k}(15)$ (at impact point, T = 15 and $Y_2$ = -5)
  $\mathbf{k} = -1$

**Step 3:**

- Enter $\mathbf{X_{2T} = 10 - 1/3T}$ & $\mathbf{Y_{2T} = 10 - T}$
- Press **MODE**, choose the **SIMUL**taneous option then **GRAPH**.
- Set the T-**RANGE**: 0→15 to stop the action at the impact point.

## Target shooting #2

A particle is moving in a circular path starting at the point (0,3) and moving in a counter-clockwise direction according to the functions:

$$X_{1T} = -3\text{Sin}T \quad \& \quad Y_{1T} = 3\text{Cos}T$$

From a specified point, ( 10,10 ) launch a projectile that will intercept this particle as it reaches the point (3,0).

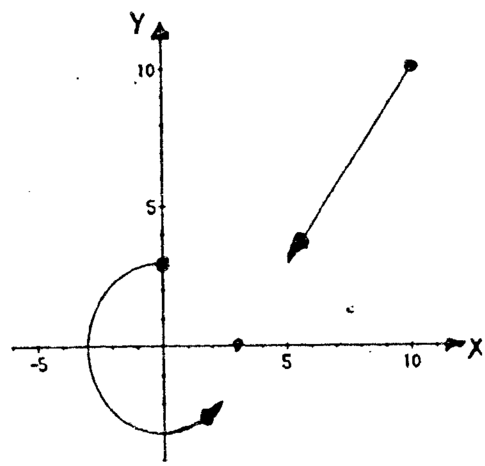
**Note:** In this problem **T** is an angle and this problem is solved below in **RAD**ian **MODE**. The **RANGE** for T should be at least one complete revolution: $0 \le T \le 2\pi$ (6.283)

**Step 1. Find T**:

- This can be done using the **TRACE** and stopping the cursor on the point (3,0). If the Tstep = .02 the value of **T** on the screen is **T** = 4.72
- This can also be done algebraicly by setting $X_{1T}$ equal to the X-coordinate of the impact point and solving for T:

$$-3\sin T = 3$$
$$\sin T = -1$$
$$T = 3\pi/2$$
$$T = 4.7124$$

- This value of T represents the radian measure of 3/4 of a revolution, which is precisely the impact point: $\frac{3}{4}(2\pi) = 3\pi/2 = 4.7124$




**Step 2. Find $X_{2T}$ & $Y_{2T}$**:

- Shooting from (10,10) to (3,0) and T = 4.7124 we set:

$X_{2T} = 10 + \mathbf{h}T = 3$ & $Y_{2T} = 10 + \mathbf{k}T = 0$

$10 + \mathbf{h}(4.7124) = 3$     $\mathbf{k}(4.7124) = -10$

$\mathbf{h} = -1.485$     $\mathbf{k} = -2.122$

**Step 3**:

- Enter **$X_{2T}$ = 10 - 1.485T** & **$Y_{2T}$ = 10 - 2.122T**
- Press **MODE**, choose the **SIMUL**taneous option then **GRAPH**.
- Set the T-**RANGE**: 0 → 4.7124 to stop the action at the impact point.

## Part E: Problem Set

1. Two planes leave airports 1000 km. apart, flying towards each other. Plane A leaves airport A at 1400h and travels at a constant speed of 200 km/h. Plane B leaves airport B at 1600h and travels at 300km/h. At what time and at what location will the two planes pass each other?

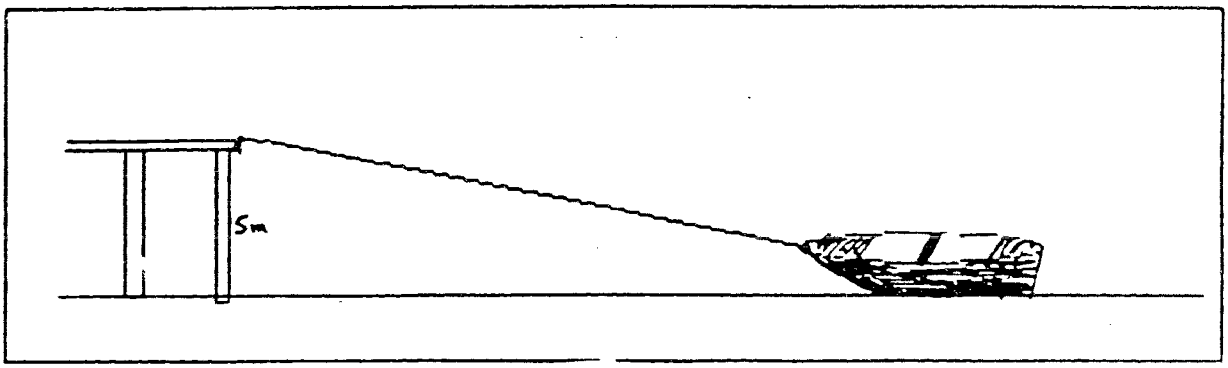
2. A baseball is struck with an initial velocity at impact of 45 m/sec. The initial height at impact is 1.1 m and the ball is struck at a 25° angle.
   **a**) Find parametric representations for the horizontal (**X1T**) and the vertical (**Y1T**) distances covered in terms of time, **T**.
   **b**) How long does it take the ball to hit the ground?
   **c**) How far does the baseball travel?
   **d**) What is the maximum height reached by the ball?
   **e**) Write a parametric representation (**X2T** & **Y2T**) for a wall 5 m high, 150 m out into center field.
   **f**) Will the baseball clear the wall? If not, can it be caught?
   **g**) How strong a wind ( m/sec ) blowing towards center field, would be necessary for the hit to be a home run?

3. A golfer is faced with a 140 m shot directly into a 5 m/sec wind. The problem in golf is to choose the best club for the shot. The best club is the highest numbered club to provide the greatest possible loft, thus minimizing the distance travelled after it hits the green. Each club imparts a different angle and a different maximum velocity to the ball at impact.

| **Club** | **Angle** | **Max. speed of ball** |
|---|---|---|
| 4-iron | 28° | 50 m/sec. |
| 5-iron | 32° | 48m/sec |
| 6-iron | 36° | 45m/sec |
| 7-iron | 39° | 40m/sec |
| 8-iron | 43° | 36m/sec |

Determine the best club and the velocity needed to drop the ball as close as possible to the hole.

4. A 25 m. rope has one end attached to a boat and the other to a pulley on a dock. The rope is being pulled in at a constant rate of 2.50 m/sec. The water is 5 m. below the pulley.



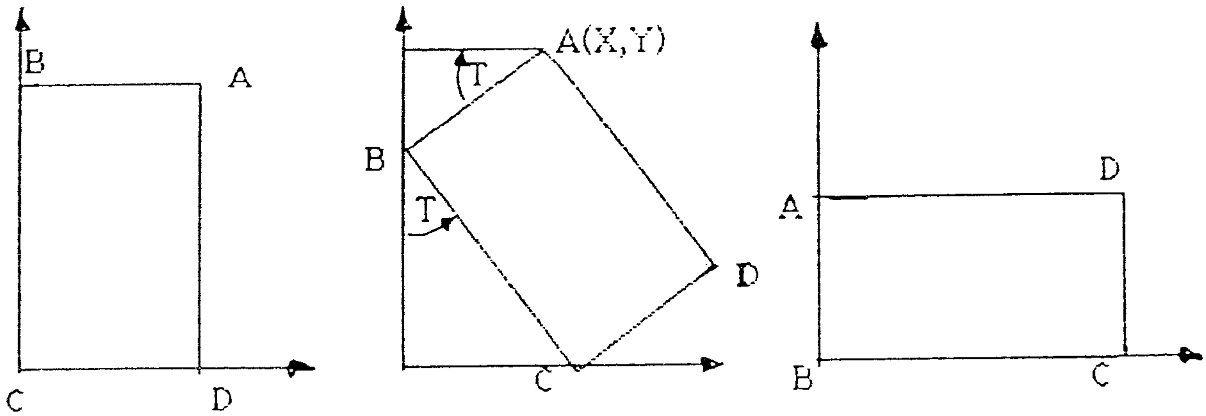

**a)** Set up two functions that will decrease, $X_{1T}$ showing the length of the rope, and $X_{2T}$ representing the distance of the boat from the dock. Graph both these functions on parallel horizontal lines.
**b)** Is the speed of the boat constant? Why or why not?
**c)** Complete the table below calculating the distance the boat travels during the 6th, 7th, and 8th seconds.

| T | $X_{2T}$ (Dist. from dock) | Dist. travelled that second |
|---|---|---|
| 5 | | XXXXXXXXXXXXXXXXXXXXXX |
| 6 | | |
| 7 | | |
| 8 | | |

5. A helicopter, flying at 18m/sec. at an altitude of 120 m.,releases a steel ball. The horizontal component of the velocity remains constant at 18 m/sec.

**a)** Set up a parametric representation of the flight of the ball.
**b)** How high is the ball after 3 sec.
**c)** How far will the ball travel horizontally before it hits the ground?

6. A rectangle **ABCD** is moved so that vertex **B** always touches the Y-axis and vertex **C** always touches the X-axis. **AB** - 3 and **AD** - 5. Let ( X, Y ) be the coordinates of point **A**, and **T** be the angle shown in the diagram.




**a**) Find a pair of parametric representations **X(T)** and **Y(T)**.
**b**) Graph the path of point **A**.
**c**) Find the maximum value of **Y**.
**d**) Solve for **Y** in terms of **X** by eliminating **T**.
**e**) Graph this function ( change **MODE** from **Parametric** to **Function** ). What is the shape of this curve?

7. A ball is tossed from a height of 2 m. at an initial velocity of 25 m/sec. at an angle of 60°. Its path is described by:

$$X_{1T} = 25T \cos 60^\circ \quad \& \quad Y_{1T} = 2 + 25T \sin 60^\circ - 4.9T^2$$

From the point ( 100,0 ) a bullet is to be fired that will travel in a straight line at 100m/sec. The problem is to fire the gun so that it makes contact with the ball exactly 3 seconds after the ball is tossed.

**a**) How long after the ball is tossed should the gun be fired?
**b**) At what angle will the bullet travel?
**c**) What are the parametric equations simulating the path of the projectile?

- Set **Range**:

**T**: 0 → 5 (TStep - 0.1)
**X**: -10 → 100 (XScl - 10)
**Y**: -10 → 60 (YScl - 10)

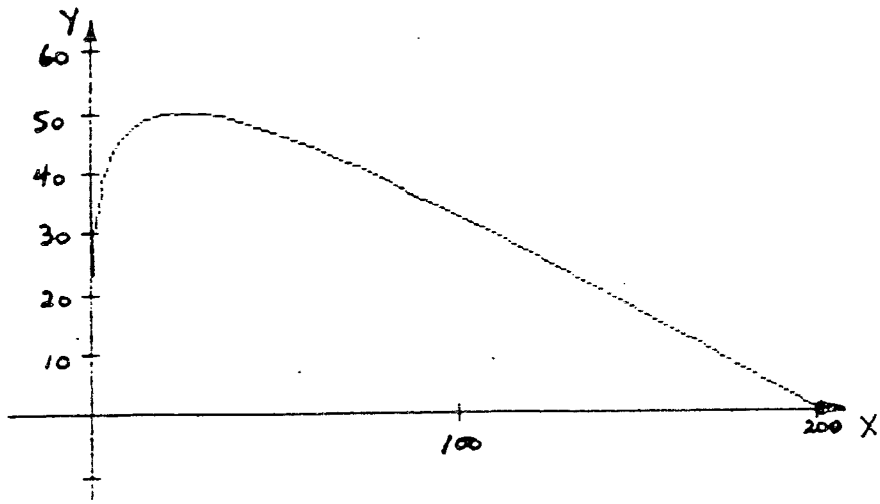
8. A car on a test track is approaching a turn. It is headed due north at 20 m/sec. when it starts to turn towards the northeast. The turn is described by the parametric equations:

$$X_{1T} = 0.2T^3 \quad \& \quad Y_{1T} = 20T - 2T^2$$

where **X** & **Y** are in meters and **T** in seconds.

- Set **Range:**

| | |
|---|---|
| T: 0 → 10 | (TStep - 0.1) |
| X: -10 → 300 | (XScl-100) |
| Y: -10 → 200 | (YScl-100) |




The origin ( 0,0 ) will be the beginning of the turn and the graph of the above equations will simulate the path of the car for the next 10 seconds.

**a)** What is his position when he starts to turn south, in other words, when he reaches his maximum northern displacement?

**b)** A projectile launched at T - 0, from the location (200, 200), at a constant vertical height, is to intercept the race car at a spot on the track with location ( 102.4, 32 ). What would be the parametric representations for the straight line path of this projectile?

**c)** What is the speed of the projectile?

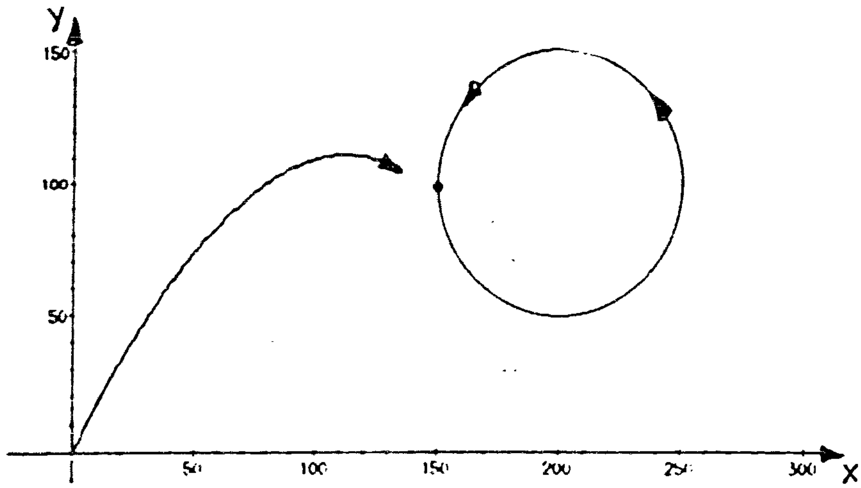
**d)** If the projectile can travel at 100m/sec. at what time should it be launched?

**e)** Set up a parametric representation simulating the path of this faster projectile?

9. A target is rotating in a circular path with a radius of 50 m. and center ( 200, 100 ) and rotates according to the equations:

$$X_{1T} - 200 + 50\text{Cos } T \quad \& \quad Y_{1T} - 100 + 50\text{Sin } T$$

- Press **MODE** and choose the **DEG**ree option
  **Note:** T - Time in secs. & one revolution is $0 \leq T \leq 360$
- A projectile is fired from the origin at T - 0, at an initial velocity of **V** - 2 m/sec. The problem is to design a gravitational effect that will cause this projectile to arc in a parabolic path and intercept the target at one of its locus points ( 150, 100 ).




**a**) Find **T**, the time it takes the target to reach ( 150, 100 ).

**b**) Find the angle Θ needed so that the constant horizontal component of the velocity of the projectile:

$$X_{2T} - VT\text{Cos } \Theta$$

satisfies the horizontal requirements.

**c**) Find the gravitational effect **k** necessary so that the vertical component of the velocity:

$$Y_{2T} = VT\text{Sin } \Theta + kT^2$$

satisfies the vertical requirements of the projectile.

**Notes:**

- A linear path can be simulated to start at a later time by translating the components to the left or right as was done with two previous problems: Problem#1 in Part D when the father left two hours after the girl and in Problem 8(d) of the problem set with the faster projectile.
- In this problem if you have a faster projectile, and since the path is parabolic, I assume that you would need to construct a program if you wanted to launch it at a later time.

# Appendix

## 1. Use of the MODE feature:

- To change the option use the cursors and the **ENTER** button.
- An option is activated if it is shaded.
- When **MODE** is pressed the following options are significant:

**a) FUNCTION** vs. **PARAM** - Function mode is the usual y = F(x) representation; parametrics is the other option. If **FUNCTION** is activated then when **Y=** is pressed there are four possible functions that can be created and **X/T** will display **X** . If **PARAM** is activated then **Y=** gives three function possibilities but in this case each is represented by a pair of functions **X(T) & Y(T)** and use of the **X/T** button will display the parameter **T** .

**b) CONNECTED vs. DOT** - The **DOT** feature is useful sometimes if a graph consisting of a series of dots is desired. The spaces between the dots can be adjusted using the **TSTEP** which is contained in the **RANGE.** This is only possible when using parametrics.

**c) SEQUENCE vs. SIMUL** - Functions can be graphed in sequence or simultaneously.

**d) GRID OFF vs. GRID ON** - The grid lines can be displayed by using this feature and are determined by the **XSCL** and **YSCL** choices made when setting up the **RANGE**

**e) RAD vs DEG** - Calculations are done with trigonometric functions in radians or degrees, whichever one is activated.

## 2. Use of the RANGE feature

- It is necessary to have some idea of what the function is going to look like so that you can choose a suitable range for the **X & Y** variables, and in the case of parametrics, for the **T** variable.

- The choices made for **XSCL & YSCL** form the cross-hatches on the coordinate axes and they also determine the grid if **GRID ON** is activated.

- If **PARAM** is activated then the choice of **TSTEP** determines the speed at which parametric graphs are drawn.

3. **Use of the ZOOM feature:**

- If you want to zoom in to get a close-up picture of a section of a graph, then if you press **ZOOM 1, BOX** will be activated. Then press **ENTER,** adjust the cursor to the top-left corner of the desired viewing rectangle and press **ENTER** again. Then dragging the cursor down and to the right will create the rectangle, and **ENTER** will activate the **ZOOM.**

- **ZOOM 6** (Standard) will always return to a view of -10 to 10 for both the **X & Y** variables

- **ZOOM 5** (Square) will adjust the ratio of the **X**-range to the **Y**-Range to a 3:2 value so that circles will look round, squares will look like squares, and a slope of 2 will look like a slope of 2. The **X**-values stay the same and the **Y**-values are changed when this feature is used.

- **ZOOM 7** (Trig) sets up a range of -360° to 360° for the **X** (or **T**) variable ( if **DEGREE** mode is activated). This is useful when graphing trigonometric functions. If **RADIAN** mode is activated then a range of $-2\pi$ to $2\pi$ (in decimal form) is set up for **X**.

4. **Use of the TRACE feature:**

- If Function mode is activated then the **TRACE** begins at the center of the X-range. If Parameter mode is activated then the **TRACE** begins at the middle of the T-range. Use of the left & right cursor keys will then move the cursor along the graph of the function. The coordinates are displayed as the cursor moves. If in parametric mode, the **TSTEP** chosen will determine the speed of the **TRACE.** If more than one function is displayed then use of the up or down cursor keys will switch to the next function on the **Y** = list.