FINITE HOMOGENEOUS 3-GRAPHS

by

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Abstract

A 3-graph consists of a vertex set together with a set of unordered triples of vertices; the members of the latter set are called edges. A bijection from one finite subset of the vertex set to another is called a local isomorphism of the 3-graph if it maps edges to edges and non-edges to non-edges. A 3-graph is called homogeneous if each of its local isomorphisms extends to an automorphism. By exploiting the classification of finite 2-transitive permutation groups which is found in the literature we show that up to isomorphism there are only four non-trivial finite homogeneous 3-graphs and we give explicit descriptions of them.

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Contents

Chapter 1

Introduction

In this thesis we classify the finite homogeneous 3-graphs. In fact, we will show that there are just four non-trivial finite homogeneous 3-graphs. Two of them are essentially the projective planes over GF(2) and GF(3). The remaining two are structures closely related to the projective lines over GF(5) and GF(3^2).

In order to describe the objects which concern us we first explain the notion of a structure which has been studied extensively by mathematical logicians since the 1950's. Let L consist of a finite set $\{\mathbf{R}_1, \ldots, \mathbf{R}_k\}$ of *relation symbols* together with a *signature* $(n_1, \ldots, n_k)$, where n_i is a natural number called the *arity* of $\mathbf{R}_i$. We call L a *finite relational language*. This is the only sort of language we shall consider although in other contexts an infinite number of relation symbols is often permitted as well as symbols representing functions and constants.

With L as above, an *L-structure* $\mathcal{M}$ is a $(k+1)$-tuple

$$(M, \mathbf{R}_1^{\mathcal{M}}, \ldots, \mathbf{R}_k^{\mathcal{M}})$$

such that

$$\mathbf{R}_i^{\mathcal{M}} \subseteq M^{n_i} \quad (1 \leq i \leq k),$$

where M is a set called the *universe of* $\mathcal{M}$ and M^n denotes the n-th cartesian power of M. The n_i-ary relation $\mathbf{R}_i^{\mathcal{M}}$ is called the *interpretation of* $\mathbf{R}_i$ in $\mathcal{M}$.

An L-structure

$$\mathcal{N} = (N, \mathbf{R}_1^{\mathcal{N}}, \ldots, \mathbf{R}_k^{\mathcal{N}})$$

is a *substructure* of $\mathcal{M}$ if $N \subseteq M$ and

$$\mathbf{R}_i^{\mathcal{N}} = \mathbf{R}_i^{\mathcal{M}} \cap N^{n_i}.$$

Thus there is a one-to-one correspondence between substructures of $\mathcal{M}$ and the subsets of M. The *restriction of $\mathcal{M}$ to $N \subseteq M$*, $\mathcal{M} \restriction N$, is the substructure of $\mathcal{M}$ on N.

Now consider an injective map $F : N \to M$. We extend F to the cartesian powers of N and their power sets in the obvious fashion:

$$F(a_1, \ldots, a_n) = (F(a_1), \ldots, F(a_n)) \quad (a_1, \ldots, a_n \in N),$$

and

$$F(X) = \{F(\overline{a}) : \overline{a} \in X\} \quad (X \subseteq N^n).$$

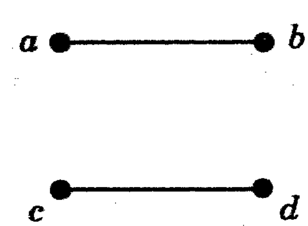
F is said to be an *embedding of $\mathcal{N}$ in $\mathcal{M}$* if

$$F(\mathbf{R}_i^{\mathcal{N}}) = \mathbf{R}_i^{\mathcal{M}} \cap (F(N))^{n_i}.$$

F is an *isomorphism* if it is an embedding and onto M. F is an *automorphism* if it is an isomorphism and $M = N$.

Consider a graph such as in figure 1.1.

Figure 1.1:



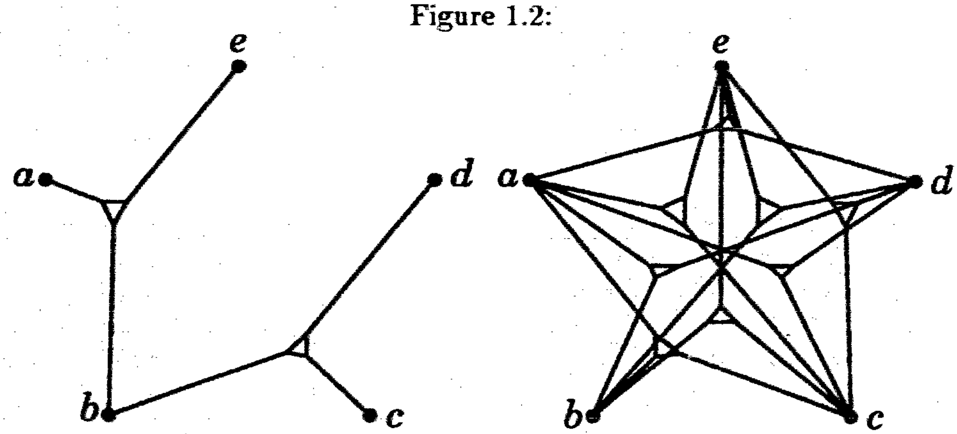
This graph, call it $\mathcal{M}$, can be regarded as an L-structure, $(M, \mathbf{R}^{\mathcal{M}})$ where the universe M is $\{a, b, c, d\}$ and the language L consists of one relation whose interpretation $\mathbf{R}^{\mathcal{M}}$ is $\{(a, b),\ (b, a),\ (c, d),\ (d, c)\}$. Often, to simplify matters when dealing with graphs, we consider the relation $\mathbf{R}^{\mathcal{M}}$ to be a subset of $[M]^2$, where $[M]^i = \{K \subseteq M : |K| = i\}$. Using this convention, the relation of the above graph becomes $\{\{a, b\},\ \{c, d\}\}$.

Now 3-graphs are like graphs except that an edge is an unordered triple rather than an unordered pair. To describe a 3-graph within the framework of model theory we let L be the language with one ternary relation symbol $\mathbf{R}$. Then a *3-graph* is an L-structure $\mathcal{M} = (M, \mathbf{R}^{\mathcal{M}})$ which satisfies:

$$\forall x \forall y \forall z \left[\mathbf{R}(x,y,z) \rightarrow (x \neq y \wedge y \neq z \wedge z \neq x \wedge \mathbf{R}(y,x,z) \wedge \mathbf{R}(z,x,y))\right].$$

However, just as for graphs (which we can think of as 2-graphs in a more general setting) because of the irreflexiveness and symmetry of the relation it is convenient for us to abuse the conventions of model theory and to let $\mathbf{R}^{\mathcal{M}}$ be a subset of $[M]^3$ instead of M^3. So in this thesis by 3-graph we will understand a structure $\mathcal{M} = (M, \mathbf{R}^{\mathcal{M}})$, where M is the *vertex set* and $\mathbf{R}^{\mathcal{M}}$ is the *edge set*. With 3-graphs so defined, we define the *complement* $\overline{\mathcal{M}}$ of the 3-graph $\mathcal{M}$ to be the structure $(M, [M]^3 - \mathbf{R}^{\mathcal{M}})$. So the complement of a 3-graph $\mathcal{M}$ is the 3-graph with the same vertex set (or universe) having edges exactly where $\mathcal{M}$ does not.

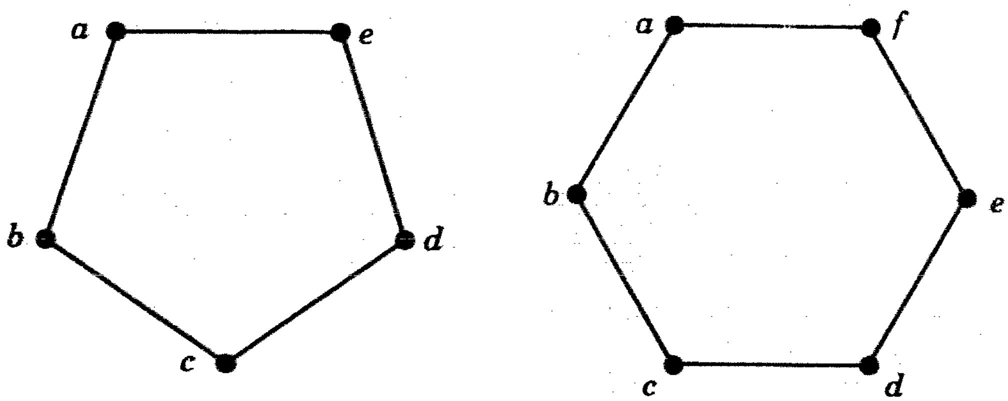
Figure 1.2:



Graphically, we will denote an edge of a 3-graph as a small triangle, with lines extending from the vertices of the triangle to the vertices belonging to the particular edge of the 3-graph. For example, in figure 1.2, on the left we have a 3-graph on $\{a, b, c, d, e\}$ with edges $\{a, b, e\}$ and $\{b, c, d\}$, while on the right we have the complement of that 3-graph.

A structure $\mathcal{M}$ is *homogeneous* if any isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. Consider the structures in figure 1.3.

Figure 1.3:



We can check that the structure on the left is homogeneous by listing all the local isomorphisms, i.e., all the isomorphisms from one (finite) substructure to another, and checking that each of them extends to an automorphism. This is tedious but presents no difficulty. For example, the isomorphism $(a,c) \to (a,d)$ can be extended to the automorphism $(a,b,c,d,e) \to (a,e,d,c,b)$.

However, this is not the case for the structure on the right. We check again the isomorphism $(a,c) \to (a,d)$, but in this case there is no way to extend it to an automorphism of the structure. Note that the complement of a homogeneous 3-graph will be homogeneous. We let $\mathrm{Hmg}(L)$ denote the class of all homogeneous structures on the language L. Two points $x, y \in M$ have the same type over a subset H of M if there is a local isomorphism that fixes H and maps x to y.

A complete first-order theory that admits quantifier elimination has a countable homogeneous model. Finite structures are homogeneous if and only if they admit quantifier elimination. This tie to quantifier elimination makes homogeneous structures a topic of interest for logicians.

Lachlan wrote a survey of the work done on homogeneous structures in [18]. We shall give a brief overview of this work. Study of homogeneous structures has been greatly facilitated by the work of Fraïssé, [10], which shows a direct correspondence between homogeneous structures and amalgamation classes. Considering a particular homogeneous structure often becomes much simpler by looking at the corresponding amalgamation class.

Extensive work has been done on classifying the homogeneous L-structures where L has a single binary relation. Henson began, with his paper "A family of countable homogeneous graphs" [13] that appeared in 1971, a classification of countable homogeneous graphs, and presented a number of related questions. Gardiner, in [11], completed the classification of the finite homogeneous graphs, and Woodrow and Lachlan completed the classification of countable homogeneous graphs in [27] and [19].

The countable homogeneous partially ordered sets were classified by Schmerl in [22] and the countable homogeneous tournaments were classified by Lachlan in [17]. Cherlin has classified the countable homogeneous directed graphs in [5] and [6].

Classification of countably infinite homogeneous structures has relied heavily on the work of Fraïssé. The study of these structures is generally approached by looking at the finite structures which they, and their corresponding amalgamation classes, omit. The technique of using amalgamation classes is not as useful when dealing with finite homogeneous structures. However, as the latter can be interpreted as permutation groups, there is a wide variety of combinatorial and group theoretic arguments that can be applied. This paper depends on the classification of finite simple groups to guarantee a full listing of the finite homogeneous 3-graphs. The goal of this paper is to prove the following theorem.

Theorem 1.1 *Let $\mathcal{M}$ be a finite, non-trivial, homogeneous 3-graph. Then the socle of* $\mathrm{Aut}\,(\mathcal{M})$ *is one of the groups:* $\mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,9)$, $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(3,3)$*; and $\mathcal{M}$ is determined up to isomorphism by the socle. The cardinalities of these homogeneous 3-graphs are 6, 10, 7 and 13 respectively.*

More detailed information on these structures can be found in §3.1.5, §3.1.7, Lemma 1.10 and §3.1.4.

Given a set M, if G is a subgroup of the symmetric group on M, then we refer to the pair as a *permutation group*, denoted $(M;G)$ or $\mathcal{G}$; and we say that G acts on M. Although G acts primarily on M we will assume that its action has been extended to cartesian powers of M and to all other sets constructed from the set M in a canonical fashion, e.g., $[M]^5$, ${}^{M}M$ (the set of functions from M into itself), etc.

The permutation group $(M;G)$ is *transitive* if there is only one orbit in the action of G on M, that is, for all $a, b \in M$, there exists $g \in G$ such that $g(a) = b$. The permutation group $(M;G)$ is *k-transitive* ($k \geq 1$) if, given any two ordered k-tuples $(x_1, \ldots, x_k)$, and $(y_1, \ldots, y_k)$ of distinct elements of M, there is some $g \in G$ such that $g(x_i) = y_i$, $1 \leq i \leq k$. Two permutation groups $(M_1;G_1)$ and $(M_2;G_2)$ are isomorphic if there exists a bijection F from M_1 to M_2 such that $G_2 = \{FgF^{-1} : g \in G_1\}$. Unless stated otherwise, isomorphic will mean isomorphic as permutation groups.

With a 3-graph $\mathcal{M}$, we associate the permutation group $\mathcal{G} = (M;G)$, where G is $\mathrm{Aut}\,(\mathcal{M})$, the subgroup of $\mathrm{Sym}(M)$ that preserves the relation $R^{\mathcal{M}}$. If $\mathrm{Aut}\,(\mathcal{M})$ is equal to $\mathrm{Sym}(M)$, then we call $\mathcal{M}$ a *trivial* 3-graph; so a 3-graph is trivial if and only if it has either all possible edges or has no edges.

When $\mathcal{M}$ is homogeneous, $\mathcal{G}$ must be transitive as, for any $x, y \in \mathcal{M}$, there must exist an automorphism of $\mathcal{M}$ that maps x to y. In fact, as $\mathcal{M}$ has only a single ternary relation, $\mathcal{G}$ will be 2-transitive, since, for any two ordered pairs of points, the map taking one to the other is a local isomorphism and hence extends to an automorphism of $\mathcal{M}$. So we have the following lemma:

Lemma 1.2 *If $\mathcal{M}$ is a homogeneous 3-graph, then the permutation group $(M; \mathrm{Aut}\,(\mathcal{M}))$ is 2-transitive.*

It is by considering 2-transitive permutation groups that we will be able to determine all the finite homogeneous 3-graphs. Cameron, in [4, page 5], considers the *socle* of a permutation group which is defined to be the product of the minimal normal subgroups of the group. He shows that, when G is a primitive permutation group acting on a finite set, the socle of G is a direct product of isomorphic simple groups. Many of the results we use concerning the 2-transitive simple groups, particularly the classical groups and their actions, are used without specific reference to their sources. We use the descriptions of these groups, drawn from the existing literature, to determine if they describe the automorphism group of a homogeneous 3-graph.

Observe that when $\mathcal{M}$ is a homogeneous 3-graph we can recover $\mathcal{M}$ from the corresponding permutation group $(M;G)$ up to complementation because G has just two orbits on unordered triples, one of which is the edge set of $\mathcal{M}$.

In the language of design theory (see [1]), homogeneous 3-graphs are triple systems satisfying a very strong symmetry property. For the four triple systems listed in Theorem 1.1 the respective values of λ are 2, 4, 1 and 2. These triple systems are surely well known to design theorists.

1.1 An end in sight

Before we start looking for specific instances of finite homogeneous 3-graphs, we will use a result of Cherlin and Lachlan to show that there can only be a finite number of such 3-graphs that are non-trivial. We will use their dichotomy theorem, as well as the definitions leading to it, from [7, pages 817-818].

Let $\mathcal{G} = (M; G)$ be a finite permutation group (not necessarily one arising from a 3-graph) and $\mathcal{F} = \{H_i : i \in I\}$ be a family of pairwise disjoint subsets of M. $\mathcal{F}$ is *mutually indiscernible* in $\mathcal{G}$ if every $\pi \in \mathrm{Sym}(\cup\mathcal{F})$ which fixes each H_i setwise extends to a member of G. $\mathcal{F}$ is *mutually quasi-indiscernible* if for every family $\{\pi_i : i \in I\}$ with $\pi_i \in \mathrm{Alt}(H_i)$ $(i \in I)$ there exists $g \in G$ such that $g\restriction H_i = \pi_i$ $(i \in I)$.

A finite permutation group $(H; G)$ is a *(twisted) coordinate system* if there is a G-invariant equivalence relation E on H such that $H/E = \{H_i : i \in I\}$ is a finite mutually (quasi-) indiscernible family on which G acts transitively, and $|H_i| \geq 5$ $(i \in I)$. The degenerate case in which $G = \mathrm{Sym}(H)$ is allowed and even typical. The H_i are the *components* of H.

Given a possibly twisted coordinate system $(H; G)$ with components H_i and k such that $0 < 2k \leq |H_i|$, define the *Grassmannian permutation group*:

$$\mathrm{Gr}_k(H; G) = (\{X \subseteq H : (\forall i \in I)(|X \cap H_i| = k)\}; G).$$

Of course, in this equation the final occurrence of G refers to the action of G on the Grassmannian set $\{X \subseteq H : (\forall i \in I)(|X \cap H_i| = k)\}$. Such abuses of notation will not cause confusion since we can infer from the context on which set G is thought of as acting. In the present case note that the action of G on the Grassmannian set is faithful.

A finite permutation group $(M; G)$ is *coordinatizable* if it is isomorphic to $\mathrm{Gr}_k(H; G)$ for some k, H, G as above. When $(M; G)$ is coordinatizable, the coordinatization is essentially unique, as shown by the following result, where $\mathrm{Soc}(G)$ denotes the socle of G. This result is stat·d without proof in [7, page 818].

Lemma 1.3 *With the above notation,* $\mathrm{Soc}(G) \cong [\mathrm{Alt}(n)]^d$, *where* $d = |I|$.

Proof: For $i \in I$, let N_i denote the group consisting of all $g \in G$ (seen as acting on H) such that g fixes $H - H_i$ pointwise and induces an even permutation on H_i. Since $\mathcal{F}$ is invariant as a set under G, $H_i \lhd G$. Since $|H_i| \geq 5$, N_i is simple and so N_i is a minimal normal subgroup of G.

Now consider an arbitrary, non-identity, normal subgroup N of G. Choose $i \in I$, $a \in H_i$, and $g \in N$ such that $g(a) \neq a$. There are two cases:
Case 1. $g(a) \in H_i$. Then $g(H_i) = H_i$. For all $n \in N_i$, $n^{-1}g^{-1}ng \in N \cap N_i$. Choosing n such that $ng(a) \neq gn(a)$, we see that $N \cap N_i \neq \{1\}$. Hence $N_i \leq N$.
Case 2. $g(a) \in H_j$ for some $j \in I - \{i\}$. Then $g(H_i) = H_j$. Choose $b \in H_i - \{a\}$. Let

$$h = g^{-1}(g(a)g(b))g(g(a)g(b)).$$

Since h is a commutator, $h \in N$. Also

$$h = (ab)(g(a)g(b)).$$

Cojugating by elements of N_i, we see that $(cd)(g(a)g(b)) \in N$ for all $c, d \in H_i$ such that $c \neq d$. Again, it is clear that $N_i \leq N$.

We have shown that the groups N_i, $i \in I$, are the only minimal normal subgroups of G which justifies our remark above. □

A finite relational structure $\mathcal{M}$ is *coordinatizable* if the corresponding permutation group $(M; \mathrm{Aut}\,(\mathcal{M}))$ is coordinatizable.

Theorem 1.4 (Dichotomy Theorem, [7], page 818) *Let a finite relational language L be fixed. There is an integer m such that for every finite $\mathcal{M} \in \mathrm{Hmg}(L)$ and every maximal equivalence relation E (where $E \neq M^2$) on M invariant under* $\mathrm{Aut}\,(\mathcal{M})$ *one of the following occurs.*

(A) $|M/E| \leq m$,

(B) $(M/E, K)$ *is coordinatizable, where* $K \leq \mathrm{Sym}(M/E)$ *is the group induced by* $\mathrm{Aut}\,(\mathcal{M})$.

Corollary 1.5 *Up to isomorphism, there are only a finite number of non-trivial finite homogeneous 3-graphs.*

Proof: In the theorem fix the language L to consist of a single ternary relation. Let m denote the integer provided by the conclusion. Consider a finite homogeneous 3-graph $\mathcal{M}$ seen as an L-structure. Since $\mathcal{M}$ is 2-transitive the only equivalence relation $E \neq M^2$, is the identity on $\mathcal{M}$. We conclude that either $|M| \leq m$ or $\mathcal{M}$ is coordinatizable.

Consider the case in which $\mathcal{M}$ is coordinatizable. There exist an integer $k > 0$ and a permutation group $(H;G)$ such that $2k \leq |H_i|$, $i \in I$, and

$$(M; \mathrm{Aut}\,(\mathcal{M})) \cong \mathrm{Gr}_k(H;G).$$

Consider a, $b \in M$ and let X_a, $X_b \subseteq H$ be the subsets of H which correspond to a, b under the isomorphism. If $k > 1$ or $|I| > 1$, then as a, b run through M, $|X_a \cap X_b|$ takes at least the values 0, 1, 2. In this case, the action of $\mathrm{Aut}\,(\mathcal{M})$ on M is not 2-transitive because $|X_a \cap X_b|$ is preserved under $\mathrm{Aut}\,(\mathcal{M})$.

Since homogeneous 3-graphs have 2-transitive automorphism groups, $k = 1$ and $|I| = 1$. In this case the $\mathrm{Gr}_k(H;G)$ is just a set with either the symmetric group or the alternating group acting on it. Hence $\mathrm{Aut}\,(\mathcal{M}) \geq \mathrm{Alt}(M)$. Since the language is that of a ternary relation it is clear that $\mathrm{Aut}\,(\mathcal{M}) = \mathrm{Sym}(M)$. So $\mathcal{M}$ is trivial if it is coordinatizable. □

1.2 Getting started

In a permutation group $(M;G)$, for any $x \in M$, we define the *stabilizer of* x, G_x, to be $\{g \in G : g(x) = x\}$. For any $A \subset M$, we define the *stabilizer of* A, G_A, to be $\{g \in G : (\forall a \in A)g(a) = a\}$. For any $A \subset M$, we define the *set stabilizer of* A, $G_{\{A\}}$, to be $\{g \in G : g(A) = A\}$. A transitive permutation group $(M;G)$ is *regular* if G_x is the identity for every $x \in M$. A transitive permutation group $(M;G)$ is *primitive* if it has no non-trivial equivalence relations that are invariant under the action of G.

Lemma 1.6 ([26], Proposition 4.4) *Let $(M;G)$ be a transitive permutation group, $a \in M$, and G be abelian. Then $G_a = \{1\}$.*

Lemma 1.7 ([26], §20) *Let $(M;G)$ be a transitive permutation group, $a \in M$, and $G_a = \{1\}$. Then*

$$F : g(a) \mapsto g \quad (g \in G)$$

is a bijection from M to G. Further, for $g \in G$, the action of g on G induced by F, namely FgF^{-1}, is multiplication by g on the left.

Remark: If G is abelian, then in the previous lemma the action of g on G is addition of g and is thought of as a translation.

Lemma 1.8 ([26], Theorem 8.8) *Let $(M;G)$ be a primitive permutation group, $N \lhd G$, and $N \neq \{1\}$. Then $(M;N)$ is transitive.*

Lemma 1.9 ([2], page 7, lemma 1.3.6) *Let $(M;G)$ be a transitive permutation group and $a \in M$. Then $(M;G)$ is k-transitive if and only if $(M - \{a\}; G_a)$ is $(k-1)$-transitive.*

We will later make use of projective spaces and the groups associated with them, so we will now make some definitions. We start with a vector space V of dimension d over a field F. For our purposes, F will typically be GF(q), so we will give some of the definitions referring to d and q, as well as for V generally.

Let $V^* = V - \{0\}$ and $x, y \in V^*$. Define an equivalence relation where x and y are equivalent if the statement $(\exists\lambda \in F^*)(x = \lambda y)$ holds. The equivalence classes so defined become the points of the *projective geometry* PG(V).

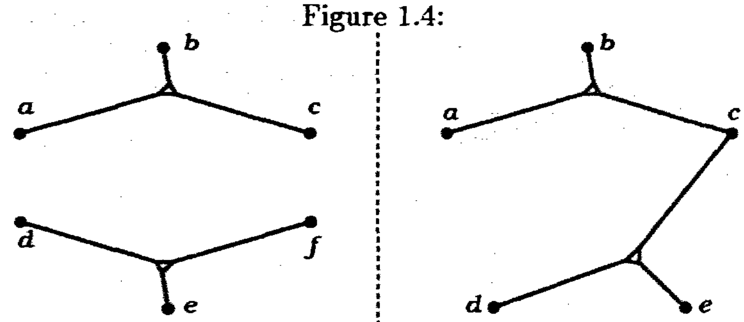
We can denote the equivalence class of $x \in V^*$ as $[x]$, and, under the mapping $x \mapsto [x]$, the image of a subspace U of V, denoted $[U]$, is a subspace of $\mathrm{PG}(V)$. It turns out that if a subspace U has dimension k, then its image $[U]$ has dimension $k-1$, and, when $V = V(d,q)$, the dimension of generated projective space is $d-1$, and $\mathrm{PG}(V) = \mathrm{PG}(d-1,q)$. See [2, §2.5] for further details.

The *general linear group on* V, $\mathrm{GL}(V)$, is the group of linear operators on V, which is isomorphic to the group of the $d \times d$ non-singular matrices with entries from F. The *special linear group*, $\mathrm{SL}(V)$ is the subgroup of $\mathrm{GL}(V)$ consisting of the elements with determinant equal to one. To get the *projective general linear group*, $\mathrm{PGL}(V)$, we start with $\mathrm{GL}(V)$ and divide by its center. Similarly, we get the *projective special linear group* $\mathrm{PSL}(V)$ from $\mathrm{SL}(V)$.

Lemma 1.10 *Let $\mathcal{M}$ be a homogeneous 3-graph of size ≥ 4 and having at least one edge. If the intersection of any two edges in $\mathcal{M}$ is one or fewer points, or, equivalently, if two points are sufficient to determine an edge, then $|M| = 7$ and $\mathcal{M}$ is unique.*

Proof: First, we claim any two edges must meet in exactly one point. Since there exists one edge that connects two points, we know that for any two points there must be an edge connecting them. We have an edge, $\{a,b,c\}$, and a fourth point d, and there exists an edge through a and d. So we know intersecting edges do occur. Now consider figure 1.4. If we

Figure 1.4:

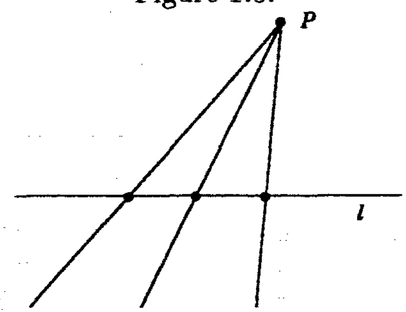


look at only (a,b,d,e), the two sides are exactly alike. As two points determine an edge, we cannot have both of these situations occuring. Since we know we do have intersecting edges in $\mathcal{M}$, the situation on the left will never occur. This establishes the claim.

If we think of the edges as lines, then we have a structure where each line has three points, two points determine a line, and two lines intersect in exactly one point. This is a projective plane with lines of size three.

Suppose there are n points. Then there are $n(n-1)/6$ lines, and there are $(n-1)/2$ lines through each point. We can look at a point P and a line l, where P is not on l, as in figure 1.5. Every line including P must cross l, but l contains only three points, so P can be included in only three lines. So $(n-1)/2 \leq 3$, and $n \leq 7$.

Figure 1.5:



If we look at the projective planes, we see that the only projective geometry having lines of size three and order greater than three and less than eight is PG(2,2), which has seven points. So now we will check to make sure it is homogeneous. The points of $\mathcal{M}$ can be identified with elements of $(\mathrm{GF}(2))^3 - \{(0,0,0)\}$, where $\{A, B, C\} \in [(\mathrm{GF}(2))^3 - \{(0,0,0)\}]^3$ is a line if $A + B + C = (0,0,0)$. The ternary relation R on this geometry is defined by $R(A,B,C) \Leftrightarrow A + B + C = (0,0,0)$ in GF(2).

Let $\mathbf{V} = \mathrm{V}(3,2)$. For any two bases u_1, u_2, u_3 and v_1, v_2, v_3 of $\mathbf{V}$, the mapping $u_i \mapsto v_i$ generates a linear transformation which gives an automorphism of $\mathcal{M}$. So, we start with an isomorphism between two substructures of $\mathcal{M}$. If one of these substructures contains three vectors that are independent, then we can consider them as a basis. This basis must map to three independent vectors in the other substructure, and this generates an automorphism. If the substructures do not contain three independent vectors, then we can arbitrarily pick vectors independent of the vectors we already have, until we have three, and then proceed as before. So any isomorphism between substructures can be extended to an automorphism, and PG(2, 2) is homogeneous. □

Take $\mathcal{M}$, a finite homogeneous 3-graph with corresponding permutation group $(M; G) = \mathcal{G}$. Recall that $\mathcal{G}$ is 2-transitive.

Lemma 1.11 *If N, the socle of a group G, is a non-abelian simple group, then*

$$N \leq G \leq \mathrm{Aut}\,(N)$$

.

Proof: Since N is normal, the elements of G will induce automorphisms of N by conjugation, so we need to show that distinct elements of G give distinct automorphisms of N. We know $C_G(N)$ is normal, from the normality of N, so either $N \leq C_G(N)$, or $C_G(N) = 1$, otherwise, $C_G(N) \cap N$ would yield a normal subgroup smaller than N, contradicting the minimality of N.

The former case would imply that N is abelian, which is not the case we are considering, so it must be that $C_G(N) = 1$.

Now suppose we have $a, b \in G$ such that for every $n \in N$, $a^{-1}na = b^{-1}nb$. Then $ba^{-1}nab^{-1} = n$ and $(ab^{-1})^{-1}n(ab^{-1}) = n$. But then $ab^{-1} \in C_G(N)$, so $ab^{-1} = 1$ and $a = b$. Thus no two distinct elements of G yield the same automorphism of N, and we can conclude that $G \leq \mathrm{Aut}(N)$. □

Our main instrument will be a theorem characterizing the socle of a 2-transitive permutation group. The theorem that follows is a revised and expanded version of a theorem by Burnside, [3, page 202].

Theorem 1.12 *Let (M, G) be a 2-transitive finite permutation group and N be the socle of $\mathcal{G}$. There are two possibilities.*

i) N is simple and non-abelian. In this case, M can be identified with a conjugacy class of subgroups of N with G acting by conjugation.

ii) N is elementary abelian. In this case, M can be identified with N in such a way that a fixed element $a \in M$ corresponds to 0, N acts by translation, and G_a acts by conjugation.

Proof: As was mentioned above, from Cameron's paper [4, page 5] we know that $\mathrm{Soc}(G)$ is the direct product of isomorphic simple groups. If these isomorphic simple groups are abelian, then certainly N is elementary abelian. Suppose otherwise, i.e., N is the direct product of isomorphic non-abelian simple groups. In this case the theorm of Burnside says that there is a normal subgroup H of G which is simple such that $C_G(H) = \{1\}$. Being simple, H must be a minimal normal subgroup, and since any two minimal normal subgroups of a group will commute, $N = H$. For the rest of part i), if N is simple, then, to explore the nature of M, we look at a mapping

$$\phi : x \mapsto G_x \cap N \; (x \in M).$$

N acts transitively, since $N \lhd G$ and (M, G) is primitive, so $G_x \cap N \neq N$. Suppose $G_x \cap N = \{1\}$. For this x, define the mapping $b \mapsto n_b$ $(b \in M)$ by letting n_b be the element of N such that $n_b(x) = b$. There is only one possibility for n_b, otherwise there is an element of $N - \{1\}$ that fixes x which contradicts $g_x \cap N = \{1\}$. The mapping is also one-to-one and onto. Using this bijection to identify M with N, we now have G acting on N. We observe that G_x acts on N by conjugation, i.e., $g(m) = gmg^{-1}$ $(g \in G_x,\ m \in N)$. Since the action of G on M is 2-transitive, the action of G_x on $N - \{1\}$ is transitive. Hence any two elements of $N - \{1\}$ are conjugate in G. It follows that N is a p-group for some prime p. Hence by [21, page 57, Theorem 4.4], N has a non-trivial center, which is a contradiction. So $G_x \cap N$ is always a non-trivial subgroup of N.

Note that $\phi(g(x)) = g(G_x \cap N)g^{-1}$. This follows from the fact that $g(G_x \cap N)g^{-1}$ must be a subgroup of N and also a subgroup of $G_{g(x)}$. So $g(G_x \cap N)g^{-1} \leq G_{g(x)} \cap N$, and, since $g(G_x \cap N)g^{-1}$ has the same number of elements as $G_{g(x)} \cap N$, they must be equal.

Since $G_x \cap N \not\trianglelefteq N$, the mapping ϕ must have range of size greater than one. On the other hand, if ϕ is not one-to-one, then we have a G-invariant block system defined on M; we can take $x, y \in M$ to be equivalent if and only if $\phi(x) = \phi(y)$, but this contradicts G being 2-transitive.

Now for part ii), fix $a \in M$, where a is arbitrary. Define the mapping $b \mapsto n_b$ $(b \in M)$ by letting n_b be the unique element of N such that $n_b(a) = b$. There is only one possibility for n_b since N acts regularly on M. Using this bijection we identify the set M with the set N. Then N acts on $M = N$ by translation, i.e., $n(m) = n + m$ $(m \in N)$. Further, the stabilizer G_a of a in G acts on $M = N$ by conjugation. Note that $G = NG_a = G_aN$, so the action of G is completely determined by the known actions of N and G_a. □

This theorem of Burnside is the key to the rest of our work. In Chapter 2, we show that case ii) does not give rise to any non-trivial finite homogeneous 3-graphs. In Chapter 3, we consider case i), and find that it gives rise to exactly four non-trivial finite homogeneous 3-graphs.

Chapter 2

Abelian Socle

We have a group G with socle $N = \underbrace{\mathbf{Z}_p \times \cdots \times \mathbf{Z}_p}_{k \text{ times}}$, and we want to know if there is a homogeneous 3-graph $\mathcal{M}$ such that $\mathrm{Aut}\,(\mathcal{M}) = G$. If there is such $\mathcal{M}$, then there is a set of triples $\mathbf{R}^{\mathcal{M}} \subseteq [M]^3$, the edge set of $\mathcal{M}$, such that G is the stabilizer of the set $\mathbf{R}^{\mathcal{M}}$ in $\mathrm{Sym}(M)$. For the rest of this section we shall suppose that we have a homogeneous 3-graph $\mathcal{M}$ whose automorphism group is G, and we shall explore the structure of $\mathcal{M}$.

We recall from Theorem 1.12 above that, since the socle is abelian, G acts on M in the following way. Fix $a \in M$, where a is arbitrary. Define $b \mapsto n_b$ ($b \in M$) by letting n_b be the unique element of N such that $n_b(a) = b$. There is only one possibility for n_b since N acts regularly on M. This bijection allows us to identify the set M with the set N. Then N acts on $M = N$ by translation, i.e., $n(m) = n + m$ ($m \in N$). Further, the stabilizer G_a of a in G acts on $M = N$ by conjugation, i.e., $g(m) = gmg^{-1}$ ($g \in G_a$, $m \in N$). Since $G = NG_a = G_aN$, knowing how N and G_a act on $M = N$ we know the action of G.

We claim that the 4-ary relation S defined by

$$x + y = z + w \tag{2.1}$$

is definable in $\mathcal{M}$. To see this we argue as follows. First, since we are dealing with a finite structure, a relation on M ($= N$) is definable in $\mathcal{M}$ if and only if the relation seen as a set of tuples is G-invariant. When x, y, z, w are moved by $n \in N$, then $n + n$ is added to each side of (2.1) and the equation still holds. On the other hand, if we move x, y, z, w by

$g \in G_a$, then we are applying an automorphism to N and again (2.1) is preserved. Since $G = NG_a$ and N and G_a both preserve (2.1), G also preserves (2.1). By essentially the same argument we see that there is a dependence relation which is definable in $\mathcal{M}$: the elements $a_1, a_2, \ldots, a_m \in M$ are *dependent* if there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathrm{GF}(p)$ not all 0 such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = 0 \text{ and } \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m = 0.$$

The dependence relation will be useful in the discussion below. We define some further notions related to the dependence relation. First, we say that $b \in M$ *depends on* $A \subseteq M$ if there is an independent tuple $\bar{a} \subseteq A$ such that $\bar{a}b$, the $(m+1)$-tuple $(a_1, a_2, \ldots, a_m, b)$, is dependent. Otherwise, b is said to be *independent over* A. Secondly, the *affine closure* $\overline{A}$ of $A \subseteq M$ is the least set X, $A \subseteq X \subseteq M$, such that no $a \in M - X$ depends on X. It is easily seen that this closure operation satisfies the usual axioms:

i) $X \subseteq \mathrm{cl}(X)$,

ii) $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$,

iii) $X \subseteq \mathrm{cl}(Y) \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$,

iv) $b \in \mathrm{cl}(X \cup \{a\}) - \mathrm{cl}(X) \Rightarrow a \in \mathrm{cl}(X \cup \{b\})$.

The *dimension* of a set X is the size of the least $Y \subseteq \mathrm{cl}(X)$ such that $\mathrm{cl}(Y) = \mathrm{cl}(X)$. Since we are dealing with a vector space over $\mathrm{GF}(p)$ once a point is fixed, the affine closure of a set of dimension $n+1$ has size p^n.

We will show that no homogeneous 3-graphs, other than trivial ones, arise from groups with abelian socles. For $p = 2$ we obtain an outright contradiction. When $p = 3$ we show that, if $\mathcal{M}$ is a homogeneous 3-graph, then either each pair in $[M]^2$ lies in a unique edge of $\mathcal{M}$, or each pair in $[M]^2$ lies in a unique edge of the complementary 3-graph. From Lemma 1.10 this means that either $\mathcal{M}$ or its complement is the projective plane $\mathrm{PG}(2,2)$. However, the automorphism group of $\mathrm{PG}(2,2)$ has socle $\mathrm{PSL}(3,2)$ which is simple and non-abelian, contradicting the case hypothesis. Finally, when $p > 3$ we show that a ternary relation is definable in $\mathcal{M}$ which is asymmetric on triples of distinct elements. Again, this contradicts our assumptions.

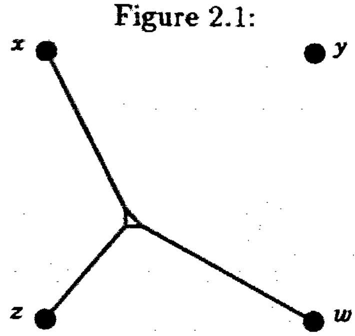
2.1 $N = \mathbf{Z}_p \times \cdots \times \mathbf{Z}_p$, $p = 2$

We first consider the case where $p = 2$; as all the elements have order two, the relation $x + y = z + w$ is equivalent to $x + y + z + w = 0$. We will use the latter form because it makes the symmetry of the relation explicit.

Lemma 2.1 *Given $x, y, z, w \in M$ such that $x + y + z + w = 0$, if the set $\{x, y, z, w\}$ has any edges on it, it has all four possible edges.*

Proof: Suppose there is exactly one edge, xzw, on these four points, as in figure 2.1. We

Figure 2.1:



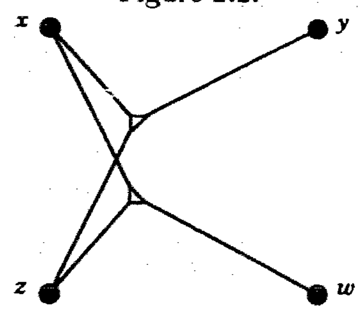
define a local isomorphism mapping $x \to x$, $y \to w$, and $w \to y$. By homogeneity this isomorphism can be extended to $\alpha \in \mathrm{Aut}\,(M)$. Since $x + y + z + w = 0$ and α preserves the relation S, $\alpha(z) = z$. However, this introduces a new edge, $xyz = \alpha(xzw)$, contradicting our assumption that there is exactly one edge on these four points.

Suppose there are exactly two edges on $\{x, y, z, w\}$, as in figure 2.2.
Consider the local isomorphism that fixes z and interchanges x and y. Again, the induced automorphism introduces an extra edge on the four points, resulting in a contradiction.

If we have exactly three edges, xyz, xwz, and yzw, we can use the same argument and again get a contradiction. Thus, we can conclude that, if there are any edges at all on $\{x, y, z, w\}$, all four possible edges must be present. □

Now let us start again with four points x, y, z, w such that $x+y+z+w = 0$ and assume there is no edge on $\{x, y, z, w\}$. Take a point a independent over $\{x, y, z, w\}$. We will study

Figure 2.2:



the edges on the affine closure E of $\{a, x, y, z\}$ in $\mathcal{M}$, which has size 8. We can identify the remaining members of E as follows. Let

$$b = a + x + z,$$

$$c = a + x + y,$$

$$d = a + x + w.$$

We can see that a, b, c, d, x, y, z, w are all distinct and so these are all the members of E. We will show that up to a permutation of the points in E and complementation there is only one possibility for the set of edges on E. The possible edge set for E will be described in Case 1.2 below.

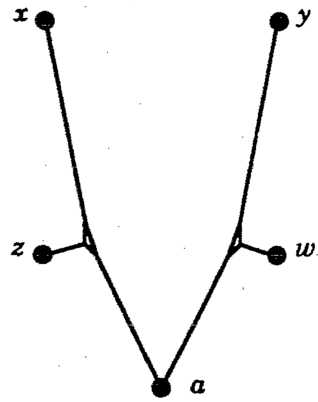
The point a must have an edge with $\{x, y, z, w\}$, otherwise, a and w would each have the same type over $\{x, y, z\}$, which contradicts the definability of the relation S. By the same token we have:

Lemma 2.2 *For any 3-set $T \subseteq \{x, y, z, w\}$, a point a, independent over $\{x, y, z, w\}$, must make at least one edge with T.*

We now consider various cases according to the number of edges which a makes with $\{x, y, z, w\}$.

Case 1. There are exactly two edges between a and $\{x, y, z, w\}$. From Lemma 2.2 we may suppose without loss of generality that the two edges are axz and awy, as shown in figure 2.3. Since axz is an edge and $a + b + x + z = 0$, by Lemma 2.1 we have all four

Figure 2.3:



edges on $\{a, b, x, z\}$. It also follows that $a + b + y + w = 0$ [all edges]. Consider the 4-set $\{a, x, y, z\}$, on which the only edge is axz. For any $\pi \in \mathrm{Sym}(\{a, x, z\})$, the map $\pi \cup \{(y, y)\}$ is a local isomorphism. Note that $\pi \cup \{(y, y)\}$ means the extension of π to $\{a, x, y, z\}$ which maps y to itself. Hence there exists $\pi^* \in \mathrm{Aut}\,(\mathcal{M})$ such that $\pi \subseteq \pi^*$ and $\pi^*(y) = y$, where $\pi \subseteq \pi^*$ indicates that the function π^* extends the function π. Since E is the affine closure of $\{a, x, y, z\}$, π^* fixes E as a set.

We now consider two subcases:

Case 1.1. The only edges between b and $\{x, y, z, w\}$ are bxz and byw. Let π fix z and switch a and x. Consider the image Z of the 4-set $\{a, b, y, w\}$ under π^*. There are four edges on Z. Hence each of the points in $Z - \{x, y\}$ makes an edge with $\{x, y\}$. Since none of a, b, z, w make an edge with $\{x, y\}$, the remaining points of Z must be c and d. But $a+b+y+w = 0$ and $c + d + x + y \neq 0$. So Z cannot be the image under π^* of $\{a, b, y, w\}$, contradiction.

Case 1.2. Otherwise. Then b makes at least one edge with $\{x, y, z, w\}$ other than bxz and bwy. Let π_1 be the local isomorphism which fixes a, y, w and switches x and z, and π_2 be the local isomorphism which fixes a, x, z and switches y and w. From the homogeneity there exist π_1^*, $\pi_2^* \in \mathrm{Aut}\,(\mathcal{M})$ extending π_1, π_2 respectively. Since $b = a + x + z$, we have $\pi_1^*(b) = \pi_2^*(b) = b$. Now the group $\langle \pi_1^*, \pi_2^* \rangle$ acts transitively on the set

$$\{\{x, y\}, \{x, w\}, \{z, y\}, \{z, w\}\}.$$

Thus, since b makes an edge with one of these pairs, it makes an edge with each of them.

Focusing for the moment on the edges through a, we observe that:

$$a+d+y+z=0 \text{ [no edges]}, \quad a+d+x+w=0 \text{ [no edges]},$$
$$a+c+x+y=0 \text{ [no edges]}, \quad a+c+z+w=0 \text{ [no edges]},$$
$$c+d+x+z=0 \text{ [no edges]}, \quad c+d+y+w=0 \text{ [no edges]}.$$

Each of the equations may be checked by substituting for b, c, d the expressions which define them. In each case the absence of edges follows from Lemma 2.1. For example, since ayz is not an edge, there is no edge on $\{a, d, y, z\}$.

We also observe that the map σ, which fixes b and x, and permutes z, w, and y cyclically, is a local isomorphism and hence extends to $\sigma^* \in \text{Aut}(\mathcal{M})$. Since $a = b+x+z$, $c = b+x+w$, and $d = b+x+y$, σ^* permutes a, c, and d cyclically. Applying σ^* we get:

$$b+c+x+w=0 \text{ [all edges]}, \quad b+c+y+z=0 \text{ [all edges]},$$
$$b+d+x+y=0 \text{ [all edges]}, \quad b+d+z+w=0 \text{ [all edges]}.$$

At this point we have determined all the edges of E which intersect both $\{a, b, c, d\}$ and $\{x, y, z, w\}$.

It remains to discover what edges, if any, there are on $\{a, b, c, d\}$. The 4-set $\{x, z, c, a\}$ spans E, and the only edge on it is $\{a, x, z\}$. Thus there is an automorphism of $\mathcal{M}$ which, while fixing E as a set, permutes x, z, a cyclically and fixes c. Since there is an edge through c and x, there is also an edge through c and a in E. From our findings above, the third point of that edge must be either b or d. So by Lemma 2.1 we have

$$a+b+c+d=0 \text{ [all edges]}.$$

We conclude that this case determines uniquely the set of edges on E.

Case 2. There are six edges between a and $\{x, y, z, w\}$. Clearly, we have

$$a+b+x+z=0 \text{ [all edges]}, \quad a+b+y+w=0 \text{ [all edges]}.$$

To obtain a contradiction suppose that b makes more than the edges bxz, byw with $\{x, y, z, w\}$. By the same argument as in Case 1.2, b makes six edges with $\{x, y, z, w\}$. But then x and y both form a complete 3-graph on four points with the edge abz. This is a contradiction of Lemma 2.1 since $a+b+x+z=0$ but $a+b+y+z\neq 0$. Therefore there are exactly two edges between b and $\{x,y,z,w\}$. This takes us back to Case 1.2 with the roles of a and b interchanged.

Case 3. There are either five edges between a and $\{x,y,z,w\}$ or three edges two of which intersect only at a. Without loss of generality we may suppose that axz, axy, and ayw are edges, and that azw is not. There is a local isomorphism α fixing a and x and switching y and z. Let $\alpha^* \in \mathrm{Aut}(\mathcal{M})$ extend α. Since $x+y+z+w=0$, $\alpha^*(w)=w$. Thus $azw=\alpha^*(ayw)$ is an edge, contradiction. So this case cannot occur.

Case 4. There are four edges between a and $\{x,y,z,w\}$. Up to a permutation of $\{x,y,z,w\}$ there are only two possibilities for the set of four edges. There must be two edges whose intersection is a, which we may take to be axz and ayw. Now the two possibilities are distinguished by whether the two remaining edges intersect in two points or one. In the former case the two remaining edges may be taken to be axy and axw. This case is ruled out by the argument of Case 3. Thus the two remaining edges which a makes with $\{x,y,z,w\}$ meet only in a. Thus without loss of generality the edges which a makes with $\{x,y,z,w\}$ are axz, ayw, axy, and azw. Consider the edges between b and $\{x,y,z,w\}$. Since

$$a+b+x+z=0 \text{ [all edges]}, \quad \text{and} \quad a+b+y+w=0 \text{ [all edges]},$$

bxz and byw are both edges. From Case 3, b cannot make either three or five edges with $\{x,y,z,w\}$. From Cases 1 and 2, if the number of edges between b and $\{x,y,z,w\}$ were two or six, then the number of edges between a and $\{x,y,z,w\}$ would be six or two respectively. We conclude that b makes exactly four edges with $\{x,y,z,w\}$. The argument for a shows that the other two edges which b makes with $\{x,y,z,w\}$ have intersection b. So there are two cases:

Case 4.1. b realizes the same type as a over $\{x,y,z,w\}$. Since there are four edges on each of the sets $\{a,b,x,z\}$, $\{a,b,x,y\}$, the mapping $(a,b,x,z)\to(a,b,x,y)$ is a local isomorphism and so extends to an automorphism of $\mathcal{M}$. Since $a+b+x+z=0$ but $a+b+x+y\neq 0$, this is a contradiction.

Case 4.2. The edges between b and $\{x,y,z,w\}$ are bxz, byw, byz, and bxw. There is a local isomorphism which fixes a and maps $x\to z\to w\to y\to x$. The induced automorphism switches b and c. Thus the edges which c makes with $\{x, y, z, w\}$ are cxy, czw, cxw, and cyz. Since $a + d + x + w = a + d + y + z = 0$, neither dxw nor dyz is an edge. Since $b + d + w + z = b + d + x + y = 0$, neither dwz nor dxy is an edge. Finally, since $c + d + x + z = c + d + y + w = 0$, neither dxz nor dyw is an edge. We conclude that there are no edges between d and $\{x, y, z, w\}$. This contradicts Lemma 2.2.

Case 5. Otherwise. From above, if any of a, b, c, d makes a number of edges with $\{x, y, z, w\}$ which is different from three, then the same is true for all of a, b, c, d. Thus in the present case each of a, b, c, d makes three edges with $\{x, y, z, w\}$. On the one hand, from Case 3 any two of the three edges which a makes with $\{x, y, z, w\}$ meet in a pair. On the other hand, from Lemma 2.2 the intersection of the three edges is a. Thus without loss of generality we may suppose that the edges which a makes with $\{x, y, z, w\}$ are axy, ayz, and azx. Since $a + b + x + z = 0$, bxz is an edge and byw is not by Lemma 2.1. To obtain a contradiction suppose that the edges which b makes with $\{x, y, z, w\}$ are bxz, bxw, bzw. (Just as a selects a triple from $\{x, y, z, w\}$ so must b.) Notice that $\{a, b, y, w\}$ is a 4-set with no edges such that $a + b + y + w = 0$. Further, the edges which x makes with this 4-set are precisely xab, xay, and xbw. Since the last two of these intersect only at x, we have a contradiction by Case 3. It follows that b realizes the same type over $\{x, y, z, w\}$ as a. Similarly, c and d also realize the same type over $\{x, y, z, w\}$ as a. We may also note that the only edges which $\{a, b\}$ makes with $\{x, y, z, w\}$ are abx and abz because

$$a + b + x + z = a + b + y + w = 0,$$

and axz is an edge, while ayw is not. In the same fashion all edges which meet $\{a, b, c, d\}$ in a pair are determined. In particular, between w and $\{a, b, c, d\}$ there is no edge between w and $\{a, b, c, d\}$.

It only remains to determine the edges on $\{a, b, c, d\}$. Since $a + b + c + d = 0$, we have all or none. If there are none, then w realizes the same type as a over $\{b, c, d\}$, contradiction. Thus we have all four edges on $\{a, b, c, d\}$.

We have shown above that up to a permutation of E there are two possibilities for the set of edges on E found in cases 1.2 and 5 respectively. However, it is easily checked that the permutation $(ay)(bw)(cx)(dz)$ maps each of these edge sets into the complement of the other. Thus up to complementation and a permutation of E there is only one possibility for the edge set on E. It is also clear that, if $\mathcal{M}$ is non-trivial, then it has affine dimension greater than four because the structure on E found above is clearly not homogeneous. It is important to observe that no 4-set in E has exactly 2 edges on it. Hence the same is true of $\mathcal{M}$.

For the rest, replacing $\mathcal{M}$ by $\overline{\mathcal{M}}$ if necessary, we may suppose that there is a closed subset

$$E = \{a, b, c, d, x, y, z, w\}$$

of M whose edge set is that found in Case 1.2. Let $e \in M - E$ be arbitrary, and F denote the affine closure of $\{e\} \cup \{x, y, z, w\}$. There are two cases:

Case I. $\mathcal{M}{\restriction}E \cong \mathcal{M}{\restriction}F$. Then there exists $f \in F - E$ such that f and b realize the same type over $\{x, y, z, w\}$, i.e., b and f both make edges with every pair from $\{x, y, z, w\}$. For every pair $\{u, v\}$ from $\{x, y, z, w\}$, on $\{b, f, u, v\}$ we have at least the edges buv and fuv. Hence at least one of bfu and bfv is an edge, otherwise we would have a 4-set with exactly two edges on it. Therefore we can choose u, v in $\{x, y, z, w\}$ such that there are four edges on $\{b, f, u, v\}$. Without loss of generality, there are four edges on $\{b, f, x, y\}$. From the discussion above we have

$$b + d + x + y = 0 \quad \text{[all edges]}.$$

Hence d and f realize the same type over $x + y + b$, contradiction.

Case II. Otherwise. Then the edge set on F is like that described in Case 5. In particular, there exist $f \in F - E$ and a 3-set from $\{x, y, z, w\}$, say $\{x, y, z\}$, such that the edges between f and $\{x, y, z, w\}$ are fxy, fyz, and fzx. There is a local isomorphism σ which fixes x, y, z and takes b to f. Let $\sigma^* \in \mathrm{Aut}(\mathcal{M})$ extend σ. Then $\sigma^*(w) = w$ since $x + y + z + w = 0$ and S is definable. Hence f and b realize the same type over $\{x, y, z, w\}$, contradiction.

2.2 $N = \mathbf{Z}_p \times \cdots \times \mathbf{Z}_p$, $p > 2$

We now consider the cases in which $p > 2$. In this situation, the relation $x + x = y + z$ is non-trivial and definable. Restricted to distinct x, y, z this relation must be either the relation of the 3-graph or the complementary relation. Replacing $\mathcal{M}$ by $\overline{\mathcal{M}}$ if necessary, we may suppose that $x + x = y + z$ defines the edge set of $\mathcal{M}$.

If $p = 3$, then $x+x = y+z$ is equivalent to $x+y+z = 0$ which shows that the relation is symmetric. In this case $\mathcal{M}$ has the property that each doubleton is included in exactly one edge. By Lemma 1.10, $\mathcal{M}$ is PG$(2,2)$. But the socle of the automorphism group of PG$(2,2)$ is PSL$(3,2)$ which is simple non-abelian. So this case yields no homogeneous 3-graphs.

If $p > 3$, then we test to see if the relation is symmetric. We let $2x = y + z$, and hence $y = 2x - z$. If the relation is symmetric, then also $2z = x + y$, and $y = 2z - x$. But then

$$2x - z = 2z - x$$

and $z = x$. So, for $p > 3$, if the relation is non-trivial, it is not symmetric, so it cannot be the relation of a 3-graph.

Chapter 3

Nonabelian Simple Socle

In this section $\mathcal{M}$ denotes a supposed finite homogeneous 3-graph whose automorphism group is G. N denotes the socle of G and is assumed to be non-abelian. As noted in Theorem 1.12, since G is 2-transitive in its action on M, N is simple, $N \leq G \leq \mathrm{Aut}\,(N)$, and G acts in the following way: M can be identified with a conjugacy class of subgroups of N and G acts by conjugation. Cameron [4, page 8] lists all triples (N, n, k) such that there is a finite k-transitive permutation group G of degree n with socle N, but no $(k+1)$-transitive permutation group of degree n with socle N. We have reproduced Cameron's list below in Table 1. The list indicates how many 2-transitive representations there are in each case, that is to say, how many possibilities there are for the conjugacy class of subgroups of N, which is naturally identified with M.

N	n	k	Remarks
$A_n, n \geq 5$	n	n	Two representations if $n = 6$
$\mathrm{PSL}(d,q), d \geq 2$	$(q^d-1)/(q-1)$	3 if $d = 2$	$(d,q) \neq (2,2),(2,3)$
		2 if $d > 2$	Two representations if $d > 2$
$\mathrm{PSU}(3,q)$	q^3+1	2	$q > 2$
${}^2B_2(q)$ (Suzuki)	q^2+1	2	$q = 2^{2a+1} > 2$
${}^2G_2(q)$ (Ree)	q^3+1	2	$q = 3^{2a+1} > 2$
$\mathrm{PSp}(2d,2)$	$2^{2d-1}+2^{d-1}$	2	$d > 2$
$\mathrm{PSp}(2d,2)$	$2^{2d-1}-2^{d-1}$	2	$d > 2$
$\mathrm{PSL}(2,11)$	11	2	Two representations
$\mathrm{PSL}(2,8)$	28	2	
A_7	15	2	Two representations
M_{11} (Mathieu)	11	4	
M_{11} (Mathieu)	12	3	
M_{12} (Mathieu)	12	5	Two representations
M_{22} (Mathieu)	22	3	
M_{23} (Mathieu)	23	4	
M_{24} (Mathieu)	24	5	
HS (Higman-Sims)	176	2	Two representations
Co_3 (Conway)	276	2	

Table 1.

We will consider all the pairs (N, n) which occur in Cameron's list and check for which of them there is a corresponding group G whose 2-transitive permutation representation of degree n yields a homogeneous 3-graph. The results of our search will be as follows. The only non-trivial finite homogeneous 3-graphs are those given by the pairs:

$$N = \mathrm{PSL}(2,5), \qquad n = 6,$$

$$N = \mathrm{PSL}(2,9), \qquad n = 10,$$

$$N = \mathrm{PSL}(3,2), \qquad n = 7,$$

$$N = \mathrm{PSL}(3,3), \qquad n = 13.$$

As well as supplying the list of possible pairs (N, n) which is the basis of our work in this section, Cameron [4] offers another useful piece of information. It turns out that, except in the case in which $N = \mathrm{PSL}(2,8)$ and $n = 28$, the action of N is also 2-transitive. Note that we may ignore the alternating groups because the 3-graphs generated by their natural action either have all possible edges or none and thus are of no interest. We now proceed to treat the remaining pairs (N, n) listed in Table 1 beginning with the cases in which N is one of the projective special linear groups.

3.1 Linear groups

Let $d \geq 2$ and $V = \mathrm{V}(d,q)$ denote the vector space of dimension d over $\mathrm{GF}(q)$. Then the special linear group $\mathrm{SL}(d,q)$ is by definition the group of all linear transformations of V into itself which have determinant 1. The center of $\mathrm{SL}(d,q)$ consists of the linear transformations of the form $\mathbf{v} \mapsto \lambda\mathbf{v}$, where $\lambda \in \mathrm{GF}(q)$ and $\lambda^d = 1$. Let $P = \mathrm{PG}(d-1,q)$ denote the projective space corresponding to V which is defined to be the set of 1-spaces of V. Clearly, $\mathrm{SL}(d,q)$ has an induced action on P. The projective special group $\mathrm{PSL}(d,q)$ is defined to be the quotient of $\mathrm{SL}(d,q)$ by its center. An equivalent definition is obtained by saying that $\mathrm{PSL}(d,q)$ is the group of permutations of P induced by $\mathrm{SL}(d,q)$ — we can check that the elements of $\mathrm{SL}(d,q)$ which fix P pointwise are just the elements of the center. If we fix a basis of V, then with respect to that basis the elements of $\mathrm{SL}(d,q)$, and hence also those of $\mathrm{PSL}(d,q)$, are represented by $d \times d$ matrices over $\mathrm{GF}(q)$ with determinant 1.

According to Table 1, for $d > 2$ there are two possible 2-transitive representations of $\mathrm{PSL}(d,q)$ of degree $(q^d-1)/(q-1)$. These are afforded by the action on the 1-spaces of V and the action on the $(d-1)$-spaces of V. From our perspective these two representations are the same because, whether we deal with 1-spaces or $(d-1)$-spaces, the resulting permutation group is the same. The mapping $A \mapsto (A^{-1})^T$ is an automorphism of the group of $d \times d$ matrices of determinant 1. The corresponding automorphism of $\mathrm{PSL}(d,q)$ interchanges the stabilizer of a 1-space with the stabilizer of its null space. When $d = 2$, then $d - 1 = 1$ and the two possible representations coincide. Thus below it will be sufficient to examine the action on 1-spaces, i.e., the action on P.

For any $\mathbf{v} \in V - \{\mathbf{0}\}$, the corresponding 1-space is $\langle\mathbf{v}\rangle \in P$. We say that distinct points $\langle\mathbf{v}_1\rangle, \ldots, \langle\mathbf{v}_k\rangle$ in P are *dependent* or *independent* according as $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is dependent or independent in V. In particular, we say that the points are *collinear* if

$$\dim(\langle\mathbf{v}_1, \ldots, \mathbf{v}_k\rangle) = 2.$$

We note the following elementary lemma.

Lemma 3.1 *i) Let $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ be independent d-tuples in P. There exists $\alpha \in \mathrm{PSL}(d,q)$ such that $\alpha(a_i) = b_i$ $(1 \leq i \leq d)$.*

ii) Let $d > 2$, or $d = 2$ and q be even. Let $a_1, a_2, a_3, a_4 \in P$ be distinct collinear points. There exists $\alpha \in \mathrm{PSL}(d,q)$ such that $\alpha(a_1) = a_1$, $\alpha(a_2) = a_2$, and $\alpha(a_3) = a_4$.

iii) If $d = 2$ and q is odd, then $\mathrm{PSL}(d,q)$ has two orbits on triples of distinct points.

From the lemma, if $d = 2$ and q is even, then the action of $\mathrm{PSL}(d,q)$ on P is 3-transitive. Hence no non-trivial homogeneous 3-graph can arise in this case. In the other cases, because there are only two orbits, O_1 and O_2 say, of N on triples of distinct points of P, the edge-set of $\mathcal{M}$ must be one of these two orbits (it does not matter which, of course). Thus there is only one possibility for G, namely, the group consisting of all $\pi \in \mathrm{Sym}(P)$ which fix O_1 and O_2 as sets. In particular, whenever $d > 2$ we can assume that the edge-set of $\mathcal{M}$ is the set of all triples $\{a,b,c\} \subseteq P$ such that a, b, c are collinear. This means that the set of lines of the projective space is invariant under G. We now proceed to consider various cases which arise. The case in which $d = 2$ and $q > 9$ will be left to the next subsection.

3.1.1 d > 3

Fix a basis of V and with respect to the basis let a, b, c, d, e be the following points of P:

$$\langle(1,0,0,0)\rangle,\ \langle(0,1,0,0)\rangle,\ \langle(0,0,1,0)\rangle,\ \langle(0,0,0,1)\rangle,\ \langle(1,1,1,0)\rangle.$$

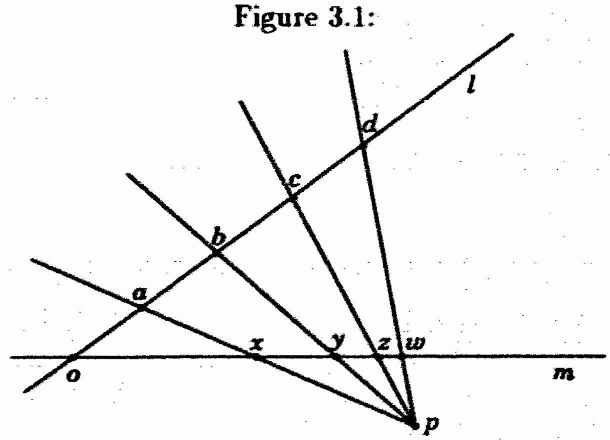
There are no edges on either of the 4-sets $\{a,b,c,d\}, \{a,b,c,e\}$. Thus, from the 3-homogeneity of $\mathcal{M}$ there exists $\alpha \in G$ which fixes a, b, and c, and maps d to e. Also, there exists f, namely $\langle(1,1,0,0)\rangle$, such that abf and cef are both edges. However, there is no f such that abf and cdf are both edges. This contradicts the invariance of the set of edges under α. Thus no homogeneous 3-graphs arise in this case.

3.1.2 d = 3, q > 3

Each line in $\mathrm{PG}(2,q)$ contains $q+1$ points. So, since $q > 3$, we have at least five points on each line. Take two lines, $l = ab$ and $m = xy$ in the projective space which meet in a point o. Let p be the point of intersection of the lines ax and by. Now consider points c, d

on l different from o, a, b, and let z, w be the unique points on m such that pcz and pdw are edges, as shown in figure 3.1. In the 3-graph $\mathcal{M}$ each of the quadruples $\{a, b, c, d\}$ and $\{x, y, z, w\}$ has all four possible edges, and there are no other edges on these eight points. From the 3- homogeneity there exists $\alpha \in G$ which switches z and w and fixes a, b, c, d, x and y. Clearly, $\alpha(p) = p$ and so α fixes the lines pc and xy setwise. Since z is the unique point of intersection of the lines pc and xy, $\alpha(z) = z$. This contradicts $\alpha(z) = w$. So, again we get no homogeneous 3-graphs.

Figure 3.1:



3.1.3 $\mathbf{d = 3,\ q = 2}$

Here we get the homogeneous 3-graph of size seven mentioned in Theorem 1.10.

3.1.4 $\mathbf{d = 3,\ q = 3}$

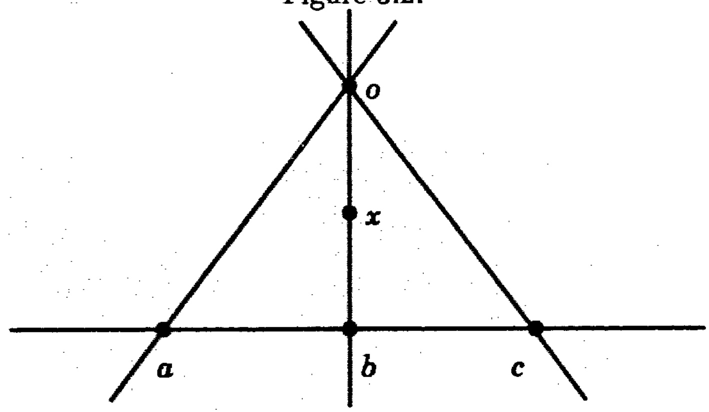
Here $\mathcal{M}$ is the projective plane over GF(3) which has 13 points. As noted above, for the edge-set of $\mathcal{M}$ we may take the set of all triples $\{a, b, c\}$ in P such that a, b, c are collinear. This set of edges is invariant under the group $\mathrm{PGL}(3, 3)$ whose definition is the same as that of $\mathrm{PSL}(3, 3)$ except that the linear transformations are now only required to be nonsingular and not necessarily to have determinant 1. From this observation we get

Lemma 3.2 *Let $a, b, c, d \in P = \mathrm{PG}(2,3)$ be distinct points no three of which are collinear, and x, y, z, u be a similar quadruple. There exists $\alpha \in \mathrm{PGL}(3,3)$ such that $\alpha(a) = x$, $\alpha(b) = y$, $\alpha(c) = z$, and $\alpha(d) = u$.*

We now check that in this case any local isomorphism of $\mathcal{M}$ extends to an automorphism. We will consider cases proceeding in increasing order of size of domain. Since $\mathrm{Aut}(\mathcal{M})$ is evidently 2-transitive we begin with size three.

Suppose that $a \to a'$, $b \to b'$, $c \to c'$ is a local isomorphism. From i) of Lemma 3.1, the local isomorphism certainly extends to an automorphism of $\mathcal{M}$ if a, b, c are not collinear. So suppose that a, b, c are collinear. Choose a point o off the line abc and let x be a third point on the line bo, as shown in figure 3.2. There are no edges on a, c, o, x, and b is the intersection of the lines ac and ox. Perform the same construction on a', b', c' to get o' and x'. From our remarks above there is an automorphism that maps a, c, o, x to a', c', o', x' respectively. Since the geometry is preserved this automorphism must also map b to b'.

Figure 3.2:



Now we consider local isomorphisms with domain of size four. If the domain consists of four points, no three collinear, then, from Lemma 3.2 the local isomorphism extends to an automorphism. For four points all on a line any automorphism which maps three of the points as required must also map the fourth point appropriately because the lines have size four. For three points on a line and a fourth point off the line, we can use the same argument as for three points on a line, letting the fourth point be the point o in the construction.

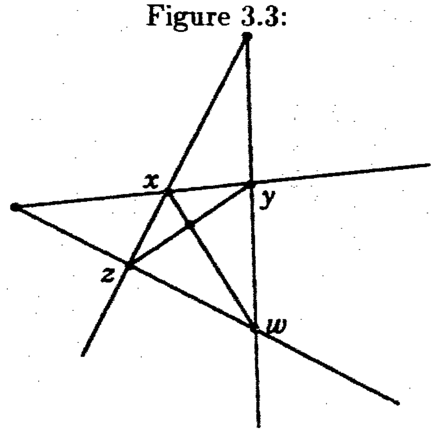
Before proceeding further we need two observations. First, consider four points x, y, z, w, no three of which are collinear. Six lines are generated by these points, as shown in figure 3.3, and the intersections of these lines give us three new points. Since each line has four points on it, there is one additional point on each line. None of the points shown can coincide, since each pair of lines intersects in a single point. So we have thirteen points defined, which is all the points in the plane. We conclude that, given any five points, at least three of them must lie on a line.

For any $X \subseteq M$ we define $\mathrm{dcl}(X)$, the *definable closure of X in M* to be

$$\{a \in M : g(a) = a \text{ for all } g \in G_X\}.$$

For our second observation, let γ be a local isomorphism of $\mathcal{M}$ with domain $X \sqcup Y$, the disjoint union of X and Y. If $Y \subseteq \mathrm{dcl}(X)$, then it is sufficient to show that there is an automorphism extending $\gamma \restriction X$. We call a subset X of M *irredundant* if for no $a \in X$ is a in $\mathrm{dcl}(X - \{a\})$. A subset which is not irredundant is said to *have redundancy.* Clearly, to prove 3-homogeneity it suffices to consider local isomorphisms with irredundant domain.

Figure 3.3:



Now consider a local isomorphism whose domain is irredundant of size at least five. From irredundancy the domain has the following character. There are three points on a line, say a, b, c. The fourth point of that line, o say, cannot be in the domain by irredundancy. Let x, y be two other points of the domain. If the line xy meets abc in any of the points a, b,

c, then the domain clearly has redundancy. Therefore xy and abc meet in o. However, now a is clearly in the definable closure of b, c, x, y. Thus there is no local isomorphism with irredundant domain of size greater than four.

3.1.5 d = 2, q = 5

In this case $\mathcal{M}$ is the projective line over GF(5), a structure of size six. We will show that N gives us a homogeneous 3-graph. Take a basis of $V = \mathrm{V}(2,5)$ and let the points of $P = \mathrm{PG}(1,5)$ be labelled as follows:

$$\infty = \langle(0,1)\rangle,\ 0 = \langle(1,0)\rangle,\ 1 = \langle(1,1)\rangle,\ 2 = \langle(1,2)\rangle,\ 3 = \langle(1,3)\rangle,\ 4 = \langle(1,4)\rangle,\ .$$

The group $N = \mathrm{PSL}(2,5)$ is generated by the mappings:

$$i \mapsto 4i,\ i \mapsto i+1,\ i \mapsto 1/i,$$

where the arithmetic is modulo 5 and

$$i \cdot \infty = \infty,\ 1/\infty = 0, \text{ and } 1/0 = \infty$$

for all i, $0 \leq i \leq 4$. Take the edge-set to be the orbit of $\{\infty, 0, 1\}$. Then the edges are:

$$\infty01,\ \infty04,\ \infty12,\ \infty23,\ \infty34,\ 013,\ 023,\ 024,\ 124,\ 123.$$

Since N acts 2-transitively, in order to show homogeneity it suffices to consider local isomorphisms which fix both 0 and 1. Since the permutation (14)(23) is in N, any local isomorphism with domain of size 3 extends to an automorphism.

For the rest, we claim that the definable closure of any 3-set is M. To see this it suffices to look at the edge $E = \{\infty, 0, 1\}$ because the edges and non-edges are interchanged by the outer automorphism of N induced by conjugation by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Now 4 is the unique point of $M - E$ which makes an edge with $\infty0$. Also, 2 and 3 are distinguished because 2 makes an edge with $\infty1$ while 3 does not. This is sufficient.

3.1.6 $\mathbf{d = 2,\ q = 7}$

Now M is the projective line over GF(7). Following the same method as in the previous case we fix a basis of $V = \mathrm{V}(2,q)$. In terms of the basis we label the points of M by ∞, 0, 1, ..., 6. Here ∞ means $\langle(0,1)\rangle$, while i means $\langle(1,i)\rangle$ for $1 \leq i \leq 6$. The orbits of $N_{\infty 0}$, the pointwise stabilizer of $\{\infty, 0\}$, are $\{0\}$, $\{\infty\}$, $\{1,2,4\}$, and $\{3,5,6\}$. The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ maps ∞, 0, 1 to 0, ∞, 6. Thus any non-empty symmetric ternary relation on M which is N-invariant contains all triples. So no homogeneous 3-graph arises in this case. The essential difference between this case and the last one is that -1 is a square (mod 5) but not (mod 7).

3.1.7 $\mathbf{d = 2,\ q = 9}$

The group PSL(2,9) is isomorphic to A_6. We find it convenient here to focus on N in its guise as an alternating group. We will describe a 2-transitive action of S_6 of degree 10. We view S_6 as the group of all permutations of the set $\Omega = \{a,b,c,d,e,f\}$. We label the partitions of this set into two 3-sets by the integers 0 through 9 as follows.

Label	Partition
0	abc | def
1	abd | cef
2	abe | cdf
3	abf | cde
4	acd | bef
5	ace | bdf
6	acf | bde
7	ade | bcf
8	adf | bce
9	aef | bcd

We study the induced action of $G = S_6$ on $M = \{0, 1, \ldots, 9\}$.

Lemma 3.3 *G acts 2-transitively on M.*

Proof: Since G clearly acts transitively on M, it is sufficient to show that G_0 is transitive on $M - \{0\}$. Let $i = \{X, Y\}$ be a partition of $\{a, b, c, d, e, f\}$ into two 3-sets. It is sufficient to show that there exists $g \in G_0$ which maps $\{X, Y\}$ to 1. Without loss of generality, $|X \cap \{a, b, c\}| = 2$. So let $X \cap \{a, b, c\} = \{u, v\}$ and $X - \{a, b, c\} = \{w\}$. Now let g permute a, b, c so that $g(u) = a$, $g(v) = b$, and permute d, e, f so that $g(w) = d$. Then $g \in G_0$ and $g(i) = 1$ as required. □

We note from the proof that A_6 also acts 2-transitively on M. For, when we created $g \in G_0$, we arbitrarily chose $g(u) = a$ and $g(v) = b$. So, we can make g an even permutation, switching the values of $g(u)$ and $g(v)$ as necessary to adjust the parity.

If we take one part X_i of a partition i, then, for a distinct partition j, the intersection of X_i with one part of j, say X_j, will have exactly one element in it, while the intersection of X_i with Y_j will have exactly two elements. Also, then $|Y_i \cap Y_j| = 1$ and $|Y_i \cap X_j| = 2$. So, from a pair of partitions i, j, we have generated a set $(X_i \cap X_j) \cup (Y_i \cap Y_j)$ that is a pair in $\Omega = \{a, b, c, d, e, f\}$. More explicitly, for $p = \{i, j\} \in [M]^2$ define $F(p)$ to be

$$\{u \in \Omega : (\exists X \in i)(\exists Y \in j)(X \cap Y = \{u\})\}.$$

By inspection $F(\{0,1\}) = \{c, d\}$. From the 2-transitivity of the action on M, F maps $[M]^2$ into $[\Omega]^2$. Now $|[\Omega]^2| = 15$ and $|[M]^2| = 45$. From the 2-transitivity of the action on Ω, $F^{-1}(X)$ has the same size for all $X \in [\Omega]^2$. Therefore $|F^{-1}(X)| = 3$ for all X. Define E to be the equivalence relation on $[M]^2$ such that pEq if and only if $F(p) = F(q)$. The equivalence classes of E will have size 3. In particular, the E-class of $\{0,1\}$ is $\{\{0,1\},\{5,7\},\{6,8\}\}$.

Looking at G_{01} the pointwise stabilizer of $\{0,1\}$ we find it is generated by the permutations (ab), (ef), $(ae)(cd)(bf)$, and hence by the permutations

$$(49)(58)(67),\ (23)(56)(78),\ (29)(34)(57).$$

Define R to be the set of all $p \in [M]^3$ such that

$$\exists q \exists r\, [(p \subset q \cup r) \wedge (qEr)].$$

We regard R as the edge set of a 3-graph with vertex set M. The elements of R will be called *edges.*

Lemma 3.4 *i) For distinct i, j, $k \in M$, $\{i, j, k\} \in R$ if and only if there exists $l \in M$ such that $\{i, j\}E\{k, l\}$.*

ii) If $\{p, q, r\}$ is an E-class, $e \in R$, and $e \subset p \cup q \cup r$, then

$$(e \subset q \cup r) \vee (e \subset r \cup p) \vee (e \subset p \cup q).$$

iii) If $\{p, q, r\}$ is an E-class, and $e \in [p \cup q \cup r]^4$, then there are either two or four edges on e.

Proof: i) It is sufficient to treat the case in which $i = 0$ and $j = 1$. Suppose there is no $l \in M$ such that $\{0,1\}E\{k,l\}$. Then $k \in \{2,3,4,9\}$. Since G_{01} acts transitively on $\{2,3,4,9\}$, it is sufficient to check the case where $k = 2$, i.e. that neither the E-class of $\{0,2\}$, nor that of $\{1,2\}$ gives rise to an edge $\{0,1,2\}$. We find that the respective E-classes of $\{0,2\}$ and $\{1,2\}$ are

$$\{\{0,2\},\ \{4,7\},\ \{6,9\}\},\ \{\{1,2\},\ \{4,5\},\ \{8,9\}\}.$$

So we have that no equivalence class generates the edge $\{0,1,2\}$, which contradicts our initial assumption.

ii) We can take p, q, r to be $\{0,1\}$, $\{5,7\}$, $\{6,8\}$ since these are the pairs in the E-class of $\{0,1\}$. Looking at G_{01} we see that 5 and 7 can be switched while holding the rest of $p \cup q \cup r$ fixed; similarly, for 6 and 8. Thus it suffices to check that $\{0,5,6\} \notin R$, which follows from the fact that $\{0,5\}$, $\{1,7\}$, $\{3,9\}$ is an E-class.

iii) We have two cases. Either e contains two pairs in the E-class in entirety, or e contains one pair, and one element from each of the remaining pairs. In the first case, e will have four edges; in the second case, e will have two edges. □

Again, we take one part X_i of a partition i, and a distinct partition j, and X_j with $|X_i \cap X_j| = 1$, $|X_i \cap Y_j| = 2$, $|Y_i \cap X_j| = 2$, and $|Y_i \cap Y_j| = 1$. This time we will map the pair $\{i,j\}$ to the set of three pairs $\{(X_i \cap X_j) \cup (Y_i \cap Y_j), X_i \cap Y_j, Y_i \cap X_j\}$. In short, we define $F' : [M]^2 \to [[\Omega]^2]^3$ by setting $F'(\{i,j\})$ equal to

$$\{F(\{i,j\})\} \cup \left\{U \in [\Omega]^2 : (\exists X \in i)(\exists Y \in j)(U = X \cap Y)\right\}.$$

Again we define an equivalence relation E' on $[M]^2$ from F', as E was defined from F. So $F'(\{01\}) = \{\{a,b\},\{c,d\},\{e,f\}\}$, and we find that the E' class of $\{0,1\}$ is $\{\{0,1\},\ \{2,3\},\ \{4,9\}\}$. Then define R' from E' as R was defined from E. In the same way as before we can show that all the properties of E and R mentioned above hold equally for E'. The triples in R' are called *co-edges*. Inspecting the E-class and the E'-class of $\{0,1\}$, it is clear that for $i \in M - \{0,1\}$,

$$\{0,1,i\} \in R \Leftrightarrow \{0,1,i\} \notin R'.$$

Thus $[M]^3$ is the disjoint union of R and R'.

We are now ready to show that the 3-graph $\mathcal{M}$ with vertex set M and edge set R is homogeneous. Let $f : X \to Y$ be an isomorphism between sub-3-graphs of $\mathcal{M}$. We have to show that f extends to an automorphism of $\mathcal{M}$. From 2-transitivity we can assume that $0, 1 \in X \cap Y$ and that $f(i) = i$ for $i \in \{0, 1\}$. We may also assume that $|X| > 2$.

Case 1. $|X| = 3$. Let $X = \{0, 1, x\}$ and $Y = \{0, 1, y\}$. By the duality of edges and co-edges we can suppose that X and Y are edges. Therefore $x, y \in \{5, 6, 7, 8\}$. Since G_{01} is transitive on $\{5, 6, 7, 8\}$, we have the desired conclusion.

Case 2. $|X| = 4$ and there are either four or no edges on X. By the duality of edges and co-edges we can assume that there are four edges on X. From Lemma 3.4,

$$X - \{0, 1\}, Y - \{0, 1\} \in \{\{5, 7\}, \{6, 8\}\}.$$

But either of the first two generators of G_{01} switches $\{5, 7\}$ and $\{6, 8\}$, while the third switches 5 and 7 keeping 6 and 8 fixed. Hence f has an extension in G_{01} as required.

Case 3. Otherwise. So $|X| \geq 4$ and, if $|X| = 4$, then X does not have four edges on it. We claim that there exists a 4-set $X_0 \subseteq X$ such that on X_0 there are exactly two edges. From Lemma 3.4 part iii), if $X \subseteq \{0, 1, 5, 6, 7, 8\}$, then any subset of size 4 of X will have either 4 or 2 edges, and if it has 4 edges, then it was covered by Case 2, so X_0 exists as claimed. For the rest, by duality we may assume that

$$X \cap \{5, 6, 7, 8\} \neq \emptyset,\ X \cap \{2, 3, 4, 9\} \neq \emptyset.$$

By applying an appropriate element of G_{01}, we can suppose that $5 \in X$. By examining the E-classes, we find that each of 2, 3, 4, 9 makes exactly one edge with $\{0, 1, 5\}$; the E-classes that generate these edges are:

$$\{\{1, 4\}, \{2, 5\}, \{3, 6\}\},$$

$$\{\{0, 9\}, \{2, 6\}, \{3, 5\}\},$$

$$\{\{1, 2\}, \{4, 5\}, \{8, 9\}\},$$

$$\{\{0, 3\}, \{4, 8\}, \{5, 9\}\},$$

$$\{\{0,7\},\ \{1,5\},\ \{2,4\}\},$$

$$\{\{0,5\},\ \{1,7\},\ \{3,9\}\}.$$

The edges formed are $\{0,3,5\}$, $\{0,5,9\}$, $\{1,2,5\}$, and $\{1,4,5\}$. So now we have our X_0 as claimed. Without loss of generality we can suppose that the two edges of X_0 intersect in $\{0,1\}$. We saw above this occurs only in the case that $X_0 \subseteq \{0,1,5,6,7,8\}$. Thus we can suppose that $X_0 = \{0,1,5,6\}$ because G_{01} induces every permutation of $\{5,6,7,8\}$ which preserves the partition $\{\{5,7\},\{6,8\}\}$. Finally, observe that every element of M realizes a different 1-type over X_0. For instance, 7 is the unique point such that there are four edges on $X_0 \cup \{7\}$, while 2 is the unique point which makes an edge with $\{1,5\}$, $\{0,6\}$, $\{5,6\}$, and with no other pair from X_0. This is sufficient.

This completes our treatment of the projective special linear groups with the exception of the case $d=2$, $q>9$ which is treated in the next section.

3.2 Groups which are too small

In this section we dispose of a number of the pairs (N, k) from Table 1 by showing that the supposed group G cannot be large enough to yield the amount of symmetry that a homogeneous 3-graph must possess. The cases addressed here are:

$N = \mathrm{PSL}(2,8)$ $(n = 28)$,

$N = \mathrm{PSL}(2,q)$ $(n = q+1,\ q > 9,$ and q odd$)$,

$N = {}^2\mathrm{B}_2(q)$ $(n = q^2+1,\ q = 2^{2a+1} > 2)$,

$N = {}^2\mathrm{G}_2(q)$ $(n = q^3+1,\ q = 3^{2a+1} > 3)$.

In these cases no homogeneous 3-graphs arise. The first observation we need shows that the automorphism group of a finite homogeneous 3-graph is fairly large.

Lemma 3.5 *Let $\mathcal{M}$ be a finite homogeneous 3-graph, $|M| = n$, and G denote the automorphism group of $\mathcal{M}$. Then*

$$|G| \geq (1/16)n(n-1)(n-2)^2 \text{ or } |G| \geq (1/16)n(n-1)^2(n-3)$$

according as n is even or odd. (Note that when n is even one of n or $n-2$ is divisible by 4; similarly for $n-1$ or $n-3$ when n is odd.)

Proof: Consider first the case in which n is even. Fix $a_0, a_1 \in M$. There are n choices for a_0, and then $n-1$ for a_1. Let X_2 be the largest orbit of the pointwise stabilizer $G_{a_0a_1}$ of $\{a_0, a_1\}$ in G, and let $a_2 \in X_2$. Let X_3 be the largest orbit of the pointwise stabilizer $G_{a_0a_1a_2}$ of $\{a_0, a_1, a_2\}$ in G and let $a_3 \in X_3$. Since there are $|X_2|$ choices for a_2, and $|X_3|$ choices for a_3, provided X_3 exists there are at least $n(n-1)|X_2||X_3|$ 4-tuples in the orbit under G of (a_0, a_1, a_2, a_3). Therefore we have $|G| \geq n(n-1)|X_2||X_3|$. Since there are only two orbits of $G_{a_0a_1}$ on $M - \{a_0, a_1\}$, the larger orbit X_2 has size at least $(n-2)/2$. Let the other be denoted X_2'. An element of $M - \{a_0, a_1, a_2\}$ can make an edge with one, neither, or both of the pairs $\{a_0, a_2\}$, $\{a_1, a_2\}$. Hence $G_{a_0a_1a_2}$ has at most four orbits on $X_2 - \{a_2\}$ and at most four on X_2'. One of the sets $X_2 - \{a_2\}$, X_2' has size $\geq (n-2)/2$. Hence $|X_3| \geq (n-2)/8$. The conclusion of the lemma is now clear when n is even. For odd n the argument is similar.

□

We now turn to the particular cases. We need the following information about the projective special linear groups:

Lemma 3.6 *Let q be a power of a prime.*

i) $|\mathrm{PSL}(2,q)|$ *is* $(q+1)q(q-1)$ *or* $(1/2)(q+1)q(q-1)$ *according as* q *is even or odd.*

ii) The index of $\mathrm{PSL}(2,q)$ *in its automorphism group is* $|\mathrm{Aut}(\mathrm{GF}(q))|$ *or* $2\cdot|\mathrm{Aut}(\mathrm{GF}(q))|$ *according as* q *is even or odd.*

Part i) appears in [21, page 166]. Part ii) follow from Part i) and the previously defined properties of $\mathrm{PSL}(2,q)$.

Consider first the case in which $N = \mathrm{PSL}(2,8)$ and $n = 28$. From Lemma 3.6 we have

$$|G| \leq (8\cdot 7\cdot 6)\cdot 3 = 1008$$

since $\mathrm{GF}(8)$ has Galois group of order 3. On the other hand from Lemma 3.5 we have

$$|G| \geq (1/16)\cdot 28\cdot 27\cdot (26)^2 = 31941.$$

So we have the desired contradiction.

Next, let $N = \mathrm{PSL}(2,q)$ and $n = q+1$, where $q > 9$ is odd. Here M may be identified with $\mathrm{PG}(1,q)$ the set of 1-spaces of $\mathrm{V}(2,q)$. Now $\mathrm{PGL}(2,q) \leq \mathrm{Aut}(N)$ and acts 3-transitively on $\mathrm{PG}(1,q)$ because the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix},$$

where $\lambda \in \mathrm{GF}(q) - \{0\}$, fixes $\langle(0,1)\rangle$ and $\langle(1,0)\rangle$, and maps $\langle(1,1)\rangle$ to $\langle(1,\lambda)\rangle$. Therefore G is a proper subgroup of $\mathrm{Aut}(N)$, and the index of G in $\mathrm{Aut}(N)$ is at least 2. Let $q = p^k$, where p is prime. From Lemma 3.6 we have

$$|G| \leq (1/2)(q+1)q(q-1)|\mathrm{Aut}(\mathrm{GF}(q))| = (k/2)(q+1)q(q-1).$$

On the other hand, since q is odd, from Lemma 3.5, we have

$$|G| \geq (1/16)(q+1)q(q-1)^2.$$

Putting these two inequalities together we observe that

$$3^k \le p^k = q \le 8k + 1.$$

It follows that $k \le 2$ and that $q \le 9$. Since this contradicts the case hypothesis, there is nothing to prove.

We now turn to the case of the Suzuki group. Here $N = {}^2B_2(q)$ and $|M| = q^2 + 1$, where $q = 2^{2a+1} > 2$. From [23, p. 869], $|N| = (q^2+1)q^2(q-1)$. From [24, Theorem 11, p. 139] the group of outer automorphisms of N is isomorphic to the Galois group of $\mathrm{GF}(q)$. Since $q = 2^{2a+1}$, the latter group has order $2a+1$. Hence

$$|G| \le (q^2+1)q^2(q-1)(2a+1).$$

On the other hand from Lemma 3.5, we have

$$|G| \ge (1/16)(q^2+1)(q^4)(q^2-2).$$

These inequalities are clearly incompatible. Thus the Suzuki groups yield no homogeneous 3-graphs.

Finally, we consider the case of the Rhee group $N = {}^2G_2(q)$ and $|M| = q^3 + 1$, where $q = 3^{2a+1} > 3$. From [20, Theorem 8.5, p. 456], $|N| = (q^3+1)q^3(q-1)$. From [20, Theorem 9.1, p. 459], the group of outer automorphisms of N is isomorphic to the Galois group of $\mathrm{GF}(q)$. We obtain a contradiction in exactly the same way as for the Suzuki groups.

3.3 Unitary groups

The treatment of the material in this section was inspired by the mimeographed notes [16] of Kantor. We begin by fixing q. Let $V = \mathrm{V}(3, q^2)$. For $\mathbf{x} \in \mathrm{GF}(q^2)$, define $\overline{\mathbf{x}}$ to be $\mathbf{x}^q$. Then $\sigma : \mathbf{x} \mapsto \overline{\mathbf{x}}$ is an automorphism and $\sigma^2 = \sigma$. The automorphism σ is extended coordinatewise to the vector space V.

We define a Hermitian form on V by:

$$(\mathbf{u}, \mathbf{v}) = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3}.$$

Elements of V will be written as row vectors relative to a chosen basis. However, when vectors are combined with matrices, they will be treated as though they were column vectors. We define $\mathbf{u}\perp\mathbf{v}$ to mean $(\mathbf{u}, \mathbf{v}) = 0$, i.e., $u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3} = 0$. For a subspace U of V, $U^\perp$ denotes the subspace $\{\mathbf{v} \in V$ such that $(U, \mathbf{v}) = 0\}$. The subspace U is *totally isotropic* if $U \subseteq U^\perp$, and U is *non-singular* if $U \cap U^\perp = \{0\}$.

Suppose that $\mathbf{u}$ and $\mathbf{v}$ generate a subspace of dimension 2, and that this subspace, $\langle \mathbf{u}, \mathbf{v} \rangle$, is totally isotropic. The subspace $\langle \mathbf{u} \rangle^\perp$ is the nullspace of $\overline{\mathbf{u}}$, so we know that $\dim(\langle \mathbf{u} \rangle^\perp) = \dim(\langle \mathbf{v} \rangle^\perp) = 2$. Since $\mathbf{u}$ and $\mathbf{v}$ must be independent vectors, $\dim(\langle \mathbf{u} \rangle^\perp \cap \langle \mathbf{v} \rangle^\perp) \leq 1$. But, if $\langle \mathbf{u}, \mathbf{v} \rangle$ is totally isotropic, then $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq (\langle \mathbf{u} \rangle^\perp \cap \langle \mathbf{v} \rangle^\perp)$, but then we have a contradiction, so it must be that a totally isotropic subspace cannot have dimension greater than 1.

Let $\mathrm{GU}(3, q)$ denote the group of all linear transformations $A : V \to V$ which satisfy:

$$(A\mathbf{u}, A\mathbf{v}) = (\mathbf{u}, \mathbf{v}).$$

These linear transformations are called *isometries*. Relative to a basis of V an isometry A can be identified with 3×3 matrix over $\mathrm{GF}(q^2)$ which satisfies $A^T\overline{A} = I$. For, if A satisfies this condition, then

$$(A\mathbf{u}, A\mathbf{v}) = \mathbf{u}^T A^T(\overline{A\mathbf{v}}) = \mathbf{u}^T(A^T\overline{A})\overline{\mathbf{v}} = \mathbf{u}^T\overline{\mathbf{v}} = (\mathbf{u}, \mathbf{v}).$$

The group $\mathrm{SU}(3, q)$ consists of the isometries of determinant 1, and $\mathrm{PSU}(3, q)$ is obtained from $\mathrm{SU}(3, q)$ by factoring out the center. The center will consist of the scalar linear transformations in $\mathrm{SU}(3, q)$. The group $\mathrm{PSU}(3, q)$ acts naturally on the subspaces of V. As long as $q > 2$, $\mathrm{PSU}(3, q)$ is a simple group and a Borel subgroup is the stabilizer of a totally isotropic 1-space [16, p. 9].

According to the Main Theorem of [9], when a group between $N = \mathrm{PSU}(3, q)$ and $\mathrm{Aut}(N)$ acts 2-transitively, the stabilizer in N of a point is a Borel subgroup. Thus M can be identified with the set of totally isotropic 1-spaces of V. Therefore $|M| = q^3 + 1$ because this is the number of totally isotropic 1-spaces. We will obtain a contradiction to the assumption that $\mathcal{M}$ is a homogeneous 3-graph by looking at the geometry induced by the unitary group.

Call l a *line* of M if it is a maximal subset of M such that in V we have

$$\dim(\langle\bigcup\{U : U \in l\}\rangle) = 2.$$

A line l of M will be the intersection of M with a line of the projective space $\mathrm{PG}(2, q^2)$, so l will be uniquely determined by any two points on it.

Lemma 3.7 *Let T, U be distinct totally isotropic 1-spaces of V. Then there exist* $\mathbf{d}$, $\mathbf{e}$, *and* $\mathbf{f}$ *in V such that*

$$T = \langle \mathbf{e} \rangle,\ U = \langle \mathbf{f} \rangle,\ (\mathbf{e}, \mathbf{f}) = (\mathbf{d}, \mathbf{d}) = 1,\ (\mathbf{d}, \mathbf{e}) = (\mathbf{d}, \mathbf{f}) = 0.$$

Moreover, $\mathbf{d}$ *is unique up to a scalar multiple.*

This lemma follows directly from a [16, p. 8, Theorem]. From the lemma, we see that the action of $\mathrm{PSU}(3, q)$ on the set of totally isotropic 1-spaces is 2-transitive. We can count the totally isotropic 1-spaces in terms of the basis **d**,**e**,**f**. Each totally isotropic 1-space not represented by **f** is represented by a vector of one of the forms:

$$\mathbf{e} + \alpha\mathbf{f} \quad (\alpha \in \mathrm{GF}(q^2),\ \alpha + \overline{\alpha} = 0) \tag{3.1}$$

and

$$\mathbf{e} + \beta\mathbf{f} + \gamma\mathbf{d} \quad (\beta, \gamma \in \mathrm{GF}(q^2),\ \beta + \overline{\beta} \neq 0,\ \beta + \overline{\beta} + \gamma\overline{\gamma} = 0). \tag{3.2}$$

There are q choices for α (the elements of $\mathrm{GF}(q)$), and $q^2 - q$ choices for β. For each choice of β there are $q + 1$ choices for γ. Thus altogether there are

$$1 + q + (q^2 - q)(q + 1) = q^3 + 1$$

totally isotropic 1-spaces of V as noted previously. Observe that the line through $\langle \mathbf{e} \rangle$ and $\langle \mathbf{f} \rangle$ consists of $\langle \mathbf{f} \rangle$ together with the 1-spaces represented by the vectors (3.1).

Lemma 3.8 *The set of lines of M is invariant under G.*

Proof: Let T, U, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$ be the same as in the previous lemma. Let m be the line through T and U. Let H be the pointwise stabilizer of $\{T, U\}$ in G. Let the 3×3 matrix A represent an element of $H \cap N$ relative to the basis $\mathbf{d}, \mathbf{e}, \mathbf{f}$. Then $A\mathbf{e}$, $A\mathbf{f}$ are scalar multiples of $\mathbf{e}$, $\mathbf{f}$ respectively. Also, $A\mathbf{e}, A\mathbf{f} \perp A\mathbf{d}$ since $\mathbf{e}, \mathbf{f} \perp \mathbf{d}$. Therefore $\mathbf{e}, \mathbf{f} \perp A\mathbf{d}$, which means that $A\mathbf{d}$ is a scalar multiple of $\mathbf{d}$. Thus A is a diagonal matrix with diagonal entries κ, λ, μ say. Since the determinant of A is 1, we have $\kappa\lambda\mu = 1$. Also, since $A\mathbf{d}$, $A\mathbf{e}$, and $A\mathbf{f}$ satisfy the identities given in lemma 3.7, we have

$$(A\mathbf{e}, A\mathbf{f}) = (\lambda\mathbf{e}, \mu\mathbf{f}) = \lambda\overline{\mu} = 1 \text{ and } (A\mathbf{d}, A\mathbf{d}) = (\kappa\mathbf{d}, \kappa\mathbf{d}) = \kappa\overline{\kappa} = 1.$$

Thus $\lambda = \mu^{-q}$ and $\kappa\mu^{-q}\mu = 1$, so $\kappa = \mu^{q-1}$. We conclude that relative to the basis $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ the elements of $H \cap N$ are represented by the diagonal matrices

$$D_\mu = \begin{bmatrix} \mu^{q-1} & 0 & 0 \\ 0 & \mu^{-q} & 0 \\ 0 & 0 & \mu \end{bmatrix} \quad (\mu \in \mathrm{GF}(q^2) - \{0\}).$$

Since $D_\mu(0, 1, \alpha) = (0, \mu^{-q}, \mu\alpha)$, the $q - 1$ points of $m - \{T, U\}$, which are represented by vectors of the form (3.1) with $\alpha \neq 0$, form an orbit of $H \cap N$. Let us now consider the action of D_μ on a point of M corresponding to a vector of the form (3.2). In particular, suppose we have a fixed point of D_μ so that

$$\langle D_\mu(1, \beta, \gamma)\rangle = \langle(\mu^{q-1}, \mu^{-q}\beta, \mu\gamma)\rangle = \langle(1, \beta, \gamma)\rangle.$$

Since neither β nor γ is zero, these equations give $\mu^{q+1} = \mu^{2q-1} = 1$ by comparing the ratios of the first two coordinates and of the last two coordinates on each side. We deduce that $\mu^3 = 1$, i.e., D_μ fixes $\langle(1, \beta, \gamma)\rangle$ for at most three values of μ. Since there are $q^2 - 1$ possible values for μ, the orbit of $\langle(1, \beta, \gamma)\rangle$ under $H \cap N$ has size at least $(q^2 - 1)/3$. Since $q > 2$ by assumption, we deduce that $(q^2 - 1)/3 > q - 1$. Therefore $m - \{T, U\}$ is the unique orbit of $H \cap N$ of size $q - 1$. Since $H \cap N \lhd H$, conjugation by any element of H induces an isomorphism of the permutation group $(M, H \cap N)$. Thus $m - \{T, U\}$ is an orbit of H, so the ternary relation of collinearity on M is G-invariant. □

Since a homogeneous 3-graph has only two types of triples of distinct elements and the complementary 3-graph is also homogeneous, we may suppose that $\{T, U, W\}$ is an edge of M if and only if T, U, W are collinear. Now homogeneity fails because there are pairs of lines which meet in M and pairs of lines which do not.

3.4 Symplectic groups

In this section we deal with the pairs (N, n) such that

$$N = \mathrm{Sp}(2d, 2) \text{ and } n \in \{2^{2d-1} + 2^{d-1},\ 2^{2d-1} - 2^{d-1}\} \quad (d > 2).$$

Fix $d > 2$. We will show that this case yields no homogeneous 3-graphs. According to the Main Theorem of [9] in this case $G = N$. Thus we need only study the 2-transitive representations of the group $\mathrm{Sp}(2d, 2)$ without worrying about the effect of possible outer automorphisms. Again we rely on the notes [16] of Kantor which give a description of the two 2-transitive representations which exist for $\mathrm{Sp}(2d, 2)$.

Define $V = V(2d+1, 2)$. Within the context of a given bilinear form $(\ ,\ ) : V \times V \to \mathrm{GF}(2)$ we define orthogonality of vectors and the related notations as we did in the case $N = \mathrm{PSU}(3, q)$, i.e., $\mathbf{u} \perp \mathbf{v}$ means $(\mathbf{u}, \mathbf{v}) = 0$, and so on.

Proofs of Lemmas 3.9, 3.10, 3.11, and 3.12 are included in the appendix.

Lemma 3.9 *There exist a symmetric bilinear form* $(\ ,\ ) : V \times V \to \mathrm{GF}(2)$ *and a mapping* $Q : V \to \mathrm{GF}(2)$ *satisfying*

$$(\mathbf{v}, \mathbf{v}) = 0 \quad (\mathbf{v} \in V), \tag{3.3}$$

$$\dim(V^{\perp}) = 1, \quad Q(\mathbf{v}) = 1 \quad (\mathbf{v} \in V^{\perp} - \{\mathbf{0}\}), \tag{3.4}$$

and

$$Q(\mathbf{u} + \mathbf{v}) = Q(\mathbf{u}) + Q(\mathbf{v}) + (\mathbf{u}, \mathbf{v}) \quad (\mathbf{u}, \mathbf{v} \in V). \tag{3.5}$$

From now on let mappings $(\ ,\ ) : V \times V \to \mathrm{GF}(2)$ and $Q : V \to \mathrm{GF}(2)$ be fixed which satisfy (3.3), (3.4), and (3.5). Further, let $\mathbf{d}$ denote the unique non-zero vector in $V^{\perp}$. A non-zero vector $\mathbf{v}$ is called *totally singular* (*t.s.*) if $Q(\mathbf{v}) = 0$. For any subspace U of V we use $\rho(U)$ to denote $U \cap U^{\perp}$. In particular, $\rho(V) = \{\mathbf{0}, \mathbf{d}\}$.

In this context the group $G = \mathrm{Sp}(2d, 2)$ is seen as the group of all linear transformations T of V which leave Q invariant in the sense that

$$Q(T\mathbf{u}) = Q(\mathbf{u}) \quad (\mathbf{u} \in V).$$

Clearly, from the identity satisfied by Q, any linear transformation which leaves Q invariant also preserves the bilinear form. (If we were only concerned with defining the group $\mathrm{Sp}(2d, 2)$, the most natural definition would be to define it as the group of all linear transformations of a $+$-hyperplane U of V which preserve the restriction of $(\ , \ )$ to U; $+$-hyperplanes are defined below.)

A hyperplane of a vector space is a subspace having dimension one less than the vector space; in this case, the hyperplanes have dimension $2d$. Consider two hyperplanes W and U not containing $\mathbf{d}$. W and U are in the same orbit of G if and only if there are bases $\{\mathbf{w}_i : 1 \le i \le 2d\}$ and $\{\mathbf{u}_i : 1 \le i \le 2d\}$ of W and U respectively such that

$$Q(\mathbf{w}_i) = Q(\mathbf{u}_i),\ (\mathbf{w}_i, \mathbf{w}_j) = (\mathbf{u}_i, \mathbf{u}_j)\ \ (1 \le i, j \le 2d).$$

Given such bases $\{\mathbf{w}_i : 1 \le i \le 2d\}$ and $\{\mathbf{u}_i : 1 \le i \le 2d\}$ there is clearly a unique $T \in G$ that will map one basis to the other, so that $T\mathbf{w}_i = \mathbf{u}_i$ for each i, $1 \le i \le 2d$. Thus to study how G acts it is useful to study the kinds of bases that subspaces of V have.

Lemma 3.10 *Let U be a subspace of V of dimension $2d-1$ not containing $\mathbf{d}$. There exist vectors $\mathbf{e}_1,\ \mathbf{e}_i,\ \mathbf{f}_i \in V\ \ (2 \le i \le d)$ such that*

$$(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{f}_j, \mathbf{f}_k) = (\mathbf{d}, \mathbf{e}_i) = (\mathbf{d}, \mathbf{f}_j) = 0\ \ (1 \le i \le d,\ 2 \le j, k \le d), \tag{3.6}$$

$$(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij}\ \ (1 \le i \le d, 2 \le j \le d), \tag{3.7}$$

$$Q(\mathbf{e}_1) = Q(\mathbf{e}_j) = Q(\mathbf{f}_j) = 0\ \ (2 \le j \le d) \tag{3.8}$$

and such that U is one of

$$\langle \{\mathbf{e}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \le i \le d\} \rangle, \tag{3.9}$$

$$\langle \{\mathbf{d} + \mathbf{e}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \le i \le d\} \rangle, \tag{3.10}$$

$$\langle \{\mathbf{e}_1, \mathbf{d} + \mathbf{e}_2, \mathbf{d} + \mathbf{f}_2\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 3 \le i \le d\} \rangle. \tag{3.11}$$

A $(2d-1)$-space (i.e., a subspace of V of dimension $2d-1$) is said to be of *type* 0, 1 or 2 according as it has the form (3.9), (3.10), or (3.11) respectively.

Lemma 3.11 *Let U be a hyperplane of V not containing $\mathbf{d}$. There exist vectors $\mathbf{e}_i$, $\mathbf{f}_i \in V$ $(1 \le i \le d)$ such that for all i, j, $1 \le i$, $j \le d$,*

$$(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{f}_i, \mathbf{f}_j) = (\mathbf{d}, \mathbf{e}_i) = (\mathbf{d}, \mathbf{f}_i) = 0 \tag{3.12}$$

$$(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij} \tag{3.13}$$

$$Q(\mathbf{e}_i) = Q(\mathbf{f}_i) = 0 \tag{3.14}$$

and U is one of the subspaces

$$\langle \mathbf{e}_i, \mathbf{f}_i : 1 \le i \le d \rangle, \tag{3.15}$$

$$\langle \{\mathbf{d} + \mathbf{e}_1, \mathbf{d} + \mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \le i \le d\} \rangle. \tag{3.16}$$

Hyperplanes of the forms (3.15), (3.16) are called *$+$-hyperplanes* and *$-$-hyperplanes* respectively. This terminology is natural because it turns out that the stabilizers in G of $+$- and $-$-hyperplanes are the orthogonal groups denoted by $\mathrm{GO}^+(2d, 2)$, $\mathrm{GO}^-(2d, 2)$ respectively. By induction on d we can verify that the number of t.s. vectors in U^+ is $2^{2d-1} + 2^{d-1} - 1$ and in U^- is $2^{2d-1} - 1$. Thus a hyperplane cannot be of both types.

Lemma 3.12 *Let U be a $(2d-1)$-space not containing $\mathbf{d}$.*

i) There are exactly two hyperplanes U_0, $U_1 \supseteq U$ with $\mathbf{d} \notin U_0, U_1$.

ii) If U is of type 0, then U_0, U_1 are both $+$-hyperplanes. If U is of type 1, then one of U_0, U_1 is a $+$-hyperplane, and the other is a $-$-hyperplane. If U is of type 2, then U_0, U_1 are both $-$-hyperplanes.

iii) G acts 2-transitively on the $+$-hyperplanes, and 2-transitively on the $-$-hyperplanes.

We now fix vectors $\mathbf{e}_i$, $\mathbf{f}_i \in V$ $(1 \le i \le d)$ such that (3.12), (3.13), and (3.14) are satisfied. Clearly, taken together with $\mathbf{d}$ these vectors determine a basis for V. In terms of this basis we define $g_i, h_i \in G$ to be the unique involutions such that

$$g_i(\mathbf{e}_i) = \mathbf{e}_i + \mathbf{f}_i + \mathbf{d},\ h_i(\mathbf{e}_i) = \mathbf{f}_i,\ h_i(\mathbf{f}_i) = \mathbf{e}_i,$$

g_i fixes $\mathbf{f}_i$, and both g_i and h_i fix all of the vectors $\mathbf{d}$, $\mathbf{e}_j$, $\mathbf{f}_j$, $(1 \le j \le d, j \ne i)$. We also let U^+, U^- denote the hyperplanes (3.15) and (3.16) respectively.

Now M can be identified either with the set of $+$-hyperplanes or with the set of $-$-hyperplanes. We consider the cases separately.

Case 1. M is the set of $+$-hyperplanes.

Lemma 3.13 *Let U_0, U_1, $U_2 \in M$ be distinct and $\rho(U_0 \cap U_1) \subseteq U_2$. Then $\rho(U_1 \cap U_2) \subseteq U_0$.*

Proof: From the 2-transitivity without loss of generality we can take $U_0 = U^+$ and

$$U_1 = g_1(U^+) = \langle \{\mathbf{d} + \mathbf{e}_1 + \mathbf{f}_1,\ \mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\}\rangle.$$

Let W denote $\langle \mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\rangle$. It is clear that $U_0 \cap U_1 = \langle \{\mathbf{f}_1\} \cup W\rangle$ and hence that $\rho(U_0 \cap U_1) = \langle \mathbf{f}_1 \rangle$. Thus our hypothesis is that U_2 is a $+$-hyperplane different from U_0, U_1, and containing $\mathbf{f}_1$. Let $\mathbf{v}$ be the unique non-zero vector in $\rho(U_1 \cap U_2)$. From the 2-transitivity of $+$-hyperplanes $Q(\mathbf{v}) = Q(\mathbf{f}_1) = 0$. Since $\mathbf{v} \in U_1$, there exists $\mathbf{w} \in W$ such that $\mathbf{v}$ is one of

$$\mathbf{w},\ \mathbf{f}_1 + \mathbf{w},\ \mathbf{d} + \mathbf{e}_1 + \mathbf{w},\ \mathbf{d} + \mathbf{e}_1 + \mathbf{f}_1 + \mathbf{w}.$$

However, since $\mathbf{v} \perp U_1 \cap U_2$, we have $\mathbf{v} \perp \mathbf{f}_1$. Hence only the first two vectors listed are possible values of $\mathbf{v}$, and so $\mathbf{v} \in U_0$ as claimed. □

We define R to be the set of all triples $(U_0, U_1, U_2) \in [M]^3$ such that $\rho(U_0 \cap U_1) \subseteq U_2$. Notice that

$$\rho(U^+ \cap g_1(U^+)) = \langle \mathbf{f}_1 \rangle,$$

and that $\mathbf{f}_1 \in g_i(U^+)$ for all i, $1 \leq i \leq d$. Also, for any $U \supseteq \langle \mathbf{e}_1, \mathbf{f}_1\rangle$ in M, $\mathbf{f}_1 \notin h_1 g_1 h_1(U)$. Thus, R is a non-trivial G-invariant subset of $[M]^3$. Supposing, towards a contradiction, that $\mathcal{M}$ is a homogeneous 3-graph, we may assume that R is the set of edges of $\mathcal{M}$.

We define a G-invariant equivalence relation E on $[M]^2$ by

$$\{U_0, U_1\} E \{U_2, U_3\} \Leftrightarrow [U_0 \neq U_1 \wedge U_2 \neq U_3 \wedge \rho(U_0 \cap U_1) = \rho(U_2 \cap U_3)].$$

Since

$$\{U^+, g_1(U^+)\} E \{g_i(U^+), g_1 g_i(U^+)\} \quad (1 < i \leq d)$$

and since for each i, $1 < i \leq d$, there exists $g \in G$ which moves $\mathbf{f}_1$ to $\mathbf{f}_i$, E is certainly non-trivial.

Lemma 3.14 *If $\{U_0, U_1\}E\{U_2, U_3\}$, then either $\{U_0, U_1\} = \{U_2, U_3\}$ or $\{U_0, U_1, U_2\}$ is an edge of M.*

Proof: From 2-transitivity we can take $U_0 = U^+$ and $U_1 = g_1(U^+)$. Suppose that $\{U_0, U_1\}E\{U_0, U\}$. It is sufficient to prove that $U = U_1$. By hypothesis $\rho(U^+ \cap U) = \langle \mathbf{f}_1 \rangle$. Since U and U^+ are distinct hyperplanes, $\dim(U^+ \cap U) = 2d - 2$. Further, since $\rho(U^+ \cap U) = \langle \mathbf{f}_1 \rangle$, we have

$$U^+ \cap U \subseteq U \cap \langle \mathbf{f}_1 \rangle^\perp = \langle \{\mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 1 < i \leq d\} \rangle.$$

Therefore $U^+ \cap U = \langle \{\mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 1 < i \leq d\} \rangle$. By inspection the only hyperplanes containing this last subspace but not $\mathbf{d}$ are the ones obtained by adjoining either $\mathbf{e}_1$ or $\mathbf{d} + \mathbf{e}_1$ to $\langle \{\mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 1 < i \leq d\} \rangle$. Hence $U = g_1(U^+)$. This completes the proof. □

We can now derive a contradiction. We consider four distinct $+$-hyperplanes: U_0, U_1, U_2, U_3. From the lemma

$$\{U_0, U_1\}E\{U_2, U_3\} \Rightarrow [\{U_0, U_1, U_2, U_3\}]^3 \subseteq R.$$

From the assumption of homogeneity this implication is an equivalence. Now take:

$$U_0 = U^+, \ U_1 = g_1(U^+), \ U_2 = g_3(U^+), \ U_3 = g_3 g_2(U^+).$$

The basis:

$$\mathbf{d}, \ \mathbf{e}_i, \ \mathbf{f}_i \ \ (1 \leq i \leq d)$$

of U^+ is moved by g_1 to

$$\mathbf{d}, \ \mathbf{d} + \mathbf{e}_1 + \mathbf{f}_1, \ \mathbf{f}_1, \ \mathbf{e}_i, \ \mathbf{f}_i \ \ (2 \leq i \leq d).$$

Hence

$$U_0 \cap U_1 = \langle \{\mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\} \rangle \text{ and } \rho(U_0 \cap U_1) = \langle \mathbf{f}_1 \rangle.$$

Switching the roles of $\mathbf{f}_1$ and $\mathbf{f}_2$ we have

$$U^+ \cap g_2(U^+) = \langle \{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 3 \leq i \leq d\} \rangle$$

and $\rho(U^+ \cap g_2(U^+)) = \langle \mathbf{f}_2 \rangle$. Now applying g_3 which maps U^+, $g_2(U^+)$ to U_2, U_3 respectively we get

$$U_2 \cap U_3 = \langle \{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{d} + \mathbf{e}_3 + \mathbf{f}_3, \mathbf{f}_3\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 4 \leq i \leq d\} \rangle$$

and hence $\rho(U_2 \cap U_3) = \langle \mathbf{f}_2 \rangle$. From all this we have:

$$\rho(U_0 \cap U_1) \subseteq U_2, U_3 \text{ and } \rho(U_2 \cap U_3) \subseteq U_0, U_1.$$

Therefore every triple from $\{U_0, U_1, U_2, U_3\}$ is an edge of M. From above this implies that $\{U_0, U_1\} E \{U_2, U_3\}$ which contradicts our finding that

$$\rho(U_0 \cap U_1) = \langle \mathbf{f}_1 \rangle \neq \langle \mathbf{f}_2 \rangle = \rho(U_2 \cap U_3).$$

Case 2. M is the set of $-$-hyperplanes. Here we follow essentially the same strategy as in Case 1. The roles of U^+ and $g_1(U^+)$ are played by U^- and $g_1(U^-)$. Since $\rho(U^- \cap g_1(U^-)) = \langle \mathbf{d} + \mathbf{f}_1 \rangle$, the role of $\mathbf{f}_1$ is played by $\mathbf{d} + \mathbf{f}_1$. We omit the details.

3.5 A_7 and PSL(2, 11)

In this section we consider the pairs (N, n):

$$N = \mathrm{PSL}(2, 11), n = 11, \text{ and } N = \mathrm{A}_7, \ n = 15.$$

Firstly, we show that the representations of $N = \mathrm{PSL}(2, 11)$ of degree 11 give rise to no homogeneous 3-graphs. A 2-transitive representation of N with degree 11 is described by Conway [8, p. 217]. Applying an outer automorphism we get the other representation mentioned by Cameron. We could proceed by studying the properties of this known structure. However, we choose to give an ad hoc argument which relies mainly on exploiting the notion of 3-homogeneity. From Lemma 3.6 we infer that $|N| = 660$ and that either $G = N$ or N has index 2 in G.

We now consider a supposed homogeneous 3-graph $\mathcal{M}$ with $G = \mathrm{Aut}(\mathcal{M})$ and $|M| = 11$. Let $R \subseteq [M]^3$ denote the edge set of $\mathcal{M}$. Let $a, b \in M$ be distinct. Let X $(= X(a, b))$, Y $(= Y(a, b))$ denote $\{c \in M : \{a, b, c\} \in R\}$, $M - (X \cup \{a, b\})$ respectively. Let G_{ab} denote the pointwise stabilizer of $\{a, b\}$ in G. From the 2-transitivity of G on M,

$$|G_{ab}| = |G|/(11 \cdot 10) \in \{6, 12\}.$$

From the 3-homogeneity of $\mathcal{M}$, X and Y are orbits of G_{ab} and so $|X|$, $|Y|$ divide $|G_{ab}|$. Since $X \cup Y = M - \{a, b\}$ has size 9, it is easy to infer that one of X and Y has size 6 and the other has size 3. Without loss of generality $|X| = 6$ and $|Y| = 3$. Let y_0, y_1, y_2 enumerate Y. As already noted, G_{ab} is transitive on Y. From the symmetry of the edge relation, $X(a, b) = X(b, a)$. There exists $g \in G$ which switches a and b, fixes y_0, and fixes the set $\{y_1, y_2\}$ setwise. It follows that $\{a, y_1, y_2\}$ is an edge if and only if $\{b, y_1, y_2\}$ is. By symmetry we see that there are now only two possibilities: $Y \cup \{a, b\}$ is a 5-set with no edges at all, or each 2-set from Y makes an edge with a and with b.

In the first possibility, we have a 5-set with no edges on it generated by an arbitrary pair, a, b. Each pair in M belongs to some such 5-set, and each such 5-set is generated by each of the 10 different pairs it contains. Two of these 5-sets cannot have a pair of points in common, so we can count how many of these 5-sets there are in M. There are $(11 \cdot 10)/2$

pairs in M, and each 5-set is generated by 10 pairs, so there would be 11/2 such 5-sets, which is impossible.

So the second possibility holds, and any 3-set, which intersects Y in a doubleton and $\{a, b\}$ in a singleton, is an edge. Clearly, G_{ab} acts as the symmetric group on Y, and there exists g_Y in the pointwise stabilizer of Y such that $g_Y(a) = b$ and $g_Y(b) = a$. We now consider two cases.

Case 1. $G = N$. Then $|G_{ab}| = 6$. It follows that $(g_Y)^2 = 1$, the identity permutation, and that the pointwise stabilizer of $Y \cup \{a, b\}$ is $\{1\}$. Hence the subgroup G_0 of G fixing each point of Y and the set $\{a, b\}$ is $\{1, g_Y\}$. Since G is not the symmetric group on M, g_Y must move an element of X. Since G_0 is normalized by G_{ab}, g_Y moves each point of X. So we have a partition

$$\{\{x, g_Y(x)\} : x \in X\}$$

of X into three 2-sets which is G_{ab}-invariant. Let these three 2-sets be $X_i = \{x_i, x_i'\}$, $0 \leq i \leq 2$. Since $G_{ab} \cong S_3$ acts transitively on $\{X_0, X_1, X_2\}$, by suitably ordering the sets X_i we can suppose that for all $g \in G_{ab}$

$$g(X_i) = X_j \Leftrightarrow g(y_i) = y_j \quad (i, j \leq 2).$$

To finish we consider the set $X(x_0, x_0')$. Since this set has size 6, it must intersect $X_1 \cup X_2$. Since there is an element of G_{ab} which fixes X_0 and switches X_1 and X_2, $X(x_0, x_0')$ intersects both X_1 and X_2. Now applying g_Y we see that

$$X_1 \cup X_2 \subseteq X(x_0, x_0').$$

And we can see that G_{ab} acts as the symmetric group on $\{X_0, X_1, X_2\}$. Since G_{ab} acts transitively on the blocks X_0, X_1, X_2, it follows that any 3-subset of $X_0 \cup X_1 \cup X_2$ which contains one of X_0, X_1, X_2 is an edge. In particular, every 3-subset of $X_0 \cup X_1$ is an edge. Hence the stabilizer of the set $X_0 \cup X_1$ in G induces S_4 on this set. Hence 24 divides $|G|$, contradiction.

Case 2. $|G_{ab}| = 12$. Clearly the pointwise stabilizer H of $\{a, b\} \cup Y$ has order 2. Let $H = \langle h \rangle$. Since G_{ab} is transitive on X, h moves each element of X. Therefore we get a partition

$$\{\{x, h(x)\} : x \in X\}$$

of X into three 2-sets which is G_{ab}-invariant. As in Case 1 let these three 2-sets be $X_i = \{x_i, x_i'\}$, $0 \leq i \leq 2$. We now study the action of G_{ab} on $Y \cup \{X_i : i \leq 2\}$. Since $G_{ab}/\langle h \rangle \cong S_3$ induces the symmetric group on Y and acts transitively on X_0, X_1, X_2, we can align the blocks X_i with the elements of Y such that for all $g \in G_{ab}$

$$g(X_i) = X_j \Leftrightarrow g(y_i) = y_j \quad (i, j \leq 2).$$

Now $g_Y h g_Y$, $(g_Y)^2$ are both in H. Therefore g_Y has order 2 or 4. If g_Y has order 4, then $h = (g_Y)^2$. Since the restriction of g_Y to X has a 4-cycle, the other two points are fixed or transposed. So $(g_Y)^2$ fixes two points of X and, since h moves each point of X, $(g_Y)^2 \neq h$. Therefore $(g_Y)^2 = 1$. It follows that h and g_Y commute. Hence g_Y preserves the partition $\{X_0, X_1, X_2\}$. If g_Y switches two of $\{X_0, X_1, X_2\}$, then for a suitable element $g \in G_{ab}$, $g^{-1} g_Y g g_Y$ induces a 3-cycle on $\{X_0, X_1, X_2\}$. This is absurd since $g^{-1} g_Y g g_Y$ is clearly in H. Therefore g_Y fixes X_i as a set for $i \leq 2$. If g_Y has the same action as h on X, then $h g_Y = (ab)$, contradiction. Now, by composing g_Y with h if necessary, we can suppose that $g_Y = (ab)(x_i x_i')$ for some $i \leq 2$. Conjugating g_Y by suitable elements of G_{ab} we see that $(ab)(x_i x_i') \in G$ for each $i \leq 2$. Taking the product of two of these elements we get an element of the pointwise stabilizer of $\{a, b\} \cup Y$ which is not in $\langle h \rangle$. This contradiction completes Case 2.

Now we turn to the other case considered in this section in which $N = A_7$ and $n = 15$. As in the previous case no homogeneous 3-graphs arise. We give an ad hoc argument similar to that made for the case in which $N = \mathrm{PSL}(2, 11)$ and $n = 11$. However, this case turns out to be a lot simpler. The only fact from group theory which we quote here is that $\mathrm{Aut}(A_7)$ is S_7 acting by conjugation. Therefore G is either A_7 or S_7, which implies that $|G| \in \{2520, 5040\}$.

We consider a supposed homogeneous 3-graph $\mathcal{M}$ with $G = \mathrm{Aut}(\mathcal{M})$ and $|M| = 15$. Let $R \subseteq [M]^3$ denote the edge set of $\mathcal{M}$. Let $a, b \in M$ be distinct. Let X $(= X(a, b))$, Y $(= Y(a, b))$ denote $\{c \in M : \{a, b, c\} \in M\}$, $M - (X \cup \{a, b\})$ respectively. Let G_{ab} denote the pointwise stabilizer of $\{a, b\}$ in G. From the 2-transitivity of G on M,

$$|G_{ab}| = |G|/(15 \cdot 14) \in \{24, 48\}.$$

From the 3-homogeneity of $\mathcal{M}$, X and Y are orbits of G_{ab} and so $|X|$, $|Y|$ divide $|G_{ab}|$. Since $X \cup Y = M - \{a, b\}$ has size 13, it is easy to infer that one of X and Y has size 1 and the other has size 12. Without loss of generality $|X| = 1$ and $|Y| = 12$. But then any two points define an edge uniquely, which occurs non-trivially only in a structure of size 7, from Lemma 1.10. So $\mathcal{M}$ cannot be a homogeneous 3-graph.

3.6 Sporadic groups

In this section we consider the pairs (N, n) in Cameron's list for which N is a sporadic simple group. These are the cases:

$$N = \mathrm{M}_{11} \text{ (Mathieu) } \quad n \in \{11, 12\},$$

$$N = \mathrm{M}_{12} \text{ (Mathieu)}, \quad n = 12$$

$$N = \mathrm{M}_{22} \text{ (Mathieu)}, \quad n = 22,$$

$$N = \mathrm{M}_{23} \text{ (Mathieu)}, \quad n = 23,$$

$$N = \mathrm{M}_{24} \text{ (Mathieu)}, \quad n = 24,$$

$$N = \mathrm{HS} \text{ (Higman-Sims)}, \quad n = 176,$$

$$N = \mathrm{Co}_3 \text{ (Conway)}, \quad n = 276.$$

3.6.1 $N = \mathrm{HS}$ (Higman-Sims), $n = 176$

We first turn to the penultimate pair in the list. Until further notice $N = \mathrm{HS}$. We consider a 2-transitive action described by Graham Higman in [15]. We will show that this group does not correspond to any homogeneous 3-graph. At the time when [15] was written the author was not certain that the group he described was the same as the group of D. G. Higman and C. C. Sims [14], although both groups had the same order and were identical in many other respects. Subsequently, the classification of finite simple groups was completed and it turned out that there is only one simple group of the order in question, 44,352,000. Thus the permutation representation described by Graham Higman is indeed a representation of the Higman-Sims group. In [15, p. 76] Higman describes the set on which the group acts 2-transitively as:

$$P = \{0, \infty\} \sqcup D \sqcup D^* \sqcup (D \times D^*)_1 \sqcup (D \times D^*)_2 \sqcup E.$$

Here D, D^* are sets of size 6, $(D \times D^*)_1$ and $(D \times D^*)_2$ are two copies of the cartesian product $D \times D^*$, and E is a set of size 90. Towards a contradiction suppose that $\mathcal{M}$ is a homogeneous 3-graph such that $M = P$ and such that the socle N of $G = \mathrm{Aut}(\mathcal{M})$ has the action described in [15].

We consider the stabilizer $H = N_{\{0,\infty\}}$ in N of the set $\{0, \infty\} \subseteq M$. According to [15, p. 78] H can be written $\langle \phi, \tau, \mathrm{S}_6 \rangle$. We focus on the action on $\{0, \infty\} \sqcup D \sqcup D^*$. The group S_6 fixes 0 and ∞ and acts as the symmetric group on each of the 6-sets D and D^*. Let $\epsilon : D \to D^*$ be a bijection, and ϵ also denote the isomorphism from $\mathrm{Sym}(D)$ onto $\mathrm{Sym}(D^*)$ which it induces. Let $\gamma : \mathrm{S}_6 \to \mathrm{Sym}(D)$ be the restriction map for D and $\gamma^* : \mathrm{S}_6 \to \mathrm{Sym}(D^*)$ be the restriction map for D^*. A crucial point about the action of S_6 is that $(\gamma^*)^{-1}\epsilon\gamma$ is an outer automorphism of S_6. The consequence of this that we need below is that the stabilizer in S_6 of a point in D^* (D) acts on D (D^*) as $\mathrm{PSL}(2,5)$ acts on the projective line over $\mathrm{GF}(5)$. In particular, we have:

$$\textit{the stabilizer in } \mathrm{S}_6 \textit{ of a point in } D^* \ (D) \textit{ acts 2-transitively on } D \ (D^*) \tag{3.17}$$

Turning to the other generators of $N_{\{0,\infty\}}$, τ stabilizes $D \sqcup D^*$ pointwise and switches 0 and ∞. Finally, ϕ fixes 0 and ∞ and switches the sets D and D^*. We note that S_6 has index 4 in H. In [15] it is also pointed out that τ centralizes S_6, while ϕ normalizes S_6 inducing an outer automorphism. Thus $N_{0\infty} = \langle \phi, \mathrm{S}_6 \rangle$, the pointwise stabilizer of $\{0, \infty\}$, is $\mathrm{Aut}(\mathrm{S}_6)$. As well as this information about H, we note from [15, p. 76] that N induces the full symmetric group S_8 on $B = \{0, \infty\} \sqcup D$.

Let $G_{\{0,\infty\}}$ denote the stabilizer in G of the sets $\{0, \infty\}$ and $D \sqcup D^*$. We claim that $G_{\{0,\infty\}}$ has exactly the same action on $D \sqcup D^*$ as $N_{\{0,\infty\}}$. If not, let $g \in G_{\{0,\infty\}}$ be a counterexample. First note that $\{D, D^*\}$ is $N_{\{0,\infty\}}$-invariant. Since g normalizes N, $\{g(D), g(D^*)\}$ is an $N_{\{0,\infty\}}$-invariant pair of 6-sets partitioning $D \sqcup D^*$. It is easy to see that $g(D)$, $g(D^*)$ are D, D^* in some order. By composing g with an element of $N_{\{0,\infty\}}$ if necessary, we may suppose that g fixes $\{0,\infty\} \sqcup D^*$ pointwise. Now choose $h \in S_6$ such that $\gamma(h)$ does not commute with $g \restriction D$. Then $g^{-1}hg \in N_{0\infty}$ fixes D^* pointwise but is not the identity on D^*, contradiction. So the claim is proved.

We now obtain a contradiction by considering what edges in $\mathcal{M}$ there can be on $\{0,\infty\} \sqcup D \sqcup D^*$. Without loss of generality, since N induces the symmetric group on $B = \{0,\infty\} \cup D$, we may suppose that no triple from B is an edge. Applying ϕ we see that no triple from $\{0,\infty\} \cup D^*$ is an edge. Consider $a, b \in D$ and $a^*, b^* \in D^*$. Applying τ, if $\{a, a^*\}$ makes an edge with one of $0, \infty$, then it also makes an edge with the other. Further, from (3.17), $\{a, a^*\}$ and $\{b, b^*\}$ are in the same orbit of $N_{0\infty}$. So, if any pair $\{a, a^*\}$ makes an edge with either 0 or ∞, then every such pair makes an edge with both 0 and ∞. Let $a \in D^*$ make an edge with a 2-set from D. From (3.17) every 2-set from D makes an edge with a. Applying S_6 and ϕ we see that, if one 3-set, which intersects one of D and D^* in a singleton and the other in a doubleton, is an edge, then all such 3-sets are edges. It follows that any permutation π of $\{0,\infty\} \sqcup D \sqcup D^*$ which fixes 0, ∞, and the partition $\{D, D^*\}$ permutes the edges on the set $\{0,\infty\} \cup D \cup D^*$. By 3-homogeneity any such permutation π extends to an automorphism of $\mathcal{M}$. This contradicts our finding above that the action of $G_{\{0,\infty\}}$ on $D \sqcup D^*$ is exactly the same as that of $N_{\{0,\infty\}}$.

3.6.2 $N = \mathrm{Co}_3$ (Conway), $n = 276$

This is the largest finite 2-transitive permutation group which arises from a sporadic simple group. As is the case for almost all the other possible simple socles, this group yields no homogeneous 3-graph. To explain why this is so we begin by introducing the basic notions needed for a description of the group in question.

For i, $0 \le i \le 22$, we use i to denote the point $\langle(i,1)\rangle$ of the projective line $\mathrm{P}(2,23)$ and ∞ to denote the remaining point $\langle(1,0)\rangle$. At the same time we think of i, $0 \le i \le 22$ as an element of GF(23). Following Conway's notation in [8] let

$$\Omega = \{i : 0 \le i < 23\} \cup \{\infty\}, \quad \Omega' = \Omega - \{\infty\}.$$

The arithmetic of Ω' is that of GF(23). For $i \in \Omega'$ we define $\infty - i = \infty$. Various subsets of Ω are denoted as follows.

$$Q = \{x^2 : x \in \mathrm{GF}(23)\}, \quad N = \Omega - Q,$$

$$N_\infty = \Omega, \quad N_i = \{n - i : n \in N\} \quad (i \in \Omega').$$

The group M_{24} is defined as the group of permutations of Ω obtained by adjoining to PSL(2, 23) the permutation $\xi : x \mapsto x^3/9$ $(x \in Q)$. It is well known and shown in [8, §3, Theorem 1] that M_{24} is 5-transitive but not 6-transitive on Ω. Now we regard $\mathbf{P}(\Omega)$, the power set of Ω, as a vector space over GF(2), where vector addition is just the symmetric difference of sets. We let $\mathbf{C}$ denote the subspace spanned by the sets N_i $(i \in \Omega)$. The space $\mathbf{C}$, which turns out to be 12-dimensional, is called the *binary Golay code*. Let $\mathbf{C}_8$ denote the set of all 8-sets (or *octads*) belonging to $\mathbf{C}$. A 12-set belonging to $\mathbf{C}$ is called an *umbral dodecad*.

Now we need to define the Leech lattice. Let $\mathbf{v}_i$ $(i \in \Omega)$ be an orthonormal basis for the Euclidean space $\mathbf{R}^{24}$. For $S \subseteq \Omega$, let $\mathbf{v}_S$ denote $\sum_{i \in S} \mathbf{v}_i$. The *Leech lattice* Λ is defined to be the additive subgroup of $\mathbf{R}^{24}$ generated by the vectors $\mathbf{v}_\Omega - 4\mathbf{v}_\infty$, and $2\mathbf{v}_C$ $(C \in \mathbf{C}_8)$. The following characterization of Λ is given in [8, §4, Theorem 2]:

Theorem 3.15 :

Part A *The vector $(x_\infty, x_0, \ldots, x_{22})$ is in Λ if and only if it satisfies each of the following conditions:*

i) the coordinates x_i are all congruent modulo 2 to m, say;

ii) the set of i for which x_i takes any given value modulo 4 is a $\mathbf{C}$-set;

iii) the coordinate-sum is congruent to $4m$ modulo 8.

Part B *For $\mathbf{x}, \mathbf{y} \in \Lambda$ the scalar product $\mathbf{x}\cdot\mathbf{y}$ is a multiple of 8, and $\mathbf{x}\cdot\mathbf{x}$ a multiple of 16.*

Let us define the length $|\mathbf{x}|$ of $\mathbf{x} \in \Lambda$ as $(\mathbf{x}\cdot\mathbf{x})/16$.

Conway defines the group $\cdot 0$ (pronounced "dotto") as the group of all Euclidean congruences of $\mathbf{R}^{24}$ which preserve Λ as a set and fix the origin. If $\pi \in \mathrm{Sym}(\Omega)$, then we extend π

to a congruence of $\mathbf{R}^{24}$ by $\pi(\mathbf{v}_i) = \mathbf{v}_{\pi(i)}$. For $S \subseteq \Omega$ we define another congruence ϵ_S of $\mathbf{R}^{24}$ by

$$\epsilon_S(\mathbf{v}_i) = \begin{cases} \mathbf{v}_i & \text{if } i \in S \\ -\mathbf{v}_i & \text{if } i \notin S. \end{cases}$$

Conway shows [8, §4, Theorem 3] that the set-stabilizer in $\cdot 0$ of the set

$$\{\{\mathbf{v}_i, -\mathbf{v}_i\} : i \in \Omega\},$$

consists of all congruences of the form $\epsilon_C \pi$, where $\pi \in \mathrm{M}_{24}$ and $C \in \mathbf{C}$. Below we shall denote this group by H although Conway called it N.

Finally, we come to the description of the simple group Co_3 and its 2-transitive representation. The group is the stabilizer in $\cdot 0$ of the vector $\mathbf{a} = (5, 1^{23}) \in \Lambda$. The notation $(5, 1^{23})$ indicates the vector that has 5 as its ∞-coordinate and 1 in every other place. The group acts faithfully and 2-transitively on the set of pairs:

$$\left\{\{\mathbf{x}, \mathbf{y}\} \in [\Lambda]^2 : \mathbf{x} + \mathbf{y} = \mathbf{a},\ |\mathbf{x}| = |\mathbf{y}| = 2\right\},$$

which we denote by Γ. There are 276 pairs in Γ. From [8, p. 242], 23 of the pairs in Γ are represented by the 23 vectors of shape $(4^2, 0^{22})$ which have non-zero ∞-coordinate. The other 253 pairs are represented by the 253 vectors of shape $(2^8 0^{16})$ with non-zero ∞-coordinate for which the set of places at which the vector is non-zero is a $\mathbf{C}$-set. We will focus attention on the vectors of the first kind. For $i \in \Omega'$, let $\mathbf{b}_i$ denote the vector with 4 in the ∞- and i-places and 0 in every other place. Let $\mathbf{c}_i$ denote $\mathbf{a} - \mathbf{b}_i$. Below we use N to denote Co_3 (denoted $\cdot 3$ in [8]). Depending on the context we can regard N as acting either on Λ or on Γ.

Towards a contradiction suppose that there is a homogeneous 3-graph $\mathcal{M}$ such that M is the set Γ and N is the socle of $G = \mathrm{Aut}(\mathcal{M})$. We define the set $\Xi = \{\{\mathbf{b}_i, \mathbf{c}_i\} : i \in \Omega'\}$.Let $G_{\{\Xi\}}$, $N_{\{\Xi\}}$ denote the set stabilizers of Ξ in G, N respectively. Let Θ, Φ denote $\{\mathbf{b}_i : i \in \Omega'\}$, $\{\mathbf{c}_i : i \in \Omega'\}$. Since $|\mathbf{b}_i - \mathbf{b}_j| = 2$ and $|\mathbf{b}_i - \mathbf{c}_j| = 3$ whenever $i, j \in \Omega'$ are distinct, the set $\{\Theta, \Phi\}$ is $N_{\{\Xi\}}$-invariant. Thus $N_{\{\Theta\}}$ has index at most 2 in $N_{\{\Xi\}}$. Since

$$\sum_{i \in \Omega'} \mathbf{b}_i = 4\mathbf{a} + 72\mathbf{v}_\infty,$$

$N_{\{\Theta\}}$ fixes $\mathbf{v}_\infty$. It follows that $N_{\{\Theta\}}$ also fixes the set $\{\mathbf{v}_i : i \in \Omega'\}$. From the remark above about the group H it is clear that the action of $N_{\{\Theta\}}$ is that induced by the action of M_{23} on Ω', where M_{23} is defined as the stabilizer in M_{24} of ∞. Now let us consider the edges that $\mathcal{M}$ places on the set Γ. Since $N_{\{\Xi\}}$ is 3-transitive, we may suppose that there are no edges on Γ at all. By homogeneity every $\sigma \in \mathrm{Sym}(\Gamma)$ extends to $g \in G_{\{\Gamma\}}$. Since G normalizes N, $G_{\{\Gamma\}}\restriction\Gamma = \mathrm{Sym}(\Gamma)$ normalizes $N_{\{\Gamma\}}\restriction\Gamma$. This is a contradiction because the only normal subgroup of $\mathrm{Sym}(\Gamma)$ is the alternating group on Γ. However, it is clear that the index of M_{23} in S_{23} is more than 2. Thus we have the desired contradiction.

3.6.3 The Mathieu Groups

Having established the needed definitions in our discussion of the Conway group, we now conclude our discussion of the sporadic goups with the Mathieu groups. From Theorem 1 of §3 in Conway's paper, [8], we know that M_{24} is 5-transitive. M_{23} is defined as the stabilizer of a single point in M_{24}, and M_{22} as the stabilizer of two points in M_{24}, so M_{23} acting on 23 points and M_{22} acting on 22 points are, respectively, 4-transitive and 3-transitive. Using information from the proof of Theorem 11 of §3, [8], we look at M_{12} as the set-stabilizer of an umbral in the action of M_{24} on 24 points. The two representations of M_{12} arise from the umbral and its complement, both of which are quintuply transitive. Fixing in addition a point in the umbral, the action on the umbral is the action of M_{11} on an 11-set and the action on the complementary umbral is the action of M_{11} on a 12-set, both of which are triply transitive. So all the permutation representations mentioned by Cameron with socle equal to one of these Mathieu groups are at least 3-transitive, and so are not of interest to us.

Chapter 4

Appendix

Lemma 4.1 (Lemma 3.9) *There exist a symmetric bilinear form* $(\ ,\ ) : V \times V \to \mathrm{GF}(2)$ *and a mapping* $Q : V \to \mathrm{GF}(2)$ *satisfying*

$$(\mathbf{v}, \mathbf{v}) = 0 \quad (\mathbf{v} \in V), \tag{4.1}$$

$$\dim(V^{\perp}) = 1, \quad Q(\mathbf{v}) = 1 \quad (\mathbf{v} \in V^{\perp} - \{\mathbf{0}\}), \tag{4.2}$$

and

$$Q(\mathbf{u} + \mathbf{v}) = Q(\mathbf{u}) + Q(\mathbf{v}) + (\mathbf{u}, \mathbf{v}) \quad (\mathbf{u}, \mathbf{v} \in V). \tag{4.3}$$

Proof: We choose a basis for $V = V(2d+1)$, $\mathbf{d}, \mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{f}_1, \ldots, \mathbf{f}_d$. On this basis, we define, for $1 \leq i, j \leq d$,

$$(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{f}_i, \mathbf{f}_j) = (\mathbf{d}, \mathbf{e}_i) = (\mathbf{d}, \mathbf{f}_i) = 0$$

and $(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij}$. Since $(,)$ is bilinear, this defines it uniquely on the whole space. Define Q on the basis by $Q(\mathbf{d}) = 1$, $Q(\mathbf{e}_i) = Q(\mathbf{f}_i) = 0$, $(1 \leq i \leq d)$. Since we are working in a field of order 2, we can consider the vectors as subsets of $\{\mathbf{d}, \mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{f}_1, \ldots, \mathbf{f}_d\}$. Then define $Q(\mathbf{u}) = 0$ if and only if the number of subsets of the form $\{\mathbf{d}\}$ or $\{\mathbf{e}_i, \mathbf{f}_i\}$ that occur in $\mathbf{u}$ is even.

Now we need to check that $Q(\mathbf{u} + \mathbf{v}) = Q(\mathbf{u}) + Q(\mathbf{v}) + (\mathbf{u}, \mathbf{v})$ $(\mathbf{u}, \mathbf{v} \in V)$. To do that, we will count all of the subsets of the form $\{\mathbf{d}\}$ or $\{\mathbf{e}_i, \mathbf{f}_i\}$ that occur in $\mathbf{u} + \mathbf{v}$ and show that, for each of these, there is precisely one corresponding 1 added to either $Q(\mathbf{u}) + Q(\mathbf{v})$

or $(\mathbf{u}, \mathbf{v})$ on the right side of the equation. If $\{\mathbf{d}\}$ or $\{\mathbf{e}_i, \mathbf{f}_i\}$ occurs in only one of $\mathbf{u}, \mathbf{v}$, then $Q(\mathbf{u}+\mathbf{v})$ and $Q(\mathbf{u})+Q(\mathbf{v})$ both have 1 added to them. If $\mathbf{e}_i$ occurs in only one of $\mathbf{u}, \mathbf{v}$, and $\mathbf{f}_i$ occurs in the other, then $Q(\mathbf{u}+\mathbf{v})$ and $(\mathbf{u}, \mathbf{v})$ both have 1 added to them. If $\{\mathbf{e}_i, \mathbf{f}_i\}$ occurs in one of $\mathbf{u}, \mathbf{v}$, and only one of $\mathbf{e}_i, \mathbf{f}_i$ occurs in the other, then both $Q(\mathbf{u})+Q(\mathbf{v})$ and $(\mathbf{u}, \mathbf{v})$ have 1 added to them, so there is no change on the right hand side. If we check the other possible arrangements, we find they also do not add to either side of the equation, so the relationship must hold. □

Lemma 4.2 (Lemma 3.10) *Let U be a subspace of V of dimension $2d-1$ not containing $\mathbf{d}$. There exist vectors $\mathbf{e}_1,\ \mathbf{e}_i,\ \mathbf{f}_i \in V\ \ (2 \le i \le d)$ such that*

$$(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{f}_j, \mathbf{f}_k) = (\mathbf{d}, \mathbf{e}_i) = (\mathbf{d}, \mathbf{f}_j) = 0 \quad (1 \le i \le d,\ 2 \le j, k \le d), \tag{4.4}$$

$$(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij} \quad (1 \le i \le d, 2 \le j \le d), \tag{4.5}$$

$$Q(\mathbf{e}_1) = Q(\mathbf{e}_j) = Q(\mathbf{f}_j) = 0 \quad (2 \le j \le d) \tag{4.6}$$

and such that U is one of

$$\langle \{\mathbf{e}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \le i \le d\} \rangle, \tag{4.7}$$

$$\langle \{\mathbf{d}+\mathbf{e}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \le i \le d\} \rangle, \tag{4.8}$$

$$\langle \{\mathbf{e}_1, \mathbf{d}+\mathbf{e}_2, \mathbf{d}+\mathbf{f}_2\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 3 \le i \le d\} \rangle. \tag{4.9}$$

Proof: Suppose we have already obtained $\mathbf{e}_1, \ldots, \mathbf{e}_k, \mathbf{f}_1, \ldots, \mathbf{f}_k \in U$ linearly independent vectors satisfying equations 4.10, 4.11, and 4.12. Let $\mathbf{u}_1, \ldots, \mathbf{u}_l, \mathbf{v}$, $(l = 2d-2k-1)$, be vectors chosen so that $U = \langle \mathbf{e}_1, \mathbf{f}_1, \ldots, \mathbf{e}_k, \mathbf{f}_k, \mathbf{u}_1, \ldots, \mathbf{u}_l \rangle$ and $V = \langle U \cup \{\mathbf{v}, \mathbf{d}\} \rangle$. Let $W = \langle \mathbf{e}_1, \mathbf{f}_1, \ldots, \mathbf{e}_k, \mathbf{f}_k \rangle$. We can choose $\mathbf{u}_1, \ldots, \mathbf{u}_l, \mathbf{v}$ to be elements of $W^\perp$. For, if $(\mathbf{u}_1, \mathbf{e}_1) = 1$, then we can replace $\mathbf{u}_1$ by $\mathbf{u}_1 + \mathbf{f}_1$ to make $(\mathbf{u}_1, \mathbf{e}_1) = 0$, and for each $\mathbf{u}_i, \mathbf{v}_i$ we can do this for all the basis vectors of W.

Since every vector of U must have a vector to which it is not orthogonal, provided that $l > 1$, we can find two of $\mathbf{u}_1, \ldots, \mathbf{u}_l$ which are not orthogonal to each other. So, without loss of generality, we get $\mathbf{e}, \mathbf{f}$ among $\mathbf{u}_1, \ldots, \mathbf{u}_l$ such that $(\mathbf{e}, \mathbf{f}) = 1$. We still have not determined $Q(\mathbf{e})$ and $Q(\mathbf{f})$. If $Q(\mathbf{e}) = 0$, then we can assume $Q(\mathbf{f}) = 0$, otherwise we substitute $\mathbf{e} + \mathbf{f}$

for $\mathbf{f}$, ($Q(\mathbf{e}+\mathbf{f}) = 0$ by equation 4.3). So we can extend the basis provided one of $Q(\mathbf{e})$ and $Q(\mathbf{f})$ is zero.

If $l > 3$, then we can repeat the process to get another pair $\mathbf{e}', \mathbf{f}'$ such that $\mathbf{e}', \mathbf{f}' \in U$ with $(\mathbf{e}', \mathbf{f}') = 1$ and $\mathbf{e}', \mathbf{f}' \in (W \cup \langle \mathbf{e}, \mathbf{f} \rangle)^{\perp}$. Either, one of the pairs gives a pair where Q of each element in the pair is zero, or $Q(\mathbf{e}) = Q(\mathbf{f}) = Q(\mathbf{e}') = Q(\mathbf{f}') = 1$. In the first case, we take that pair to extend the basis; in the second case we extend the basis with the pair $\{\mathbf{e}+\mathbf{e}', \mathbf{f}+\mathbf{e}'\}$.

Now, if we are able to get $d-1$ pairs in the basis, where $Q(\mathbf{e}_i) = Q(\mathbf{f}_i) = 0$, then we will have one vector left, call it $\mathbf{u}$. All the other vectors in our basis must be orthogonal to $\mathbf{u}$, so $(\mathbf{u}, \mathbf{v}) = 1$. If $Q(\mathbf{u}) = 0$, then we take $\mathbf{u}$ as the last vector in our basis, and U is of type 4.7. If $Q(\mathbf{u}) = 1$, then we take $\mathbf{u}+\mathbf{d}$ as the last vector and U is of type 4.8. If we could not get the last pair $\mathbf{e}, \mathbf{f}$ to have $Q(\mathbf{e}) = Q(\mathbf{f}) = 0$, then we take $\mathbf{d}+\mathbf{e}, \mathbf{d}+\mathbf{f}$ as our last pair. Now if the last vector $\mathbf{u}$ has $Q(\mathbf{u}) = 0$, then we take $\mathbf{u}$ as the last vector in the basis, and U is of type 4.9. If $Q(\mathbf{u}) = 1$, then we have to take $\mathbf{u}+\mathbf{d}$ as the last vector, but, if that happens, we could have taken $\mathbf{e}+\mathbf{u}, \mathbf{f}+\mathbf{u}$ as the last pair in our basis, which again is a case where U is of type 4.8. □

Lemma 4.3 (Lemma 3.11) *Let U be a hyperplane of V not containing $\mathbf{d}$. There exist vectors $\mathbf{e}_i$, $\mathbf{f}_i \in V$ $(1 \leq i \leq d)$ such that for all i, j, $1 \leq i$, $j \leq d$,*

$$(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{f}_i, \mathbf{f}_j) = (\mathbf{d}, \mathbf{e}_i) = (\mathbf{d}, \mathbf{f}_i) = 0, \tag{4.10}$$

$$(\mathbf{e}_i, \mathbf{f}_j) = \delta_{ij}, \tag{4.11}$$

$$Q(\mathbf{e}_i) = Q(\mathbf{f}_i) = 0 \tag{4.12}$$

and U is one of the subspaces

$$\langle \mathbf{e}_i, \mathbf{f}_i : 1 \leq i \leq d \rangle, \tag{4.13}$$

$$\langle \{\mathbf{d}+\mathbf{e}_1, \mathbf{d}+\mathbf{f}_1\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\} \rangle. \tag{4.14}$$

Proof: The proof of Lemma 4.3 proceeds similarly to the proof of Lemma 4.2, only it is simpler, since U in this case has an even number of basis vectors, so they pair off nicely.

Again, we start with an independent set satisfying the conditions we want, and in the same manner as before, we add pairs to it to form our basis for U.

Once we have $d-1$ pairs in our basis for U, where $Q(\mathbf{e}_i) = Q(\mathbf{f}_i) = 0$, then we will have one pair of vectors left, call them $\mathbf{e}, \mathbf{f}$. All the other vectors in our basis must be orthogonal to $\mathbf{e}$, so $(\mathbf{e}, \mathbf{f}) = 1$. If $Q(\mathbf{e}) = 0$ and $Q(\mathbf{f}) = 0$, then we take $\mathbf{e}, \mathbf{f}$ as the last pair in the basis; if $Q(\mathbf{e}) = 0$ and $Q(\mathbf{f}) = 1$, then we take $\mathbf{e}, \mathbf{e}+\mathbf{f}$ as the last pair in the basis; and if $Q(\mathbf{e}) = 1$ and $Q(\mathbf{f}) = 0$, then we take $\mathbf{e}+\mathbf{f}, \mathbf{f}$ as the last pair in the basis. In any of these three cases, U will be a subspace of the form 4.13. If both $Q(\mathbf{e}) = 1$ and $Q(\mathbf{f}) = 1$, then we take $\mathbf{d}+\mathbf{e}, \mathbf{d}+\mathbf{f}$ as the last pair in our basis and U is a subspace of the form 4.14. □

Lemma 4.4 (Lemma 3.12) *Let U be a $(2d-1)$-space not containing* $\mathbf{d}$.

i) There are exactly two hyperplanes U_0, $U_1 \supseteq U$ with $\mathbf{d} \notin U_0, U_1$.

ii) If U is of type 0, then U_0, U_1 are both $+$-hyperplanes. If U is of type 1, then one of U_0, U_1 is a $+$-hyperplane, and the other is a $-$-hyperplane. If U is of type 2, then U_0, U_1 are both $-$-hyperplanes.

iii) G acts 2-transitively on the $+$-hyperplanes, and 2-transitively on the $-$-hyperplanes.

Proof: i) and ii) Suppose $U = \langle\{\mathbf{e}\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\}\rangle$. There is an $\mathbf{f} \in V$, not in U, $\mathbf{f} \neq \mathbf{d}$, such that $(\mathbf{e}, \mathbf{f}) = 1$, $\mathbf{f} \in \langle\{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\}\rangle^{\perp}$, and $Q(\mathbf{f}) = 0$. We can assume that if this basis of U is extended to a subspace of V not including $\mathbf{d}$, the extension will be by one of the vectors $\mathbf{f}$, $\mathbf{e}+\mathbf{f}$, $\mathbf{d}+\mathbf{f}$, or $\mathbf{d}+\mathbf{e}+\mathbf{f}$. The first two cases yield the same $2d$-space, and we let $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{f}_1 = \mathbf{f}$; the second two cases yield a second $2d$-space, and we let $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{f}_1 = \mathbf{d}+\mathbf{e}+\mathbf{f}$. So we have two possible $+$-hyperplanes.

Suppose $U = \langle\{\mathbf{d}+\mathbf{e}\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 2 \leq i \leq d\}\rangle$. Again, we have $\mathbf{f}$ as in the previous situation. If U is extended by $\mathbf{f}$ or by $\mathbf{d}+\mathbf{e}+\mathbf{f}$, then let $\mathbf{e}_1 = \mathbf{d}+\mathbf{e}+\mathbf{f}$ and $\mathbf{f}_1 = \mathbf{f}$, and we have a $+$-hyperplane. If U is extended by $\mathbf{e}+\mathbf{f}$ or by $\mathbf{d}+\mathbf{f}$ then let $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{f}_1 = \mathbf{f}$, and we have a $-$-hyperplane.

Suppose $U = \langle\{\mathbf{e}, \mathbf{d}+\mathbf{e}', \mathbf{d}+\mathbf{f}'\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 3 \leq i \leq d\}\rangle$. This time $\mathbf{f} \in \langle\{\mathbf{e}', \mathbf{f}'\} \cup \{\mathbf{e}_i, \mathbf{f}_i : 3 \leq i \leq d\}\rangle^{\perp}$. If U is extended by $\mathbf{f}$ or by $\mathbf{e}+\mathbf{f}$, let $\mathbf{e}_1 = \mathbf{e}'$, $\mathbf{f}_1 = \mathbf{f}'$, $\mathbf{e}_2 = \mathbf{e}$, and $\mathbf{f}_2 = \mathbf{f}$, and we have a $-$-hyperplane. If U is extended by $\mathbf{d}+\mathbf{f}$ or by $\mathbf{d}+\mathbf{e}+\mathbf{f}$, let $\mathbf{e}_1 = \mathbf{e}'$, $\mathbf{f}_1 = \mathbf{f}'$, $\mathbf{e}_2 = \mathbf{e}$, and $\mathbf{f}_2 = \mathbf{d}+\mathbf{e}+\mathbf{f}$, and we have a different $-$-hyperplane.

iii) Let U_0, U_1 and U_0', U_1' be two pairs of distinct $+$-hyperplanes. Let $W = U_0 \cap U_1$ and $W' = U_0' \cap U_1'$. Both W and W' have dimension $2d-1$, and, by ii), they are both of type 0, as they each have two distinct extensions that are $+$-hyperplanes. We know there is a $g \in G$ that maps the basis of W to the basis of W', and that will also map the extension U_0 of W to the extension U_0' of W'. Now, since U_1 is the unique second extension of W that is in V and does not include $\mathbf{d}$ and U_1' is similarly the unique second extension of W', g must map U_1 to U_1' as well. The proof for $-$-hyperplanes is similar. $\square$

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