HIGHLY EXTENDABLE GRAPHS WITHOUT SHORT CYCLES

by

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Abstract

A graph is said to be k-extendable if it is matchable and every matching of size k extends to a perfect matching. The notion of extendability has been studied by a number of authors. Most highly extendable graphs that have appeared in the literature have high edge density. Indeed, it is a nontrivial problem to find graphs that have high extendability and whose girth is at least five. In light of these facts it seems to be interesting to look for constructions yielding graphs without short cycles whose extendability would also be large. This is in focus of this thesis the main result of which is a constructive proof of the existence of highly extendable graphs whose girth is greater than a given number.

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Chapter 1

Introduction

A *matching* (or a set of *independent edges*) in a graph G is a loop-free subset of the edge set of G, no two edges of which share a common endpoint. M is a *perfect matching* if it covers all vertices of G. A graph containing a perfect matching is *matchable*. If a matching M is contained in a matching N, then we say that M *extends* to N.

The notion of extendability seems to have its earliest roots in works of Kotzig [11, 12, 13], Hetyei [9], Hartfield [8], Brualdi and Perfect [3], and some other authors. In recent years, Plummer investigated this concept in a number of papers (e.g., [15, 16, 17, 18]). Other papers dealing with extendability include [2], [4] and [10].

Definition 1 Let k be a nonnegative integer and let G be a graph satisfying the following properties.

- G has at least $2k + 2$ vertices,
- G is matchable and
- every matching of size k in G extends to a perfect matching of G.

Then we say that G is *k-extendable.*

It appears from this definition that a highly extendable graph should tend to have an abundance of edges since it must contain a perfect matching extending any given matching of a fixed size. Indeed, not only do some of the most natural examples of highly extendable graphs have relatively high edge density (here, the *edge density* of a graph G is defined to be the number $m/\binom{n}{2}$ where m and n denote the number of edges and the number of vertices in G, respectively), for example, both K_{2k+2} and $K_{k+1,k+1}$ are k-extendable for $k \geq 0$, but also some of the earlier results on the extendability hint at a correlation between a high edge density of a graph and its extendability. A survey of these results is presented in Chapter 2 of this thesis.

Thus, it is interesting to ask how closely are the extendability and the edge density of a graph related. A result in this direction was achieved by Györi and Plummer [7], who proved that the Cartesian product of a k-extendable graph and an l-extendable graph is, in general, $(k+l+1)$-extendable. This gives rise to interesting constructions of highly extendable graphs with arbitrarily low edge density (see Propositions 10 and 12). However, the graphs constructed in this way will necessarily contain short cycles. This provokes the question whether or not we can construct highly extendable graphs which would not only have low edge density, but which would also have large girth (where the *girth* of a graph G is the length of a shortest cycle in G). It is this question that is investigated in the thesis, and which is answered in the affirmative.

Hence, when trying to construct graphs with desired properties, one encounters the problem of how to reconcile the requirement that graphs have high extendability (which has a tendency to force the graphs to contain "many" edges) with the condition that graphs do not contain short cycles (which forces the graphs to have relatively "few" edges).

It seems that the latter requirement is "less restrictive" or, at least, that we have a better grasp of it, and can be satisfied "more easily". In light of this, the most difficult task appears to be that of being able to construct a sufficiently rich class of highly extendable graphs, so that one can find in it subclasses containing only graphs without short cycles. Thus, Chapter 3 is crucial for our main construction (Construction 27).

The graphs used in this construction are inspired by the Cartesian product of cycles from Proposition 12 and can be viewed as "skew" products of certain Cayley graphs.

The thesis is organized as follows. Chapter 2 contains some motivational earlier results suggesting a connection between extendability and edge density, as well as two propositions which inspired our main construction. In Chapter 3, we prove that members of a certain class of graphs have high extendability, and in Chapter 4, we present a class of graphs, members of which have high girth. These results enable us to prove in Chapter 5 that Construction 27 given there produces graphs with both high girth and high extendability. The last two chapters contain some concluding statements.

Chapter 2

Motivational Results

Graphs in this thesis may contain multiple edges and loops. The vertex and the edge sets of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. The *degree* of a vertex u, denoted $\deg(u)$, is the number of edges adjacent to u, with each loop counted twice. The *minimum degree* of a graph G, denoted $\delta(G)$, is the minimum over all degrees of vertices in G.

If S is a subset of the vertex set of G, then $G[S]$ is the *subgraph* of G *induced* by S, that is, $V(G[S]) = S$ and $G[S]$ contains exactly those edges of G of whose both endpoints lie in S. The graph $G[V(G) \setminus S]$ is also denoted by $G \setminus S$.

A *walk* of *length* $n \geq 0$ is an alternating sequence $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$ of vertices u_i and edges e_i of G having the property that u_{i-1} and u_i are the two endpoints of the edge e_i for all i. If $u_0 = u_n$, then the walk is *closed*. A walk in which no vertex occurs more than once is a *path*. A closed path of length at least 1 is called a *cycle*. If G is *simple* (i.e., without loops and multiple edges), we will also use $u_0, u_1, u_2, \ldots, u_n$ instead of $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$.

A subgraph F of G which is a union of disjoint cycles and for which $V(F) = V(G)$ is called a *2-factor* of G. If F is in fact a cycle itself, then F is a *hamiltonian cycle*.

Theorems 2–8 in this chapter are earlier results from [15],[18] and [4] and they suggest the existence of a correlation between the edge-density of a graph and its extendability. The proofs presented here are in most cases more detailed versions of the original proofs.

The first of these results, Theorem 2, deals with simple graphs and says, in effect, that if a simple graph G has very large edge density (and its edges are distributed so that the degrees of all vertices are big), then G is highly extendable.

Theorem 2 (**[15]**) *Let G be a simple graph with $|V(G)| \geq 4$ being even and let $n > 0$. Assume that $\delta(G) \geq \frac{1}{2}|V(G)| + n$. Then G is n-extendable.*

Proof: As $|V(G)| \geq 4$ and $\delta(G) \geq \frac{1}{2}|V(G)|$, G contains a hamiltonian cycle by Dirac's Theorem ([5]). As $|V(G)|$ is even, we conclude that G contains a perfect matching. Hence, by the hypothesis, $|V(G)| > \delta(G) \geq \frac{1}{2}|V(G)| + n$ and $\frac{1}{2}|V(G)| > n$. Thus, G contains at least $2n + 2$ vertices. Finally, let M be a matching of size n in G and let $G' = G \setminus V(M)$. Then

$$\delta(G') \geq \delta(G) - |V(M)| = \delta(G) - 2n \geq \frac{1}{2}|V(G)| - n = \frac{1}{2}(|V(G)| - 2n) =$$

$$= \frac{1}{2}(|V(G)| - |V(M)|) = \frac{1}{2}|V(G')|.$$

Since $|V(G)|$ is even, so is $|V(G')|$. As we know that $|V(G)| \geq 2n + 2$, we get $|V(G')| \geq 2$. If $|V(G')| = 2$, then

$$\delta(G') \geq \frac{1}{2}|V(G')| = 1$$

and G' contains an edge. If $|V(G')| > 2$, then we can again apply Dirac's Theorem to G' to see that G' contains a hamiltonian cycle and, therefore (as $|V(G')|$ is even), a perfect matching. In both cases, we conclude that M extends to a perfect matching of G.∎

Before we can state and prove Theorems 3–5, we need a few more definitions.

We say that a vertex v is a *neighbor* of a vertex u if G contains an edge whose endpoints are u and v. In this case, we write $u \sim_G v$ (or $u \sim v$ if it is obvious from the context what graph we have in mind).

Let $\approx$ be a binary relation on $V(G)$ defined by $u \approx v$ if and only if G contains a path connecting u to v. It is easily seen that $\approx$ is an equivalence relation. If S is a class of equivalence under $\approx$, then we say that $G[S]$ is a *connected component* of G. A graph that has only one connected component is *connected* (note that G is connected if and only if there is a uv-path in G for every pair u, v of vertices of G). If G is connected, S is a subset of $V(G)$ and $V \setminus S$ is not connected, then S is called a *cutset*. If, for a vertex u, $\{u\}$ is a cutset, then u is called a *cut-vertex*. If G is a connected graph on at least $n+1$ vertices and if G contains no cutset of size less than n, then G is *n-connected.*

Theorems 3 and 4 are needed for the proof of Theorem 5, which can be viewed as going in the direction opposite to that of Theorem 2; it shows that if a graph is n-extendable, then it must also be $(n+1)$-connected and thus cannot have few edges.

Theorem 3 ([15]) *If G is 1-extendable and connected, then G is 2-connected.*

Proof: Suppose the contrary and let v be a cut-vertex in G with $G_1, G_2, \ldots, G_k$ being the components of $G \setminus \{v\}$, $(k \geq 2)$. For every $i, 1 \leq i \leq k$, there is a vertex u_i in G_i such that $u_i \sim v$. Let M be a perfect matching in G extending $\{u_1 v\}$ (see Fig. 2.1).

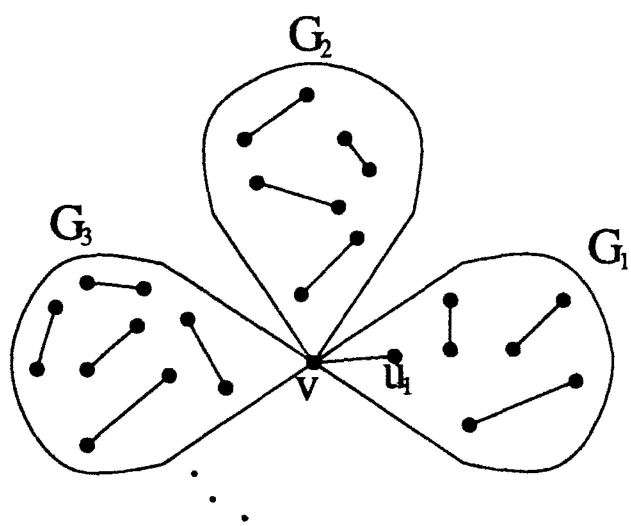

Figure 2.1.

Since v is a cut-vertex, M induces perfect matchings of the graphs $G_1 \setminus \{u_1\}, G_2, G_3 \ldots,$ G_k. In particular, $|V(G_1)|$ is odd and $|V(G_2)|$ is even. Applying similar reasoning to $\{u_2 v\}$ forces $|V(G_1)|$ to be even and $|V(G_2)|$ to be odd, a contradiction.■

Theorem 4 **([15])** *Let $n \geq 2$ and let G be an n-extendable graph. Then G is $(n-1)$-extendable.*

Proof: Suppose the statement were false and let M be a matching of size $n-1$ in G which does not extend to a perfect matching. Let N be a perfect matching in G (G contains a perfect matching since it is n-extendable) and consider the symmetric difference $M \triangle N$. As $|N| > |M|$, at least one of the components of $M \triangle N$ is an alternating path of odd length, whose first and last edges are both in N. Let P be such a path. Then $M \triangle P$ is a matching in G of size $|M|+1 = n$. By the hypothesis, it extends to a perfect matching N'. Moreover, as $|V(G)| \geq 2n+2$, $M \triangle P$ is not a perfect matching of G and there must be $e \in N'$ such that $e \notin M \triangle P$. Then e is independent of all edges of M and $M \cup \{e\}$ is a matching of size n in G. This extends to a perfect matching M' by the assumption. But M' is a perfect matching extension of M, a contradiction.■

Theorem 5 **([15])** *Let $n > 0$ and let G be a connected n-extendable graph. Then G is $(n+1)$-connected.*

Proof: Induction on n. If $n = 1$, then the result follows by Theorem 3.

For $n > 1$, suppose G is not $(n+1)$-connected. As G is n-extendable, G is $(n-1)$-extendable by Theorem 4. Hence G is n-connected by the induction hypothesis. Thus there is a cutset S of size n in G. Let $G_1, G_2, \ldots, G_k$ be the components of $G \setminus S$ (note that $k \geq 2$).

First, we will show that $|V(G_i)| < n$ for all $i, 1 \leq i \leq k$. If not, then there is an i such that $|V(G_i)| \geq n$. Without loss of generality, we may assume that $i = 1$. Then, by Menger's Theorem (a variation due to Dirac [6]), there are n vertex-disjoint paths in G connecting S with G_1. In particular, since in fact all these paths must lie within $G[S \cup V(G_1)]$, there are n independent edges, each of which has one endpoint in S and the other in G_1. Let M be the matching formed by these edges. As $|M| = n$ and G is n-extendable, M extends to a perfect matching M' of G (see Fig. 2.2). Also, the endpoints of the edges in M cover S and hence every edge in $M' \setminus M$ lies fully within one of the components G_i. In particular,

$$|V(G_1)| \equiv n \pmod 2. \tag{2.1}$$

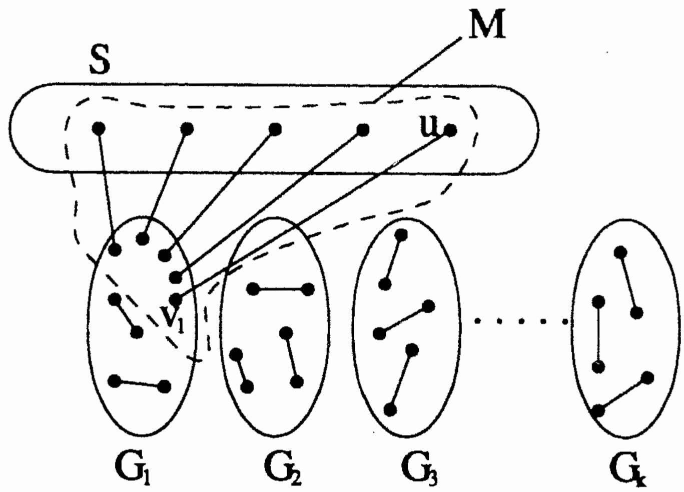

Figure 2.2.

On the other hand, let $e = uv_1 \in M$ with $u \in S, v_1 \in G_1$. Since G is n-connected and $|S \setminus \{u\}| = n-1$, $S \setminus \{u\}$ is not a cutset of G and there is a vertex $v_2 \in G_2$ such that $u \sim v_2$. Then

$$N = (M \cup \{uv_2\}) \setminus \{uv_1\}$$

is a matching of size n covering S. Again, it extends to a perfect matching N' of G and again each edge of $N' \setminus N$ lies fully within one of G_i (see Fig. 2.3).

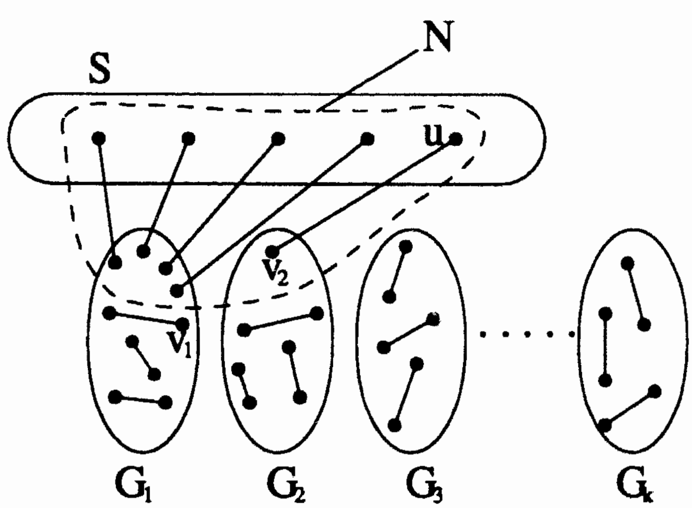

Figure 2.3.

Thus,

$$|V(G_1)| \equiv n-1 \pmod 2. \tag{2.2}$$

But (2.1) and (2.2) yield a contradiction.

Thus indeed $|V(G_i)| < n$ for all $i, 1 \leq i \leq k$. Next, we will show that in fact $|V(G_i)| = 1$ for all $i, 1 \leq i \leq k$. If not, then, say, $|V(G_1)| = m \geq 2$. Let $V(G_1) = \{u_1, u_2, \ldots, u_m\}$ and set $R_1 = \{u_1, u_2, \ldots, u_{m-1}\}$. We have

$$|V(G) \setminus (S \cup V(G_1))| = |V(G)| - (|S| + |V(G_1)|) \geq (2n+2) - (n+m) = n - m + 2 > 0.$$

Thus, we can choose a set $R_2 \subseteq V(G) \setminus (S \cup V(G_1))$, with $|R_2| = n - m + 1$. Then

$$|R_1 \cup R_2| = |R_1| + |R_2| = (m-1) + (n-m+1) = n.$$

Again, by Menger's Theorem there are n vertex-disjoint paths connecting S to $R_1 \cup R_2$. If M is the set containing the first edge of each of these paths, then M is a matching of size n in G. Moreover, M contains $m-1$ edges joining S to $V(G_1)$ and $n-m+1$ edges joining S to $V(G) \setminus (S \cup V(G_1))$. In particular, there is a unique vertex u in G_1 not covered by M. Since no matching in G extending M can cover u, M does not extend to a perfect matching of G. This contradicts the hypothesis as $|M| = n$.

Hence, we conclude that $|V(G_i)| = 1$ for all $i, 1 \leq i \leq k$. As G is n-extendable, it must contain a perfect matching, and this forces $k \leq n$. Then

$$|V(G)| = |S| + k \leq n + n = 2n < 2n + 2,$$

contradicting the fact G is n-extendable. Thus G must be $(n+1)$-connected.■

The set of neighbors of a fixed vertex u in a graph G is denoted by $\Gamma_G(u)$ (or $\Gamma(u)$ when no ambiguity may arise).

In the previous theorem, we saw that n-extendability of a graph G implies $(n+1)$-connectivity. This, in turn, means that such a graph has minimal degree at least $n+1$. The next theorem strengthens this result a bit further by showing that if G does indeed contain a vertex u of degree $n+1$, then this severely restricts the possible structure of $G[\Gamma(u)]$.

Theorem 6 ([15]) *If $n > 0$, G is connected and n-extendable, and u is a vertex in G with* $\deg(u) = n + 1$, *then $\Gamma(u)$ is an independent set.*

Proof: Let $\Gamma(u) = \{v_1, v_2, \ldots, v_{n+1}\}$ and assume that $\Gamma(u)$ is not independent.

If $|V(G)| > 2n + 2$, then

$$|V(G) \setminus (\Gamma(u) \cup \{u\})| > (2n + 2) - ((n + 1) + 1) = n,$$

and we can choose a set $W = \{w_1, w_2, \ldots w_{n+1}\}$ of size $n + 1$ such that

$$W \subseteq V(G) \setminus (\Gamma(u) \cup \{u\}). \tag{2.3}$$

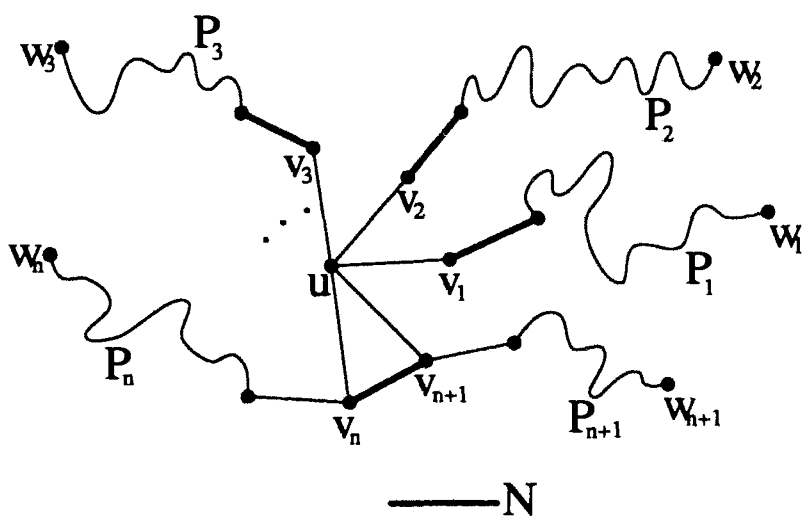

Figure 2.4.

By Theorem 5, G is $(n + 1)$-connected, and we can use Menger's Theorem. Thus, there are $n + 1$ vertex-disjoint paths $P_1, P_2, P_3 \ldots, P_{n+1}$ connecting W to $\Gamma(u)$. Say, P_i connects w_i to v_i for $i, 1 \leq i \leq n + 1$. Obviously, if M is the set containing the last edge $v_i w_i'$ of each $P_i, 1 \leq i \leq n + 1$, then M is a matching (of size $n + 1$) in G. Because $\Gamma(u)$ is not independent, we may assume that $\Gamma(u)$ contains the edge $v_n v_{n+1}$. Therefore,

$$N = (M \cup \{v_n v_{n+1}\}) \setminus \{v_n w_n', v_{n+1} w_{n+1}'\}$$

is a matching of size n in G (see Fig. 2.4).

By the hypothesis, this must extend to a perfect matching of G. But this is impossible as u is not covered by N whereas all of its neighbors are.

Now, consider the case $|V(G)| = 2n+2$. In this case, we get $|V(G)\setminus(\Gamma(u)\cup\{u\})| = n$ and can find $W = \{w_1, w_2, \dots, w_n\}$ of size n satisfying (2.3) (in fact, the equality is attained in (2.3)). A similar reasoning as before shows that there are $n+1$ vertex-disjoint paths $P_0, P_1, P_2, \dots, P_n$ connecting $\{u\} \cup W$ to $\Gamma(u)$. Suppose P_0 connects u to v_{n+1}, while P_i connects w_i to v_i, for $1 \le i \le n$. As $|V(G)| = 2n+2$, each of these paths consists, in fact, of a single edge. Hence the set M consisting of these edges is a (perfect) matching in G. If v_{n+1} is connected to another vertex in $\Gamma(u)$, say to v_n, then we set

$$N = (M \cup \{v_n v_{n+1}\}) \setminus \{v_n w_n, v_{n+1} u\}$$

and, as before, arrive at a contradiction.

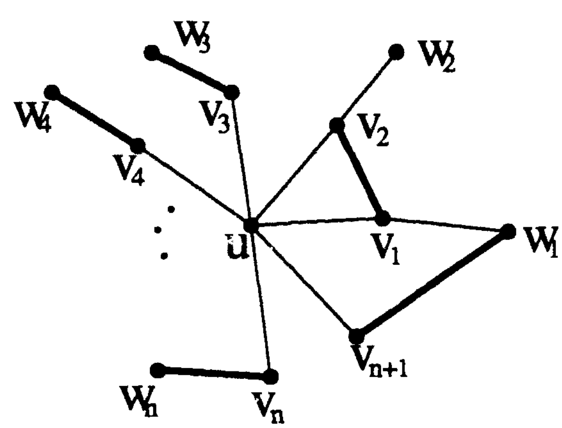

Figure 2.5.

So suppose that v_{n+1} has no neighbor in $\Gamma(u)$. Since G is $(n+1)$-connected, $\deg(v_{n+1}) \ge n+1$. But

$$|V(G) \setminus \Gamma(u)| = (2n+2) - (n+1) = n+1.$$

In particular, $\Gamma(v_{n+1}) \supseteq W$. As $\Gamma(u)$ is not independent, we may assume that $v_1 v_2$ is an edge in G. Now, $w_1 \in \Gamma(v_{n+1})$. Thus

$$(M \cup \{v_{n+1}w_1, v_1v_2\}) \setminus \{uv_{n+1}, v_1w_1, v_2w_2\}$$

is a matching of size n in G which does not extend to a perfect matching of G, a contradiction (see Fig. 2.5).■

Before we can proceed, we have to introduce another piece of notation. By $c(G)$ and $c_o(G)$ we denote the number of connected components and the number of connected components with an odd number of vertices in G, respectively.

Another concept related to extendability is the concept of the *toughness* of a graph. The toughness of a connected graph G is defined to be

$$\min\{\frac{|S|}{c(G \setminus S)} : S \text{ is a cutset of } G\}.$$

We denote the toughness of G by $\text{tough}(G)$. Thus toughness measures "how hard" it is to find a relatively small set of vertices in G which would disconnect G into relatively many connected components. In this way, toughness can be viewed as indirectly related to the number of edges in a graph (since graphs having "many edges" are "more likely" to be difficult to disconnect). In light of this, the next result, as the preceding ones, hints on a correlation between the edge-density and extendability of a graph.

Theorem 7 ([18]) *Let G be a connected graph with $|V(G)|$ even. Let $n > 0$ and let $|V(G)| \geq 2n + 2$. Then G is n-extendable if* $\text{tough}(G) > n$.

Proof: We will prove this theorem by contradiction. First, note that Tutte's Theorem on perfect matchings [19], combined with $\text{tough}(G) > n \geq 1$, implies matchability of G. Now, suppose G is not n-extendable and let $N = \{u_iv_i : 1 \leq i \leq n\}$ be a matching of size n which does not extend to a perfect matching. Let

$$G_1 = G \setminus \{u_i, v_i : 1 \leq i \leq n\}.$$

Then G_1 does not contain a perfect matching. By Tutte's Theorem, there is a cutset $S_1 \subseteq V(G_1)$ with $|S_1| < c_o(G_1 \setminus S_1)$.

If $|S_1| = c_o(G_1 \setminus S_1) - 1$, then

$$|V(G)| \equiv |V(G_1)| \equiv |S_1| + c_o(G_1 \setminus S_1) \equiv 1 \pmod 2,$$

contradicting the assumption that $|V(G)|$ is even. Thus, in fact, $|S_1| \leq c_o(G_1 \setminus S_1) - 2$. If we set

$$S_2 = S_1 \cup \{u_i, v_i : 1 \leq i \leq n\},$$

then $G \setminus S_2 = G_1 \setminus S_1$ and, therefore, $c(G \setminus S_2) = c(G_1 \setminus S_1)$. Also, $|S_2| = |S_1| + 2n$.

Thus, we have

$$\text{tough}(G) = \min\{\frac{|S|}{c(G \setminus S)} : S \subseteq V(G), S \text{ is a cutset of } G\} \leq \frac{|S_2|}{c(G \setminus S_2)} =$$

$$= \frac{|S_1| + 2n}{c(G_1 \setminus S_1)} \leq \frac{|S_1| + 2n}{c_o(G_1 \setminus S_1)} \leq \frac{|S_1| + 2n}{|S_1| + 2} \leq \frac{n(|S_1| + 2)}{|S_1| + 2} = n.$$

But this contradicts the hypothesis that $\text{tough}(G) > n$.■

Without a proof, we will give one more result relating the notions of extendability and edge density in graphs.

A *surface* is a compact connected 2-manifold. If we view a graph G, in a natural way, as a topological space (with edges being represented by closed line segments, vertices by points and the incidence between a vertex and an edge being given by the containment of the point in the boundary of the line segment), then by an *imbedding* of G in a surface Σ we mean the image $\hat{G}$ of G under a continuous injection $j : G \to \Sigma$. The connected components of $\Sigma \setminus \hat{G}$ are then called *faces* of $\hat{G}$. The set of faces of $\hat{G}$ is denoted by $F(\hat{G})$. If each face of $\hat{G}$ is homeomorphic to an open disc, we say that the imbedding is *cellular*. In this case, the *length* of a face is the number of edges one encounters when one traverses along its boundary (note that if an edge is traversed twice along the boundary of a face, then it contributes 2 to its length). The number $|V(G)| - |E(G)| + |F(\hat{G})|$ is an invariant for the set of cellular imbeddings $\hat{G}$ in Σ and is called the *Euler characteristic* of Σ and denoted by $\chi(\Sigma)$.

Theorem 8 (**[4]**) *If Σ is a surface different from the sphere and G is a connected graph imbedded in Σ, then the extendability of G is less than $2 + \lfloor\sqrt{4 - 2\chi(\Sigma)}\rfloor$.*■

The following remarks explain how this result suggests a connection between the extendability of a simple graph and its number of edges.

Let G be a simple graph and let $\hat{G}$ be a cellular imbedding of G in Σ (there always exists a Σ in which G imbeds cellularly). Then

$$|V(G)| - |E(G)| + |F(\hat{G})| = \chi(\Sigma). \tag{2.4}$$

If, for each positive integer n, f_n denotes the number of faces of length n in the imbedding, then (because $f_1 = f_2 = 0$ since G is simple)

$$f_3 + f_4 + f_5 + f_6 + \cdots = |F(\hat{G})|$$

and, since each edge contributes 2 to the total length of all faces,

$$3f_3 + 4f_4 + 5f_5 + 6f_6 + \cdots = 2|E(G)|,$$

implying

$$3(f_3 + f_4 + f_5 + f_6 + \cdots) \leq 2|E(G)|.$$

Thus $3|F(\hat{G})| \leq 2|E(G)|$. Substituting this in (2.4) yields

$$|V(G)| - |E(G)| + \frac{2}{3}|E(G)| \geq \chi(\Sigma)$$

and, after simplifying,

$$|E(G)| \leq 3|V(G)| - 3\chi(\Sigma).$$

So we see that the ratio $|E(G)|/|V(G)|$ can be bounded by a function of $\chi(\Sigma)$. As we have seen in Theorem 8, a similar result holds true for the extendability of a graph. In this way, Theorem 8 suggests a similarity between the behavior of edge density and extendability with respect to imbeddability.

All results shown so far were, each of them in a specific way, suggesting that a highly extendable graph should be "likely" to be rich in its number of edges. The following two propositions will, however, show that there exist highly extendable graphs with arbitrarily low edge density.

The propositions themselves are mentioned here because of their proofs rather than because of their content. In fact, they are both trivial corollaries of the main result of [7]. But the proofs presented here are interesting not only because of their relative simplicity but especially since, as it turns out, they admit a generalization which will lead to our main result.

But first, we need the following definition.

Definition 9 Given a positive integer n, the *n-dimensional cube* Q_n is the graph whose vertices are the elements of the set $\{0,1\}^n$ and in which two vertices are joined by an edge if and only if they differ in exactly one coordinate.

Proposition 10 *Let n be a positive integer. Then Q_n is $(n-1)$-extendable.*

Proof: The proof will be by induction on n. When $n = 1$, the result is obvious (Q_1 is just an edge and is matchable). So assume $n > 1$. Obviously, $|V(Q_n)| \geq 2(n-1)+2 = 2n$. Let $M = \{u_iv_i : 1 \leq i \leq n-1\}$ be a matching of size $n-1$ in Q_n and let $S = \{u_i, v_i : 1 \leq i \leq n-1\}$. Since $|M| < n$, there exists $j, 1 \leq j \leq n$, such that for every $i, 1 \leq i \leq n-1$, the coordinate in which u_i and v_i differ is not the j-th one. Let

$$V_i = \{u \in V(Q_n) : j\text{-th coordinate of } u \text{ is } i\}, i = 0, 1.$$

There are two cases.

Case 1. $S \subseteq V_i$ for some i. Let ϕ be the involution on $V(Q_n) = \{0,1\}^n$ defined by

$$\phi(m_1, m_2, \ldots, m_{j-1}, m_j, m_{j+1}, \ldots, m_n) = (m_1, m_2, \ldots, m_{j-1}, 1-m_j, m_{j+1}, \ldots, m_n).$$

If we set

$$M' = \{\phi(u_i)\phi(v_i) : 1 \leq i \leq n-1\}$$

and

$$S' = \{\phi(u_i), \phi(v_i) : 1 \leq i \leq n-1\}$$

then M' is a matching in Q_n and $S' \subseteq V_{1-i}$. Thus $S \cap S' = \emptyset$, and $M \cup M'$ is also a matching in Q_n. Furthermore, for every $u \in V(Q_n)$ it holds true that $u \notin S \cup S'$ if and only if $\phi(u) \notin S \cup S'$. Therefore, if we set

$$A_0 = \{u \in V(Q_n) \setminus (S \cup S') : j\text{−th coordinate of } u \text{ is } 0\}$$

and

$$A_1 = \{\phi(u) : u \in A_0\},$$

then A_0, A_1 is a partition of $V(Q_n) \setminus (S \cup S')$. But $N = \{u\phi(u) : u \in A_0\}$ is a perfect matching of $Q_n[A_0 \cup A_1]$. Then $M \cup M' \cup N$ is a perfect matching in Q_n extending M (see Fig. 2.6).

Case 2. $S \cap V_0 \neq \emptyset$ and $S \cap V_1 \neq \emptyset$. By the choice of j, each edge of M is either fully in $Q_n[V_0]$ or fully in $Q_n[V_1]$. Thus $M = M_0 \cup M_1$, where M_i contains the edges of M that lie in $Q_n[V_i], i = 0, 1$.

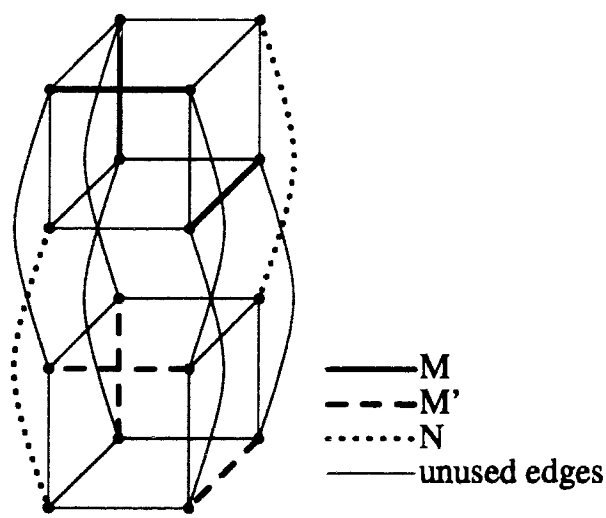

Figure 2.6.

Since $S \cap V_i \neq \emptyset$ for $i = 0, 1$, $|M_0| \leq n-2$ and $|M_1| \leq n-2$. As $Q_n[V_0]$ and $Q_n[V_1]$ are isomorphic to Q_{n-1}, we see that, by the induction hypothesis, M_i extends to a perfect matching N_i of $Q_n[V_i]$ for $i = 0, 1$. Then, $N_0 \cup N_1$ is a perfect matching extension of M in Q_n.∎

The previous result can, in fact, be strengthened to a wider class of graphs.

Definition 11 Let G_1 and G_2 be two graphs. Then their *cartesian product* is the graph $G_1 \times G_2$ with

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

and

$$E(G_1 \times G_2) = (E(G_1) \times V(G_2)) \cup (E(G_2) \times V(G_1)),$$

where if $e \in G_i$ has endpoints v_1, v_2 and $u \in G_{3-i}, i \in \{1, 2\}$, then (e, u) connects (v_1, u) and (v_2, u).

Remark Note that the cartesian product is associative (up to isomorphism of graphs).

The graph which is itself the cycle on n vertices is denoted by C_n. With this notation, we have the following result.

Proposition 12 *Let n be a positive integer and let $G = C_{4m_1+2} \times C_{4m_2+2} \times \ldots \times C_{4m_n+2}$ for some positive integers $m_1, m_2, \ldots, m_n$. Then G is $(n-1)$-extendable.*

Proof: Obviously $|V(G)| \geq 2(n-1) + 2 = 2n$. We may assume that $V(G) = Z_{4m_1+2} \times Z_{4m_2+2} \times \ldots \times Z_{4m_n+2}$ and that vertices $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots v_n)$ are joined by an edge in G if there exists $i, 1 \leq i \leq n$, such that ($|u_i - v_i| = 1$ and $u_j = v_j$ whenever $j \neq i$). The proof will again be by the induction on n. When $n = 1$, then G is a cycle of an even length $4m_1 + 2$ and is matchable. So we may assume that $n > 1$. Certainly, G is matchable. Let $M = \{u_i v_i : 1 \leq i \leq n-1\}$ be a matching of size $n - 1$ in G and let S and j be as in the previous proposition. Partition the set $V(G)$ into the sets

$$V_i = \{u \in V(G) : j\text{−th coordinate of } u \text{ is } i\}, 0 \leq i \leq 4m_j + 1.$$

Again, there are 2 cases.

Case 1. $S \subseteq V_i$ for some i. We may assume $S \subseteq V_0$. We define a mapping $\phi : V(G) \to V(G)$ by

$$\phi(u_1, u_2, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n) = (u_1, u_2, \ldots, u_{j-1}, u_j + 1, u_{j+1}, \ldots, u_n).$$

Note the following two facts.

- The set $F = \{u, \phi(u), \phi^2(u), \ldots, \phi^{4m_j+1}(u), \phi^{4m_j+2}(u) = u : u \in V_0\}$ is a 2-factor of G. In particular, $u \sim_G \phi(u)$ for every $u \in V(G)$.
- If uv is an edge in G and $i \geq 0$, then $\phi^i(u)\phi^i(v)$ is also an edge in G.

Now, if we set

$$M' = \{\phi^{2m_j+1}(u_i)\phi^{2m_j+1}(v_i) : 1 \leq i \leq n-1\},$$

then M' is a matching in G. Let

$$S' = \{\phi^{2m_j+1}(u_i), \phi^{2m_j+1}(v_i) : 1 \leq i \leq n-1\}.$$

Since $S \subseteq V_0$, $S' \subseteq V_{2m_j+1}$, and so $S \cap S' = \emptyset$. Thus $M \cup M'$ is also a matching in G. Moreover, $F \setminus (S \cup S')$ consists of cycles of length $4m_j + 2$ (with cycles $u, \phi(u), \phi^2(u), \ldots, \phi^{4m_j+2}(u) = u$ for all $u \in V_0 \setminus S$) and paths of odd length $2m_j - 1$ (with the pair of paths $\phi(u), \phi^2(u), \ldots, \phi^{2m_j}(u)$ and $\phi^{2m_j+2}(u), \phi^{2m_j+3}(u), \ldots, \phi^{4m_j+1}(u)$ for each $u \in S$).

Hence $F \setminus (S \cup S')$ contains a perfect matching N. Then $M \cup M' \cup N$ is a perfect matching of G extending M.

Case 2. There is no i such that $S \subseteq V_i$. The proof of this case is essentially the same as that of Case 2 of the previous proposition (with M partitioned into $4m_j + 2$ sets) and is omitted.■

Remark It should probably be noted here that the proof of Proposition 12 could be simplified. In fact, there is a simpler proof of this proposition which goes through even when the stipulation that the lengths of the cycles be congruent to 2 (mod 4) is replaced by a weaker requirement that the lengths of the cycles be even.

We chose to present the proof given above since it more closely resembles the proof of the main result of this thesis, where the corresponding simplification is impossible.

Chapter 3

Extendability

In this chapter we present some theorems giving sufficient conditions for high extendability of graphs.

Lemma 13 *Let $k > 0$ and $s > \binom{2k}{2}$. Let $G_1, G_2, \ldots, G_s$ be graphs on the same non-empty vertex set and let $G = G_1 \cup G_2 \cup \ldots \cup G_s$. Assume further that the following conditions are met:*

a) For every pair u, v of distinct vertices in G there is at most one $i, 1 \leq i \leq s$, such that u and v lie in the same connected component of G_i.

b) Each G_i is matchable.

c) If H_1 and H_2 are distinct connected components of some G_i and uv is an edge in G connecting H_1 to H_2, then $G[(V(H_1) \cup V(H_2)) \setminus \{u, v\}]$ is matchable.
Then G is k-extendable.

Proof: First, we have to show that $|V(G)| \geq 2k + 2$. Since $V(G) \neq \emptyset$, there is some $u \in V(G)$. As $V(G) = V(G_i)$ for $1 \leq i \leq s, u \in V(G_i)$ for all i. Since each G_i is matchable, there is a vertex v_i, $v_i \neq u$, in G_i with $u \sim_{G_i} v_i$, $1 \leq i \leq s$. By a), all v_i are distinct. Consequently,

$$|V(G)| \geq |\{u, v_1, v_2, \ldots, v_s\}| = s + 1 \geq \binom{2k}{2} + 2 \geq 2k + 1.$$

By b), all $|V(G_i)|$ (and hence $|V(G)|$) are even. We conclude that indeed $|V(G)| \geq 2k + 2$. G must also contain a perfect matching because each G_i is matchable.

Let $M = \{u_iv_i : 1 \leq i \leq k\}$ be a matching of size k in G. The set $N = \{u_i, v_i : 1 \leq i \leq k\}$ contains $2k$ elements and, therefore, there are exactly $\binom{2k}{2}$ pairs $\{u, v\}$ of elements of this set. By a) and since $s > \binom{2k}{2}$, there is $j, 1 \leq j \leq s$, such that no two elements of N lie in the same connected component of G_j.

Now, for $1 \leq i \leq k$, let H_{u_i} and H_{v_i} be the components of G_j containing u_i and v_i, respectively. Then all H_x are distinct and, by c), $G[(V(H_{u_i}) \cup V(H_{v_i})) \setminus \{u_i, v_i\}]$ contains a perfect matching M_i for all i. The set $M \cup \bigcup_{i=1}^{k} M_i$ is a perfect matching of $G[\bigcup_{i=1}^{k}\{V(H_{u_i}), V(H_{v_i})\}]$.

Furthermore, as G_j is matchable by b), all of its connected components are also matchable. But $G_j \setminus \bigcup_{i=1}^{k}\{V(H_{u_i}), V(H_{v_i})\}$ is a disjoint union of such components and, hence, is matchable and contains a perfect matching, say M'.

Now, $(M \cup M') \cup \bigcup_{i=1}^{k} M_i$ is a perfect matching i G extending M. This ends the proof of this lemma.■

The next corollary is an example how Lemma 13, which looks quite technical, can be applied to an interesting class of graphs, a certain subclass of the class of Cayley graphs, to show that graphs in this class have high extendability.

Definition 14 Let H be a group and let S be a subset of $H \setminus \{1_H\}$ such that $s^{-1} \in S$ whenever $s \in S$. The *Cayley graph* $X(H, S)$ on H with respect to S is the graph with the vertex set H and with the edge set defined by the following rule: $h_1, h_2 \in H$ are connected by an edge in $X(H, S)$ if and only if $h_1^{-1}h_2 \in S$ ($\Leftrightarrow h_2^{-1}h_1 \in S$).

Corollary 15 *Let $X(H, S)$ be a Cayley graph on an abelian group H and let m be the number of elements of order 2 in S. Assume that*

a) the order of each element in S is even,

b) $s_1, s_2 \in S$, $s_1 \neq s_2^{\pm 1}$ and $s_1^i = s_2^j$ imply that $s_1^i = s_2^j = 1$ and

c) $(|S| + m)/2 > \binom{2k}{2}$ for some $k > 0$.

Then $X(H, S)$ is k-extendable.

Proof: For $s \in S$, let G_s be the subgraph of $X(H, S)$ containing the edges arising from the element s (that is, the edges h_1h_2 where $h_1^{-1}h_2 = s^{\pm 1}$). Obviously, for every s and t in S, $G_s = G_t$ if and only if $s = t^{\pm 1}$. Thus, the set $\{G_s : s \in S\}$ contains

$$(|S|-m)/2+m=(|S|+m)/2>\binom{2k}{2}$$

distinct graphs. The union of these graphs is $X(H,S)$ and $V(G_s)=V(X(H,S))\neq\emptyset$ for all $s\in S$. Hence, it suffices to show that the conditions a)–c) of Lemma 13 are satisfied. To see that condition a) of Lemma 13 is satisfied, let u and v be two distinct vertices in G.

If there are s_1 and s_2 in S such that u and v are in the same connected component of G_{s_i}, for $i=1,2$, then

$$u=vs_1^i=vs_2^j$$

for some i and j. This implies that

$$uv^{-1}=s_1^i=s_2^j.$$

If G_{s_1} and G_{s_2} are distinct, then $s_1\neq s_2^{\pm 1}$ and, by b),

$$s_1^i=s_2^j=1.$$

This contradicts the assumption that u and v are distinct. Thus condition a) of Lemma 13 is satisfied.

As the order of each element s in S is even, each G_s is either a perfect matching in $X(H,S)$ (if the order of s is 2) or a 2-factor consisting of cycles of even length (if the order of s is larger than 2). In both cases, G_s is matchable, and condition b) from Lemma 13 is met.

Finally, let H_1 and H_2 be distinct connected components of some G_s and let uv be an edge in G connecting H_1 with H_2. Without loss of generality, we may assume that $u\in H_1$ and $v\in H_2$. Then $v=ut$ for some $t\in S$. Now,

$$H_1=u,us,us^2,\dots,us^{m-1},u$$

and

$$H_2 = v, vs, vs^2, \ldots, vs^{m-1}, v,$$

where m is the order of s in H. As H is abelian, we get

$$vs^i = uts^i = us^it \text{ for all } i, 1 \leq i \leq m.$$

Hence, in G, the vertices us^i and vs^i are connected by an edge e_i for all $i, 1 \leq i \leq m$. The edges $e_i, 1 \leq i \leq m-1$, form a perfect matching of

$$X(H,S)[(V(H_1) \cup V(H_2)) \setminus \{u,v\}]$$

and condition c) of Lemma 13 is met.

We conclude that $X(H,S)$ is k-extendable.■

Another corollary, which already suggests resemblance with the statements about the extendability of Q_n and $C_{4m_1+2} \times C_{4m_2+2} \times \ldots \times C_{4m_n+2}$ is the following.

Corollary 16 *Let $k > 0$ and let $s > \binom{2k}{2}$. Let $G_1, G_2, \ldots, G_s$ be graphs on the same non-empty vertex set and let $G = G_1 \cup G_2 \cup \ldots \cup G_s$. For each $i, 1 \leq i \leq s$, let G_i be a 2-factor of G consisting only of cycles whose lengths are congruent to 2* (mod 4). *Further, assume that*

a) if $i \neq j$, C is a cycle in G_i and C' is a cycle in G_j, then $|V(C) \cap V(C')| \leq 1$ and

b) for all $i, 1 \leq i \leq s$, if $C_1 = u_0, u_1, \ldots, u_{4p+1}, u_0$ and $C_2 = v_0, v_1, \ldots, v_{4q+1}, v_0$ are two distinct cycles in G_i such that $u_0 \sim_G v_0$, then $u_{2p+1} \sim_G v_{2q+1}$.
Then G is k-extendable.

Proof: Again, we only need to check the conditions a)–c) of Lemma 13. Condition a) of Lemma 13 is directly implied by a) of this corollary. Condition b) in Lemma 13 is satisfied since every G_i consists of even length cycles only. To see that c) of Lemma 13 is also met, let C_1 and C_2 be distinct connected components of some G_i. Consequently, C_1 and C_2 are cycles. Say,

$$C_1 = u_0 u_1 \dots u_{4p+1} u_0$$

and

$$C_2 = v_0 v_1 \dots v_{4q+1} v_0.$$

If $e = uv$ is an edge connecting C_1 to C_2, we may without loss of generality assume that $u = u_0$ and $v = v_0$. Then, by b), the edge $f = u_{2p+1} v_{2q+1}$ is in G. Since

$$(C_1 \cup C_2) \setminus \{u_0, v_0, u_{2p+1}, v_{2q+1}\}$$

is composed of four distinct paths, two of them having length $2p - 1$ and other two being of length $2q - 1$, it is matchable and, consequently, condition c) of Lemma 13 is satisfied.

Thus, G is k-extendable.■

In this special case where each G_i consists exclusively of cycles of length congruent to $2 \pmod 4$, we can, with a further condition posed on G, prove that $s > k$ can be used instead of $s > \binom{2k}{2}$.

Proposition 17 *Let $k \geq 0$ and let $s > k$. Let $G_1, G_2, \dots, G_s$ be graphs on the same non-empty vertex set and let $G = G_1 \cup G_2 \cup \dots \cup G_s$. Let, for each $1 \leq i \leq s$, G_i be a 2-factor of G consisting of cycles all of whose lengths are congruent to $2 \pmod 4$. Assume that a) and b) of the previous corollary hold true and that*

c) For every i, $G \setminus E(G_i)$ can be written as a vertex-disjoint union of graphs $H_1^i, H_2^i, \dots, H_{p_i}^i$ where, for all $j, 1 \leq j \leq p_i$, and for all cycles C in G_i, $|C \cap H_j^i| \leq 1$. Then G is k-extendable.

Remark This proposition is a generalization of Proposition 12 (and also Proposition 10). The numbers s and k in this proposition correspond to the numbers n and $n-1$ of Proposition 12, respectively, and the graph G_i here corresponds to the subgraph of $C_{4m_1+2} \times C_{4m_2+2} \times \dots \times C_{4m_n+2}$ containing the i-th "parallel class" of edges of $C_{4m_1+2} \times C_{4m_2+2} \times \dots \times C_{4m_n+2}$ in Proposition 12.

Finally, the graph H_j^i corresponds to the subgraph of $C_{4m_1+2} \times C_{4m_2+2} \times \ldots \times C_{4m_n+2}$ induced by the set $Z_{4m_1+2} \times Z_{4m_2+2} \times \ldots \times Z_{4m_{i-1}+2} \times \{j\} \times Z_{4m_{i+1}+2} \times \ldots \times Z_{4m_n+2}$.

Before we embark on the proof of this proposition, we will first prove the following simple observation.

Observation 18 *Let $s > k > 0$ and let G be as in Proposition 17. Let $i \in \{1, 2, \ldots, s\}$ and let $G \setminus E(G_i) = H_1^i \cup H_2^i \cup \ldots \cup H_{p_i}^i$, where $H_1^i, H_2^i, \ldots, H_{p_i}^i$ are as in Proposition 17. Then each H_j^i, $1 \le j \le p_i$, satisfies the hypothesis of Proposition 17 with G, s and k being replaced by $H_j^i, s' = s - 1$ and $k' = k - 1$, respectively.*

Proof: Fix $j, 1 \le j \le p_i$. Obviously

$$H_j^i = G_1[V(H_j^i)] \cup \ldots \cup G_{i-1}[V(H_j^i)] \cup G_{i+1}[V(H_j^i)] \cup \ldots \cup G_s[V(H_j^i)].$$

Thus, setting

$$G_1' = G_1[V(H_j^i)], G_2' = G_2[V(H_j^i)], \ldots G_{i-1}' = G_{i-1}[V(H_j^i)],$$

$$G_i' = G_{i+1}[V(H_j^i)], \ldots, G_{s-1}' = G_s[V(H_j^i)]$$

gives $H_j^i = G_1' \cup G_2' \cup \ldots \cup G_{s'}'$.

Clearly, each G_i' is a 2-factor of H_j^i consisting of cycles whose lengths are congruent to $2 \pmod 4$, and $V(G_i') = V(H_j^i) \neq \emptyset$. Furthermore, conditions a) and b) (of Corollary 16) are obviously satisfied by the graphs $G_i', 1 \le i \le s'$. Finally, for every $l, 1 \le l \le s'$, $H_j^i \setminus E(G_l')$ is the union of the graphs

$$H_1^{l'}[V(H_j^i)], H_2^{l'}[V(H_j^i)], H_3^{l'}[V(H_j^i)], \ldots, H_{p_{l'}}^{l'}[V(H_j^i)],$$

where $l' = l$ or $l' = l + 1$ according as whether $l < i$ or $l > i$. This union is vertex-disjoint since $H_1^{l'}, H_2^{l'}, \ldots H_{p_{l'}}^{l'}$ are vertex-disjoint. Now, if C is a cycle in G_l', then it is also a cycle in $G_{l'}$. Thus, by hypothesis, $|C \cap H_m^{l'}| \le 1$ for $1 \le m \le p_{l'}$ and, consequently, $|C \cap H_m^{l'}[V(H_j^i)]| \le 1$ for $1 \le m \le p_{l'}$.■

We are ready to prove the proposition.

Proof of Proposition 17: We will proceed by induction on s.

When $s = 1$, k is necessarily 0, and we only have to show that G contains a perfect matching. But in this case $G = G_1$ and is matchable.

So assume $s > 1$. First, we have to show $|V(G)| \geq 2k + 2$. Let $u \in V(G)$. Then $u \in V(G_i)$ for all $i, 1 \leq i \leq s$. As each G_i is a 2-factor, it contains a cycle C_i with $u \in C_i$. By condition a), $|V(C_i) \cap V(C_j)| \leq 1$ for all $i \neq j, 1 \leq i, j \leq s$. As each C_i contains u, we have $(V(C_i) \setminus \{u\}) \cap (V(C_j) \setminus \{u\}) = \emptyset$ for all $i \neq j$. Thus,

$$|V(G)| \geq |\bigcup_{i=1}^{s} V(C_i)| = 1 + |\bigcup_{i=1}^{s} V(C_i) \setminus \{u\}| = 1 + \sum_{i=1}^{s}(|V(C_i)| - 1) \geq 2s + 1 \geq 2k + 2.$$

Also, G is matchable as each G_i is matchable. Now, let

$$M = \{u_i v_i : 1 \leq i \leq k\}$$

be a matching in G. As $|M| = k < s$ and no edge of G is contained in more than one G_i, there exists $i, 1 \leq i \leq s$, such that none of the edges of M belongs to G_i. Without loss of generality, we may assume $i = 1$. Therefore, all edges of M lie in

$$G \setminus E(G_1) = H_1^1 \cup H_2^1 \cup \ldots \cup H_{p_1}^1.$$

We consider the two possible cases.

Case 1. There exists $i, 1 \leq i \leq p_1$, such that all edges of M lie in H_i^1. Again, we may assume $i = 1$. By assumption c), each of the cycles in G_1 has at most one vertex in common with H_1^1. Thus, if for $1 \leq j \leq k$ we let C_{u_j} (C_{v_j}) be the cycle in G_1 containing u_j (v_j), then these cycles are pairwise distinct (and, therefore, pairwise disjoint). As in the proof of the previous corollary, we conclude from condition b) that for each $j, 1 \leq j \leq k$, the edge $u_j v_j$ extends to a perfect matching M_j of $G[V(C_{u_j}) \cup V(C_{v_j})]$. Also,

$$G_1[V(G) \setminus \bigcup_{j=1}^{k}(V(C_{u_j}) \cup V(C_{v_j}))]$$

is a union of (even-length) cycles and contains a perfect matching M'. Now $M' \cup (\bigcup_{j=1}^{k} M_j)$ is a perfect matching of G extending M.

Case 2. There is no $i, 1 \leq i \leq p$, such that all edges of M lie in H_i^1. Hence each H_i^1 contains at most $k-1$ edges of M and, by Observation 18, we may apply the induction hypothesis to each H_i^1 to show that H_i^1 contains a perfect matching M_i such that M_i contains all edges of M which lie in H_i^1.

Then $\bigcup_{i=1}^{p_1} M_i$ is a perfect matching extension of M.■

Chapter 4

Girth

This chapter deals with graphs that do not contain short cycles. Similarly as in the previous chapter, where we presented sufficient conditions for a graph to be highly extendable, we will here give conditions that are sufficient to make sure that a graph has no short cycles.

For a graph G, by the *girth* of G (denoted by girth(G)) we mean the length of a shortest cycle in G (if G contains no cycle then girth$(G) = \infty$).

Let G be a graph and let e be an edge in G connecting (not necessarily distinct) vertices u and v. Then by the *contraction* of the edge e we mean the following procedure.

Step 1. Delete the edge e from G.

Step 2. Identify the vertices u and v in G.

It is readily seen that the graph resulting from a sequence of contractions is independent of the order in which these contractions are performed and depends only on the set of edges that are being contracted. Thus, if $S \subseteq E(G)$, then we can talk about the contraction of the set S.

One easily observes that if e is an edge in G, then the contraction of e does not increase the girth of G unless (e is a loop and the contraction of e results in a loopless graph). From this, we see that the next statement holds true.

Observation 19 *Let $G = G_1 \cup G_2 \cup \ldots \cup G_s$ where $V(G) = V(G_i)$ for all $i, 1 \leq$*

$i \leq s$, and where $s \geq 1$. Assume that $E(G_i) \cap E(G_j) = \emptyset$ for all pairs i, j such that $1 \leq i, j \leq s$ and $i \neq j$. For each i, let G'_i be obtained from G by contracting all edges not in G_i. Then

$$\text{girth}(G) \geq \min_{1 \leq i \leq s} \{\text{girth}(G'_i)\}. \blacksquare$$

This observation will serve as a motivation for the results that follow. In these results, we restrict our attention to the class of graphs described in the next definition.

Definition 20 Let G be a graph with $V(G) = T_1 \times T_2 \times \ldots \times T_s$ for some sets $T_1, T_2, \ldots, T_s, s \geq 1$, in which every pair of adjacent vertices differ in exactly one coordinate. Then we say that G is *orthogonal.* Also, we denote by $E_i, 1 \leq i \leq s$, the set of edges of G whose endpoints differ in the i-th coordinate.

We will see that for orthogonal graphs the conclusion of the above observation can be strengthened and this will prove useful in our main construction. Before we can state the first proposition inspired by Observation 19, we need the definition of a projection of an orthogonal graph.

Definition 21 Let G be an orthogonal graph, $V(G) = T_1 \times T_2 \times \ldots \times T_s$. Then the *projection* $\pi_i(G)$ *of* G *onto* T_i is the graph obtained from G by first deleting all edges not in E_i and then identifying all vertices having the same i-th coordinate. We then think of $\pi_i(G)$ as having the vertex set T_i (the edge set of $\pi_i(G)$ is E_i).

Proposition 22 *Let G be an orthogonal graph with $V(G) = T_1 \times T_2 \times \ldots \times T_s$. Then*

$$\text{girth}(G) \geq \min_{1 \leq i \leq s} \{\text{girth}(\pi_i(G))\}.$$

Remark Note that if G is a cartesian product of cycles, then the proposition is a (trivial) special case of Observation 19. For example, if $G = C_3 \times C_4$ and $i = 2$, then $\pi_2(G) = 3C_4$ and can be viewed as being obtained from G by the contraction of the edges of the copies of C_3 in G. In fact, Proposition 22 can, with a little effort, be proven using Observation 19, but we will not follow this path and rather give an independent proof as it is illuminating in the light of the statements that will follow.

Proof of Proposition 22: Let

$$C = u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_0$$

be a shortest cycle in G. Then $e_1 \in E_j$ for some $j, 1 \leq j \leq s$. Let

$$1 = i_1 < i_2 < \cdots < i_m \leq n$$

be all those indices i for which $e_i \in E_j$.

For $0 \leq i \leq n-1$, let $u_{i,j}$ be the j-th coordinate of u_i. Then, by the definition of the graph $\pi_j(G)$, the edge e_{i_k} connects the vertices $u_{i_k-1,j}$ and $u_{i_k,j}$ in $\pi_j(G)$ for $1 \leq k \leq m$ (here, $u_{i_0} = u_{i_m}$).

Moreover,

$$u_{i_k,j} = u_{i_k+1,j} = u_{i_k+2,j} = u_{i_k+3,j} = \cdots = u_{i_{k+1}-1,j}$$

for $1 \leq k \leq m$. Thus (noting that $u_{0,j} = u_{i_m,j}$),

$$C' = u_{0,j}, e_{i_1}, u_{i_1,j}, e_{i_2}, u_{i_2,j}, \ldots, u_{i_{m-1},j}, e_{i_m}, u_{0,j}$$

is a closed walk (without repeated edges) of length m in $\pi_j(G)$. Hence,

$$\text{girth}(\pi_j(G)) \leq |C'| = m \leq n = \text{girth}(G)$$

and the proposition is proven.■

The following strengthening of the above proposition is crucial because it will serve as a tool for constructing classes of (regular) graphs with arbitrarily large girth which will also have properties that will enable us to show that such graphs have high extendability.

Lemma 23 *Let G be an orthogonal graph with $V(G) = T_1 \times T_2 \times \ldots \times T_s$, for some $s \geq 1$. Let*

$$\mathcal{G} = \{\pi_i(G[T_1 \times T_2 \times \ldots \times T_i \times \{u_{i+1}\} \times \{u_{i+2}\} \times \ldots \times \{u_s\}]) : 1 \leq i \leq s,$$

$$u_j \in T_j \text{ for } i < j \leq s\}.$$

Then

$$\text{girth}(G) \geq \min_{G' \in \mathcal{G}}\{\text{girth}(G')\}.$$

Proof: Let C be a shortest cycle in G and let j be the largest index such that $E(C) \cap E_j \neq \emptyset$. Then $V(C) \subseteq T_1 \times T_2 \times \ldots \times T_j \times \{u_{j+1}\} \times \{u_{j+2}\} \times \ldots \times \{u_s\}$ for some $u_{j+1} \in T_{j+1}, u_{j+2} \in T_{j+2}, \ldots, u_s \in T_s$. The graph

$$G' = \pi_j(G[T_1 \times T_2 \times \ldots \times T_j \times \{u_{j+1}\} \times \{u_{j+2}\} \times \ldots \times \{u_s\}])$$

can be viewed as being obtained from G by first deleting the vertices not in $T_1 \times T_2 \times \ldots \times T_j \times \{u_{j+1}\} \times \{u_{j+2}\} \times \ldots \times \{u_s\}$ and edges not in E_j and then identifying the vertices having the same j-th coordinate. By the choice of j, none of the deleted vertices is in C and not all edges of C are deleted. Thus, the projection of C onto the j-th coordinate is a closed walk without repeated edges of length greater than 0 and is contained in G'. Therefore, $\text{girth}(G') \leq \text{girth}(G)$.

As $G' \in \mathcal{G}$, the conclusion follows.■

To get a flavor of how this lemma improves Proposition 22, consider the following graph G. $V(G) = Z_n \times Z_m$ for some $n, m \geq 3$ and

$$E(G) = (\bigcup_{i=0}^{m-1} E(H_i)) \cup (\bigcup_{i=0}^{n-1} E(H_i')),$$

where H_i, for $0 \leq i \leq m-1$, is the cycle $(0,i),(1,i),(2,i),\ldots,(n-1,i),(0,i)$ and where each H_i' is a certain graph with $V(H_i') = \{i\} \times Z_m$. Let's first estimate the girth of G using Lemma 23. The collection $\mathcal{G}$ contains m copies of the graph C_n and a copy of the graph $H = \pi_2(G)$. By the lemma,

$$\text{girth}(G) \geq \min\{n, \text{girth}(H)\}.$$

Since if m is sufficiently large compared to n, then H_i' can be chosen so that H contains no short cycles, the appropriate choice of H_i' will lead to $\text{girth}(G) \geq n$.

However, if we estimate the girth of G using Proposition 22, we can only get $\mathrm{girth}(G) \geq 2$ as the graph $\pi_1(G)$ of the proposition is a copy of mC_n and has girth 2.

If u and v are two vertices in G, then $d_G(u,v)$ (or simply $d(u,v)$ when no confusion may arise) is the *distance* between u and v in G, i.e., the length of a shortest uv-path in G. We set $d_G(u,v) = \infty$ if there is no path connecting u and v in G.

Now, finally, we can prove the main theorem of this chapter.

Theorem 24 *Let G and $\mathcal{G}$ be as in the previous lemma and let G satisfy the following extra requirement.*

a) If $G' \in \mathcal{G}, V(G') = T_i$, e_1 and e_2 are multiple edges in G' connecting the vertices u' and v', and the endpoints of e_1 (e_2) in G are u_1 (u_2) and v_1 (v_2), then $d(u_1,u_2) > g$ and $d(v_1,v_2) > g$.

Also, let $\mathcal{H}$ be the set of the graphs which are obtained from graphs of $\mathcal{G}$ by identifying multiple edges. Then

$$\mathrm{girth}(G) \geq \min\{g, \min_{G' \in \mathcal{H}}\{\mathrm{girth}(G')\}.$$

Proof: Let C be a shortest cycle in G and let u be a vertex in C. If the length of C is larger than g, the conclusion follows. So we may assume $|C| \leq g$. In this case, for every vertex v in C, we have $d(u,v) \leq g/2$. Let $\tilde{G}$ be obtained from G by deleting all those edges of G with at least one endpoint at distance greater than $g/2$ from u. Let $\tilde{\mathcal{G}}$ be obtained from $\tilde{G}$ as $\mathcal{G}$ is from G. By the previous lemma, we have

$$\mathrm{girth}(\tilde{G}) \geq \min_{G' \in \tilde{\mathcal{G}}}\{\mathrm{girth}(G')\}.$$

Let $\tilde{\mathcal{H}}$ be obtained from $\tilde{\mathcal{G}}$ as $\mathcal{H}$ is from $\mathcal{G}$. The way $\tilde{G}$ was chosen and the additional condition a) placed on G in this theorem imply that graphs in $\tilde{\mathcal{G}}$ do not have multiple edges and, consequently, $\tilde{\mathcal{H}} = \tilde{\mathcal{G}}$. On the other hand, each graph in $\tilde{\mathcal{H}}$ is a subgraph of some graph in $\mathcal{H}$. Combining these two observations, we get

$$\min_{G' \in \tilde{\mathcal{G}}}\{\mathrm{girth}(G')\} = \min_{G' \in \tilde{\mathcal{H}}}\{\mathrm{girth}(G')\} \geq \min_{G' \in \mathcal{H}}\{\mathrm{girth}(G')\}.$$

Thus,

$$\text{girth}(\tilde{G}) \geq \min_{G' \in \mathcal{H}} \{\text{girth}(G')\}.$$

But $\tilde{G}$ contains C, a shortest cycle in G. This means that girth(G) = girth$(\tilde{G})$ and the conclusion follows.■

Let $H_1, H_2, \ldots, H_s$ be groups. Further, let $f_1 \in H_1 \setminus \{1_{H_1}\}$ and let, for $2 \leq i \leq s$, f_i be a function from $H_1 \times H_2 \times \ldots \times H_{i-1}$ to $H_i \setminus \{1_{H_i}\}$. We will construct a simple orthogonal graph G on the vertex set $H_1 \times H_2 \times \ldots \times H_s$ as follows.

The vertex $(u_1, u_2, \ldots, u_s)$ will be connected to the vertices $(u_1 f_1^{\pm 1}, u_2, u_3, \ldots, u_s)$ and to the vertices $(u_1, u_2, \ldots, u_{i-1}, u_i f_i^{\pm 1}(u_1, u_2, \ldots, u_{i-1}), u_{i+1}, u_{i+2}, \ldots, u_s)$, for $2 \leq i \leq s$.

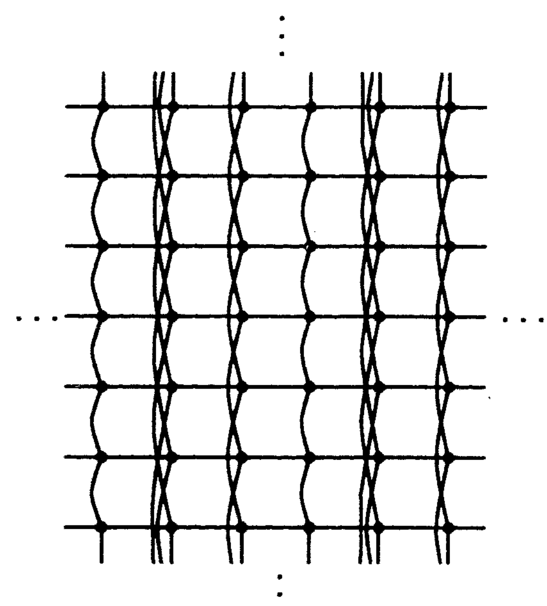

Figure 4.1.

Example Let $H_1 = Z_{3n}$ and $H_2 = Z_m$ for some $n, m > 0$. Let $f_1 = 1$ (note that 1 is not the identity element of Z_{3n}) and let

$f_2(a) = 1$ if $a \equiv 0 \pmod 3$,

$f_2(a) = 3$ if $a \equiv 1 \pmod 3$ and

$f_2(a) = 2$ if $a \equiv 2 \pmod 3$.

Figure 4.1 indicates what the resulting G looks like.

Corollary 25 *Let $H_1, H_2, \ldots, H_s, f_1, f_2, \ldots, f_s$ and G be as above. Assume that the order of f_1 in H_1 is greater than g and that the girths of the graphs $X(H_i, \{f_i^{\pm 1}(v) : v \in H_1 \times H_2 \times \ldots \times H_{i-1}\})$ are larger than g for $2 \leq i \leq s$. Assume further that the following condition is satisfied.*

b) For every pair $u = (u_1, u_2, \ldots, u_s), v = (v_1, v_2, \ldots, v_s)$ of distinct vertices of G and for every $i, 2 \leq i \leq s$, if $u_j = v_j$ for all $j, i \leq j \leq s$, and $f_i(u_1, u_2, \ldots, u_{i-1}) = f_i^{\pm 1}(v_1, v_2, \ldots, v_{i-1})$ then $d(u, v) > g$.

Then $\text{girth}(G) \geq g$.

Proof: Again, let $\mathcal{G}$ be as in Lemma 23 and Theorem 24. Condition b) implies that a) of the previous theorem is met. Thus G satisfies the assumptions of the previous theorem and hence

$$\text{girth}(G) \geq \min\{g, \min_{G' \in \mathcal{H}}\{\text{girth}(G')\}\},$$

where $\mathcal{H}$ is as in Theorem 24.

Thus, it suffices to prove that $\min_{G' \in \mathcal{H}}\{\text{girth}(G')\} \geq g$. But, by the definition of $\mathcal{H}$, each graph in $\mathcal{H}$ is isomorphic to either $X(H_1, \{f_1, f_1^{-1}\})$ or to one of the graphs $X(H_i, \{f_i^{\pm 1}(v) : v \in H_1 \times H_2 \times \ldots \times H_{i-1}\}), 2 \leq i \leq s$. By the assumption, all these graphs have girth greater than g and the corollary follows.■

Remark Note that if each $f_i, 2 \leq i \leq s$, is injective and $f_i(u) \neq f_i^{-1}(v)$ for all $u, v \in H_1 \times H_2 \times \ldots \times H_{i-1}$ then $\mathcal{G} = \mathcal{H}$ and, using Lemma 23, the assumption b) can be dropped.

Chapter 5

Main Result

In this chapter, we present our main construction and prove that it produces highly extendable graphs that do not contain any short cycles.

Lemma 26 *Let n and k be positive integers. Then there exists a finite group $H_{n,k}$ containing elements $a_1, a_2, \ldots, a_k$ ($a_i \neq a_j^{\pm 1}$ for $i \neq j$) such that, for every $m \leq n$ and for every m-tuple $(b_1, b_2, \ldots, b_m)$ of elements of $H_{n,k}$ of the form a_i, a_i^{-1}, if $b_1 b_2 \ldots b_m = 1_{H_{n,k}}$, then there exists i, $1 \leq i \leq m-1$, for which $b_{i+1} = b_i^{-1}$. Moreover, the elements a_i can be chosen so that their orders are odd.*

Remark Note that the lemma says that the Cayley graph $X(H_{n,k}, \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_k^{\pm 1}\})$ has girth larger than n.

Proof: We will show that $H_{n,k}$ can be chosen from among groups used in [14]. To this end, let $SL_2(Z)$ (or $SL_2(Z_p)$ for p prime) denote the group of 2×2 matrices over Z (or over Z_p) which have determinant equal to 1 and let $\phi_p : SL_2(Z) \rightarrow SL_2(Z_p)$ denote the homomorphism reducing the entries of elements of $SL_2(Z)$ mod p.

Take a sufficiently large positive integer s and $k+1$ distinct pairs $(m_i, q_i), 1 \leq i \leq k+1$, of numbers satisfying $\gcd(m_i, q_i) = 1$, $0 \leq m_i \leq s/2$ and $0 \leq q_i \leq s/2$ for all i (in particular, $(0,0)$ is not among these pairs). Let b_i, c_i be such that

$$C_i = \begin{bmatrix} m_i & c_i \\ q_i & b_i \end{bmatrix}$$

has determinant equal to 1. We may assume that $|c_i| < s/2$ and $|b_i| < s/2$. For $1 \leq i \leq k+1$, set

$$g_i = C_i \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} C_i^{-1}.$$

In [14], it was shown that for any prime p the Cayley graph of $SL_2(Z_p)$ with respect to $\{\phi_p(g_i), \phi_p(g_i^{-1}) : 1 \leq i \leq k\}$ is $2k$-regular and has girth larger than $\frac{2}{3}\log_s(p/2) - 1$. Hence, if we choose p so that $\frac{2}{3}\log_s(p/2) - 1 \geq n$, set $a_i = \phi_p(g_i)$ for all i, and let $H_{n,k} = SL_2(Z_p)$, then $H_{n,k}$ and a_i will be as this lemma requires except that possibly the order of some a_i could be even. Now, we will show that p can be chosen so that the orders of all a_i are odd. In order to do this, choose p as above satisfying the additional requirement that it be greater than s. Consider, for a positive integer t, the t-th power of g_i:

$$g_i^t = C_i \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}^t C_i^{-1} = C_i \begin{bmatrix} 1 & ts \\ 0 & 1 \end{bmatrix} C_i^{-1} =$$

$$= \begin{bmatrix} m_i & c_i \\ q_i & b_i \end{bmatrix} \begin{bmatrix} 1 & ts \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_i & -c_i \\ -q_i & m_i \end{bmatrix} = \begin{bmatrix} -stm_iq_i + 1 & stm_i^2 \\ -stq_i^2 & stm_iq_i + 1 \end{bmatrix}.$$

Hence, $\phi_p(g_i)^t = \phi_p(g_i^t) = 1_{SL_2(Z_p)}$ if and only if

$$tsm_iq_i \equiv tsm_i^2 \equiv tsq_i^2 \equiv 0 \pmod{p}.$$

As p is prime, s, m_i and q_i are smaller than p and at least one of m_i, q_i is nonzero, the conditions are satisfied exactly when t is a multiple of p. Thus, the order of $\phi_p(g_i)$ in $SL_2(Z_p)$ is p, an odd number.∎

In the rest of the thesis we assume that, for every pair n, k, the group $H_{n,k}$ is fixed. The elements a_i of $H_{n,k}$ are referred to as *natural generators* for the group $H_{n,k}$.

Now we are ready to give the main construction which will feature certain orthogonal graphs. These graphs turn out to be special instances of graphs dealt with in Corollary 25 and, thus, do not contain short cycles. We will also be able to prove, in a series of claims, that they satisfy the conditions of Proposition 17, implying that they are also highly extendable.

Construction 27 *Let k and n be given positive integers. We construct a graph $G = G_{k,n}$ as follows.*

- *First let $s > k$ and define recursively groups $H'_1, H'_2, \ldots, H'_s$; let H'_1 be Z_{2n+1} and, for $2 \leq i \leq s$, let $H'_i = H_{n,|H'_1||H'_2|\ldots|H'_{i-1}|}$. When $i \geq 2$, the natural generators for H'_i will be denoted by $a_j^{(i)}$, $1 \leq j \leq |H'_1||H'_2|\ldots|H'_{i-1}|$.*
- *For $i, 1 \leq i \leq s$, let the group H_i be defined by $H_i = H'_i \times Z_2$.*
- *For $i, 2 \leq i \leq s$, let $f'_i : H'_1 \times H'_2 \times \ldots \times H'_{i-1} \rightarrow H'_i$ be a bijection between $H'_1 \times H'_2 \times \ldots \times H'_{i-1}$ and the natural generators $\{a_j^{(i)} : j \in \{1, 2, \ldots, |H'_1||H'_2| \ldots |H'_{i-1}|\}\}$.*
- *For $i, 2 \leq i \leq s$, let $f_i : H_1 \times H_2 \times \ldots \times H_{i-1} \rightarrow H_i$ be defined by $f_i((b_1, n_1), (b_2, n_2), \ldots, (b_{i-1}, n_{i-1})) = (f'_i(b_1, b_2, \ldots, b_{i-1}), 1)$.*
- *Let $f_1 = (1, 1) \in H_1 = H'_1 \times Z_2$.*
- *Set $V(G) = H = H_1 \times H_2 \times \ldots \times H_s$.*
- *For every $u = (u_1, u_2, \ldots, u_s)$ and $v = (v_1, v_2, \ldots, v_s)$ in $V(G)$, let uv be an edge in G if there is a unique $i, 1 \leq i \leq s$, for which $u_i \neq v_i$ and, for this i, $u_i = v_i f_i^{\pm 1}(v_1, v_2, \ldots v_{i-1}) (= v_i f_i^{\pm 1}(u_1, u_2, \ldots u_{i-1}))$ if $i \neq 1$ and $u_1 = v_1 f_1^{\pm 1}$ if $i = 1$.*

Let, for $1 \leq i \leq s$, G_i be defined by $V(G_i) = V(G)$ and

$$E(G_i) = \{uv : u \text{ and } v \text{ differ in the } i\text{-th coordinate and } uv \text{ is an edge in } G\}.$$

Claim 28 *For every $i, 1 \leq i \leq s$, G_i is a 2-factor of G, all of whose cycles have lengths congruent to 2 modulo 4.*

Proof: If $i = 1$, then $f_1 = (1,1)$ has order $4n+2$ in the group $H_1 = H'_1 \times Z_2$ and the result follows. If $i > 1$, then G_i is a 2-factor of G and the length of each cycle in G_i is equal to the order of the element $(f'_i(b), 1)$ in H_i for some $b \in H'_1 \times H'_2 \times \ldots \times H'_{i-1}$. By the definition of f'_i,

$$f'_i(b) = a_j^{(i)}$$

for some $j, 1 \leq j \leq |H'_1||H'_2|\ldots|H'_{i-1}|$. Hence the order of $f'_i(b)$ in H'_i is odd and, therefore, the order of $(f'_i(b), 1)$ in H_i is congruent to $2 \pmod 4$. This ends the proof.■

Claim 29 *For every i, j with $i \neq j$ and $1 \leq i, j \leq s$, if C is a cycle in G_i and C' is a cycle in G_j, then $|V(C) \cap V(C')| \leq 1$.*

Proof: Suppose the contrary and let C and C' be cycles in G_i and G_j ($i \neq j, 1 \leq i, j \leq s$), respectively, with $|V(C) \cap V(C')| \geq 2$. Then there are two distinct vertices u and v in $V(C) \cap V(C')$. By the definition of G_i, $u, v \in V(C)$ implies that u and v can only differ in the i-th coordinate. Similarly, from $u, v \in V(C')$ we get that u, v can only differ in the j-th coordinate, a contradiction.■

Claim 30 *For all $i, 1 \leq i \leq s$, if $C_1 = u_0, u_1, u_2, \ldots, u_{4p+1}, u_0$ and $C_2 = v_0, v_1, v_2, \ldots, v_{4q+1}, v_0$ are two distinct cycles in G_i such that $u_0 \sim_G v_0$, then $u_{2p+1} \sim_G v_{2q+1}$.*

Proof: Let $u_0 = (u'_1, u'_2, \ldots, u'_s)$. Then $u_1 = u_0 c$ where

$$c = (1_{H_1}, 1_{H_2}, \ldots, 1_{H_{i-1}}, f_i^{\epsilon}(u'_1, u'_2, \ldots, u'_{i-1}), 1_{H_{i+1}}, \ldots, 1_{H_s})$$

for some $\epsilon \in \{1, -1\}$ if $i \neq 1$ and

$$c = (f_1^{\epsilon}, 1_{H_2}, 1_{H_3}, \ldots, 1_{H_s})$$

for some $\epsilon \in \{1, -1\}$ if $i = 1$.

Then $u_2 = u_0c^2, u_3 = u_0c^3$ and so on. In particular, as C_1 is a cycle of length $4p+2$, the order of c in H is $4p+2$. Thus, if $i \neq 1$, then the order of $f_i^\epsilon(u_1', u_2', \ldots, u_{i-1}')$ in H_i is also $4p+2$. But

$$f_i^\epsilon = ((f_i')^\epsilon(b_1, b_2, \ldots, b_{i-1}), 1),$$

where

$$(u_1', u_2', \ldots, u_{i-1}') = ((b_1, n_1), (b_2, n_2), \ldots (b_{i-1}, n_{i-1})).$$

Consequently, for the order of $f_i^\epsilon(u_1', u_2', \ldots, u_{i-1}')$ to be $4p+2$, the order of $f_i'(b_1, b_2, \ldots, b_{i-1})$ (in H_i') must be $2p+1$. Thus

$$c^{2p+1} = (1_{H_1}, 1_{H_2}, \ldots, 1_{H_{i-1}}, ((f_i')^{(2p+1)\epsilon}(b_1, b_2, \ldots, b_{i-1}), 1), 1_{H_{i+1}}, \ldots, 1_{H_s}) =$$

$$= (1_{H_1}, 1_{H_2}, \ldots, 1_{H_{i-1}}, (1_{H_i'}, 1), 1_{H_{i+1}}, 1_{H_{i+2}}, \ldots, 1_{H_s}). \tag{5.1}$$

A similar argument shows that the same conclusion holds when $i = 1$. Now, let $d \in H$ be such that $v_1 = v_0d$. The same reasoning as above gives

$$d^{2q+1} = (1_{H_1}, 1_{H_2}, \ldots, 1_{H_{i-1}}, (1_{H_i'}, 1), 1_{H_{i+1}}, 1_{H_{i+2}}, \ldots, 1_{H_s}) = c^{2p+1}.$$

Let $g \in H$ be such that $v_0 = u_0g$. Similarly to c and d, we have that g is either

$$(1_{H_1}, 1_{H_2}, \ldots, 1_{H_{j-1}}, f_j^{\epsilon'}(u_1', u_2', \ldots, u_{j-1}'), 1_{H_{j+1}}, \ldots, 1_{H_s})$$

or

$$(f_1^{\epsilon'}, 1_{H_2}, 1_{H_3}, \ldots, 1_{H_s})$$

for some $\epsilon' \in \{1, -1\}$.

From (5.1) it follows that if $u_0 = ((b_1, n_1), (b_2, n_2), \ldots, (b_s, n_s))$, then

$$u_{2p+1} = ((b_1, n_1), (b_2, n_2), \ldots, (b_{i-1}, n_{i-1}), (b_i, 1 - n_i), (b_{i+1}, n_{i+1}), \ldots, (b_s, n_s)).$$

The definitions of $f_j, 1 \leq j \leq s$, and of the graph G imply that $u_{2p+1}, u_{2p+1}g$ is also an edge in G. Therefore, it suffices to show that $v_{2q+1} = u_{2p+1}g$. But $u_{2p+1}g = u_0c^{2p+1}g$. Again, from (5.1) we see that every element of H, in particular, g, commutes with c^{2p+1}, and we get

$$u_{2p+1}g = u_0c^{2p+1}g = u_0gc^{2p+1} = u_0gd^{2q+1} = v_0d^{2q+1} = v_{2q+1}.\blacksquare$$

Claim 31 *For every $i, 1 \leq i \leq s$, $G \setminus E(G_i)$ can be written as a vertex-disjoint union of graphs $G'_1, G'_2, \ldots, G'_p$, where $p = |H_i|$ and for all j, $|C \cap G'_j| \leq 1$ whenever C is a cycle in G_i.*

Proof: Let $H_i = \{a_k : 1 \leq k \leq |H_i|\}$. For $1 \leq k \leq |H_i|$, let

$$G'_k = G[H_1 \times H_2 \times \ldots \times H_{i-1} \times \{a_k\} \times H_{i+1} \times \ldots \times H_s].$$

Obviously, $G'_k, 1 \leq k \leq |H_i|$, are vertex-disjoint and $\bigcup_{k=1}^{|H_i|} G'_k$ does not contain any edges of G_i. Moreover, if uv is an edge in $G \setminus E(G_i)$, where $u = (u_1, u_2, \ldots, u_s), v = (v_1, v_2, \ldots, v_s)$, then $u_i = v_i = a_l$ for some l, $1 \leq l \leq |H_i|$, and, therefore, $uv \in G'_l$. Thus, $G \setminus E(G_i)$ is a vertex-disjoint union of the graphs $G'_k, 1 \leq k \leq |H_i|$.

Finally, if C is a cycle in G_i and $u \neq v$ are two vertices in C, then $u_i \neq v_i$ (where u_i and v_i are the i-th coordinates of u and v, respectively) and, consequently, u and v cannot lie in the same G'_j. Thus, $|C \cap G'_j| \leq 1$, for all j.$\blacksquare$

Remark Note that in the proof of this claim we only needed the fact that G is orthogonal.

Claim 32 *Let $u = (u_1, u_2, \ldots, u_s), v = (v_1, v_2, \ldots, v_s)$ be two distinct vertices in G and let $2 \leq i \leq s$. Assume further that $u_j = v_j$ for all $j, i \leq j \leq s$, and that $f_i(u_1, u_2, \ldots, u_{i-1}) = f_i^{\pm 1}(v_1, v_2, \ldots v_{i-1})$. Then $d(u, v) > n$.*

Proof: By the definition of f_i, $f_i(u_1, u_2, \ldots, u_{i-1}) = f_i^{\pm 1}(v_1, v_2, \ldots, v_{i-1})$ implies that

$$(u_1, u_2, \ldots, u_{i-1}) = ((b_1, n_1), (b_2, n_2), \ldots, (b_{i-1}, n_{i-1}))$$

and

$$(v_1, v_2, \ldots, v_{i-1}) = ((b_1, m_1), (b_2, m_2), \ldots, (b_{i-1}, m_{i-1}))$$

for some bs, ns and m. Moreover, since $u_j = v_j$ for $j \geq i$ and since u and v are distinct by the assumption, there must be l $(1 \leq l \leq i-1)$ such that $n_l \neq m_l$.

Now, if xy is an edge in G, x and y differ in the l-th coordinate and $l > 1$, then $x_l y_l$ is an edge in $G' = X(H_l, \{f_l^{\pm 1}(z) : z \in H_1 \times H_2 \times \ldots \times H_{l-1}\})$ by the definition of G (here x_l and y_l are the l-th coordinates of x and y, respectively). Similarly, if $l = 1$, then $x_l y_l$ is an edge in $G' = X(H_1, \{(1,1), (2n,1)\})$. Let $u = w_1, w_2, w_3, \ldots, w_{r_1} = v$ be a shortest uv-walk in G and let, for $1 \leq j \leq r_1$, w'_j be the l-th coordinate of w_j.

If $1 \leq t_1 < t_2 < \cdots < t_{r_2} < r_1$ are all those numbers t for which $w'_t \neq w'_{t+1}$, then $w'_{t_j} w'_{t_j+1}$ is an edge in G' for all $j, 1 \leq j \leq r_2$.

Moreover,

$$u_l = w'_1 = w'_2 = \cdots = w'_{t_1} \neq w'_{t_1+1} = w'_{t_1+2} = \cdots = w'_{t_2} \neq w'_{t_2+1} = w'_{t_2+2} = \cdots$$

$$\cdots = w'_{t_3} \neq w'_{t_3+1} = w'_{t_3+2} = \cdots = w'_{t_{r_2}} \neq w'_{t_{r_2}+1} = w'_{t_{r_2}+2} = \cdots = w'_{r_1} = v_l.$$

Thus,

$$u_l = w'_{t_1}, w'_{t_2}, \ldots, w'_{t_{r_2}}, w'_{r_1} = v_l$$

is a walk of length r_2 in G'.

Since $n_l \neq m_l$ and each $f_l(z)$ is either of the form $(f'_l(b), 1)$ for some $b \in H'_1 \times H'_2 \times \ldots \times H'_{l-1}$ (if $l > 1$) or of the form $(\pm 1, 1)$ (if $l = 1$), r_2 must be odd. Hence, if $w'_j = (c_j, o_j)$ for $1 \leq j \leq r_1$, then $c_1 = c_{r_1} = b_l$ and, thus, $b_l = c_{t_1}, c_{t_2}, \ldots, c_{t_{r_2}}, c_{r_1} = b_l$ is a closed walk of length r_2 in G'', where $G'' = X(H'_l, \{(f'_l)^{\pm 1}(z) : z \in H'_1 \times H'_2 \times \ldots \times H'_{l-1}\})$ if $l > 1$ and $G'' = X(H'_1, \{1, 2n\})$ if $l = 1$.

Then (as r_2 is odd) G'' must contain a cycle of length $r_3, r_3 \leq r_2$. But by the choice of f'_l in Construction 27, the girth of G'' is strictly greater than n when $l > 1$ while when $l = 1$, then G'' is a $(2n+1)$-cycle and, again, girth$(G'') > n$. Thus $n < r_3$, and we get

$$n < r_3 \le r_2 \le r_1 - 1 = d(u,v).\blacksquare$$

Now we are able to prove the main result.

Theorem 33 *The graph $G = G_{k,n}$ is k-extendable and has girth at least n, for every k and n.*

Proof: Obviously $s > k \ge 0$, $G = G_1 \cup G_2 \cup \ldots \cup G_s$, and $V(G_i) = V(G) \ne \emptyset$ for all i. This, together with Claims 28,29,30 and 31, shows that the assumptions of Proposition 17 are satisfied and, hence, G is k-extendable.

For the proof of the girth, we will first show that for $2 \le i \le s$

$$\mathrm{girth}(X(H_i, \{f_i^{\pm 1}(v) : v \in H_1 \times H_2 \times \ldots \times H_{i-1}\})) > n. \tag{5.2}$$

Suppose not and let $C = (b_0, m_0), (b_1, m_1), (b_2, m_2), \ldots, (b_{r-1}, m_{r-1}), (b_0, m_0)$ be a cycle of length $r \le n$ in $X(H_i, \{f_i^{\pm 1}(v) : v \in H_1 \times H_2 \times \ldots \times H_{i-1})$ (note that $r \ge 3$ because G is a simple graph). Then $b_0, b_1, b_2, \ldots, b_{r-1}, b_0$ is a closed walk of length r in $X(H_i', \{(f_i')^{\pm 1}(v) : v \in H_1' \times H_2' \times \ldots \times H_{i-1}'\})$. But $\mathrm{girth}(X(H_i', \{(f_i')^{\pm 1}(v) : v \in H_1' \times H_2' \times \ldots \times H_{i-1}'\})) > n$. Thus, there is $j, 0 \le j \le r-1$, such that $b_j = b_{j+2}$ (indices are taken mod r). Then if $h' \in H_i'$ is such that $b_{j+1} = b_j h'$, then $b_{j+2} = b_j (h')^{-1}$. From the construction of G, it follows that $(b_{j+1}, m_{j+1}) = (b_j, m_j)h$ and $(b_{j+2}, m_{j+2}) = (b_{j+1}, m_{j+1})h^{-1}$, where $h = (h', 1) \in H_i$. In particular, $(b_j, m_j) = (b_{j+2}, m_{j+2})$, contradicting the fact that C is a cycle having length at least 3.

$$\text{Also, the order of } f_1 \text{ in } H_1 \text{ is } 4n+2 \text{ and is strictly greater than } n. \tag{5.3}$$

Claim 32 implies that condition b) of Corollary 25 is satisfied (with n instead of g). (5.2) and (5.3) show that two of the remaining conditions of Corollary 25 are also met (again with n in place of g). All other conditions of Corollary 25 are satisfied trivially. Thus, $\mathrm{girth}(G) \ge n$.$\blacksquare$

Remark In Chapter 2, we mentioned some known results concerning n-extendable graphs. Two of them, Theorems 2 and 7, give conditions that imply n-extendability. Here we note that neither of these theorems can be applied to prove that the graphs we constructed are n-extendable. This is immediately seen in the case of Theorem 2 as $\delta(G_{k,n}) \ll |V(G_{k,n})|/2$. For Theorem 7, we note that if G is a bipartite graph, then tough$(G) \leq 1$. This is seen as follows. Let G be a bipartite graph with parts S_1 and S_2 and assume $|S_1| \geq |S_2|$. Then $c(G \setminus S_2) = |S_1|$ and we get

$$\frac{|S_2|}{c(G \setminus S_2)} = \frac{|S_2|}{|S_1|} \leq 1. \blacksquare$$

We conclude this chapter with a statement regarding the structure of graphs from Construction 27.

Observation 34 *The graphs $G_{k,n}$ are bipartite.*

Proof: Partition the set $V(G_{k,n})$ into two subsets V_0 and V_1, where

$$V_i = \{((b_1, n_1), (b_2, n_2), \ldots, (b_s, n_s)) : n_1 + n_2 + \cdots + n_s \equiv i \pmod 2\}.$$

From the definition of the edges of $G_{k,n}$, it is readily seen that every edge in $G_{k,n}$ connects a vertex of V_0 with a vertex of V_1. $\blacksquare$

Chapter 6

Order Estimate

The number of vertices of a graph G is called the *order* of G. We will prove the following

Theorem 35 *The graphs $G_{k,n}$ from Construction 27 can be constructed such that their orders do not exceed $(21n)^{(k+1)(13n)^k}$.*

Before we can prove this result we need some lemmas.

Lemma 36 *Given positive integers n and k, the group $H_{n,k}$ can be constructed with order not exceeding $256(4k)^{12n}$.*

Proof: Since $H_{n,k} = SL_2(Z_p)$ for some prime p and since $|SL_2(Z_p)| \leq p^4$, it suffices to show that in Lemma 26 p can be chosen so that $p \leq 4(4k)^{3n}$. To see this, let s be as in that lemma. That is, $s > 0$ is large enough so that one can choose $k+1$ distinct pairs $(m_i, q_i), 1 \leq i \leq k+1$, with $\gcd(m_i, q_i) = 1, 0 \leq m_i \leq s/2, 0 \leq q_i \leq s/2$. As we can set $m_i = 1$ for $1 \leq i \leq k+1$ and $q_i = i$ for $1 \leq i \leq k+1$, we see that any $s \geq 2k+2$ will suffice. As $k > 0$, we have $4k \geq 2k+2$ and we conclude that s can be any number satisfying $s \geq 4k$. So we set $s = 4k$. By Lemma 26, p can be any prime satisfying the following two requirements.

- $\frac{2}{3}\log_s(p/2) - 1 \geq n$,
- $p > s$.

If q is a number greater than or equal to $2(4k)^{3n}$, then we see that these two conditions are satisfied; for the first one we get

$$\frac{2}{3}\log_s(q/2) - 1 \geq \frac{2}{3}\log_s((4k)^{3n}) - 1 = \frac{2}{3}\log_{4k}((4k)^{3n}) - 1 = \frac{2}{3}.3n - 1 = 2n - 1 \geq n,$$

while the second condition is met trivially as $s = 4k$. Since for every positive number m there is a prime between m and $2m$, we know that there is a prime number p with

$$4(4k)^{3n} \geq p \geq 2(4k)^{3n}.$$

Putting $H_{n,k} = SL_2(Z_p)$ gives the desired result.■

From now on we asssume that $H_{n,k}$ are in fact chosen so that $|H_{n,k}| \leq 256(4k)^{12n}$.

Lemma 37 *Let n be a positive integer and let the groups H_i' be defined recursively as follows. H_1' is any group of order m_1, for some $m_1 > 0$. If the orders of $H_1', H_2', \dots, H_{i-1}'$ are $m_1, m_2, \dots, m_{i-1}$, we set $H_i' = H_{n,m_1 m_2 \dots m_{i-1}}$. Then*

$$m_i = |H_i'| \leq (256(4m_1)^{12n})^{(13n)^{i-2}}$$

for all $i > 1$.

Proof: First, we will recursively define numbers $n_i, i > 0$. For $i = 1$, we set $n_1 = m_1$ and for $i > 1$, we set $n_i = 256(4n_1 n_2 \dots n_{i-1})^{12n}$.

Now, we will prove by induction that $m_i \leq n_i$ for all $i > 0$. When $i = 1$, we have $m_1 = n_1$. So assume $i > 1$ and $m_j \leq n_j$ for all $j, 1 \leq j < i$. Then by Lemma 36

$$m_i = |H_i'| = |H_{n,m_1 m_2 \dots m_{i-1}}| \leq 256(4m_1 m_2 \dots m_{i-1})^{12n}.$$

Therefore, using the induction hypothesis, we get

$$m_i \leq 256(4n_1 n_2 \dots n_{i-1})^{12n} = n_i.$$

Thus, it suffices to show that for $i > 1$, $n_i \leq (256(4m_1)^{12n})^{(13n)^{i-2}}$. We will again show this by induction.

When $i = 2$, we have

$$n_2 = 256(4n_1)^{12n} = 256(4m_1)^{12n} = (256(4m_1)^{12n})^{(13n)^{2-2}}.$$

For $i > 2$, we get

$$n_i = 256(4n_1n_2\ldots n_{i-1})^{12n} = 256(4n_1n_2\ldots n_{i-2})^{12n}(n_{i-1})^{12n}.$$

But $256(4n_1n_2\ldots n_{i-2})^{12n} = n_{i-1}$ by the definition, and we get

$$n_i = n_{i-1}(n_{i-1})^{12n} = (n_{i-1})^{12n+1} \leq (n_{i-1})^{13n} \leq$$

$$\leq ((256(4m_1)^{12n})^{(13n)^{i-3}})^{13n} = (256(4m_1)^{12n})^{13n^{i-2}}. \blacksquare$$

Proof of Theorem 35: In the construction of the graphs $G_{k,n}$, the groups $H_i', 1 \leq i \leq s$, are as in Lemma 37 with $m_1 = |H_1'| = |Z_{2n+1}| = 2n+1$. Then, by the conclusion of that lemma,

$$m_i \leq (256(8n+4)^{12n})^{(13n)^{i-2}} \leq (256(12n)^{12n})^{(13n)^{s-2}}$$

for all $i, i > 1$. For $i = 1$ the last inequality is satisfied trivially. Then, for each $i, 1 \leq i \leq s$, we have

$$|H_i| = |H_i' \times Z_2| = |H_i'| \cdot 2 \leq 2(256(12n)^{12n})^{(13n)^{s-2}}.$$

As $V(G_{k,n}) = H_1 \times H_2 \times \ldots \times H_s$, we obtain

$$|V(G_{k,n})| \leq (2(256(12n)^{12n})^{(13n)^{s-2}})^s = 2^s(256(12n)^{12n})^{s(13n)^{s-2}}.$$

Since s can be any number greater than k, we may choose $s = k+1$ to get

$$|V(G_{k,n})| \leq 2^{k+1}(256(12n)^{12n})^{(k+1).(13n)^{k-1}} \leq (512(12n)^{12n})^{(k+1).(13n)^{k-1}} \leq$$

$$\leq ((1.75)^{12n}(12n)^{12n})^{(k+1).(13n)^{k-1}} = (21n)^{12n(k+1).(13n)^{k-1}} \leq (21n)^{(k+1)(13n)(13n)^{k-1}} \leq$$

$$\leq (21n)^{(k+1)(13n)^k}. \blacksquare$$

Remark The bound given in Theorem 35 is much greater than the order of a smallest graph with minimum degree k and girth n, which is known to be of order $(k-1)^{n/2}$ (for the upper bound, see, for example, Theorem 1, p. 68 in [1]). Therefore, it might be interesting to ask how close one can get to this bound with graphs that are, in addition, k-extendable.

Chapter 7

Surfaces and Extendability

Some observations relating the extendability of graphs with their imbeddability are introduced in this chapter.

Lemma 38 *Let Σ be a surface. Then there exists a number k such that every graph cellularly imbeddable in Σ and having girth at least k fails to be 2-extendable.*

Proof: Suppose $\hat{G}$ is a cellular imbedding of G in Σ and let $f_i, i > 0$, be as in Theorem 8. Since $\text{girth}(G) \geq k$, we get $f_i = 0$ for $1 \leq i \leq k-1$ and, therefore, we obtain

$$f_k + f_{k+1} + f_{k+2} + \cdots = |F(\hat{G})|$$

and

$$kf_k + (k+1)f_{k+1} + (k+2)f_{k+2} + \cdots = 2|E(G)|,$$

implying

$$k(f_k + f_{k+1} + f_{k+2} + \cdots) \leq 2|E(G)|$$

and

$$k|F(\hat{G})| \leq 2|E(G)|. \tag{7.1}$$

Suppose further that G is 2-extendable. By Theorem 5, G is 3-connected. Hence $\delta(G) \geq 3$. But $|V(G)|\delta(G) \leq 2|E(G)|$, implying

$$3|V(G)| \leq 2|E(G)|. \tag{7.2}$$

Now, consider the Euler formula $|V(G)| - |E(G)| + |F(\hat{G})| = \chi(\Sigma)$. Substituting $|V(G)| \leq \frac{2}{3}|E(G)|$ of (7.2) into this formula yields

$$\frac{2}{3}|E(G)| - |E(G)| + |F(\hat{G})| \geq \chi(\Sigma)$$

and so

$$|F(\hat{G})| - \chi(\Sigma) \geq \frac{1}{3}|E(G)|.$$

Substituting $|E(G)| \geq \frac{k}{2}|F(\hat{G})|$ from (7.1) gives

$$|F(\hat{G})| - \chi(\Sigma) \geq \frac{k}{6}|F(\hat{G})|$$

and so

$$\chi(\Sigma) \leq \frac{6-k}{6}|F(\hat{G})|.$$

Now, if $k > 6$, then $(6-k)/6 < 0$ and $\frac{6-k}{6}|F(\hat{G})| \leq (6-k)/6$. Therefore, for $k > 6$, $\chi(\Sigma) \leq (6-k)/6$, and so $k \leq 6 - 6\chi(\Sigma)$. Thus, for G to be 2-extendable, k has to satisfy $k \leq \max\{6, 6 - 6\chi(\Sigma)\}$. The conclusion follows.■

For a surface Σ, let $k(\Sigma)$ denote the smallest number k satisfying the conclusion of Lemma 38. Construction 27 shows that for every $k > 0$ there is a 2-extendable graph $G_{2,k}$ with girth at least k. Since there exists a surface Σ such that $G_{2,k}$ cellularly imbeds in Σ, we see that $k(\Sigma)$ is an unbounded function. It might be of interest to try to determine this function. In general, this problem appears to be difficult. We show here the value of the function $k(\Sigma)$ for the simplest surface — the sphere. We start with an auxiliary observation.

Observation 39 *The regular dodecahedron is 2-extendable.*

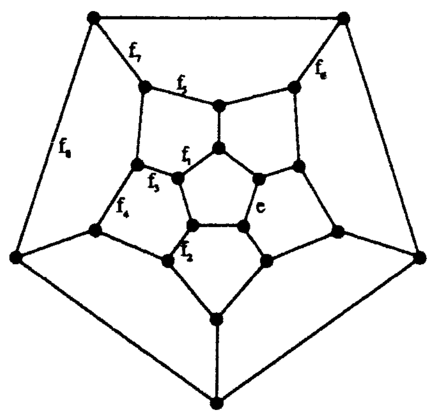

Figure 7.1.

Proof: Certainly, the regular dodecahedron has more than $2.2+2=6$ vertices. There are eight essentially different matchings of size 2 in this graph; they are the matchings $M_i = \{e, f_i\}, 1 \leq i \leq 8$ depicted in Figure 7.1. Figure 7.2 shows that all these matchings can be extended to perfect matchings of the regular dodecahedron (the perfect matching depicted on the lefthand side of this picture extends the matchings M_1, M_2, M_4, M_5, M_6 and M_8 while the perfect matching on the righthand side is an extension of M_3 and M_7 (and, also, of M_1, M_2 and M_6).■

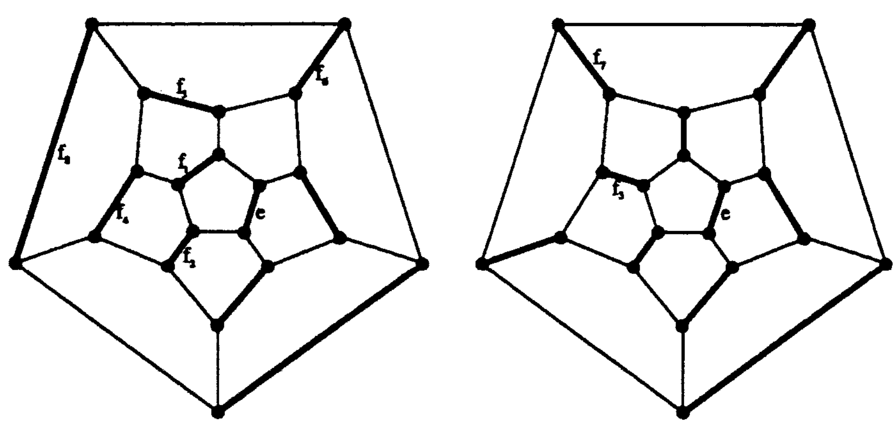

Figure 7.2.

Observation 40 $k(S_0) = 6$.

Proof: From the previous observation we know that the regular dodecahedron, which imbeds cellularly in the sphere, is 2-extendable. Since its girth is 5, we have $k(S_0) \geq 6$. On the other hand, as in the previous lemma, we get $\chi(S_0) \leq \frac{6-k}{6}|F(\hat{G})|$ if G is cellularly imbeddable in S_0 and has girth at least k. Thus, if $k \geq 6$, we have (as $\chi(S_0) = 2$) $2 \leq 0$, a contradiction. Hence, $k \leq 5$ and, therefore, $k(S_0) \leq 6$. We conclude that in fact $k(S_0) = 6$.■

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