CP Violation, Anomalous Discrete Symmetry and Dynamical Fermion Mass

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
in the Department
of
Physics

SIMON FRASER UNIVERSITY
August 10, 1993

APPROVAL

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Degree: Doctor of Philosophy

Title of thesis: CP Violation, Anomalous Discrete Symmetry and Dynamical Fermion Mass

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CP Violation, Anomalous Discrete Symmetry

and Dynamical Fermion Mass

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Abstract

This thesis has its origin in a series of theoretical investigations on the aspects of the standard model in particle physics. It summarizes some relatively independent but intrinsically related research work that I have done in the past four years. Diverse topics are presented and used to prove the relevance of certain theoretical models to the particle phenomena in nature.

I investigate a measure of CP violation in strong interactions. In the presence of nontrivial topological gauge configurations, the θ-term in QCD has a profound effect: it breaks the CP symmetry. The CP-violating amplitude is shown to be determined by the vacuum tunneling process, where the semiclassical method makes most sense. I discuss the important issue of whether or not the instanton dynamics satisfies the anomalous Ward identity (AWI). The strong CP violation measure, when complying with the vacuum alignment, is proportional to the topological susceptibility. To solve the IR divergence problem of the instanton computation, I present a 'classically gauged' Georgi-Manohar model and derive an effective potential which uniquely determines an explicit $U(1)_A$ symmetry breaking sector. The CP violation effects are analyzed in this model. It is shown that the strong CP problem and the $U(1)$ problem are closely related. Some possible solutions to both problems are discussed.

I examine an interesting scenario to solve the cosmological domain wall problem from the viewpoint of particle physics. The effective potential for Higgs fields is calculated in the presence of the QCD axial anomaly. It is shown that some discrete symmetries such as CP and Z_2 can be anomalous due to a so-called K-term induced by instantons. It is pointed out that Z_2 domain-wall problem in the two-doublet standard model can be resolved by two types of solutions: the CP-conserving one and the CP-breaking one. In the first case, there exist two Z_2 -related local minima whose energy splitting is provided by the instanton effect. In the second case, there is only one unique vacuum so that the domain walls do not form at all. The consequences of this new source of CP violation are discussed.

I study the behavior of the self-mass for a quark with a current mass larger than Λ_{QCD}, as a function of its Euclidean momentum and mass, in QCD. An expression for the Bethe-Salpeter kernel of the Schwinger-Dyson (SD) equation valid in both the infrared and ultraviolet regions is obtained based on a renormalization group analysis. The resulting SD equation is solved numerically. It is found that the quark constituent mass at zero momentum is substantially enhanced due to its effective gauge interaction. The solution in the ultraviolet region agrees well with the known asymptotic solution. The self-mass scales exactly as the on-shell current mass at a fixed momentum.

A dynamical mechanism that may yield a natural chiral symmetry for fermions is presented. The small fermion mass generated by various dynamical interactions is obtained.

Acknowledgments

I wish to specially thank my senior supervisor, Prof. K. S. Viswanathan, who guided my steps in this thesis research. I have enjoyed his fruitful way of supervision and his pleasant partnership ever since I came to Canada from China four years ago. His great efforts to expose me to wide area of physics helped me understand some fundamental research topics. I thank him for his extreme accessibility and genuine willingness in collaboration. Invariably for years, I had a privilege to drop in his office every day, getting together with him for a few hours, discussing physics with grace and freedom. I also greatly benefited from his insights on planning my future. Without Prof. Viswanathan, obviously this work would never have been done.

I remain grateful to Prof. Richard Woloshyn for his generous support in the final stages of my graduate studies. He made many useful suggestions on topics of my research and encouraged me at times, especially when there was a difficulty. Prof. Woloshyn is a very busy person. I owe troubles to him every time when I intruded on his schedules, remembering his genial words 'I have never been too busy to talk to you'.

Prof. Dandi Wu, one of my instructors and collaborators, on many occasions, straightened me out with his broad phenomenological knowledge. He was actively involved in the early stages of my thesis research, when the strong CP problem was our common concern. Although this topic was not, unfortunately, pursued further and is not included in the thesis, the working experience with him was enlightening and most exciting.

I would like to gratefully acknowledge Prof. Roberto Peccei for many conversations and written communications we had in the past. I sincerely appreciate his opinion on many issues of particle physics, and on my work, sometimes in the form of constructive criticism.

Last, but not least, I would like to thank my wife, Elizabeth, for her enduring patience and devoted support, who should by rights be worthy of dedication of this thesis.

To all goes my heartfelt gratitude.

Dedication

To my wife and my parents

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Chapter 1

Introduction

1.1 Motivations

There are two obvious trends in today's theoretical research in particle physics. One is to explore the fundamental force for all kinds of interactions: strong interactions, electromagnetic interactions, weak interactions and gravity. The ultimate goal is the establishment of the unification of these four interactions. Though tremendous efforts have been made along this direction, it seems that we are still far away from a complete understanding of the problem, let alone a solution to it. A direct extension of an idea that led to the unification of the electromagnetic and weak interactions does not appear to be as successful when strong and electroweak interactions are unified. A solution based on the superstring theory has not been fully satisfactory.

The other direction is less ambitious but has much richer phenomenological contents. It studies various aspects of the standard model which has been extremely successful in the past couple of decades in describing particle phenomena at relevant energy scales or temperatures. The standard model is based on the gauge symmetry group $SU(3) \times SU(2) \times U(1)$ and consists of quantum chromodynamics (QCD) and the Glashow-Salam-Weinberg electroweak theory, which are a theoretical syntheses of our understanding of the particle spectroscopy over many years. The foundation of the standard model is the gauge invariance principle and that the gauge symmetry may be hidden in the manner of Higgs mechanism. Theoretical predictions based on this model have been tested and so far there has been no single definite experiment that betrays it. The extreme success of the standard model indicates that a theoretical framework of strong and electroweak interactions has been well established, a point of view shared by most physicists today. Therefore, it is clearly desirable to explore theoretical aspects of the standard model beyond the tree-level, mean field approximation, in order to test the validity of the standard model in many domains.

The thesis is devoted to studies of such a task, in particular, to investigating the various quantum effects in gauge field theory. Historically, the prototype of an underlying theory may have been built from observing essential features of relevant phenomena, often from the existence of manifest symmetries. Parameters such as coupling constants, masses, mixing angles and phases are put in by hand to fit experimental data, and the model stays as a classical theory and descriptive. However, some experiments may probe purely quantum phenomena and cannot be accounted for in the classical theory. In addition, the quantum effects of a theory often not only make quantitative corrections to a certain process but predict phenomena that cannot been obtained from a classical theory. Therefore, the verification of a mature theory, applied to the standard model, requires a full study of physical phenomena including those that caused by the quantum effects.

Among these effects, what seems most interesting is symmetry breaking. There are many symmetries, either exact or approximate, in the standard model at the classical level: gauge symmetries, global symmetries, discrete symmetries and accidental symmetries. Some appear broken in nature. It is hence desirable to examine if they can be broken by quantum effects and what consequences the symmetry breaking may produce. The standard way to carry out this task is to consider the generating functional for a given theory, instead of the lagrangian which only gives a classical description on the theory. A complete evaluation of the path integral

$$Z[J] = \int [\mathrm{d}\phi]\, \mathrm{e}^{i\int d^4x[\mathcal{L}(\phi)+J\phi]} \tag{1.1}$$

gives both the classical and the quantum considerations, where ϕ is the general field excitation, $\mathcal{L}(\phi)$ is the lagrangian and J is the external source. However, an exact integration of (1.1) would be extremely difficult if not impossible.

In this thesis, I shall discuss various quantum effects associated with fundamental fermions which have non-abelian gauge interactions. These fermions are referred to as quarks in most chapters. One needs to consider an integration in (1.1) over fermions. It turns out that even this is not so trivial, and approximate methods have to be employed. Often encountered approximations are the one-loop approximation, the semi-classical approximation and the ladder approximation. It is found that these gauge interactions indeed *significantly* change the symmetry structure of the theory and the qualitative behavior of the theory in certain domains.

1.2 Outline of Topics

In Chapter 2, I study CP symmetry in strong interactions. In a gauge theory, a mysterious term called θ-term

$$\theta F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{1.2}$$

where $F_{\mu\nu}$ is the gauge field strength tensor and $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, can be added into the lagrangian since it also satisfies gauge invariance. Although this term is formally P- and CP-odd, it does not cause any P or CP violating consequences at the classical level since it is a total divergence which yields a surface term in the classical action. However, in a non-abelian gauge theory, the θ-term breaks CP symmetry explicitly in the presence of a special type of quantum effect, the instanton effect. The CP-violating amplitude is shown to be determined by the quantum tunneling between two topologically different vacua. I also discuss a long-standing puzzle on the conventional way to estimate the strong CP violation by use of current algebra, which does not by itself exhibit the topological feature of this effect. The anomalous Ward identity does not, as claimed, put a constraint on how the measure of strong CP violation should behave. I show, however, that it is unambiguously proportional to the topological susceptibility when a vacuum alignment is done properly. An effective chiral model of the constituent quarks is proposed to study the relation between the measure of strong CP violation and the $U(1)$ problem. CP violating processes are calculated in the effective theory.

In Chapter 3, a Z_2 discrete symmetry in the electroweak sector of the two-Higgs standard model is discussed. It has been long realized that spontaneously broken discrete symmetry, which may be an attractive possibility in building models of particle physics, can lead to a grave domain wall problem in the context of cosmology. When several degenerate ground states are approached homogeneously in an expanding universe, adjacent domains filled by different vacua are separated by domain walls. The energy density of these walls turns out to be so high that existence of just one in the observable part of universe leads to unacceptable cosmological consequences. We shall argue, however, that there is a quite natural and appealing way out in the context of particle physics: the discrete symmetry can be *anomalous.* In this case the ground states which appear, at the classical level and to all orders of perturbation theory, to be degenerate, are found instead to be separated by a finite energy difference when they are compared non-perturbatively in the quantized theory. The effective potential for Higgs bosons is calculated in the presence of the QCD axial anomaly. It is shown that some discrete symmetries such as CP and Z_2 can be *explicitly* broken by a so-called K-term induced by instantons. We also point out that the quantum effect may drive a spontaneous CP violation which would be impossible in the two-Higgs model with a Z_2 symmetry.

Chapter 4, however, investigates not so much on the symmetry issue, rather, the behavior of the quark mass in both the ultraviolet and infrared regions. Dynamical chiral symmetry breaking in QCD has been studied extensively and yet has not been completely understood. The difficulty to obtain an infrared solution to the Schwinger-Dyson (SD) equation arises when the validity of the ladder approximation is not justified in the infrared region. In this chapter, I study a slightly different aspect of the solution to the SD equation however, approaching to the same problem: the self-mass for a very massive quark. When the quark current mass is much larger than Λ_{QCD}, a renormalization group equation (RGE) allows us to derive an approximation to the integral kernel valid both in the UV and in the IR regions. A complete solution to the non-linear SD equation is then obtained by numerical means. We find that self-mass acquires a substantial enhancement due to gauge interactions when the current mass becomes small, especially when the external momentum tends to zero. It is then expected based on extrapolation, that the constituent mass for a light quark (defined as the self-mass at zero momentum) can be very large compared with its current mass. In addition, it gives a complete description of the self-mass for the very massive quarks such as the charm and the bottom quarks, and a picture of the transition going from the heavy quarks to the less massive quarks.

I shall show, in Chapter 5, a converse possibility that the quantum effects may restore the chiral symmetry or yield an approximate chiral invariance in certain models.

All in all, let us begin the journey.

Chapter 2

Measure of Strong CP Violation

2.1 Introduction to Strong CP

The discovery of instantons [2.0] has been associated with some of the most interesting developments in strong interaction theory. It has led to a resolution [2.1] of the long-standing $U(1)$ problem [2.2], and also pointed to the existence in QCD [2.3] of vacuum tunneling and of a vacuum angle θ, which combining with the phase of the determinant of the quark mass matrix, signals the CP violation in strong interactions. The difficulty in understanding the very different hierarchies of the strong CP violation and weak CP violation in the standard model has been targeted as the so-called strong CP problem (for a review, see Ref. [2.4]).

The theoretical understanding of weak CP violation is well-established in the framework of Kobayashi-Maskawa mechanism [2.5] in spite of the challenge in higher-precision experimental measurements. It has been shown [2.6] that the determinant of the commutator of the up-type and down-type quark mass matrices $[M^u, M^d] \equiv iC$

given by

$$\det C = -2\mathcal{J}_{\mathrm{weak}}(m_t - m_c)(m_c - m_u)(m_u - m_t)(m_b - m_s)(m_s - m_d)(m_d - m_b) \quad (2.1)$$

where

$$\mathcal{J}_{\mathrm{weak}} \equiv \sin^2\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_1 \cos\theta_2 \cos\theta_3 \sin\delta \quad (2.2)$$

is the unique measure of the weak CP violation. All CP-violating effects in weak interaction must be proportional to $\det C$. Even though the CP-violating phase $\sin\delta$ can be of order 1, the physical amplitude is naturally suppressed by the product of Carbibo mixing angles.

However, the measure of CP violation in QCD, which we shall denote as $\mathcal{J}_{\mathrm{strong}}$, is not so clear. It has long been realized that θ_{QCD} and phases of quark masses are not independent parameters in QCD. In the presence of the chiral anomaly [2.7], they are related through the chiral transformations of quark fields. Thus $\mathcal{J}_{\mathrm{strong}}$ must be proportional to a combination

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \arg\det M \quad (2.3)$$

which is invariant under chiral rotations. It is well-known that if one of quarks is massless, $\bar{\theta}$ can take any arbitrary value since one can make arbitrary rotations on the chiral field. This suggests that the $\bar{\theta}$- dependence of $\mathcal{J}_{\mathrm{strong}}$ disappears in the chiral limit. Thus in the case of $L = 2$ where L is the number of light quarks, $\mathcal{J}_{\mathrm{strong}}$ has a form

$$\mathcal{J}_{\mathrm{strong}} = m_u m_d K \sin\bar{\theta} \quad (2.4)$$

where we have written $\sin\bar{\theta}$ instead of $\bar{\theta}$ to take care of the periodicity in $\bar{\theta}$. Is there any other common factor that we can extract from strong CP effects? Or, is K in (2.4) only a kinematical factor which varies with different physical processes?

To answer the question, we need to know whether there is any other condition under which the strong CP violation vanishes. Recently, the reanalysis of strong CP effects has shed some light on this issue. Several authors [2.8] have pointed out by studying an effective lagrangian that the strong CP violation should vanish if the chiral anomaly is absent. We regard their work as constructive and enlightening. However, how to realize such a feature in QCD with quarks is not apparent in their approaches. In QCD theory, indeed, if the chiral anomaly is absent, the phases of quark masses can be rotated away without changing the θ-term. But it is not clear why $\theta_{\rm QCD}$ itself does not lead to CP violation in strong interactions. In addition, the presence of the chiral anomaly in a gauge theory may not be directly related to CP violation. One example is QED. It is well-known that QED is a CP-conserving theory even if it is chirally anomalous, and, in principle, could have a θ-term and a complex electron mass term.

In this chapter, however, we show that the measure of strong CP violation does acquire a factor referred to as the measure of the non-triviality of the non-abelian gauge vacuum. It is simply due to the fact that the θ-term is a total divergence whose integration over space-time yields a surface term. It can be dropped unless there are non-trivial gauge configurations at the boundary. K in (2.4) will be shown to be the vacuum tunneling amplitude between different vacua characterized by the winding numbers

$$\nu = \int d^4x F\tilde{F} \equiv \frac{g^2}{32\pi^2}\int d^4x F_{\mu\nu}\tilde{F_{\mu\nu}} \tag{2.5}$$

where a semiclassical method makes most sense to deal with it. To probe the property of the K-factor, we proceed to consider a classically gauged linear σ-model. A derivation of a $U(1)_A$ sector of the model can be made by taking into account the fermion zero modes in the instanton fields. Contrary to the conventional result [2.9, 2.11] where K has a singularity in quark masses such that $\mathcal{J}_{\mathrm{strong}}$ is a linear function of the quark mass, our model clearly shows that K is to be explained as the mass difference between the $U(1)$ particle and pions. Thus, $\mathcal{J}_{\mathrm{strong}}$ has a form

$$\mathcal{J}_{\mathrm{strong}} = m_u m_d (m_\eta^2 - m_\pi^2) \sin \bar{\theta}. \tag{2.6}$$

In the context of the effective model, the strong CP effects can be explicitly calculated and various solutions to the strong CP problem will be discussed with new insights.

This chapter is organized as follows. In sect. 2.2, we discuss a long-standing problem raised by Crewther [2.9, 2.10] on whether or not the instanton is consistent with the anomalous Ward identity (AWI). We find that the AWI does not put any constraint on the topological susceptibility $\langle\langle \nu^2 \rangle\rangle$ in QCD. The AWI is automatically satisfied by instanton dynamics if the singularity in the chiral limit of some fermionic operator is taken care of. Sect. 2.3 deals with an instanton computation of $\langle\langle \nu^2 \rangle\rangle$ in the dilute gas approximation. The vacuum alignment equations of the quark condensates are derived based on the path integral formalism. Upon making alignment among strong CP phases, we rederive an effective CP-violating lagrangian. In sect. 2.4 we present a classically gauged linear σ-model. In the semiclassical approximation, the instanton fields are integrated out. An effective one-loop potential is obtained by integrating over fermions in the instanton background where the fermion zero modes are essential to yield an explicit $U(1)_A$ symmetry breaking. The strong CP effects and the $U(1)$ particle mass are calculated in the model. Sect. 2.5 devotes to discussions on various possible solutions to the strong CP problem. Sect. 2.6 is reserved for conclusions.

2.2 Does Instanton Satisfy the AWI?

Let us leave our discussion on $\mathcal{J}_{\rm strong}$ aside for the moment and turn to a problem which turns out to be key to understanding both strong CP violation and the $U(1)$ problem. It is pointed out long ago that instanton physics, in some ways, suffers from difficulties. It is well-known that the integration over the instanton size is infrared divergent. It is further argued by Witten [2.12] that the semiclassical method based on the instanton solution of Yang-Mills equation is in conflict with the most successful idea of $\frac{1}{N_c}$ expansion in QCD. The reason is that instanton effects are of order $e^{-\frac{1}{g^2}}$ or e^{-N_c}, for g^2 is of order $\frac{1}{N_c}$ in the large N_c limit, which is smaller than any finite power of $\frac{1}{N_c}$ obtained by summing Feynman diagrams. These problems, as they stand now, indeed reflect various defects in the instanton calculation (we will come back to these points in later sections).

However, there was another type of objection initiated by Crewther [2.9] followed by others [2.10], which would be even more serious if it were correct. For many years Crewther has emphasized that the breakdown of $U(1)_A$ symmetry by the chiral anomaly and the instanton is related to the breakdown of the $SU(L) \times SU(L)$ symmetry. This relation is represented by the so-called anomalous Ward identity. He claimed that the instanton dynamics failed to satisfy the AWI and one would still expect the unwanted $U(1)_A$ goldstone boson. It was further shown that the topological susceptibility defined as

$$\langle\langle \nu^2 \rangle\rangle = \int d^4x \langle T\, iF\tilde{F}(x)\, iF\tilde{F}(0)\rangle \tag{2.7}$$

when the AWI is satisfied, must be equal to $m\langle\bar{\psi}\psi\rangle$ (m is the quark mass. We have assumed that all quarks have equal masses). As we shall see in sect. 3, $\langle\langle \nu^2 \rangle\rangle$ is to be identified as the measure of strong CP violation. If Crewther is right, it would seem that the strong CP has no direct relation with the topological vacuum structure.

To see where the problem lies, we carefully follow a path integral derivation of the AWI. Consider a fermion bilinear operator $\bar{\psi}_L\psi_R$ with chirality +2 (a sum over flavor indices is understood). Its vacuum expectation value (VEV) is formally given

$$\begin{aligned}\langle\overline{\psi}_L\psi_R\rangle &= \frac{1}{V}\langle\int d^4x\overline{\psi}_L\psi_R(x)\rangle \\ &= \frac{1}{V}\frac{1}{Z}\int \mathcal{D}(A,\bar{\psi},\psi)\int d^4x\overline{\psi}_L\psi_R(x)e^{-S[A,\bar{\psi},\psi]}\end{aligned} \tag{2.8}$$

where the QCD action in Euclidean space is

$$S[A,\bar{\psi},\psi] = \int d^4x\bar{\psi}\,\not{D}\psi + m\bar{\psi}\psi + \tfrac{1}{4}F^2 - i\theta F\tilde{F} \tag{2.9}$$

and Z is the normalization factor, V is the volume of space-time. Under an infinitesimal $U(1)_A$ transformation

$$\psi_R \rightarrow e^{i\alpha(x)}\psi_R \quad ; \quad \psi_L \rightarrow e^{-i\alpha(x)}\psi_L \tag{2.10}$$

the measure $\mathcal{D}(A,\bar{\psi},\psi)$ will change because of the chiral anomaly. However, the integral (2.8) will not change since (2.10) is only a matter of changing integration variables. (2.8) then becomes

$$\begin{aligned}\langle\bar{\psi}_L\psi_R\rangle &= \frac{1}{VZ}\int \mathcal{D}(A,\bar{\psi},\psi)\int d^4x e^{2i\alpha(x)}\bar{\psi}_L\psi_R(x)\exp\{-S[A,\bar{\psi},\psi] + \\ & \quad i\alpha(x)\int d^4x[\partial_\mu J^5_\mu - 2m\bar{\psi}\gamma_5\psi - 2L\,F\tilde{F}]\}\end{aligned} \tag{2.11}$$

where the $U(1)_A$ current is $J^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$ and L is the number of light quarks. The independence of $\langle\bar{\psi}_L\psi_R\rangle$ on $\alpha(x)$ implies the vanishing of the first derivative which yields the AWI

$$\begin{aligned}\int d^4x\partial_\mu\langle T\,J^5_\mu(x)\,\bar{\psi}_L\psi_R(0)\rangle = 2m\int d^4x\langle T\,\bar{\psi}i\gamma_5\psi(x)\,\bar{\psi}_L\psi_R(0)\rangle + \\ 2L\int d^4x\langle T\,iF\tilde{F}(x)\,\bar{\psi}_L\psi_R(0)\rangle - 2i\langle\bar{\psi}_L\psi_R\rangle.\end{aligned} \tag{2.12}$$

Crewther's arguments go as follows. If there is no $U(1)_A$ goldstone boson coupling to J^5_μ, the l.h.s. of Eq. (2.12) vanishes. In the chiral limit, the first term of the r.h.s. would vanish too. Thus one has when $m \to 0$

$$L \int d^4x \langle T\, F\tilde{F}(x)\, \bar{\psi}_L\psi_R(0)\rangle = \langle \bar{\psi}_L\psi_R\rangle. \tag{2.13}$$

The instanton dynamics assumes that the integration over the gauge field is separated into a sum over gauge configurations characterized by the integer winding number ν in (2.5), i. e. $\int [dA] = \sum_\nu \int [dA]_\nu$ and $\langle \bar{\psi}_L\psi_R\rangle = \sum_\nu \int \langle \bar{\psi}_L\psi_R\rangle_\nu$. Eq. (2.13) would then imply

$$(L\nu - 1)\langle \bar{\psi}_L\psi_R\rangle_\nu = 0. \tag{2.14}$$

By assuming the spontaneous chiral symmetry breaking caused by $\langle \bar{\psi}_L\psi_R\rangle \neq 0$, (2.14) cannot be satisfied if ν is an integer. Moreover, by noting that

$$\frac{d\langle \bar{\psi}_L\psi_R\rangle}{d\theta} = i \int d^4x \langle T\, F\tilde{F}(x)\, \bar{\psi}_L\psi_R(0)\rangle \tag{2.15}$$

one obtains

$$(-i\frac{d}{d\theta} - \frac{1}{L})\langle \bar{\psi}_L\psi_R\rangle = 0 \quad \Rightarrow \quad \langle \bar{\psi}_L\psi_R\rangle_\theta = \langle \bar{\psi}_L\psi_R\rangle_{\theta=0}\, e^{i\frac{\theta}{L}} \tag{2.16}$$

which is unacceptable because the θ-dependence of $\langle \bar{\psi}_L\psi_R\rangle$ would have a wrong periodicity $2\pi L$. Along the same line, one could derive the AWI for operator $\bar{\psi}_R\psi_L$ and $F\tilde{F}$ and combine them with (2.12) to obtain

$$\langle\langle \nu^2\rangle\rangle = \frac{m^2}{L^2} \int d^4x \langle T\, \bar{\psi} i\gamma_5\psi(x)\, \bar{\psi} i\gamma_5\psi(0) + \frac{m}{L^2}\langle \bar{\psi}\psi\rangle. \tag{2.17}$$

Assuming that the first term in the r.h.s of (2.17) is of order $O(m^2)$, one would conclude that $\langle\langle \nu^2\rangle\rangle$ was a linear function of m, which, again, contradicts with the instanton computation.

We argue, however, that all these inconsistencies arise from dropping the first term of the r.h.s.of (2.12) in the chiral limit or treating it as a higher order term. The $U(1)_A$ fermion operator $\bar{\psi}i\gamma_5\psi$, when the fermion fields are integrated out *first* as they should be, may observe a $\frac{1}{m}$ singularity in certain gauge configurations. To see this, we first calculate the VEV of $\bar{\psi}i\gamma_5\psi$ in a fixed background field A_μ. Upon the fermion integration, one has

$$\langle\bar{\psi}i\gamma_5\psi\rangle^A = Tr\frac{i\gamma_5}{\not{D}(A)+m} = \frac{1}{m}T(m^2) \tag{2.18}$$

where

$$T(m^2) = Tr\frac{i\gamma_5 m^2}{-\not{D}^2+m^2} = Tr\frac{i\gamma_5 m^2}{-D^2+\frac{1}{2}g\sigma_{\mu\nu}F_{\mu\nu}+m^2}. \tag{2.19}$$

It is easy to check that $\frac{d}{dm^2}T(m^2) \equiv 0$, i. e. $T(m^2)$ is independent of m^2. Thus it can be calculated in the limit $m^2 \to \infty$ [2.19]

$$\begin{aligned}\lim_{m^2\to\infty} T(m^2) &= -iL\int d^4x\; tr\gamma_5(\tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu})^2\int\frac{d^4p}{(2\pi)^4}\frac{m^2}{(p^2+m^2)^3}\\ &= iLF\tilde{F}\end{aligned} \tag{2.20}$$

and therefore

$$\langle\bar{\psi}i\gamma_5\psi\rangle^A = -iL\frac{F\tilde{F}}{m}. \tag{2.21}$$

It has a pole at $m=0$. It is clear that $m\langle\bar{\psi}i\gamma_5\psi\rangle^A$ may be finite in the limit $m \to 0$ if $F\tilde{F}$ is nontrivial. Performing the fermion integration for the first term of r.h.s. of (2.12), we obtain

$$\begin{aligned}&m\int d^4x\langle T\,\bar{\psi}i\gamma_5\psi(x)\,\bar{\psi}_L\psi_R(0)\rangle\\ &=\int d^4x\langle T\,Tr\left(\frac{im\gamma_5}{\not{D}+m}\right)(x)\,Tr\left(\frac{1+\gamma_5}{2(\not{D}+m)}\right)(0)\rangle\end{aligned}$$

$$-\langle Tr\left(\frac{im\gamma_5}{\not{D}+m}\frac{1+\gamma_5}{2}\frac{1}{\not{D}+m}\right)\rangle \tag{2.22}$$

$$= -L\int d^4x\langle T\, iF\tilde{F}(x)\; Tr\left(\frac{1+\gamma_5}{2}\frac{1}{\not{D}+m}\right)(0)\rangle$$
$$-i\langle Tr\left(\tfrac{1}{2}(1+\gamma_5)\frac{m}{-\not{D}^2+m^2}\right)\rangle. \tag{2.23}$$

Identifying the second term in (2.23) with $\langle\bar{\psi}_L\psi_R\rangle$, we find that the r. h. s. of (2.12) vanishes identically for any m. This is not surprising since if we had considered a *global* $U(1)_A$ transformation instead of a local one in (2.10) at the beginning, we would have come up with the same conclusion immediately. Similarly, (2.17) is an identity to be satisfied (trivially) by any dynamics which respects the basic rule of the fermion quantization (and of course the anomaly relation. If there were no anomaly, the second term of r.h.s. of (2.12) would be absent. The cancellation would be incomplete indicating the existence of a massless excitation coupling with J^5_μ. Thus the chiral anomaly is essential to solve the $U(1)$ problem.).

There is a delicate problem about taking the chiral limit. One may ask what if the quark mass term is simply absent in the lagrangian in the first place. Crewther's problem seems to come back if the first term of the r. h. s. of (2.12) is not present. Actually this is where the puzzle comes about. In this case, however, a nonvanishing value of the quark condensate is not well-defined. It relates to a general feature of the spontaneous symmetry breaking mechanism. For example, in the ϕ^4-theory with spontaneous breaking of the reflection symmetry ($\phi\rightarrow-\phi$), the VEV of ϕ is calculated

$$\langle\phi\rangle = \frac{1}{Z}\int d\phi\,\phi\, e^{-\int d^4x(\partial_\mu\phi)^2+\frac{\lambda}{4}(\phi^2-v^2)^2}. \tag{2.24}$$

Since the action is perfectly reflection-symmetric and ϕ is an odd operator under reflection, we have $\langle\phi\rangle\equiv 0$. Mathematically this is true because of the equal weight of the degenerate vacua. But what is of physical interest is a situation where one of the degenerate vacua is *chosen* as the ground state. The way to do it is to introduce a source term $\int d^4xJ\phi$ into the action which breaks the symmetry explicitly. The degeneracy of the vacua in the absence of the source implies that $\langle\phi\rangle_J$ is a multi-valued function of J at $J = 0$. The VEV's of ϕ crucially depends on the way that J tends to zero. In particular, $\langle\phi\rangle_{J\to 0^+} = -\langle\phi\rangle_{J\to 0^-} \neq 0$.

The same procedure should follow for the spontaneous chiral symmetry breaking in QCD. In order to define the quark condensate $\langle\bar\psi_L\psi_R\rangle$, one ought to add the source term $\int d^4xJ\bar\psi_L\psi_R(x)$ to the action. Then a $U(1)_A$ transformation changes the source term as well

$$\int d^4xJ\bar\psi_L\psi_R \rightarrow \int d^4xJe^{2i\alpha}\bar\psi_L\psi_R. \tag{2.25}$$

We also need to take this change into account because $\langle\bar\psi_L\psi_R\rangle$ defined by the way that $J\rightarrow 0$ would be different from the one defined by $Je^{2i\alpha}\rightarrow 0$. By differentiating $\langle\bar\psi_L\psi_R\rangle$ with respect to α we obtain an equation exactly the same as (2.12) except that m is replaced by J. For the same reason as we have discussed, the r. h. s. of the equation is identically zero for any value J (even in the limit $J\rightarrow 0$). There is no $U(1)_A$ goldstone boson, and, in general, (2.13), (2.14) and (2.16) do not hold.

We have shown that the AWI for the isosinglet current J^5_μ is trivially satisfied by QCD dynamics including the axial anomaly. (2.17) is an identity satisfied by any dynamics if the singularity of the singlet operator $\bar\psi i\gamma_5\psi$ in the zero mass limit is appropriately handled. It does not put any constraint on how the topological susceptibility $\langle\langle\nu^2\rangle\rangle$ should behave as a function of the quark mass. Thus, it does not, from the context of the field theory, rule out the instanton computation. However, this should not be confused with the case of the AWI's for non-singlet currents where the assumption on the lowest lying resonances has to be made. For a non-singlet axial current $J_\mu^a = \bar\psi\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi$ (λ^a's are generators of $SU(L)$, $a = 1, \cdots, L^2 - 1$), the corresponding AWI reads

$$m^2 \int d^4x \langle T\, \bar\psi i\gamma_5 \tfrac{\lambda_a}{2}\psi(x)\; \bar\psi i\gamma_5 \tfrac{\lambda_b}{2}\psi(0)\rangle - \delta^{ab}\frac{m}{L}\langle\bar\psi\psi\rangle = 0. \tag{2.26}$$

It can be readily checked by integrating the fermion fields that (2.26) is satisfied in QCD. Unlike the singlet current in (2.18)

$$\langle\bar\psi i\gamma_5 \tfrac{\lambda_a}{2}\psi\rangle^A = Tr\frac{\lambda_a}{2}\frac{i\gamma_5}{\not D + m} = 0 \tag{2.27}$$

because the λ^a's are traceless. *Assuming* that pions are lowest lying resonances which dominate the correlation function, one obtains

$$m^2 \int d^4x \langle T\bar\psi i\gamma_5 \frac{\lambda_a}{2}\psi(x)\; \bar\psi i\gamma_5 \frac{\lambda_b}{2}\psi(0)\rangle_{\text{res.}} = F_\pi^2 m_\pi^2 \delta^{ab} \tag{2.28}$$

leading to $F_\pi^2 m_\pi^2 = -\frac{1}{L} m\langle\bar\psi\psi\rangle$. Can we do the same analysis for the singlet operator

$$m^2 \int d^4x \langle T\, \bar\psi i\gamma_5\psi(x)\; \bar\psi i\gamma_5\psi(0)\rangle_{\text{res.}} = ? \tag{2.29}$$

such that we may get a phenomenological value for $\langle\langle\nu^2\rangle\rangle$ from (2.17) without resorting to instanton computations? It is, however, very difficult to do that. For the axial singlet operator, we cannot generally assume pion dominance. In fact, $m\bar\psi i\gamma_5\psi$ does not couple to pions because the λ^a's commute with the identity [2.11]. In addition, $\bar\psi i\gamma_5\psi$ has pole behavior at $m = 0$ whose residue is $F\tilde F$. It may couple to a gauge ghost [2.13] as well as glue balls and the $U(1)_A$ particle. It may also exhibit a non-zero subtraction constant in the spectral dispersion representation [2.14], which by itself is not surprising in the presence of anomaly. All these factors may overlap with each other, causing double counting. This has made an estimation of (2.29) extremely difficult if not impossible.

In summary, the AWI and the low energy phenomenology may not put a constraint on the topological susceptibility. Therefore, it leaves us a task of calculating $\langle\langle\nu^2\rangle\rangle$ and the measure of strong CP violation from instanton dynamics. To avoid the infrared divergence, we further relate $\langle\langle\nu^2\rangle\rangle$ to the $U(1)_A$ particle mass in an effective theory.

2.3 Effective CP-Violating Lagrangian in QCD

In Sect. 2.2 we have shown that the axial singlet operator $\bar{\psi}i\gamma_5\psi$ is related to $F\tilde{F}$ in a fixed gauge background. When the gauge fields are integrated out, (2.21) becomes a relation on VEV's. It can be easily proven that such a relation is true for each flavor. In general, when the quark mass is complex, one derives

$$\begin{aligned}
&-i(m_i e^{i\varphi_i}\langle\bar{\psi}^i_L\psi^i_R\rangle - m_i e^{-i\varphi_i}\langle\bar{\psi}^i_R\psi^i_L\rangle) \\
&= -i(m_i e^{i\varphi_i}\langle Tr\frac{1}{2}\frac{1+\gamma_5}{\not{D}+m_i e^{i\varphi_i\gamma_5}}\rangle - m_i e^{-i\varphi_i}\langle Tr\frac{1}{2}\frac{1-\gamma_5}{\not{D}+m_i e^{i\varphi_i\gamma_5}}\rangle) \\
&= \langle iF\tilde{F}\rangle
\end{aligned} \tag{2.30}$$

where φ_i is the phase of the ith quark mass ($i = 1, \cdots, L$), no sum over i is understood in (2.30). Now define

$$\langle\bar{\psi}^i_L\psi^i_R\rangle \equiv -\frac{C_i}{2}e^{i\beta_i} \quad ; \quad \langle\bar{\psi}^i_R\psi^i_L\rangle \equiv -\frac{C_i}{2}e^{-i\beta_i} \tag{2.31}$$

or

$$\langle\bar{\psi}^i\psi^i\rangle \equiv -C_i\cos\beta_i \quad ; \quad \langle\bar{\psi}^i i\gamma_5\psi^i\rangle \equiv C_i\sin\beta_i \tag{2.32}$$

where $C_i > 0$ and β_i is the phase of the ith quark condensate. Eq. (2.30) yields

$$\langle iF\tilde{F}\rangle = -m_i C_i\sin(\varphi_i+\beta_i). \tag{2.33}$$

$$(i = 1, 2, \cdots, L)$$

which is to be referred to as the vacuum alignment equation (VAE) [2.16]. It can also be derived directly by taking vacuum expectation values on both sides of the anomaly relation [2.20](see a detailed discussion on the vacuum alignment in Appendix A). Eq. (2.33) means that if the first moment of the topological charge is non-zero in the presence of instanton, the quark condensate develops a phase β_i different from $-\varphi_i$. If the phase of the fermion mass φ_i is zero as it can always be made so by making a chiral rotation, the fermion condensate has a non-trivial phase $\beta_i \neq 0$ i. e. develops an imaginary part which is determined by the topological structure of the theory. This of course would not happen in a theory like QED where only the trivial topological configuration exists. We shall see that it is the combination $\varphi_i + \beta_i$'s that determine the CP violating amplitude in strong interactions.

$\langle F\tilde{F}\rangle$ can be calculated from instanton dynamics in the dilute gas approximation (DGA) [2.15]. The vacuum to vacuum amplitude in the presence of the θ-term is given

$$Z(\bar{\theta}) = \sum_{\nu=0,\pm 1,\cdots}^{\infty} \int \mathcal{D}(A, \bar{\psi}, \psi) e^{i\bar{\theta}\nu} e^{-\int d^4x \sum_i \bar{\psi}^i (\not{D} + m_i)\psi^i + \frac{1}{4}F^2} \tag{2.34}$$

where we have not explicitly included the gauge fixing and the ghost terms. Inclusion of them must be understood when a practical computation is performed. The phase of the quark masses have been rotated away and $\bar{\theta} = \theta_{\rm QCD} + \sum_i \varphi_i$. In the DGA,

$$Z(\bar{\theta}) = \sum_{n_+=0}^{\infty} \sum_{n_-=0}^{\infty} \frac{1}{n_+!} \frac{1}{n_-!} (Z_+)^{n_+} (Z_-)^{n_-} = e^{Z_+ + Z_-} \tag{2.35}$$

where Z_+ (Z_-) is the single instanton (anti-instanton) amplitude

$$\begin{aligned} Z_+ &= e^{i\bar{\theta}} \int d^4z \frac{d\rho}{\rho^5} C_{N_c} \left(\frac{8\pi^2}{g^2(\rho)}\right)^{2N_c} e^{-\frac{8\pi^2}{g^2(\rho)}} d(M\rho) \\ Z_- &= Z_+^* \end{aligned} \tag{2.36}$$

with

$$C_{N_c} = \frac{4.6\exp(-1.68N_c)}{\pi^2(N_c-1)!(N_c-2)!}.$$

The factor $d(M\rho)$ in (2.36) is connected with the so-called fermion determinant, which introduces important physics. It was first discovered by 't Hooft [2.17] that there exists a zero mode of the operator $\not{D}$ in the instanton field. Thus we expect $d(M\rho) \propto \det M$ (M is the quarks mass matrix). For small quark masses, $d(M\rho)$ is equal to [2.17, 2.18]

$$\begin{aligned} d(M\rho) &= \prod_{i=1}^{L} f(m_i\rho) \\ f(x) &= 1.34x(1 + x^2 \ln x + \cdots), \quad x \ll 1. \end{aligned} \tag{2.37}$$

Combining (2.37) and (2.36) with (2.35) one obtains

$$Z(\bar{\theta}) = \exp[2V \cos\bar{\theta}\, m_1 m_2 \cdots m_L K(L)] \tag{2.38}$$

where $K(L)$ is of dimension $4 - L$

$$K(L) \cong (1.34)^L \int \frac{d\rho}{\rho^{5-L}} C_{N_c} \left(\frac{8\pi^2}{g^2(\rho)}\right)^{2N_c} e^{-\frac{8\pi^2}{g^2(\rho)}}. \tag{2.39}$$

The first moment $\langle iF\tilde{F}\rangle$ is calculated by taking an average of the topological charge over 4-space

$$\begin{aligned} \langle iF\tilde{F}\rangle &= \frac{1}{V}\langle \int d^4x\, iF\tilde{F}\rangle = \frac{1}{V}\frac{d}{d\bar{\theta}} \ln Z(\bar{\theta}) \\ &= -2m_u m_d \cdots m_L K(L) \sin\bar{\theta} \end{aligned} \tag{2.40}$$

and the topological susceptibility is equal to

$$\langle\langle \nu^2 \rangle\rangle = \frac{1}{V}\frac{d^2}{d\bar{\theta}^2} \ln Z(\bar{\theta}) = -2m_u m_d \cdots m_L K(L) \cos\bar{\theta}. \tag{2.41}$$

Clearly, when $\bar{\theta}$ is small we have

$$\langle iF\tilde{F}\rangle = \langle\langle \nu^2 \rangle\rangle \bar{\theta}. \tag{2.42}$$

The vacuum alignment in QCD can be readily made through the VAE (2.33). By defining the quark field, one can change the phase of the quark mass φ_i and phase of the quark condensate β_i. However, $\varphi_i + \beta_i$'s will not change under the redefinition. They are only functions of $\bar{\theta}$ as shown in (2.33). One may choose $\beta_i = 0$ $(i = 1, \cdots, L)$ such that the vacuum is CP-conserving

$$\langle \bar{\psi}^i i\gamma_5 \psi^i \rangle = 0. \qquad (i = 1, 2, \cdots, L) \tag{2.43}$$

Then the phases of the quark masses are no longer arbitrary. They are uniquely determined by the vacuum alignment equation (2.33),

$$\varphi_i = -\frac{\langle\langle \nu^2 \rangle\rangle}{m_i C}\bar{\theta} \qquad (i = 1, 2, \cdots, L) \tag{2.44}$$

$$\theta_{\mathrm{QCD}} = \bar{\theta} - \sum_i \varphi_i = \left(1 - \sum_i \frac{\langle\langle \nu^2 \rangle\rangle}{m_i C}\right)\bar{\theta}$$

where we have assumed φ_i's are small and C_i's are all equal to C. To be aligned with the vacuum, the strong CP phase $\bar{\theta}$ must be distributed among the θ-term and the quark mass terms according to their determined weights. The effective CP-violating part of the QCD lagrangian reads

$$\mathcal{L}_{\mathrm{CP}}^{\beta_i=0} = i\theta_{\mathrm{QCD}} F\tilde{F} - \frac{2}{C} m_u m_d \cdots m_L K(L) \bar{\theta} \bar{\psi} i\gamma_5 \psi. \tag{2.45}$$

with θ_{QCD} given in (2.44).

It is worth emphasizing that the effective CP-violating interactions in (2.45) are only valid in the CP-conserving vacuum where β_i's are zero. One can alternatively choose a certain pattern of the phase distribution and ask in what direction the vacuum is to align with it. In general, the vacuum angles are not zero and should be determined by the VAE (2.33). For example, we can choose $\varphi_i = 0$ $(i = 1, \cdots, L)$ such that $\mathcal{L}_{\mathrm{CP}}^{\beta_i=0} = i\bar{\theta} F\tilde{F}$. In this case, the vacuum condensates are complex $\beta_i = -\frac{\langle\langle \nu^2 \rangle\rangle}{m_i C}\bar{\theta}$.

A physical CP-violating amplitude gets contributions from both the CP-violating part of the lagrangian and the CP-violating part of the quark condensate. A proof of the equivalence of different chiral frames on strong CP effects is given in Ref. [2.21] where it is shown that the vacuum alignment equation (2.33) plays an essential role.

Does the left-over θ-term in the effective lagrangians play any role in computing the strong CP effects? So far there have been only two CP violating processes available: $\eta \to 2\pi$ and the electric dipole moment (EDM) of the neutron. The latter process depends on a computation on the effective CP-odd π-N coupling [2.23]. Both of them would involve in an evaluation of the commutator $[Q_5^a, F\tilde{F}]$ if the θ-term were to contribute

$$\begin{aligned} \left\langle \pi^a \pi^b \right| \theta_{\rm QCD} F\tilde{F} \left|\eta\right\rangle &= -\frac{i\theta_{\rm QCD}}{F_\pi} \left\langle \pi^b \right| [Q_5^a,\, F\tilde{F}] \left|\eta\right\rangle ; \\ \left\langle \pi^a N \right| \theta_{\rm QCD} F\tilde{F} \left|N'\right\rangle &= -\frac{i\theta_{\rm QCD}}{F_\pi} \left\langle N \right| [Q_5^a,\, F\tilde{F}] \left|N'\right\rangle \end{aligned} \tag{2.46}$$

where we have used the soft-pion theorem. It is obvious that $[Q_5^a, F\tilde{F}] = 0$ since Q_5^a is a non-singlet charge and thus the canonical commutation relation applies. The θ-term in our particular choice of the effective lagrangian and the vacuum can be ignored. However, it is emphasized that this should not be considered as a general statement. The whole point has to do with the vacuum alignment. What really matters is the correlation relation between ϕ_i's and β_i's given by (2.33).

The above statement can be exemplified in the following. For simplicity, let us assume $m_u = m_d = \cdots = m_L = m$ (for a non-equal mass case, a redefinition of pion fields is needed. See more in Appendix A) and $L = 3$ where pions and η are all light pseudoscalars and the soft-pion theorem applies. The amplitude of $\eta \to 2\pi$ is readily calculated when β_i's are zero

$$A(\eta \to 2\pi) = \left\langle \pi^0 \pi^0 \right| \mathcal{L}_{\rm CP}^{\beta_i=0} \left|\eta\right\rangle = \bar{\theta} \left(\frac{-i}{F_\pi}\right)^3 \left\langle [Q_5^3, [Q_5^3, [Q_5^8,\, \bar{\psi} i\gamma_5 \psi]]] \right\rangle$$

$$= \frac{4}{\sqrt{3}}\frac{1}{F_\pi^3} m_u m_d m_s K\bar{\theta} \tag{2.47}$$

In deriving (2.47), we have dropped $F\tilde{F}$ term. In a chiral frame where ϕ_i's are zero, we can still drop the θ-term. But the CP-conserving part of the lagrangian will contribute because the vacuum condensates are CP violating

$$\begin{aligned} A(\eta \to 2\pi) &= -\left\langle \pi^0\pi^0 \middle| m\bar{\psi}\psi \left|\eta\right\rangle = -m\left(\frac{-i}{F_\pi}\right)^3 \left\langle [Q_5^3,[Q_5^3,[Q_5^8,\bar{\psi}\psi]]]\right\rangle \right. \\ &= -\frac{2}{\sqrt{3}}\frac{1}{F_\pi^3} mC\sin\beta = \frac{4}{\sqrt{3}}\frac{1}{F_\pi^3} m_u m_d m_s K\bar{\theta} \end{aligned} \tag{2.48}$$

where $\beta_i = -\frac{\langle\langle\nu^2\rangle\rangle}{m_i C}\bar{\theta}$. Both (2.41) and (2.48) yield the same result.

We conclude that the measure of strong CP violation is given by the topological susceptibility

$$\mathcal{J}_{\text{strong}} = -\frac{1}{2}\langle\langle\nu^2\rangle\rangle\bar{\theta} = m_1 m_2 \cdots m_L K(L)\bar{\theta} \tag{2.49}$$

However, $K(L)$ is still an unknown factor, in addition, the integral in (2.39) is simply divergent for large instanton density. This is the shortcoming of all instanton computations if one uses the dilute gas approximation (DGA). More seriously, as we shall see below, $K(L)$ is to be related to the mass of the $U(1)_A$ particle. If $K(L)$ is of order e^{-N_c} as argued by Witten [2.12], it would be in conflict with Witten and Veneziano's solution to the $U(1)_A$ problem [2.13] in which the mass of $U(1)_A$ particle is of $O(\frac{1}{N_c})$. This suggests that we should not take the expression for $K(L)$ in (2.39) too seriously since it is divergent after all. Furthermore, the DGA may not be valid in the IR region of the QCD theory. It has been suggested in Ref. [2.24] that the instanton liquid can in principle avoid the IR problem, and gives rise to a description on the $U(1)_A$ particle mass consistent with Witten and Veneziano's scenario. Nevertheless, some main features of the instanton computation do not depend on the detail of the topological configurations. For instance, those mass factors appearing in (2.49) will not change since they are the direct result of Atiyah-Singer index theorem [2.25] on the fermion zero modes.

2.4 Effective Chiral Model

2.4.1 The Model and Quantum Corrections

We consider an effective chiral theory where meson degrees of freedom are explicitly introduced. The virtue of the model is that it reflects all flavor symmetries in strong interactions as described by QCD and the mesons as independent field excitations couple to fermions through Yukawa couplings. Unlike a conventional effective theory [2.26] in which the nucleons are involved, the model that we will be discussing contains quarks, gluons and mesons. It is a linear version of the gauged sigma model suggested by Georgi and Manohar [2.27], which describes strong interactions in the intermediate energy region between the scale of the chiral symmetry breaking and the scale of the quark confinement.

The model reads

$$\begin{aligned} \mathcal{L} &= -\bar{\psi}\,\not{D}\psi - \frac{1}{4}F^2 + i\theta F\tilde{F} - f\bar{\psi}_L\phi\psi_R - f\bar{\psi}_R\phi^\dagger\psi_L - \\ & \quad Tr\partial_\mu\phi\partial_\mu\phi^\dagger - V_0(\phi\phi^\dagger) - V_m(\phi,\phi^\dagger) \end{aligned} \tag{2.50}$$

where ϕ is a complex $L \times L$ matrix, $V_0(\phi\phi^\dagger)$ is the most general form of a potential invariant under $U(L) \times U(L)$ (renormalizable)

$$V_0(\phi\phi^\dagger) = -\mu^2 Tr\phi\phi^\dagger + \frac{1}{2}(\lambda_1 - \lambda_2)(Tr\phi\phi^\dagger)^2 + \lambda_2 Tr(\phi\phi^\dagger)^2 \tag{2.51}$$

and

$$V_m(\phi,\phi^\dagger) = -\frac{1}{4} m e^{i\chi} Tr\phi - \frac{1}{4} m e^{-i\chi} Tr\phi^\dagger. \tag{2.52}$$

(2.50) needs some explanations. Under $U(L)_L \times U(L)_R$, the quark fields as well as the complex meson field transform as

$$\begin{aligned} \psi_L &\to U_L\psi_L \quad , \quad \psi_R \to U_R\psi_R; \\ \phi &\to U_L\phi U_R^\dagger \quad , \quad \phi^\dagger \to U_R\phi^\dagger U_L^\dagger. \end{aligned} \tag{2.53}$$

In the absence of V_m, $\mathcal{L}$ is invariant *classically* under (2.53) but broken down to $SU(L)_L \times SU(L)_R \times U(1)_V$ by the chiral anomaly. V_m, replacing the quark mass (m now is of dimension 3), serves as an explicit symmetry breaking and must be treated as a perturbation. f is the Yukawa coupling, chosen to be real by redefining ϕ. Under $U(1)_A$ transformation

$$\begin{aligned} \psi_L &\to e^{i\omega}\psi_L \quad , \quad \psi_R \to e^{-i\omega}\psi_R; \\ \phi &\to e^{2i\omega}\phi \quad , \quad \phi^\dagger \to e^{-2i\omega}\phi^\dagger. \end{aligned} \tag{2.54}$$

the θ-term and V_m change as $\theta \to \theta - 2L\omega$, $\chi \to \chi + 2\omega$. But $\bar{\theta} = \theta + L\chi$ remains unchanged. Except for the meson sector, the gauge interaction in (2.50) looks identical to QCD. One may wonder if we are double counting the degrees of freedom. It is explained in [2.27] that these quarks and gluons are not the same as in QCD. In particular, quarks are supposed to acquire *constituent* masses about $360 MeV$, which is huge compared to the current mass in QCD. The gauge coupling g_s between quarks and gluons in the effective theory is found to be

$$\alpha_s \cong 0.28 \tag{2.55}$$

much less than its QCD counterpart. This may explain why the nonrelativistic quark model works since the quarks inside a proton could be treated as weakly interacting objects.

However, the drawback of the model is that it has a very serious $U(1)$ problem. Indeed, if one calculates the physical spectrum from $V_0 + V_m$, one finds L^2 would-be goldstone modes. In addition, the nontrivial topological structure of the theory has been totally overlooked. The classical excitations such as instantons have not been accounted for in the model, which, according to the original idea of 't Hooft [2.1], are crucial to solving the $U(1)$ problem.

We therefore consider the quantum correction to the lagrangian (2.50) in the presence of non-trivial classical gauge fields known as instantons. We argue that the effective gauge coupling α_s in (2.55) is obtained *only if* those classical extrema to the action have been effectively summed over by semiclassical methods. We find that the 1-loop quantum fluctuations around instantons lead to a dramatic change in the $U(1)_A$ sector of the model. The $U(1)$ particle acquires an extra mass from the vacuum tunneling effects, which, in turn, results in the so-called strong CP problem.

The effective action of the meson field is calculated as

$$\begin{aligned} Z &= \int \mathcal{D}(\phi,\phi^\dagger) e^{-S_0[\phi,\phi^\dagger]} \int \mathcal{D}(A,\bar{\psi},\psi) e^{-S[\bar{\psi},\psi;A;\phi,\phi^\dagger]} \\ &= \int \mathcal{D}(\phi,\phi^\dagger) e^{-S_{\text{eff}}[\phi,\phi^\dagger]} \end{aligned} \tag{2.56}$$

where

$$S_{\text{eff}}[\phi,\phi^\dagger] = S_0[\phi,\phi^\dagger] + \Delta S[\phi,\phi^\dagger] \tag{2.57}$$

and the quantum correction is given

$$\Delta S[\phi,\phi^\dagger] = -\ln \int \mathcal{D}(A,\bar{\psi},\psi) e^{-S[\bar{\psi},\psi;A;\phi,\phi^\dagger]} \equiv -\ln \tilde{Z}[\phi,\phi^\dagger]. \tag{2.58}$$

The calculation of $\tilde{Z}[\phi, \phi^\dagger]$ in the instanton background follows the standard derivation of the vacuum-to-vacuum amplitude as in [2.17]

$$\tilde{Z}[\phi, \phi^\dagger] = \sum_\nu \int \mathcal{D}A_{cl} e^{i\theta\nu - S[A_{cl}]} (\mathrm{Det}\,\mathcal{M}_A)^{-1/2} \mathrm{Det}\,\mathcal{M}_\psi \mathrm{Det}\,\mathcal{M}_{gh} \tag{2.59}$$

where

$$\begin{aligned} \mathcal{M}_A &= -D^2 - 2F \\ \mathcal{M}_{gh} &= -D^2 \\ \mathcal{M}_\psi &= \not{D} + \frac{f}{2}(\phi + \phi^\dagger) + \frac{f}{2}(\phi - \phi^\dagger)\gamma_5. \end{aligned} \tag{2.60}$$

If only the effective potential is of concern, ϕ and $\phi^\dagger$ in $\mathcal{M}_\psi$ are to be taken as constant fields. The fermion determinant, as usual, needs special treatment:

$$\mathrm{Det}\,\mathcal{M}_\psi = \mathrm{Det}^{(0)}\mathcal{M}_\psi \mathrm{Det}'\mathcal{M}_\psi. \tag{2.61}$$

$\mathrm{Det}^{(0)}$ denotes contributions from the subspace of zero modes of $\not{D}$. In a single instanton field, $\not{D}$ has a zero mode with chirality -1 ($\gamma_5 = -1$) [2.19]. Thus we have

$$\mathrm{Det}^{(0)}\mathcal{M}_\psi = \det\left[\frac{f}{2}(\phi + \phi^\dagger) + \frac{f}{2}(\phi - \phi^\dagger)(-1)\right] = \det(f\phi^\dagger)$$

where det only acts upon flavor indices. The prime in $\mathrm{Det}'\mathcal{M}_\psi$ reminds us of excluding zero modes from the eigenvalue product. Since $[\not{D}, \gamma_5] \neq 0$, $\mathcal{M}_\psi$ cannot be diagonalized in the basis of eigenvectors of $\not{D}$. The nonvanishing eigenvalues always appear in pair, i. e. if $\not{D}\varphi_n = \lambda_n\varphi_n$ where $\lambda_n \neq 0$, then $\not{D}\gamma_5\varphi_n = -\gamma_5\not{D}\varphi_n = -\lambda_n\gamma_5\varphi_n$, namely both λ_n and $-\lambda_n$ are eigenvalues of $\not{D}$. In addition, γ_5 takes φ_n to φ_{-n}. Therefore

$$\begin{aligned} \mathrm{Det}'\mathcal{M}_\psi &= \det \prod_{\lambda_n > 0} \begin{pmatrix} i\lambda_n + \frac{f}{2}(\phi + \phi^\dagger) & \frac{f}{2}(\phi - \phi^\dagger) \\ \frac{f}{2}(\phi - \phi^\dagger) & -i\lambda_n + \frac{f}{2}(\phi + \phi^\dagger) \end{pmatrix} \\ &= \det \prod_{\lambda_n > 0} (\lambda_n^2 + f^2\phi\phi^\dagger) = \mathrm{Det}'^{1/2}(-\not{D}^2 + f^2\phi\phi^\dagger). \end{aligned} \tag{2.62}$$

Now we are ready to make the DGA. We need to further assume a weak-field approximation of ϕ and $\phi^\dagger$. This can be justified since ϕ and $\phi^\dagger$ fluctuate about their VEV's, which are about $300MeV$. The large fluctuations are exponentially suppressed by $\exp(-\lambda_1|\phi|^4)$. In the DGA

$$\tilde{Z}[\phi,\phi^\dagger] = \mathrm{Det}^{1/2}(-\partial^2 + f^2\phi\phi^\dagger)\exp(\tilde{Z}_+ + \tilde{Z}_-) \tag{2.63}$$

where

$$\begin{aligned}\tilde{Z}_+[\phi,\phi^\dagger] &= e^{i\theta}\det(f\phi^\dagger)\int dz\frac{d\rho}{\rho^5}C_{N_c}\left(\frac{8\pi^2}{g^2(\rho)}\right)^{2N_c} e^{-\frac{8\pi^2}{g^2(\rho)}}.\\ &\quad \det\left[1.34\rho\left(1+f^2\phi\phi^\dagger\ln f^2\phi\phi^\dagger+\cdots\right)\right]\\ &\cong VK(L)e^{i\theta}\det(f\phi^\dagger)\end{aligned} \tag{2.64}$$

$$\tilde{Z}_-[\phi,\phi^\dagger] = \tilde{Z}_+^\dagger[\phi,\phi^\dagger]$$

and $K(L)$ is given in (2.39).

Combining (2.63) with (2.58), and noticing that $\ln\mathrm{Det}(-\partial^2+f^2\phi\phi^\dagger)$ contains terms which can be absorbed into the tree-level lagrangian by redefinition of bare parameters, we obtain the following effective lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= -\bar{\psi}\,\not{D}_s\psi - \frac{1}{4}F_s^2 - (f\bar{\psi}_L\phi\psi_R + h.c.) - Tr(\partial_\mu\phi\partial_\mu\phi^\dagger) - \\ &\quad V_0(\phi\phi^\dagger) - V_m(\phi,\phi^\dagger) - V_k(\phi,\phi^\dagger)\end{aligned} \tag{2.65}$$

where

$$V_k(\phi,\phi^\dagger) = -K(L)f^Le^{i\theta}\det\phi^\dagger - K(L)f^Le^{-i\theta}\det\phi \tag{2.66}$$

Several remarks on (2.65) are in order. The presence of V_k in (2.65) is the direct result of fermion zero modes in the instanton field. It is invariant under $SU(L)_L\times SU(L)_R\times$

$U(1)_V$ but not invariant under $U(1)_A$. Under $U(1)_A$ rotation (2.54), $e^{i\theta}\det\phi \to e^{i(\theta-2\omega L)}\det\phi$. Thus V_k takes over the role of the θ-term and respects the anomaly relation. Again, $\bar{\theta} = \theta + \chi L$ remains invariant. The prototype of V_k was suggested long time ago by several authors [2.28] and re-discussed by t' Hooft [2.29] in the context of instanton. It is different from a model originally proposed by Di Vecchia [2.32] and recently analyzed in Ref. [2.8], although physical contents of both models may be similar. The gauge interactions between quarks and gluons are still present in (2.65) as required in the nonrelativistic quark model. However, they differ from QCD in that the gauge coupling g_s has a smaller value, and most importantly, the gauge field A_s now possesses a *trivial* topology at infinity. The gauge interaction sector in (2.65) is very analogous to QED: the fermion chiral anomaly still exists, but any θ-term $\int d^4x\theta F_s\tilde{F}_s$ in the action would be simply a vanishing surface term and can be dropped.

2.4.2 U(1) Particle Mass and Strong CP Violation

We would like to discuss the physical spectrum of the model (2.65) (this part has been worked out in Ref. [2.29]) and show how the strong CP effects can be calculated effectively. To simplify the problem, we take $L = 2$ and u and d quarks have equal masses. In this case, η is identified as the $U(1)$ particle and there will not be a mixing between π^0 and η.

The complex meson field ϕ contains eight particle excitations σ, η, π_a and α_a $(a = 1,2,3)$:

$$\phi = \frac{1}{2}(\sigma + i\eta) + \frac{1}{2}(\vec{\alpha} + i\vec{\pi})\cdot\vec{\tau} \tag{2.67}$$

where $\tau^{1,2,3}$ are the Pauli matrices. In terms of physical fields, V_0, V_m and V_k can be rewritten as

$$V_0(\phi\phi^\dagger) = -\frac{\mu^2}{2}(\sigma^2+\eta^2+\vec{\alpha}^2+\vec{\pi}^2)+\frac{\lambda_1}{8}(\sigma^2+\eta^2+\vec{\alpha}^2+\vec{\pi}^2)^2+ \frac{\lambda_2}{2}[(\sigma\vec{\alpha}+\eta\vec{\pi})^2+(\vec{\alpha}\times\vec{\pi})^2] \quad (2.68)$$

$$V_m(\phi,\phi^\dagger) = -\frac{1}{4}me^{i\chi}(\sigma+i\eta)-\frac{1}{4}me^{-i\chi}(\sigma-i\eta) \quad (2.69)$$

$$V_k(\phi,\phi^\dagger) = -\frac{1}{2}Kf^2(\sigma^2-\eta^2-\vec{\alpha}^2+\vec{\pi}^2)\cos\theta - K(\sigma\eta-\vec{\alpha}\cdot\vec{\pi})\sin\theta \quad (2.70)$$

Assuming, for convenience,

$$\langle\phi\rangle = \frac{1}{2}\langle\sigma+i\eta\rangle = \frac{1}{2}ve^{i\varphi} \qquad (v>0). \quad (2.71)$$

we get, by taking the extremum of $V_0+V_m+V_k$ with respect to v and φ

$$v^2 = \frac{2\mu^2}{\lambda_1}+\frac{2m}{\lambda_1 v}\cos(\chi+\varphi)-\frac{2Kf^2}{\lambda_1}\cos(\theta-2\varphi) \quad (2.72)$$

and

$$m\sin(\chi+\varphi)-Kf^2v\sin(\theta-2\varphi)=0. \quad (2.73)$$

Eq. (2.73) plays a role of the vacuum alignment in the effective theory. If we take $\varphi = 0$ as we wish, (2.73) implies a consistency constraint on χ and θ: They are not separately independent parameters. They can expressed in terms of the physical parameter $\bar{\theta} = \theta + 2\chi$ as

$$\sin\chi \cong -\frac{Kf^2v}{m+2Kf^2v}\sin\bar{\theta} \quad (2.74)$$

$$\sin\theta \cong -\frac{m}{m+2Kf^2v}\sin\bar{\theta} \quad (2.75)$$

where we have assumed that $\sin\chi$ is very small ($<< 1$).

Rewriting $\mathcal{L}_{\text{eff}}$ in terms of the shifted field $\phi \longrightarrow \langle\phi\rangle + \phi$, we get

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= -\bar{\psi}(\not{D}_s + \frac{1}{2}fv)\psi - \frac{1}{4}F_s^2 - (f\bar{\psi}_L\phi\psi_R + h.c.) - Tr(\partial_\mu\phi\partial_\mu\phi^\dagger) - \\ & \frac{1}{2}(\sigma,\eta)M_{\sigma\eta}^2\begin{pmatrix}\sigma\\ \eta\end{pmatrix} - \frac{1}{2}(\vec{\alpha},\vec{\pi})M_{\alpha\pi}^2\begin{pmatrix}\vec{\alpha}\\ \vec{\pi}\end{pmatrix} - \frac{\lambda_1 v}{2}\sigma(\sigma^2+\eta^2+\vec{\alpha}^2+\vec{\pi}^2) - \\ & \lambda_2 v\vec{\alpha}\cdot(\sigma\vec{\alpha}+\eta\vec{\pi}) - \frac{\lambda_1}{8}(\sigma^2+\eta^2+\vec{\alpha}^2+\vec{\pi}^2)^2 - \frac{\lambda_2}{2}(\sigma\vec{\alpha}+\eta\vec{\pi})^2 - \frac{\lambda_2}{2}(\vec{\alpha}\times\vec{\pi})^2\end{aligned} \tag{2.76}$$

where the meson mass matrices are given

$$\begin{aligned}M_{\sigma\eta}^2 &= \begin{pmatrix}\lambda_1 v^2 + \frac{m}{v}\cos\chi & -\frac{1}{2}Kf^2\sin\theta \\ -\frac{1}{2}Kf^2\sin\theta & \frac{m}{v}\cos\chi + 2Kf^2 cos\theta\end{pmatrix}\\ M_{\alpha\pi}^2 &= \begin{pmatrix}\lambda_1 v^2 + \frac{m}{v}\cos\chi + 2Kf^2 cos\theta & \frac{1}{2}Kf^2\sin\theta \\ \frac{1}{2}Kf^2\sin\theta & \frac{m}{v}\cos\chi\end{pmatrix}.\end{aligned} \tag{2.77}$$

The quark acquires a large constituent mass

$$m_Q = \frac{1}{2}fv \cong \frac{f\mu^2}{\lambda_1} + \frac{fm}{\lambda_1 v} + \frac{Kf^3}{\lambda_1}. \tag{2.78}$$

It is interesting to note that m_Q arises from three parts: the spontaneous chiral symmetry breaking (from V_0), the explicit chiral symmetry breaking (from V_m) and the instanton induced symmetry breaking (from V_k). The instanton does *spontaneously* break chiral symmetry $SU(L)_L \times SU(L)_R$ [2.30]. The mass spectrum of mesonic states can be read off from diagonalizing (2.77). The mixing probability is proportional to $(Kf^2\sin\theta)^2 = m^2\sin^2\chi$ which is of high orders thus hardly affects the physical masses

$$\begin{aligned} m_\eta^2 &= \frac{m}{v}\cos\chi + 2Kf^2 cos\theta, \qquad m_\pi^2 = \frac{m}{v}\cos\chi; \\ m_\sigma^2 &= \lambda_1 v^2 + \frac{m}{v}\cos\chi, \qquad m_{\vec{\alpha}}^2 = \lambda_2 v^2 + \frac{m}{v}\cos\chi + 2Kf^2\cos\theta.\end{aligned} \tag{2.79}$$

(2.79) clearly shows how the instanton induced V_k leads to a mass splitting between pions and the $U(1)$ particle η. When $\bar{\theta}$ thus θ is small,

$$m_\eta^2 - m_\pi^2 = 2Kf^2, \tag{2.80}$$

and in the chiral limit $m \to 0$, $m_\pi^2 \to 0$ but $m_\eta^2 \to 2Kf^2$. We conclude that the $U(1)$ problem is solved in the framework of the effective theory if $2Kf^2$ is big enough.

The CP-violating effects originate from the mixing between the scalar and pseudoscalars eventhough the mixing is negligible in computing the meson masses. To diagonalize the quadratic terms in (2.76), we define the physical meson fields (the primed fields)

$$\sigma = \sigma' \cos\gamma + \eta \sin\gamma \quad , \quad \eta = -\sigma' \sin\gamma + \eta cos\gamma; \tag{2.81}$$

$$\vec{\alpha} = \vec{\alpha}' \cos\gamma' + \vec{\pi} \sin\gamma' \quad , \quad \vec{\pi} = -\vec{\alpha} \sin\gamma' + \vec{\pi} \cos\gamma' \tag{2.82}$$

such that the off-diagonal elements in (2.77) vanish. The mixing angles γ and γ' are determined

$$\gamma = \frac{Kf^2 \sin\theta}{m_\sigma^2 - m_\eta^2} = \frac{1}{2}\frac{m_\pi^2}{m_\sigma^2 - m_\eta^2}\left(1 - \frac{m_\pi^2}{m_\eta^2}\right)\bar{\theta} \tag{2.83}$$

$$\gamma' = -\frac{Kf^2 \sin\theta}{m_\alpha^2 - m_\pi^2} = -\frac{1}{2}\frac{m_\pi^2}{m_\alpha^2 - m_\pi^2}\left(1 - \frac{m_\pi^2}{m_\eta^2}\right)\bar{\theta} \tag{2.84}$$

which meet the criteria that the mixing and thus the strong CP violation must disappear as $m_\pi^2 \to 0$ or $m_\eta^2 = m_\pi^2$ or $\bar{\theta} = 0$. In terms of the physical fields, the CP-violating part of the effective potential is identified (for simplicity we drop the prime notations)

$$\begin{aligned} V_{\rm CP} = {} & \frac{\lambda_1 v}{2} \sin\gamma\, \eta(\sigma^2 + \eta^2 + \vec{\alpha}^2 + \vec{\pi}^2) + \lambda_2 v \cos\gamma' \sin(\gamma - \gamma')\vec{\alpha}\cdot(\eta\vec{\alpha} - \sigma\vec{\pi}) + \\ & \lambda_2 v \sin\gamma \cos(\gamma - \gamma')\vec{\pi}\cdot(\sigma\vec{\alpha} + \eta\vec{\pi}) \end{aligned} \tag{2.85}$$

and the Yukawa coupling between quarks and mesons contains CP-violating part too

$$\mathcal{L}_{\rm Yukawa} = -\frac{1}{2}\bar{\psi}(\sin\gamma + i\gamma_5\cos\gamma)\psi\eta - \frac{1}{2}\bar{\psi}(\sin\gamma' + i\gamma_5\cos\gamma')\vec{\tau}\psi\cdot\vec{\pi}. \tag{2.86}$$

The Feynman rules for CP-violating vertices and the typical CP-violating $qq \to qq$ amplitude are shown in Figure 2.1.

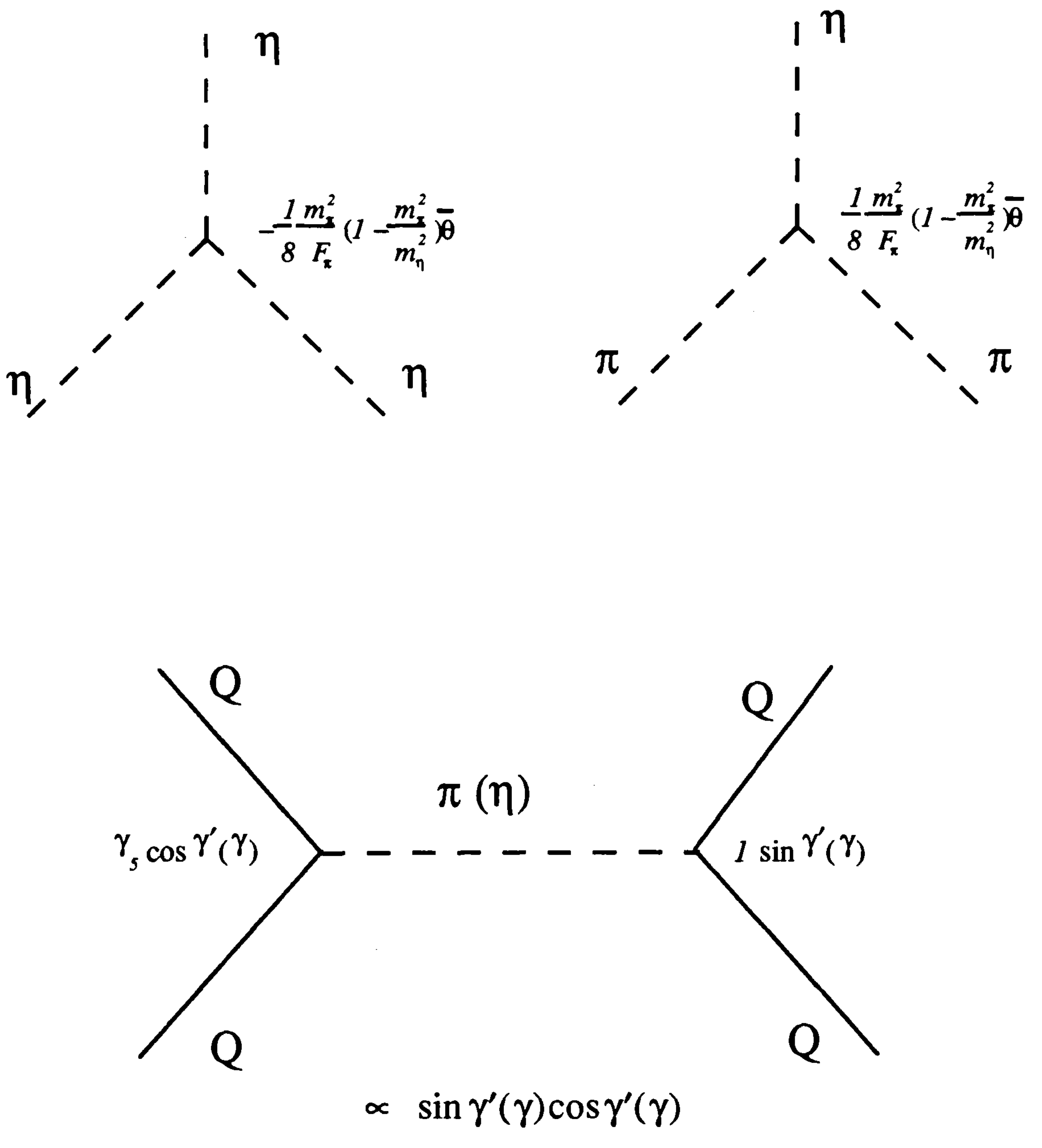

Figure 2.1: Feynman rules for η^3 and $\eta\pi^2$ couplings. The CP-violating $qq \to qq$ scattering. We have assumed that $m_\sigma^2 \gg m_\eta^2$, $m_\alpha^2 \gg m_\pi^2$ and $v = 2F_\pi$.

The amplitude of $\eta \rightarrow 2\pi$ decays reads from (2.85)

$$A(\eta \rightarrow 2\pi) = \frac{1}{4}\frac{m_\pi^2}{F_\pi}\left(1 - \frac{m_\pi^2}{m_\eta^2}\right)\bar{\theta} \tag{2.87}$$

where $F_\pi = \frac{v}{2}$. It is worth noting that (2.87) does not have a direct comparison with the QCD calculation (2.47) and (2.48) where we worked in the case $L = 3$ and η is one of the would-be goldstone bosons. In (2.87), however, η has been referred to as the $U(1)$ particle (the isosinglet).

2.4.3 EDM for Constituent Quark

The CP-violating Yukawa coupling in (2.86) results in an important strong CP effect: the EDM of the constituent quark. It can be examined by computing the effective interaction of the type

$$\mu_{\rm EDM}\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi F_{\mu\nu}^{\rm em} \tag{2.88}$$

when an external electromagnetic field $A_\mu^{\rm em}$ is introduced. The coefficient $\mu_{\rm EDM}$ is defined as the EDM of the quark. Since (2.88) is not invariant under chiral rotation, the EDM can be converted into the magnetic moment if the fermion field is chiral (the chirality flip). When $m_Q \neq 0$, we have to check the phase of the constituent quark mass m_Q since only the phase difference between the quark mass and the effective interaction makes sense. In our convention, m_Q is real at tree-level. At higher level, the mass acquires infinite renormalization. The renormalizability of our model guarantees that the renormalized mass will not develop a γ_5-dependent part. However, m_Q may acquire a finite renormalization which may contain a γ_5-part at high orders. But that phase would be too small to cancel (2.88).

In the background of EM field, the charged quarks and pions coupling to $A_\mu^{\rm em}$ through the covariant derivative $D_\mu^{\rm em}$

$$-\bar{\psi}_Q \not{D}_Q^{\rm em}\psi_Q - \left|D_\mu^{\rm em}\pi^+\right|^2 \tag{2.89}$$

where

$$D_{\mu,Q}^{\rm em} = \partial_\mu + eQA_\mu^{\rm em} \tag{2.90}$$

and Q is the electric charge of the particle. Following Schwinger's formalism [2.33] on the derivation of the anomalous magnet moment of electron, we obtain the effective interactions

$$\begin{aligned}\int d^4x L_{\rm eff}^{\rm em} &= -\int d^4x \sum_{Q=u,d} \bar{\psi}_Q(\not{D}_Q^{\rm em} + m_Q)\psi_Q \\ &-\frac{f^2}{2!}\int d^4x d^4y \sum_{Q=u,d} \bar{\psi}_Q(x)e^{i\gamma'\gamma_5}S_{\pi^0\pi^0}S_F^Q(x,y)e^{i\gamma'\gamma_5}\psi_Q(y) \\ &-\frac{f^2}{2!}\int d^4x d^4y \bar{u}(x)e^{i\gamma'\gamma_5}S_{\pi^+\pi^-}S_F^d(x,y)e^{i\gamma'\gamma_5}u(y) \\ &-\frac{f^2}{2!}\int d^4x d^4y \bar{d}(x)e^{i\gamma'\gamma_5}S_{\pi^+\pi^-}S_F^u(x,y)e^{i\gamma'\gamma_5}d(y)\end{aligned} \tag{2.91}$$

where $S_{\pi\pi}$'s and S_F^Q's are pion and quark propagators in the background of $A_\mu^{\rm em}$,

$$\begin{aligned}S_{\pi^0\pi^0} &= \frac{1}{\partial^2 - m_\pi^2} \quad , \quad S_{\pi^+\pi^-} = \frac{1}{(D_\mu^{\rm em})^2 - m_\pi^2}; \\ S_F^Q &= \frac{1}{\not{D}_Q^{\rm em} + m_Q}.\end{aligned} \tag{2.92}$$

Because $\frac{e^2}{4\pi} << 1$, we can expand these propagators perturbatively in e

$$S_F^Q = \frac{\not{D}_Q^{\rm em} - m_Q}{(D_Q^{\rm em})^2 - m_Q^2}\left(1 + \frac{\frac{1}{2}eQ\sigma_{\mu\nu}F_{\mu\nu}^{\rm em}}{(D_Q^{\rm em})^2 - m_Q^2} + \cdots\right) \tag{2.93}$$

$$S_{\pi^+\pi^-} = \frac{1}{\partial^2 - m_\pi^2}\left(1 + \frac{eA_\mu^{\rm em}\partial_\mu + e\partial_\mu A_\mu^{\rm em}}{\partial^2 - m_\pi^2} + \cdots\right) \tag{2.94}$$

where the ellipses denote $O(e^2)$. The extraction of the effective interaction of (2.88) is done with the aid of Feynman diagrams in Fig. 2.2. The contributions from the second term in (2.91) correspond to Fig. 2.2(a), the third to Fig. 2.2(b) and the fourth to Fig. 2.2(c). Summing them up, we get

$$\mu^u_{\rm EDM} = \mu^d_{\rm EDM} = \frac{ef^2}{32\pi^2}\sin 2\gamma' m_Q \left[-\frac{2}{3}\frac{1}{m_Q^2 - m_\pi^2} + \frac{m_Q^2}{(m_Q^2 - m_\pi^2)^2}\ln\frac{m_Q^2}{m_\pi^2}\right]. \quad (2.95)$$

A detailed computation can be found in Appendix B. The EDM of neutron is obtained by applying the $SU(6)$ quark model,

$$\mu^{\rm neutron}_{\rm EDM} = \frac{4}{3}\mu^d_{\rm EDM} - \frac{1}{3}\mu^u_{\rm EDM} \cong \frac{e}{2m_Q}\frac{f^2}{16\pi^2}\sin 2\gamma' \ln\frac{m_Q^2}{m_\pi^2} \quad (2.96)$$

where we have used $m_Q^2 \ll m_\pi^2$ and γ' is given in (2.84).

2.5 Solutions to Strong CP Problem

In above, we have studied extensively the measure of strong CP violation and its physical effects from the viewpoint of QCD and of an effective chiral theory. $\mathcal{J}_{\rm strong}$ is a product of quark masses, $\bar\theta$ and the instanton amplitude $K(L)$. It should vanish when any one of them vanishes. The most stringent experimental constraint on $\mathcal{J}_{\rm strong}$ comes from the EDM of neutron, which has been measured at a very high precision [2.34]

$$\mu^{\rm neutron}_{\rm EDM} < 1.2 \times 10^{-25} ecm. \quad (2.97)$$

this implies

$$\mathcal{J}_{\rm strong} < 10^{-16} GeV^4. \quad (2.98)$$

At a typical hadron energy scale, one would suspect $\mathcal{J}_{\rm strong} \simeq \Lambda^4_{\rm QCD} \simeq 10^{-4} \sim 10^{-6} GeV^4$, enormously larger than the upper limit. This is so-called strong CP

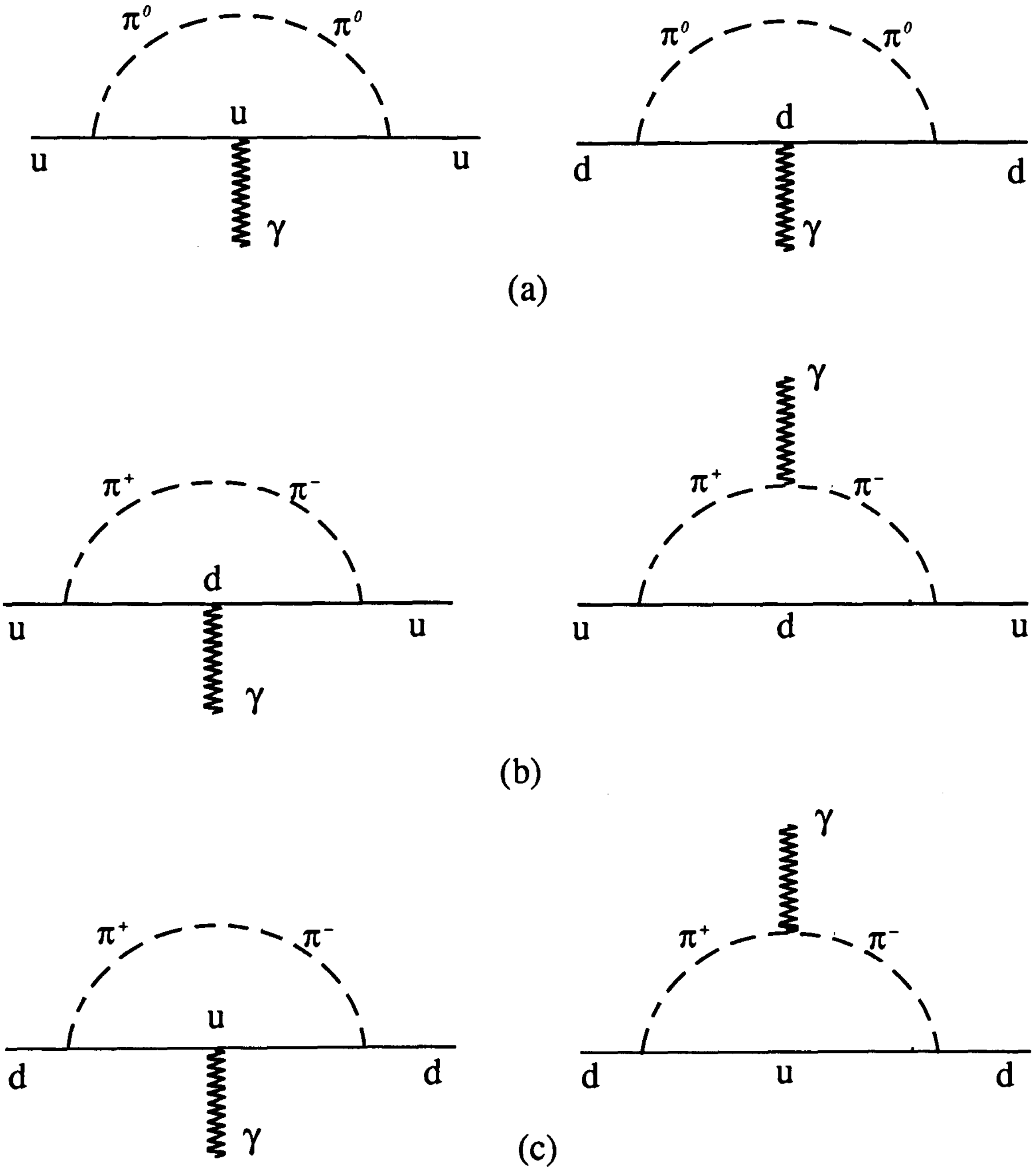

Figure 2.2: Diagrammatic representation of Schwinger's formulation on the EDM for the constituent quark.

problem. It has puzzled us for more than a decade, ever since the instanton was discovered.

2.5.1 $m_u = 0$ Scenario

When $m_u = 0$ thus $\mathcal{J}_{\rm strong} = 0$, the strong CP problem is most neatly and elegantly solved. In the meantime, the U(1) problem can be solved by instanton without resorting to other assumptions. There is an additional $U(1)_A$ symmetry associated with u quark. $m_u = 0$, unlike setting $\bar{\theta} = 0$, does increase the symmetry of the system and thus does not violate 't Hooft's naturalness principle. However, $m_u = 0$ seems to contradict with phenomenology where $m_u^{\rm exp} \simeq 5 \sim 10 MeV$ [2.35].

However, there is a loophole in this argument [2.36]. The instanton *explicitly* breaks $U(1)_A$, as well as $U(1)_A^u$ associated with the massless u quark if all other light quarks are *massive*. The instanton is acting as a flavor-changing force, as a result, u quark acquires a radiative mass from other flavors! This is again due to the existence of the zero modes of $\not{D}$ in the nontrivial instanton field. In the presence of a massless fermion, the vacuum tunneling effect is suppressed unless we insert an operator that contains enough grassmann fields to eliminate all the zero modes. In the $\nu = \pm 1$ sector, the only operator which survives is $\bar{u}u$. To see how it works, let's recall the partition function $Z(\theta)$ in (2.38). $\langle \bar{u}u \rangle$ is calculated by taking the average over space-time

$$\begin{aligned} \langle \bar{u}u \rangle_{\rm instanton} &= \frac{1}{V}\langle \int d^4x \bar{u}u(x) \rangle = -\frac{1}{V}\frac{d}{dm_u}\ln Z(\bar{\theta}) \\ &= -2m_d \cdots m_L K(L) \end{aligned} \tag{2.99}$$

where we have rotated $\bar{\theta}$ to zero as we can when $m_u = 0$. (2.99) implies that $U(1)_A^u$ symmetry is broken by instanton. Of course we would not have the goldstone boson since it is referred to as an explicit breaking. We should not confuse the condensate $\langle\bar{u}u\rangle$ caused by the spontaneous symmetry breaking with $\langle\bar{u}u\rangle_{\text{instanton}}$. The former can be non-zero even if all quarks are massless while the latter vanishes if d quark mass is zero. The instanton induced u quark mass can be roughly estimated [2.31] in the case $L = 2$ where $K(2)$ is related to m_η^2,

$$\begin{aligned} m_u^{\text{instanton}} &\cong -\pi\alpha_s(\bar{\rho})C_F\bar{\rho}^2\langle\bar{u}u\rangle_{\text{instanton}} \\ &= \frac{4}{3}\pi\alpha_s(\bar{\rho})\bar{\rho}^2F_\pi^2\frac{m_\eta^2 - m_\pi^2}{m_Q^2}m_d \cong 4\,MeV \end{aligned} \tag{2.100}$$

where we take $\bar{\rho} \simeq (\frac{1}{3}\Lambda_{\text{QCD}})^{-1}$, $K = -\frac{1}{2f^2}(m_\eta^2 - m_\pi^2)$ and $f = \frac{2m_Q}{F_\pi}$. $m_u^{\text{instanton}}$ must be viewed as an explicit mass because of its proportionality to m_d. What seems remarkable is that the order of magnitude of $m_u^{\text{instanton}}$ is consistent with the phenomenological value. The massless u quark is still the most favorable solution to the strong CP problem.

2.5.2 Peccei-Quinn Symmetry

Another possibility of rendering $\mathcal{J}_{\text{strong}} = 0$ is that $\bar{\theta} = 0$ for some dynamical reason. This is realized if the phase of the quark masses $\theta_{QFD} = \sum_i \varphi_i$ is equal to $-\theta_{\text{QCD}}$. A decade ago, Peccei and Quinn [2.37] suggested that the strong CP problem may be naturally solved if one or more quarks acquire current mass entirely through the Higgs mechanism where the lagrangian of quarks and scalars exhibits an adjoint chiral symmetry: the Peccei- Quinn symmetry.

For simplicity, let us examine a toy model of a single quark

$$\mathcal{L}_{\text{toy}} = -\bar{\psi}\,\not{D}\psi - \frac{1}{4}F^2 + i\theta F\tilde{F} - (f\bar{\psi}_L\psi_R\phi + h.c.) - \partial_\mu\phi\partial_\mu\phi^* - V_0(\phi,\phi^*) \tag{2.101}$$

where

$$V_0(\phi, \phi^*) = -\mu^2 \phi\phi^* + \frac{1}{4}\lambda(\phi\phi^*)^2. \tag{2.102}$$

(2.101) is invariant under the PQ symmetry

$$\psi_R \to e^{i\alpha}\psi_R \quad , \quad \psi_L \to e^{-i\alpha}\psi_L;$$
$$\phi \to e^{-2i\alpha}\phi \quad , \quad \phi^* \to e^{2i\alpha}\phi^*. \tag{2.103}$$

The PQ symmetry is broken at the quantum level by the chiral anomaly, and effectively

$$\mathcal{L}_{\text{toy}} \to \mathcal{L}_{\text{toy}} - 2i\alpha F\tilde{F}. \tag{2.104}$$

Choosing $\alpha = \frac{\theta}{2}$ yields $\bar{\theta} = 0$.

The effective potential of the scalar fields can be calculated in a similar way to (2.65)

$$V_{\text{eff}}(\phi, \phi^*) = -\mu^2 \phi\phi^* + \frac{1}{4}\lambda(\phi\phi^*)^2 - Kf^* e^{-i\theta} \det\phi^* - Kfe^{i\theta} det\phi \tag{2.105}$$

where K is the instanton amplitude. The last two terms in the effective potential breaks the PQ symmetry. The VEV's of ϕ and ϕ^* are found to be

$$\langle f\phi \rangle = ve^{-i\theta} \quad ; \quad \langle f^*\phi^* \rangle = ve^{i\theta} \tag{2.106}$$

and

$$v^2 = \frac{2\mu^2|f|^2}{\lambda} + \frac{2K|f|^4}{\lambda v}. \tag{2.107}$$

Thus the fermion mass reads from the Yukawa interaction $m = fve^{-i\theta}$ and

$$\bar{\theta} = \theta + \arg\langle f\phi \rangle = 0. \tag{2.108}$$

The axion [2.38] mass is readily derived from (2.105) by diagonalizing the quadratic terms

$$m^2_{\text{axion}} = \frac{2K|f|^2}{v}. \tag{2.109}$$

Unfortunately, we have not been able to discover this particle yet so far.

2.6 Summary

We have studied the measure of CP violation in strong interactions. It arises from the nontrivial topological structure of Yang-Mills fields, a non-zero vacuum angle $\bar{\theta}$ as well as nonvanishing quark current masses. The instanton dynamics makes most sense in dealing with the topological gauge configurations where the semiclassical method applies. It has been shown that the instanton dynamics, as a consistent field theory, automatically satisfies the so-called anomalous Ward identity. Crewther's original complaints on the topological susceptibility and θ-periodicity of the fermion operator are a result of inconsistently handling the singularities in some fermion operators. We conclude that QCD theory itself does not put any constraint on the instanton computation.

In the presence of the chiral anomaly, there is no would-be $U(1)_A$ goldstone particle. By studying an effective chiral theory, we find that the instanton leads to an explicit $U(1)_A$ symmetry breaking. If the instanton is to solve the U(1) problem, the measure of the strong CP violation is connected to the mass of the U(1) particle. It may be natural to think that strong CP problem is the side effect of the U(1) problem and both problems cannot be solved simultaneously in the context of QCD.

However, we point out that the massless u quark scenario to solve the strong CP problem may not be such a silly idea. The u quark may acquire a mass from the d

quark through the instanton interaction in which the fermion zero modes plays an essential role. In any case, with the failure so far to observe axions experimentally, the strong CP problem is wide open to new mechanisms [2.39].

Chapter 3

Anomalous Discrete Symmetry

3.1 Domain Wall Problem

One of the most significant applications of cosmological arguments to fundamental particle physics is the observation by Kobsarev, Okun and Zeldovich (KOZ) [3.1] that spontaneously broken discrete symmetry, which may be an attractive possibility in building models of particle physics, can lead to grave difficulties in the context of cosmology. The reason is the following. Spontaneously broken discrete symmetry implies the existence of several degenerate ground states or vacua. At high temperature, the symmetry is restored as shown by finite temperature field theory. As the temperature drops in an expanding universe (the big bang), the symmetry is broken. But this symmetry breaking occurs independently in all causally unconnected regions of the universe, and therefore in each of such regions at the time of symmetry-breaking phase transition, different choices of the vacuum configurations can arise. Adjacent domains filled by different vacua are separated by domain walls.

The energy per unit area of a domain wall is set by a microphysical parameter; call it ϵ (as in the case of ferromagnetic domain walls). Then the energy in the domain wall will be at least of order $R^2\epsilon$, where R is the radius of the universe, corresponding to an energy density $\rho_{dw} \approx \epsilon/R$. This decrease in energy density is much slower than ordinary radiation or matter. Thus stable domain walls will quickly come to dominate the mass of the universe, and continue to do so.

This implies that a theory with spontaneous breaking of a discrete symmetry is in gross disagreement with cosmological observation. This conclusion is rather disappointing, since spontaneously broken symmetries are an important ingredient in many interesting models of particle physics. It would be nice if we could find a way to save some of those models.

Recently, Preskill, Trivedi, Wilczek and Wise (PTWW) [3.2] have reported an interesting scenario to solve the cosmological domain wall problem. They have pointed out that because some discrete symmetry can be anomalous due to the QCD axial anomaly and instantons, a non-perturbative communication between the Higgs sector and the QCD sector leads to a tiny but cosmologically significant splitting of the vacuum degeneracy. Incorporating PTWW's idea, Krauss and Rey [3.3] have shown that certain models of spontaneous CP violation can in principle avoid the domain wall problem provided that CP is slightly broken by θ_{QCD} in strong interactions. In this chapter, we examine the idea by computing the effective potential for Higgs bosons in the presence of QCD chiral anomaly. We show that the instanton dynamics for light quarks does break Z_2 symmetry of the two-doublet standard model. However, it may also lead to a spontaneous CP symmetry breaking.

3.2 A Simple Model

To illustrate how the anomalous discrete symmetry arises, let us first consider a simple model with spontaneous CP violation. The prototype of this model was first considered by T. D. Lee [3.4]. The lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial\varphi)^2 - \lambda^2(\varphi^2 - \eta^2)^2 + \bar{\psi}(\not{\partial} + m - if\gamma_5\varphi)\psi \tag{3.1}$$

where the Higgs field φ belongs to a real representation. The minimum of the potential corresponds to $\langle\varphi\rangle = \pm\eta$ and CP symmetry is spontaneously broken. It was first pointed out by KOZ that the degeneracy of CP conjugate vacua $\langle\varphi\rangle = \eta$ and $\langle\varphi\rangle = -\eta$ results in a serious domain wall problem in cosmology [3.1]. However, the situation is quite different if the fermion field ψ suffers from non-abelian gauge interactions. In that case, (3.1) can be extended to include, for example, color interactions (φ is of course colorless)

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \lambda^2(\varphi^2 - \eta^2)^2 + \bar{\psi}(\not{D} + m - if\gamma_5\varphi)\psi - \frac{1}{4}F^2 - i\theta F\tilde{F} \tag{3.2}$$

where $F\tilde{F} = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$. Though CP is explicitly broken by the θ-term if $\theta \neq 0, \pi$, the domain wall problem persists at the tree level because the vacua $\langle\varphi\rangle = \pm\eta$ are still degenerate. However, we show that the degeneracy of the vacua will be lifted by taking into account the chiral anomaly or the instanton effect.

The effective action of the Higgs field is calculated as

$$\mathbf{Z} = \int \mathcal{D}(\varphi)e^{-S_0[\varphi]}\int \mathcal{D}(A,\bar{\psi},\psi)e^{-S[\bar{\psi},\bar{\psi};A;\varphi]} = \int \mathcal{D}\varphi e^{-S_{eff}[\varphi]} \tag{3.3}$$

where

$$S_{eff}[\varphi] = S_0[\varphi] + \Delta S[\varphi] \tag{3.4}$$

and the quantum correction is given

$$\Delta S[\varphi] = -\ln \int \mathcal{D}(A;\bar{\psi},\psi)e^{-S[\bar{\psi},\psi;A;\varphi]} \equiv -\ln \tilde{\mathbf{Z}}. \tag{3.5}$$

The calculation of $\tilde{\mathbf{Z}}[\varphi]$ in the instanton field follows the standard semiclassical approximation method as illustrated in, e. g. , Ref.[3.5]

$$\tilde{\mathbf{Z}}[\varphi] = \sum_{\nu} \int \mathcal{D}[A_{cl}]_{\nu} e^{-S[A_{cl}]} \det{}^{-1/2} M_A \det M_{\psi} \det M_{gh} \tag{3.6}$$

where

$$\begin{aligned} M_A &= -D^2 - 2F \\ M_{gh} &= -D^2 \\ M_{\psi} &= \not{D} + m - if\gamma_5\varphi \end{aligned} \tag{3.7}$$

and ν stands for the winding number of the non-trivial topological gauge configuration. If the effective potential is of concern, we can take φ in M_{ψ} as a constant field. The new physics comes from the zero modes of the fermion determinant in the instanton field A_{cl}. We factorize $\det M_{\psi}$ as follows

$$\det M_{\psi} = \det{}^{(0)} M_{\psi} \det{}' M_{\psi} \tag{3.8}$$

where $\det^{(0)}$ denotes contributions from the subspace of zero modes of $\not{D}$. According to the index theorem [3.6], $\not{D}$ has a zero mode with chirality -1 ($\gamma_5 = -1$) in a single instanton field [3.7]. Thus we have

$$\det{}^{(0)} M_{\psi} = m + if\varphi. \tag{3.9}$$

The prime in $\det' M_{\psi}$ reminds us to exclude zero modes from the eigenvalue product. Since $[\not{D}, \gamma_5] \neq 0$, M_{ψ} cannot be diagonalized in the basis of eigenvectors of $\not{D}$. The non-vanishing eigenvalues of $\not{D}$ always appear in pairs, i. e. if $\not{D}\varphi_n = \lambda_n\varphi_n$ where $\lambda_n \neq 0$, then $\not{D}\gamma_5\varphi_n = -\gamma_5\not{D}\varphi_n = -\lambda_n\gamma_5\varphi_n$, namely both λ_n and $-\lambda_n$ are eigenvalues of $\not{D}$. In addition, γ_5 takes φ_n to φ_{-n}. Therefore

$$\begin{aligned} \det{}'M_\psi &= \prod_{\lambda_n>0} \det\begin{pmatrix} i\lambda_n + m & -if\varphi \\ -if\varphi & -i\lambda_n + m \end{pmatrix} = \prod_{\lambda_n>0} (\lambda_n^2 + m^2 + f^2\varphi^2) \\ &= \det{}'^{1/2}(-\not{D}^2 + m^2 + f^2\varphi^2), \end{aligned} \tag{3.10}$$

i. e. $\det' M_\psi$ is a function of φ^2 which does not break the discrete symmetry. It is to be emphasized that the above analysis does not depend on the details of the instanton dynamics. It is the result of using the index theorem, which represents the general feature of the chiral anomaly in a gauge theory.

Though we could proceed to analyze in general the effective potential based on Eqs. (3.9) and (3.10), we still would like to obtain the concrete form of V_{eff} in the dilute gas approximation (DGA) [3.8]. In the DGA,

$$\tilde{\mathbf{Z}}[\varphi] = \det(-\partial^2 + m^2 + f^2\varphi^2)\exp(\tilde{\mathbf{Z}}_+ + \tilde{\mathbf{Z}}_-) \tag{3.11}$$

where

$$\begin{aligned} \tilde{\mathbf{Z}}_+[\varphi] &= VKe^{i\theta}(m + f\varphi) \\ \tilde{\mathbf{Z}}_-[\varphi] &= VKe^{-i\theta}(m - f\varphi) \end{aligned} \tag{3.12}$$

and

$$K = 1.34C_{N_c}\int \frac{d\rho}{\rho^4}\left(\frac{8\pi^2}{g^2(\rho)}\right)^{2N_c} e^{-\frac{8\pi^2}{g^2(\rho)}}. \tag{3.13}$$

ρ is the instanton density, $C_{N_c} = \frac{N_c^2-1}{2N_c}$, N_c is the number of colors. In deriving (3.12), we have assumed that $m + f\langle\varphi\rangle$ is small compared to $\Lambda_{\rm QCD}$. Noticing that $\ln\det(-\partial^2 + m^2 + f^2\varphi^2)$ contains terms which can be absorbed into the tree level lagrangian by redefining λ^2 and η, we obtain the following effective potential (strictly speaking in the large N_c limit)

$$V_{eff} = \lambda^2(\varphi^2 - \eta^2)^2 + Ke^{i\theta}(m + if\varphi) + Ke^{-i\theta}(m - if\varphi). \tag{3.14}$$

Clearly, the last two terms (we shall call them the K-term) *explicitly* break CP symmetry when $\theta \neq 0$, for they are not invariant under $T\varphi T^{-1} = -\varphi$. The split in the energy density between the CP conjugate vacua $\langle\varphi\rangle = \eta$ and $\langle\varphi\rangle = -\eta$ is given

$$\Delta E_{vac} = |V_{eff}(\eta) - V_{eff}(-\eta)| = 4Kf\sin\theta|\langle\varphi\rangle|. \tag{3.15}$$

Therefore, domain walls created at the scale $\langle\varphi\rangle$ will feel an energy difference between the two sides of the wall. The false vacuum at some space point will begin to decay towards the true vacuum.

3.3 Two-Higgs-Doublet Model

Another perhaps more interesting example to observe the anomalous discrete symmetry is to consider the two Higgs doublets model, which is the simplest allowed extension of the standard model. It has been known for many years that experimentally there are no flavor-changing neutral current (FCNC) weak interactions, none with anything like the strength of the familiar charged-current weak interactions. The observed suppression of the FCNC is so dramatic numerically that one finds it hard to believe that it comes about because the parameters of the theory just happen to take certain values. Glashow and Weinberg [3.9] propose that the conservation of flavors by the neutral currents is *natural*, and that it follows from the symmetry structure of the theory, and does not depend on the values taken by the parameters of the theory.

The condition for a natural neutral flavor conservation (NFC) puts stringent restrictions on the system of Higgs bosons. It requires a certain form of the Yukawa couplings. Generally the Yukawa interactions that respect $SU(2) \times U(1)$ symmetry have the following terms

$$\sum_i \bar{Q}_L f_U^i U_R \phi_i + \sum_i \bar{Q}_L f_D^i D_R \tilde{\phi}_i + \text{H.C.} \tag{3.16}$$

where each Higgs boson ϕ_i $(i = 1, 2)$ can couple with both the charge $\frac{2}{3}$ quarks (U_R) and the charge $-\frac{1}{3}$ quarks (D_R), f_U^i and f_D^i are 3×3 Yukawa coupling matrices in flavor space, $\tilde{\phi}_2 = i\sigma_2 \phi_2^*$. When the Higgs bosons develop the vacuum expectation values, the quarks obtain the mass matrices

$$M_U = \sum_i f_U^i \langle \phi_i \rangle \quad ; \quad M_D = \sum_i f_D^i \left\langle \tilde{\phi}_i \right\rangle . \tag{3.17}$$

It is clear that when M_U and M_D are diagonalized by redefining the quark fields in flavor indices, the Yukawa coupling matrices are not in general diagonalized. A neutral Higgs boson H may have off-diagonal interactions such as $d + H \to s$, then its exchange can produce an effective $\Delta S = 2$ Fermi interaction

$$\bar{s} + d \to H \to s + \bar{d} \tag{3.18}$$

which may result in a contradiction with the observed small K_1^0-K_2^0 mass difference. Therefore, under the requirement of naturalness, it is essential that each Higgs boson only couples with one type quarks, either U_R or D_R, but not both.

Such an arrangement can be most naturally implemented by imposing some additional symmetries. One way is to have a global symmetry, for instance, an additional $U(1)$ symmetry under which U_R and ϕ_1 transform non-trivially but D_R and ϕ_2 do not. It, however, turns out that when a continuous symmetry such as a $U(1)$ is spontaneously broken, it is always accompanied with a massless physical excitation (known as the axion on general grounds) which is nowhere to be found so far. The other way, as we shall consider below, is to impose a Z_2 discrete symmetry under which

$$\phi_1 \to \phi_1, \quad \phi_2 \to -\phi_2; \qquad U_R \to U_R, \quad D_R \to -D_R. \tag{3.19}$$

The most general, renormalizable Higgs potential and Yukawa interactions which respect (3.19) read

$$\begin{aligned} V_0(\phi_1, \phi_2) &= -m_1^2\phi_1^\dagger\phi_1 - m_2^2\phi_2^\dagger\phi_2 + a_{11}(\phi_1^\dagger\phi_1)^2 + a_{22}(\phi_2^\dagger\phi_2)^2 \\ &\quad + a_{12}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + b_{12}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) - \left[\lambda(\phi_1^\dagger\phi_2)^2 + \lambda^*(\phi_2^\dagger\phi_1)^2\right] \end{aligned} \tag{3.20}$$

and

$$\mathcal{L}_Y = \bar{Q}_L f_U U_R \phi_1 + \bar{Q}_L f_D D_R \tilde{\phi}_2 + \text{h.c.} \tag{3.21}$$

where a natural neutral flavor conservation is achieved at the tree level since the Yukawa coupling matrices will be diagonal in the basis in which the mass matrices are diagonal. The hermicity of V_0 requires coefficients in (3.20) except for λ to be real. We shall examine the CP violation (SCPV) in this model in the next section. We first choose f_U and f_D to be real, $\theta_{\rm QCD} = 0$ in the QCD sector and a real but positive λ (the reason for it will be stated in the next section) in order to study a CP-conserving theory. When ϕ_1 and ϕ_2 acquire VEV's, Z_2 symmetry in (3.19) is spontaneously broken, which poses dangers for cosmology. PTWW have argued that when the non-perturbative QCD effect turns on, it breaks Z_2 symmetry and solves the domain wall problem.

To see explicitly how PTWW's idea works, we attempt to compute the K-term in the effective potential following the same procedure as in the previous model. We will first consider one generation of light quarks consisting of u and d ($m_u, m_d \ll \Lambda_{\rm QCD}$) to simplify the problem. The Higgs coupling to light quarks can be rewritten in a form

$$\mathcal{L}_m = (\,\bar{u}_L \quad \bar{d}_L\,)\, H \begin{pmatrix} u_R \\ d_R \end{pmatrix} + (\,\bar{u}_R \quad \bar{d}_R\,)\, H^\dagger \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{3.22}$$

where

$$H \equiv \begin{pmatrix} f_d \phi_2^0 * & f_u \phi_1^+ \\ -f_d \phi_2^- & f_u \phi_1^0 \end{pmatrix}. \tag{3.23}$$

Thus it is easy to identify

$$M_\psi = \not{D} + \frac{1}{2}(H + H^\dagger) + \frac{1}{2}(H - H^\dagger)\gamma_5 \tag{3.24}$$

where $\det M_\psi$ runs over color, spinor as well as flavor indices,

$$\det{}^{(0)} M_\psi = \begin{cases} \det H^\dagger = f_u f_d \phi_1^\dagger \phi_2 & \text{for a single instanton} \\ \det H = f_u f_d \phi_2^\dagger \phi_1 & \text{for a single anti-instanton} \end{cases} \tag{3.25}$$

and

$$\begin{aligned} \det{}' M_\psi &= \det{}'^{1/2}(-\not{D}^2 + HH^\dagger) \\ &= \prod_{\lambda_n > 0} [\lambda_n^4 + \lambda_n^2 (f_u^2 \phi_1^\dagger \phi_1 + f_d^2 \phi_2^\dagger \phi_2) + f_u^2 f_d^2 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1)]. \end{aligned} \tag{3.26}$$

It is clear that $\det' M_\psi$ can be absorbed into $V_0(\phi_1, \phi_2)$ in (3.20) but $\det^{(0)} M_\psi$ constitutes the so-called the K-term which breaks Z_2 symmetry. The effective potential reads

$$V_{eff}(\phi_1, \phi_2) = V_0(\phi_1, \phi_2) + K f_u f_d (\phi_1^\dagger \phi_2 + \phi_2^\dagger \phi_1) \tag{3.27}$$

where

$$K = (1.34)^2 C_{N_c} \int \frac{d\rho}{\rho^3} \left(\frac{8\pi^2}{g^2(\rho)} \right)^{2N_c} \exp\left(-\frac{8\pi^2}{g^2(\rho)} \right). \tag{3.28}$$

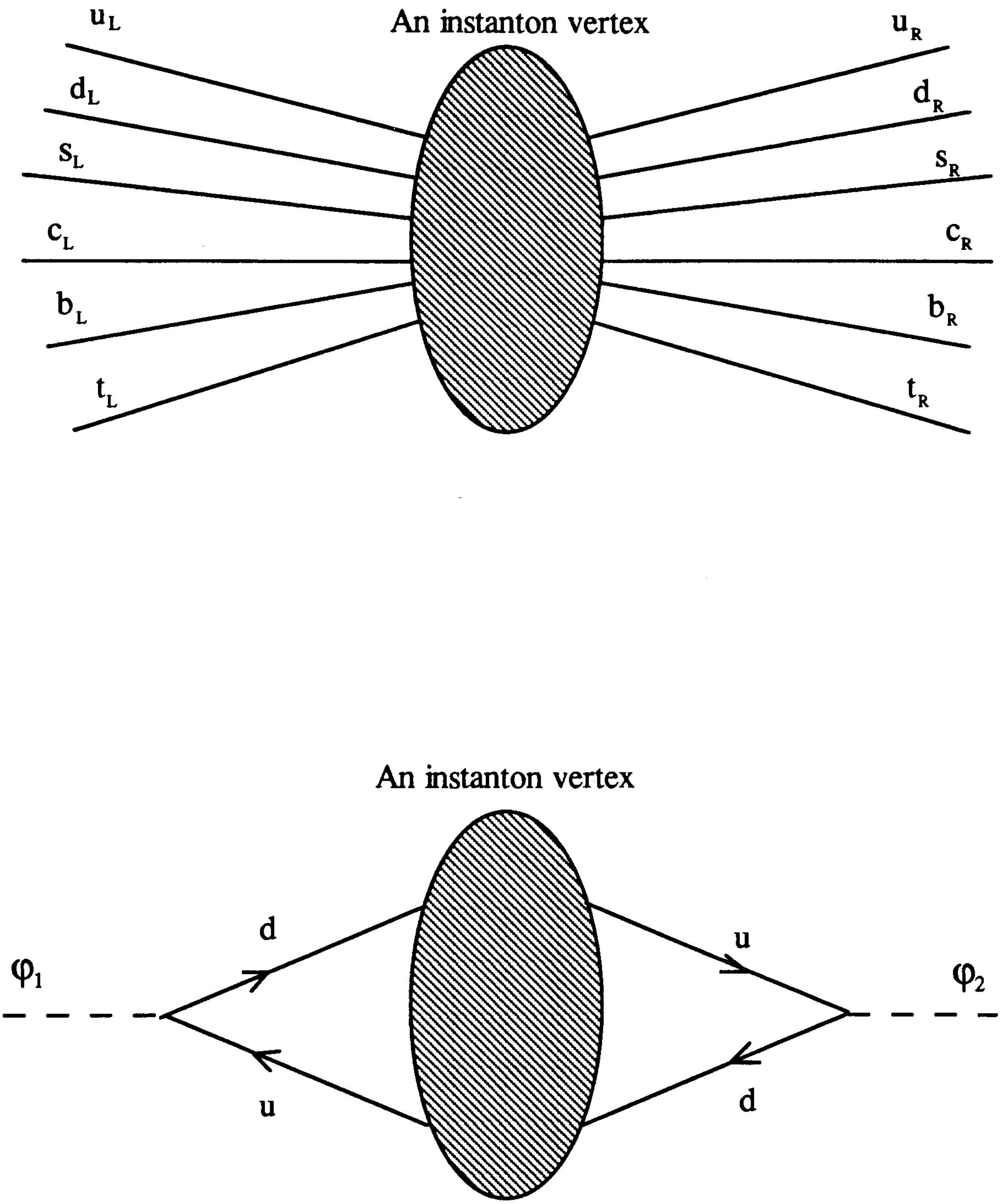

Figure 3.1: The instanton vertex and the instanton-induced coupling between φ_1 and φ_2.

K is of dimension 2. The derivation of the K-term is illustrated in terms of the Feynman diagrammatics in Figure. 3.1.

When $\lambda > 0$, it can be readily shown that the Z_2-related (v_1, v_2) and $(v_1, -v_2)$ (where v_1 and v_2 are real) are local minima of $V_{eff}(\phi_1, \phi_2)$. However, they are not degenerate because of the K-term. The difference in the energy density between these two vacua (v_1, v_2) and $(v_1, -v_2)$ is given by

$$\Delta E_{vac} = |V_{eff}(v_1, v_2) - V_{eff}(v_1, -v_2)| \simeq 4K f_u f_d v_1 v_2 = 4K m_u m_d. \tag{3.29}$$

K is the vacuum-to-vacuum amplitude in the instanton field. It is also the amplitude of the axial $U(1)$ symmetry breaking in QCD needed to solve the $U(1)$ problem. It has been estimated in [3.11] in connection with the $U(1)$ particle mass

$$K \sim (m_\eta^2 - m_\pi^2). \tag{3.30}$$

Thus $\Delta E_{vac} \simeq 10^{-4} \sim 10^{-5}\text{GeV}^4$, which is tiny but significant enough to solve the domain wall problem associated with Z_2 symmetry [3.2].

The main feature of the cosmological argument is the following. The symmetry breaking sets in at the weak scale, and the energy barrier between the nearly degenerate vacua are generically also of this magnitude. The energy difference between the vacua, on the contrary, is set by the strong scale – and actually, as we have seen, further suppressed by light quark masses. Furthermore, this energy difference depends sharply on the temperature [3.12] and vanishes rapidly as this temperature exceeds the strong scale, thus we expect that at the temperatures where the symmetry is spontaneously broken, the energy difference has negligible dynamical import. At and below these temperatures, until the strong interactions kick in, the cosmology will develop as if it were heading towards a domain-wall dominated universe. However, at the strong scale the energy difference will cease to be negligible. It creates a pressure difference driving the domain walls into the false vacuum regions, and ultimately squeezing them out.

3.4 Induced Weak CP Violation

3.4.1 The CP-Violating Phase

When $\lambda < 0$, it turns out that neither (v_1, v_2) nor $(v_1, -v_2)$ is a minimum. In fact, they are both local maxima of V_{eff}. The true vacuum configuration, denoted by $(v_1, v_2 e^{i\alpha})$, which minimizes the effective potential acquires a non-trivial phase α ($\alpha \neq 0, \pi$). The domain wall problem associated with Z_2 is hence automatically resolved since the Z_2-related configuration $(v_1, -v_2 e^{i\alpha})$ is no longer the minimum of the effective potential.

However, what interests us is that the existence of the relative phase between $\langle\varphi_1\rangle$ and $\langle\varphi_2\rangle$ breaks CP symmetry in weak interactions [3.13]. To see how it is possible, we calculate the α-dependent terms in the effective potential

$$V_{eff}(\alpha) = 2|\lambda| v_1^2 v_2^2 \cos 2\alpha + 2K f_u f_d v_1 v_2 \cos\alpha. \tag{3.31}$$

By minimizing $V_{eff}(\alpha)$ with respect to α one obtains

$$\cos\alpha = -\frac{K m_u m_d}{4|\lambda| v^4} \frac{\xi^2}{(1+\xi^2)^2} \qquad \text{if } \lambda < 0 \tag{3.32}$$

where $v = \sqrt{v_1^2 + v_2^2}$, which is 250 GeV (the electroweak scale) and $\xi = \frac{v_1}{v_2}$. Typically, ξ takes values from 10^{-2} to O(1). As long as the scalar fields couple only weakly to themselves, we must have $|\lambda| << 1$, say $|\lambda| \sim 10^{-2}$. Then $\cos\alpha$ in (3.32) is estimated 10^{-8} to 10^{-12}. Contrary to one's suspicion that Z_2 symmetry in the two-doublet model actually forbids a CP violation, it can occur when Z_2 is explicitly broken by the quantum effects. The strength of the effect is dynamically determined by the instanton amplitude factor K.

Does this new source of CP violation lead to any observable effects in electroweak interactions? Obviously, the phases of quark masses and Yukawa couplings originating from SCPV can be rotated away by making appropriate hypercharge transformation. Thus the CP-breaking Cabbibo-Kobayashi-Maskawa (CKM) matrix does not arise in this model. The CP nonconservation is entirely due to neutral Higgs boson exchanges, i.e. through the mixing between scalar fields and pseudoscalar fields while the mixing probability is proportional to $\sin\alpha\cos\alpha$ which is about 10^{-8} to 10^{-12}. All CP-violating processes are to be suppressed by this factor. Its contribution to $K_L \rightarrow 2\pi$ can be neglected since this process involves charged flavor changing. The electric dipole moment of the neutron (NEDM) will receive suppression factors, a 10^{-8} from Higgs propagators if Higgs bosons are of 100GeV and a 10^{-8} to 10^{-12} from the mixings. Thus the NEDM is estimated to be 10^{-28} to 10^{-32}e·cm, which can be six orders of magnitude larger than the standard model prediction based on the CKM mechanism (which is 10^{-34}e·cm) and may be detectable in future experimental measurements. The current experimental upper limit on the NEDM is about 10^{-26}e·cm, two more orders of magnitude precision in experiment is required to test this model. It may not be sufficient to generate the electroweak baryogenesis based on the weak phase transition since the instanton effect is greatly suppressed at temperature characteristic of the weak scale [3.14]. Even though there are several ways of enhancing the CP violating effects by, for example, allowing a larger difference between v_1 and v_2, or having nearly degenerate masses for Higgs bosons, it would seem unnatural to accommodate them in the standard model from a theoretical viewpoint.

3.4.2 Origin and Generalization

There are many theoretical questions raised regarding this model that we intend to address below. What is the origin of this new source of *weak* CP violation? Is it a spontaneous CP symmetry breaking so that the Z_2 domain wall problem is replaced by a CP domain wall problem? Is there a strong CP problem in this model with the appearance of a non-zero α?

To answer these questions, we must first understand what has happened when λ changes sign. In the presence of strong interactions, the phase of the scalar coupling λ is not a *physical* parameter. One can always make an appropriate hypercharge transformation (or a redefinition of the related fields)

$$\begin{aligned} \phi_1 &\rightarrow e^{i\beta/4}\phi_1 \quad , \quad \phi_2 \rightarrow e^{-i\beta/4}\phi_2 \; ; \\ U_R &\rightarrow e^{-i\beta/4}U_R \quad , \quad D_R \rightarrow e^{-i\beta/4}D_R \end{aligned} \tag{3.33}$$

where β is the phase of λ ($\lambda = |\lambda|e^{i\beta}$), such that

$$-\left[\lambda(\phi_1^\dagger\phi_2)^2 + \lambda^*(\phi_2^\dagger\phi_1)^2\right] \longrightarrow -|\lambda|\left[(\phi_1^\dagger\phi_2)^2 + (\phi_2^\dagger\phi_1)^2\right] \tag{3.34}$$

without changing any other terms in the weak interaction sector. Therefore, only the absolute value of λ is defined as the physical couping constant. A positive λ (where $\beta = 0$) and a negative λ (where $\beta = \pi$) do not make any physical distinctions – they are simply equivalent. CP symmetry is conserved for any value of β, as is well known. However, when the quarks suffer from the non-abelian strong interactions, the transformation in (3.33) induces a change in the strong interaction sector, more specifically, in the θ term

$$\theta F\tilde{F} \longrightarrow \left(\theta + \frac{n_G\beta}{2}\right) F\tilde{F} \tag{3.35}$$

where n_G is the number of quark generations. This can be also seen without making the transformation in (3.33). One can keep the original phase of λ and compute the vacuum configuration by minimizing $V_0(\phi_1, \phi_2)$. The VEV at the classical level is $(v_1, v_2 e^{i\beta/2})$ and the quark mass matrices are

$$M_U = f_U v_1 \quad , \quad M_D = f_D v_2 e^{i\beta/2} \, . \tag{3.36}$$

Rotate the down-type quark fields to get rid of $e^{i\beta/2}$, then the θ term becomes $(\theta + \frac{n_G \beta}{2}) F\tilde{F}$. Hence, if initially $\theta = 0$ and $\beta = \pi$ (where λ is negative) as earlier, at the classical level, one has an effective θ term $(\frac{n_G \pi}{2}) F\tilde{F}$, which *maximally* violate CP symmetry in strong interactions if $n_G = 1$ or 3. Of course, $(v_1, v_2 e^{i\pi/2})$ is not CP-violating in the weak sector since the mixing probability is proportional to $\cos\frac{\pi}{2}\sin\frac{\pi}{2} = 0$. Only when the quantum effects or the instanton effects are taken into account, the weak CP violation arises from balancing between the induced K-term and the λ-term in the potential, and the VEV's develop a relative phase α slightly different from $\frac{\pi}{2}$

$$\alpha \simeq \frac{\pi}{2} - \frac{K m_u m_d}{4|\lambda| v^4} \frac{\xi^2}{(1+\xi^2)^2} \tag{3.37}$$

where $\cos\alpha \sin\alpha \neq 0$. Therefore, the weak CP violation is possible only when CP is explicitly broken in the strong interactions. Since both a α and a $-\alpha$ are degenerate solutions as $\cos\alpha$ in (3.32) is an even function of α, the weak CP violation in this model should be referred to as a spontaneous symmetry breaking, which may also cause a domain wall problem in cosmology.

To further clarify these issues, we need to extend our discussion to a more realistic case where there may be any number of generations, the Yukawa coupling matrices f_U and f_D can be complex to incorporate an explicit CP violation in the manner of CKM mechanism. The phase of $\det f_U$ and $\det f_D$ can be rotated away by redefining the right-handed quark fields while θ changes correspondingly according to the anomaly relation. The phase of λ can be removed by making a transformation of the type (3.33). Now the θ term has a coefficient

$$\theta_{\text{QCD}} = \theta + \arg\det f_U + \arg\det f_D + \frac{n_G\beta}{2}\,. \tag{3.38}$$

We parametrize ϕ_1 and ϕ_2 in terms of their phase fields $\alpha_1(x)$ and $\alpha_2(x)$ as

$$\phi_1 \longrightarrow v_1 e^{i\alpha_1(x)} \qquad ; \qquad \phi_2 \longrightarrow v_2 e^{i\alpha_2(x)} \tag{3.39}$$

and denote the relative phase field by $\alpha(x) \equiv \alpha_1(x) - \alpha_2(x)$. The α_1- and α_2-dependence of the Yukawa couplings can be removed by making *local* chiral rotations. Because of the chiral anomaly, the θ-term becomes

$$(\theta_{\text{QCD}} + n_G\alpha(x))\, F\tilde{F}\,. \tag{3.40}$$

The effective potential for $\alpha(x)$ can be calculated without resorting to an explicit instanton computation [3.15]

$$V_{eff} = \langle\langle\nu^2\rangle\rangle_{\text{QCD}} \cos(\theta_{\text{QCD}} + n_G\alpha) - 2|\lambda| v_1^2 v_2^2 \cos 2\alpha \tag{3.41}$$

where the topological susceptibility $\langle\langle\nu\rangle\rangle_{\text{QCD}}$ is defined by

$$\langle\langle\nu^2\rangle\rangle_{\text{QCD}} = \int d^4x \langle \mathrm{T} iF\tilde{F}(x)\, iF\tilde{F}(0)\rangle. \tag{3.42}$$

By minimizing (3.41) one obtains

$$-3\langle\langle\nu^2\rangle\rangle_{\text{QCD}} \sin(\theta_{\text{QCD}} + 3\alpha) + 4|\lambda| v_1^2 v_2^2 \sin 2\alpha = 0 \tag{3.43}$$

and

$$-9\langle\langle\nu^2\rangle\rangle_{\text{QCD}} \cos(\theta_{\text{QCD}} + 3\alpha) + 8|\lambda| v_1^2 v_2^2 \cos 2\alpha > 0 \tag{3.44}$$

for stability, where we have taken $n_G = 3$. Bearing in mind that $|\lambda| v_1^2 v_2^2 \gg \langle\langle \nu^2 \rangle\rangle_{\text{QCD}}$, one needs $|\sin 2\alpha| \ll 1$ and a not too small value for $|\sin(\theta_{\text{QCD}} + 3\alpha)|$ in order for (3.43) to have a solution. Clearly, when $\theta_{\text{QCD}} = 0$, the only solutions are $\alpha = 0, \pi$ where $\cos 2\alpha > 0$ ($\alpha = \pm\frac{\pi}{2}$ are not minima since they do not satisfy the stability condition $\cos 2\alpha > 0$). This corresponds to the solution provided by PTWW, i.e. both (v_1, v_2) and $(v_1, -v_2)$ are local minima whose energy splitting is caused by the instanton effects. Again, a weak CP violation is not possible in this case.

However, when $\theta_{\text{QCD}} = \pm\frac{\pi}{2}$, which corresponds to a negative λ, a non-trivial solution α or $\pi - \alpha$ is found where

$$\alpha \simeq \frac{3\langle\langle \nu^2 \rangle\rangle_{\text{QCD}}}{4|\lambda| v_1^2 v_2^2}. \tag{3.45}$$

The non-vanishing relative phase between $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ indicates a spontaneous CP violation in the Higgs sector. The CP-violating mixing probability is proportional to $\cos\alpha \sin\alpha$. Expectedly, as it is a general feature of the spontaneously broken discrete symmetry, the vacua characterized by α or $\pi - \alpha$ are degenerate in energy density. In the general case where θ_{QCD} takes arbitrary values, a non-trivial solution always exists. It will lead to a mixture of an explicit CP violation and an induced spontaneous CP breaking. The order of magnitude of the CP angle α, however, is mainly determined by the strength of the SCPV, typically by the ratio of $\langle\langle \nu^2 \rangle\rangle_{\text{QCD}}$ to $|\lambda| v_1^2 v_2^2$, if $\sin(\theta_{\text{QCD}} + 3\alpha)$ is of O(1). The Z_2 domain wall problem is of course resolved by admitting a CP-violating solution. And better yet, when $\theta_{\text{QCD}} \neq \pm\frac{\pi}{2}$, the solution is unique because $\sin(\theta_{\text{QCD}} + 3\alpha)$ and $\sin 2\alpha$ do not have a common periodic structure. The domain walls associted with CP violation simply do not form.

3.5 Discussions

We have studied a two-Higgs-doublet standard model with a Z_2 symmetry. In the presence of strong interactions, particularly, the QCD θ term, the analysis of the domain wall structure changes drastically. Depending on different values of the QCD θ-parameter, the theory exhibits different dynamical phases. A CP violation in the Higgs sector is in general possible, with strength typically 10^{-8}-10^{-12}. The cosmological domain walls associated with Z_2 or CP need not to form.

PTWW have analyzed a special case when $\theta_{QCD} = 0$, in which, as is extensively studied in Chapter 2, CP is conserved in strong interactions. However, without a plausible solution to the strong CP problem, choosing this particular value for θ_{QCD} is very unnatural from a theoretical point of view. They further point out that incorporation of Peccei-Quinn symmetry into this model so that θ_{QCD} can dynamically relax to zero from any initial value does not reproduce the desired results. It comes with other cosmological problems known as axion domain walls. In spite of lack of an understanding of the strong CP problem, we have unbiasedly studied a general case where θ_{QCD} is arbitrary and shown an interesting possibility to have a spontaneous CP violation in the weak interaction sector. This new CP violation is compatible with that in the CKM matrix and serves as a supplement to it, especially in the NEDM. The problem of this scenario is of course the strong CP problem, which, I believe, is an independent problem and should be solved separately from the Z_2 domain wall problem. An investigation on this issue is in progress.

Chapter 4

Self-mass for Massive Quark

4.1 Introduction

Dynamical chiral symmetry breaking in QCD has been studied extensively [4.1] and yet has not been completely understood. A rigorous proof of this phenomenon based on a reliable non-perturbative scheme has not so far been given. The standard method to study this problem is to start with the Schwinger-Dyson (SD) equation [4.2]. However, some difficulties arise when applying this equation. The most serious one is the use of a single gluon exchange (the ladder approximation). Although the SD equation incorporates non-perturbative features, the ladder approximation of the integral kernel is essentially a perturbative scheme and its validity must be justified by the smallness of the effective gauge coupling.

Some efforts have been made in order to establish the validity of the ladder approximation by making use of the asymptotic freedom in QCD. Instead of the complete solution of the SD equation, the asymptotic solutions when $p^2 \to \infty$ have been discussed where the effective coupling is small [4.3]. Two types of solutions, the irregular and regular ones, have been found by linearizing the differential SD equation in the ultraviolet region. In the presence of a current mass, a linear combination of both solutions satisfies the UV boundary condition while the irregular solution is dominant in the UV region. It is then speculated that the regular solution which goes to zero faster than the irregular one in the UV region may become important, and perhaps becomes dominant in the infrared region if the current mass is small. As a result, the light quark may acquire a dynamical mass much larger than its current mass in the IR region. This has been referred to as the dynamical chiral symmetry breaking. However, the above statement stays as a speculation as long as we do not have an appropriate approximation to make the SD equation solvable in the IR region. Some attempts on this issue have been made by assuming a certain behavior of the effective coupling constant in the IR region [4.4]. In general, there has not been a complete solution to the SD equation valid both in the UV and in the IR regions with approximations based on QCD.

In this chapter, we study a slightly different aspect of the solution to the SD equation however approaching to the same problem: the self-mass for a very massive quark. When the quark current mass is much larger than Λ_{QCD}, a renormalization group equation (RGE) allows us to derive an approximation to the integral kernel valid both in the UV and in the IR regions. A complete solution to the non-linear SD equation is then obtained by numerical means. To observe how the gauge interaction affects the result, we rescale all dimensionful quantities such as the self-mass, the momentum and the current mass by the current mass. As an effect of the renormalization, the interaction strength and the rescaled quark self-mass are functions of the current mass. We find that it does go up in the IR region when the current mass becomes small, especially when $p^2 \to 0$. It is then expected based on extrapolation that the constituent mass for a light quark (defined as the self-mass at $p^2 \to 0$) can be very large compared with its current mass. Certainly, we cannot quite approach the point where the quark current mass is as small as a few MeV's. Our method is only applicable when the current mass is bigger than Λ_{QCD}. Thus at best the strange quark may be accounted for in our method. We do not claim that our result is a proof of the dynamical chiral symmetry breaking. However, it may approach to the same limit from a different point of view, as much as the approach based on the momentum extrapolation from the UV region to the IR region, if not better. In addition, it gives a complete description of the self-mass for the very massive quarks such as the charm and the bottom quarks, and a picture of the transition going from the heavy quarks to the less massive quarks.

The plan of this chapter is as follows. In section 4.2 we derive the renormalized SD equation in the framework of QCD. Section 4.3 illustrates the RGE-improved integral kernel for the massive quark. The numerical solutions for different current masses are found and discussed in section 4.4.

4.2 Renormalized SD Equation

The standard form for the quark propagator is defined in the momentum space as

$$S(p) = \frac{1}{\not{p}A(p^2) - B(p^2)} \tag{4.1}$$

where $A(p^2)$ is the wave function and $B(p^2)$ is referred to as the quark self-mass. In the absence of gauge interactions $B(p^2)$ is equal to the quark current mass m_0. When the gauge interaction is turned on, $B(p^2)$ receives corrections. In general, $B(p^2)$ is a function of the momentum and the quark current mass m_0. The quark propagator satisfies the following Schwinger-Dyson equation

$$S(p) = \not{p} - m_0 + ig^2C_2(N)\int \frac{d^4k}{(2\pi)^4}\Gamma^\mu(p,k)S(k)D_{\mu\nu}(p-k)\gamma^\nu \tag{4.2}$$

where g is the gauge coupling constant, $C_2(N) = \frac{N^2-1}{2N}$ for $SU(N)$ gauge group, Γ^μ is the complete quark-antiquark vertex and $D_{\mu\nu}$ stands for the complete gluon propagator. As we can see from (4.1), if we are to solve $S(p)$ from (4.2), $\not{p}A(p^2)$ may dominate $B(p^2)$ in the UV region and we have to look for the subleading behavior of $S(p)$ in order to determine the self-mass. This complication may be simplified by use of the Ward identities (for simplicity, we consider all quarks having the same bare mass)

$$(p-k)^\mu\Gamma^i_{5\mu}(p,k) = -2im_0\Gamma^i_5(p,k) + \gamma_5\frac{\lambda_i}{2}S^{-1}(p) + S^{-1}(k)\frac{\lambda_i}{2}\gamma_5 \tag{4.3}$$

where $\Gamma^i_{5\mu}$'s are the vertices of the colorless axial-vector currents $J^i_{\mu 5} = \bar{\psi}\gamma_\mu\gamma_5\frac{\lambda_i}{2}\psi$, Γ^i_5's are the vertices of the colorless pseudoscalar densities $J^i_5 = \bar{\psi}\gamma_5\frac{\lambda_i}{2}\psi$, λ_i's are the $SU(N_f)$ matrices. The vertices $\Gamma^i_{5\mu}$ and Γ^i_5 satisfy equations of the Bethe-Salpeter type:

$$\Gamma^i_{5\mu}(p,k)_{\alpha\beta} = \frac{\lambda_i}{2}\left(\gamma_\mu\gamma_5\right)_{\alpha\beta} + \int\frac{d^4q}{(2\pi)^4}K(p,k,q)_{\alpha\beta\alpha'\beta'} \left[S(q)\Gamma^i_{5\mu}(q,q+p-k)S(q+p-k)\right]_{\alpha'\beta'} \;; \tag{4.4}$$

$$\Gamma^i_5(p,k)_{\alpha\beta} = \frac{\lambda_i}{2}\left(i\gamma_5\right)_{\alpha\beta} + \int\frac{d^4q}{(2\pi)^4}K(p,k,q)_{\alpha\beta\alpha'\beta'} \left[S(q)\Gamma^i_5(q,q+p-k)S(q+p-k)\right]_{\alpha'\beta'} \tag{4.5}$$

where $K(p,k,q)_{\alpha\beta\alpha'\beta'}$ is the 2PI fermion-antifermion scattering kernel.

Substituting (4.4) and (4.5) into (4.3) we get

$$\begin{aligned} i(\not{p}-\not{k})\gamma_5 &+ \int\frac{d^4q}{(2\pi)^4}K(p,k,q)\left[S(q+p-k)\gamma_5 + \gamma_5 S(q)\right] \\ &= 2m_0\gamma_5 + \gamma_5 S^{-1}(p) + S^{-1}(k)\gamma_5\,. \end{aligned} \tag{4.6}$$

By taking the limit $p - k \to 0$ and defining $q = \frac{1}{2}(p+k)$ we obtain

$$\gamma_5 B(q^2) = m_0\gamma_5 + \int \frac{\mathrm{d}^4 k}{(2\pi)^4} K(q,k) \left[S(k)\gamma_5 B(k^2) S(k)\right] . \tag{4.7}$$

Eq. (4.7) may be visualized by a skeleton diagram shown in Figure 4.1.

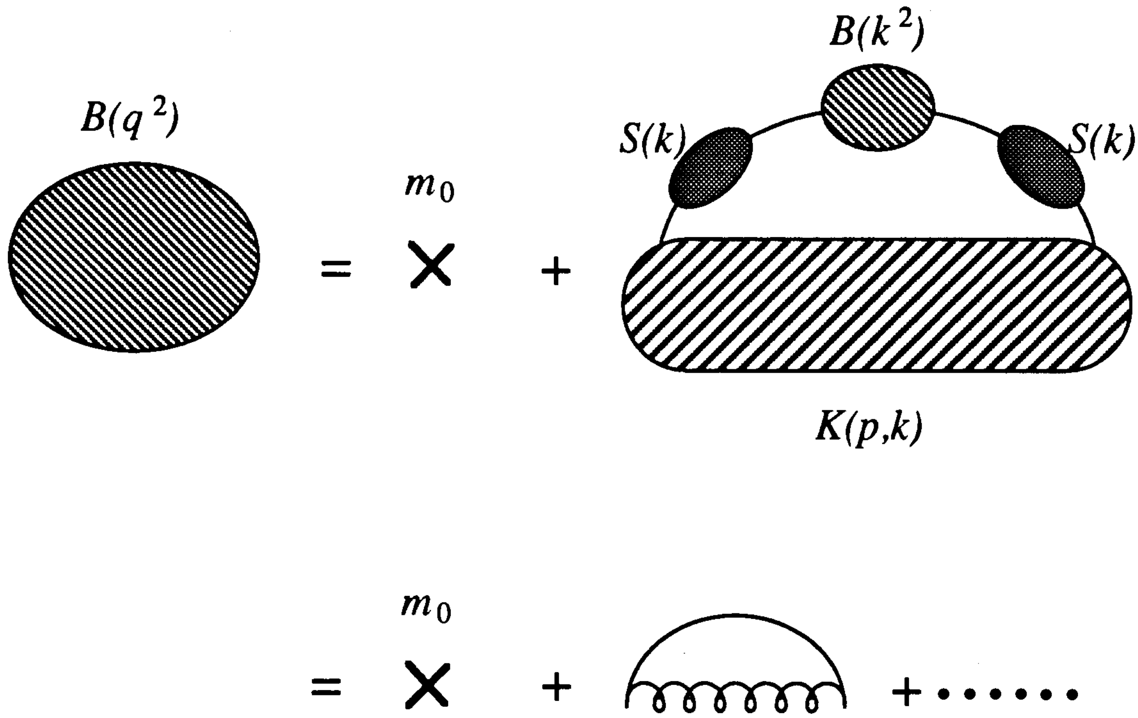

Figure 4.1: The skeleton diagrams of the SD equation and their perturbative expansions

Eq. (4.7) stands for the most general relation which the bare quark self-mass must satisfy. So far we have not made any approximations. To actually solve the equation, we need to know $A(q^2)$ and $K(q,k)$, which, in turn, satisfy another set of equations involving $B(q^2)$ and $S(q)$. This seems a hopeless circle unless we make some approximations. If the coupling constant in $K(q,k)$ is small, we can expand the kernel perturbatively in the sense of the Hartree-Fock approximation. The whole point of studying the SD equation in this perturbative scheme is that it still represents a resummation of infinitely many ladder diagrams which cannot be done in a pure perturbative calculation.

It is then clear that we have to renormalize the SD equation such that the quantities appearing in Eq. (4.7) become renormalized in order to carry out the perturbative expansion. Let us consider the renormalized functions

$$S_R(k) = Z_\psi^{-1}(\mu,\Lambda)S(k,\Lambda) \tag{4.8}$$

$$K_R(q,k) = Z_\psi^2(\mu,\Lambda)K(q,k,\Lambda)\,, \tag{4.9}$$

where the bare quantities depend on an ultraviolet cutoff Λ, $Z_\psi(\mu,\Lambda)$ is the renormalization constant for the fermion propagator and μ is the renormalization point. The bare current mass m_0, of course, is also dependent on Λ. Substituting (4.8) and (4.9) in (4.7) one then obtains the renormalized SD equation

$$\gamma_5 B_R(p^2) = Z_\psi(\mu,\Lambda)m_0\gamma_5 + \int \frac{\mathrm{d}^4k}{(2\pi)^4} K_R(p,k)\left[S_R(k)\gamma_5 B_R(k^2)S_R(k)\right]\,. \tag{4.10}$$

where the bare mass $m_0(\Lambda)$ is related to the renormalized mass in the leading-logarithmic approximation

$$m_0(\Lambda) = m(\mu)\left[\frac{\ln(\mu^2/\Lambda^2_{\mathrm{QCD}})}{\ln(\Lambda^2/\Lambda^2_{\mathrm{QCD}})}\right]^d \tag{4.11}$$

and $d = 3C_2(N)/\beta_0$ where $\beta_0 = 11 - 2N_f/3$. It is noteworthy that the renormalized quantities in (4.10) are in general functions of the momentum, the renormalized coupling constant $g(\mu)$, the renormalized current mass $m(\mu)$, the renormalized gauge parameter $\xi(\mu)$ and the renormalization point μ. $B_R(p^2)$, for example, is just a short hand notation. Below, we further drop the subscript "R" but it should be understood implicitly throughout the rest of the chapter.

4.3 Bethe-Salpeter Kernel for Massive Quark

Our goal is to solve the renormalized SD equation for the self-mass $B(p^2)$ of the heavy quark whose current mass is larger than Λ_{QCD}. We expand the Bethe-Salpeter kernel $K(p,k)$ perturbatively if the relevant coupling is small. A lowest few diagrams in the straightforward expansion in terms of the renormalized coupling $g(\mu)$ are depicted in Figure 4.2 and the contributions are

$$K\,(p,k;g(\mu),m(\mu),\xi(\mu);\mu)_{\alpha\beta\alpha'\beta'} = C_2(N)(\gamma^\mu)_{\beta\beta'}(\gamma^\nu)_{\alpha\alpha'}d_{\mu\nu}(p-k)$$
$$g^2(\mu)\left[1+\mathcal{O}\left(g^2(\mu)\ln\frac{4\pi\mu^2}{a_1p^2+a_2k^2+a_3(p-k)^2+a_4m^2}\right)\right] \qquad (4.12)$$

where $d_{\mu\nu}$ is the free gluon propagator and a_i's are some kinetic constants. In general, the renormalized gauge coupling $g(\mu)$ is not small in strong interactions, the series [...] in (4.12) does not converge and the perturbative expansion makes no sense. However, thanks to the asymptotic freedom in QCD, $g(\mu)$ can be very small if the renormalization point μ is chosen to be much larger than Λ_{QCD}. Recall that in the leading-logarithmic approximation

$$\frac{g^2(\mu)}{4\pi} = \frac{4\pi}{\beta_0 \ln \mu^2/\Lambda^2_{QCD}} \,. \qquad (4.13)$$

The kernel at different renormalization points are related through the RGE. Thus one can expand $K(p,k)$ at a very high scale where the effective coupling is very small and calculate $K(p,k)$ at a desired scale from the RGE. In the meantime, we also have to get rid of a large multiplicative logarithmic factor in (4.12) to guarantee that the higher order terms are negligible compared to 1. It is thus clear that μ^2 must be scaled to the biggest among p^2, k^2,$(p-k)^2$ and m^2 so that the logarithmic factor is small, in addition, it must be much larger than Λ^2_{QCD} so that $g^2(\mu)$ is also very small. Since there is always a factor $1/(p-k)^2$ in the gluon free propagator, the integral in

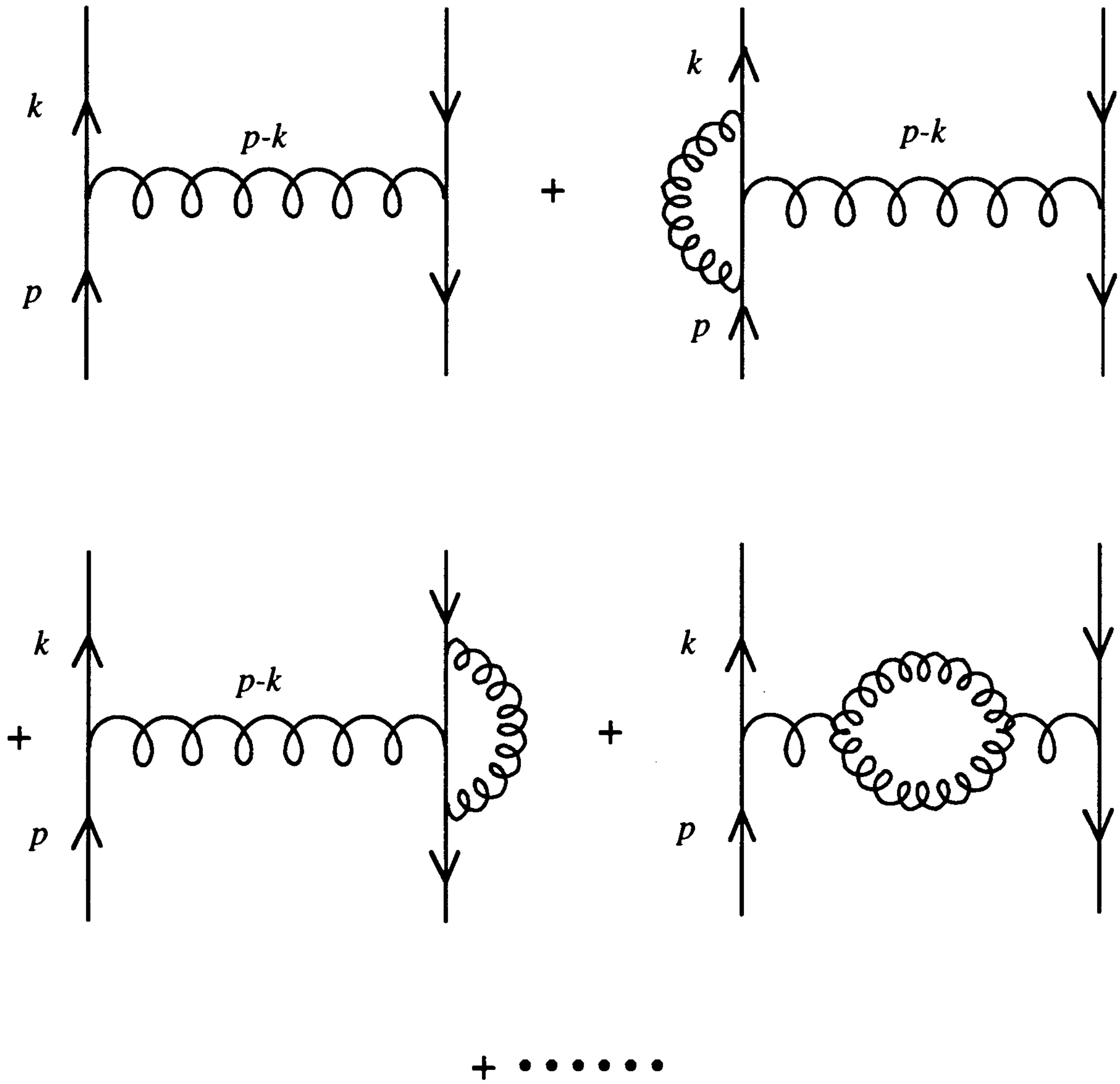

Figure 4.2: The perturbative expansion of the Bethe-Salpeter kernel $K(p, k)$

(4.12) gets its main contribution around $(p-k)^2 \sim 0$. Therefore we should expand the kernel at a scale characteristic of $\max\{p^2, k^2, m^2\}$ which itself must be larger than $\Lambda^2_{\rm QCD}$ and use the RGE to obtain a kernel at the physical scale.

The RGE analysis proceeds as follows [4.6]. The renormalized kernel satisfies an RGE

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} + \delta(g)\xi\frac{\partial}{\partial\xi} - 2\gamma_F(g)\right\} K\left(p,k;g(\mu),m(\mu),\xi(\mu);\mu\right) = 0 \tag{4.14}$$

which basically tells us how the kernel changes as the renormalization point μ changes. The solution to (4.14) is known and is given by

$$\begin{aligned} &K\left(p,k;g(\mu),m(\mu),\xi(\mu);\mu\right) \\ &= K\left(p,k;\bar{g}(t),\bar{m}(t),\bar{\xi}(t);\mu e^t\right)\exp\left[-2\int_{g(\mu)}^{\bar{g}(t)} dx\frac{\gamma_F(x)}{\beta(x)}\right] \end{aligned} \tag{4.15}$$

where

$$t = \int_{g(\mu)}^{\bar{g}(t)} \frac{dx}{\beta(x)} \quad , \qquad \bar{m}(t) = m(\mu)\left[\int_{g(\mu)}^{\bar{g}(t)} dx\frac{\gamma_m(x)}{\beta(x)}\right] , \tag{4.16}$$

$$\bar{\xi}(t) = \xi(\mu)\left[\int_{g(\mu)}^{\bar{g}(t)} dx\frac{\delta(x)}{\beta(x)}\right] .$$

Let us emphasize that $\bar{g}(t)$, $\bar{m}(t)$ and $\bar{\xi}(t)$ are functions of the new scale μe^t and $\Lambda_{\rm QCD}$, and are *not* functions of μ. The other useful relation is based on dimensional grounds when rescaling the dimensionful parameters. For example, the following relation holds ($t = \ln p/p_0$)

$$K\left(p,k;g(\mu),m(\mu),\xi(\mu);\mu\right) = e^{2t}K\left(p_0, e^{-t}k; g(\mu), e^{-t}m(\mu), \xi(\mu); e^{-t}\mu\right) . \tag{4.17}$$

To obtain a RGE-improved integral kernel, we choose a rescaling factor e^t where

$$t = \frac{1}{2}\ln\frac{\max\{p^2,k^2,m^2\}}{\mu^2} \tag{4.18}$$

and the on-shell current mass is defined by

$$m = m(\mu = m)\ . \tag{4.19}$$

Combining (4.15) and (4.17), one gets [4.7]

$$\begin{aligned} &K\left(p,k;g(\mu),m(\mu),\xi(\mu);\mu\right) \\ &= K\left(p,k;\bar{g}(t),\bar{m}(t),\bar{\xi}(t);\mu e^{t}\right)\exp\left[-2\int_{g(\mu)}^{\bar{g}(t)}\mathrm{d}x\frac{\gamma_F(x)}{\beta(x)}\right] \qquad (4.20)\\ &= e^{2t}K\left(e^{-t}p,e^{-t}k;\bar{g}(t),e^{-t}\bar{m}(t),\bar{\xi}(t);\mu\right)\exp\left[-2\int_{g(\mu)}^{\bar{g}(t)}\mathrm{d}x\frac{\gamma_F(x)}{\beta(x)}\right]\ . \qquad (4.21)\end{aligned}$$

Now we can calculate the effective kernel $K\left(e^{-t}p,e^{-t}k;\bar{g}(t),e^{-t}\bar{m}(t),\bar{\xi}(t);\mu\right)$ instead and substitute it back in (4.21). According to (4.18), the largest among $e^{-t}p$, $e^{-t}k$ and $e^{-t}\bar{m}(t)$ is μ, thus there is no large logarithmic factor in the expansion. The effective coupling $\bar{g}(t)$ is given in the leading-logarithmic approximation

$$\frac{\bar{g}^2(t)}{4\pi} = \frac{4\pi}{\beta_0 \ln\max\left(p^2,k^2,m^2\right)/\Lambda_{\mathrm{QCD}}^2} \tag{4.22}$$

which can be very small if $\max\left(p^2,k^2,m^2\right) \gg \Lambda_{\mathrm{QCD}}^2$. This requirement can be achieved if $p^2 \gg \Lambda_{\mathrm{QCD}}^2$ or $m^2 \gg \Lambda_{\mathrm{QCD}}^2$. k^2, however, is the integral variable which must run from 0 to ∞. Therefore, we can neglect the high order terms and write the effective kernel to a good approximation

$$\begin{aligned} K\left(e^{-t}p,e^{-t}k;\bar{g}(t),e^{-t}\bar{m}(t),\bar{\xi}(t);\mu\right)_{\alpha\beta\alpha'\beta'} &= C_2(N)(\gamma^\mu)_{\beta\beta'}(\gamma^\nu)_{\alpha\alpha'} \\ &\quad d_{\mu\nu}(e^{-t}p - e^{-t}k)\bar{g}^2(t)\ . \qquad (4.23)\end{aligned}$$

Substituting (4.23) into (4.21) we obtain a RGE-improved kernel

$$K(p,k) = C_2(N)(\gamma^\mu)_{\beta\beta'}(\gamma^\nu)_{\alpha\alpha'}d_{\mu\nu}(p-k)\bar{g}^2(t)\exp\left[-2\int_{g(\mu)}^{\bar{g}(t)}\mathrm{d}x\frac{\gamma_F(x)}{\beta(x)}\right]\ . \tag{4.24}$$

In the region where $p^2 > m^2 \gg \Lambda^2_{\rm QCD}$,

$$\bar{g}^2(t) = \bar{g}^2(p^2)\theta(p^2 - k^2) + \bar{g}^2(k^2)\theta(k^2 - p^2) \tag{4.25}$$

which has been discussed by many authors [4.5] in studying the UV behavior of the solution. In the region where $p^2 < m^2$, but still $m^2 \gg \Lambda^2_{\rm QCD}$

$$\bar{g}^2(t) = \bar{g}^2(m^2)\theta(m^2 - k^2) + \bar{g}^2(k^2)\theta(k^2 - m^2) \tag{4.26}$$

which combining with (4.25) gives a complete description of the kernel for the massive quarks in the entire momentum region.

4.4 Numerical Solutions

In above, we have derived a Bethe-Salpeter kernel suitable for heavy quarks based on the RGE analysis. The same analysis can be made on the wave function $A(p^2)$. The leading-logarithmic corrections to $A(p^2)$ are proportional to $\bar{g}^2(s)$ where $s = \frac{1}{2}\ln \max(p^2, m^2)/\Lambda^2_{\rm QCD}$. However, when we work in the Landau gauge, these corrections are absent and the problem can be further simplified. In the Landau gauge, we have

$$A(p^2) = 1\ , \qquad Z_\psi = 1 \tag{4.27}$$

and the anomalous dimension γ_F is equal to zero. Substituting the kernel and (4.27) into the SD equation, we obtain two coupled integral equations which are valid in different regions respectively. We would like to observe how the effective gauge interaction affects the quark self-mass when the quark current mass changes. Thus it is instructive to measure all dimensionful quantities in unit of the on-shell current mass

m, i. e. we define new variables

$$x=\frac{p^2}{m^2}\,,\quad y=\frac{k^2}{m^2}\,,\quad \Delta^2=\frac{\Lambda^2}{m^2}\,,\quad B(x)=\frac{B\left(\frac{p^2}{m^2}\right)}{m}\,. \tag{4.28}$$

Clearly, if there are no interactions, $B(x)=1$. In addition, we are interested in the self-mass function at the scale of the on-shell current mass. Thus we choose the renormalization point $\mu=m(\mu=m)=m$. Putting everything together, we obtain the following integral equations in the Euclidean space

$$\begin{aligned} B(x) &= B_0(\Delta)+\frac{\lambda_0}{x}\int_0^x \mathrm{d}y\frac{yB(y)}{y+B^2(y)} \qquad (4.29)\\ &+\lambda_0\int_x^1 \mathrm{d}y\frac{B(y)}{y+B^2(y)}+\int_1^{\Delta^2}\mathrm{d}y\lambda(y)\frac{B(y)}{y+B^2(y)} \qquad (x<1) \end{aligned}$$

$$\begin{aligned} B(x) &= B_0(\Delta)+\frac{\lambda(x)}{x}\int_0^1 \mathrm{d}y\frac{yB(y)}{y+B^2(y)} \qquad (4.30)\\ &+\frac{\lambda(x)}{x}\int_1^x \mathrm{d}y\frac{yB(y)}{y+B^2(y)}+\int_x^{\Delta^2}\mathrm{d}y\lambda(y)\frac{B(y)}{y+B^2(y)} \qquad (x>1) \end{aligned}$$

where

$$\lambda_0=\frac{d}{\ln m^2/\Lambda_{\mathrm{QCD}}^2}\,,\qquad \lambda(x)=\frac{d}{\ln x+\ln m^2/\Lambda_{\mathrm{QCD}}^2}=\frac{d\lambda_0}{\lambda_0\ln x+d} \tag{4.31}$$

$$B_0(\Delta)=\left(\frac{\ln m^2/\Lambda_{\mathrm{QCD}}^2}{\ln \Lambda^2/\Lambda_{\mathrm{QCD}}^2}\right)^d=\left(\frac{d}{\lambda_0\ln\Delta^2+d}\right)^d$$

where $d=3C_2(N)/\beta_0$. Eqs. (4.29) and (4.30) are coupled equations since (4.29) contains an integration from 1 to Δ^2 which requires the solution for $x>1$ while (4.30) contains an integration from 0 to 1 which requires the solution for $x<1$. It is easily seen from (4.29) and (4.30) that $B(x)$ is continuous at $x=1$ because both equations give the same limit $B(1)$. The derivative $B'(x)$, however, is not continuous. The discrepancy between the limits from $x<1$ and $x>1$

$$\Delta B'(x)\big|_{x=1}\equiv B'(x)\big|_{x\to 1^-}-B'(x)\big|_{x\to 1^+}\propto\lambda_0^2 \tag{4.32}$$

is of higher order and is an artifact of the leading-logarithmic approximation that we have used to model the kernel.

When $x \to \infty$, an asymptotic solution to (4.30) can be obtained analytically as is done by many authors by converting (4.30) into a differential equation. The regularity of $B(x)$ when $x \to \infty$ allows one to linearize the differential equation in the UV region. The solution reads

$$B(x) = \left(\frac{d}{\lambda_0}\right)^d \left(\frac{d}{\lambda_0} + \ln x\right)^{-d} + c_2 \frac{1}{x} \left(\frac{d}{\lambda_0} + \ln x\right)^{d-1} \qquad (x \to \infty) \qquad (4.33)$$

where c_2 is the coefficient of the regular solution which cannot be determined by the UV boundary condition. It is expected that when x becomes small, the regular solution is important in the self-mass. The first term in the r. h. s. of (4.33) goes to zero much more slowly than the regular solution when $x \to \infty$ and is referred to as the irregular solution. It completely dominates the self-mass in the UV region.

To solve the coupled integral equations (4.29) and (4.30) numerically in the entire region, clearly, we have to specify the integration cutoff Δ^2 since practically we cannot reach $\Delta^2 = \infty$. This will bring us some ambiguities of choosing Δ and we have to estimate the computation errors with it. Fortunately, the asymptotic solution (4.33) provides a useful hint as to how the solution should behave when x is very large. We can actually use the irregular solution to perform the integration from a very large x, say $X = \mathrm{e}^{40}$, to Δ^2 both in (4.29) and in (4.30)

$$\int_{\mathrm{e}^{40}}^{\Delta^2} \mathrm{d}y \lambda(y) \frac{B(y)}{y + B^2(y)} = -\left(\frac{d}{\lambda_0}\right)^d \left[\left(\frac{d}{\lambda_0} + \ln \Delta^2\right)^{-d} - \left(\frac{d}{\lambda_0} + 40\right)^{-d}\right] \qquad (4.34)$$

and add this term to the inhomogeneous term $B_0(\Delta)$. The Δ^2-dependence in the new inhomogeneous term cancels as can be seen from (4.31) and (4.34). The accuracy of the computation is controlled by X and the total number of integration points, and we find $X = e^{40}$ is sufficient to require the relative errors to be less than 10^{-3}. We further change the momentum variable from x to $\ln(1+x)$ in order to deal with the huge integration interval from 0 to e^{40}. The IR behavior of the solution, which we are particularly interested in, is not deformed very much from this transformation since $\ln(1+x)$ behaves like x when x is very small. We then integrate a trial function from 0 to 40 and iterate the results till the desired accuracy is reached.

Figure 4.3 illustrates the numerical solutions to the SD equation as a function of the momentum and λ_0 or the current mass m. We truncate the graphs at $\ln(1+x) = 6$ in order to make the IR behavior of the solution more obvious. The discontinuity of the slope at $\ln(1+x) = \ln 2$ where $x = 1$ is present as expected because (4.29) and (4.30) are not smoothly connected. The corner grows sharper when λ_0 becomes larger which is also predicted by (4.32). The overall feature of the curves is that the quark self-mass monotonically decreases as the momentum gets large and eventually goes to zero when $x \to \infty$. The speed is controlled by the current mass, the smaller m is, the faster it goes down. There is a value of the momentum around the current mass, above which the self-mass is less than the current mass while below which the self-mass is enhanced from the current mass by interactions. The constituent mass defined as the self-mass at $x = 0$ can substantially differ from the current mass when m is around a few hundred MeV. The enhancement can be as high as 60% when $m = 1.3\Lambda_{\rm QCD} \simeq 270$MeV for example.

There is another point in Figure 4.3 that we would like to address. The different curves seem to cross at the same point where $\ln(1+x) = 1$ or $x \simeq 1.7$. This can be a matter of the normalization condition of the self-mass implied in the SD equation. It by itself is not surprising in the perturbation theory where we can normalize the renormalized self-mass as $B_R(p,\mu)|_{\mu=\sqrt{1.7}m} = m$. In our case, we do not need to

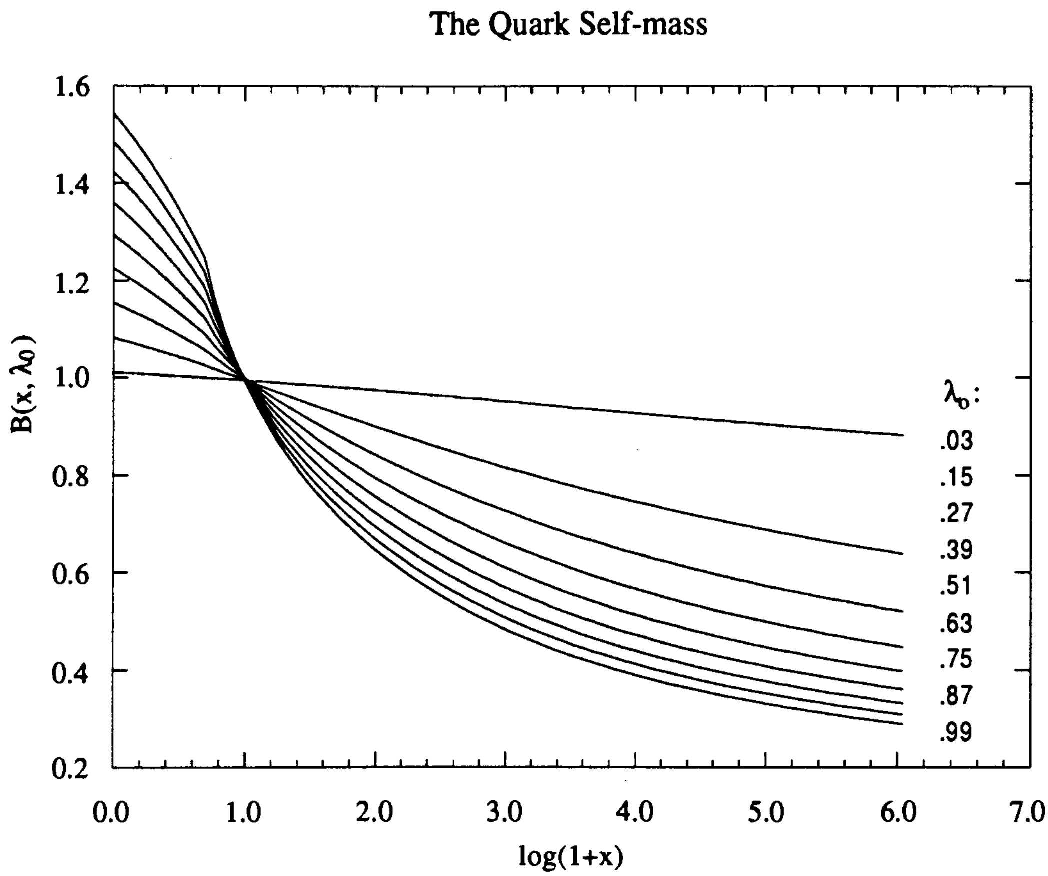

Figure 4.3: The quark self-mass functions for massive quarks

impose such a condition since we simply look for a solution to the SD equation. It is likely that the SD equation has incorporated a similar condition implicitly.

4.5 Conclusions and Discussions

In this chapter, we have studied a renormalized Schwinger-Dyson equation for the quark self-mass. An RGE-improved kernel valid in both UV and IR regions is obtained provided that the quark current mass is larger than $\Lambda_{\rm QCD}$. The numerical solution to the SD equation as function of the momentum and the quark current mass is found. The quark self-mass exhibits a momentum dependence, and the relative splitting from the mechanical mass in the lagrangian depends on the current mass. For a very heavy quark, the self-mass repeats the current mass and strong interactions are negligible. However, a dynamical contribution to the light quark self-mass can be very important and the self-mass can look very differently from the mechanical mass.

Inclusion of the asymptotic solution of the self-mass at a large momentum and a small current mass may bring us a more complete picture. The asymptotic behavior of $B(p^2, m^2)$ as $p^2 \gg \Lambda^2_{\rm QCD}$ has been known for a long time [4.3]. It has been argued based on the asymptotic solution that $B(p^2, m^2)$ should grow as p^2 gets small and a large self-mass in the IR region is suspected even when the current mass is very small. What we have done in this chapter seems to approach the same limit, however with p^2 and m^2 interchanged. Indeed, a more careful study of Eq. (4.12) reveals that the condition for an RGE-improved kernel is

$$p^2 + m^2 \gg \Lambda^2_{\rm QCD} \,. \tag{4.35}$$

Either a large p^2 or a large m^2 satisfies (4.35). Thus the self-mass $B(p^2, m^2)$ as function of two variables p^2 and m^2 is known in the region far above the line satisfying

$p^2 + m^2 = \Lambda_{\mathrm{QCD}}^2$ in the p^2-m^2 plane. Different points in this region represent different physical situations, but the real IR region where $p^2 + m^2 < \Lambda_{\mathrm{QCD}}^2$ is not reached by both approaches.

Chapter 5

Smallness of Fermion Mass

5.1 Introduction

The smallness of fermion mass indicates an approximate chiral symmetry in quantum field theory. In particular, in strong interactions, the chiral symmetry for light quarks has its Goldstone realization, i.e. the spontaneous symmetry breaking. However, the understanding of the origin of the small fermion masses is a subtle problem. To a large extent, this is due to the advent of the standard model. Such a theory employs the Higgs mechanism to break the gauge symmetry while generating various fermion masses via Yukawa interactions. As a result, the fermion mass has a form $m_i = f_i v$ (i=various fermions), where v ($\sim$ 250 GeV) is the order parameter of the electro-weak symmetry breaking and f_i is the Yukawa coupling. To account for the mass spectrum of light fermions like u, d quarks and leptons, the corresponding Yukawa couplings must be as small as $\mathcal{O}(10^{-5})$. It is worth trying to understand why we have such a very special choice of these parameters.

We will show, however, that chiral invariance in the massive fermion field can be a natural consequence, if the fermion couples to a massless non-self-interacting scalar field Φ at the Lagrangian level. Such a scalar field system possesses a trivial symmetry of shifting Φ by an arbitrary constant. This symmetry is broken by the presence of the Yukawa coupling with the massive fermion. As an effect of the fermion loop corrections, the effective potential for the scalar fields does not have the same shifting symmetry as appearing in the lagrangian. The minimum of the potential corresponds to a particular scalar vacuum expectation value (VEV). We find that it is always such that after we shift the scalar field about its VEV, the resulting fermion mass vanishes. A chiral symmetry is restored for the total Lagrangian involving the effective potential for scalar fields. This kind of dynamical restoration of a symmetry has been discussed by Peccei and Quinn [5.1] in explaining the strong CP problem.

5.2 Effective Potential

Let us illustrate this result first for a toy model consisting of a N-component single fermion flavor and a singlet complex scalar field. N may be referred to as the number of colors N_c in the case of strong interactions. A more realistic implication of this model on gauge theories will be discussed later. The Lagrangian is

$$\mathcal{L} = \bar{\Psi} i \partial\!\!\!/ \Psi - m \bar{\Psi}_L \Psi_R - m^* \bar{\Psi}_R \Psi_L - f \bar{\Psi}_L \Psi_R \Phi - f^* \bar{\Psi}_R \Psi_L \Phi^* + \partial_\mu \Phi^* \partial^\mu \Phi. \quad (5.1)$$

The fermion mass m may have its origin from electro-weak symmetry breaking and f is an arbitrary complex constant. We can write the generating functional for Green's function as

$$Z[J, J^*] = \int [d\bar{\Psi}][d\Psi][d\Phi][d\Phi^*] \exp\left\{ i \int d^4x \left[\mathcal{L}(\Psi, \Phi) + J\Phi + J^*\Phi^* \right] \right\}. \quad (5.2)$$

One integrates over the fermion fields to obtain

$$Z[J,J^*] \sim \int [d\Phi][d\Phi^*]\exp\left\{\mathrm{i}\int d^4x\,[\partial_\mu\Phi^*\partial^\mu\Phi + J\Phi + J^*\Phi^*] + N\mathrm{Tr}\ln S_F^{-1}(x,\Phi)\right\}, \tag{5.3}$$

where

$$S_F^{-1}(x,\Phi) = \mathrm{i}\not\partial - \mathrm{Re}(f\Phi+m) - \mathrm{i}\gamma_5\mathrm{Im}(f\Phi+m), \tag{5.4}$$

and "Tr" traces over the space-time and the spinor indices. To obtain the effective action, we need to expand the integrand in (5.3) about its stationary point. In the leading large-N expansion (i.e. the leading loop expansion) [5.2] we have

$$\Gamma[\Phi_c,\Phi_c^*] = -\mathrm{i}N\mathrm{Tr}\ln S_F^{-1}(\Phi_c,\Phi_c^*)\ . \tag{5.5}$$

The minus sign in (5.5) is standard for fermion loops [5.3] since the integration over Grassmann fields always gives a det other than a $\det^{-1/2}$ for the bosonic integration. The effective potential $V(\Phi_c,\Phi_c^*)$ is obtained from $\Gamma[\Phi_c,\Phi_c^*]$ by taking Φ_c and Φ_c^* to be constants. It is important to note, from Eq.(5.4), that Φ field always appears through the combination $f\Phi+m$. Thus one defines

$$f\varphi = f\Phi + m\ . \tag{5.6}$$

The effective potential is then

$$V(\varphi_c,\varphi_c^*) = \mathrm{i}N\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{tr}\ln[-\not k - \mathrm{Re}(f\varphi_c) - \mathrm{i}\gamma_5\mathrm{Im}(f\varphi_c)]\ , \tag{5.7}$$

where "tr" denotes trace over only spinor indices. We then find that

$$\begin{aligned}\frac{\partial V(\varphi_c,\varphi_c^*)}{\partial\varphi_c} &= \frac{1}{2}fN\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{tr}(1+\gamma_5)\frac{[(f\varphi_c+f^*\varphi_c^*)-\gamma_5(f\varphi_c-f^*\varphi_c^*)]}{k^2-|f|^2\varphi_c\varphi_c^*}\\ &= -4N|f|^2\varphi_c^*\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-|f|^2\varphi_c\varphi_c^*},\end{aligned} \tag{5.8}$$

and

$$\frac{\partial V(\varphi_c, \varphi_c^*)}{\partial \varphi_c^*} = -4N\,|f|^2\,\varphi_c \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - |f|^2\,\varphi_c\varphi_c^*} \,. \tag{5.9}$$

From this expression, one learns that the effective potential has an extremum at $\varphi_c = \varphi_c^* = 0$. To confirm that this indeed corresponds to a minimum, we need to obtain the explicit form of the effective potential by integrating (5.8) and (5.9) over the field variables φ_c and φ_c^* (i.e. by the tadpole techniques invented by Weinberg [5.4]). Note that (5.8) and (5.9) are ultraviolet divergent. We may renormalize the effective potential by adding counterterms and imposing the following renormalization conditions

$$\left.\frac{\partial^2 V_R(\varphi_c, \varphi_c^*)}{\partial \varphi_c^* \partial \varphi_c}\right|_{\varphi_c = \varphi_c^* = 0} = 0\,; \tag{5.10}$$

$$\left.\frac{\partial^4 V_R(\varphi_c, \varphi_c^*)}{\partial \varphi_c^{*2} \partial \varphi_c^2}\right|_{\varphi_c = \varphi_c^* = M} = 0\,. \tag{5.11}$$

The renormalized effective potential is

$$V_R(\varphi_c, \varphi_c^*) = \frac{3}{2}\lambda_0\,|f|^4\,(\varphi_c\varphi_c^*)^2 + \frac{1}{2}\,|f|^4\,(\varphi_c\varphi_c^*)^2 \ln \frac{M^2}{|f|^2\,\varphi_c\varphi_c^*}\,, \tag{5.12}$$

where $\lambda_0 = 2N/16\pi^2$ is the standard factor for the loop expansion.

It is noted that the effective potential is not bounded from below as the scalar field gets large. The logarithmic term reverses sign when $|f|^2\,\varphi_c\varphi_c^*$ is greater than M^2. However, there is no reason to believe the perturbative one-loop effective potential should be still valid when φ_c is large compared to M (in fact, M can be as large as the grand unification scale), for the two-loop contribution would dominate over the one-loop contribution when φ_c is large. Eq. (5.12) is only meaningful when the scalar field is small and the perturbative expansion is justified. More specifically, (5.12) should be understood as the asymptotic form of the effective potential in the limit of

$\varphi_c \to 0$ for any finite λ_0. One learns from the above expression that a local minimum occurs at $\varphi_c = \varphi_c^* = 0$, or equivalently, Φ and Φ^* develop non-zero VEVs

$$\langle\Phi\rangle = -\frac{m}{f}\,; \qquad \langle\Phi^*\rangle = -\frac{m^*}{f^*}\,. \tag{5.13}$$

The behavior of the effective potential at large φ_c has to be obtained by a nonperturbative method. Here, however, we simply assume that it is bounded from below. Even if it is not, we have to compute the life time of the local minimum and compare it with the age of universe. In any case, if the system perturbs around the local minimum, in terms of the shifted scalar variables Φ', Φ'^* defined by $\Phi' \equiv \Phi - \langle\Phi\rangle$ and $\Phi'^* \equiv \Phi^* - \langle\Phi^*\rangle$, the fermion acquires an additional mass $-m$ and the effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = \bar{\Psi}\mathrm{i}\partial\!\!\!/\Psi - f\bar{\Psi}_L\Psi_R\Phi' - f^*\bar{\Psi}_R\Psi_L\Phi'^* + V_R\left(|f|^2\,\Phi'^*\Phi'\right)\,. \tag{5.14}$$

The fermion has become massless! Indeed, under the chiral transformations

$$\begin{aligned} \Psi_L &\to e^{i\alpha}\Psi_L\,, & \Psi_R &\to e^{-i\alpha}\Psi_R\,; \\ \Phi' &\to e^{2i\alpha}\Phi'\,, & \Phi'^* &\to e^{2i\alpha}\Phi'^*\,; \end{aligned} \tag{5.15}$$

the Lagrangian (5.14) is invariant. The importance of this result is that the parameters m and f in the original Lagrangian are arbitrary. Because of the dynamical mechanism involving the fermion loop expansion, the scalar field picks up a vacuum expectation value such that the resulting fermion mass is zero. In addition, there remains no information on the original mass in the effective theory.

5.3 Gauge Theories and Composite Models

The generalization to gauge theories needs a more careful study. In the presence of gauge interactions among fermions and gauge bosons (the scalar is a singlet of the gauge group and free of gauge interactions), the effective action is (in Euclidean space)

$$\Gamma[\varphi_c, \varphi_c^*] \sim -\ln \int [\mathrm{d}A_\mu] \det [i\not{D} - \mathrm{iRe}(f\varphi_c) + \gamma_5 \mathrm{Im}(f\varphi_c)]\, \mathrm{e}^{-S[A]} , \tag{5.16}$$

where

$$S[A_\mu] = -\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \theta\frac{1}{4}\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} + \text{gauge fixing terms} \tag{5.17}$$

is the gauge field action. Even though the exact integration over the gauge fields is not possible, a general analysis of the effective potential can be made. To proceed further, we need to know the properties of eigenmodes of the hermitian operator $i\not{D}$. In a QED-like theory, the vacuum of the gauge configuration is trivial, $i\not{D}$ has only non-zero eigenmodes λ_n which appear in pair (λ_n and $-\lambda_n$). Thus the effective potential has a form

$$V_{\text{QED-like}}(\varphi_c, \varphi_c^*) = V\left(|f|^2 \varphi_c^* \varphi_c\right) , \tag{5.18}$$

and we suspect that $\varphi_c = \varphi_c^* = 0$ is still a minimum point. However, in the theory like QCD, because of the nontriviality of the vacuum, the integration in (5.16) should sum over all instanton configurations [5.5]. For one-(anti-)instanton configuration, for example, $i\not{D}$ has one zeromode with chirality +1 (−1) [5.6]. To one-instanton approximation, we have

$$V_{\text{QCD}}(\varphi_c, \varphi_c^*) = -K\left[\mathrm{e}^{i\theta} f\varphi_c + \mathrm{e}^{-i\theta} f^* \varphi_c^*\right] + V_0\left(|f|^2 \varphi_c^* \varphi_c\right) , \tag{5.19}$$

where K is a small positive constant which involves an evaluation of the fermion determinant in the instanton background [5.7]. The term proportional to $\mathrm{e}^{i\theta}$ arises from the fermion zero eigenmode in the $q = 1$sector, while the one with $\mathrm{e}^{-i\theta}$ is from the $q = -1$ sector (q is the winding number of the gauge field configuration). Clearly that $\varphi_c = \varphi_c^* = 0$ is no longer an extremum point of $V_{\rm QCD}$. The VEV's of φ_c and φ_c^* are determined by

$$\begin{aligned}\frac{\partial V_{\rm QCD}(\varphi_c,\varphi_c^*)}{\partial\varphi_c} &= -Ke^{i\theta}f + V_0'\left(|\langle f\varphi_c\rangle|^2\right)|f|^2\langle\varphi_c^*\rangle = 0\,, &(5.20)\\ \frac{\partial V_{\rm QCD}(\varphi_c,\varphi_c^*)}{\partial\varphi_c^*} &= -Ke^{-i\theta}f^* + V_0'\left(|\langle f\varphi_c\rangle|^2\right)|f|^2\langle\varphi_c\rangle = 0\,. &(5.21)\end{aligned}$$

As K is small, $\langle\varphi_c\rangle$ differs from 0 only slightly. We can expand V_0' around 0, to the first order,

$$V_0'\left(|\langle f\varphi_c\rangle|^2\right) = V_0'(0) + V_0''(0)\,|\langle f\varphi_c\rangle|^2 + \mathcal{O}\left(|\langle f\varphi_c\rangle|^4\right)\,. \qquad (5.22)$$

Eqs. (5.20) and (5.21) yield then

$$\begin{aligned}\langle\varphi_c\rangle &= e^{-i\theta}\frac{K^{1/3}}{fV_0''(0)^{1/3}}\,;\\ \langle\varphi_c^*\rangle &= e^{i\theta}\frac{K^{1/3}}{f^*V_0''(0)^{1/3}}\,, \qquad (5.23)\end{aligned}$$

Upon shifting the scalar field around its VEV, we obtain an effective fermion mass

$$m_{\rm eff} = e^{-i\theta}\frac{K^{1/3}}{V_0''(0)^{1/3}}\,, \qquad (5.24)$$

which is induced by the instanton interactions. It, again, does not depend on the original mass parameter m.

A composite model may have a feature similar to the model with fundamental scalar fields. The consideration of a composite scalar field is more interesting in the sense that the gauge interaction of fermions may lead to an effective four-fermion interaction in the low-energy region [5.8]. We may apply the above arguments to the following Lagrangian

$$\mathcal{L} = \bar\Psi i\partial\!\!\!/\Psi - m\bar\Psi_L\Psi_R - m^*\bar\Psi_R\Psi_L + G^2\bar\Psi_L\Psi_R\bar\Psi_R\Psi_L\,. \qquad (5.25)$$

where $G^2 > 0$ is an effective coupling constant. With the help of an auxiliary field ϕ [5.9] we can rewrite (5.25) as

$$\begin{aligned} \mathcal{L} &= \bar{\Psi} i \partial\!\!\!/ \Psi - m\bar{\Psi}_L\Psi_R - m^*\bar{\Psi}_R\Psi_L \\ &\quad - f\bar{\Psi}_L\Psi_R\phi - f^*\bar{\Psi}_R\Psi_L\phi^* - \frac{1}{G^2}\,|f|\,\phi^*\phi\,, \end{aligned} \tag{5.26}$$

where f is an arbitrary complex parameter. Following the same steps as in the case of the fundamental scalar field, the effective potential for the composite field ϕ is derived by integrating out the fermions. In terms of the combination $f\varphi = f\phi + m$, it is given by

$$\begin{aligned} V_{\text{Composite}}(\varphi_c, \varphi_c^*) &= -\frac{m^*}{G_R^2} f\varphi_c - \frac{m}{G_R^2} f^*\varphi_c^* + \frac{1}{G_R^2}\,|f|\,\varphi_c^*\varphi_c \\ &\quad + \frac{1}{4}\lambda_0\,|f|^4\,(\varphi_c^*\varphi_c)^2 + \frac{1}{2}\lambda_0\,|f|^4\,(\varphi_c^*\varphi_c)^2 \ln\frac{\Lambda^2}{|f|^2\,\varphi_c^*\varphi_c}\,, \end{aligned} \tag{5.27}$$

where G_R^2 is the "renormalized" quantity defined by the fine tuning condition [5.10]

$$\frac{1}{G_R^2} = \frac{1}{G^2} - \lambda_0\Lambda^2\,, \tag{5.28}$$

and Λ^2 is the Euclidean cutoff. Note that some coefficients of terms in (5.27) are different from those in Eq.(5.12). This arises from the different renormalization conditions because the four-fermion interaction is not essentially renormalizable. In the weak-coupling theory with $G_R^2 > 0$, $\varphi_c = \varphi_c^* = 0$ is the minimum point of the potential if the first two terms in (5.27) are absent. Inclusion of them, in virtue of (5.19), produces a non-zero fermion mass which is proportional to $(\frac{m}{G_R^2})^{1/3}$. In the strong-coupling theory with $G_R^2 < 0$, we have the spontaneous chiral symmetry breaking [5.10]. The first two terms in (5.27) constitute an explicit symmetry breaking correction to the non-zero VEV's of φ_c and φ_c^*.

5.4 Summary

We have shown in a class of models that the effective fermion mass spectrum may greatly differ from the one in the original Lagrangian [5.11]. An approximate chiral symmetry may be a natural consequence of the mass cancellation mechanism arising from the quantum effect of the fermion loops. This may shed light on the smallness of the light quark and the lepton masses, as compared with the electro-weak scale. However, the role of the scalar field (which can be fundamental or composite) in the context of the standard model needs further study.

Chapter 6

Envoi

In this thesis, I have studied the quantum effects in the standard model of particle physics with emphasis on the symmetry structure of the theory. I have shown various fascinating possibilities on CP violation, anomalous discrete symmetry and dynamical fermion mass. Many problems tackled are not quite yet close to their ultimate solutions. Sometimes it is due to the fact that we have not asked the right questions of nature. Let me quote what Richard Feynman had to say about quantum theory. In what surely must be poetry, he writes:

> We have always had a great deal of difficulty
> understanding the world view
> that quantum mechanics represents.
> At least I do,
> because I'm an old enough man
> that I haven't got to the point
> that this stuff is obvious to me.
> Okay, I still get nervous with it...

You know how it always is,
every new idea,
it takes a generation or two
until it becomes obvious
that there's no real problem.
I cannot define the real problem,
therefore I suspect there is no real problem,
but I'm not sure
there's no real problem.

The symmetries find their heaven in the classical theory and enjoy violations in quantum theory. The experiments will have to prove whether or not this belief is true. On this hopeful note, let me close.

Appendix A

QCD Vacuum Alignment

In this appendix, we shall discuss in detail the issue of the vacuum alignment in QCD, which is crucial to identify the measure of strong CP violation as pointed out in Chapter 2.

A.1 Vacuum Alignment Equation

In QCD lagrangian of strong interactions, there are two possible sources of CP violation: the complex quark mass terms and the θ-term. It has been long realized that they are related to each other by chiral transformations associated with the quark fields. The physical effects of CP violation only depend on a chiral-rotation invariant $\bar{\theta}$ defined as

$$\bar{\theta} = \theta_{QCD} + \theta_{QFD} = \theta_{QCD} + \sum_{i}^{L_f} \phi_i \tag{A.1}$$

where θ_{QCD} is the coefficient of the θ-term, ϕ_i is the phase of the ith quark mass term, and L_f is the number of light quarks[1]. However, there is another source of

[1]The inclusion of heavy quarks will not change our discussion significantly if they are in the normal phase. Otherwise see Ref. [A.4].

CP-violating angles, the phases of the quark condensates, which arise from dynamical chiral symmetry breaking (DCSB)

$$\langle \bar{\psi}_L^i \psi_R^i \rangle = -\frac{C_i}{2} e^{i\beta_i} \; ; \; \langle \bar{\psi}_R^i \psi_L^i \rangle = -\frac{C_i}{2} e^{-i\beta_i} \tag{A.2}$$

where ψ is the quark field and C_i's, β_i's are real. The QCD vacuum orientation is characterized by a set of phases of the quark condensates. If the vacuum angle $\beta_i \neq 0$ it follows that $\langle \bar{\psi}^i i\gamma_5 \psi^i \rangle = C_i \sin\beta_i \neq 0$ which may also break CP symmetry since $\langle \bar{\psi} i\gamma_5 \psi \rangle$ is a P-odd and CP-odd quantity. It has been proven by Vafa and Witten that when $\bar{\theta} = 0$ and $\phi_i = 0$ for all i's, the parity symmetry in a vector-like theory such as QCD is not spontaneously broken [A.1] therefore $\beta_i = 0$ or π [A.2] for all i's. When $\bar{\theta} \neq 0$, on the other hand, one generally expects that the CP-violating interactions in the lagrangian may result in a CP-asymmetric physical vacuum. The purpose of this appendix is to study the vacuum orientation in the presence of strong CP violation and its potential effects on CP-violating processes in strong interactions. We find that the phases of quark condensates can be completely determined as functions of $\bar{\theta}$ and ϕ_i's via a vacuum alignment equation. Thus β_i's are not spontaneously generated either even when the CP symmetry is explicitly violated by $\bar{\theta} \neq 0$.

Obviously, the quark condensates (A.2) cannot be referred to as fundamental parameters of the theory since they are subject to chiral transformations. In fact β_i's can be set to any values if we make appropriate chiral transformations for the quark fields. Such transformations also change the phases of the quark masses, as well as the coefficient of the θ-term because of the chiral anomaly. But what is important is the correlation between the vacuum orientation and the distribution of the strong CP phases among θ-term and quark mass terms, which is to be determined by the vacuum alignment. The effective CP-violating interactions in low energy hadron physics (for instance, in current algebra) highly depend on this correlating feature. As we shall see below, the sole $\bar{\theta}$-dependence of the strong CP effects is proven only when the orientations of the vacuum are properly considered. In addition, it is of interest to study DCSB in the presence of strong CP violation in its own right.

One way of relating the phases of the quark condensates with θ_{QCD} and ϕ_i's is to consider the so-called anomalous Ward identity [A.3]

$$\frac{1}{2}\partial^\mu J_\mu^{(i)5} = F\tilde{F} + im_i e^{i\phi_i}\bar{\psi}_L^i\psi_R^i - im_i e^{-i\phi_i}\bar{\psi}_R^i\psi_L^i \tag{A.3}$$

$$(i = 1, 2, \ldots, L_f)$$

where $J_\mu^{(i)5} = \bar{\psi}^i\gamma_\mu\gamma_5\psi^i$, $F\tilde{F} = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ and $F^{\mu\nu}$ the non-abelian gauge field strength tensor. Taking the vacuum expectation values (VEV) on both sides of (A.3) yields [A.4]

$$\left\langle F\tilde{F}\right\rangle = -im_i[e^{i\phi_i}\left\langle\bar{\psi}_L^i\psi_R^i\right\rangle - e^{-i\phi_i}\left\langle\bar{\psi}_R^i\psi_L^i\right\rangle] \tag{A.4a}$$

$$= -m_i C_i \sin(\phi_i + \beta_i) \qquad (i = 1, 2, \cdots, L_f). \tag{A.4b}$$

In deriving (A.4a) we have assumed that the VEV's of the divergence of the gauge invariant current vanish. Eq.(A.4b) is the master equation of this appendix. It is important to point out that if the DCSB does not occur, (A.4a) would vanish identically and there is no constraint on those phases. Indeed even though the quark condensate can be non-zero due to the explicit chiral symmetry breaking (ECSB) i. e. the quark current masses, it does not contribute to (A.4a) because it possesses a phase opposite to the phase of the quark mass ϕ_i and renders the RHS of (A.4a) zero. This can be easily seen by taking the free-quark limit in which the condensate is calculated as

$$\left\langle\bar{\psi}_L^i\psi_R^i\right\rangle = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}Tr\frac{1+\gamma_5}{\not{k} - m_i e^{i\phi_i\gamma_5}} = -m_i\Lambda^2(m_i)e^{-i\phi_i} \tag{A.5}$$

where Λ^2 is real. The substitution of (A.5) into (A.4a) yields $\langle F\tilde{F}\rangle = 0$. Therefore, C_i's in (A.4a) should be understood as the purely *dynamical* condensates originating from DCSB. The *kinematical* part of the condensates that is induced by the ECSB and has a phase $-\phi_i$ has been subtracted out in (A.4a). It is the DSCB combining with the topological structure of QCD Yang-Mills fields (the instanton effect) that makes the strong CP phases non-trivial and relates them to each other.

Eq.(A.4b) is not immediately useful to us since it has an unknown quantity $\langle F\tilde{F}\rangle$. It involves no quark fields thus is independent of the chiral transformation. It is conceivable that $\langle F\tilde{F}\rangle$ is solely a function of $\bar{\theta}$ (not of ϕ_i's and β_i's separately). A rigorous proof can be made by summing over instanton configurations in QCD θ vacua. For simplicity, consider QCD with a single quark field ψ. The VEV's of $F\tilde{F}$ is given by [A.5]

$$\begin{aligned}\left\langle F\tilde{F}\right\rangle &= \frac{1}{VT}\left\langle \int d^4x F\tilde{F}\right\rangle \qquad\qquad (A.6)\\ &= \frac{1}{VT}\frac{1}{N}\sum_{\nu=0,\pm1,\cdots} e^{i\bar{\theta}\nu}\nu\int [dA_\mu]_\nu \det(i\not{D}_\nu + im)\exp(-\int d^4x FF)\end{aligned}$$

where N is the normalization factor, VT is the volume of Euclidean space-time, and ν is the winding number of the instanton field configuration, and the fermion determinant results from the integration over the quark field. We have made an appropriate chiral transformation such that the quark mass is real and $\theta_{QCD} = \bar{\theta}$ (we can always do so because the generating functional is invariant under the redefinition of integral variables). It is shown that when $\nu > 0$ (< 0) $i\not{D}_\nu$ has $|\nu|$ zero modes with negative (positive) chirality [A.6]. We thus obtain

$$\begin{aligned}\left\langle F\tilde{F}\right\rangle &= mA_1(m^2)\frac{-i}{2}(e^{i\bar{\theta}} - e^{-i\bar{\theta}})\\ &+m^2A_2(m^2)\frac{-i}{2}(e^{i2\bar{\theta}} - e^{-i2\bar{\theta}}) + \cdots + m^{|\nu|}A_{|\nu|}(m^2)\frac{-i}{2}(e^{i|\nu|\bar{\theta}} - e^{-i|\nu|\bar{\theta}}) + \cdots\end{aligned}$$

$$= mA_1(m^2)\sin\bar{\theta} + m^2A_2(m^2)\sin 2\bar{\theta} + \cdots = K(m,\theta)\sin\bar{\theta} \tag{A.7}$$

where $A_{|\nu|}(m^2)$'s are given in Euclidean space

$$A_{|\nu|}(m^2) = \frac{1}{VTN}e^{-|\nu|\frac{8\pi^2}{g^2}}\int [dA_\mu]_\nu \prod_{\lambda_r>0}[\lambda_r^2(A)+m^2]\exp[-\int d^4xFF]. \tag{A.8}$$

Here $\lambda_r(A)$'s are non-zero eigenvalues of $i\not{D}_\nu$. Clearly $A_{|\nu|}(m^2)$'s are some real functions of m^2 and do not vanish as $m \to 0$. If $\bar{\theta}$ is small as it must be, $\langle F\tilde{F}\rangle \simeq K(m)\bar{\theta} = K(m)\theta_{QCD} + K(m)\theta_{QFD}$.

Combining (A.7) with (A.4b), we derive the so-called vacuum alignment equation (VAE) [A.4, A.8], which determines the orientation of the QCD vacuum in the presence of strong CP violation

$$K(m)\bar{\theta} = m_iC_i(\phi_i + \beta_i) + O(m^2;\bar{\theta}^2) \tag{A.9}$$

$$(i = 1,\ 2,\ \cdots,\ L_f).$$

Eq.(A.9) has proven that β_i's are not spontaneously generated even when $\bar{\theta} \neq 0$. The conclusion of Vafa and Witten's theorem [A.1] can be extended to the case where parity symmetry is explicitly violated. A similar result has been worked out previously [A.8] from different points of view. If m_i's vanish, β_i's can be arbitrary. This is referred to as the degeneracy of QCD vacua when the ECSB is absent. Any vacuum characterized by a set of the vacuum angles β_i's is as good as any other and the orientation of the DCSB is arbitrary. However, the importance of (A.9) is that when the ECSB is turned on, the ground state must align with it in such a way that (A.9) is satisfied. Though both ϕ_i and β_i are not physical parameters and can be changed through chiral rotations, their sum is uniquely determined by the physical parameter $\bar{\theta}$. When one is chosen the other is completely determined through making the vacuum alignment. As is emphasized by Dashen [A.9], a misaligned vacuum, whose orientation angles do not satisfy (A.9), may cause an inconsistency such as the goldstone bosons (pions) acquiring negative mass squared.

Once the DCSB and the ECSB align with each other, an absolute rotation of the whole system is of no concern. Thus a chiral transformation is allowed only if the corresponding change of the vacuum orientation has been taken into account. We can have two ways to make the vacuum alignment. We may choose one particular vacuum, for example, by requiring the quark condensate to be real $\beta_i = 0$ $(i = 1, 2, \cdots, L_f)$ and ask what perturbation (the ECSB) is aligned with it. Recalling that C_i's are dynamical condensates and thus $C_i = C_j = C$ $(i, j = 1,\ 2,\ \cdots,\ L_f)$, we obtain by solving (A.9) for ϕ_i's, to $O(m; \bar{\theta})$

$$(A) \qquad \phi_i = \frac{K(m)}{m_i}\bar{\theta} \;\; ; \;\; \beta_i = 0 \qquad (i = 1,\ 2,\ \cdots,\ L_f)$$

$$\theta_{QFD} = \sum_i \phi_i = \frac{K(m)}{\bar{m}}\bar{\theta} \quad ; \quad \theta_{QCD} = \bar{\theta} - \theta_{QFD} = \frac{K(m) - \bar{m}}{\bar{m}}\bar{\theta} \tag{A.10}$$

where $\bar{m} = (\sum_i \frac{1}{m_i})^{-1}$ and the CP-violating lagrangian

$$\mathcal{L}^{CP}_{(A)} = -\sum_i m_i \phi_i \bar{\psi}^i i\gamma_5 \psi^i + \theta_{QCD} F\tilde{F} = -\frac{K(m)}{\bar{m}}\bar{\theta}\,\bar{\psi} i\gamma_5 I \psi + \frac{K(m) - \bar{m}}{\bar{m}}\bar{\theta} F\tilde{F} \tag{A.11}$$

where I is an identity matrix. We shall call the solution (A.10) basis (A). Another way is to assume a certain pattern of the ECSB and to ask which one of the degenerate vacua corresponds to the perturbation. For example, we may choose the quark mass terms real $\phi_i = 0$ $(i = 1, 2, \cdots, L_f)$ and determine the vacuum angle β_i's. Again, from (A.9) we have

$$(B) \qquad \phi_i = 0 \quad ; \quad \beta_i = \frac{K(m)}{m_i}\bar{\theta} \quad (i = 1,\ 2,\ \cdots,\ L_f)$$

$$\theta_{QFD} = \sum_i \phi_i = 0 \quad ; \quad \theta_{QCD} = \bar{\theta} \tag{A.12}$$

and

$$\mathcal{L}^{CP}_{(B)} = \bar{\theta} F\tilde{F}. \tag{A.13}$$

Solution (A.12) is to be called basis (B). We would like to emphasize again that by performing the chiral rotation on quark fields one has distributed the strong CP phases among the θ-term and quark mass terms, and obtained different lagrangians, each of which corresponds to a certain vacuum orientation. When calculating the strong CP effects we must take this into consideration to assure the correct result.

A.2 $\eta \to 2\pi$ Decays

As an illustration, we compute the CP-violating $\eta \to 2\pi$ decays in two bases with a given $\bar{\theta}$. In basis (A) where the condensates are real, we apply the soft-pion theorem to extracting η and π's[2]

$$\begin{aligned} \mathcal{A}(\eta \to \pi^+\pi^-) &= \left\langle \pi^+\pi^- \left| \mathcal{L}^{CP}_{(A)} \right| \eta \right\rangle \\ &= -\frac{K(m)}{\bar{m}}\bar{\theta}\left(\frac{-i}{F_\pi}\right)^3 \left\langle [Q_5^8, [Q_5^+, [Q_5^-, \bar{\psi} i\gamma_5 I \psi]] \right\rangle \\ &\simeq -\frac{K(m)}{\bar{m}}\bar{\theta}\frac{1}{F_\pi}\left\langle \bar{\psi}\{\tfrac{\lambda^8}{2}, \{\tfrac{\lambda^+}{2}, \{\tfrac{\lambda^-}{2}, I\}\psi \right\rangle = \frac{K(m)}{\bar{m}}\bar{\theta}\frac{1}{F_\pi^3}\frac{2}{\sqrt{3}}C \end{aligned} \tag{A.14}$$

where the pion decay constant $F_\pi \approx 93MeV$ and we have used $[Q_5^a, F\tilde{F}] = 0$. The broken generators of $SU(3)_A$ corresponding to light pseudoscalars are given by

$$Q_5^a = \int d^3x \psi^\dagger \gamma_5 \frac{\lambda^a}{2}\psi(x) \qquad (a = 1, 2, \cdots, 8) \tag{A.15}$$

where λ^a's are Gell-Mann matrices and $\lambda^\pm = 1/\sqrt{2}(\lambda_1 \mp i\lambda_2)$. (A.14) has been first derived by Crewther, Di Vicchia, Veneziano and Witten (CDVW) [A.10] and later by

[2] We have worked in the context of $SU(3)_L \times SU(3)_R$ where η and π's are all light pseudoscalars.

Shifman, Vainshtein and Zakharov [A.7] in a different context. However, there have been doubts about the calculation since it does not explicitly exhibit the use of the topological non-triviality of the θ-vacuum. More concretely, one may shift the strong CP phases from θ_{QFD} to θ_{QCD} through chiral rotations and compute the amplitude, as one does in (A.14),

$$\left\langle \pi^+\pi^- \left| \mathcal{L}^{CP}_{(B)} \right| \eta \right\rangle = \bar{\theta}\left(\frac{-i}{F_\pi}\right) \left\langle \pi^- \left| [Q_5^-, F\tilde{F}] \right| \eta \right\rangle \tag{A.16}$$

which is zero if one imposes the canonical commutation relation by which Q_5^a commutes with gauge fields. This contradiction has triggered a serious doubt on whether or not the strong CP phases lead to any physical effects at all [A.11].

We believe that this concern is not necessary. The vacuum alignment equation (VAE) has incorporated the non-perturbative features of QCD vacuum into the game. Both $\mathcal{L}^{CP}_{(A)}$ and $\mathcal{L}^{CP}_{(B)}$ are solutions of the VAE and should, if one does things correctly, result in the same conclusion. In basis (B) the quark masses are real but the condensates are complex. The vacuum does not respect CP symmetry. In this case even though $\mathcal{L}^{CP}_{(B)}$ does not contribute to the amplitude as shown in (A.16), the CP conserving part of the lagrangian may do. Moreover, when the quarks have non-degenerate masses (mass splitting), the condensates are of the form

$$\left\langle \bar{\psi}_L \psi_R \right\rangle = -\frac{C}{2}\beta \equiv -\frac{C}{2}(I + i\delta) \tag{A.17a}$$

with

$$\beta = \begin{pmatrix} e^{i\beta_u} & & \\ & e^{i\beta_d} & \\ & & \ddots \end{pmatrix} \; ; \; \delta \simeq \begin{pmatrix} \beta_u & & \\ & \beta_d & \\ & & \ddots \end{pmatrix} \tag{A.17b}$$

if β_i's are small. Apparently (A.17a) is not invariant under $SU(L_f)_V$ transformations if β_i's are not all equal. In other words, the vector charges defined as generators of

$SU(L_f)_V$ do not annihilate the vacuum completely or

$$Q^a |0\rangle \neq 0. \tag{A.18}$$

Clearly, the subgroup of $SU(L_f)_L \times SU(L_f)_R$ which leaves the vacuum invariant must satisfy

$$U_L^+ \beta U_R = \beta \quad or \quad U_R = \beta^{-1} U_L \beta \tag{A.19}$$

where U_L and U_R are left and right unitary representations of $SU(L_f)$. The broken generators, which excite the glodstone bosons known as pions, are those of the coset of the unbroken group. From (A.19) it is easy to understand that the broken group is not $SU(L_f)_A$ any more but to be rotated to $\beta^{-1} SU(L_f)_A \beta$ generated by δ. The pion generators, denoted by $\tilde{Q}_5^a$, are thus

$$\tilde{Q}_5^a = \int d^3x \{\psi^\dagger \gamma_5 \tfrac{\lambda^a}{2} \psi(x) + \psi^\dagger [\tfrac{\lambda^a}{2}, i\delta] \psi(x)\} + O(\delta^2), \tag{A.20}$$

i. e. , the pions are mixing of P-odd and P-even components.

Now that $\mathcal{L}_{(B)}^{CP} = \bar{\theta} F \tilde{F}$ has no contribution to the amplitude, we have

$$\begin{aligned}
\mathcal{A}(\eta \to \pi^+\pi^-) &= \left\langle \pi^+\pi^- \left| -\bar{\psi} m \psi \right| \eta \right\rangle \\
&= -(\frac{-i}{F_\pi})^3 \left\langle [\tilde{Q}_5^8, [\tilde{Q}_5^+, [\tilde{Q}_5^-, \bar{\psi} m \psi] \right\rangle \\
&= -\frac{i}{F_\pi^3} \{ \left\langle \bar{\psi} [[\tfrac{\lambda^8}{2}, i\delta], \{\tfrac{\lambda^+}{2}, \{\tfrac{\lambda^-}{2}, m\}\}] \psi \right\rangle + \left\langle \bar{\psi} \{\tfrac{\lambda^8}{2}, [[\tfrac{\lambda^+}{2} i\delta], \{\tfrac{\lambda^-}{2}, m\}]\} \psi \right\rangle \\
&\quad + \left\langle \bar{\psi} \{\tfrac{\lambda^8}{2}, \{\tfrac{\lambda^+}{2}, [[\tfrac{\lambda^-}{2}, i\delta], m]\}\} \psi \right\rangle - \left\langle \bar{\psi} \gamma_5 \{\tfrac{\lambda^8}{2}, \{\tfrac{\lambda^+}{2}, \{\tfrac{\lambda^-}{2}, m\}\}\} \psi \right\rangle \}
\end{aligned} \tag{A.21}$$

where m is the diagonal mass matrix which is real in this basis. The first three terms in parenthesis come from the modification of the pion generators and the last term reflects the complexity of the condensates which is absent in basis (A). They are of the same order of m and $\bar{\theta}$. Manipulations of these commutators yield, to $O(m; \bar{\theta})$

$$A(\eta \to \pi^+\pi^-) = \frac{-i}{F_\pi^3} \{ \frac{i}{\sqrt{3}} (\beta_u - \beta_d)(-m_u \langle \bar{u} u \rangle + m_d \left\langle \bar{d} d \right\rangle)$$

$$-\frac{1}{\sqrt{3}}(m_u + m_d)(\langle\bar{u}\gamma_5 u\rangle + \left\langle\bar{d}\gamma_5 d\right\rangle)\} = \frac{K(m)}{\bar{m}}\bar{\theta}\frac{1}{F_\pi^3}\frac{2}{\sqrt{3}}C. \qquad (A.22)$$

In deriving the final step of (A.22) we have substituted in $\langle\bar{\psi}^i\psi^i\rangle = -C\cos\beta_i \simeq -C$ and $\langle\bar{\psi}^i i\gamma_5\psi^i\rangle = C\sin\beta_i \simeq C\beta_i$ and solution (A.2). We therefore confirm that CDVW's result is independent of chiral frames.

We conclude that the study of the vacuum orientation of the dynamical chiral symmetry breaking provides us an improved understanding of strong CP violation [A.11]. It has been shown that $\eta \to 2\pi$ decay occurs if $\bar{\theta}$ is non-zero. More precise experimental measurements on the decay rate is encouraged to constrain $\bar{\theta}$.

Appendix B

EDM for Dirac Fermion

B.1 EDM Basics

The electric dipole moment (EDM) of elementary particles has been extensively studied since CP violation was discovered in nature. It is suggested that the non-zero EDM of a fundamental particle can be another evidence of CP nonconservation and put a stringent constraint on the various CP violation mechanisms. In general, the EDM μ of a Dirac fermion with an electric charge e_f is defined by the coefficient of the P- and T-odd effective interaction of the type

$$\tfrac{1}{2}e_f\mu\bar{\Psi}i\gamma_5\sigma_{\mu\nu}F^{\mu\nu}\Psi \tag{B.1}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $F_c^{\mu\nu}$ is the field strength tensor of an external (classical) electromagnetic field A_c^μ. This type of the effective interactions, together with those for the anomalous magnetic moment (AMM), can be written in a more general form

$$\tfrac{1}{2}e_f\mu\bar{\Psi}e^{i\alpha_2\gamma_5}\sigma_{\mu\nu}F^{\mu\nu}\Psi = \tfrac{1}{2}e_f\bar{\Psi}\mu(\cos\alpha_2 + i\gamma_5\sin\alpha_2)\sigma_{\mu\nu}F^{\mu\nu}\Psi \tag{B.2}$$

with $e_f\mu\cos\alpha_2$ and $e_f\mu\sin\alpha_2$ to be further identified with the AMM and the EDM respectively.

However, this kind of naive identification cannot be physical since the phase α_2 is subject to change by a redefinition of the quark field. Under a global chiral transformation

$$\Psi \to e^{-i\frac{\phi}{2}\gamma_5}\Psi \quad ; \quad \bar{\Psi} \to \bar{\Psi}e^{-i\frac{\phi}{2}\gamma_5} \tag{B.3}$$

(B.2) is transformed to

$$\frac{1}{2}e_f\mu\,\bar{\Psi}e^{i(\alpha_2-\phi)\gamma_5}\sigma_{\mu\nu}F^{\mu\nu}\Psi. \tag{B.4}$$

In particular, when we choose $\phi = \alpha_2$, the EDM is rotated away completely and the AMM gets its maximum value μ.

It is obvious that if the theory possesses a chiral symmetry, i.e. an invariance under (B.3), the effective coupling can indeed be rotated away (the chirality flip). We therefore reach the conclusion that the EDM of a massless Dirac fermion must vanish. For a massive fermion, a chiral rotation like (B.3) will change the phase of the mass term and in general affect the CP-violating amplitude. However, the relative phase of the mass term and the effective interaction (B.2) does not change under the chiral transformation. We show below that a physical definition of the fermion EDM only depends on this relative phase.

We write the Dirac equation , including the quantum effects such as those in the effective interaction (B.2), as follows

$$[i\,\not{D} - me^{i\alpha_1\gamma_5} + \tfrac{1}{2}e_f\mu\, e^{i\alpha_2\gamma_5}\sigma_{\mu\nu}F^{\mu\nu}]\Psi = 0 \tag{B.5}$$

with

$$i\,\not{D} = i\,\not{\partial} - e_f\,\not{A}$$

where m and μ are real, and α_1 and α_2 are some arbitrary phases. The vector potential A_c^μ may be chosen such that $A_c^0 = 0$. In momentum space Eq. (B.5) leads to

$$i\frac{\partial\Psi}{\partial t} = [(\vec{\alpha}\cdot\vec{P}) + \beta m\cos\alpha_1 + i\beta\gamma_5 m\sin\alpha_1 - i\beta\vec{\alpha}\cdot\vec{E}e_f\mu cos\alpha_2$$

$$-\beta\vec{\Sigma}\cdot\vec{B}e_f\mu cos\alpha_2 + \beta\vec{\Sigma}\cdot\vec{E}e_f\mu\sin\alpha_2 - i\beta\vec{\alpha}\cdot\vec{B}e_f\mu\sin\alpha_2]\Psi = 0 \qquad \text{(B.6)}$$

where $\vec{P} = \vec{p} - e_f\vec{A}_c$. To explore the physical interpretation of (B.6), we study a stationary solution of energy E,

$$\Psi = e^{-iEt}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \qquad \text{(B.7)}$$

Eq. (B.6) reads

$$(E - m\cos\alpha_1 + e_f\mu\vec{\sigma}\cdot\vec{B}')\varphi = [(\vec{\sigma}\cdot\vec{P}) + im\sin\alpha_1 - ie_f\mu\vec{\sigma}\cdot\vec{E}']\chi \qquad \text{(B.8}a\text{)}$$

$$(E + m\cos\alpha_1 - e_f\mu\vec{\sigma}\cdot\vec{B}')\chi = [(\vec{\sigma}\cdot\vec{P}) - im\sin\alpha_1 + ie_f\mu\vec{\sigma}\cdot\vec{E}']\varphi \qquad \text{(B.8}b\text{)}$$

where

$$\vec{B}' = \cos\alpha_2\vec{B} - \sin\alpha_2\vec{E} \quad ; \quad \vec{E}' = -\sin\alpha_2\vec{B} + \cos\alpha_2\vec{E}. \qquad \text{(B.9)}$$

Solving (B.8a) and (B.8b) for φ, one expresses χ in terms of φ from (B.8b) and substitutes it into (B.8a). Noticing that the commutator of $[\vec{\sigma}\cdot\vec{P}+im\sin\alpha_1 - ie_f\mu\vec{\sigma}\cdot\vec{E}']$ and $[E + m\cos\alpha_1 - e_f\mu\vec{\sigma}\cdot\vec{B}']^{-1}$ is of the order of $O(e_f^2)$, one obtains

$$(E^2 - m^2)\varphi = [(\vec{p} - e_f\vec{A})^2 - e_f\vec{\sigma}\cdot\vec{B}$$

$$-2e_f m\mu\vec{\sigma}\cdot\vec{B}\cos(\alpha_1 - \alpha_2) - 2e_f m\mu\vec{\sigma}\cdot\vec{E}\sin(\alpha_1 - \alpha_2)]\varphi + O(e_f^2). \qquad \text{(B.10)}$$

The spin dependences of the energy are through the magnetic interaction $\vec{\sigma}\cdot\vec{B}$ and the electric interaction $\vec{\sigma}\cdot\vec{E}$. As is expected, the Dirac equation only recognizes the relative phase $\alpha_1 - \alpha_2$, which is invariant under the chiral transformation. In the non-relativistic approximation, we obtain the magnetic moment μ_M and the EDM $\mu_{\rm EDM}$ as follows

$$|\mu_{\rm M}| = 2\frac{e_f}{2m} + 2e_f\mu\cos(\alpha_1 - \alpha_2), \tag{B.11}$$

$$|\mu_{\rm EDM}| = 2e_f\mu\sin(\alpha_1 - \alpha_2). \tag{B.12}$$

Some simple observations can be made from Eqs. (B.11) and (B.12). The magnitudes of the AMM and the EDM for a Dirac fermion are closely related by the relation $|\mu_{\rm EDM}/\mu_{\rm AMM}| = \tan(\alpha_1 - \alpha_2)$, both of which arise from the quantum corrections to the tree-level Dirac equation. Thus, on general grounds, the EDM is purely a quantum effect which should be absent at the tree level. Even though the fermion mass does not explicitly enter Eq. (B.12), the phase of the mass term α_1 does have an important effect on the EDM. In fact $|\mu_{\rm EDM}|$ depends on the difference between α_1 and α_2. When the mass term in (B.5) is absent, α_1 is a free parameter and can be chosen to cancel α_2, yielding a vanishing EDM. Needless to say, the mass m in Eq. (B.5) should be understood to be the renormalized effective mass. In order to have a non-zero CP-violating EDM, one has to calculate μ, α_2 as well as the phase of the effective mass α_1 from various CP violation sources and make sure that $\mu \neq 0$ and $\alpha_1 - \alpha_2 \neq 0$ or π.

B.2 Schwinger's Formalism

There are different ways to explicitly compute the EDM for a Dirac fermion. One usually goes to the momentum representation, calculates the Feynman diagrams and looks for terms which has a structure $i\gamma_5\sigma_{\mu\nu}k^\nu$ where k^ν is the momentum of the photon. Here I will present a simpler and more direct way based on Schwinger's formalism. To illustrate the character of this computation, I will exemplify it by considering the terms in (2.91) in Chapter 2. When a photon is emitted from the fermion line, we need to replace the fermion propagator in (2.91) by (2.93). Now, remember that we are looking for a term with structure like in (2.88). $\not{D}$ anticommutes with γ_5 so that the $\not{D}_Q^{\rm em}$-term in (2.93) does not contribute. In addition, we only keep terms up to $O(e^2)$. Thus, we only need to substitute S_F^Q in (2.91) by

$$-\frac{m_Q}{(\partial^2 - m_Q^2)^2}\frac{1}{2}eQ\sigma_{\mu\nu}F_{\mu\nu}\ . \tag{B.13}$$

Do not forget a $\delta^4(x-y)$ for each propagator. The relevant part in the RHS of (2.91) becomes

$$\int d^4x d^4y \bar{\psi}(x) e^{i\gamma'\gamma_5} \frac{1}{\partial^2 - m_\pi^2}\delta^4(x-y)\frac{1}{(\partial^2 - m_Q^2)^2}\delta^4(x-y) \cdot \frac{1}{2} e m_Q Q \sigma_{\mu\nu}F_{\mu\nu}\psi(y)\ . \tag{B.14}$$

Using

$$(2\pi)^4\delta^4(x-y) = \int d^4k e^{ik(x-y)} \tag{B.15}$$

one obtains

$$\begin{aligned} &\frac{1}{(2\pi)^8}\int d^4x d^4y e^{i(k_1+k_2)(x-y)} d^4k_1 d^4k_2 \bar{\psi}(x) e^{2\gamma'\gamma_5} \\ &\quad\cdot \frac{1}{k_1^2 - m_\pi^2}\frac{1}{(k_2^2 - m_Q^2)^2}\frac{1}{2} e m_Q Q\sigma_{\mu\nu}F_{\mu\nu}\psi(y) \\ &\simeq \int d^4x \bar{\psi}(x) e^{i\gamma'\gamma_5}\frac{1}{2} e m_Q Q \sigma_{\mu\nu}F_{\mu\nu}\psi(x) \\ &\quad\cdot \frac{1}{(2\pi)^4}\int d^4k \frac{1}{k^2 - m_\pi^2}\frac{1}{(k^2 - m_Q^2)^2}\ . \end{aligned} \tag{B.16}$$

Performing the momentum integration yields the desired result.

When a photon is emitted from a meson line, the same procedure follows except that one needs to consider the following

$$\bar{\psi}(x) e^{2\gamma'\gamma_5}\left(\partial_\mu A_\mu + A_\mu \partial_\mu\right)\psi(y)\ . \tag{B.17}$$

The above expression can be rewritten

$$\bar{\psi}(x)e^{2\gamma'\gamma_5}\left(\overleftarrow{\partial_\mu} + \overrightarrow{\partial_\mu}\right)\psi(y)A_\mu \,. \tag{B.18}$$

Using the Gordon identity

$$\begin{aligned} & i\bar{\psi}(x)\left(\overleftarrow{\partial_\mu} + \overrightarrow{\partial_\mu}\right)\psi(y) \\ &= 2m\bar{\psi}(x)\gamma_\mu\psi(y) + \bar{\psi}(x)\sigma_{\mu\nu}\left(\overleftarrow{\partial_\nu} - \overrightarrow{\partial_\nu}\right)\psi(y) \end{aligned} \tag{B.19}$$

one obtains

$$-i\bar{\psi}(x)e^{2\gamma'\gamma_5}\sigma_{\mu\nu}\left(\partial_\nu A_\mu - A_\mu\partial_\nu\right)\psi(y) \tag{B.20}$$

where $\partial_\nu A_\mu - A_\mu\partial_\nu = (\partial_\nu A_\mu)$. Since $\sigma_{\mu\nu}$ is anti-symmetric in μ and ν, (B.20) becomes

$$i\bar{\psi}(x)e^{2\gamma'\gamma_5}\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\psi(y) \,. \tag{B.21}$$

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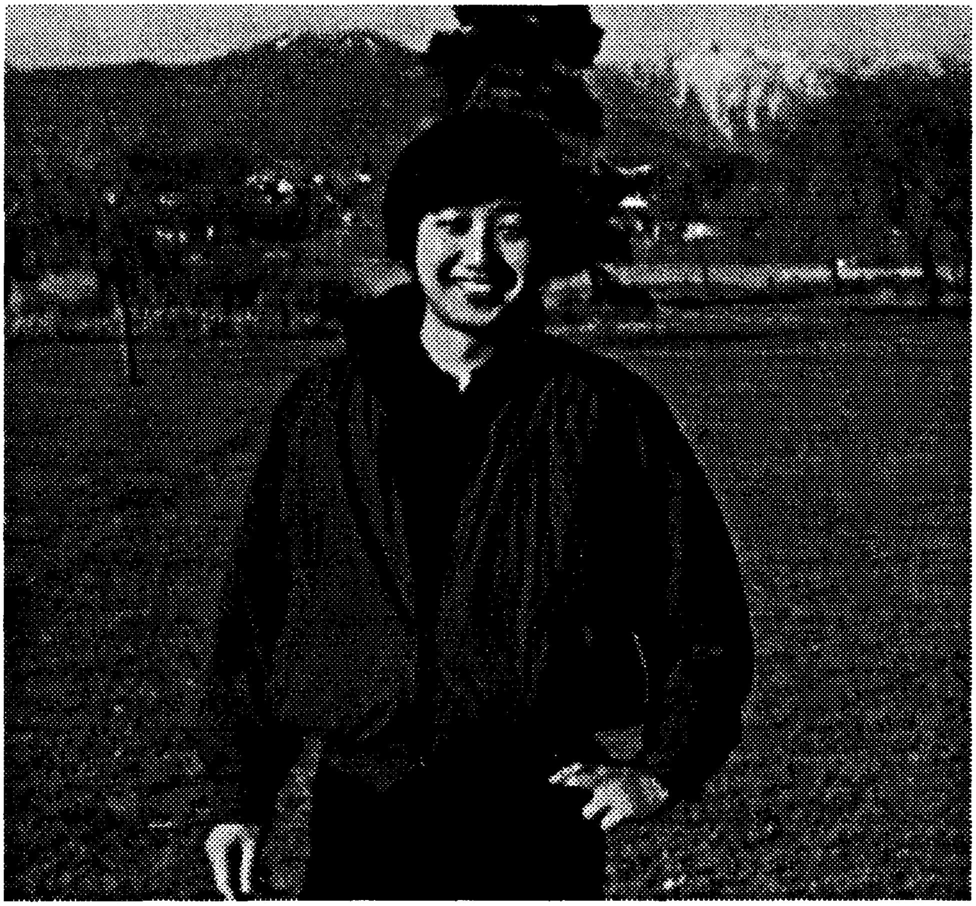

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