A STUDY OF SUBSPACES OF BOUNDED SEQUENCES, SEQUENTIAL COMPLETENESS, AND METHODS OF ALMOST CONVERGENCE

by

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A STUDY OF ~~BOUNDE~~ SUBSPACES OF BOUNDED SEQUENCES, SEQUENTIAL COMPLETENESS AND METHODS OF ALMOST CONVERGENCE

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ABSTRACT

The two main purposes of this thesis are:

(i) To investigate the sequential completeness of ℓ_1 with respect to weak topologies generated by subspaces of m whose β-dual is ℓ_1 ;

(ii) To introduce a class of summability methods that contains the method of almost convergence and to study its properties.

Chapter 1 is of an introductory nature. In Chapter 2 we obtain a characterization of those subspaces of m whose β-dual is ℓ_1, and then obtain several external characterizations of those subspaces of m that generate sequentially complete weak topologies on ℓ_1. In Chapter 3 we introduce a new class of summability methods that contains the method of almost convergence, and then study the properties of the subspaces of m generated by these methods. In Chapter 4, by establishing the sequential completeness of ℓ_1 under suitable weak topologies, we obtain consistency theorems for the summability methods introduced in Chapter 3.

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CHAPTER 1

PRELIMINARIES

§1. Introduction.

Using the notion of Banach limits, Lorentz [13] introduced the concept of almost convergence and developed a significant theory. Further studies related to almost convergence have since been carried out in [11], [16], [19] and [4]. Replacing the Banach limits by T-Banach limits (3.2 Definition 3), we define a new class of summability methods, which we call the T-almost convergence methods. A main purpose of this thesis is to study properties leading to the establishment of a bounded consistency theorem for these methods.

The bounded consistency theorem is one of the most important results of summability theory. The first proof of this famous theorem, requiring seven pages of calculations, was given by Brudno [7]. The result was merely stated by Mazur and Orlicz in [15], though a special case was given by Banach [2, p. 95]. The challenge of constructing a shorter proof was met by Petersen [17] by giving a streamlined version of Brudno's proof. Observing the basic relationship between this theorem and the sequential completeness of ℓ_1 under appropriate weak topologies, Bennett and Kalton [5] constructed a functional analytic proof. The same observation led them to extend the theorem to the space of almost convergence sequences [4].

The relationship between the bounded consistency theorem and the sequential completeness of ℓ_1 leads us to study the dual structure of ℓ_1 with some subspaces of m . In doing so we are able to characterize the class of subspaces of m whose β-dual is ℓ_1 . As a consequence of this characterization, we also answer some open questions raised in [24].

§2. Sequence spaces.

The primary aim of this and the remaining sections is to collect together the basic definitions and results of sequence space theory and summability theory, of which we shall make frequent use in the rest of the thesis. A detailed study of these materials can be found in [10] and [24].

We denote by ω the set of all real sequences. The set ω, under the usual operations of pointwise addition and scalar multiplication, becomes a vector space over $\mathbb{R}$. Any subspace E of ω is called a sequence space. An arbitrary member (x_n) of ω is sometimes denoted by x only. For x in ω, we write $|x|$ to mean $(|x_n|)$. The pointwise multiplication of two sequences x and y is denoted by $x.y$; i.e., $x.y = (x_n y_n)$. The matrix multiplication of two sequences is denoted by xy; i.e., $xy = \sum_{n=1}^{\infty} x_n y_n$.

We also adopt the following notation:

$e, e^k \in \omega$ are given by

$e = (1,1,.......)$

$e^k = (0,...,0,1,0,...)$ with the one in the kth position;

φ is the linear span of $\{e^k | k \in \mathbb{N}\}$;

$m = \{x \in \omega | \; \|x\|_\infty = \sup_n |x_n| < \infty\}$;

$c = \{x \in \omega \mid \lim_n x_n \text{ exists}\}$;

$c_o = \{x \in c \mid \lim_n x_n = 0\}$;

$ac = \{x \in \omega \mid \lim_p (x_{n+1}+x_{n+2}+...+x_{n+p})/p \text{ exists uniformly in } n\}$

$$ac_o = \{x \in ac \mid \lim_p (x_{n+1}+x_{n+2}+\dots+x_{n+p})/p = 0 \text{ uniformly in } n\}.$$

$$\ell_1 = \{x \in \omega \mid \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

We consider only sequence spaces containing φ. For $x \in \omega$, we write

$$P_n x = (x_1, x_2, \dots, x_n, 0, \dots).$$

For any subset M of $\mathbb{N}$, we denote the characteristic function of M by χ_M; i.e.,

$$(\chi_M)_k = \begin{cases} 1 & \text{if } k \in M \\ 0 & \text{if } k \in N \setminus M. \end{cases}$$

DEFINITION 1. A sequence space E is called monotone if $\chi_M . x \in E$ for every $x \in E$ and every $M \subseteq N$.

For a subset S of ω, $\langle S \rangle$ denotes the linear span of S. If E and F are two subspaces of ω, then $E \oplus F$ denotes the direct sum of E and F.

§3. Topologies on sequence spaces.

DEFINITION 1. A sequence space E with a locally convex topology τ is called a K-space provided that the linear functionals

$$x \to x_n \quad (n = 1,2,\ldots)$$

are continuous on E . If, in addition, (E,τ) is complete and metrizable, then (E,τ) is called an FK-space.

DEFINITION 2. A K-space (E,τ) is called an AD-space if φ is dense in E .

DEFINITION 3. A K-space (E,τ) is called an AK-space if $(P_n x)$ converges to x for every $x \in E$.

An FK-space has a topology generated by an increasing sequence of seminorms. If E, F are two FK-spaces with $E \subseteq F$, then the FK-topology of E is finer than the FK-topology of F restricted to E . In particular, the topology of an FK-space is unique.

The topological dual of a K-space (E,τ) is usually denoted by E' . For some important K-spaces, E' cannot be represented as a sequence space. To deal with this situation Köthe and Toeplitz [12] introduced the α-dual and β-dual of sequence spaces.

DEFINITION 4. Let E be a sequence space and define

(i) $E^{\alpha} = \{x \in \omega \mid \sum_{n=1}^{\infty} |x_n y_n| < \infty$ for every $y \in E\}$, and

(ii) $E^{\beta} = \{x \in \omega \mid \sum_{n=1}^{\infty} x_n y_n$ converges for every $y \in E\}$.

Then E^{α} and E^{β} are called the α- and β- dual of E, respectively.

There is a natural way of defining K-space topologies by considering dual pairs of sequence spaces. For a given sequence space E, let F denote a subspace of E^{β} with $\varphi \subseteq F$. Then E and F form a dual system under the bilinear functional $< x,y >$, where

$$< x,y > = \sum_{n=1}^{\infty} x_n y_n, \; x \in E, \; y \in F .$$

Any K-space topology on E is said to be compatible with the dual system $< E,F >$ if $E' = F$. The weak topology $\sigma(E,F)$ is the smallest compatible topology on E. For each $\sigma(F,E)$-bounded subset K of F, define the seminorm p_K on E by

$$p_K(x) = \sup_{y \in K} | < x,y > | .$$

If $\mathcal{F}$ is a family of $\sigma(F,E)$ bounded subsets of F, then the topology on E generated by the collection of seminorms $\{p_K | K \in \mathcal{F}\}$ is called the topology of uniform convergence on members of $\mathcal{F}$. The topology of uniform convergence on convex $\sigma(F,E)$-compact subsets of F is called the Mackey topology and denoted by $\tau(E,F)$. The Mackey topology is the largest compatible topology on E. The topology of uniform convergence on $\sigma(F,E)$-bounded subsets of F is called the strong topology and denoted by $\beta(E,F)$.

The following important results concerning dual systems can be found in [23].

PROPOSITION 1. Let $< E,F >$ be a dual pair of sequence spaces. If A is a convex subset of E, then the $\sigma(E,F)$-closure of A coincides with the $\tau(E,F)$-closure of A.

PROPOSITION 2. Let $< E,F >$ be a dual pair of sequence spaces and let τ be a compatible topology on E. Suppose (x^n) is a τ-Cauchy sequence in E. If (x^n) is $\sigma(E,F)$-convergent to x in E, then (x^n) is τ-convergent to x in E.

If $< E,F >$ is a dual pair of sequence spaces, then Proposition 1 implies that $(E,\tau(E,F))$ is an AD-space. The following result concerning dual pairs is known as the Grothendieck criterion.

THEOREM 1. Let $< E,F >$ be a dual pair, and let $\mathcal{F}$ be a family of $\sigma(F,E)$ bounded subsets of F. Suppose the topology τ (on E) of uniform convergence on members of $\mathcal{F}$ is compatible with the dual pair $< E,F >$. Then (E,τ) is complete if every linear functional on F, which is $\sigma(F,E)$-continuous on members of $\mathcal{F}$, belongs to E.

A comprehensive study of dual systems including the proof of Theorem 1 is contained in [23].

A topological space X is called separable if X has a countable dense subset.

PROPOSITION 3. Every AD-space is separable.

Proof. Let E be an AD-space. We claim that $D = \{x = (x_k) \in \varphi | x_k \in \mathbb{Q}$ for every $k \in \mathbb{N}\}$ is a countable dense subset of E. For each finite subset M of $\mathbb{N}$, let $D_M = \{x \in D | x_k = 0$ for $k \notin M\}$. Then D_M is countable and, moreover, $D = \cup\{D_M \mid M$ is a finite subset of $\mathbb{N}\}$. Since the collection of finite subsets of $\mathbb{N}$ is countable, D is also countable. Now let $x \in E$, and let p be a continuous seminorm on E. Let $\varepsilon > 0$. Since E is AD, there exists $y \in \varphi$ such that $p(x-y) < \frac{\varepsilon}{2}$. Since $y \in \varphi$, there exists $m \in \mathbb{N}$ such that $y = \sum_{k=1}^{m} y_k e^k$. For each k ($\leq m$), let $(y_{kn})_{n=1}^{\infty}$ be a sequence in $\mathbb{Q}$ such that $\lim_n y_{kn} = y_k$ in $\mathbb{R}$. Since E is a topological vector space, $\lim_n y_{kn}\, e^k = y_k e^k$ in E and hence $\lim_n \sum_{k=1}^{m} y_{kn} e^k = \sum_{k=1}^{m} y_k e^k = y$ in E. Thus there exists $n_o \in \mathbb{N}$ such that $p(\sum_{k=1}^{m} y_{kn_o} e^k - y) < \varepsilon/2$. Therefore,

$$p(\sum_{k=1}^{m} y_{kn_o} e^k - x) \leq p(\sum_{k=1}^{m} y_{kn_o} e^k - y) + p(y-x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Since $\sum_{k=1}^{m} y_{kn_o} e^k \in D$, it follows that D is a dense subset of E.

§4. Topological properties of K-spaces.

The following fundamental result characterizes compact subsets of a K-space. The proof is given in [9, p. 1010].

THEOREM 1. Let (E,τ) be a K-space. Then M is a relatively compact (respectively, compact) subset of E if and only if M is a relatively sequentially compact (respectively, sequentially compact) subset of E.

THEOREM 2. Let τ_1, τ_2 be two K-space topologies on a sequence space E. Then the following statements are equivalent:

(i) E has the same convergent sequences with respect to τ_1 and τ_2;

(ii) E has the same Cauchy sequences with respect to τ_1 and τ_2;

(iii) E has the same null sequences (sequences converging to 0) with respect to τ_1 and τ_2;

(iv) E has the same compact sets with respect to τ_1 and τ_2.

The proof of ((i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)) is given in [22, p. 343]. Applying Theorem 1, one can easily show that ((i) $\Rightarrow$ (iv) $\Rightarrow$ (iii)).

We denote by m_o the linear span of all sequences taking only the values zero and one. It is easy to check that m_o is dense in $(m, \| \|_\infty)$. Now we state the well-known Schur's lemma. The proof of this lemma is given in [23, p. 4].

THEOREM 3. A sequence (x^n) in ℓ_1 is $\sigma(\ell_1, m_o)$-convergent if and only if (x^n) is ℓ_1-norm convergent.

The following theorem, characterizing relatively compact subsets of ℓ_1, is stated in [5, p. 563] without proof. We give an elementary proof.

<u>THEOREM 4</u>. An ℓ_1-norm bounded subset K of ℓ_1 is relatively compact if and only if $\lim_n \sup_{x \in K} \sum_{i=n}^{\infty} |x_i| = 0$.

Proof. (Necessity) Suppose a bounded subset K of ℓ_1 is relatively compact. Assume that $\lim_n \sup_{x \in K} \sum_{i=n}^{\infty} |x_i| \neq 0$. Then there exists an $\varepsilon > 0$, a strictly increasing sequence (n_m) of positive integers and a sequence (x^m) in K such that

$$(1) \quad \sum_{i=n_m}^{\infty} |x_i^m| > \varepsilon .$$

Since K is relatively sequentially compact, there exists a subsequence (x^{m_k}) of (x^m) such that (x^{m_k}) converges in ℓ_1. Let $\lim_k x^{m_k} = x$. Since $x \in \ell_1$, there exists $p \in \mathbb{N}$ such that

$$(2) \quad \sum_{i=p}^{\infty} |x_i| < \varepsilon/2 .$$

Since (x^{m_k}) converges to x in ℓ_1 there exists $k_o (>p) \in \mathbb{N}$ such that

$$(3) \quad \sum_{i=1}^{\infty} |x_i^{m_k} - x_i| < \varepsilon/2 \quad \text{for } k \geq k_o .$$

Now, for $k \geq k_o$,

$$\sum_{i=n_{m_k}}^{\infty} |x_i^{m_k}| \le \sum_{i=p}^{\infty} |x_i^{m_k}| \quad (\text{since } p < k_o \le k \le n_{m_k})$$

$$\le \sum_{i=p}^{\infty} |x_i^{m_k} - x_k| + \sum_{i=p}^{\infty} |x_k|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{by (2) and (3).}$$

This contradicts (1). Hence $\lim_n \sup_{x \in K} \sum_{i=n}^{\infty} |x_i| = 0$.

(Sufficiency) Suppose K is a bounded subset of ℓ_1 such that $\lim_n \sup_{x \in K} \sum_{i=n}^{\infty} |x_i| = 0$. Let (x^n) be a sequence in K. Since (x^n) is pointwise bounded, there exists a subsequence (x^{n_k}) of (x^n) such that (x^{n_k}) converges pointwise to a member x of ω. Since (x^n) is ℓ_1-norm bounded, $x \in \ell_1$. To show that (x^{n_k}) converges to x in $(\ell_1, \| \|_1)$, let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that, for $k \in \mathbb{N}$,

$$(4) \quad \sum_{i=p}^{\infty} |x_i^{n_k}| < \varepsilon/3 \quad \text{and} \quad \sum_{i=p}^{\infty} |x_i| < \varepsilon/3 .$$

Also we can choose $k_o \in \mathbb{N}$ such that

$$(5) \quad \sum_{i=1}^{p-1} |x_i^{n_k} - x_i| < \varepsilon/3 \quad \text{for } k \ge k_o.$$

Now, for $k \ge k_o$,

$$\sum_{i=1}^{\infty} |x_i^{n_k} - x_i| \le \sum_{i=1}^{p-1} |x_i^{n_k} - x_i| + \sum_{i=p}^{\infty} |x_i^{n_k}| + \sum_{i=p}^{\infty} |x_i|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{by (4) and (5)} .$$

For a given FK-space E , the sets S_E and W_E are defined by:

$$S_E = \{x \in E \mid x = \sum_{k=1}^{\infty} e^k x_k\} ;$$

$$W_E = \{x \in E \mid f(x) = \sum_{k=1}^{\infty} f(e^k) x_k \text{ for every } f \in E'\}.$$

The following results concerning FK-spaces containing c_o are given in [5, p. 565].

<u>THEOREM 5</u>. An FK-space E contains c_o if and only if $(f(e^k)) \in \ell_1$ for every $f \in E'$.

<u>THEOREM 6</u>. For any FK-space E containing c_o, $c_o \subseteq S_E \subseteq W_E$.

Let X be a vector space over $\mathbb{R}$ with two homogeneous norms $\| \ \|$ and $\| \ \|^*$. Also assume that $\| \ \|$ is finer than $\| \ \|^*$. Then $(X, \| \ \|, \| \ \|^*)$ is called a two-norm space. A sequence (x_n) in X is said to be two-norm convergent to a member x in X if $\sup_n \|x_n\| < \infty$ and $\lim_n \|x_n - x\|^* = 0$. A linear functional f on X is called a two-norm linear functional if $\lim_n f(x_n) = 0$ for every (x_n) in X such that (x_n) is two-norm convergent to 0 . The following result regarding two-norm linear functionals is given in [1, p. 130].

<u>THEOREM 7</u>. Let $(X, \| \|, \| \|^*)$ be a two-norm space. Then f is a two-norm linear functional on X if and only if f is in the closure of the dual of $(X, \| \|^*)$ in $(X, \| \|)'$.

§5. Infinite matrices.

Given an infinite matrix $A = (a_{nk})$, we define the set ω_A to be $\{x \in \omega \mid \sum_{k=1}^{\infty} a_{nk}x_k$ converges for every $n \in \mathbb{N}\}$. For $x \in \omega_A$, we write $y = Ax$ to mean that $y_n = (Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$ for each n. Given a sequence space E and an infinite matrix A, we define the set E_A to be $\{x \in \omega_A \mid Ax \in E\}$. It is easy to verify that E_A is a sequence space. When $E = c$, this set is called the convergence domain of A. If $x \in c_A$, $\lim_n (Ax)_n$ exists and we denote this limit by $\lim_A x$.

Zeller [25] proved that, for any FK-space E, E_A is also an FK-space. Bennett [3] proved that E_A is a separable FK-space if E is a separable FK-space. For convenience, we write W_A for W_{c_A}.

Let $A = (a_{nk})$ be an infinite matrix. If $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$, we say that A has a finite norm and write $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}|$. A matrix A is called regular if c_A contains c and $\lim_A x = \lim_n x_n$ for every $x \in c$. A main theorem of summability theory is the Silverman-Toeplitz theorem which characterizes regular matrices. The proof of this theorem can be found in [24, p. 6].

<u>THEOREM 1</u>. A matrix A is regular if and only if the following conditions hold:

(i) $\|A\| < \infty$;

(ii) $\lim_n a_{nk} = 0$ for $k = 1,2,\ldots$;

(iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = 1$.

The proof of the following result is given in [18, p. 568].

<u>THEOREM 2</u>. Let A be a matrix such that

(i) $\|A\| < \infty$, and

(ii) $\lim_n a_{nk} = 0$ for $k = 1,2,\ldots$. Then $W_A \cap m = c_{o_A} \cap m$.

The following associative laws for matrices are given in [24, p. 8]. We frequently use them in Chapter 3.

<u>THEOREM 3</u>. Let A, B and C be matrices with finite norms. Let $t \in \ell_1$ and $x \in m$. Then the following laws hold:

(i) $t(Ax) = (tA)x$. (Here $t(Ax) = \sum_{n=1}^{\infty} t_n (Ax)_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} t_n a_{nk} x_k$ and $(tA)x = \sum_{k=1}^{\infty} (tA)_k x_k = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} t_n a_{nk} x_k$) ;

(ii) $(AB)C = A(BC)$;

(iii) $(AB)x = A(Bx)$.

CHAPTER 2

SEQUENTIAL COMPLETENESS

§1. Introduction.

In many situations the sequence spaces under consideration are not complete. It is known that important results in the general theory can be established under the weaker hypothesis of sequential completeness (e.g., the uniform boundedness theorem). Furthermore, in their papers ([5], [4]) Bennett and Kalton observed that the bounded consistency theorem is implied by the sequential completeness of ℓ_1 under suitable weak topologies. Two different methods are generally used to establish the sequential completeness of ℓ_1 under such topologies. The first one uses elementary gliding hump arguments, while the second uses more sophisticated functional analysis methods involving Orlicz-Pettis type results. Both rely on some structural properties of the subspace of m which generates the weak topology on ℓ_1 .

In this chapter we obtain a characterization of those subspaces of m whose β-dual is ℓ_1 , and then obtain an external characterization of those subspaces of m that generate sequentially complete weak topologies on ℓ_1 . As a consequence of these results, we answer some open questions about FK-spaces raised in [24].

§2. Definitions and basic results.

DEFINITION 1. A sequence space (E,τ) is called sequentially complete if every Cauchy sequence in E τ-converges to a member of E.

The following result is essentially contained in [24, p. 253].

PROPOSITION 1. Let $\langle E,F \rangle$ be a dual pair of sequence spaces. Then a sequence (a^n) of members of E is $\sigma(E,F)$-Cauchy if and only if $F \subseteq c_A$, where $A = (a_{nk})$ is the infinite matrix whose nth row is a^n.

Proof. Suppose (a^n) is $\sigma(E,F)$-Cauchy, and let $x \in F$. Then $(\sum_{k=1}^{\infty} a_k^n x_k)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $(\sum_{k=1}^{\infty} a_k^n x_k)_{n=1}^{\infty} \in c$. This means that $Ax \in c$ and hence $x \in c_A$. Thus $F \subseteq c_A$.

Suppose $F \subseteq c_A$. Then $(\sum_{k=1}^{\infty} a_k^n x_k)_{n=1}^{\infty} \in c$ for every $x \in F$. This implies that (a^n) is $\sigma(E,F)$-Cauchy.

PROPOSITION 2. Let $\langle E,F \rangle$ be a dual pair of sequence spaces, and suppose $(E,\sigma(E,F))$ is sequentially complete. Then $F^\beta = E$.

Proof. Since $\langle E,F \rangle$ is a dual pair, $E \subseteq F^\beta$. Let $x \in F^\beta$. Then $\sum_{k=1}^{\infty} x_k y_k$ converges for every $y \in F$. This implies that $(P_n x)_{n=1}^{\infty}$ is is $\sigma(E,F)$-Cauchy since $\varphi \subseteq E$. Since $(E,\sigma(E,F))$ is sequentially complete, $(P_n x)_{n=1}^{\infty}$ is $\sigma(E,F)$-convergent to x in E. Hence $x \in E$ and thus $F^\beta \subseteq E$.

The following proposition states a well known result for monotone sequence spaces (see 1.2 Definition 1). The proof can be found in [10, p. 188].

PROPOSITION 3. Let $< E,F >$ be a dual pair of sequence spaces such that $F^{\beta} = E$. If F is monotone, then $(E,\sigma(E,F))$ is sequentially complete.

The following result is generally known for normed spaces.

PROPOSITION 4. Let $(X,\| \|)$ be a normed space, and let Y be a subspace of X'-the dual space of X. Then every norm bounded $\sigma(X,Y)$-Cauchy sequence (x_n) in X is $\sigma(X,\bar{Y})$-Cauchy. Here $\bar{Y}$ is the closure of Y with respect to the usual norm topology on X'. Moreover, if (x_n) is $\sigma(X,Y)$-convergent, then (x_n) is $\sigma(X,\bar{Y})$-convergent.

Proof. Suppose (x_n) is a norm bounded $\sigma(X,Y)$-Cauchy sequence in X. Let $g \in \bar{Y}$ and $\varepsilon > 0$. Then there exists $h \in Y$ such that

$\|g-h\| < \dfrac{\varepsilon}{4 \sup_n \|x_n\|}$. Choose $n_o \in \mathbb{N}$ such that $|h(x_n-x_m)| < \varepsilon/2$ for $n,m \geq n_o$.

Thus, for $n,m \geq n_o$,

$$|g(x_n-x_m)| \leq |(g-h)(x_n-x_m)| + |h(x_n-x_m)| < \|g-h\|\|x_n-x_m\| + \varepsilon/2$$

$$\leq \|g-h\|.2 \sup_n\|x_n\| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Hence (x_n) is $\sigma(X,\bar{Y})$-Cauchy.

The last part can be proved by a similar argument.

§3. Weak topologies on ℓ_1.

Proposition 2 of the previous section implies that the β-dual of every subspace E of m which generates a sequentially complete weak topology on ℓ_1 must be ℓ_1. But $E^\beta = \ell_1$ is not a sufficient condition for sequential completeness of the corresponding weak topology on ℓ_1. For instance, $(\ell_1, \sigma(\ell_1, c))$ is not sequentially complete, though $c^\beta = \ell_1$. It seems difficult to obtain an internal characterization of such subspaces of m. The following theorem, however, characterizes subspaces of m whose β-dual is ℓ_1, and consequently we obtain a useful external characterization of subspaces of m generating sequentially complete weak topologies on ℓ_1.

THEOREM 1. Let E be a subspace of m containing φ. Then the following are equivalent:

(i) $E^\beta = \ell_1$;

(ii) every $\sigma(\ell_1,E)$-bounded sequence in ℓ_1 is ℓ_1-norm bounded ;

(iii) every $\sigma(\ell_1,E)$-bounded subset of ℓ_1 is ℓ_1-norm bounded ;

(iv) every $\sigma(\ell_1,E)$-Cauchy sequence in ℓ_1 is ℓ_1-norm bounded ;

(v) every $\sigma(\ell_1,E)$-Cauchy sequence in ℓ_1 is $\sigma(\ell_1,\varphi)$-convergent .

Proof. ((i) $\Rightarrow$ (ii)). Suppose $E^\beta = \ell_1$, and let (x^n) be a $\sigma(\ell_1, E)$-bounded sequence in ℓ_1. Suppose $\sup_n \|x^n\| = \infty$. Since (x^n) is $\sigma(\ell_1,E)$-bounded and $\varphi \subseteq E$,

$$(1) \quad \sup_n \sum_{k=1}^{p} |x_k^n| < \infty \quad \text{for } p = 1,2,\ldots .$$

Let $k_1 = 1$. Choose $n_1 \in \mathbb{N}$ such that $\|x^{n_1}\|_1 > (2+1) \sup_n |x_1^n| + 2+1$, and then $k_2 (>k_1) \in \mathbb{N}$ such that $\sum_{k=k_2+1}^{\infty} |x_k^{n_1}| < 1$. Note that $\sum_{k=2}^{k_2} |x_k^{n_1}| > 2|x_1^{n_1}| + 2$, i.e., $\sum_{k=k_1+1}^{k_2} |x_k^{n_1}| > 2 \sum_{k=1}^{k_1} |x_k^{n_1}| + 2$. Since

$\sup_n \sum_{k=1}^{k_2} |x_k^n| < \infty$ by (1), we can choose $n_2 (>n_1) \in \mathbb{N}$ such that

$\|x^{n_2}\|_1 > (2^2+1) \sup_n \sum_{k=1}^{k_2} |x_k^n| + 2^2+1$, and then $k_3 (>k_2) \in \mathbb{N}$ such that

$\sum_{k=k_3+1}^{\infty} |x_k^{n_2}| < 1$. Note that $\sum_{k=k_2+1}^{k_3} |x_k^{n_2}| > 2^2 \sum_{k=1}^{k_2} |x_k^{n_2}| + 2^2$. We can proceed to choose strictly increasing sequences (k_r) and (n_r) of positive integers such that:

$$\text{(2)} \quad M_r = \sum_{k=k_r+1}^{k_{r+1}} |x_k^{n_r}| > 2^r \sum_{k=1}^{k_r} |x_k^{n_r}| + 2^r ;$$

$$\text{(3)} \quad \sum_{k=k_{r+1}+1}^{\infty} |x_k^{n_r}| < 1 .$$

From (2) $(\frac{1}{M_r})_{r=1}^{\infty} \in \ell_1$. Let $y_k = \frac{x_k^{n_r}}{rM_r}$ for $k_r < k \le k_{r+1}$. Then (y_k) is a sequence of real numbers such that

$$\text{(4)} \quad \sum_{k=k_r+1}^{k_{r+1}} |y_k| = \frac{1}{r} .$$

(4) implies that $(y_k) \notin \ell_1$. Let $z \in E$. Then, for any $r \in \mathbb{N}$,

$$\left|\sum_{k=1}^{k_{r+1}} x_k^{n_r} z_k\right| \le \left|\sum_{k=1}^{\infty} x_k^{n_r} z_k\right| + \sum_{k=k_{r+1}+1}^{\infty} |x_k^{n_r} z_k| \le \left|\sum_{k=1}^{\infty} x_k^{n_r} z_k\right| + \|z\|_\infty$$

by (3). Since (x^n) is $\sigma(\ell_1,E)$-bounded, $\sup_r \left|\sum_{k=1}^{\infty} x_k^{n_r} z_k\right| < \infty$ and hence

$\sup_r \left|\sum_{k=1}^{k_{r+1}} x_k^{n_r} z_k\right| < \infty$. Thus $\sum_{r=1}^{\infty} \frac{1}{rM_r} \left(\sum_{k=1}^{k_{r+1}} x_k^{n_r} z_k\right)$ converges since

$\left(\frac{1}{rM_r}\right)_{r=1}^{\infty} \in \ell_1$.

But $$\left|\sum_{r=1}^{\infty} \frac{1}{rM_r} \left(\sum_{k=1}^{k_r} x_k^{n_r} z_k\right)\right| \le \|z\|_\infty \sum_{r=1}^{\infty} \frac{1}{rM_r} \left(\sum_{k=1}^{k_r} |x_k^{n_r}|\right)$$

$$\le \|z\|_\infty \sum_{r=1}^{\infty} \frac{1}{r2^r} \text{ (by (2))} < \infty .$$

Hence $\sum_{r=1}^{\infty} \frac{1}{rM_r} \left(\sum_{k=k_r+1}^{k_{r+1}} x_k^{n_r} z_k\right)$ converges. Now we show that $\sum_{k=1}^{\infty} y_k z_k$ is Cauchy. Let $\varepsilon > 0$. Choose $r_o \in \mathbb{N}$ such that:

(5) $\left|\sum_{r=\ell}^{m} \frac{1}{rM_r} \left(\sum_{k=k_r+1}^{k_{r+1}} x_k^{n_r} z_k\right)\right| < \varepsilon/3$ for $\ell, m \ge r_o$;

(6) $\frac{1}{r_o} < \frac{\varepsilon}{3\|z\|_\infty}$.

Let $p,q \in \mathbb{N}$ such that $k_{r_o} < p \le q$. Then there exist $s,t \in \mathbb{N}$ such that $k_s < p \le k_{s+1}$ and $k_t < q \le k_{t+1}$. Note that $r_o \le s \le t$.

Case 1. $s = t$.

$$\left|\sum_{k=p}^{q} y_k z_k\right| \le \sum_{k=p}^{q} |y_k z_k| \le \sum_{k=k_s+1}^{k_{s+1}} |y_k z_k| \le \|z\|_\infty \frac{1}{s} \quad \text{(by (4))} \le \frac{\|z\|_\infty}{r_o} < \varepsilon/3$$

$$\text{(by (6))} < \varepsilon .$$

Case 2. $t = s+1$.

$$\left|\sum_{k=p}^{q} y_k z_k\right| \le \sum_{k=p}^{k_{s+1}} |y_k z_k| + \sum_{k=k_{s+1}+1}^{q} |y_k z_k| \le \sum_{k=k_s+1}^{k_{s+1}} |y_k z_k| + \sum_{k=k_{s+1}+1}^{k_{s+2}} |y_k z_k|$$

$$\le \|z\|_\infty \frac{1}{s} + \|z\|_\infty \frac{1}{s+1} \quad \text{(by (4))} \le \|z\|_\infty \frac{2}{r_o} < \frac{2\varepsilon}{3} \quad \text{(by (6))} < \varepsilon .$$

Case 3. $t > s+1$.

$$\left|\sum_{k=p}^{q} y_k z_k\right| \le \sum_{k=p}^{k_{s+1}} |y_k z_k| + \left|\sum_{k=k_{s+1}+1}^{k_t} y_k z_k\right| + \sum_{k=k_t+1}^{q} |y_k z_k|$$

$$\le \sum_{k=k_s+1}^{k_{s+1}} |y_k z_k| + \left|\sum_{r=s+1}^{t-1} \Big(\sum_{k=k_r+1}^{k_{r+1}} y_k z_k\Big)\right| + \sum_{k=k_t+1}^{k_{t+1}} |y_k z_k|$$

$$\le \|z\|_\infty \cdot \frac{1}{s} + \left|\sum_{r=s+1}^{t-1} \frac{1}{rM_r} \Big(\sum_{k=k_r+1}^{k_{r+1}} x_k^{n_r} z_k\Big)\right| + \|z\|_\infty \cdot \frac{1}{t} \quad \text{(by (4)}$$

$$\text{and since } \; y_k = \frac{x_k^{n_r}}{rM_r} \; \text{ for } \; k_r < k \le k_{r+1})$$

$$< \|z\|_\infty \cdot \frac{1}{r_o} + \frac{\varepsilon}{3} + \|z\|_\infty \cdot \frac{1}{r_o} \quad \text{(by (5) since } \; r_o < s+1 \le t-1)$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{(by (6))} = \varepsilon .$$

Thus $\sum_{k=1}^{\infty} y_k z_k$ is Cauchy and hence convergent. Since (z_k) is arbitrary in E, $(y_k) \in E^{\beta}$. This contradicts that $E^{\beta} = \ell_1$ since $(y_k) \notin \ell_1$.

((ii) ⇒ (iii)) and ((iii) ⇒ (iv)) are obvious.

((iv) ⇒ (v)). Suppose condition (iv) holds. Let (x^n) be $\sigma(\ell_1, E)$-Cauchy. Then (x^n) is $\sigma(\omega,\varphi)$-Cauchy. Hence there exists $x \in \omega$ such that (x^n) is $\sigma(\omega,\varphi)$-convergent to x. Since $\sup_n \|x^n\|_1 < \infty$,

$\sum_{k=1}^{m} |x_k| = \sum_{k=1}^{m} \lim_n |x_k^n| \le \sup_n \|x^n\|_1 < \infty$ for $m \in N$, and hence $x \in \ell_1$.

Thus (x^n) is $\sigma(\ell_1,\varphi)$-convergent.

((v) ⇒ (i)). Suppose condition (v) holds, and let $x \in E^{\beta}$. Then $\sum_{k=1}^{\infty} x_k y_k$ converges for every $y \in E$ and hence $(P_n x)_{n=1}^{\infty}$ is $\sigma(\ell_1, E)$-Cauchy. Thus $(P_n x)_{n=1}^{\infty}$ is $\sigma(\ell_1,\varphi)$-convergent. This implies that $x \in \ell_1$ and hence $E^{\beta} \subseteq \ell_1$. Since $E \subseteq m$, $\ell_1 \subseteq E^{\beta}$.

<u>COROLLARY 1</u>. Let E be a subspace of m containing φ. Then the following are equivalent:

(i) $E^{\beta} = \ell_1$;

(ii) for every matrix $A = (a_{nk})$ such that $E \subseteq c_A$, $\|A\| < \infty$.

Proof. ((i) ⇒ (ii)). Assume $E^{\beta} = \ell_1$ and suppose $A = (a_{nk})$ is a matrix such that $E \subseteq c_A$. Then $(a^n)_{n=1}^{\infty}$ is a sequence in ℓ_1, where

$a^n = (a_{nk})_{k=1}^{\infty}$. By Proposition 1 of §2, (a^n) is $\sigma(\ell_1,E)$-Cauchy. By Theorem 1 ((i) ⇒ (iv)), $\|A\| = \sup_n \|a^n\|_1 < \infty$.

((ii) ⇒ (i)). Assume condition (ii) and let $t \in E^\beta$. Define a matrix $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} t_k & \text{if } 1 \le k \le n \\ 0 & \text{if } k > n . \end{cases}$$

Since $t \in E^\beta$, $E \subseteq c_A$ and hence $\|A\| < \infty$. This implies that $t \in \ell_1$ so that $E^\beta \subseteq \ell_1$. Since $E \subseteq m$, $\ell_1 \subseteq E^\beta$.

<u>COROLLARY 2</u>. Let E be a subspace of m containing φ such that ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete, and let $A = (a_{nk})$ be an infinite matrix. If $E \subseteq c_A$, then $\|A\| < \infty$ and (a^n) is $\sigma(\ell_1,E)$-convergent, where $a^n = (a_{nk})_{k=1}^{\infty}$.

Proof. Since ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete, $E^\beta = \ell_1$ by Proposition 2 of §2. Hence $\|A\| < \infty$ by Corollary 1. Also, by Proposition 1 of §2, (a^n) is $\sigma(\ell_1,E)$-Cauchy. Since ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete, (a^n) is $\sigma(\ell_1,E)$-convergent.

Now we use Theorem 1 to obtain an external characterization of those subspaces of m generating sequentially complete weak topologies on ℓ_1.

THEOREM 2. Let E be a subspace of m containing φ. Then ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete if and only if

(i) $E^\beta = \ell_1$, and

(ii) $E \subseteq c_A \Rightarrow E \subseteq c_{o_A}$, whenever A is an infinite matrix such that $\|A\| < \infty$ and such that each column of A belongs to c_o.

Proof. (Necessity). Suppose ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete. Then $E^\beta = \ell_1$ by Proposition 2 of §2. Let $A = (a_{nk})$ be an infinite matrix such that $\|A\| < \infty$ and such that each column of A belongs to c_o. Suppose $E \subseteq c_A$. Then $(a^n)_{n=1}^\infty$ is $\sigma(\ell_1,E)$-Cauchy by Proposition 1 of §2, where $a^n = (a_{nk})_{k=1}^\infty$. Since ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete, (a^n) is $\sigma(\ell_1,E)$ convergent. But (a^n) pointwise converges to 0, and hence (a^n) is $\sigma(\ell_1,E)$-convergent to 0. This implies that $E \subseteq c_{o_A}$.

(Sufficiency). Suppose conditions (i) and (ii) hold and let (x^n) be a $\sigma(\ell_1,E)$-Cauchy sequence in ℓ_1. Then $\sup_n \|x^n\|_1 < \infty$, and (x^n) is $\sigma(\ell_1,\varphi)$-convergent to a member x of ℓ_1, by Theorem 1((i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v)). Let $a_{nk} = (x_k^n - x_k)$ for $n,k \in \mathbb{N}$ and $A = (a_{nk})$. Then $\|A\| < \infty$ and each column of A belongs to c_o. Since $(x^n - x)_{n=1}^\infty$ is $\sigma(\ell_1,E)$-Cauchy, $E \subseteq c_A$ by Proposition 1 of §2. Thus $E \subseteq c_{o_A}$. This implies that $(x^n - x)_{n=1}^\infty$ $\sigma(\ell_1,E)$-converges to 0, and hence (x^n) $\sigma(\ell_1,E)$-converges to x.

COROLLARY 1. Let E be a subspace of m containing φ, If $E^{\beta} = \ell_1$ and $E \subseteq c_o$, then ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete.

Proof. Let A be an infinite matrix such that $\|A\| < \infty$ and such that each column of A belongs to c_o. First we show that $c_o \subseteq c_{o_A}$. Let $x = (x_k) \in c_o$ and $\varepsilon > 0$. Choose $k_o \in \mathbb{N}$ such that $|x_k| < \frac{\varepsilon}{2\|A\|}$ for $k \geq k_o$, and then $n_o \in \mathbb{N}$ such that $\sum_{k=1}^{k_o} |a_{nk}| < \frac{\varepsilon}{2\|x\|_\infty}$ for $n \geq n_o$. Thus, for $n \geq n_o$,

$$\left|\sum_{k=1}^{\infty} a_{nk}x_k\right| \leq \sum_{k=1}^{k_o-1} |a_{nk}x_k| + \sum_{k=k_o}^{\infty} |a_{nk}x_k|$$

$$\leq \|x\|_\infty \sum_{k=1}^{k_o-1} |a_{nk}| + \frac{\varepsilon}{2\|A\|} \sum_{k=k_o}^{\infty} |a_{nk}|$$

$$< \|x\|_\infty \frac{\varepsilon}{2\|x\|_\infty} + \frac{\varepsilon}{2\|A\|} \cdot \|A\| = \varepsilon .$$

This implies that $\lim_A x = 0$ and hence $c_o \subseteq c_{o_A}$. Since $E \subseteq c_o$, $E \subseteq c_{o_A}$. Thus ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete by Theorem 2.

PROPOSITION 1. Let E be a subspace of m containing φ. Then ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete if and only if ℓ_1 is $\sigma(\ell_1, \bar{E}^{\infty})$-sequentially complete and $E^{\beta} = \ell_1$.

Proof. (Necessity). Suppose ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete. Then $E^{\beta} = \ell_1$ by Proposition 2 of §2, and hence any $\sigma(\ell_1, E)$-Cauchy sequence (x^n) in ℓ_1 is ℓ_1-norm bounded by Theorem 1 ((i) ⇒ (iv)). Thus $(\ell_1,\sigma(\ell_1,E))$ and $(\ell_1,\sigma(\ell_1,\bar{E}^{\infty}))$ have the same Cauchy sequences by Proposition 4 of §2. By 1.4 Theorem 2, $(\ell_1,\sigma(\ell_1,E))$ and $(\ell_1,\sigma(\ell_1,\bar{E}^{\infty}))$ have the same convergent sequences. This implies that ℓ_1 is $\sigma(\ell_1,\bar{E}^{\infty})$-sequentially complete.

(Sufficiency). Suppose ℓ_1 is $\sigma(\ell_1,\bar{E}^{\infty})$-sequentially complete and $E^{\beta} = \ell_1$. Using the same argument as above we can conclude that $(\ell_1,\sigma(\ell_1,E))$ and $(\ell_1,\sigma(\ell_1,\bar{E}^{\infty}))$ have the same Cauchy sequences and the same convergent sequences. This implies that ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete.

<u>DEFINITION 1</u>. Let E be a subspace of m containing φ such that ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete. Further assume that $e \notin \bar{E}^{\infty}$. Let $G = \bar{E}^{\infty} \oplus <\{e\}>$. For each $x \in G$, there exist $y \in \bar{E}^{\infty}$ and $\alpha \in R$ such that $x = y + \alpha e$. α is called the E-limit of x and we write E-lim $x = \alpha$.

Remark. $c \subseteq G$ since $\varphi \subseteq E$.

The following consistency theorem holds for E-limits.

<u>THEOREM 3</u>. Let E be a subspace of m containing φ such that ℓ_1 is $\sigma(\ell_1,E)$-sequentially complete. Further assume that $e \notin \bar{E}^{\infty}$. Let $G = E \oplus <\{e\}>$. If A is a regular matrix such that $G \subseteq c_A$, then E-lim $x = \lim_A x$ for every $x \in G$.

Proof. $(\ell_1, \sigma(\ell_1, \bar{E}^{\infty}))$ is sequentially complete by Proposition 1.

Since $\bar{E}^{\infty} \subseteq G \subseteq c_A$, $\bar{E}^{\infty} \subseteq c_{o_A}$ by Theorem 2. This implies that

$\lim_A x = 0$ for every $x \in \bar{E}^{\infty}$. Let $x \in G$ and E-lim $x = \alpha$. Then

there exists $y \in \bar{E}^{\infty}$ such that $x = y + \alpha e$, and hence

$$\lim_A x = \lim_A y + \lim_A \alpha e = \alpha = \text{E-lim } x .$$

The following theorem gives another external characterization of subspaces of m generating sequentially complete weak topologies on ℓ_1. A similar result was proved by J.J. Sember in [20] and we follow essentially the same argument.

<u>THEOREM 4</u>. Let E be a subspace of m containing φ. Then the following are equivalent:

(i) ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete ;

(ii) If F is any separable FK space containing E, then $c_o \oplus E \subseteq W_F$.

Proof. ((i) ⇒ (ii)). Since $(\ell_1, \sigma(\ell_1, E))$ is sequentially complete, $E^{\beta} = \ell_1$ by Proposition 2 of §2. Let F be a separable FK-space containing E. By Theorem 5 ((i) ⇒ (iv)) of [6, p. 517] it follows that $E \subseteq W_F$. Now we show that $c_o \subseteq W_F$. Let $f \in F'$. Then

$f(x) = \sum_{k=1}^{\infty} x_k f(e_k)$ for every $x \in E$, since $E \subseteq W_F$. This implies that $(f(e_k))_{k=1}^{\infty} \in \ell_1$, since $E^{\beta} = \ell_1$. It follows that $c_o \subseteq F$ by

1.4 Theorem 5. Since F is an FK space containing c_o, $c_o \subseteq W_F$ by 1.4 Theorem 6.

((ii) ⇒ (i)). We first show that condition (ii) implies that $E^\beta = \ell_1$. To this end suppose $E \subseteq c_A$, where A is an infinite matrix. Since c_A is a separable FK-space condition (ii) implies that $c_o \subseteq c_A$. Since $c_o^\beta = \ell_1$, Corollary 1 of Theorem 1 implies that $\|A\| < \infty$. Now the same corollary implies that $E^\beta = \ell_1$.

To show that $(\ell_1, \sigma(\ell_1, E))$ is sequentially complete, let A be a matrix such that $\|A\| < \infty$ and such that each column of A belongs to c_o. Suppose $E \subseteq c_A$. Then condition (ii) implies that $E \subseteq W_A$ since c_A is a separable FK-space. But $W_A \cap m = c_{o_A} \cap m$ by 1.5 Theorem 2. Thus $E \subseteq c_{o_A}$ and hence $(\ell_1, \sigma(\ell_1, E))$ is sequentially complete by Theorem 2.

<u>COROLLARY 1</u>. Let E be a separable FK-space such that $E \subseteq m$. If ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete, then $E = c_o$.

Proof. It follows from Theorem 4 that $E \oplus c_o \subseteq W_E$. Since $c_o \subseteq E \subseteq m$, the FK-topology on E is finer than the uniform topology on E. Hence $W_E \subseteq c_o$ so that $E = W_E = c_o$.

<u>COROLLARY 2</u>. Let A be a matrix such that $\|A\| < \infty$ and such that each column of A belongs to c_o. If $c_{o_A} \neq c_o$, then c_{o_A} contains an unbounded sequence.

Proof. By 1.5 Theorem 2, $W_A \cap m = c_{o_A} \cap m$. Since $c_o \subseteq c_A$, it follows from Theorem 3 of [5, p. 568] that ℓ_1 is $\sigma(\ell_1, c_{o_A} \cap m)$-sequentially complete. Suppose $c_{o_A} \subseteq m$. Then ℓ_1 is $\sigma(\ell_1, c_{o_A})$-sequentially complete. Since c_{o_A} is a separable FK-space, Corollary 1 implies that $c_{o_A} = c_o$. This contradiction shows that $c_{o_A} \not\subseteq m$.

A. Wilansky asked the following questions in [24, p. 260, 300].

1. Is there an FK-space smaller than c_o whose β-dual is ℓ_1 ?
2. Is c_o the only FK-space which is AD and whose β-dual is ℓ_1 ?

The following corollaries give a partial answer to 1 and an affirmative answer to 2.

<u>COROLLARY 3.</u> If E is a separable FK-space such that $E \subseteq c_o$ and $E^\beta = \ell_1$, then $E = c_o$.

Proof. ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete by Corollary 1 of Theorem 2. Thus, by Corollary 1 of Theorem 4, $E = c_o$.

<u>COROLLARY 4.</u> Let E be an FK-space. If E is AD and $E^\beta = \ell_1$, then $E = c_o$.

Proof. The condition $E^\beta = \ell_1$ implies that $E \subseteq m$. Thus the FK-topology on E is finer than the uniform norm topology on E. Hence $c_o = \bar{\varphi}^{\infty} \supseteq \bar{\varphi} = E$ ($\bar{\varphi}$ is the closure of φ in E with respect to the FK-topology). Since E is AD, it follows from 1.3, Proposition 3 that E is separable. Thus Corollary 3 implies that $E = c_o$.

THEOREM 5. Let E be a monotone subspace of m containing φ. Then the following are equivalent:

(i) ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete;

(ii) If F is any separable FK-space containing E, then

$$c_0 \oplus E \subseteq S_F .$$

Proof.((i) ⇒ (ii)). Since ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete, $E^\beta = \ell_1$ by Proposition 2 of §2. Since E is monotone, Theorem 6 of [6, p. 519] can be applied (see the remark of p. 519) to give the condition $E \subseteq S_F$. We can apply the same argument as in the proof of Theorem 4 to show that $c_0 \subseteq E$.

((ii) ⇒ (i)). It follows from the same argument as in the proof of Theorem 4 that $E^\beta = \ell_1$. Since E is monotone, ℓ_1 is $\sigma(\ell_1, E)$-sequentially complete by Proposition 3 of §2.

CHAPTER 3

T-ALMOST CONVERGENCE

§1. Introduction.

Lorentz, in [13], introduced the concept of almost convergence. One of his equivalent forms of a bounded sequence being almost convergent was

$$\lim_p (x_{n+1} + x_{n+2} + \dots + x_{n+p})/p \text{ exists uniformly in } n .$$

It is easy to observe that this formulation is also equivalent to

$$\lim_p (T_o x + T_o^2 x + \dots + T_o^p x)_n/p \text{ exists uniformly in } n ,$$

where $T_o = (t_{nk})$ is the infinite matrix defined by

$$t_{nk} = \begin{cases} 1 & \text{if } k = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

In this chapter we replace the matrix T_o by a more general matrix T, and then study the sequence spaces that are generated by T in the same way that the space of almost convergence sequences is generated by T_o. Also, for these sequence spaces, we establish several results already known for the special case of almost convergence.

We apply some of the basic techniques in [4] to obtain these results. Some of the details are more difficult than those of [4]. We need considerable preparation, for example, to establish Theorem 2 of §5.

§2. Definitions and basic results.

DEFINITION 1. A continuous linear function $L: m \to \mathbb{R}$ is called an extended limit if $L(x) = \lim_n x_n$ for every $x = (x_n) \in c$.

PROPOSITION 1. Extended limits exist.

Proof. Let $L: c \to R$ be defined by $L(x) = \lim_n x_n$. Since $|L(x)| = |\lim_n x_n| \le \|x\|_\infty$, L is continuous. By the Hahn Banach theorem L can be extended continuously over m.

REMARK. In general the norm of an extended limit is taken to be one. We drop this condition from our definition since it does not serve any useful purpose in our work.

DEFINITION 2. An infinite matrix $T = (t_{nk})$ of non-negative entries is called lifting if

(i) $t_{nk} = 0$ for $n \ge k$, and

(ii) $\sum_{k=1}^{\infty} t_{nk} = 1$ for $n \in \mathbb{N}$.

REMARK. Every lifting matrix is regular.

DEFINITION 3. Let T be a lifting matrix. An extended limit L is called a T-Banach limit if $L(x) = L(Tx)$ for every $x \in m$.

In the rest we assume that $T = (t_{nk})$ is a lifting matrix. The existence of T-Banach limits will be shown later. We denote by Λ_T the set of all T-Banach limits and also use the following notations;

$$U_T = \{x - Tx \mid x \in m\};$$

$$Tac = \{x \in m \mid L(x) = L'(x) \quad \text{for} \quad L, L' \in \Lambda_T\};$$

$$Tac_o = \{x \in Tac \mid L(x) = 0 \quad \text{for} \quad L \in \Lambda_T\}.$$

It is easy to verify that U_T, Tac and Tac_o are linear subspaces of m. For each $x \in Tac$, $L(x)$ assumes a common value for every T-Banach limit L. We denote this common value by T-Lim x and say that x is T-almost convergent to T-Lim x. Also note that T-Lim x is a linear functional on Tac.

<u>PROPOSITION 2</u>. Let T be a lifting matrix. Recall that $U_T = \{x - Tx \mid x \in m\}$. Then

(i) $U_T = \{x - T^n x \mid x \in m, n \in \mathbb{N}\}$, and

(ii) U_T is a linear subspace of Tac_o with $\varphi \subseteq U_T$.

Proof. (i) For $x \in m$ and $n \in \mathbb{N}$,

$$x - T^n x = (I - T^n)x$$

$$= (I - T)(I + T + \dots + T^{n-1})x \quad \text{by} \quad 1.5, \text{ Theorem } 3(iii).$$

Since $(I + T + \dots + T^{n-1})x \in m$, $x - T^n x \in U_T$.

(ii) For $x \in m$ and $L \in \Lambda_T$, $L(x - Tx) = L(x) - L(Tx) = 0$, and hence $x - Tx \in Tac_o$. Thus $U_T \subseteq Tac_o$. Since $t_{ij} = 0$ for $i \geq j$, $(I - T)e^1 = (1,0,0,\dots)$, $(I - T)e^2 = (-t_{12},1,0,0,\dots),\dots,(I - T)e^n = (-t_{1n},-t_{2n},\dots,-t_{n-1,n},1,0,0,\dots),\dots$. Hence $\varphi \subseteq U_T$.

PROPOSITION 3. Let T be a lifting matrix. Then the following statements are true:

(i) $T\text{-Lim } x = \lim_n x_n$ for every $x = (x_n) \in c$;

(ii) $c \subsetneqq \text{Tac}$ and $c_o \subsetneqq \text{Tac}_o$;

(iii) $\text{Tac} = \text{Tac}_o \oplus < \{e\} >$;

(iv) Tac and Tac_o are closed linear subspaces of m .

Proof. (i) follows directly from the definitions.

(ii) $c \subseteq \text{Tac}$ and $c_o \subseteq \text{Tac}_o$ follow from the definitions. Now we show that $\text{Tac}_o \not\subseteq c$. By Proposition 2 (ii), $U_T = \{(I-T)x \mid x \in m\} \subseteq \text{Tac}_o$. Since $\sum_{k=1}^{\infty} |(I-T)_{nk}|$ is not uniformly convergent in n, $U_T \not\subseteq c$ (see [14, p. 10]). Hence $\text{Tac}_o \not\subseteq c$. Thus $c \neq \text{Tac}$ and $c_o \neq \text{Tac}_o$.

(iii) For each $x \in \text{Tac}$, $x = (x - (T - \text{Lim } x)e) + (T - \text{Lim } x)e$. Since $(x - (T - \text{Lim } x)e) \in \text{Tac}_o$, $x \in \text{Tac}_o \oplus < \{e\} >$ and hence $\text{Tac} \subseteq \text{Tac}_o \oplus < \{e\} >$. Since $c \subseteq \text{Tac}$ (by (i)) and $\text{Tac}_o \subseteq \text{Tac}$, $\text{Tac}_o \oplus < \{e\} > \subseteq \text{Tac}$.

(iv) Suppose (x^n) is a sequence in Tac such that (x^n) is convergent to x in $(m, \| \|_\infty)$. Then, for every $L \in \Lambda_T$, $(L(x^n))_{n=1}^{\infty}$ is convergent to $L(x)$ in R . Since $x^n \in \text{Tac}$, $L(x^n) = T - \text{Lim } x^n$

for $n \in N$. Hence $L(x) = \lim_n (T - \mathrm{Lim}\, x^n)$ for every $L \in \Lambda_T$. Thus $x \in \mathrm{Tac}$ and $T - \mathrm{Lim}\, x = \lim_n (T - \mathrm{Lim}\, x^n)$. Therefore, Tac is closed in $(m, \|\ \|_\infty)$. The same argument can be used for Tac_o.

PROPOSITION 4. Let L be a continuous linear functional on $(m, \|\ \|_\infty)$, and let T be a lifting matrix. Then L is a T-Banach limit if and only if (i) $L(e) = 1$, and (ii) $L(U_T) = \{0\}$.

Proof. (Necessity). Suppose L is a T-Banach limit. Then (i) follows from the definition of T-Banach limit. Let $x \in m$. Then $L(x - Tx) = L(x) - L(Tx) = 0$ and hence (ii) holds.

(Sufficiency). Suppose (i) and (ii) hold for a continuous linear functional L on $(m, \|\ \|_\infty)$. Then $L(\varphi) = \{0\}$ since $\varphi \subseteq U_T$ by Proposition 2(ii). Hence $L(c_o) = L(\bar{\varphi}^{\infty}) = \{0\}$. Since $L(e) = 1$, it follows that $L(x) = \lim_n x_n$ for $x \in c$. Thus L is an extended limit. Also condition (ii) implies that $L(x) = L(Tx)$ for every $x \in m$. Hence L is a T-Banach limit.

§3. A characterization of T-almost convergent sequences.

Modifying the technique used in [4] to establish a characterization of almost convergent sequences, we obtain a similar characterization for T-almost convergent sequences (Theorem 1). First we state the following lemma, which can be found in [4 , p. 26].

LEMMA 1. For every $x \in m\setminus c_o$, there exists an extended limit L such that $L(x) \neq 0$.

THEOREM 1. Let $A = (a_{nk})$ be a regular matrix such that $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}| = 0$ (assume $a_{no} = 0$ for every n), and let $x \in m$. Let T be a lifting matrix. Then $x \in Tac$ and $T\text{-Lim } x = \alpha$ if and only if $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = \alpha$ uniformly in n.

Proof. First we show that $x \in Tac_o$ if and only if $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = 0$ uniformly in n. Suppose $x = (x_n) \in Tac_o$. Let (n_p) be any sequence of positive integers. Define the linear map $\psi: m \to m$ by $[\psi(y)]_p = \sum_{k=1}^{\infty} a_{pk}(T^k y)_{n_p}$. Then

$$|[\psi(y)]_p| = \left|\sum_{k=1}^{\infty} a_{pk}(T^k y)_{n_p}\right| \leq \sum_{k=1}^{\infty} |a_{pk}| \|T^k y\|_\infty$$

$$\leq \|y\|_\infty \sum_{k=1}^{\infty} |a_{pk}| \text{ (since } \|T\| = 1) \leq \|y\|_\infty \|A\| .$$

Hence ψ is continuous and, moreover,

(1) $\lim_p [\psi(e)]_p = \lim_p \sum_{k=1}^{\infty} a_{pk}(T^k e)_{n_p} = \lim_p \sum_{k=1}^{\infty} a_{pk}$

(since $Te = e$) $= 1$ (since A is regular).

Let $y = (y_n) \in m$. Then

$$|[\psi(y - Ty)]_p| = |\sum_{k=1}^{\infty} a_{pk}[T^k(y - Ty)]_{n_p}| = |\sum_{k=1}^{\infty} a_{pk}(T^k y)_{n_p} - \sum_{k=1}^{\infty} a_{pk}(T^{k+1}y)_{n_p}|$$

(since each series is absolutely convergent) =

$$|\sum_{k=1}^{\infty} (a_{pk} - a_{p,k-1})(T^k y)_{n_p}| \le \|y\|_{\infty} \sum_{k=1}^{\infty} |a_{pk} - a_{p,k-1}| \to 0 \text{ as } p \to \infty .$$

This implies that $\psi(y - Ty) \in c_o$ and hence

(2) $\psi(U_T) \subseteq c_o$.

If L is an extended limit, we have (i) $L o \psi(e) = 1$, (by (i)), and (ii) $L o \psi(U_T) = \{0\}$ (by (2)), where o denotes the composition of two functions. Thus Proposition 4 of §2 implies that

(3) $L o \psi$ is a T-Banach limit.

It follows that $L(\psi(x)) = 0$, since $x \in Tac_o$. Since L is an arbitrary extended limit, by Lemma 1, $\psi(x) \in c_o$ so that $\lim_p \sum_{k=1}^{\infty} a_{pk}[T^k x]_{n_p} = 0$. Since (n_p) is an arbitrary sequence of positive integers, $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = 0$ uniformly in n .

Conversely, suppose $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = 0$ uniformly in n.

Since $\sum_{k=1}^{\infty} |a_{pk}| \|T^k x\|_\infty \le \|x\|_\infty \sum_{k=1}^{\infty} |a_{pk}|$ (since $\|T\| = 1$) $\le \|x\|_\infty \|A\| < \infty$,

(4) $\sum_{k=1}^{\infty} a_{pk} T^k x$ is a convergent series in $(m, \| \|_\infty)$ for each p.

Hence the hypothesis is equivalent to

(5) $\lim_p (\sum_{k=1}^{\infty} a_{pk} T^k x) = 0$ in $(m, \| \|_\infty)$..

Thus, for each T-Banach limit L,

$$|L(x)| = |\lim_p \sum_{k=1}^{\infty} a_{pk} L(x)| \quad (\text{since } \lim_p \sum_{k=1}^{\infty} a_{pk} = 1)$$

$$= |\lim_p \sum_{k=1}^{\infty} a_{pk} L(T^k x)| \quad (\text{since } L(T^k x) = L(x) \text{ for every } k)$$

$$= |\lim_p L(\sum_{k=1}^{\infty} a_{pk} T^k x)| \quad (\text{by (4) and since } L \text{ is continuous})$$

$$= 0 \quad (\text{by (5) and since } L \text{ is continuous}).$$

This implies that $x \in Tac_o$.

Now suppose $x \in Tac$ and $T\text{-Lim}\, x = \alpha$. By Proposition 3 (iii) of §2, there exists $y \in Tac_o$ such that $x = y + \alpha e$. Since

$\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k y)_n = 0$ uniformly in n, $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n$

$= \lim_p [\sum_{k=1}^{\infty} a_{pk}(T^k y)_n + \sum_{k=1}^{\infty} a_{pk}\alpha] = \alpha$ uniformly in n.

Conversely, suppose $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = \alpha$ uniformly in n. Then

$\lim_p \sum_{k=1}^{\infty} a_{pk} T^k (x - \alpha e)_n = \lim_p [\sum_{k=1}^{\infty} a_{pk}(T^k x)_n - \sum_{k=1}^{\infty} a_{pk}\alpha] = 0$ uniformly in n

and hence $x - \alpha e \in Tac_o$. This implies that $x \in Tac$ and $T\text{-}Lim\ x = \alpha$.

<u>REMARK</u>. (3) assures the existence of T-Banach limits.

<u>COROLLARY 1</u>. Let T be a lifting matrix. Then $x \in Tac$ and $T\text{-}Lim\ x = \alpha$ if and only if $\lim_p \frac{1}{p}(Tx + \ldots + T^p x)_n = \alpha$ uniformly in n.

Proof. Choose $A = (a_{nk})$ such that $a_{nk} = \frac{1}{n}$ for $1 \leq k \leq n$, and $a_{nk} = 0$ for $k > n$. Then A is regular and $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}| = \lim_n \frac{2}{n} = 0$. Now apply Theorem 1.

<u>COROLLARY 2</u>. Let T be a lifting matrix. Then $Tac_o = \bar{U}_T^{\infty}$.

Proof. Let $x \in Tac_o$. Then $x - \frac{Tx + \ldots + T^p x}{p} = \frac{(x - Tx) + \ldots + (x - T^p x)}{p} \in U_T$ (by Proposition 2(i) of §2) and

$\|x - (x - \frac{Tx + \ldots + T^p x}{p})\|_{\infty} = \|\frac{Tx + \ldots + T^p x}{p}\|_{\infty} \to 0$ as $p \to \infty$

by Corollary 1. Hence $x \in \bar{U}_T^{\infty}$ so that $Tac_o \subseteq \bar{U}_T^{\infty}$. Since Tac_o is closed in m and $U_T \subseteq Tac_o$, $\bar{U}_T^{\infty} \subseteq Tac_o$.

<u>THEOREM 2</u>. Let $A = (a_{nk})$ be a regular matrix, and let $x \in m$. Let T be a lifting matrix. If $\lim_p \sum_{k=1}^{\infty} a_{pk}(T^k x)_n = \alpha$ uniformly in n, then $x \in Tac$ and $T\text{-}Lim\ x = \alpha$.

Proof. The proof is the same as the proof of the sufficiency of Theorem 1.

§4. Some examples.

EXAMPLE 1. First we consider the case when $T = (t_{nk}) = T_o$, i.e.,

$$t_{nk} = \begin{cases} 1 & \text{if } k = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that, for this matrix, Tac = ac (the space of almost convergent sequences). Moreover, we can easily verify that $U_T = bs$ (the space of bounded series) and hence $\overline{bs}^{\infty} = ac_o$.

Now we are in a position to give an easy proof of a principal result in [13, Theorem 7, p. 176].

THEOREM 1. Let $A = (a_{nk})$ be a regular matrix. Then $ac \subseteq c_A$ if and only if $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}| = 0$ (assume $a_{n,o} = 0$). Moreover, when A has this property, T_o - Lim $x = \lim_A x$ for every $x \in ac$.

Proof. (Necessity). Suppose $ac \subseteq c_A$. Then, for every $x \in m$, $(x - T_o x) \in c_A$ and hence $A[(I - T_o)x] \in c$. By 1.5, Theorem 3(iii), $[A(I - T_o)]x \in c$. Hence, by Schur's Lemma (1.4, Theorem 3),

$$\lim_n \sum_{k=1}^{\infty} |[A(I - T_o)]_{nk}| = \sum_{k=1}^{\infty} |\lim_n [A(I - T_o)]_{nk}|, \text{ i.e., } \lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}|$$

$$= \sum_{k=1}^{\infty} |\lim_n (a_{nk} - a_{n,k-1})| = 0$$ since A is regular.

(Sufficiency) suppose $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}| = 0$. Let $x \in ac_o$.

Then $\lim_n \sum_{k=1}^{\infty} a_{nk}x_{k+1} = \lim_n \sum_{k=1}^{\infty} a_{nk}(T_o^k x)_1 = 0$ by Theorem 1 of §3. But

$\lim_n \sum_{k=1}^{\infty} (a_{nk} - a_{n,k-1})x_k = 0$, since $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k-1}| = 0$. Hence

$\lim_n \sum_{k=1}^{\infty} a_{nk}x_k = 0$ so that $x \in c_{o_A}$. This implies that $ac \subseteq c_A$ (since

$e \in c_A$) and that $T_o\text{-Lim}\, x = \lim_A x$ for every $x \in ac$.

<u>COROLLARY 1</u>. Let $A = (a_{nk})$ be a regular matrix. Then

$\{x \in m | \lim_p \sum_{k=1}^{\infty} a_{pk}x_{k+n}$ exists uniformly in $n\} = ac$ if and only if

$\lim_p \sum_{k=1}^{\infty} |a_{pk} - a_{p,k-1}| = 0.$

Proof. To prove the necessity, let $x \in ac$. Then $(0,x_1,x_2,\ldots) \in ac$ and hence $\lim_p \sum_{k=1}^{\infty} a_{pk}x_k$ exists so that $x \in c_A$. Thus, by Theorem 1,

$\lim_p \sum_{k=1}^{\infty} |a_{pk} - a_{p,k+1}| = 0$. The sufficiency follows from Theorem 1 of §3.

<u>EXAMPLE 2</u>. We consider the case when $T_1 = (t_{nk})$ is given by

$$t_{nk} = \begin{cases} 1 & \text{if } k = n+2 \\ 0 & \text{if } k \neq n+2. \end{cases}$$

Then, by Corollary 1 of Theorem 1, of §3, $x \in T_1 ac_o$ if and only if

$\lim_p \frac{1}{p}(T_1 x + \ldots + T_1^p x)_n = 0$ uniformly in n, i.e.,

$\lim_p \frac{1}{p}(x_{n+2} + x_{n+4} + \ldots + x_{n+2p}) = 0$ uniformly in n. Thus $((-1)^n) \notin T_1 ac_o$.
Note that $(-1)^n \in ac_o$.

EXAMPLE 3. Let

$$\begin{aligned}
J_1 &= \{1, 2, 4, 7, 11, \ldots\} \\
J_2 &= \{3, 5, 8, 12, \ldots\} \\
J_3 &= \{6, 9, 13, \ldots\} \\
J_4 &= \{10, 14, \ldots\} \\
J_5 &= \{15, 20, \ldots\} \\
&\vdots \\
J_n &= \{\tfrac{n(n+1)}{2}, \tfrac{n(n+1)}{2} + n, \tfrac{n(n+1)}{2} + n + n+1, \ldots\} \\
&\vdots
\end{aligned}$$

Note that the J_ns are pairwise disjoint.

Let $T = (t_{nk})$ be defined by

$$t_{nk} = \begin{cases} 1 & \text{if } n,k \text{ are two consecutive numbers of one of } J_i\text{s} \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to check that each row of T contains only one non-zero entry which is equal to 1 and lies above the main diagonal. Let us denote J_i; $i = 1,2,\ldots$ by $\{j_1^i, j_2^i, \ldots\}$. If $n \in \mathbb{N}$, then there exist $i,k \in \mathbb{N}$ such that $n = j_k^i$. For $x \in Tac_o$, $(Tx)_n = \sum_{\ell=1}^{\infty} t_{n\ell} x_\ell = x_{j_{k+1}^i}$;

$(T^2x)_n = \sum_{\ell=1}^{\infty} t_{n\ell}(Tx)_\ell = (Tx)_{j_{k+1}^i} = x_{j_{k+2}^i}$; ...; $(T^p x)_n = x_{j_{k+p}^i}$. Hence

$x \in Tac_o$ if and only if $\lim_p \frac{1}{p}(x_{j_{k+1}^i} + x_{j_{k+2}^i} + \ldots + x_{j_{k+p}^i}) = 0$ uniformly in i and k. Let $x = (x_k)$ be defined by $x_1 = 1$, $x_2 = x_3 = -1$, $x_4 = x_5 = x_6 = 1$, $x_7 = x_8 = x_9 = x_{10} = -1$, ... Then $x \in Tac_o$ but $x \notin ac_o$.

§5. Duality between ℓ_1 and Tac_o.

For every lifting matrix T, Tac_o and ℓ_1 form a dual pair of sequence spaces with interesting properties. In this section we study some of these properties. We start with the following proposition.

PROPOSITION 1. Let $T = (t_{nk})$ be a lifting matrix and $y \in \ell_1$. Then

(i) $yT \in \ell_1$ and $(yT)_k = y_1t_{1k} + y_2t_{2k} + \dots y_{k-1}t_{k-1,k}$

(thus $(y(I-T))_k = y_k - y_1t_{1k} - y_2t_{2k} - \dots - y_{k-1}t_{k-1,k}$), and

(ii) $\| |y|T\|_1 = \|y\|_1$, where $|y| = (|y_k|)$.

Proof. (i) $\sum_{k=1}^{\infty} |(yT)_k| = \sum_{k=1}^{\infty} |\sum_{i=1}^{\infty} y_i t_{ik}|$

$$\le \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |y_i| t_{ik}$$

$$= \sum_{i=1}^{\infty} |y_i| \sum_{k=1}^{\infty} t_{ik} = \sum_{i=1}^{\infty} |y_i| < \infty \quad \text{since } y \in \ell_1 .$$

Hence $yT \in \ell_1$.

$$\text{Also } (yT)_k = \sum_{i=1}^{\infty} y_i t_{ik} = y_1t_{1k} + y_2t_{2k} + \dots + y_{k-1}t_{k-1,k}$$

since $t_{ik} = 0$ for $i \ge k$.

$$\text{(ii)} \quad \| |y|T\|_1 = \sum_{k=1}^{\infty} |\sum_{i=1}^{\infty} |y_i| t_{ik}| = \sum_{i=1}^{\infty} |y_i| \sum_{k=1}^{\infty} t_{ik} = \sum_{i=1}^{\infty} |y_i| = \|y\|_1 .$$

<u>THEOREM 1</u>. Let $T = (t_{nk})$ be a lifting matrix and suppose (x^n) is a sequence in ℓ_1. Then the following are equivalent:

(i) (x^n) is $\sigma(\ell_1, Tac_o)$-convergent to x in ℓ_1;

(ii) (x^n) is $\sigma(\ell_1, U_T \oplus c_o)$-convergent to x in ℓ_1;

(iii) $\sup_n \|x^n\|_1 < \infty$ and x is a sequence such that $\lim_n \|(x^n - x)(I-T)\|_1 = 0$.

Proof. ((i) ⇒ (ii)) This is obvious since $U_T \oplus c_o \subseteq Tac_o$.

((ii) ⇒ (i)) Assume (ii). Since $c_o^\beta = \ell_1$, $(U_T \oplus c_o)^\beta = \ell_1$, thus $\sup_n \|x^n\|_1 < \infty$ by 2.3, Theorem 1 ((i) ⇒ (iv)). Now by 2.2, Proposition 4, (x^n) is $\sigma(\ell_1, Tac_o)$-convergent to x (since $\overline{U_T \oplus c_o}^{\infty} = Tac_o$, by Corollary 2 of Theorem 1 of §3).

((ii) ⇒ (iii)) Assume (ii). Again, since $c_o^\beta = \ell_1$, $(U_T \oplus c_o)^\beta = \ell_1$, thus $\sup_n \|x^n\|_1 < \infty$ by 2.3, Theorem 1 ((ii) ⇒ (iv)). Moreover, since (x^n) is $\sigma(\ell_1, U_T)$-convergent to x, $\lim_n \sum_{k=1}^{\infty} (x_k^n - x_k)[(I-T)y]_k = 0$ for every $y \in m$. Since $x^n - x \in \ell_1$, $\|I-T\| < \infty$, and $y \in m$, by 1.5, Theorem 3 (i),

$$\sum_{k=1}^{\infty} [(x^n - x)(I-T)]_k y_k = \sum_{k=1}^{\infty} (x_k^n - x_k)[(I-T)y]_k \to 0 \text{ as } n \to \infty \text{ for every}$$

$y \in m$. Thus $\|(x^n - x)(I-T)\|_1 \to 0$ as $n \to \infty$ by 1.4, Theorem 3.

((iii) ⇒ (ii)) Assume (iii). Since $\lim_n \|(x^n - x)(I-T)\|_1 = 0$, $\lim_n [(x^n - x)(I-T)]_k = 0$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}$,

$$[(x^n-x)(I-T)]_k = (x_k^n-x_k) - (x_1^n-x_1)t_{1k} - \dots - (x_{k-1}^n-x_{k-1})\, t_{k-1,k}$$

by Proposition 1(i). Thus $\lim_n[(x^n-x)(I-T)]_1 = \lim_n(x_1^n-x_1) = 0$ and

$$\lim_n[x^n-x)(I-T)]_2 = \lim_n[(x_2^n-x_2) - (x_1^n-x_1)t_{12}] = 0 ,$$

and hence $\lim_n(x_2^n-x_2) = 0$. By induction, we can easily show that $\lim_n(x_k^n-x_k) = 0$ for every k. Now, for each $p \in \mathbb{N}$,

$$\sum_{k=1}^{p} |x_k| = \lim_n \sum_{k=1}^{p} |x_k^n| \le \sup_n\|x^n\| < \infty .$$

Hence $x \in \ell_1$ and thus (x^n) is $\sigma(\ell_1,\varphi)$-convergent to x. By 2.2, Proposition 4, (x^n) is $\sigma(\ell_1,c_o)$-convergent to x. Moreover, since

$$\lim_n\|(x^n-x)(I-T)\|_1 = 0, \quad \lim_n \sum_{k=1}^{\infty} [(x^n-x)(I-T)]_k y_k = 0 \text{ for every } y \in m.$$

By 1.5, Theorem 3(i),

$$\sum_{k=1}^{\infty} (x_k^n-x_k)[(I-T)y]_k = \sum_{k=1}^{\infty} [(x^n-x)(I-T)]_k y_k .$$

Hence $\lim_n \sum_{k=1}^{\infty} (x_k^n-x_k)[(I-T)y]_k = 0$ for every $y \in m$. Thus (x^n) is $\sigma(\ell_1,U_T)$-convergent to x.

<u>REMARK</u>. Condition (iii) of Theorem 2 identifies $\sigma(\ell_1,Tac_o)$ with a two norm topology. For details concerning this type of topology we refer the reader to [1].

<u>COROLLARY 1</u>. Let T be a lifting matrix. Then $(\ell_1,\sigma(\ell_1,Tac_o))$ and $(\ell_1,\sigma(\ell_1,U_T\oplus c_o))$ are sequentially complete.

Proof. Suppose (x^n) is $\sigma(\ell_1, Tac_o)$-Cauchy. Then (x^n) is $\sigma(\ell_1, c_o)$-Cauchy and hence there exists $x \in \ell_1$ such that (x^n) is $\sigma(\ell_1, c_o)$-convergent to x since, by 2.2, Proposition 3, $(\ell_1, \sigma(\ell_1, c_o))$ is sequentially complete. Without loss of generality we can assume that $x = 0$. Also, by Theorem 1(iii), $(x^n(I-T))_{n=1}^{\infty}$ is Cauchy in $(\ell_1, \|\cdot\|_1)$ and hence there exists $y \in \ell_1$ such that $\lim_n \|x^n(I-T)-y\|_1 = 0$. Thus, for $k \in \mathbb{N}$, $y_k = \lim_n [x^n(I-T)]_k = \lim_n (x_k^n - x_1^n t_{1k} - \ldots - x_{k-1}^n t_{k-1,k})$ (by Proposition 1(i)) $= 0$ since (x^n) is $\sigma(\ell_1, c_o)$-convergent to 0. Moreover, since $Tac_o^{\beta} = \ell_1$, $\sup_n \|x^n\|_1 < \infty$ by 2.3, Theorem 1((i) $\Rightarrow$ (iv)). By Theorem 1, (x^n) is $\sigma(\ell_1, Tac_o)$-convergent to 0.

The same argument can be used for $(\ell_1, \sigma(\ell_1, U_T \oplus c_o))$. •

<u>COROLLARY 2</u>. Let T be a lifting matrix. If A is a regular matrix such that $Tac \subseteq c_A$, then $T\text{-Lim}\, x = \lim_A x$ for every $x \in Tac$.

Proof. Apply 2.3, Theorem 3, letting $E = Tac_o$ and $G = Tac$.

<u>COROLLARY 3</u>. Let T be a lifting matrix. Then $(Tac_o, \|\ \|_\infty)$ is not separable.

Proof. By Proposition 3 of §2, $c_o \underset{\neq}{\subset} Tac_o$. Now apply 2.3, Theorem 4.

<u>COROLLARY 4</u>. Let T be a lifting matrix. Then, for a subset C of ℓ_1, the following are equivalent:

(i) C is $\sigma(\ell_1, Tac_o)$-relatively compact;

(ii) C is $\sigma(\ell_1, U_T \oplus c_o)$-relatively compact.

(iii) C is ℓ_1-norm bounded and $C(I-T)$ is relatively compact in $(\ell_1, \| \ \|_1)$, where $C(I-T) = \{x(I-T) \mid x \in C\}$.

Proof. ((i) $\Leftrightarrow$ (ii)) A subset of a K-space is relatively compact if and only if it is relatively sequentially compact, by 1.4, Theorem 1. Hence it follows from Theorem 1 that (i) and (ii) are equivalent.

((i) $\Rightarrow$ (iii)) Suppose C is $\sigma(\ell_1, Tac_o)$-relatively compact. Then C is $\sigma(\ell_1, Tac_o)$-bounded. Since $Tac_o^\beta = \ell_1$, it follows from 2.3, Theorem 1 ((i) $\Rightarrow$ (ii)) that C is ℓ_1-norm bounded. Suppose (x^n) is a sequence in C. Then there exists a subsequence (x^{n_i}) of (x^n) such that (x^{n_i}) is $\sigma(\ell_1, Tac_o)$-convergent to a member x in ℓ_1. By Theorem 1((i) $\Rightarrow$ (iii)), $\lim_i \|(x^{n_i}-x)(I-T)\|_1 = 0$. Thus $(x^{n_i}(I-T))_{i=1}^\infty$ is ℓ_1-norm convergent to $x(I-T)$, and hence $C(I-T)$ is relatively compact in $(\ell_1, \| \ \|_1)$.

((iii) $\Rightarrow$ (i)) Assume condition (iii) and suppose (x^n) is a sequence in C. Then there exists a subsequence (x^{n_i}) of (x^n) such that $(x^{n_i}(I-T))_{i=1}^\infty$ is ℓ_1-norm convergent. Hence $(x^{n_i}(I-T))_{i=1}^\infty$ is Cauchy in $(\ell_1, \| \ \|_1)$. Since (x^{n_i}) is ℓ_1-norm bounded, it follows from

Theorem 1((iii) ⇒ (i)) that (x^{n_i}) is $\sigma(\ell_1,Tac_o)$-Cauchy. Since ℓ_1 is $\sigma(\ell_1,Tac_o)$-sequentially complete by Corollary 1, (x^{n_i}) is $\sigma(\ell_1,Tac_o)$-convergent.

<u>COROLLARY 5</u>. Let T be a lifting matrix, and suppose C is a $\sigma(\ell_1,Tac_o)$-relatively compact subset of ℓ_1. Then the convex hull of $\hat{C}$ of C is also $\sigma(\ell_1,Tac_o)$-relatively compact.

Proof. Suppose C is a $\sigma(\ell_1,Tac_o)$-relatively compact subset of ℓ_1. Then C is ℓ_1-norm bounded and $C(I-T)$ is relatively compact in $(\ell_1,\| \|_1)$, by Corollary 4 ((i) ⇒ (iii)). Hence the convex hull $\hat{C}$ of C is ℓ_1-norm bounded and $\hat{C}(I-T)$ is relatively compact in $(\ell_1,\| \|_1)$, since $\hat{C}(I-T)$ is the convex hull of $C(I-T)$. Thus $\hat{C}$ is $\sigma(\ell_1,Tac_o)$-relatively compact, by Corollary 4((iii) ⇒ (i)).

<u>REMARK</u>. Corollary 5 implies that $\tau(Tac_o,\ell_1)$ is the topology of uniform convergence on $\sigma(\ell_1,Tac_o)$-compact sets.

We use the following lemmas to establish some topological properties of $(Tac_o,\tau(Tac_o,\ell_1))$.

<u>LEMMA 1</u>. Let T be a lifting matrix. Suppose a sequence (x^n) in ℓ_1 is $\sigma(\ell_1,Tac_o)$-convergent to x. Then $(|x^n|)$ is $\sigma(\ell_1,Tac_o)$-convergent to $|x|$, where $|x^n| = (|x_k^n|)_{k=1}^{\infty}$ and $|x| = (|x_k|)$.

Proof. Let (x^n) be a sequence in ℓ_1 such that (x^n) is $\sigma(\ell_1,Tac_o)$-convergent to x. Then, by Theorem 1((i) ⇒ (iii)),

(1) $\lim_n \|(x^n - x)(I-T)\|_1 = 0$ and $\sup_n \|x^n\|_1 < \infty$.

Case 1. $x = 0$. Then

(1)' $\lim_n \|x^n(I-T)\|_1 = 0$ and $\sup_n \|x^n\|_1 < \infty$.

Let $M = \{n \mid \| |x^n|(I-T)\|_1 > \|x^n(I-T)\|_1\}$. If M is finite, then there exists $n_o \in \mathbb{N}$ such that $\| |x^n|(I-T)\|_1 \le \|x^n(I-T)\|_1$ for every $n \ge n_o$. Thus $\lim_n \| |x^n|(I-T)\|_1 = 0$ by (1)'. Also $\sup_n \| |x^n| \|_1 = \sup \|x^n\|_1 < \infty$ by (1)'. Hence $(|x^n|)$ is $\sigma(\ell_1, Tac_o)$-convergent to 0 by Theorem 1((iii) ⇒ (i)). Suppose M is infinite. Then the members of M form a strictly increasing sequence of positive integers. Let us denote this by $(n_k)_{k=1}^{\infty}$. Let, for $k \in \mathbb{N}$,

$$\varepsilon_k = \| |x^{n_k}|(I-T)\|_1 - \|x^{n_k}(I-T)\|_1, \text{ i.e.,}$$

$$\varepsilon_k = \sum_{i=1}^{\infty} [\, \big| |x_i^{n_k}| - |x_1^{n_k}| t_{1i} - \dots - |x_{i-1}^{n_k}| t_{i-1,i} \big| - |x_i^{n_k} - x_1^{n_k} t_{1i} - \dots - x_{i-1}^{n_k} t_{i-1,i}| \,]$$

(see Proposition 1(i)).

First notice that

$$(2) \quad \sum_{i=1}^{\infty} (|x_i^{n_k}| + |x_1^{n_k}| t_{1i} + \dots + |x_{i-1}^{n_k}| t_{i-1,i})$$

$$= \sum_{i=1}^{\infty} |x_i^{n_k}| + \sum_{i=1}^{\infty} (|x_1^{n_k}| t_{1i} + \dots + |x_{i-1}^{n_k}| t_{i-1,i})$$

$$= \|x^{n_k}\|_1 + \| |x^{n_k}| T\|_1 \quad \text{(by Proposition 1(i))}$$

$$= 2\|x^{n_k}\|_1 \quad \text{(by Proposition 1(ii)}.$$

Similarly,

$$(3)\quad \sum_{i=1}^{\infty}(|x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i}) = 0.$$

Let $P_k = \{i \,|\, |x_i^{n_k}| \geq |x_1^{n_k}|t_{1i} + \dots + |x_{i-1}^{n_k}|t_{i-1,i}\}$. Then, for $i \in P_k$,

$$(4)\quad |x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}| \geq |x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i}$$

$$= \big||x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i}\big|.$$

Let $Q_k = \mathbb{N}\setminus P_k$. Then

$$\varepsilon_k = \sum_{i\in Q_k}\big[\big||x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i}\big|-|x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}|\big]$$

$$+ \sum_{i\in P_k}\big[\big||x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i}\big|-|x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}|\big]$$

$$\leq \sum_{i\in Q_k}(|x_1^{n_k}|t_{1i}+\dots+|x_{i-1}^{n_k}|t_{i-1,i}-|x_i^{n_k}|) +$$

$$\sum_{i\in P_k}\big[|x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}|-(|x_i^{n_k}|-|x_1^{n_k}|t_{1i}-\dots-|x_{i-1}^{n_k}|t_{i-1,i})\big] \text{ (by (4))}$$

$$= \sum_{i=1}^{\infty}(|x_1^{n_k}|t_{1i}+\dots+|x_{i-1}^{n_k}|t_{i-1,i}-|x_i^{n_k}|) + \sum_{i\in P_k}|x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}|$$

$$\leq 0 + \sum_{i=1}^{\infty}|x_i^{n_k}-x_1^{n_k}t_{1i}-\dots-x_{i-1}^{n_k}t_{i-1,i}| \text{ (by (3))}$$

$$= \|x^{n_k}(I-T)\|_1 \text{ (by Proposition 1(i))} \to 0 \text{ as } k\to\infty \text{ by (1)}'.$$

i.e., $\| |x^{n_k}|(I-T)\|_1 - \|x^{n_k}(I-T)\|_1 \to 0$ as $k \to \infty$. Also, for $n \notin M = \{n_k\} | k \in \mathbb{N}\}$, $\| |x^n|(I-T)\|_1 \le \|x^n(I-T)\|_1$. Hence it follows from (1)' that $\lim_n \| |x^n|(I-T)\|_1 = 0$. Also $\sup_n \| |x^n| \|_1 = \sup_n \|x^n\|_1 < \infty$. Thus $(|x^n|)$ is $\sigma(\ell_1, Tac_o)$-convergent to 0 by Theorem 1((iii) ⇒ (i)).

Case 2. x is any member in ℓ_1. Let $\varepsilon > 0$. Since $x \in \ell_1$, there exists $m \in \mathbb{N}$ such that

$$(5) \quad \sum_{k=m}^{\infty} |x_k| < \varepsilon/16.$$

Since (x^n) is pointwise convergent to x, there exists $n_o \in \mathbb{N}$ such that

$$(6) \quad \sum_{k=1}^{m-1} |x_k^n - x_k| < \varepsilon/8 \quad \text{for } n \ge n_o .$$

Since $(x^n - x)_{n=1}^{\infty}$ is $\sigma(\ell_1, Tac_o)$-convergent to 0, $(|x^n - x|)_{n=1}^{\infty}$ is $\sigma(\ell_1, Tac_o)$-convergent to 0, and hence $\lim_n \| |x^n - x|(I-T)\|_1 = 0$ by Theorem 1((i) ⇒ (iii)). Also, by (1), $\lim_n \|(x^n - x)(I-T)\|_1 = 0$. Thus there exists $n_1 (> n_o)$ such that:

$$(7) \quad \|(x^n - x)(I-T)\|_1 < \varepsilon/8 \quad \text{for } n \ge n_1 ;$$

$$(8) \quad \| |x^n - x|(I-T)\|_1 < \varepsilon/8 \quad \text{for } n \ge n_1 .$$

For $n \geq n_1$,

$$\|(|x^n|-|x|)(I-T)\|_1 = \sum_{k=1}^{\infty} |(|x_k^n|-|x_k|)-(|x_1^n|-|x_1|)t_{1k}-\ldots-(|x_{k-1}^n|-|x_{k-1}|)t_{k-1,k}|$$

(by Proposition I(i))

$$= \sum_{k=1}^{m-1} |(|x_k^n|-|x_k|)-(|x_1^n|-|x_1|)t_{1k}-\ldots-(|x_{k-1}^n|-|x_{k-1}|)t_{k-1,k}| +$$

$$\sum_{k=m}^{\infty} |(|x_k^n|-|x_k|)-(|x_1^n|-|x_1|)t_{1k}-\ldots-(|x_{m-1}^n|-|x_{m-1}|)t_{m-1,k}$$

$$-\ldots-(|x_{k-1}^n|-|x_{k-1}|)t_{k-1,k}|$$

$$\leq \sum_{k=1}^{m-1} (|x_k^n-x_k|+|x_1^n-x_1|t_{1k}+\ldots+|x_{k-1}^n-x_{k-1}|t_{k-1,k})$$

$$+ \sum_{k=m}^{\infty} |(|x_k^n|-|x_k|)-(|x_m^n|-|x_m|)t_{mk}-\ldots-(|x_{k-1}^n|-|x_{k-1}|)t_{k-1,k}|$$

$$+ \sum_{k=m}^{\infty} (|x_1^n|-|x_1|)t_{1k}+\ldots+(|x_{m-1}^n|-|x_{m-1}|)t_{m-1,k}|$$

$$\leq \sum_{k=1}^{m-1} (|x_k^n-x_k|+|x_1^n-x_1|t_{1k}+\ldots+|x_{k-1}^n-x_{k-1}|t_{k-1,k})$$

$$+ \sum_{k=m}^{\infty} (|x_1^n-x_1|t_{1k}+\ldots+|x_{m-1}^n-x_{m-1}|t_{m-1,k})$$

$$+ \sum_{k=m}^{\infty} |(|x_k^n|-|x_k|)-(|x_m^n|-|x_m|)t_{mk}-\ldots-(|x_{k-1}^n|-|x_{k-1}|)t_{k-1,k}|$$

$$= \sum_{k=1}^{m-1} |x_k^n - x_k| + [\sum_{k=1}^{m-1} (|x_1^n - x_1| t_{1k} + \ldots + |x_{k-1}^n - x_{k-1}| t_{k-1,k})$$

$$+ \sum_{k=m}^{\infty} (|x_1^n - x_1| t_{1k} + \ldots + |x_{m-1}^n - x_{m-1}| t_{m-1,k})]$$

$$+ \sum_{k=m}^{\infty} |(|x_k^n| - |x_k|) - (|x_m^n| - |x_m|) t_{mk} - \ldots - (|x_{k-1}^n| - |x_{k-1}|) t_{k-1,k}|$$

$$= \sum_{k=1}^{m-1} |x_k^n - x_k| + \|(|x_1^n - x_1|, |x_2^n - x_2|, \ldots, |x_{m-1}^n - x_{m-1}|, 0, 0, \ldots) T\|_1$$

$$+ \sum_{k=m}^{\infty} |(|x_k^n| - |x_k|) - (|x_m^n| - |x_m|) t_{mk} - \ldots - (|x_{k-1}^n| - |x_{k-1}|) t_{k-1,k}|$$

(by Proposition 1(i))

$$\leq 2 \sum_{k=1}^{m-1} |x_k^n - x_k| + \sum_{k=m}^{\infty} ||x_k^n| - |x_m^n| t_{mk} - \ldots - |x_{k-1}^n| t_{k-1,}|$$

$$+ \sum_{k=m}^{\infty} (|x_k| + |x_m| t_{mk} + \ldots + |x_{k-1}| t_{k-1,k})$$ (by Proposition 1(ii))

$$= 2 \sum_{k=1}^{m-1} |x_k^n - x_k| + \sum_{k=m}^{\infty} || x_k^n| - |x_m^n| t_{mk} - \ldots - |x_{k-1}^n| t_{k-1,k}|$$

$$+ \sum_{k=m}^{\infty} |x_k| + \sum_{k=m}^{\infty} (|x_m| t_{mk} + \ldots + |x_{k-1}| t_{k-1,k})$$

$$= 2 \sum_{k=1}^{m-1} |x_k^n - x_k| + \sum_{k=m}^{\infty} ||x_k^n| - |x_m^n| t_{mk} - \ldots - |x_{k-1}^n| t_{k-1,k}|$$

$$+ \sum_{k=m}^{\infty} |x_k| + \|(0, 0, \ldots, 0, |x_m|, |x_{m+1}|, \ldots) T\|_1$$ (by Proposition 1(i))

$$= 2 \sum_{k=1}^{m-1} |x_k^n - x_k| + \sum_{k=m}^{\infty} \big||x_k^n| - |x_m^n| t_{mk} - \cdots - |x_{k-1}^n| t_{k-1,k}\big| + 2 \sum_{k=m}^{\infty} |x_k|$$

(by Proposition 1(ii))

$$< \frac{\varepsilon}{4} + \sum_{k=m}^{\infty} \big||x_k^n| - |x_m^n| t_{mk} - \cdots - |x_{k-1}^n| t_{k-1,k}\big| + \frac{\varepsilon}{8} \quad \text{by (6) and (5)}$$

since $n \geq n_1 > n_o$.

i.e., (9) $\|(|x^n| - |x|)(I-T)\|_1 < \sum_{k=m}^{\infty} \big||x_k^n| - |x_m^n| t_{mk} - \cdots - |x_{k-1}^n| t_{k-1,k}\big| + \frac{\varepsilon}{2}$

for $n \geq n_1$.

Now

$$\||x^n - x|(I-T)\|_1 = \sum_{k=1}^{\infty} \big||x_k^n - x_k| - |x_1^n - x_1| t_{1k} - \cdots - |x_{k-1}^n - x_{k-1}| t_{k-1,k}\big|$$

(by Propositon 1(i))

$$\geq \sum_{k=m}^{\infty} \big||x_k^n - x_k| - |x_1^n - x_1| t_{1k} - \cdots - |x_{k-1}^n - x_{k-1}| t_{k-1,k}\big|$$

$$\geq \sum_{k=m}^{\infty} \big||x_k^n - x_k| - |x_m^n - x_m| t_{mk} - \cdots - |x_{k-1}^n - x_{k-1}| t_{k-1,k}\big|$$

$$- \sum_{k=m}^{\infty} (|x_1^n - x_1| t_{1k} + \cdots + |x_{m-1}^n - x_{m-1}| t_{m-1,k})$$

$$\geq \sum_{k=m}^{\infty} \big||x_k^n - x_k| - |x_m^n - x_m| t_{mk} - \cdots - |x_{k-1}^n - x_{k-1}| t_{k-1,k}\big|$$

$$- \sum_{k=1}^{m-1} |x_k^n - x_k| \quad \left(\text{since } \sum_{k=m}^{\infty} t_{ik} \leq 1 \text{ for } i \in \mathbb{N}\right) ,$$

Let $\alpha(n,k) = ||x_k^n - x_k| - |x_m^n - x_m| t_{mk} - \ldots - |x_{k-1}^n - x_{k-1}| t_{k-1,k}|$.

Then $\alpha(n,k) = ||x_k^n - x_k| - \sum_{j=m}^{k-1} |x_j^n - x_j| t_{jk}|$

$$\geq \max\{(|x_k^n - x_k| - \sum_{j=m}^{k-1} |x_j^n - x_j| t_{jk}),\ (\sum_{j=m}^{k-1} |x_j^n - x_j| t_{jk}) - |x_k^n - x_k|\}$$

Hence $\alpha(n,k) \geq \max\{(|x_k^n| - |x_k| - \sum_{j=m}^{k-1} (|x_j^n| + |x_j|) t_{jk}),$

$$(\sum_{j=m}^{k-1} (|x_j^n| - |x_j|) t_{jk}) - |x_k^n| - |x_k|\}$$

i.e., $\alpha(n,k) \geqq \max\{(|x_k^n| - \sum_{j=m}^{k-1} |x_j^n| t_{jk}) - (|x_k| + \sum_{j=m}^{k-1} |x_j| t_{jk})$,

$$- (|x_k^n| - \sum_{j=m}^{k-1} |x_j^n| t_{jk}) - (|x_k| + \sum_{j=m}^{k-1} |x_j| t_{jk})\}.$$

Hence $\alpha(n,k) \geq ||x_k^n| - \sum_{j=m}^{k-1} |x_j^n| t_{jk}| - (|x_k| + \sum_{j=m}^{k-1} |x_j| t_{jk})$.

Thus $\| |x^n - x| (I-T) \|_1 \geq \sum_{k=m}^{\infty} ||x_k^n| - |x_m^n| t_{mk} - \ldots - |x_{k-1}^n| t_{k-1,k}|$

$$- \sum_{k=m}^{\infty} (|x_k| + |x_m| t_{mk} + \ldots + |x_{k-1}| t_{k-1,k}) - \sum_{k=1}^{m-1} |x_k^n - x_k|$$

$$= \sum_{k=m}^{\infty} ||x_k^n| - |x_m^n| t_{mk} - \ldots - |x_{k-1}^n| t_{k-1,k}| - 2 \sum_{k=m}^{\infty} |x_k| - \sum_{k=1}^{m-1} |x_k^n - x_k|$$

(since $\sum_{k=m}^{\infty} (|x_k| + |x_m| t_{mk} + \ldots + |x_{k-1}| t_{k-1,k}) = \sum_{k=m}^{\infty} |x_k|$

$$+ \sum_{k=m}^{\infty} (|x_m|t_{mk}+\ldots+|x_{k-1}|t_{k-1,k}) = \sum_{k=m}^{\infty} |x_k| + \|(0,0,\ldots$$

$$\ldots,0,|x_m|,|x_{m+1}|,\ldots)T\|_1 = 2 \sum_{k=m}^{\infty} |x_k| \quad \text{by Proposition 1)}$$

$$\geq \sum_{k=m}^{\infty} \big||x_k^n| - |x_m^n|t_{mk} - \ldots - |x_{k-1}^n|t_{k-1,k}\big| - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \text{ (by (5) and (6),}$$

since $n \geq n_1 > n_o$).

Since $n \geq n_1$, $\||x^n - x|(I-T)\|_1 < \varepsilon/8$ by (8) and hence

$$\sum_{k=m}^{\infty} \big||x_k^n| - |x_m^n|t_{mk} - \ldots - |x_{k-1}^n|t_{k-1,k}\big| < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \frac{\varepsilon}{2} .$$

Thus $\|(|x^n| - |x|)(I-T)\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $n \geq n_1$ by (9).

Hence $\lim_n \|(|x^n| - |x|)(I-T)\|_1 = 0$. Also $\sup_n \||x^n|\|_1 = \sup_n \|x^n\|_1 < \infty$ by (1).

Thus $(|x^n|)$ is $\sigma(\ell_1, Tac_o)$-convergent to $|x|$ by Theorem 1((iii) $\Rightarrow$ (i)).

LEMMA 2. Let T be a lifting matrix. If a subset C of ℓ_1 is $\sigma(\ell_1, Tac_o)$-relatively compact, then $C \cup |C|$ is $\sigma(\ell_1, Tac_o)$-relatively compact, where $|C| = \{(|x_n|) \mid (x_n) \in C\}$.

Proof. Suppose a subset C of ℓ_1 is $\sigma(\ell_1, Tac_o)$-relatively compact. Let (x^n) be a sequence in $C \cup |C|$. Then there exists a subsequence $(x^{n_k})_{k=1}^{\infty}$ of (x^n) such that (x^{n_k}) is in C, or (x^{n_k}) is in $|C|$.

If (x^{n_k}) is in C, then there exists a subsequence $(x^{n_{k_i}})_{i=1}^{\infty}$ of (x^{n_k}) such that $(x^{n_{k_i}})_{i=1}^{\infty}$ is $\sigma(\ell_1,Tac_o)$-convergent since C is $\sigma(\ell_1,Tac_o)$-relatively compact. If (x^{n_k}) is in $|C|$, then there exists a sequence (y^k) in C such that $|y^k| = x^{n_k}$ for each k. Since C is $\sigma(\ell_1,Tac_o)$-relatively compact, there exists a subsequence $(y^{k_i})_{i=1}^{\infty}$ of (y^k) such that (y^{k_i}) is $\sigma(\ell_1,Tac_o)$-convergent. By Lemma 1, $(|y^{k_i}|)_{i=1}^{\infty} = (x^{n_{k_i}})_{i=1}^{\infty}$ is $\sigma(\ell_1,Tac_o)$-convergent. Hence $|C| \cup C$ is $\sigma(\ell_1,Tac_o)$-relatively compact.

<u>LEMMA 3</u>. Let T be a lifting matrix. If a subset of C of ℓ_1 is $\sigma(\ell_1,Tac_o)$-relatively compact, then $P(C) = \{P_n x \mid x \in C \text{ and } n \in \mathbb{N}\}$ is $\sigma(\ell_1,Tac_o)$-relatively compact.

Proof. Suppose a subset C of ℓ_1 is $\sigma(\ell_1,Tac_o)$-relatively compact. Then $C \cup |C|$ is $\sigma(\ell_1,Tac_o)$-relatively compact by Lemma 2. Thus $(C \cup |C|)(I-T)$ is relatively compact in $(\ell_1, \|\ \|_1)$ by Corollary 4 ((i) ⇒ (iii)) of Theorem 1. By 1.4, Theorem 4,

$$\lim_n \sup_{x \in C \cup |C|} \|x(I-T) - P_n(x(I-T))\|_1 = 0.$$

Let $\varepsilon > 0$. Then there exists $n_o \in N$ such that

$$(1) \qquad \sup_{x \in C \cup |C|} \|x(I-T) - P_n(x(I-T))\|_1 < \varepsilon/2 \quad \text{for } n \geq n_o .$$

We claim that $\sup_{x \in P(C)} \|x(I-T) - P_n(x(I-T))\|_1 < \varepsilon$ for $n \geq n_o$. Let $m \in \mathbb{N}$, $n \geq n_o$, and $x \in C$.

Case 1. $m \leq n$.

$$\|(P_m x)(I-T) - P_n[(P_m x)(I-T)]\|_1 = \sum_{k=n+1}^{\infty} |x_1 t_{1k}+\ldots+x_m t_{mk}| \quad \text{(by Proposition 1(i))}$$

$$\leq \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_m|t_{mk})$$

$$\leq \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_n|t_{nk}) \quad \text{(since } m \leq n\text{)}.$$

Now $\left|\sum_{k=n+1}^{\infty} (|x_k|-|x_1|t_{1k}-\ldots-|x_{k-1}|t_{k-1,k})\right|$

$$= \left|\sum_{k=n+1}^{\infty} |x_k| - \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_{k-1}|t_{k-1,k})\right|$$

$$= \left|\sum_{k=n+1}^{\infty} |x_k| - \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_n|t_{nk}) - \sum_{k=n+1}^{\infty} (|x_{n+1}|t_{n+1,k}+\ldots+|x_{k-1}|t_{k-1,k})\right|$$

$$= \left|\sum_{k=n+1}^{\infty} |x_k| - \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_n|t_{nk}) - \|(0,0,\ldots,0,|x_{n+1}|,\ldots)T\|_1\right|$$

(by Proposition 1(i))

$$= \sum_{k=n+1}^{\infty} (|x_1|t_{1k}+\ldots+|x_n|t_{nk}) \quad \text{(by Proposition 1(ii))}.$$

Hence

$$\|(P_m x)(I-T) - P_n[(P_m x)(I-T)]\|_1 \leq \left|\sum_{k=n+1}^{\infty} (|x_k|-|x_1|t_{1k}-\ldots-|x_{k-1}|t_{k-1,k})\right|$$

$$\leq \sum_{k=n+1}^{\infty} \left||x_k|-|x_1|t_{1k}-\ldots-|x_{k-1}|t_{k-1,k}\right|$$

$$= \||x|(I-T) - P_n(|x|(I-T))\|_1 \quad \text{(by Proposition 1(i))}$$

$< \varepsilon/2$ by (1).

Case 2. $m > n$.

$$\|(P_m x)(I-T) - P_n[(P_m x)(I-T)]\|_1$$

$$= \sum_{k=n+1}^{m} |[(P_m x)(I-T)]_k| + \sum_{k=m+1}^{\infty} |[(P_m x)(I-T)]_k|$$

$$= \sum_{k=n+1}^{m} |x_k - x_1 t_{1k} - \ldots - x_{k-1} t_{k-1,k}| + \sum_{k=m+1}^{\infty} |x_1 t_{1k} + \ldots + x_m t_{mk}|$$

(by Proposition 1(ii))

$$\le \sum_{k=n+1}^{\infty} |x_k - x_1 t_{1k} - \ldots - x_{k-1} t_{k-1,k}| + \sum_{k=m+1}^{\infty} (|x_1| t_{1k} + \ldots + |x_m| t_{mk}) .$$

Now, as in Case 1, we can show that

$$\sum_{k=m+1}^{\infty} (|x_1| t_{1k} + \ldots + |x_m| t_{mk}) = \left| \sum_{k=m+1}^{\infty} (|x_k| - |x_1| t_{1k} - \ldots - |x_{k-1}| t_{k-1,k}) \right| .$$

Thus

$$\|(P_m x)(I-T) - P_n[(P_m x)(I-T)]\|_1$$

$$\le \sum_{k=n+1}^{\infty} |x_k - x_1 t_{1k} - \ldots - x_{k-1} t_{k-1,k}| + \left| \sum_{k=m+1}^{\infty} (|x_k| - |x_1| t_{1k} - \ldots - |x_{k-1}| t_{k-1,k}) \right|$$

$$\le \sum_{k=n+1}^{\infty} |x_k - x_1 t_{1k} - \ldots - x_{k-1} t_{k-1,k}| + \sum_{k=m+1}^{\infty} \big| |x_k| - |x_1| t_{1k} - \ldots - |x_{k-1}| t_{k-1,k} \big|$$

$$= \|x(I-T) - P_n(x(I-T))\|_1 + \| |x|(I-T) - P_m(|x|(I-T)) \|_1 \quad \text{(by Proposition 1(i))}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ by (1) since } m > n \ge n_o .$$

Hence $\lim_n \sup_{x \in P(C)} \|x(I-T) - P_n(x(I-T))\|_1 = 0$. Thus $P(C)(I-T)$ is relatively compact in $(\ell_1, \| \|_1)$, by 1.4, Theorem 4. Since C is $\sigma(\ell_1, Tac_o)$-relatively compact, C is ℓ_1-norm bounded by Corollary 4 ((i) $\Rightarrow$ (iii)) of Theorem 1. Thus $P(C)$ is also ℓ_1-norm bounded. It follows from the same Corollary that $P(C)$ is $\sigma(\ell_1, Tac_o)$-relatively compact.

The following theorem was proved for ac_o in [4, Theorem 4, p. 30].

<u>THEOREM 2</u>. Let T be a lifting matrix. Then $(Tac_o, \tau(Tac_o, \ell_1))$ is an AK-space.

Proof. Let C be a $\sigma(\ell_1, Tac_o)$-relatively compact subset of ℓ_1. Then $P(C) = \{P_n x | x \in C \text{ and } n \in \mathbb{N}\}$ is $\sigma(\ell_1, Tac_o)$-relatively compact by Lemma 3. Let $y = (y_k) \in Tac_o$. Then

$$\sup_n \sup_{x \in C} \left| \sum_{k=1}^{n} x_k y_k \right| = \sup_{x \in P(C)} \left| \sum_{k=1}^{\infty} x_k y_k \right| .$$

Thus the family $P_n : (Tac_o, \tau(Tac_o, \ell_1)) \to (Tac_o, \tau(Tac_o, \ell_1))$, $n = 1,2,\ldots$ is equicontinuous. Now we claim that the set $S = \{x \in Tac_o | (P_n x)_{n=1}^{\infty}$ is $\tau(Tac_o, \ell_1)$-convergent to $x\}$ is $\tau(Tac_o, \ell_1)$-closed. Suppose a net (x^λ) in S is $\tau(Tac_o, \ell_1)$-convergent to x in Tac_o. Let $\| \|$ be a $\tau(Tac_o, \ell_1)$-continuous seminorm on Tac_o, and let $\varepsilon > 0$. Since the family P_n, $n = 1,2,\ldots$ is equicontinuous and $\lim_\lambda \|x^\lambda - x\| = 0$, there exists λ_o such that

(1) $\sup_n \|P_n x^{\lambda_o} - P_n x\| < \varepsilon/3$, and $\|x^{\lambda_o} - x\| < \varepsilon/3$.

Since $x^{\lambda_o} \in S$, there exists $n_o \in N$ such that

(2) $\|P_n x^{\lambda_o} - x^{\lambda_o}\| < \varepsilon/3$ for $n \geq n_o$.

Now

$$\|P_n x - x\| \leq \|P_n x - P_n x^{\lambda_o}\| + \|P_n x^{\lambda_o} - x^{\lambda_o}\| + \|x^{\lambda_o} - x\|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{for } n \geq n_o \text{ by (1) and (2).}$$

Hence $(P_n x)_{n=1}^{\infty}$ is $\tau(Tac_o, \ell_1)$-convergent to x so that $x \in S$. Thus S is $\tau(Tac_o, \ell_1)$-closed. By 1.3, Proposition 1, $\bar{\varphi}^{\tau(Tac_o, \ell_1)} = \bar{\varphi}^{\sigma(Tac_o, \ell_1)} = Tac_o$. Thus $S = Tac_o$ since $\varphi \subseteq S$.

<u>LEMMA 4</u>. Let T be a lifting matrix. Then $(\ell_1, \|x(I-T)\|_1)$ is a normed space and $(\ell_1, \|x(I-T\|_1)' \subseteq Tac_o$.

Proof. To claim that $\|x(I-T)\|_1$ is a norm on ℓ_1 it suffices to show that $x(I-T) = 0 \Rightarrow x = 0$. Suppose $x(I-T) = 0$. Then $(x(I-T))_1 = x_1 = 0$; $(x(I-T))_2 = x_2 - x_1 t_{12} = 0$, hence $x_2 = 0$. Inductively we can easily show that $x_k = 0$ for all k.

Since $x \to x(I-T)$ is a continuous linear function from $(\ell_1, \| \|_1)$ into $(\ell_1, \| \|_1)$ and $(\ell_1, \| \|_1)$ is AK, $\|((P_n x) - x)(I-T)\|_1 \to 0$ as $n \to \infty$ for $x \in \ell_1$. Hence $(\ell_1, \|x(I-T)\|_1)$

is also an AK-space. Now let $f \in (\ell_1, \|x(I-T)\|_1)'$. Then, for every $x \in \ell_1$, $f(x) = \sum_{k=1}^{\infty} x_k f(e^k)$ since $(\ell_1, \|x(I-T)\|_1)$ is AK. Let $y_k = f(e^k)$ for each k. We claim that $y = (y_k) \in Tac_o$. Since $f \in (\ell_1, \|x(I-T)\|_1)'$,

$$(1) \quad \left|\sum_{k=1}^{\infty} x_k y_k\right| = |f(x)| \leq \|x(I-T)\|_1 \|f\| \quad \text{for} \quad x \in \ell_1 .$$

Let $p \in \mathbb{N}$. Then the nth row of $\frac{T+\ldots+T^p}{p}$ is in ℓ_1 for each n, and hence

$$\left|\left(\frac{T+\ldots+T^p}{p}\, y\right)_n\right| = \left|\sum_{k=1}^{\infty}\left(\frac{T+\ldots+T^p}{p}\right)_{nk} y_k\right|$$

$$\leq \left\|\left[\left(\frac{T+\ldots+T^p}{p}\right)_{nk}\right]_{k=1}^{\infty} (I-T)\right\|_1 \|f\| \text{ (by (i))}$$

$$= \sum_{k=1}^{\infty}\left|\left(\frac{T+\ldots+T^p}{p}\right)_{nk} - \left(\frac{T^2+\ldots+T^{p+1}}{p}\right)_{nk}\right| \|f\|$$

$$= \sum_{k=1}^{\infty}\left|\left(\frac{T-T^{p+1}}{p}\right)_{nk}\right| \|f\|$$

$$\leq \frac{2}{p}\|f\| \quad \left(\text{since } \sum_{k=1}^{\infty}|(T)_{nk}| = \sum_{k=1}^{\infty}|(T^{p+1})_{nk}| = 1\right).$$

Thus $\lim_p \left|\left(\frac{(T+\ldots T^p}{p}\, y\right)_n\right| = 0$ uniformly in n. By Corollary 1 of Theorem 1 of §3, $y \in Tac_o$.

<u>THEOREM 3</u>. Let T be a lifting matrix. Then $(Tac_o, \tau(Tac_o, \ell_1))$ is complete.

Proof. To show that $(Tac_o, \tau(Tac_o, \ell_1))$ is complete we use Grothendieck's criterion (see 1.3, Theorem 1.) Let f be a linear functional on ℓ_1 which is $\sigma(\ell_1,Tac_o)$-continuous on each $\sigma(\ell_1,Tac_o)$-compact set, and suppose (x^n) is a sequence in ℓ_1 which is convergent to 0 in the two-norm topology $(\ell_1,\|x\|_1,\|x(I-T)\|_1)$. Then, by Theorem 1((iii) $\Rightarrow$ (i)), (x^n) is $\sigma(\ell_1,Tac_o)$-convergent to 0. Hence $\{x_n|n \in \mathbb{N}\}$ is $\sigma(\ell_1,Tac_o)$-relatively compact so that f is $\sigma(\ell_1,Tac_o)$-continuous on $\{x_n|n \in \mathbb{N}\}$. Thus $(f(x_n))_{n=1}^{\infty}$ is convergent to 0 in $\mathbb{R}$. Therefore, f is continuous in the two norm-topology $(\ell_1,\| \|_1\|x(I-T)\|_1)$. Hence f lies in the closure of $(\ell_1,\|x(I-T)\|_1)'$ in $(\ell_1,\| \|_1)'$ (i.e., $(m,\| \|_\infty)$) by 1.4, Theorem 2. Since $(\ell_1,\|x(I-T)\|_1)' \subseteq Tac_o$ (by Lemma 4) and Tac_o is closed in $(m,\| \|_\infty)$, $f \in Tac_o$. Hence $(Tac_o,\tau(Tac_o,\ell_1))$ is complete by Grothendieck's criterion.

CHAPTER 4

CONSISTENCY THEOREMS FOR T-ALMOST CONVERGENCE

§1. Introduction.

The main purpose of this chapter is to establish the bounded consistency theorem for T-almost convergence. The bounded consistency theorem is a principal result in the theory of summability. Two different approaches to establish this theorem for almost convergence can be found in [4] and [21]. It seems difficult to construct a proof for T-almost convergence parallel to these proofs. In proving this theorem we first establish the sequential completeness of ℓ_1 under certain weak topologies. To do this we apply a gliding hump argument together with a technique called "the principle of aping sequences". Erdős and Piranian developed this technique in [8] and derived the classical bounded consistency theorem as a quick application. As we expected, it was necessary to penetrate deep into the structure of T-almost convergent sequences to establish the theorem. This made some arguments rather long and difficult. Finally, employing some techniques already developed, we obtain a result for T-almost convergent sequences (Theorem 3 of section 3) which is unknown even for convergent sequences.

§2. Notations and basic results.

Recall the definition (3.2, Definition 2) of a lifting matrix $T = (t_{jk})$ in Chapter 3. When T is a lifting matrix, T^n (nth power of T) is defined for $n \in \mathbb{N}$, and for convenience we write $T^n = (t^n_{jk})$ for $n = 0,1,2...$ with $T^o = I$ and $T^1 = T$. It should be noticed that t^n_{jk} is not the n^{th} power of t_{jk}. Under these notations we obtain the following proposition.

PROPOSITION 1. Let $T = (t_{jk})$ be a lifting matrix. Then the following hold:

(i) $t^n_{jk} = 0$ for $k < j+n$;

(ii) $\sum_{k=1}^{\infty} t^n_{jk} = 1$ for $n = 0,1,2,...$ and $j = 1,2,...$;

(iii) $\sum_{k=1}^{q} t^m_{jk} \leq \sum_{k=1}^{q} t^n_{jk}$ (equivalently, $\sum_{k=q+1}^{\infty} t^m_{jk} \geq \sum_{k=q+1}^{\infty} t^n_{jk}$)

for $m > n$ and $q,j = 1,2,...$.

Proof. (i) Clearly it is true for $n = 0,1$. Suppose $t^n_{jk} = 0$ for every j,k such that $k < j+n$. Then $t^{n+1}_{jk} = \sum_{p=1}^{\infty} t_{jp}t^n_{pk} = 0$ for $k < j+n+1$, since $t_{jp} = 0$ for $p \leq j$ and $t^n_{pk} = 0$ $k < p+n$. So (i) follows by induction.

(ii) This follows from the fact that $T^n e = e$.

(iii) It is sufficient to show that $\sum_{k=1}^{q} t^{n+1}_{jk} \leq \sum_{k=1}^{q} t^n_{jk}$.

If $q < j+n+1$, then $\sum_{k=1}^{q} t_{jk}^{n+1} = 0$ by (i). Suppose $q \geq j+n+1$.

Then $\sum_{k=1}^{q} t_{jk}^{n+1} = \sum_{k=j+n+1}^{q} t_{jk}^{n+1}$ by (i). Also, for $k \geq j+n+1$,

$t_{jk}^{n+1} = (T^nT)_{jk} = \sum_{i=1}^{\infty} t_{ji}^{n} t_{ik} = \sum_{i=j+n}^{k-1} t_{ji}^{n} t_{ik}$ (since $t_{ji}^{n} = 0$ for $i < j+n$

and $t_{ik} = 0$ for $i > k-1$ by (i)). Hence

$$\sum_{k=j+n+1}^{q} t_{jk}^{n+1} = \sum_{k=j+n+1}^{q} \sum_{i=j+n}^{k-1} t_{ji}^{n} t_{ik}$$

$$\leq \sum_{k=j+n+1}^{q} \sum_{i=j+n}^{q} t_{ji}^{n} t_{ik} \quad \text{(since } k \leq q\text{)}$$

$$= \sum_{i=j+n}^{q} t_{ji}^{n} \sum_{k=j+n+1}^{q} t_{ik}$$

$$\leq \sum_{i=j+n}^{q} t_{ji}^{n} \quad \text{(by (ii))}$$

$$= \sum_{k=j+n}^{q} t_{jk}^{n} = \sum_{k=1}^{q} t_{jk}^{n} \quad \text{(by (i))}.$$

Thus $\sum_{k=1}^{q} t_{jk}^{n+1} \leq \sum_{k=1}^{q} t_{jk}^{n}$.

§3. Main results.

In this section we obtain several consistency theorems for T-almost convergent sequences by establishing the following theorem. The proof of this theorem is difficult and uses a complicated gliding hump argument based on the properties of lifting matrices and T-almost convergent sequences.

THEOREM 1. Let T be a lifting matrix, and let $B = (b_{jk})$ be an infinite matrix such that $\|B\| < \infty$ and such that every column of B belongs to c_o. Then ℓ_1 is $\sigma(\ell_1,(Tac_o)_B \cap m)$-sequentially complete.

Proof. Let $B = (b_{jk})$ be an infinite matrix such that $\|B\| < \infty$ and such that every column of B belongs to c_o, and suppose $A = (a_{jk})$ is an infinite matrix with the same properties as B such that $(Tac_o)_B \cap m \subseteq c_A$. Since $c_o \subseteq c_{o_B} \subseteq (Tac_o)_B$, $[(Tac_o)_B \cap m]^{\beta} = \ell_1$ and hence, because of 2.3, Theorem 2, it suffices to prove that $(Tac_o)_B \cap m \subseteq c_{o_A}$.

Suppose there exists $x = (x_k) \in (Tac_o)_B \cap m$ such that $\lim_A x \neq 0$. We may assume that $\lim_A x = 1$. Let $y = Bx$ and $z = Ax$. Then $y \in Tac_o$ and $z \in c$. We construct a bounded sequence $u = (u_k)$ such that $u.x \in (Tac_o)_B \setminus c_A$. This leads to a contradiction since $(Tac_o)_B \cap m \subseteq c_A$.

Let $k_1 \in \mathbb{N}$. Choose $n_1 \in \mathbb{N}$ such that;

$$(b_1) \quad \sum_{k=1}^{k_1} (|a_{jk}|+|b_{jk}|) < \frac{1}{2} \text{ for } j \geq n_1 ;$$

(c_1) $\sum_{k=1}^{n_1} t_{1k} \geq \frac{1}{2}$.

For $1 \leq j \leq n_1$, notice that $\sum_{k=1}^{n_1} t_{jk}^{o} = t_{jj}^{o} = 1$, and that $\sum_{k=1}^{n_1} t_{jk}^{p} = 0$

if $p \geq n_1$ (by Proposition 1(i) of §2). For $1 \leq j \leq n_1$, let i_{j1}

$(0 \leq i_{j1} < n_1)$ be the largest integer such that

(e_1) $\sum_{k=1}^{n_1} t_{jk}^{i_{j1}} \geq \frac{1}{2}$.

Let

(f_1) $i_{j1} = 0$ for $j > n_1$.

Notice that

(g_1) $i_{11} \geq 1$ (by (c_1) and (e_1)), and $i_{j1} < n_1$ for $j = 1,2,\ldots$.

Choose k_2 $(>k_1) \in \mathbb{N}$ such that

(a_2) $\sum_{k=k_2}^{\infty} (|a_{jk}|+|b_{jk}|) < 1$ for $j \leq n_1$.

Now choose n_2 $(>n_1) \in \mathbb{N}$ such that:

(b_2) $\sum_{k=1}^{k_2} (|a_{jk}|+|b_{jk}|) < \frac{1}{2^2}$ for $j \geq n_2$;

(c_2) $\sum_{k=1}^{n_2} t_{jk}^{i_{j1}+n_1} \geq \frac{1}{2} + \frac{1}{2^2}$ for $1 \leq j \leq n_1$.

By Proposition 1(i) of §2, $\sum_{k=1}^{n_2} t_{n_1 k}^{i_{n_1 1}+n_1} = \sum_{k=2n_1+i_{n_1 1}}^{n_2} t_{n_1 k}^{i_{n_1 1}+n_1}$ (> 0 by (c_2)),

and hence

(d_2) $2n_1 \le n_2$.

For $1 \le j \le n_2$, notice that $\sum_{k=1}^{n_2} t_{jk}^{o} = t_{jj}^{o} = 1$, and that $\sum_{k=1}^{n_2} t_{jk}^{p} = 0$ if

$p \ge n_2$ (by Proposition 1(i) of §2). For $1 \le j \le n_2$, let $i_{j2} (0 \le i_{j2} \le n_2)$

be the largest integer such that

(e_2) $\sum_{k=1}^{n_2} t_{jk}^{i_{j2}} \ge \frac{1}{2} + \frac{1}{2^2}$.

Let

(f_2) $i_{j2} = 0$ for $j > n_2$.

Notice that

(g_2) $i_{j2} \ge i_{j1} + n_1$ for $1 \le j \le n_1$ (by (c_2) and (e_2)), and $i_{j2} < n_2$

for $j = 1,2,\ldots$.

Choose $k_3 (> k_2) \in \mathbb{N}$ such that

(a_3) $\sum_{k=k_3}^{\infty} (|a_{jk}|+|b_{jk}|) < \frac{1}{2}$ for $j \le n_2$.

Now choose $n_3 (> n_2) \in \mathbb{N}$ such that;

(b_3) $\sum_{k=1}^{k_3} (|a_{jk}|+|b_{jk}|) < \frac{1}{2^3}$ for $j \ge n_3$;

(c_3) $\sum_{k=1}^{n_3} t_{jk}^{i_{j2}+n_2} \ge \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$ for $1 \le j \le n_2$.

By Proposition 1(i) of §2, $\sum_{k=1}^{n_3} t_{n_2k}^{i_{n_22}+n_2} = \sum_{k=2n_2+i_{n_22}}^{n_3} t_{n_2k}^{i_{n_22}+n_2}$ (> 0 by (c_3)),

and hence

(d_3) $2n_2 \le n_3$.

For $1 \le j \le n_3$, notice that $\sum_{k=1}^{n_3} t_{jk}^{o} = t_{jj}^{o} = 1$, and that $\sum_{k=1}^{n_3} t_{jk}^{p} = 0$ if $p \ge n_3$ (by Proposition 1(i) of §2). For $1 \le j \le n_3$, let $i_{j3} (0 \le i_{j3} < n_3)$ be the largest integer such that

(e_3) $\sum_{k=1}^{n_3} t_{jk}^{i_{j3}} \ge \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$.

Let

(f_3) $i_{j3} = 0$ for $j > n_3$.

Notice that

(g_3) $i_{j3} \ge i_{j2} + n_2$ for $1 \le j \le n_2$ (by (c_3) and (e_3)), and $i_{j3} < n_3$ for $j = 1,2,\ldots$.

We proceed inductively to construct strictly increasing sequences $(k_r)_{r=1}^{\infty}$, $(n_r)_{r=1}^{\infty}$ of positive integers, and increasing sequences

$(i_{jr})_{r=1}^{\infty}$; $j = 1,2,\ldots$ of non-negative integers such that:

(i) $$\max_{n_r \le j \le n_{r+1}} \left[\sum_{k=1}^{k_r} (|a_{jk}|+|b_{jk}|) + \sum_{k=k_{r+2}}^{\infty} (|a_{jk}|+|b_{jk}|) \right] = \frac{1}{2^{r-1}}$$

(see (b_1), (b_2), (b_3), (a_2), and (a_3));

(ii) $\sum_{k=1}^{n_r} t_{jk}^{i_{jr}} \ge \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^r}$ (equivalently, $\sum_{k=n_r+1}^{\infty} t_{jk}^{i_{jr}} \le \frac{1}{2^{r+1}} + \frac{1}{2^{r+2}} + \ldots$) for $1 \le j \le n_r$, $r = 1,2,\ldots$ (see (e_1), (e_2), and (e_3));

(iii) $2n_r \le n_{r+1}$ for $r = 1,2,\ldots$ (see (d_2) and (d_3));

(iv) $i_{j,r+1} \ge i_{jr} + n_r$ for $1 \le j \le n_r$, $r = 1,2,\ldots$ (see (g_2) and (g_3));

(v) $i_{jr} < n_r$ for $j,r = 1,2,\ldots$ (see (g_1), (g_2), and (g_3));

(vi) $i_{jr} = 0$ for $j > n_r$, $r = 1,2,\ldots$ (see (f_1), (f_2), and (f_3)).

Define bounded sequences $u = (u_j)$ and $v = (v_j)$ such that $u_j = \sin\sqrt{r}$ for $k_r \le j < k_{r+1}$ and $v_j = \sin\sqrt{r}$ for $n_r \le j < n_{r+1}$.
First we show that $(Bx).v$ $(=y.v) \in Tac_o$.

By the definition of u and v,

(1) $\|u\|_\infty = \|v\|_\infty \le 1$.

Let $1 > \varepsilon > 0$. Since $y \in Tac_o$, it follows from Corollary 1 of 3.3, Theorem 1 that there exists $p_o \in N$ such that $\|(\frac{T+T^2+\ldots+T^p}{p})y\|_\infty < \frac{\varepsilon}{20}$ for $p \ge p_o$. This means that

(2) $\left|\sum_{k=1}^{\infty}(t_{jk}+t_{jk}^2+\ldots+t_{jk}^p)y_k\right| < p\varepsilon/20$ for $p \geq p_o$; $j = 1,2,\ldots$,

Choose $r(> 2) \in \mathbb{N}$ such that:

(3) $n_r > \max\{20p_o/\varepsilon \ , \ 20p_o\|y\|_\infty/\varepsilon\}$;

(4) $|\sin\sqrt{m} - \sin\sqrt{m-1}| < \varepsilon/20\|y\|_\infty$ for $m \geq r$;

(5) $\sum_{k=r}^{\infty} 1/2^k < \varepsilon/20\|y\|_\infty$.

Note that in the rest of the proof r is a fixed integer satisfying (3), (4), and (5). Let $p \in \mathbb{N}$ such that

(6) $p > \max\{20(\frac{p_o+n_r}{\varepsilon}), 20\|y\|_\infty(\frac{p_o+n_r}{\varepsilon})\}$,

Now we claim that, for p ($\in\mathbb{N}$) satisfying (6),

$\left|\sum_{k=1}^{\infty}(t_{jk}+t_{jk}^2+\ldots+t_{jk}^p)y_k v_k\right|/p < \varepsilon$ for every $j \in N$. Let $j \in \mathbb{N}$.

Case 1. $i_{jr} \geq 1$. Then

(1)' $j \leq n_r$ by (vi).

Since $p > n_r$ (by (6)) and $i_{jr} < n_r$ (by (v)), $p > i_{jr}$. Let $s(\geq r) \in \mathbb{N}$ such that $i_{js} < p \leq i_{j,s+1}$. Then

(2)' $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right|$

$$\le \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jr}})y_k v_k\right| + \left[\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jr}+1}+\ldots+t_{jk}^{i_{j,r+1}})y_k v_k\right|\right.$$

$$+\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,r+1}+1}+\ldots+t_{jk}^{i_{j,r+2}})y_k v_k\right|+\ldots+\left.\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,s-1}+1}+\ldots+t_{jk}^{i_{js}})y_k v_k\right|\right]$$

$$+\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{js}+1}+\ldots+t_{jk}^{p})y_k v_k\right|$$

(3)' $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jr}})y_k v_k\right| \le \|y.v\|_{\infty}.i_{jr}$ (by Proposition 1(ii) of §2)

$< \|y.v\|_{\infty}.n_r$ (by (v))

$\le \|y\|_{\infty}.n_r$ (by (1))

$< p\varepsilon/20$ (since $n_r < p\varepsilon/20\|y\|_{\infty}$ (by (6)).

For $r \le m \le s$ and $i_{jm} < q \le i_{j,m+1}$,

(4)' $\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k v_k\right|$

$$\le \left|\sum_{k=1}^{n_m+1}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k v_k\right| + \left|\sum_{k=n_{m+1}+1}^{\infty}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k v_k\right|.$$

For $r \le m \le s$, $n_{m+1} > n_r \ge j$ (by (1)') and hence

$$\sum_{k=n_{m+1}+1}^{\infty} t_{jk}^{i_{j,m+1}} \leq \sum_{k=m+2}^{\infty} 1/2^k \text{ (by (ii))} < \sum_{k=r}^{\infty} 1/2^k \text{ (since } r \leq m) < \frac{\varepsilon}{20\|y\|_\infty}$$

(by (5)). Also, by Proposition 1(iii)) of §2,

$$(5)' \quad \sum_{k=n_{m+1}+1}^{\infty} t_{jk}^{i_{jm}+1} \leq \sum_{k=n_{m+1}+1}^{\infty} t_{jk}^{i_{jm}+2} \leq \ldots \leq \sum_{k=n_{m+1}+1}^{\infty} t_{jk}^{i_{j,m+1}} < \frac{\varepsilon}{20\|y\|_\infty} \text{ for } r \leq m \leq s.$$

Hence, for $r \leq m \leq s$ and $i_{jm} < q \leq i_{j,m+1}$,

$$(6)' \quad \left|\sum_{k=n_{m+1}+1}^{\infty} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q}) y_k v_k\right| < \|y.v\|_\infty \, (q - i_{jm}) \cdot \frac{\varepsilon}{20\|y\|_\infty}$$

$$\leq (q - i_{jm}) \frac{\varepsilon}{20} \quad \text{(by (1))}.$$

Also, for $r \leq m \leq s$ and $i_{jm} < q \leq i_{j,m+1}$,

$$(7)' \quad \left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q}) y_k v_k\right|$$

$$= \left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q}) y_k v_{n_{m-1}} + \sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q})(v_k - v_{n_{m-1}}) y_k\right|$$

$$\leq |v_{n_{m-1}}| \left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q}) y_k\right| + \left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1} + \ldots + t_{jk}^{q})(v_k - v_{n_{m-1}}) y_k\right|.$$

For $j \leq n_{m-1}$, $i_{jm} \geq n_{m-1}$ by (iv). Hence $j + i_{jm} + 1 > n_{m-1}$ for every $j \in \mathbb{N}$.

Since $t_{jk}^{i_{jm}+1} = t_{jk}^{i_{jm}+2} = \ldots = t_{jk}^{i_{j,m+1}} = 0$ for $k < j + i_{jm} + 1$ by

Proposition 1(i) of §2,

$$(8)' \quad \left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})(v_k - v_{n_{m-1}})y_k\right|$$

$$= \left|\sum_{k=n_{m-1}}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})(v_k - v_{n_{m-1}})y_k\right| \text{ for } r \le m \le s$$

and $i_{jm} < q \le i_{j,m+1}$.

By the definition of (v_k), for $r \le m \le s$ and $n_{m-1} \le k \le n_{m+1}$,

$$(9)' \quad |v_k - v_{n_{m-1}}| \le \max\{|\sin\sqrt{m} - \sin\sqrt{m-1}|, |\sin\sqrt{m+1} - \sin\sqrt{m-1}|\}$$

$$< \frac{2\varepsilon}{20\|y\|_\infty} \text{ by (4) (since } m \ge r).$$

Hence, for $r \le m \le s$ and $i_{jm} < q \le i_{j,m+1}$,

$$(10)' \quad \left|\sum_{k=n_{m-1}}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})(v_k - v_{n_{m-1}})y_k\right|$$

$$< \|y\|_\infty \cdot \frac{2\varepsilon}{20\|y\|_\infty} \cdot \sum_{k=n_{m-1}}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q}) \text{ (by (9)')}$$

$$\le \frac{2}{20}(q - i_{jm})\varepsilon \text{ (by Proposition 1(ii) of §2).}$$

For $r \le m \le s$ and $i_{jm} < q \le i_{j,m+1}$,

$$(11)' \quad |v_{n_{m-1}}|\left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right|$$

$$\le \left|\sum_{k=1}^{\infty} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k - \sum_{k=n_{m+1}+1}^{\infty} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right| \text{ (by (1))}$$

$$\le \left|\sum_{k=1}^{\infty} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right| + \left|\sum_{k=n_{m+1}+1}^{\infty} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right|.$$

For $r \le m \le s$ and $i_{jm} < q \le i_{j,m+1}$,

(12)' $$\left|\sum_{k=n_m+1}^{\infty}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right| < \|y\|_\infty \cdot \frac{\varepsilon}{20\|y\|_\infty} \cdot (q - i_{jm}) \text{ (by (5)')}$$

$$= (q - i_{jm})\frac{\varepsilon}{20}.$$

For $r \le m \le s$ and $i_{jm} < q \le i_{j,m+1}$,

(13)' $$\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})y_k\right|$$

$$= \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{q})y_k - \sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jm}})y_k\right|$$

$$\le \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{q})y_k\right| + \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jm}})y_k\right| .$$

For $r \le m \le s$, $j \le n_r$ (by (1)') $\le n_m$ and hence $i_{j,m+1} \ge i_{jm} + n_m$ (by (iv)) $> 2i_{jm}$ (by (v)). Thus

(14)' $$\frac{i_{j,m+1}}{i_{j,m+1}-i_{jm}} < 2 \quad\text{and}\quad \frac{i_{jm}}{i_{j,m+1}-i_{jm}} < 1 \quad\text{for}\quad r \le m \le s.$$

For $r \le m \le s$, $j \le n_r$ (by (1)') $\le n_m$ and hence $i_{j,m+1} \ge n_m$ (by (iv)) $\ge n_r > p_o$ (by (3)). Thus, by (2),

(15)' $$\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{j,m+1}})y_k\right| < i_{j.m+1}\cdot\frac{\varepsilon}{20}$$

$$= \frac{i_{j,m+1}}{i_{j,m+1}-i_{jm}} \cdot (i_{j,m+1}-i_{jm})\cdot\frac{\varepsilon}{20}$$

$$< (i_{j,m+1}-i_{jm})\cdot\frac{\varepsilon}{10} \quad \text{for } r \le m \le s \text{ by (14)'}.$$

If $i_{jm} \geq p_o$ $(r \leq m \leq s)$, $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jm}})y_k\right| < i_{jm}\cdot\frac{\varepsilon}{20}$ (by

(2)) $= \frac{i_{jm}}{i_{j,m+1}-i_{jm}}\cdot(i_{j,m+1}-i_{jm})\cdot\frac{\varepsilon}{20} < (i_{j,m+1}-i_{jm})\cdot\frac{\varepsilon}{20}$ (by (14)').

If $i_{jm} < p_o$ $(r \leq m \leq s)$, $\frac{i_{jm}}{i_{j,m+1}-i_{jm}} < \frac{p_o}{n_m}$ (note that since

$j \leq n_r$, (by (1)') $\leq n_m$, $i_{j,m+1}-i_{jm} \geq n_m$ by (iv)) $\leq \frac{p_o}{n_r} < \frac{\varepsilon}{20\|y\|_\infty}$ (by (3)) and

hence $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jm}})y_k\right| \leq \|y\|_\infty\cdot i_{jm}$ (by Proposition 1(ii) of §2)

$= \|y\|_\infty\cdot\frac{i_{jm}}{i_{j,m+1}-i_{jm}}\cdot(i_{j,m+1}-i_{jm}) < \|y\|_\infty\cdot\frac{\varepsilon}{20\|y\|_\infty}\cdot(i_{j,m+1}-i_{jm})$

$= (i_{j,m+1}-i_{jm})\frac{\varepsilon}{20}$. Thus

(16)' $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{jm}})y_k\right| < (i_{j,m+1}-i_{jm})\frac{\varepsilon}{20}$ for $r \leq m \leq s$.

From (13)', (15)', and (16)' with $q = i_{j,m+1}$,

(17)' $\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{i_{j,m+1}})y_k\right| < (i_{j,m+1}-i_{jm})\cdot\frac{3\varepsilon}{20}$ for $r \leq m \leq s-1$.

From (11)', (17)', and (12)' with $q = i_{j,m+1}$,

(18)' $|v_{n_{m-1}}|\left|\sum_{k=1}^{n_{m+1}}(t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{i_{j,m+1}})y_k\right| < (i_{j,m+1}-i_{jm})\cdot\frac{4\varepsilon}{20}$ for $r \leq m \leq s-1$.

From (8)' and (10)',

(19)' $$\left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{q})(y_k - y_{n_{m-1}})y_k\right|$$

$$< \frac{2}{20}(q - i_{jm})\varepsilon \quad \text{for} \quad r \le m \le s \quad \text{and} \quad i_{jm} < q \le i_{j,m+1}.$$

From (7)', (18)', and (19)' with $q = i_{j,m+1}$,

(20)' $$\left|\sum_{k=1}^{n_{m+1}} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{i_{j,m+1}})y_k v_k\right| < (i_{j,m+1} - i_{jm})\frac{6\varepsilon}{20} \quad \text{for} \quad r \le m \le s-1.$$

From (4)', (20)', and (6)' with $q = i_{j,m+1}$,

(21)' $$\left|\sum_{k=1}^{\infty} (t_{jk}^{i_{jm}+1}+\ldots+t_{jk}^{i_{j,m+1}})y_k v_k\right| < (i_{j,m+1} - i_{jm})\frac{7\varepsilon}{20} \quad \text{for} \quad r \le m \le s-1.$$

Hence

(22)' $$\left|\sum_{k=1}^{\infty} (t_{jk}^{i_{jr}+1}+\ldots+t_{jk}^{i_{j,r+1}})y_k v_k\right| + \left|\sum_{k=1}^{\infty} (t_{jk}^{i_{j,r+1}+1}+\ldots+t_{jk}^{i_{j,r+2}})y_k v_k\right|$$

$$+\ldots+\left|\sum_{k=1}^{\infty} (t_{jk}^{i_{j,s-1}+1}+\ldots+t_{jk}^{i_{js}})y_k v_k\right|$$

$$< (i_{j,r+1} - i_{jr})\cdot\frac{7\varepsilon}{20} + (i_{j,r+2} - i_{j,r+1})\frac{7\varepsilon}{20} + \ldots + (i_{js} - i_{j,s-1})\frac{7\varepsilon}{20}$$

$$= (i_{js} - i_{jr})\cdot\frac{7\varepsilon}{20}$$

$$< \frac{7}{20}p\varepsilon \quad (\text{since} \quad i_{js} < p).$$

Since $p > p_o$ (by (6)),

(23)' $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k\right| < p\varepsilon/20$ (by (2)).

If $i_{js} \geq p_o$, $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{js}})y_k\right| < i_{js}.\varepsilon/20$ (by (2)) $< p\varepsilon/20$ (since $p > i_{js}$). If $i_{js} < p_o$, $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{js}})y_k\right| \leq \|y\|_{\infty}.i_{js}$ (by Proposition 1(ii) of §2) $< \|y\|_{\infty}p_o = \|y\|_{\infty}.p.p_o/p < \|y\|_{\infty}.p.\frac{\varepsilon}{20\|y\|_{\infty}}$ (by (6)) $= p\varepsilon/20$.

Hence

(24)' $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{i_{js}})y_k\right| < p\varepsilon/20.$

From (13)', (23)' and (24)' with $m = s$ and $q = p$ (note that $i_{js} < p \leq i_{j,s+1}$),

(25)' $\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{js}+1}+\ldots+t_{jk}^{p})y_k\right| < 2p\varepsilon/20.$

From (11)', (25)' and (12)' with $m = s$ and $q = p$,

(26)' $|v_{n_{s-1}}|\left|\sum_{k=1}^{n_{s+1}}(t_{jk}^{i_{js}+1}+\ldots+t_{jk}^{p})y_k\right| < 3p\varepsilon/20.$

From (7)', (26)', and (19)' with $m = s$ and $q = p$,

(27)' $\left|\sum_{k=1}^{n_{s+1}}(t_{jk}^{i_{js}+1}+\ldots+t_{jk}^{p})y_kv_k\right| < 5p\varepsilon/20.$

From (4)', (27)' and (6)' with $m = s$ and $q = p$, (28)' $\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{js}+1}+\ldots+t_{jk}^{p})y_k v_k\right| < 6p\varepsilon/20.$

From (2)', (3)', (22)' and (28)',

$$* \qquad \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right| < 14p\varepsilon/20 < p\varepsilon.$$

Case 2. $i_{jr} = 0$. Let t be the smallest integer such that $i_{jt} \geq 1$. Then

(1)" $t > r$, $i_{j,t-1} = 0$, and $j \leq n_t$ (by (vi)).

For $1 \leq q \leq i_{jt}$,

(2)" $\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right|$

$$\leq \left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right| + \left|\sum_{k=n_t+1}^{\infty}(t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right|.$$

Since $j \leq n_t$ (by (1)"), $\sum_{k=n_t+1}^{\infty} t_{jk}^{i_{jt}} \leq \sum_{k=t+1}^{\infty} 1/2^k$ (by (ii)) $< \sum_{k=r}^{\infty} 1/2^k$

(since $t > r$ by (1)") $< \frac{\varepsilon}{20\|y\|_\infty}$ by (5). Also, by Proposition 1(iii) of §2,

(3)" $\sum_{k=n_t+1}^{\infty} t_{jk} \leq \sum_{k=n_t+1}^{\infty} t_{jk}^{2} \leq \ldots \leq \sum_{k=n_t+1}^{\infty} t_{jk}^{i_{jt}} < \frac{\varepsilon}{20\|y\|_\infty}$.

Hence, for $1 \leq q \leq i_{jt}$

(4)" $\left|\sum_{k=n_t+1}^{\infty}(t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right| < \|y.v\|_\infty \cdot q \cdot \frac{\varepsilon}{20\|y\|_\infty} \leq \frac{1}{20}q\varepsilon$ by (1).

As same as (7)' in Case 1, we can show that

$$(5)'' \quad \left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^q)y_k v_k\right|$$

$$\le |v_{n_{t-2}}|\left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^q)y_k\right| + \left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^q)(v_k-v_{n_{t-2}})y_k\right|$$

for $1 \le q \le i_{jt}$.

Since $t > r$ (by (1)") and $r > 2$, $t-2 \ge 1$. If $j \le n_{t-2}$, then $i_{j,t-1} \ge i_{j,t-2} + n_{t-2}$ by (iv). This is a contradiction since $i_{j,t-1} = 0$ (by (1)") and $n_{t-2} > 0$. Thus $j > n_{t-2}$ and hence $t_{jk} = t_{jk}^2 = \ldots = t_{jk}^{i_{jt}} = 0$ for $k < n_{t-2}$ (by Proposition 1(i) of §2).

Therefore, for $1 \le q \le i_{jt}$

$$(6)'' \quad \left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^q)(v_k-v_{n_{t-2}})y_k\right| = \left|\sum_{k=n_{t-2}}^{n_t}(t_{jk}+\ldots+t_{jk}^q)(v_k-v_{n_{t-2}})y_k\right|,$$

Since $t-1 \ge r$ (by (1)"), as same as (10)' in Case 1, we can show that

$$(7)'' \quad \left|\sum_{k=n_{t-2}}^{n_t}(t_{jk}+\ldots+t_{jk}^q)(v_k-v_{n_{t-2}})y_k\right| < \frac{2}{20}\cdot q\varepsilon \quad \text{for} \quad 1 \le q \le i_{jt}\ .$$

As same as (11)' in Case 1, we can show that

$$(8)'' \quad |v_{n_{t-2}}|\left|\sum_{k=1}^{n_t}(t_{jk}+\ldots+t_{jk}^q)y_k\right|$$

$$\le \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^q)y_k\right| + \left|\sum_{k=n_t+1}^{\infty}(t_{jk}+\ldots+t_{jk}^q)y_k\right| \quad \text{for} \quad 1 \le q \le i_{jt}.$$

For $1 \le q \le i_{jt}$, by (3)",

(9)" $$\left|\sum_{k=n_t+1}^{\infty} (t_{jk}+\ldots+t_{jk}^{q})y_k\right| < \|y\|_\infty . q . \frac{\varepsilon}{20\|y\|_\infty} = \frac{1}{20} q\varepsilon .$$

Now if $p_o \le q$ ($\le i_{jt}$),

(10)" $$\left|\sum_{k=1}^{\infty} (t_{jk}+\ldots+t_{jk}^{q})y_k\right| < \frac{1}{20} q\varepsilon \quad \text{by (2).}$$

Therefore, if $p_o \le q$ ($\le i_{jt}$), from (8)", (10)", and (9)",

(11)" $$\left|v_{n_{t-2}}\right| \left|\sum_{k=1}^{n_t} (t_{jk}+\ldots+t_{jk}^{q})y_k\right| < \frac{2}{20} q\varepsilon ,$$

from (6)" and (7)",

(12)" $$\left|\sum_{k=1}^{n_t} (t_{jk}+\ldots+t_{jk}^{q})(v_k - v_{n_{t-2}})y_k\right| < \frac{2}{20} q\varepsilon ,$$

from (5)", (11)" and (12)",

(13)" $$\left|\sum_{k=1}^{n_t} (t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right| < \frac{4}{20} q\varepsilon \quad \text{and}$$

from (2)", (13)", and (4)",

(14)" $$\left|\sum_{k=1}^{\infty} (t_{jk}+\ldots+t_{jk}^{q})y_k v_k\right| < \frac{5}{20} q\varepsilon .$$

To show that $\left|\sum_{k=1}^{\infty} (t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right| < p\varepsilon$ we consider the cases $p > i_{jt}$ and $p \le i_{jt}$ separately. First let $p > i_{jt}$. Then there exists $s\ (\ge t) \in N$ such that $i_{js} < p \le i_{j,s+1}$. Thus

$$(15)'' \quad \left|\sum_{k=1}^{\infty}(t_{jk}+\dots+t_{jk}^{p})y_k v_k\right| \le \left|\sum_{k=1}^{\infty}(t_{jk}+\dots+t_{jk}^{i_{jt}})y_k v_k\right|$$

$$+\left[\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jt}+1}+\dots+t_{jk}^{i_{j,t+1}})y_k v_k\right| + \left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,t+1}+1}+\dots+t_{jk}^{i_{j,t+2}})y_k v_k\right|+\dots\right.$$

$$\left.\dots+\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,s-1}+1}+\dots+t_{jk}^{i_{js}})y_k v_k\right|\right] + \sum_{k=1}^{\infty}(t_{jk}^{i_{js}+1}+\dots+t_{jk}^{p})y_k v_k|.$$

As same as (22)' and (28)' in Case I we can show that:

$$(16)'' \quad \left|\sum_{k=1}^{\infty}(t_{jk}^{i_{jt}+1}+\dots+t_{jk}^{i_{j,t+1}})y_k v_k\right| + \left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,t+1}+1}+\dots+t_{jk}^{i_{j,t+2}})y_k v_k\right|$$

$$+\dots+\left|\sum_{k=1}^{\infty}(t_{jk}^{i_{j,s-1}+1}+\dots+t_{jk}^{i_{js}})y_k v_k\right| < \frac{7}{20}\,p\varepsilon;$$

$$(17)'' \quad \left|\sum_{k=1}^{\infty}(t_{jk}^{i_{js}+1}+\dots+t_{jk}^{p})y_k v_k\right| < \frac{6}{20}\,p\varepsilon\ .$$

If $i_{jt} > p_o$, from (14)" with $q = i_{jt}$, $\left|\sum_{k=1}^{\infty}(t_{jk}+\dots+t_{jk}^{i_{jt}})y_k v_k\right|$

$< \frac{5}{20}\cdot i_{jt}\cdot\varepsilon < \frac{5}{20}\,p\varepsilon$ (since $p > i_{jt}$), and if $i_{jt} \le p_o$, then

$\left|\sum_{k=1}^{\infty}(t_{jk}+\dots+t_{jk}^{i_{jt}})y_k v_k\right| \le \|y.v\|_\infty\cdot i_{jt}$ (by Proposition 1(ii) of §2) $\le$

$\|y.v\|_\infty\cdot p_o \le \|y\|_\infty\cdot p\cdot p_o/p$ (by (1)) $< \|y\|_\infty\cdot p\cdot\frac{\varepsilon}{20\|y\|_\infty}$ (by (6)) $= \frac{1}{20}\,p\varepsilon$.

Hence

$$(18)'' \quad \left|\sum_{k=1}^{\infty}(t_{jk}+\dots+t_{jk}^{i_{jt}})y_k v_k\right| < \frac{5}{20}\,p\varepsilon.$$

From (15)", (18)", (16)", and (17)",

$$** \quad \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right| < \frac{18}{20}\,p\varepsilon < p\varepsilon.$$

Now let $p \le i_{jt}$. Then, from (14)" with $q = p$ ($> p_o$ by (6)),

$$*** \quad \left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right| < \frac{5}{20}\,p\varepsilon < p\varepsilon.$$

Thus it follows from *, **, and *** (* is at the end of Case 1) that

$$\left|\sum_{k=1}^{\infty}(t_{jk}+\ldots+t_{jk}^{p})y_k v_k\right| < p\varepsilon \quad \text{for} \quad p > \max\{20(\frac{p_o+n_r}{\varepsilon}),\ 20\|y\|_\infty(\frac{p_o+n_r}{\varepsilon})\} \text{ and } j \in \mathbb{N}.$$

This implies that

$$[(\frac{T+\ldots+T^p}{p})(y.v)]_j \to 0 \quad \text{as} \quad p \to \infty \quad \text{uniformly in } j .$$

Hence $y.v(= (Bx).v) \in Tac_o$ by Corollary 1 of 3.3, Theorem 1.

Now we show that $x.u \in (Tac_o)_B \setminus c_A$. Let $\varepsilon > 0$. Choose $m_o \in \mathbb{N}$ such that:

(7) $\dfrac{1}{2^{m_o-1}} < \dfrac{\varepsilon}{4\|x\|_\infty}$;

(8) $|\sin\sqrt{m} - \sin\sqrt{m-1}| < \min\{\dfrac{\varepsilon}{2\|A\|.\|x\|_\infty}, \dfrac{\varepsilon}{2\|B\|x\|_\infty}\}$ for $m \ge m_o$.

Let $j \ge n_{m_o}$. Then there exists $m(\ge m_o) \in \mathbb{N}$ such that $n_m \le j < n_{m+1}$. Now

(9) $|[B(x.u)]_j - [(Bx).v]_j|$

$$= |\sum_{k=1}^{\infty} b_{jk}x_k u_k - (\sum_{k=1}^{\infty} b_{jk}x_k).v_j|$$

$$= |\sum_{k=1}^{k_m} b_{jk}x_k u_k + \sum_{k=k_m+1}^{k_{m+1}-1} b_{jk}x_k u_k + \sum_{k=k_{m+1}}^{k_{m+2}-1} b_{jk}x_k u_k + \sum_{k=k_{m+2}}^{\infty} b_{jk}x_k u_k$$

$$- (\sum_{k=1}^{\infty} b_{jk}x_k).v_j|$$

$$= |\sum_{k=1}^{k_m} b_{jk}x_k u_k + \sin\sqrt{m} \sum_{k=k_m+1}^{k_{m+1}-1} b_{jk}x_k + \sin\sqrt{m+1} \sum_{k=k_{m+1}}^{k_{m+2}-1} b_{jk}x_k$$

$$+ \sum_{k=k_{m+2}}^{\infty} b_{jk}x_k u_k - (\sum_{k=1}^{\infty} b_{jk}x_k)\sin\sqrt{m}| \quad \text{(by the definition of } (u_k) \text{ and } (v_k))$$

$$= |\sum_{k=1}^{k_m} b_{jk}x_k(u_k - \sin\sqrt{m}) + (\sin\sqrt{m+1} - \sin\sqrt{m}) \sum_{k=k_{m+1}}^{k_{m+2}-1} b_{jk}x_k + \sum_{k=k_{m+2}}^{\infty} b_{jk}x_k (u_k - \sin\sqrt{m})|.$$

Since $\|u\| \leq 1$ (by (1)), $|u_k - \sin\sqrt{m}| \leq 2$ for $k \in \mathbb{N}$, and hence

(10) $$|\sum_{k=1}^{k_m} b_{jk}x_k(u_k - \sin\sqrt{m}) + \sum_{k=k_{m+2}}^{\infty} b_{jk}x_k(u_k - \sin\sqrt{m})|$$

$$\leq 2\|x\|_\infty (\sum_{k=1}^{k_m} |b_{jk}| + \sum_{k=k_{m+2}}^{\infty} |b_{jk}|)$$

$$< 2\|x\|_\infty \frac{1}{2^{m-1}} \quad \text{(by (i) since } n_m \leq j < n_{m+1})$$

$$\leq 2\|x\|_\infty \frac{1}{2^{m_o-1}} \quad \text{(since } m_o \leq m)$$

$$< 2\|x\|_\infty \frac{\varepsilon}{4\|x\|_\infty} \quad \text{(by (7))} = \frac{\varepsilon}{2}.$$

Also

$$(11)\quad \left|(\sin\sqrt{m+1} - \sin\sqrt{m}) \sum_{k=k_{m+1}}^{k_{m+2}-1} b_{jk}x_k\right|$$

$$\leq |\sin\sqrt{m+1} - \sin\sqrt{m}|.\|B\|.\|x\|_\infty$$

$$< \frac{\varepsilon}{2\|B\|\|x\|_\infty}\|B\|\|x\|_\infty \quad \text{(by (8) since } m \geq m_o)$$

$$= \frac{\varepsilon}{2}.$$

From (9), (10), and (11) we have $|[B(x.u)]_j - [(Bx).v]_j| < \varepsilon$ for $j \geq n_{m_o}$. Hence $\lim_j |[B(x.u)]_j - [(Bx).v]_j| = 0$ so that $B(x.u) - (Bx).v \in c_o$ ($\subseteq Tac_o$). Since $(Bx).v \in Tac_o$, $B(x.u) \in Tac_o$ and hence $x.u \in (Tac_o)_B$.

Replacing B by A, we can similarly show that $A(x.u) - (Ax).v \in c_o$. Since $\lim_A x = 1$ and (v_k) oscillates between 1 and -1, $A(x.u)$ oscillates between 1 and -1, and hence $x.u \notin c_A$.

We now establish our first consistency theorem for T-almost convergent sequences.

<u>THEOREM 2</u>. Let $T = (t_{jk})$ and $S = (s_{jk})$ be lifting matrices and let $A = (a_{jk})$ and $B = (b_{jk})$ be regular matrices. Suppose $(Sac)_B \cap m \subseteq (Tac)_A$. Then $S\text{-}\lim_B x = T\text{-}\lim_A x$ for $x \in (Sac)_B \cap m$.

Proof. Let $T = (t_{jk})$ and $S = (s_{jk})$ be lifting matrices and let $A = (a_{jk})$ and $B = (b_{jk})$ be regular matrices. Suppose $(Sac)_B \cap m \subseteq (Tac)_A$. First we show that $(Sac_o)_B \cap m \subseteq (Tac_o)_A$. Let $x \in (Sac_o)_B \cap m$. Then $x \in (Tac)_A$ and hence $Ax \in Tac$. Let

T-LimAx = $\alpha(x)$. Then, by Corollary 1 of 3.3, Theorem 1,

$\lim_p \left(\frac{T(Ax)+\ldots+T^p(Ax)}{p}\right)_j = \alpha(x)$ uniformly in j. This is equivalent to

$\lim_p \left[\left(\frac{TA+\ldots+T^pA}{p}\right)x\right]_j = \alpha(x)$ uniformly in j by 1.5, Theorem 3(iii). In

particular, $\lim_p \left[\left(\frac{TA+\ldots+T^pA}{p}\right)x\right]_1 = \alpha(x)$. i.e.,

$\lim_p \sum_{i=1}^{\infty} \left(\frac{TA+\ldots+T^pA}{p}\right)_{1i} x_i = \alpha(x)$. Since this is true for every

$x \in (Sac_o)_B \cap m$ and $[(Sac_o)_B \cap m]^\beta = \ell_1$, $\left[\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right]_{i=1}^{\infty} \in \ell_1$ for

each p and, moreover, the sequence $\left[\left(\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right)_{i=1}^{\infty}\right]_{p=1}^{\infty}$ in ℓ_1 is

$\sigma(\ell_1,(Sac_o)_B \cap m)$-Cauchy. Since ℓ_1 is $\sigma(\ell_1,(Sac_o)_B \cap m)$-sequentially

complete by Theorem 1, $\left[\left(\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right)_{i=1}^{\infty}\right]_{p=1}^{\infty}$ is $\sigma(\ell_1,(Sac_o)_B \cap m)$-convergent to a member (y_k) in ℓ_1. To show that $(y_k) = 0$, it is sufficient

to show that $\left[\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right]_{i=1}^{\infty} \to 0$ point-wise as $p \to \infty$. Let $i \in \mathbb{N}$

and $\varepsilon > 0$. Since $A = (a_{jk})$ is regular, $\lim_k a_{ki} = 0$ and hence there

exists $k_o \in \mathbb{N}$ such that

(1) $|a_{ki}| < \frac{\varepsilon}{2}$ for $k \geq k_o$.

For $p \geq \frac{2\|A\|k_o}{\varepsilon}$,

$$\left|\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right|$$

$$= \left|\sum_{k=1}^{\infty} t_{1k}a_{ki}+\ldots+\sum_{k=1}^{\infty} t^p_{1k}a_{ki}\right|/p$$

$$= \left|\sum_{k=2}^{\infty} t_{1k}a_{ki}+\ldots+\sum_{k=k_o}^{\infty} t^{k_o-1}_{1k}a_{ki}+\ldots+\sum_{k=p+1}^{\infty} t^p_{1k}a_{ki}\right|/p \quad \text{(by Proposition 1(i) of §2)}$$

$$\leq \left|\sum_{k=2}^{\infty} t_{1k}a_{ki}+\ldots+\sum_{k=k_o}^{\infty} t^{k_o-1}_{1k}a_{ki}\right|/p + \left|\sum_{k=k_o+1}^{\infty} t^{k_o}_{1k}a_{ki}+\ldots+\sum_{k=p+1}^{\infty} t^p_{1k}a_{ki}\right|/p$$

$$\leq \sup_k|a_{ki}|\left(\sum_{k=2}^{\infty} t_{1k}+\ldots+\sum_{k=k_o}^{\infty} t^{k_o-1}_{1k}\right)/p + \sup_{k>k_o}|a_{ki}|\left(\sum_{k=k_o+1}^{\infty} t^{k_o}_{1k}+\ldots+\sum_{k=p+1}^{\infty} t^p_{1k}\right)/p$$

$$\leq \|A\|\frac{k_o}{p} + \frac{\varepsilon}{2}\cdot\frac{p-k_o+1}{p} \quad \text{(by Proposition 2(ii) of §2, and by (1))}$$

$$< \|A\|\frac{k_o}{p} + \frac{\varepsilon}{2}$$

$$\leq \|A\|\cdot\frac{\varepsilon}{2\|A\|} + \frac{\varepsilon}{2} \quad \left(\text{since } p \geq \frac{2\|A\|k_o}{\varepsilon}\right) = \varepsilon .$$

Hence $\left[\left(\frac{TA+\ldots+T^pA}{p}\right)_{1i}\right]_{i=1}^{\infty} \to 0$ pointwise as $p \to \infty$. Thus $(y_k) = 0$.

This implies that $\alpha(x) = 0$, and hence $Ax \in Tac_o$ so that $x \in (Tac_o)_A$.

Therefore, $(Sac_o)_B \cap m \subseteq (Tac_o)_A$.

Now let $x \in (Sac)_B \cap m$. Then $x - (S\text{-}\lim_B x)e \in (Sac_o)_B \cap m \subseteq (Tac_o)_A$, and hence $T\text{-}\lim_A(x-(S\text{-}\lim_B x)e) = 0$. i.e., $T\text{-}\lim_A x = S\text{-}\lim_B x$.

The following corollary is a statement, analogous to the original bounded consistency theorem, for T-almost convergent sequences.

COROLLARY 1. Let T be a lifting matrix, and let A and B be regular matrices. Suppose $(Tac)_B \cap m \subseteq c_A$. Then $\lim_A x = T\text{-}\lim_B x$ for $x \in (Tac)_B \cap m$.

Proof. Suppose $(\mathrm{Tac})_B \cap m \subseteq c_A$. Since $c_A \subseteq (\mathrm{Tac})_A$, $(\mathrm{Tac})_B \cap m \subseteq (\mathrm{Tac})_A$, and hence it follows from Theorem 2 that $T\text{-}\operatorname{Lim}_A x = T\text{-}\operatorname{Lim}_B x$ for $x \in (\mathrm{Tac})_B \cap m$. But $\lim_A x = T\text{-}\operatorname{Lim}_A x$ for $x \in c_A$, and hence $\lim_A x = T\text{-}\operatorname{Lim}_B x$ for $x \in (\mathrm{Tac})_B \cap m$.

When $T_o = t_{jk}$ is given by $t_{jk} = \begin{cases} 1 \text{ if } k = j+1 \\ 0 \text{ otherwise} \end{cases}$, Corollary 1 reduces to the following, which was first obtained by Bennett and Kalton [4].

<u>COROLLARY 2</u>. Let A and B regular matrices and suppose $(ac)_B \cap m \subseteq c_A$. Then $\lim_A x = T_o - \operatorname{Lim}_B x$ for $x \in (ac)_B \cap m$.

Before stating our next result, let us recall the following notation. If E is an FK-space containing φ, then we write

$$W_E = \{x \in E \mid P_n x \to x \text{ weakly in } E\} .$$

<u>THEOREM 3</u>. Let T be a lifting matrix, and let $B = (b_{jk})$ be an infinite matrix such that $\|B\| < \infty$ and such that every column of B belongs to c_o. Suppose E is an FK-space containing c_o. Then ℓ_1 is $\sigma(\ell_1, (\mathrm{Tac}_o)_B \cap (W_E \cap m))$-sequentially complete.

Proof. Let $B = (b_{jk})$ be an infinite matrix such that $\|B\| < \infty$ and such that every column of B belongs to c_o, and let E be an FK-space containing c_o. Suppose $A = (a_{jk})$ is an infinite matrix with the same properties as B such that $(\mathrm{Tac}_o)_B \cap (W_E \cap m) \subseteq c_A$. Since $c_o \subseteq (\mathrm{Tac}_o)_B \cap (W_E \cap m)$, $[(\mathrm{Tac}_o)_B \cap (W_E \cap m)]^\beta = \ell_1$, and hence, as in the proof of Theorem 1, it suffices to prove that $(\mathrm{Tac}_o)_B \cap (W_E \cap m) \subseteq c_{o_A}$.

Suppose there exists $x = (x_k) \in (Tac_o)_B \cap (W_E \cap m)$ such that $\lim_A x \neq 0$. We may assume that $\lim_A x = 1$. As in the proof of Theorem 1, we construct a bounded sequence $u = (u_k)$ such that $u.x \in (Tac_o)_B \cap (W_E \cap m) \setminus c_A$. This leads to a contradiction, since $(Tac_o)_B \cap (W_E \cap m) \subseteq c_A$. In constructing (u_k) we only change the choice of (k_r) in the proof of Theorem 1 such that:

(a) the change does not affect the proof of $u.x \in (Tac_o)_B \setminus c_A$;

(b) $u.x \in (W_E \cap m)$.

Now we state this modification of the choice of (k_r).

Let (p_n) be an increasing sequence of seminorms which generates the FK-topology on E. Since $c_o \subseteq E$, the uniform norm topology on c_o is finer than the FK-topology on E restricted to c_o. Thus we may assume that

(1) $p_n(y) \leq \|y\|_\infty$ for $n \in \mathbb{N}$ and $y \in c_o$.

Since $x \in W_E \cap m$, x belong to the weak closure (in E) of the convex hull $P(x)$ of the set $\{P_n x | n \in \mathbb{N}\}$. It follows from 1.3, Proposition 1 that the closure of $P(x)$ in E with respect to the FK-topology coincides with the weak closure of $P(x)$ in E. Hence there exists a sequence (x^t) in φ such that:

(2) $\|x^t\|_\infty \leq \|x\|_\infty$ for $t \in \mathbb{N}$;

(3) $x^t \to x$ in E with respect to the FK-topology (hence (x^t) is Cauchy in E with respect to the FK-topology).

It follows from (3) that we can choose $t_1 \in \mathbb{N}$ such that

$$(\alpha_1) \quad p_1(x^t - x^s) < \frac{1}{2^2} \quad \text{for } t,s \geq t_1 .$$

Choose $k_1 \in \mathbb{N}$ such that:

$$(\gamma_1) \quad x_k^{t_1} = 0 \quad \text{for } k \geq k_1 .$$

Now we choose n_1 and $(i_{j1})_{j=1}^{\infty}$ as same as in the proof of Theorem 1.

(3) implies that (x^t) is pointwise Cauchy, and hence it follows from (3) that we can choose $t_2\ (> t_1) \in \mathbb{N}$ such that:

$$(\alpha_2) \quad p_2(x^t - x^s) < \frac{1}{2^3} \quad \text{for } t,s \geq t_2 ;$$

$$(\beta_2) \quad \sum_{k=1}^{k_1} |x_k^t - x_k^s| < \frac{1}{2^3} \quad \text{for } t,s \geq t_2 .$$

Now we choose $k_2 (> k_1) \in \mathbb{N}$ such that:

$$(\gamma_2) \quad x_k^{t_2} = 0 \quad \text{for } k \geq k_2 ;$$

$$(a_2) \quad \sum_{k=k_2}^{\infty} (|a_{jk}| + |b_{jk}| < 1 \quad \text{for } j \leq n_1 .$$

We choose n_2 and $(i_{j2})_{j=1}^{\infty}$ as same as in the proof of Theorem 1.

We proceed to construct strictly increasing sequences (t_r), (k_r), and (n_r) of positive integers and increasing sequences $(i_{jr})_{r=1}^{\infty}$, $j = 1,2,\ldots$ of non-negative integers. These sequences, in addition to conditions (i) to (vi) in the proof of the Theorem 1, satisfy the following conditions.

(vii) $p_r(x^t - x^s) < \frac{1}{2^{r+1}}$ for $s,t \geq t_r$, $r = 1,2,\ldots$ (see (α_1) and (α_2));

(viii) $\sum_{k=1}^{k_{r-1}} |x_k^t - x_k^s| < \frac{1}{2^{r+1}}$ for $s,t \geq t_r$, $r = 2,3,\ldots$, (see (β_2));

(ix) $x_k^{t_r} = 0$ for $k \geq k_r$, $r = 1,2,\ldots$ (see (γ_1) and (γ_2)).

Define bounded sequences $u = (u_j)$ and $v = (v_j)$ as same as in the proof of Theorem 1. i.e., $u_j = \sin\sqrt{r}$ if $k_r \leq j < k_{r+1}$ and $v_j = \sin\sqrt{r}$ if $n_r \leq j < n_{r+1}$. Now we show that $(x^{t_r}.u)_{r=1}^{\infty}$ is Cauchy in E with respect to the FK-topology. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Choose m ($> n$) such that:

(4) $\sum_{k=m}^{\infty} \frac{1}{2^k} < \varepsilon/3$;

(5) $|\sin\sqrt{p} - \sin\sqrt{p-1}| < \frac{\varepsilon}{3\|x\|_\infty}$ for $p \geq m$.

Now, for $q > p > m$,

(6) $u.x^{t_p} - u.x^{t_q} = \sum_{r=p}^{q-1} (u.x^{t_r} - u.x^{t_{r+1}})$.

For $p \leq r < q$, $x_k^{t_r} = 0$ for $k \geq k_r$ and $x_k^{t_{r+1}} = 0$ for $k \geq k_{r+1}$ by (ix), and hence

(7) $u.x^{t_r} - u.x^{t_{r+1}} = (u.x^{t_r} - u.x^{t_{r+1}}).(\chi_{[1,k_{r-1}]} + \chi_{(k_{r-1},k_r)}) - u.x^{t_{r+1}}.\chi_{[k_r,k_{r+1})}$

$= (u.x^{t_r} - u.x^{t_{r+1}}).\chi_{[1,k_{r-1}]} + \sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{(k_{r-1},k_r)}$

$- \sin\sqrt{r}\, x^{t_{r+1}}.\chi_{[k_r,k_{r+1})}$ (by the definition of (u_k))

$$= (u.x^{t_r} - u.x^{t_{r+1}}).\chi_{[1,k_{r-1}]} + \sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{(k_{r-1},k_{r+1})}$$

$$- \sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{[k_r,k_{r+1})} - \sin\sqrt{r}\, x^{t_{r+1}}.\chi_{[k_r,k_{r+1})}$$

$$= (u.x^{t_r} - u.x^{t_{r+1}}).\chi_{[1,k_{r-1}]} + \sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{(k_{r-1},k_{r+1})}$$

$$+ (\sin\sqrt{r-1} - \sin\sqrt{r})x^{t_{r+1}}.\chi_{[k_r,k_{r+1})} \quad \text{(since}$$

$$x_k^{t_r} = 0 \quad \text{for} \quad k \geq k_r).$$

For $y \in \varphi$, by (1),

$$(8) \quad p_n(y) \leq \|y\|_\infty \leq \|y\|_1 .$$

Hence, for $(m <)\ p \leq r < q$,

$$(9) \quad p_n[(u.x^{t_r} - u.x^{t_{r+1}}).\chi_{[1,k_{r-1}]}] \leq \sum_{k=1}^{k_{r-1}} |u_k(x_k^{t_r} - x_k^{t_{r+1}})|$$

$$\leq \sum_{k=1}^{k_{r-1}} |x_k^{t_r} - x_k^{t_{r+1}}| \quad (\text{since } \|u\| \leq 1)$$

$$< \frac{1}{2^{r+1}} \quad \text{by (viii)}.$$

For $(m <)\ p \leq r < q$,

$$(10) \quad p_n[(\sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{(k_{r-1},k_{r+1})}]$$

$$\leq p_n[(x^{t_r} - x^{t_{r+1}}).(\chi_{[1,k_{r+1})} - \chi_{[1,k_{r-1}]})] \quad (\text{since } |\sin\sqrt{r-1}| \leq 1)$$

$$\le p_n[(x^{t_r}-x^{t_{r+1}})\cdot\chi_{[1,k_{r+1})}] + p_n[(x^{t_r}-x^{t_{r+1}})\cdot\chi_{[1,k_{r-1}]}]$$

$$\le p_r[x^{t_r}-x^{t_{r+1}})] \quad (\text{since } r \ge p > m > n \text{ and } x_k^{t_r}, x_k^{t_{r+1}} = 0 \text{ for } k \ge k_{r+1}$$

$$\text{by (ix)}) + \sum_{k=1}^{r-1} |x_k^{t_r}-x_k^{t_{r+1}}| \quad (\text{by (8)})$$

$$< \frac{1}{2^{r+1}} + \frac{1}{2^{r+1}} \text{ (by (vii) and (viii))} = \frac{1}{2^r}.$$

Also,

(11) $$p_n\Big(\sum_{r=p}^{q-1} (\sin\sqrt{r-1} - \sin\sqrt{r})x^{t_{r+1}}\cdot\chi_{[k_r,k_{r+1})}\Big)$$

$$\le \Big\|\sum_{r=p}^{q-1} (\sin\sqrt{r-1} - \sin\sqrt{r})x^{t_{r+1}}\cdot\chi_{[k_r,k_{r+1})}\Big\|_\infty \quad (\text{by (1)})$$

$$\le \sup_{p\le r<q} |\sin\sqrt{r-1} - \sin\sqrt{r}| \; \|x^{t_{r+1}}\|_\infty$$

$$\le \frac{\varepsilon}{3\|x\|_\infty} \cdot \|x\|_\infty = \frac{\varepsilon}{3} \text{ by (5) and (2) (since } p > m).$$

From (6) and (7), for $q > p > m$,

$$p_n(u.x^{t_p}-u.x^{t_q})$$

$$= p_n\Big(\sum_{r=p}^{q-1} [(u.x^{t_r}-u.x^{t_{r+1}})\cdot\chi_{[1,k_{r-1}]} + \sin\sqrt{r-1}(x^{t_r}-x^{t_{r+1}})\cdot\chi_{(k_{r-1},k_{r+1})}$$

$$+ (\sin\sqrt{r-1} - \sin\sqrt{r})x^{t_{r+1}}\cdot\chi_{[k_r,k_{r+1})}]$$

$$\leq \sum_{r=p}^{q-1} (p_n[(u.x^{t_r} - u.x^{t_{r+1}}).\chi_{[1,k_{r-1}]}] + p_n[(\sin\sqrt{r-1}(x^{t_r} - x^{t_{r+1}}).\chi_{(k_{r-1},k_{r+1})}])$$

$$+ \; p_n(\sum_{r=p}^{q-1} (\sin\sqrt{r-1} - \sin\sqrt{r})x^{t_{r+1}}.\chi_{[k_r,k_{r+1})})$$

$$< \sum_{r=p}^{q-1} (\frac{1}{2^{r+1}} + \frac{1}{2^r}) + \frac{\varepsilon}{3} \quad \text{(by (9), (10), and (11))}$$

$$< 2 \sum_{r=m}^{\infty} \frac{1}{2^r} + \frac{\varepsilon}{3} \quad \text{(since } p > m)$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{(by (4))}.$$

Hence $(u.x^{t_r})$ is Cauchy in E with respect to the FK-topology, and thus converges to $u.x$ since $(u.x^{t_r})$ pointwise converges to $u.x$ (by (3)).

To show that $u.x \in W_E$, let $f \in E'$. Then it follows from 1.4, Theorem 5 that $(f(e^k)) \in \ell_1$ since $c_o \subseteq E$. For convenience, let us write $u.x^{t_r} = y^r$ for each r. Then $\lim_r y^r = u.x$ in E with respect to the FK-topology, and hence

$$(12) \quad f(u.x) = \lim_r f(y^r) = \lim_r \sum_{k=1}^{\infty} f(e^k)y_k^r \quad \text{(since } y^r \in \varphi \text{ for each } r).$$

Now we show that $f(u.x) = \sum_{k=1}^{\infty} f(e^k)u_k x_k$. Let $\varepsilon > 0$. Since $(f(e^k)) \in \ell_1$, there exists $n \in \mathbb{N}$ such that

$$(13) \quad \sum_{k=n}^{\infty} |f(e^k)| < \frac{\varepsilon}{4\|x\|_\infty}.$$

Since (y^r) is point-wise convergent to $u.x$, we can choose $r_o \in \mathbb{N}$ such that

$$(14) \quad \sum_{k=1}^{n-1} |f(e^k)(y_k^r - u_k x_k)| < \frac{\varepsilon}{2} \quad \text{for } r \geq r_o .$$

For $r \in \mathbb{N}$, since $y^r = u.x^{t_r}$, $\|y^r\|_\infty \le \|x^{t_r}\|_\infty \le \|x\|_\infty$ (by (2), and since $\|u\| \le 1$), and hence

(15) $|y_k^r - u_k x_k| \le |y_k^r| + |u_k x_k| \le 2\|x\|_\infty$ for $k, r \in \mathbb{N}$.

Now, for $r \ge r_o$,

$$\left|\sum_{k=1}^{\infty} f(e^k)(y_k^r - u_k x_k)\right| \le \sum_{k=1}^{n-1} |f(e^k)(y_k^r - u_k x_k)| + \sum_{k=n}^{\infty} |f(e^k)||y_k^r - u_k x_k|$$

$$< \frac{\varepsilon}{2} + 2\|x\|_\infty \cdot \frac{\varepsilon}{4\|x\|_\infty} \quad \text{(by (14), (15), and (13))}$$

$$= \varepsilon .$$

Hence $\lim_r \sum_{k=1}^{\infty} f(e^k) y_k^r = \sum_{k=1}^{\infty} f(e^k) u_k x_k$ and thus, by (12),

$f(u.x) = \sum_{k=1}^{\infty} f(e^k) u_k x_k$. Therefore, $u.x \in W_E$.

Now using the same proof of Theorem 1, we can show that $u.x \in (Tac_o)_B \setminus c_A$.

When $T_o = (t_{jk})$ is given by $t_{jk} = \begin{cases} 1 \text{ if } k = j+1 \\ 0 \text{ otherwise} \end{cases}$, Theorem 3 reduces to the following.

<u>COROLLARY 1</u>. Let B be an infinite matrix such that $\|B\| < \infty$ and such that each column of B belong to c_o . Suppose E is an FK-space containing c_o . Then ℓ_1 is $\sigma(\ell_1, (ac_o)_B \cap (W_E \cap m))$-sequentially complete.

When $B = I$, the Corollary 1 reduces to the following, which was first obtained by Bennett and Kalton [4].

COROLLARY 2. If E is an FK-space containing c_o, then ℓ_1 is $\sigma(\ell_1,(ac_o) \cap W_E)$-sequentially complete.

Now we establish the original bounded consistency theorem.

COROLLARY 2. (The bounded consistency theorem [9]).
Let A and B regular matrices, and suppose $c_B \cap m \subseteq c_A$. Then $\lim_A x = \lim_B x$ for every $x \in c_B \cap m$.

Proof. Let A and B regular matrices, and suppose $c_B \cap m \subseteq c_A$. Letting $E = c_B$, it follows from Corollary 1 that ℓ_1 is $\sigma(\ell_1,(ac_o)_B \cap (W_B \cap m))$-sequentially complete. Since $W_B \cap m = c_{o_B} \cap m$ (by 1.5, Theorem 2), $(ac_o)_B \cap (W_B \cap m) = c_{o_B} \cap m$ and hence ℓ_1 is $\sigma(\ell_1,c_{o_B} \cap m)$-complete. Since $c_{o_B} \cap m \subseteq c_A$, it follows from 2.3, Theorem 2 that $c_{o_B} \cap m \subseteq c_{o_A}$. Now let $x \in c_B \cap m$. Then

$x-(\lim_B x)e \in c_{o_B} \cap m \subseteq c_{o_A}$, and hence $\lim_A(x-(\lim_B x)e) = 0$.
i.e., $\lim_A x = \lim_B x$.

Finally we show that Theorem 3 is still true if we replace Tac_o by c_o.

THEOREM 4. Let $B = (b_{jk})$ be an infinite matrix such that $\|B\| \leq \infty$ and such that every column of B belongs to c_o. Suppose E is an FK-space containing c_o. Then ℓ_1 is $\sigma(\ell_1,c_{o_B} \cap W_E \cap m)$ sequentially complete.

Proof. Let $B = (b_{jk})$ be an infinite matrix such that $\|B\| < \infty$ and such that every column of B belongs to c_o, and let E be an

FK-space containing c_o. Suppose $A = (a_{jk})$ is an infinite matrix with the same properties as B such that $c_{o_B} \cap W_E \cap m \subseteq c_A$. Since $c_o \subseteq c_{o_B} \cap W_E \cap m$, $(c_o \cap W_E \cap m)^\beta = \ell_1$, and hence, as in the proof of Theorem 1, it suffices to prove that $c_{o_B} \cap W_E \cap m \subseteq c_{o_A}$.

Suppose there exists $x = (x_k) \in c_{o_B} \cap W_E \cap m$ such that $\lim_A x \neq 0$. We may assume that $\lim_A x = 1$. As in the proof of Theorem 3, we construct a bounded sequence $u = (u_k)$ such that $u.x \in (c_{o_B} \cap W_E \cap m) \setminus c_A$. This leads to a contradiction, since $c_{o_B} \cap W_E \cap m \subseteq c_A$.

As same as in the proof of Theorem 3, let (p_n) be an increasing sequence of seminorms which generates the FK-topology on E and (x^t) a sequence in φ such that

(1) $p_n(y) \leq \|y\|$ for $n \in N$ and $y \in c_o$;

(2) $\|x^t\|_\infty \leq \|x\|_\infty$;

(3) $x^t \to x$ in E with respect to the FK-topology.

Now, similar to the proof of Theorem 3, we can inductively construct strictly increasing sequences (t_r), (k_r), and (n_r) such that:

(i) $$\max_{n_r \leq j \leq n_{r+1}} \left[\sum_{k=1}^{k_r} (|a_{jk}| + |b_{jk}|) + \sum_{k=k_{r+1}}^{\infty} (|a_{jk}| + |b_{jk}|) = \frac{1}{2^{r-1}} \right.$$

(ii) $p_r(x^t - x^s) < \frac{1}{2^{r+1}}$ for $s,t \geq t_r$, $r = 1,2,\ldots$;

(iii) $\sum_{k=1}^{k_{r-1}} |x_k^t - x_k^s| < \frac{1}{2^{r+1}}$ for $s,t \geq t_r$, $r = 2,3,\ldots$;

(iv) $x_k^{t_r} = 0$ for $k \geq k_r$, $r = 1,2,\ldots$.

Define bounded sequences $u = (u_j)$ and $v = (v_j)$ as same as in the proof of Theorem 3, i.e., $u_j = \sin\sqrt{r}$ if $k_r \leq j < k_{r+1}$ and $v_j = \sin\sqrt{r}$ if $n_r \leq j < n_{r+1}$. Now as same as in the proof of Theorem 3 we can show that $u.x \in W_E \cap m$.

Since $x \in c_{o_B} \cap W \cap m$, $Bx \in c_o$ and hence $(Bx).v \in c_o$. Now as same as in the last part (from (7) to the end) of Theorem 1, we can show that $B(x,u) - (B.x).v \in c_o$ (hence $B(x.u) \in c_o$) and $x.u \notin c_A$. Therefore, $x.u \in c_{o_B} \setminus c_A$.

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