FIXED POINT THEOREMS IN METRIC SPACES

by

Andrew F. Sound

B.S., California Lutheran College, 1982

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

September, 1984

APPROVAL

Name: Andrew F. Sound

Degree: Master of Science

Title of Thesis: Fixed Point Theorems in Metric Spaces

Examining Committee:

Chairman: A.R. Freedman

C.Y. Shen
Senior Supervisor

C. Kim

J.J. Sember

A. Das
External Examiner

Date Approved: September 13, 1984

ABSTRACT

Let T be a mapping of a metric space X into itself. We call x a fixed point of T if $Tx=x$. The central question of this thesis is what conditions on T or X will guarantee that T has a fixed point.

The most famous result of fixed-point theory is that of Banach. Different ways to prove Banach's theorem and its various generalisations are presented. In most cases, two kinds of proof are shown: the original and a newer, simpler method.

Brouwer's and Schauder's theorems are also discussed, along with some miscellaneous results, including common fixed points for a sequence of mappings and a converse to Banach's theorem.

TABLE OF CONTENTS

I. Introduction

Fixed point theorems have been around for some years. The earliest one we discuss in this thesis is Brouwer's theorem, which Poincare proved in an equivalent form in 1886 [24]. The most famous result is Banach's fixed point theorem, published in 1922 [4].

Banach's theorem deals with mappings of a metric space into itself that shrink distances under a "Lipschitz condition". In other words, $T:X \dashrightarrow X$ is such that, for some k, $0 \le k < 1$,

$$(1) \qquad d(Tx,Ty) \le kd(x,y)$$

for all x and y in X. If X is complete, says Banach, T will have a fixed point in X. In chapter 2, we give the classical proof of this, together with a couple of more recent proofs that involve the use of Cantor's intersection theorem.

Chapters 3 and 4 deal with various generalisations of Banach's theorem. In the first of these, we remove the Lipschitz condition, that is, we consider $T:X \dashrightarrow X$ where T satisfies

$$(2) \qquad d(Tx,Ty) < d(x,y)$$

for all $x,y \in X$, $x \ne y$. Under compactness of X (or a weaker condition on T), T will have a fixed point in X. We also consider mappings for which, for some $\varepsilon>0$, $0<d(x,y)<\varepsilon$ implies (2), as well as the case where (2) is true for some iteration T^n of T, n possibly depending on x and y.

A good deal of work is being done with non-expansive mappings, i.e. those where

$$d(Tx,Ty) \leq d(x,y).$$

Non-expansive mappings are generally considered in subsets K of Banach spaces with $T(K) \subseteq K$. In Chapter 4, we show that if K is a weakly compact, convex set that has a property called "normal structure", then T has a fixed point in K.

The main thrust of Chapter 5 is an alternate method of dealing with the types of mappings in Chapters 3 and 4. This method, due to J.S.W. Wong, involves considering a new function

$$\phi(x) = d(x,Tx).$$

In compact X and under suitable conditions on T, we can show that ϕ reaches its minimum on X and this usually gives us the fixed point of T. The proofs of this kind are generally much simpler than those considered in chapters 3 and 4.

Chapter 6 is mainly about Schauder's theorem, which says that if C is a compact, convex subset of a Banach space, and T:C-->C is continuous, then T has a fixed point in C. We first consider Brouwer's theorem, which is really Schauder's theorem restricted to finite-dimensional vector spaces. Then we use a sequence of mappings that converge uniformly to T, each mapping being a mapping of a finite-dimensional subspace into itself. This will accomplish the proof of Schauder's theorem.

Chapter 7 deals briefly with two interesting aspects of fixed point theory not yet covered. One result is a generalisation of Banach's theorem for a sequence of mappings, only some of which satisfy (1). The other is a partial converse of Banach's theorem.

Some Notation and Definitions

A mapping T is called a self mapping of X if $T:X \to X$. If $C \subseteq X$, and $T(C) \subseteq C$, then we call C T-invariant. A fixed point, of course, is a point x such that $Tx=x$. We call T^n an iteration of T, where T^n is defined inductively as $T^1=T$, and $T^n=T(T^{n-1})$. We assume T^0 is the identity. This definition will apply everywhere except in one part of Chapter 7, where another definition is given. The weak topology on a normed space X is the weakest topology that makes all the bounded linear functionals on X continuous.

We will use cl(A) to denote the closure of a set A, and diam(A) to denote its diameter. ($\text{diam}(A)=\sup\{d(x,y): x,y \in A\}$). If x and y are two points in a vector space, then we will call the segment between them seg(x,y). R will mean the set of real numbers, and R^n the vector space that is the product of n copies of R.

II. Banach's Fixed Point Theorem

The classical fixed-point theorem is due to S. Banach [4]. It deals with a class of mappings called contractions.

<u>DEFINITION</u> 1: Let (X,d) be a metric space. A mapping T:X-->X is called a <u>contraction</u> if there exists a constant k such that

$$0 \leq k < 1, \text{ and}$$

$$\text{for every } x,y \text{ in } X,\ d(Tx,Ty) \leq kd(x,y).$$

The second condition is sometimes called a <u>Lipschitz</u> condition, and we will call k a <u>Lipschitz constant</u>.

As it turns out, in the context of a complete metric space, each contraction mapping has a fixed point.

<u>THEOREM</u> 2.1(Banach): If (X,d) is a complete metric space, and T:X-->X a contraction mapping, then T has a unique fixed point in X.

<u>PROOF</u>: Define the sequence $\{x_n\}$ inductively as $x_n=T^n x_0$ with x_0 an arbitrary element of X. Clearly, for each positive integer n,

$$d(x_{n+1},x_n) \leq k^n d(x_1,x_0).$$

Looking at $d(x_m,x_n)$, for $m \geq n$, we see by the triangle inequality and by the formula for the sum of a geometric series:

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n)$$

$$\le k^{m-1} d(x_1, x_0) + k^{m-2} d(x_1, x_0) + \dots + k^n d(x_1, x_0)$$

$$\le \{k^n/(1-k)\} d(x_1, x_0).$$

Thus $\{x_n\}$ is a Cauchy sequence, and by the completeness of X, it converges to a point z in X.

Now consider Tz. For $n \geq 1$,

$$d(x_{n+1}, Tz) \le k\ d(x_n, z) < d(x_n, z),$$

which means $d(x_n, Tz) \to 0$ as $n \to \infty$, so z=Tz and z is a fixed point of T.

It is easy to show that any fixed point of a contraction is unique. Suppose there were two fixed points, z_1 and z_2. Then

$$d(z_1, z_2) = d(Tz_1, Tz_2) \le k\ d(z_1, z_2).$$

Since $k<1$, $d(z_1, z_2) \le k\ d(z_1, z_2)$ only if $z_1 = z_2$.□

This proof leads to an immediate corollary of Theorem 2.1, which has practical applications as regards actually calculating the fixed point of a contraction.

<u>COROLLARY</u> 2.1.1: Let (X,d) be a complete metric space, T:X-->X a contraction mapping, and z its fixed point. Then for any x_0 in X, the sequence $\{x_n\}$, where $x_n = T^n x_0$, converges to the fixed point z. Furthermore, for any n,

$$d(x_n, z) \le \{k^n/(1-k)\} d(x_0, Tx_0).$$

<u>PROOF</u>: Convergence of $\{x_n\}$ has already been established.

The distance criterion is established as follows:

$$d(x_n,z) = d(T^n x_0, T^n z) \le k^n d(x_0,z).$$

Furthermore,

$$d(x_0,z) \le d(x_0,Tx_0) + d(Tx_0,z) \le d(x_0,Tx_0) + k\, d(x_0,z)$$

So a little algebra gives us

$$d(x_0,z) \le d(x_0,Tx_0)/(1-k).$$

Thus, combining these two results, we get the required inequality.□

There is another method of proving Theorem 2.1, which D.W. Boyd and J.S.W. Wong published in 1969 [7]. It involves the use of Cantor's intersection theorem, which we state here. See Goldberg, [16], p.158 for a proof.

<u>THEOREM</u> 2.2 (Cantor): Let (X,d) be a complete metric space, and for each positive integer n, let F_n be a non-empty, closed and bounded subset of X, such that

(1) $F_{n+1} \subseteq F_n$, for each n, and

(2) $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$.

Then $\bigcap_1^\infty F_n$ contains exactly one point.

The basic idea of the proof using Cantor's theorem is to define a new function $\phi(x)=d(x,Tx)$, which has the property that $\phi(x)=0$ if and only if x is a fixed point of T.

<u>Alternate Proof of Theorem</u> 2.1 (Boyd & Wong, 7): Define a new function $\phi(x)=d(x,Tx)$, for $x \in X$. This is continuous, since

$$\phi(x) - \phi(a) = d(x,Tx) - d(a,Ta)$$

$$\le d(x,a) + d(a,Ta) + d(Ta,Tx) - d(a,Ta)$$

$$\le d(x,a) + kd(x,a)$$

$$= (1+k)\ d(x,a),$$

and similarly for $\phi(a)-\phi(x)$, so $|\phi(x)-\phi(a)| \le (1+k)d(x,a)$, and ϕ is continuous.

Furthermore, for any $x \in X$, $\phi(T^n x) \le k^n \phi(x)$, so $\phi(T^n x) \to 0$ as $n \to \infty$. Now, for each positive integer m, define a set $C_m = \{x \in X: \phi(x) \le 1/m\}$. By continuity of ϕ, these sets are all closed, and since $\phi(T^n x) \to 0$, each set is also non-empty. We obtain boundedness from the following estimate of the diameter of each C_m:

Let $x, y \in C_m$. Then

$$d(x,y) \le d(x,Tx) + d(Tx,Ty) + d(Ty,y)$$

$$\le \phi(x) + kd(x,y) + \phi(y)$$

$$\le (2/m) + kd(x,y)$$

So $d(x,y) \le 2/\{m(1-k)\}$ for each $x,y \in C_m$, and C_m is bounded, with $\mathrm{diam}(C_m) \to 0$ as $m \to \infty$. Thus the sets $\{C_m\}$ satisfy the conditions of Theorem 2.2, and $\bigcap_1^\infty C_m$ contains precisely one point, which we call z. Clearly, z is the unique fixed point of T, since

$$z \in \bigcap C_m \iff \phi(z)=0 \iff Tz=z.$$

$\square$

Of course, the proof of Cantor's Theorem requires the use of Cauchy sequences as much as Banach's proof of Theorem 2.1, so this alternate method of proof doesn't really gain us much in that direction. On the other hand, the idea of using the function ϕ instead of T generalises to a method of proof that can be used on a wide class of mappings that are less restrictive than the contractions. This method of proof tends to be simpler than the usual methods, and sometimes eliminates the necessity of using the Axiom of Choice. We will return to this topic in Chapter 5.

I. I. Kolodner [23] presents another interesting method of proving Theorem 2.1 using Cantor's intersection theorem. In this proof, we start out by looking at a complete metric space that is also bounded, and then show how this result generalises to the unbounded case. Kolodner's theorem, you will note, says a little bit more than Theorem 2.1.

<u>THEOREM</u> 2.3: Let U be a self-mapping of a bounded, complete metric space X with diameter D, and suppose there exists a positive integer p such that $T=U^p$ is a contraction with Lipschitz constant k. Then

(i) U has a unique fixed point z in X,

(ii) for any s with $0 \leq s \leq (p-1)$,

if $x \in U^{pn+s}(X)$, then $d(z,x) \leq k^n D$, and

(iii) if $\{x_n\}$ is a sequence in X such that

$x_n \in U^n(X)$, then $x_n \to z$.

<u>PROOF</u>: Clearly, the sets $T^n(X)$ form a descending sequence, and $\text{diam}(T^n(X)) \leq k^n D \to 0$ as $n \to \infty$. These two conditions will still be satisfied if we take the closure of each element of the sequence, so the sequence $\{cl(T^n(X))\}$ satisfies all the

hypotheses of Cantor's theorem. This means that $\bigcap_1^\infty cl(T^n(X))$ contains only one point, which we will call z.

Consider the two sets of fixed points,

$$S_1 = \{x \in X:\ Ux = x\}, \text{ and}$$

$$S_p = \{x \in X:\ Tx = x\}.$$

Clearly, $S_1 \subseteq S_p \subseteq \bigcap cl(T^n(X))$. On the other hand, T is continuous, so we have

$$\begin{aligned} T(\{z\}) &= T(\bigcap cl(T^n(x))) \\ &\subseteq \bigcap T(cl(T^n(X))) \\ &\subseteq \bigcap(cl(T^{n+1}(X))) \\ &= \{z\} \end{aligned}$$

So $Tz=z$, and thus $S_p=\{z\}$. Also, notice that $T(Uz)=U(Tz)=U(z)$, so $Uz \in S_p=\{z\}$, and hence $Uz=z$.

This takes care of (i). Part (ii) follows because $U^{pn+s}(X) \subseteq T^n(X)$, and $diam(T^n(X)) \leq k^n D$. Part (iii), of course, follows immediately from part (ii).□

Conveniently enough, we can reduce the general case to the bounded case if we consider the closed ball Y(x) centred at an arbitrary point x in X, with a radius of $d(x,Tx)/(1-k)$. This radius is motivated by the following considerations: If $y \in Y(x)$, where Y(x) is the ball we just defined, then

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx) + d(Tx,Ty) \\ &\leq d(x,Tx) + kd(x,y) \\ &\leq d(x,Tx) + kd(x,Tx)/(1-k) \\ &= d(x,Tx)/(1-k). \end{aligned}$$

This means that $T(Y(x)) \subseteq Y(x)$.

Also, it happens that, if z should be a fixed point of T, then

$$d(x,z) \leq d(x,Tx) + d(Tx,Tz)$$
$$\leq d(x,Tx) + kd(x,z)$$

So $d(x,z) \leq d(x,Tx)/(1-k)$, which means that any fixed point of T will be in $Y(x)$.

Then if we consider only the restriction of U to $Y(x)$, we can be assured of the existence of a fixed point that is unique in $Y(x)$, and hence in all of X. We therefore have the following corollary to Theorem 2.3:

COROLLARY 2.3.1: Let U be a self-mapping of a complete metric space X such that there exists a positive integer p such that $T=U^p$ is a contraction with Lipschitz constant k. Then

(i) U has a unique fixed point z in X,

(ii) if $w \in U^{pn}(Y(x))$, for some $Y(x)$ as defined above, then

$$d(z,w) \leq 2k\ d(x,U^p x)/(1-k), \text{ and}$$

(iii) if $\{x_n\}$ is such that $x_n \in U^{pn}(Y(x))$, then $x_n \dashrightarrow z$.

PROOF: (i) was shown above. The difference between (ii) and (iii) here and in Theorem 2.3 is that here, we don't know that $U(Y(x)) \subseteq Y(x)$. (ii) follows from

$$d(z,w) \leq d(T^n z, T^n x) + d(T^n x, w)$$
$$\leq k^n d(z,x) + k^n d(x, T^{-n} w)$$

$$\leq 2k^n d(x,Tx)/(1-k),$$

since z is in $Y(x)$ and w is in $T^n(Y(x))$. Again, (iii) is an immediate consequence of (ii).□

III. Fixed Point Theorems on Contractive Mappings

Now that we have seen Banach's Theorem, it might be interesting to see how far we can generalise it. Probably the most obvious idea would be to remove the Lipschitz condition, or in other words, to make a definition like:

DEFINITION 1: A mapping $T:X \to X$ is said to be contractive if for each $x,y \in X$ with $x \neq y$, we have

$$d(Tx,Ty) < d(x,y)$$

(The literature, unfortunately, doesn't make a clear distinction between this kind of contractive mapping and those with a Lipschitz condition. Either kind is said to be contractive or a contraction, as the sentence requires. Perhaps it is best to call the contractions of Chapter 2 Banach contractions if the meaning is not clear from the context.)

M. Edelstein [13] was apparently the first to look at contractions of this type, and managed to prove that, in the context of a compact metric space, these contractions will indeed have fixed points.

The method of proof is basically to consider only those pairs of points (x,y) with the property that

$$d(Tx,Ty) < Rd(x,y) \qquad (*)$$

for suitable $0<R<1$. Then from compactness we know that every sequence $\{T^n x\}$ has a subsequence, $\{T^{n_k} x\}$, that converges, say to

$z \in X$. Clearly, $T^{n_k}(Tx) \to Tz$. If we chose R so that $d(Tz,T^2z) < Rd(z,Tz)$ (as we can easily do), then it turns out that $(T^{n_k}z, T^{n_k+1}z)$ eventually satisfies (*). In that case, we can use the arguments similar to those used in the proof of Theorem 2.1. The details follow. Note that instead of assuming compactness, we use a characteristic of $\{T^n x\}$ that is always true in a compact metric space, in order to gain a degree more generality.

<u>THEOREM</u> 3.1 (Edelstein, 14): Let X be a metric space and $T: X \to X$ a contractive mapping. Suppose there exists $x_0 \in X$ such that the sequence $\{T^n x_0\}$ has a convergent subsequence. Then T has a unique fixed point in X.

<u>PROOF</u>: Let $\{T^{n_k} x_0\}$ be the subsequence that converges to z, say. Since T is contractive, it is clear that $\{T^{n_k+1} x_0\} \to Tz$. Let us assume that $z \neq Tz$, and derive a contradiction.

To do this, we define a new function r. Let

$$Y = \{(x,y) \in X \times X : x \neq y\} = X \times X - \Delta,$$

where Δ is the diagonal of $X \times X$. Define a function r from Y to the reals by

$$r(x,y) = \frac{d(Tx,Ty)}{d(x,y)} .$$

It's easy to see that r is continuous (under the product topology), since $d(Tx,Ty)$ and $d(x,y)$ are both continuous, and $d(x,y)$ is non-zero on Y.

This being the case, we can find a real R such that

$$r(z,Tz) < R < 1,$$

and a Y-neighborhood U of (z,Tz) with the property that

$$(x,y)\in U \implies r(x,y)<R \implies d(Tx,Ty) < Rd(x,y)$$

We can also find two open disks, S_1 and S_2, centred at z and Tz respectively, and having radius p small enough that $p<d(z,Tz)/3$ and $S_1 \times S_2 \subseteq U$.

However, it turns out that $\{T^{n_k} x_0\}$ is eventually in S_1, and $\{T^{n_k+1} x_0\}$ is eventually in S_2. That is, there is a natural number N such that

$$j \geq N \implies T^{n_j} x_0 \in S_1 \text{ and } T^{n_j+1} x_0 \in S_2.$$

This means that for any $j \geq N$, $d(T^{n_j} x_0, T^{n_j+1} x_0)>p$. It also means $(T^{n_j} x_0, T^{n_j+1} x_0)$ is in U. And in that case we have

$$d(T^{n_j+1} x_0, T^{n_j+2} x_0) < Rd(T^{n_j} x_0, T^{n_j+1} x_0).$$

On the other hand, from contractivity we see

$$d(T^{n_j} x_0, T^{n_j+1} x_0) \leq d(T^{n_{j-1}+1} x_0, T^{n_{j-1}+2} x_0).$$

Combining these two, and iterating, we have

$$\begin{aligned} d(T^{n_j} x_0, T^{n_j+1} x_0) &\leq d(T^{n_{j-1}+1} x_0, T^{n_{j-1}+2} x_0) \\ &< R\, d(T^{n_{j-1}} x_0, T^{n_{j-1}+1} x_0) \\ &\leq R\, d(T^{n_{j-2}+1} x_0, T^{n_{j-2}+2} x_0) \\ &\;\;\vdots \\ &< R^{n_j-n_N} d(T^{n_N} x_0, T^{n_N+1} x_0). \end{aligned}$$

Since $R<1$, the last term goes to 0 as $j \to \infty$. But we have said that $d(T^{n_j} x_0, T^{n_j+1} x_0)>p$, so this is impossible. Thus it must be that $Tz=z$.

Finally, it is easy to see that the fixed point is unique. For if z_1 and z_2 are distinct fixed points of T, then

$$d(z_1,z_2) = d(Tz_1,Tz_2) < d(z_1,z_2)$$

which is impossible.□

An observation we might make here is that from uniqueness of the fixed point, we can see that, for any $x \in X$, every convergent subsequence of $\{T^n x\}$ must converge to the fixed point, and, more importantly, if $x \in X$ is such that $\{T^n x\}$ has a convergent subsequence, then $T^n x \to z$ also. We prove this as a corollary.

<u>COROLLARY</u> 3.1.1: Let X and T be as in Theorem 3.1. Then if $x \in X$ is such that $\{T^n x\}$ has a convergent subsequence, then $T^n x \to z$, the fixed point of T.

<u>PROOF</u>: Let $\{T^{n_k} x\}$ be the convergent subsequence. Then for each $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $j \geq N \implies d(z, T^{n_j} x) < \varepsilon$. But in that case, for $n \geq n_N$, we have

$$\begin{aligned} d(z, T^n x) &= d(T^{n-n_N} z, T^{n-n_N}(T^{n_N} x)) \\ &\leq d(z, T^{n_N} x) \\ &< \varepsilon, \end{aligned}$$

whence $T^n x \to z$.□

Also, as we mentioned earlier, in a compact metric space, all sequences have convergent subsequences, so we can say:

<u>COROLLARY</u> 3.1.2: If X is a compact metric space and $T: X \to X$ is a contractive mapping, then there exists a unique fixed point to which the sequence $\{T^n x\}$ converges for each $x \in X$.

It's also easy to see that we could replace the completeness condition on X in Theorem 2.1 with a condition on T similar to that in Theorem 3.1, but it's not clear we would really gain all that much.

A slight generalisation of the "contractive" concept is "ε-contractivity", defined as

DEFINITION 2: A mapping T:X-->X is called ε-contractive if, for all x,y in X, $0<d(x,y)<\varepsilon \Longrightarrow d(Tx,Ty)<d(x,y)$.

It should be noted that ε-contractive mappings are continuous, so the function r:Y-->R as defined in the proof of Theorem 3.1 is still continuous.

Interestingly enough, ε-contractive mappings do not always have fixed points. For example, the function $f(x)=-(x^2+|x|)/2x$ on the compact space $[-2,-1]\cup[1,2]$ is ε-contractive, but has no fixed points. On the other hand, $f^2(1)=1$, and in fact we can show that ε-contractive mappings have "periodic points", i.e., there exists a point z and a positive integer k such that $T^k z=z$. The proof of this follows the lines of Theorem 3.1, except that in the beginning we have to have the subsequence of $\{T^n x_0\}$ converge to the point where $d(T^n x_0,z)<\varepsilon$. This is how "periodicity", so to speak, arises.

THEOREM 3.2 (Edelstein, 14): Let T:X-->X be ε-contractive and suppose there exists $x_0 \in X$ such that $\{T^n x_0\}$ has a subsequence that converges to, say, z. Then z is a periodic point of T.

<u>PROOF</u>: Letting $\{T^{n_k} x_0\}$ be the convergent subsequence, find I such that $i \geq I$ implies $d(T^{n_i} x_0, z) < \varepsilon/4$. In that case, $d(T^{n_{i+1}} x_0, T^{n_{i+1}-n_i} z) < \varepsilon/4$ also, and we have

$$d(z, T^{n_{i+1}-n_i} z) \leq d(z, T^{n_{i+1}} x_0) + d(T^{n_{i+1}} x_0, T^{n_{i+1}-n_i} z)$$
$$< \varepsilon/2.$$

From now on, we will let $K = n_{I+1} - n_I$, and $w = T^K z$. It will turn out that w must equal z, which proves our assertion.

Suppose $w \neq z$. Then we can again define r as we did before, and find a Y-neighbourhood of (z,w) such that

$$(x,y) \in U \Longrightarrow r(x,y) < R, \text{ where } r(w,z) < R < 1.$$

Again we choose S_1 and S_2, open disks centred at z and w respectively, each with radius $p < d(z,w)/3$ and $S_1 \times S_2 \subseteq U$. As before we see that $\{T^{n_i} x_0\}$ is eventually in S_1, and $\{T^{n_i+K} x_0\}$ is eventually in S_2. This means that $(T^{n_i} x_0, T^{n_i+K} x_0) \in S_1 \times S_2 \subseteq U$ for all i greater than some N.

Thus,

$$d(T^{n_i+1} x_0, T^{n_i+K+1} x_0) < R\, d(T^{n_i} x_0, T^{n_i+K} x_0),$$

and as before we have, for $i \geq N$,

$$d(T^{n_i} x_0, T^{n_i+K} x_0) < R^{n_i-n_N} d(T^{n_N} x_0, T^{n_i+K} x_0).$$

This converges to 0 as $i \to \infty$. But again this is incompatible with the radii of S_1 and S_2. Thus we know $z = w = T^K z$, and z is a periodic point. □

We might remark that if T is $\mathcal{E}$-contractive, and $z \in X$ is such that $T^k z = z$ but $Tz \neq z$, then

$$d(z,Tz) = d(T^k z, T^{k+1} z) < d(z,Tz)$$

if $d(z,Tz) < \mathcal{E}$. Under this last condition, then, the periodic point is actually a fixed point. Edelstein gives other conditions for this occurence. For example, $\mathcal{E}$-contractive self-mappings of a compact, convex subset of R^n will always have fixed points. This, of course, takes care of our earlier example.

Just as a matter of interest, we note that in a compact space, there are only finitely many periodic points of a given $\mathcal{E}$-contractive mapping. This is because, if $T^m x = x$ and $T^n y = y$, $m,n>0$, and if $0 < d(x,y) < \mathcal{E}$, then $d(x,y) = d(T^{mn}x, T^{mn}y) < d(x,y)$. Thus, every pair of periodic points is a distance of at least $\mathcal{E}$ apart. Then by compactness, there can only be finitely many.

After Edelstein published his contractive and $\mathcal{E}$-contractive results, D. F. Bailey [2] invented yet another class of mappings called "weakly contractive". The proof of his fixed point theorem requires another related idea, that of "proximal" points.

DEFINITION 3: Let X be a metric space and T a self-mapping of X. If $x,y \in X$ are such that for every $\mathcal{E} > 0$, there exists $n = n(\mathcal{E})$ such that $d(T^n x, T^n y) < \mathcal{E}$, then x and y are said to be *proximal under T*.

If T is such that for every $x,y \in X$, $d(x,y) > 0$, there exists

$n=n(x,y)$ such that $d(T^n x, T^n y) < d(x,y)$, then T is said to be <u>weakly contractive</u>.

If $\varepsilon > 0$ and T is such that for every $x,y \in X$ with $\varepsilon > d(x,y) > 0$, there exists $n(x,y)$ such that $d(T^n x, T^n y) < d(x,y)$, then T is said to be <u>ε-weakly contractive</u>.

A simple example of a weakly contractive function is the function $f(x)=\sqrt{x}$, on the space $[0.1,1]$. Since $f^n(x) \to 1$ for all x, f is weakly contractive, but is clearly not contractive.

<u>LEMMA</u> 1: Let X be a compact metric space, and $T:X \to X$ be continuous. If for some $x \in X$ and some positive integer k, x is proximal to $T^k x$, then there exists $z \in X$ such that $T^k z = z$.

<u>PROOF</u>: Let $\{n_i\}$ be a sequence of positive integers such that $n_{i+1} > n_i$ and

$$d(T^{n_i} x, T^{n_i+k} x) \to 0$$

as $i \to \infty$. Since $T^{n_i} x$ has a convergent subsequence, without loss of generality, we can assume $T^{n_i} x \to z$. By continuity, $T^{n_i+k} x \to T^k z$. But since $d(T^{n_i} x, T^{n_i+k} x) \to 0$, it must be that $T^k z = z$. □

<u>LEMMA</u> 2: Let X be compact, $T:X \to X$ be continuous and weakly contractive. Then every pair of points in X is proximal under T.

<u>PROOF</u>: Suppose this is not the case, that is, there exists

a pair $\{x,y\}$ of points that are not proximal under T. In that case $T^n x \neq T^n y$ for every positive n, so by weak contractiveness, we can find a sequence $\{n_i\}$ of positive integers with the property that

$$d(x,y) > d(T^{n_1}x, T^{n_1}y) > \ldots > d(T^{n_i}x, T^{n_i}y) > \ldots$$

Assume also that each n_i is the smallest possible that fulfills this requirement. If that is true, then $k<n_i$ implies that $d(T^k x, T^k y) > d(T^{n_i}x, T^{n_i}y)$. This is also true for any subsequence of $\{n_i\}$, and in particular, it is true of the subsequence $\{n_{i'}\}$ where $T^{n_{i'}}x \to a$ and $T^{n_{i'}}y \to b$. We will call this subsequence $\{n_i\}$ from now on.

Since x and y are not proximal, $a \neq b$. However, if we choose any positive integer k, we see that $n_i + k \leq n_{i+k}$. This means that

$$\begin{aligned} d(T^k a, T^k b) &= \lim d(T^{n_i+k}x,\ T^{n_i+k}y) \\ &\geq \lim d(T^{n_{i+k}}x, T^{n_{i+k}}y) \\ &= d(a,b). \end{aligned}$$

This is not compatible with weak contractivity, so x and y must be proximal.□

<u>THEOREM</u> 3.3: Let X be compact, $T:X \to X$ continuous and weakly contractive. Then T has a unique fixed point in X.

<u>PROOF</u>: By Lemma 2, Tx is proximal to x for any $x \in X$, so by Lemma 1, a fixed point exists. It is unique because if x and y were distinct fixed points, then we could find some n such that $d(x,y) = d(T^n x, T^n y) < d(x,y)$.□

It's probably not startling that if T is ε-weakly contractive, we can again show the existence of finitely many periodic points. The proof follows the lines of Lemma 2 (starting with $d(x,y)<\varepsilon$) and Theorem 3.3 exactly.

IV. Non-Expansive Mappings

The next generalisation of Banach's Theorem we might make is the following:

DEFINITION 1: Let X be a metric space. A mapping $T:X\to X$ is said to be non-expansive if, for each $x,y\in X$,

$$d(Tx,Ty) \leq d(x,y).$$

Much of the work currently being done on fixed point theorems is being done on non-expansive mappings in Banach spaces. The question we are concerned with is what kind of subsets of Banach spaces have the "fixed point property" for non-expansive mappings, i.e., for which subsets K of a Banach space will every non-expansive mapping $T:K\to K$ be guaranteed to have a fixed point. Clearly, we have lost any hope of uniqueness, since the identity is always non-expansive.

Compactness of K is certainly not enough, since a rotation of the unit circle in R^2 is non-expansive, but of course has no fixed points. It turns out that in a convex, compact set, each non-expansive mapping has a fixed point. This is a consequence of Schauder's fixed-point theorem, which we will discuss later on. A certain amount of labour has been expended recently to generalise the compactness to weak compactness. The first results on this question came in 1965 when Browder [8, 9], Kirk [20], and Gohde [15] all announced more or less similar results, Kirk's being the most general. The theorem we prove below is due to Kirk (1980, [22]). First, however, we need the concept of normal structure.

DEFINITION 2: Let X be a Banach space and $K \subseteq X$. Let diam(K) be the diameter of K. A point x of K is said to be a diametral point of K if

$$\sup\{d(x,k): k \in K\} = \operatorname{diam}(K).$$

A convex set K is said to have normal structure if every bounded convex subset H of K containing more than one point has a non-diametral point, i.e., if, for each suitable $H \subseteq K$, there exists $x \in H$ such that $\sup\{d(x,h): h \in H\} < \operatorname{diam}(H)$.

We will also note the following well-known properties of weakly compact sets:

LEMMA 1: A closed, convex set is weakly closed.

LEMMA 2: A family of sets is said to have the finite intersection property if each of its finite sub-families has a non-empty intersection. A set K is (weakly-)compact if and only if every family of (weakly-)closed subsets of K that has the finite intersection property also has a non-empty intersection.

For a proof of Lemma 1, see Taylor & Lay, [28], Theorem III.6.3. Lemma 2 is an elementary property of compactness. See Willard, [29], 17.4, for example.

Finally, we are ready to prove the following theorem:

<u>THEOREM</u> 4.1 (Kirk, 22): Let X be a Banach space, and K a nonempty, weakly compact convex subset of X, and suppose K has normal structure. Then every non-expansive mapping T:K-->K has a fixed point.

The method of proof here is to show first that K has a minimal T-invariant, non-empty, closed, convex subset H. That is, H is T-invariant, non-empty, closed, and convex, and has no proper subset with these properties. We prove this first as a lemma, and then show H contains only one point.

<u>LEMMA</u> 3: Let K, X, and T be as in Theorem 4.1. Then K contains a minimal T-invariant, non-empty, closed, convex subset.

<u>PROOF</u>: Let $\mathcal{H}$ be the set of all non-empty, T-invariant, closed convex subsets of K, partially ordered by inclusion. If $\mathcal{L}$ is any nonempty linearly ordered chain in $\mathcal{H}$, then the "minimal element" of $\mathcal{L}$ will be the intersection of all the elements of $\mathcal{L}$. To see this, we note that the intersection is closed, convex, and invariant, so the question is, will it be non-empty? But all closed convex sets are weakly closed, and a linearly ordered chain certainly has the finite intersection property, and thus

by Lemma 2 and weak compactness of K, the answer is yes. Then by Zorn's lemma, $\mathcal{H}$ has a minimal element.□

<u>PROOF</u> of Theorem 4.1: Let H be the minimal subset guaranteed by Lemma 3. K is weakly compact, hence bounded, so H is bounded also. Now suppose H has more than one point. Then by normal structure on K, there is a point $z \in H$ such that

$$r = \sup\{d(z,h): h \in H\} < \operatorname{diam}(H).$$

In that case, the set

$$C = \{x \in H : H \subseteq B(x;r)\},$$

where $B(x;r)$ is the closed ball about x with radius r, is non-empty. It can easily be seen that C is closed and convex, and if we can show that C is also T-invariant, then by minimality of H, we have C=H. So we need to show that C is T-invariant.

For a set A, let conv(A) be the smallest closed convex set that contains A, i.e., if $A \subseteq B$ and B is closed and convex, then $\operatorname{conv}(A) \subseteq B$. Now since $T(H) \subseteq H$, $\operatorname{conv}(T(H)) \subseteq H$ also, and thus $T(\operatorname{conv}(T(H))) \subseteq T(H) \subseteq \operatorname{conv}(T(H))$. So conv(T(H)) is T-invariant. Then by the minimality of H, $H = \operatorname{conv}(T(H))$. But by non-expansiveness of T, if $z \in C$, we have

$$\begin{aligned} H \subseteq B(z;r) &\implies T(H) \subseteq B(Tz;r) \\ &\implies \operatorname{conv}(T(H)) \subseteq B(Tz;r) \\ &\implies H \subseteq B(Tz;r), \end{aligned}$$

and so $Tz \in C$ also. Thus C is T-invariant, and so C=H.

On the other hand, consider diam(C). If $z_1, z_2 \in C$, then $z_1 \in H \subseteq B(z_2; r)$, so $d(z_1, z_2) \leq r$, diam(C)$\leq r$, and we have

$$\text{diam}(C) \leq r < \text{diam}(H) = \text{diam}(C),$$

which is impossible. So it must be that H has only one point.□

The reason we left the part relating to Zorn's Lemma as a separate lemma is to emphasise that this is the only place where we used weak compactness. Thus it is possible to change the theorem slightly.

<u>THEOREM</u> 4.2 (Kirk, 20): Let X be a reflexive Banach space and K be a nonempty, bounded, closed, and convex subset of X, K having normal structure. If T:K-->K is a non-expansive mapping, then T has a fixed point in K.

This is the theorem that Kirk originally proved. To show the existence of H, we need to use a characterisation of reflexivity due to Smulian [27].

<u>LEMMA</u> 4: A Banach space X is reflexive if and only if every bounded descending sequence of non-empty, closed, convex subsets of X has a non-empty intersection.

Fixed Points Without Normal Structure

One interesting aspect of all this is the property of "normal structure". Normal structure is a fairly natural property, and it is apparent that it holds for all suitable subsets of R^n. An example of a space that does not have normal structure is the following, from Karlovitz, [19].

For a given real number B, let ℓ^2_B be the real space ℓ^2 with the following norm:

$$|x|_B = \max\{|x|_\infty, |x|_2/B\}.$$

In the space $\ell^2_{\sqrt{2}}$, consider the set

$$C=\{x \in \ell^2 : x(i) \geq 0 \text{ for each } i, |x|_2 \leq 1\}$$

which is bounded and convex.

(Note: ℓ^2_B is a space of sequences, so x(i) represents the i-th element of the sequence x. "x_n" would represent the n-th sequence in a sequence of elements of ℓ^2.)

It's not difficult to show that diam(C)=1. Furthermore, if for each natural n, we let x_n be a sequence such that $x_n(n)=1$ and $x_n(i)=0$ for $i \neq n$, then $\lim |x_n - y|_B = 1$ for any $y \in C$. Thus, it is easy to see that since $x_n \in C$ for every n, there is no non-diametral point in C, so C does not have normal structure.

Karlovitz shows, however, that $\ell^2_{\sqrt{2}}$ still has the fixed point property for non-expansive mappings. In fact, Baillon and Schoneberg [3] manage to prove the fixed-point property for ℓ^2_B with $B \leq 2$. To do this, they make use of a generalisation of normal structure called asymptotic normal structure.

<u>DEFINITION</u> 3: A Banach space X has <u>asymptotic normal structure</u> if for every bounded, closed and convex subset C of X that has more than one point, and for each sequence $\{x_n\}$ in C satisfying $x_n - x_{n+1} \to 0$, there is a point $x \in X$ such that $\liminf |x_n - x| < \text{diam}(C)$.

It should be fairly obvious that any space with normal structure also has asymptotic normal structure, since

$$\liminf |x_n - x| \leq \sup\{|y-x| : y \in C\} < \text{diam}(C),$$

for x a non-diametral point of C. It turns out that asymptotic normal structure is enough to give us the fixed-point property.

<u>THEOREM</u> 4.3 (Baillon & Schoneberg, 3): Let X be a reflexive Banach space and $K \subseteq X$ be closed, bounded, convex, and non-empty, and have asymptotic normal structure. Then every non-expansive mapping $T: K \to K$ has a fixed point in K.

The method of proof is this: By the same arguments we used in Lemma 4 (or Lemma 3, with suitable changes to the present hypothesis), we can deduce the existence of a minimal, closed, bounded, non-empty, convex, T-invariant subset H of K. We can also construct a sequence $\{x_n\}$ in H with the two properties that $x_n - Tx_n \to 0$ and $x_n - x_{n+1} \to 0$ as $n \to \infty$. From this second property, and asymptotic normal structure, we see that there exists a point $x \in K$ such that $\liminf |x_n - x| < \text{diam}(H)$. (This is assuming that H has more than one point.) Unfortunately, using the first property and minimality of H, we can show that $|x_n - x| \to \text{diam}(H)$ for all x in H. This last point we prove as Lemma 6. First, however, we need to state another well-known lemma, which is really just a special case of the Hahn-Banach Extension Theorem.

<u>LEMMA</u> 5 (Hahn-Banach): If X is a normed space, and $x_0 \in X$, then there exists a bounded linear functional f on X such that

$$f(x_0) = |x_0|, \text{ and}$$

$$f(x) \le |x|, \text{ for all } x \text{ in } X.$$

<u>LEMMA</u> 6 (Karlovitz, 19): Let X, K, and T be as in Theorem 4.3, and suppose H is a minimal T-invariant, non-empty, closed, bounded, convex subset of K. If $\{x_n\}$ is a sequence in H with the property that $x_n - Tx_n \to 0$, then $|x_n - x| \to \text{diam}(H)$ for all x in H.

<u>PROOF</u>: Fix $y \in H$ and let $s = \lim\sup|y - x_n|$. Define a set

$$D = \{x \in H: \lim\sup|x - x_n| \le s\}$$

which is easily shown to be non-empty, closed, and convex. If it is also T-invariant, then we know D=H. And it is, for if x is in D,

$$|Tx - x_n| \le |Tx - Tx_n| + |Tx_n - x_n| \le |x - x_n| + |Tx_n - x_n|.$$

$|Tx_n - x_n| \to 0$, so $\lim\sup|Tx - x_n| \le \lim\sup|x - x_n| \le s$.

Now take a subsequence $\{x_{n'}\}$ such that $|y - x_{n'}|$ converges, to s', say. We want to show that for <u>any</u> x in H, $|x - x_{n'}| \to s'$. Suppose not. Then there must exist some $z \in H$ such that $|z - x_{n'}| \not\to s'$. Hence, there must be a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$

such that $|z-x_{n''}|\to t$, for some $t\neq s'$.

Then the set $E=\{x\in H: \limsup|x-x_{n'}|\leq \min(t,s')\}$ is non-empty, since it contains either y or z, but not both. However, by arguments similar to those above, we can prove that E=H, which is a contradiction. So $\lim|x-x_{n'}|=s'$ for all x in H.

Now we show that s'=diam(H). The set

$$F=\{u\in H: |u-x|\leq s' \text{ for each } x \text{ in } H\}$$

is clearly closed and convex. It is also non-empty, as follows. Since H is bounded and X is reflexive, $\{x_{n'}\}$ has a weakly convergent subsequence $\{x_{n''}\}$ whose limit we will call z. (See Taylor & Lay, 38, Th III.10.6. Note that this is the only place in the Lemma where we use reflexivity.) From the Hahn-Banach extension theorem, for any $x\neq z$, there is a bounded linear functional f such that $f(x-z)=|x-z|$ and $f(y)\leq|y|$ for all $y\in X$. Since $x_{n''}\to z$, weakly, $|f(x_{n''})-f(z)|\to 0$. In that case,

$$\begin{aligned}|x-z| = f(x-z) &= f(x-x_{n''}) + f(x_{n''}-z)\\ &\leq |x-x_{n''}| + |f(x_{n''})-f(z)|\\ &\to s' + 0,\end{aligned}$$

since $|x-x_{n''}|\to s'$. Thus, $|z-x|\leq s'$ for all x in X, and $z\in F$. Now all we need to do is show that F is T-invariant. Since $T(H)\subseteq H$, conv(T(H))=H, by minimality. Hence, for arbitrary u in H and $\varepsilon>0$, we can find v in H so that $|u-v|<\varepsilon$ and $v=\sum_{i=1}^{n}\lambda_i Tx_i$, where $\sum_i^n\lambda_i=1$, $0\leq\lambda_i\leq 1$, and $x_i\in H$ for each positive integer i. In that case we see that if w is in F, then so is Tw, for

$$\begin{aligned}|Tw-u| &\leq |Tw-v| + |v-u|\\ &< |\sum_{i=1}^{n}\lambda_i(Tw-Tx_i)| + \varepsilon\end{aligned}$$

$$\leq \sum \lambda_i |Tw-Tx_i| + \varepsilon$$

$$\leq \sum \lambda_i |w-x_i| + \varepsilon$$

$$\leq s' + \varepsilon.$$

Thus it must be that $|Tw-u| \leq s'$ for all $w \in F$ and $u \in H$, so F is T-invariant, and F=H. Clearly then $s'=\mathrm{diam}(H)$.

The effect of all this is to show that, for any $x \in H$, if $\{x_{n'}\}$ is a subsequence of $\{x_n\}$, and $|x-x_{n'}|$ converges, then it converges to diam(H). Since $\{x_n\}$ is bounded, this means $\lim|x-x_n|=\mathrm{diam}(H)$. □

Now we are ready to prove the theorem.

<u>PROOF of Theorem</u> 4.3 [3]: The only thing that remains to be done is to construct the sequence $\{x_n\}$ in H such that $x_n-Tx_n \to 0$ and $x_n-x_{n+1} \to 0$. In order to accomplish this, we make use of Banach's fixed point theorem.

Fix $z \in H$, and for each natural number n, define a function U_n so that

$$U_n x = (z/n) + \{1-(1/n)\}Tx.$$

Since T is non-expansive, it is easy to see that U_n is a Banach contraction with Lipschitz constant $\{1-(1/n)\}$. Also, $x \in H \Longrightarrow Tx \in H$, and by convexity of H, $U_n x \in H$ also. Thus, the requirements of Theorem 1 are met, and there must be a fixed point x_n of U_n, so

$$x_n = (z/n) + \{1-(1/n)\}Tx_n.$$

The sequence $\{x_n\}$ of fixed points is our desired sequence. In the first place,

$$x_n-Tx_n = (z-Tx_n)/n \to 0 \text{ as } n \to \infty.$$

Also,

$$| x_n - x_{n+1} | = \left| \frac{1}{n(n+1)}(z-Tx_n) + \left(1- \frac{1}{n+1}\right)(Tx_n - Tx_{n+1}) \right|$$

$$\leq \{1/n(n+1)\}|z-Tx_n| + \{1- 1/(n+1)\}|x_n - x_{n+1}|,$$

by non-expansiveness, and rearranging and dividing gives us

$$|x_n - x_{n+1}| \leq |z-Tx_n|/n \text{ --> } 0 \text{ as } n\text{-->} \;,$$

as desired.□

Once again, of course, we can eliminate the requirement that X be reflexive if instead we require K to be weakly compact, yielding:

<u>THEOREM</u> 4.4: Let X be a Banach space, $K \subseteq X$ be weakly compact, nonempty, and convex, and also have asymptotic normal structure. Then every non-expansive mapping T:K-->K has a fixed point in K.

The proof is identical, except for a slight difference in the lemma,

<u>LEMMA</u> 7: Let X, K, and T be as in Theorem 4.4, and H as in Lemma 6. If $\{x_n\}$ is a sequence in H with the property that $x_n - Tx_n$-->0, then $|x_n - x|$-->diam(H) for all x in H.

<u>PROOF</u>: As before, except the existence of the weakly convergent subsequence $\{x_{n''}\}$ comes from weak compactness (see

Taylor & Lay, 28, Th. III.10.10).□

Finally, we briefly note some further results of Baillon and Schoneberg. It turns out that, for $B \geq 1$, the space ℓ^2_B described in the example above has normal structure if and only if $B < \sqrt{2}$, and has asymptotic normal structure if and only if $B < 2$. Also the space ℓ^2_2 has the fixed point property for non-expansive mappings even though it does not have asymptotic normal structure. On the other hand, an example has been found of a weakly compact, convex subset of a Banach space that has non-expansive self-mappings without fixed points. See Alspach [1] and Schechtman [25].

V. Fixed Points Using Wong's Methods

In section 1, you will recall, we proved Banach's fixed-point theorem using a function $\phi(x)=d(x,Tx)$, and said that later on we would be able to prove more fixed-point theorems using the same idea. To do this, we will reduce the various properties of T to properties of the related function ϕ. For example, if $\phi(x)=d(x,Tx)$, then one property ϕ has is that $\phi(x)=0$ if and only if $x=Tx$. We will call this property _T-invariance_. We also define the following property of ϕ.

DEFINITION 1: Let X be a metric space, T a self-mapping of X, and ϕ a real-valued function of X. ϕ is said to be _weakly contractive with respect to T_ if for each $x \in X$ with $\phi(x)>0$, there exists $n=n(x)$, a positive integer, such that $\phi(T^n x)<\phi(x)$.

It's easy to see that if T is a Banach contraction then $\phi(x)= d(x,Tx)$ is weakly contractive with respect to T. In fact, this is still true if T is just a contractive mapping or even if T is only weakly contractive.

The common property of weak contractiveness leads to a useful theorem, of which several of our earlier results become simple corollaries.

THEOREM 5.1 (Wong, 30): Let X be a compact metric space and T a self-mapping of X. If there exists a continuous, nonnegative real-valued function ϕ on X which is T-invariant and weakly contractive with respect to T, then T has a fixed point in X.

PROOF: Since ϕ is continuous and X is compact, ϕ reaches its minimum on X. Let $\phi(z)$ be the minimum, and assume $\phi(z)>0$. Then there exists n(z) such that $\phi(T^n z)<\phi(z)$, which is impossible if $\phi(z)$ is a minimum. So $\phi(z)=0$, and by invariance, z is a fixed point of T.▯

After we make the obvious remark that if T is continuous, $\phi(x)=d(x,Tx)$ is also continuous, and that contractive mappings are always continuous, we can go on to state the following corollaries.

COROLLARY 5.1.1 (Corollary 3.1.2): Let X be a compact metric space, T:X-->X a contractive mapping. Then T has a fixed point in X.

COROLLARY 5.1.2 (Theorem 3.3): Let X be a compact metric space, T:X-->X a weakly contractive, continuous mapping. Then T has a fixed point in X.

PROOF (Wong, 30): Let $\phi(x)=d(x,Tx)$. By weak contractiveness of T, ϕ is weakly contractive with respect to T, and clearly also continuous and T-invariant. Thus, by the theorem, T has a fixed point in X. ▯

We can also derive a theorem to take care of our "ε-contractive" conditions. First we make the obvious definition of "ε-weakly contractive with respect to T", namely that a real valued function ϕ is said to be <u>ε-weakly contractive with respect to</u> a function T if and only if $0<\phi(x)<\varepsilon \Longrightarrow$ there exists $n=n(x)$ such that $\phi(T^n x)<\phi(x)$. Then our theorem becomes:

<u>THEOREM</u> 5.2: Let T be a self-mapping of X, a compact metric space. Suppose there exists a continuous, nonnegative real-valued function ϕ that is ε-weakly contractive with respect to T and that is T^k-invariant for some natural number k. Then if there is an $x\in X$ such that $\phi(x)<\varepsilon$, T^k has a fixed point in X.

<u>PROOF</u> (Wong, 30): Again, ϕ reaches its miniumum on X. If $\phi(z)$ is the minimum, clearly $\phi(z)<\varepsilon$. If $\phi(z)>0$, then there exists m such that $\phi(T^m z)<\phi(z)$, which is impossible if $\phi(z)$ is a minimum. $\square$

<u>COROLLARY</u> 5.2.1 (see Theorem 3.2): Let X be a compact metric space, and let $T:X \dashrightarrow X$ be ε-contractive. Then T has a periodic point in X.

<u>COROLLARY</u> 5.2.2: Let X be a compact metric space, and let $T:X \dashrightarrow X$ be continuous and ε-weakly contractive. Then T has a periodic point in X.

<u>PROOF</u>: Since X is compact, for any $x\in X$ there exist positive integers k,m such that $d(T^m x, T^{m+k} x)<\varepsilon$. In that case,

positive integers k,m such that $d(T^m x, T^{m+k}x)<\varepsilon$. In that case, let $\phi(x)=d(x,T^k x)$, and $z=T^m x$ is such that $\phi(z)<\varepsilon$. Obviously, ϕ is T^k-invariant and ε-weakly contractive.□

Another weakening of the contractive fixed points idea we considered previously was to eliminate compactness of X, but require instead that $\{T^n x\}$ had a convergent subsequence, for some $x\in X$. We can obtain results using Wong's methods, but we need another condition on the function ϕ.

DEFINITION 2: A real-valued function ϕ on X is said to be regular with respect to some self-mapping T of X if $\{\phi(T^n x)\}$ converges for every $x\in X$. For any $\varepsilon>0$, ϕ is ε-regular with respect to T if $\{\phi(T^n x)\}$ converges for every $x\in X$ with $\phi(x)<\varepsilon$.

THEOREM 5.3: Let X be a metric space, and let $T:X\to X$ be continuous. Suppose there exists a nonnegative continuous functional ϕ which is T-invariant, weakly contractive, and regular with respect to T. Then if the sequence $\{T^n x\}$ has a convergent subsequence for some $x\in X$, T has a fixed point in X.

PROOF: Let $\{T^{n_k}x\}$ be the convergent subsequence and say it converges to z. Assuming $\phi(z)>0$, there exists N such that $\phi(T^N z)<\phi(z)$. Then using regularity and continuity of ϕ, we see

$$\begin{aligned}\phi(z) &= \lim \phi(T^{n_k}x) && \text{(continuity)}\\ &= \lim \phi(T^{n_k+N}x) && \text{(regularity)}\\ &= \phi(\lim T^{n_k+N}x) && \text{(continuity)}\end{aligned}$$

$$= \phi(T^N(\lim T^{n_k}x)) \qquad \text{(cont. of T)}$$

$$= \phi(T^N z) \qquad \text{(defn. of z)}$$

$$< \phi(z) \qquad \text{(weak-contr.)}$$

This is impossible, so $\phi(z)=0$, and z is a fixed point. □

<u>COROLLARY</u> 5.3.1 (Theorem 3.1): Let X be a metric space and $T:X \to X$ be a contractive mapping. If there exists $x \in X$ such that the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point in X.

<u>COROLLARY</u> 5.3.2 (Wong, 30): Let X be a metric space and $T:X \to X$ be a weakly contractive mapping that is also non-expansive. If there exists $x \in X$ such that the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point in X.

<u>PROOF</u>: T is continuous since it is non-expansive. Let $\phi(x)=d(x,Tx)$, which is weakly contractive with respect to T, T-invariant, and continuous. Since **R** is complete and for any x, the sequence $\{\phi(T^n x)\}=\{d(T^n x,T^{n+1}x)\}$ is non-increasing (by non-expansiveness), ϕ is regular. Then by Theorem 5.3, T has a fixed point in X. □

Diminishing Orbital Diameters

We have now taken care of all the theorems proved in Chapters 2 and 3. The situation for non-expansive mappings is not quite as simple, unfortunately. Theorem 4.1 does not respond well to these methods, although we can obtain some results on weakly compact subsets, as we shall see in the next section of this chapter.

On the other hand, L.P. Belluce and W.A. Kirk obtained a few interesting results on non-expansive mappings by placing additional restrictions on the mapping itself, rather than on its domain. These restrictions involve various conditions of "shrinking orbits".

DEFINITION 3: Let X be a metric space, and T a self-mapping of X. Call the set denoted by

$$O(T^n x) = \{T^n x,\ T^{n+1} x,\ T^{n+2} x,\ \dots\}$$

the orbit of $T^n x$ (for $n \geq 0$). Define $r(x) = \lim(\mathrm{diam}(O(T^n x)))$, the limiting orbital diameter of x. If $r(x) < \mathrm{diam}(O(x))$ for every $x \in X$ with $\mathrm{diam}(O(x)) > 0$, then we say that T has diminishing orbital diameters.

Belluce and Kirk proved in 1967 (5, using a result of Edelstein, 14) that in a compact space, a non-expansive mapping with diminishing orbital diameters has a fixed point. Kirk [21] showed that it was sufficient that the mapping be continuous with diminishing orbital diameters. We will first present Kirk's proof, and then use Wong's method on it.

THEOREM 5.4: Let X be a compact metric space, and let $T:X \to X$ be continuous with diminishing orbital diameters. Then T has a fixed point z in X, and every sequence $\{T^n x\}$ in X has a subsequence that converges to a fixed point of T.

PROOF (Kirk, 21): For any $x \in X$, let $L(x)$ be the set of all limits of convergent subsequences of $\{T^n x\}$. X is compact, so $L(x)$ is non-empty. Since T is continuous, $T(L(x)) \subseteq L(x)$, and from the definition, $L(x)$ is easily seen to be closed. By Zorn's lemma, we can find $K \subseteq L(X)$ that is a minimal closed, nonempty, T-invariant subset of $L(x)$. Let $x_0 \in K$ and assume $x_0 \neq Tx_0$. Then $O(x_0) \subseteq K$, and, again by continuity, $cl(O(x_0))$ is T-invariant. That means $cl(O(x_0))=K$. However, by diminishing orbital diameters, there is an N such that $cl(O(T^N x_0))$ is a proper non-empty, closed, T-invariant subset of $cl(O(x_0))$, and hence of K. This is a contradiction, so $x_0=Tx_0$.□

We might note that this proof requires the use of Zorn's lemma, which is not necessary if we say that T is also non-expansive. As well, Belluce and Kirk [5] also obtain results in weakly compact settings, the method being similar to that used in Theorem 4.1.

A simple example of a mapping with diminishing orbital diameters is the fuction $f(x)=\sqrt{x}$ on the space [0,1]. Since $f^n(x) \to 1$ for all $x>0$, f has diminishing orbital diameters, but because 0 is included, it is not non-expansive or indeed even weakly contractive.

In order to use Wong's methods on this kind of mapping, we need to examine a slightly weaker version of continuity.

DEFINITION 4: Let X be a metric space and ϕ a real-valued function on X. We call ϕ lower semi-continuous if, for each real r, the set

$$\phi^{-1}(r,\infty) = \{x \in X: \phi(x)>r\}$$

is open in X.

The utility of this is that a lower semi-continuous function always reaches its minimum on a compact set. (see Dugundji, 12, XI.2.4). What we are going to do is to show that the function $\phi(x)=\text{diam}(O(x))$ is lower semi-continuous. For this, we need the following lemma.

LEMMA 1: For each positive integer n, let ϕ_n be a continuous real-valued mapping of X. Then

$$\phi(x) = \sup\{\phi_n(x): \text{n a positive integer}\}$$

is lower semi-continuous.

PROOF [12]: $\phi(x)>a$ if and only if at least one $\phi_n(x)>a$, so $\phi^{-1}(a,\infty) = \bigcup_1^{\infty} \phi_n^{-1}(a,\infty)$. □

THEOREM 5.5: Let X be a compact metric space, and T a self-mapping of X. If there exists a lower semi-continuous function $\phi: X \dashrightarrow R^+$ which is T-invariant and weakly contractive with respect to T, then T has a fixed point in X.

PROOF: This is just a slight generalisation of Theorem 5.1, and the proof is identical, except we use our statement about the minimum of a lower semi-continuous function on a compact set. □

PROOF of Theorem 5.4 as a corollary of Theorem 5.5: If we let $\phi(x)=\text{diam}(O(x))=\sup\{d(T^i x, T^j x): i,j\geq 0\}$, then ϕ is obviously T-invariant, and by the lemma, it is also lower semi-continuous. From diminishing orbital diameters, $\text{diam}(O(T^n x))<\text{diam}(O(x))$, for some n, so ϕ is weakly contractive with respect to T.□

There are a couple of interesting points to notice about this result. First of all, of course, we have eliminated the use of Zorn's lemma. Also, the example we just used ($f(x)=\sqrt{x}$) is an example of a non-weakly contractive function f with a related functional that is weakly contractive with respect to f.

Kirk [21] defines another type of mapping with diminishing orbital diameters, requiring that the mapping T satisfy a "uniform Lipschitz condition", i.e., there exists a constant C (which may be greater than 1) such that for every positive integer n and every $x,y\in X$,

$$d(T^n x, T^n y)\leq C\, d(x,y).$$

If T satisfies this condition, then $\text{diam}(O(x))$ turns out to be a continuous function on X.

LEMMA 2: Let T be a self-mapping of a metric space X, and suppose that T satisfies a uniform Lipschitz condition with constant C. Then diam(O(x)) is a continuous function on X.

PROOF (Kirk, 21): Let $x_n \to x_0$, with each $x_n \in X$. We will show that $\text{diam}(O(x_n)) \to \text{diam}(O(x_0))$.

Of course, $\text{diam}(O(x_n)) = \sup\{d(T^i x_n, T^j x_n): i,j \geq 0\}$. Then for any $\varepsilon > 0$, there is a positive integer N such that $n \geq N$ implies

$$d(x_n, x_0) < \varepsilon/C.$$

This means that

$$d(T^k x_n, T^k x_0) < \varepsilon$$

for any k. This fact, plus the triangle inequality, shows that for $n \geq N$ and any j,k,

$$|d(T^j x_n, T^k x_n) - d(T^j x_0, T^k x_0)| \leq 2\varepsilon.$$

Thus, we can say

$$|\text{diam}(O(x_n)) - \text{diam}(O(x_0))| \leq 2\varepsilon,$$

for $n \geq N$, and we see that $\text{diam}(O(x_n)) \to \text{diam}(O(x_0))$.□

Then we can prove this corollary of Theorem 5.3:

COROLLARY 5.3.3 (Kirk, 21): Let X be a metric space, and let $T: X \to X$ satisfy a uniform Lipschitz condition with constant C, and also have diminishing orbital diameters. If there exists $x \in X$ such that $\{T^n x\}$ has a convergent subsequence, then T has a fixed point in X.

PROOF (Wong, 30): Let $\phi(x) = \text{diam}(O(x))$. By the lemma, ϕ is continuous. By diminishing orbital diameters, ϕ is weakly contractive and regular with respect to T. Clearly, ϕ is

T-invariant, so by Theorem 5.3, T has a fixed point in X.∎

Non-Expansive Mappings

As we mentioned above, a result similar to (but weaker than) Theorem 4.1 is possible using Wong's method of proof. We first note that, as in the norm topology, a weakly lower semi-continuous function reaches its minimum on a weakly compact set. If we let T be a non-expansive function and require also that T be weakly continuous, then $\phi(x)=d(x,Tx)$ is weakly lower semi-continuous, which we prove below as a lemma.

LEMMA 3: Let X be a normed space, and let T:X-->X be weakly continuous. Then the function $\phi(x)=d(x,Tx)$ is weakly lower semi-continuous.

PROOF: We will show that $\phi^{-1}(a,\infty)$ is weakly open, by showing that $\phi^{-1}(-\infty,a]$ is weakly closed. Let M be a directed set and $\{x_m\}$ a net ordered by M. (For an explanation of nets, see Willard, 29, section 11) Suppose $x_m \to x_o$, weakly, and for each m, $\phi(x_m) \leq a$. We show that $\phi(x_o) \leq a$, as well.

By the Hahn-Banach extension theorem, there exists a bounded linear functional f on X such that

$$f(x_o - Tx_o) = |x_o - Tx_o|, \text{ and}$$

$$f(x) \leq |x|, \text{ for all } x \text{ in } X.$$

Since T is weakly continuous and x_m converges weakly to x_o, $x_m - Tx_m \to x_o - Tx_o$, weakly, also. This means that $f(x_m - Tx_m) \to$

$f(x_0 - Tx_0)$ (strongly). For each m, $f(x_m - Tx_m) \leq |x_m - Tx_m| \leq a$, so $a \geq \lim f(x_m - Tx_m) = f(x_0 - Tx_0) = |x_0 - Tx_0|$. $\square$

This means, in particular, that if K is a weakly compact subset of X, and T:K-->K is weakly continuous, then the function $|x-Tx|$ reaches its minimum on K. For our theorem, we will also need the following definition.

<u>DEFINITION</u> 5: A normed space X is called <u>strictly convex</u> if for each x,y in X, $|x+y|=|x|+|y|$ implies x and y are linearly dependent.

A simple calculation shows that in a strictly convex space, a vector z is on the segment between x and y ($z \in seg(x,y)$) if and only if $d(x,y)=d(x,z)+d(z,y)$.

<u>THEOREM</u> 5.6: Let X be a strictly convex normed space, and let K be a weakly compact, convex subset of X. If T is a weakly continuous, non-expansive self-mapping of K, then T has a fixed point in K.

<u>PROOF</u>: Let $\phi(x)=d(x,Tx)$. Then by the lemma, ϕ reaches its minimum on K. Let $\phi(z)$ be this minimum, and consider seg(z,Tz).

If we choose any $x \in seg(z,Tz)$, then we see that

$$\begin{aligned}\phi(x) = d(x,Tx) &\leq d(x,Tz) + d(Tz,Tx)\\ &\leq d(x,Tz) + d(z,x)\\ &= d(z,Tz) = \phi(z).\end{aligned}$$

Since $\phi(z)$ is a minimum, this means that $d(x,Tx)=d(x,Tz)+d(Tz,Tx)$, and by strict convexity, $Tz \in seg(x,Tx)$.

However, we can also say

$$
\begin{aligned}
d(Tz,T^2z) &\leq d(Tz,Tx) + d(Tx,T^2z) \\
&\leq d(z,x) + d(x,Tz) \\
&= d(z,Tz)
\end{aligned}
$$

Again, this means that $Tx \in seg(Tz,T^2z)$. Combining this and $Tz \in seg(x,Tx)$ shows us that $Tz \in seg(z,T^2z)$, and hence that

$$
\begin{aligned}
d(z,T^2z) &= d(z,Tz) + d(Tz,T^2z) \\
&= 2\, d(z,Tz).
\end{aligned}
$$

We can continue this process inductively and eventually show that, for any positive integer n, $d(z,T^nz)=n\, d(z,Tz)$. Weakly compact sets are bounded, so this is impossible unless $d(z,Tz)=0$. $\square$

VI. Brouwer's and Schauder's Theorems

The various types of mappings we have discussed thus far have at least one thing in common, namely that they are all continuous. It's interesting to note that any continuous mapping, under suitable restrictions on its domain, will have a fixed point. Our goal in this section will be to prove Schauder's theorem, which says that a continuous mapping on any compact, convex subset of a Banach space will have a fixed point. Our method in this is first to show that in R^n, any continuous self-mapping of the unit ball will have a fixed point, and later to generalise this statement to Schauder's theorem.

Brouwer's Fixed Point Theorem

Brouwer's fixed-point theorem involves mappings of the unit ball of R^n into itself. For convenience, we will call $B^n=\{x\in R^n: |x|\leq 1\}$, the closed unit ball of R^n and the surface of B^n will be $S^n=\{x\in R^n: |x|=1\}$. By I we will mean the closed unit interval, i.e. $I=[0,1]$. Also, we establish the following sets of definitions:

<u>DEFINITION</u> 1: Let X and Y be topological spaces, with $X\subset Y$ and $X\neq Y$. Then a function $r:Y\dashrightarrow X$ is called a <u>retraction</u> if r is

continuous and $r(x)=x$ for all x in X. If a retraction exists, X is called a <u>retract</u> of Y.

<u>DEFINITION</u> 2: Let X and Y be topological spaces, and $f,g:X \to Y$ two continuous functions. Then f and g are <u>homotopic</u> if there exists a continuous function $\Phi: X \times I \to Y$ with the property that $\Phi(x,0)=f(x)$ and $\Phi(x,1)=g(x)$. Such a Φ is called a <u>homotopy</u>. A function f is called <u>nullhomotopic</u> if it is homotopic to a constant function. A space is called <u>contractible</u> if the identity map is nullhomotopic.

One might wonder if the two concepts of "contractible" and "retract" are related. In the case we're interested in, it turns out that they are.

<u>LEMMA</u> 1: S^n is a retract of B^n if and only if the identity on S^n is nullhomotopic.

<u>PROOF</u> [12]: Suppose a retraction $r:B^n \to S^n$ exists. Then $r|S^n$ (the restriction of r to S^n) is the identity on S^n, and if we let

$$\Phi(x,t) = r(tx) \text{ for } x \in S^n,\ t \in I,$$

Φ will be a homotopy of the identity to the constant function $r(0)$.

On the other hand, if a homotopy Φ exists, we can again set

$$r(tx) = \Phi(x,t)$$

Then $r:B^n \to S^n$ is well-defined (since $r(0)=\Phi(x,0)$, a constant), and is clearly a retraction.□

As it happens, this lemma is rather a moot point, since, as we will show later, S^n is not a retract of B^n. In fact, this statement is equivalent to Brouwer's Fixed Point Theorem, which we give below.

THEOREM 6.1 (Brower's Fixed Point Theorem): Every continuous mapping $f:B^n \rightarrow B^n$ has a fixed point in B^n.

LEMMA 2: Brouwer's fixed point theorem is equivalent to the proposition that S^n is not a retract of B^n.

PROOF([17], Thm. 4.1.5. cf [12], XVI.2.2): Suppose a retraction r did exist. Then -r is continuous, and has no fixed points.

On the other hand, suppose that there exists some continuous function $f:B^n \rightarrow B^n$ with no fixed points. Then define $r:B^n \rightarrow S^n$ so that r(x) is the point where the directed ray from x to f(x) intersects S^n. This point can be calculated by using the inner product, and, as long as $x \neq f(x)$, r will be well-defined and continuous. Thus, r is a retraction.□

Now we have to show that no retraction exists, or equivalently, that the identity on S^n is not nullhomotopic. To do this, following Dugundji [12], we need the concept of the "degree" of a self-mapping of S^n. In the plane, the degree of a function f is simply the number of times that f(x) "rotates" as x makes one rotation around the circle. This is obviously an integer, since f(x) has to end up back wherever it started. One way to count the rotations is to divide the circle into a finite number of arcs, starting at some arbitrary point and working our way clockwise around the circle. This is called a <u>triangulation</u> of S^2. Each arc should be small enough that its image does not contain as much as half the circle, so we might require that the diameter of each image be less than 1, say. Call the endpoints of the arcs p_1, p_2, ... p_n, $p_{n+1}=p_1$, in clockwise order. For each p_i, p_{i+1}, we choose the shorter of the two arcs between $f(p_i)$ and $f(p_{i+1})$, which we will denote $arc(f(p_i),f(p_{i+1}))$. If this arc runs clockwise, call it positive. Otherwise, it is negative.

Now choose any point x_0 on S^2 such that $x_0 \neq f(p_i)$, for any p_i. For each i, if $x_0 \in arc(f(p_i),f(p_{i+1}))$, then f(x) must have passed through x_0 while x moved from p_i to p_{i+1}. The sign of $arc(p_i,p_{i+1})$ tells which direction f(x) was travelling at the time. In this way, the number of positive $arcs(f(p_i),f(p_{i+1}))$ containing x_0, minus the number of negative arcs it's in, gives us the "net" number of rotations that f(x) makes, in the course of one clockwise rotation of x. This number will be the "degree" of f. It turns out that the degree is independent of either the specific triangulation we use or the point x we choose to count with.

According to the discussion above, it is clear that the degree of the identity is 1 and that of a constant function is 0. We will show later that the degrees of two homotopic functions must be equal. This of course will show that the identity is not nullhomotopic, proving Brouwer's fixed point theorem.

First, however, we need to generalise the idea of degree to n-space.

DEFINITION 3: If $\{p_0, p_1, \ldots, p_n\}$ is a set of n+1 points in R^n, then the convex hull of this set is called an n-simplex. If we establish a definite order for the simplex, then we call it an ordered n-simplex, $S=(p_0, p_1, \ldots, p_n)$. The points p_0, p_1, etc., we will call the vertices of S. The convex hull of any subset of S containing n members is called a face of S. We can also speak of the determinant of the ordered simplex. If $(x_1^i, x_2^i \ldots x_n^i)$ are the coordinates of p_i, then

$$\det(p_0, \ldots, p_n) = \begin{vmatrix} x_1^0 & \cdots & x_n^0 & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ x_1^n & & x_n^n & 1 \end{vmatrix}$$

The sign of an ordered simplex S is merely the sign of det(S). If det(S)=0, then S is said to be degenerate.

Geometrically, it can be seen that S is degenerate if and only if all the vertices of S lie on the same (n-1)-hyperplane.

This gives us the following lemma, which will prove useful:

LEMMA 3: Let S and S' be two non-degenerate n-simplexes that have a common face (i.e., $S=(p_0,p_1,\ldots,p_n)$ and $S'=(p_0',p_1,\ldots,p_n)$. Let L be the $(n-1)$-hyperplane that contains the common face. Then S and S' have the same sign if and only if p_0 and p_0' are on the same side of L (i.e., the line segment between p_0 and p_0' does not intersect L).

PROOF: The points on the segment between p_0 and p_0' are of the form $\lambda p_0+(1-\lambda)p_0'$, $0\leq\lambda\leq 1$. If one of these points, call it p_0'', is in L, then $S''=(p_0'', \ldots, p_n)$ is in L, so S'' is degenerate. But $\det(S'')=\lambda\det(S)+(1-\lambda)\det(S')$, and this could only be 0 if $\det(S)$ and $\det(S')$ were of opposite signs, since neither is degenerate.□

DEFINITION 4: Let $\{p_0, p_1, \ldots, p_{n-1}\}$ be an ordered set of n vectors that all lie on S^n. If it happens that the convex hull of $\{p_0, p_1, \ldots, p_{n-1}\}$ does not contain the origin, then we can project the hull from there to the sphere. We will call this projection an ordered spherical (n-1)-simplex. The determinant and sign of an ordered spherical simplex will be that of the ordered n-simplex $(p_0, p_1, \ldots, p_{n-1}, 0)$. We will call a spherical simplex degenerate if its determinant is 0.
A triangulation of S^n is a covering of S^n by finitely many spherical n-simplexes such that no two overlap except at a face

and each face of any simplex is shared by exactly one other simplex.

If we are considering a continuous function $f:S^n \to S^n$, and we have a specific triangulation T in mind, we can discuss the "image" of each S in T, in the sense that $f(S)=(f(p_0), \ldots, f(p_{n-1}))$. Since S^n is compact, we can always find a triangulation T such that, for each $S=(p_0, p_1, \ldots, p_{n-1})$ in T, $\text{diam}\{f(p_0), f(p_1), \ldots, f(p_{n-1})\}<1$. This condition will assure us that, for each spherical simplex in T, the convex hull of $\{f(p_0), \ldots, f(p_{n-1})\}$ will not contain the origin, so it can be projected into a spherical simplex. Call a triangulation of this type an "f-triangulation". In each case, let f(S) inherit the order of S, although the sign may be different. Also note that the image set of spherical simplexes won't necessarily be a triangulation or even a cover of S^n. It is also likely to contain degenerate or overlapping spherical simplexes.

Now we are ready to define the degree of a function.

DEFINITION 5: Let $f:S^n \to S^n$ be continuous and suppose T is an f-triangulation of S^n. Order each simplex in T so that it is positive. Choose any point x on S^n that is not on a face of f(S) for any S in T. Let p(x) be the number of positive spherical simplexes f(S) that contain x, and n(x) the number of negative image spherical simplexes containing x. Then let $D(f,T,x)=p(x)-n(x)$.

This number $D(f,T,x)$ will be what we use to define the degree of a function. We now show that $D(f,T,x)$ is independent of both T and x.

LEMMA 4: Let $f:S^n \dashrightarrow S^n$ be continuous and let T be an f-triangulation of S^n. Order each simplex in T so that it is positive. Then $D(f,T,x)$ is the same for any x on S^n that is not contained in any face of any $f(S)$, $S \in T$.

PROOF: There are two cases to consider, depending on whether $f(T)=\{f(S): S \in T\}$ contains any degenerate simplexes. CASE 1, no degenerate simplexes: Let y and z be any two suitable points on S^n. We can connect them with a curve that enters or leaves any simplex only by way of one of its faces. Consider what happens to $D(f,T,x)$ as x moves along the curve from y to z.

Clearly, $D(f,T,x)$ cannot change except when x passes through a face of an $f(S)$. Suppose the curve does pass through the image of a given face $F=(p_1, \ldots, p_{n-1})$. Then F is shared by exactly two spherical simplexes, say $S=(p_0, p_1, \ldots p_{n-1})$ and $S'=(p_0', p_1, \ldots, p_{n-1})$. Let L be the hyperplane that contains the common face $(f(p_1), \ldots, f(p_{n-1}))$. Assuming that $f(p_0)$ and $f(p_0')$ are on the same side of L, the image simplexes would seem to have the same sign. However, since S and S' are in the triangulation, p_0 and p_0' must be on opposite sides of the common face, and so S and S' must have opposite signs. Thus, we need to re-order one of them, say S', to make them both positive, in accordance with our hypothesis. Let S" be the re-ordered simplex. Then f(S") is of opposite sign to f(S') and therefore of opposite sign to f(S) also. Finally, when we are passing through the common face of f(S) and f(S') we are entering (or leaving) both f(S) and f(S'), so we are gaining or losing one positive and one negative simplex simultaneously. Thus, D(f,T,x) remains unchanged.

A similar argument applies if $f(p_0)$ and $f(p_0')$ are on opposite sides of L, except in that case, we would be gaining a positive or negative simplex at the same time as losing another of the same sign.

CASE 2, degenerate simplexes: In this case, we could move certain vertices by a small enough amount that we don't affect the number of simplexes that contain y or z or change the sign of any non-degenerate simplex, but by enough that all the new simplexes will be non-degenerate. Using the case above, we see that D(f,T,x) will still remain unchanged. □

LEMMA 5: Let $f:S^n \text{-->} S^n$ be continuous, and let T and T' be f-triangulations of S^n. Then D(f,T,x) = D(f,T',x).

PROOF: If we add an additional vertex to T, we can build a new triangulation that will use the new vertex, and it will clearly still be an f-triangulation. Also, we can build the new triangulation so that it contains at least one of the simplexes of the old triangulation. Then if we choose an x contained in the image of this simplex, D(f,T,x) will be unchanged. In this way, we could incrementally change T and T' into a common triangulation T", and in that case, we see $D(f,T,x)=D(f,T",x)=D(f,T',x)$.□

The last two lemmas show that the degree of a function depends on neither the triangulation we use nor the point we choose to count simplexes on. From now on, we can refer to $D(f,T,x)$ as $D(f)$. It is easy to see that the degree of the identity is 1, and that the degree of a constant function must be 0. Our next lemma shows that homotopic functions always have the same degree. Indeed, if Φ is a homotopy of two continuous functions f and g, then we can define functions f_t, $t \in I$, such that $f_t(x)=\Phi(x,t)$. If we do that, we can speak of a function $D(t)=D(f_t)$. This function turns out to be continuous, and since it has only integer values, it must be constant, which will prove Brouwer's theorem for us. First, however, we need the following lemma.

<u>LEMMA</u> 6: Let $f:S^n \to S^n$ be continuous, and let T be an f-triangulation. Then there is an $\varepsilon>0$ such that if f^* is another continuous function on S with the property that $|f(p)-f^*(p)|<\varepsilon$ for every vertex p of T, then $D(f)=D(f^*)$.

<u>PROOF</u>: Fix $x \in S^n$, not on any face of an $f(S)$, $S \in T$. Let $\varepsilon < \mathrm{Min}\{d(x,f(p)): p \text{ a vertex in } T\}$. Clearly, $D(f,T,x)$ is unchanged by moving the vertices of $f(T)$ by an amount smaller than ε, and small enough that no simplex containing x changes

sign. Thus, $D(f)=D(f^*)$, by Lemma 4.▯.

LEMMA 7: If $f,g:S^n \to S^n$ are homotopic, then $D(f)=D(g)$.

PROOF: Let $\Phi:S^n \times I \to S^n$ be the homotopy of f and g. For each $t \in I$, define a function $f_t(x)=\Phi(x,t)$. Since $S^n \times I$ is compact and Φ is continuous, Φ is also uniformly continuous. Thus we can find $\mathcal{E}>0$ such that $d(x,y)<\mathcal{E}$ implies $d(f_t(x),f_t(y))<1$ for all t in I. This means there exists a triangulation T that is an f-triangulation for every t in I. We will use this triangulation from now on.

Now choose any t in I. By Lemma 6, there is an $\mathcal{E}$ such that if $|f_t(p)-f^*(p)|<\mathcal{E}$ for every vertex p of T, then $D(f_t)=D(f^*)$. However, again by uniform continuity, there is a δ such that $|t-t'|<\delta$ implies $|f_t(x)-f_{t'}(x)|<\mathcal{E}$ for every x in S^n. Thus, for every t in I, there is a δ such that $|t-t'|<\delta$ implies that $D(f_t)=D(f_{t'})$. If we define a function $D^*:I \to Z$ (the integers) by $D^*(t)=D(f_t)$, then D^* is a continuous integral function, and thus a constant function, so $D(f)=D(g)$.▯

Finally, of course, the proof of Brouwer's function is trivial. We have shown in Lemma 2 that it is equivalent to the non-contractibility of the unit ball. By Lemma 1, this is equivalent to the identity on S not being nullhomotopic. This last is proved by Lemma 7 and the fact that the degrees of the identity and the constant map are not equal.

Also, Brouwer's theorem is easily generalised to apply to any compact, convex subset of a finite-dimensional space. In the first place, it is clear that if X is a topological space with the property that every continuous function has a fixed point, then every space Y homeomorphic to X has the same property. Also, any n-dimensional normed space is homeomorphic to R^n. (See Dugundji, 12, p. 413) And it can be shown that any compact convex subset of R^n is homeomorphic to B^n (or at least to B^m, with m<n). Thus, we have the following easy corollary to Theorem 6.1:

COROLLARY 6.1.1: If C is a compact, convex subset of a finite-dimensional Banach space, and f:C-->C is continuous, then f has a fixed point in C.

Schauder's Fixed Point Theorem

Schauder's Fixed Point Theorem essentially removes the "finite-dimensional" restriction from Brouwer's Theorem. The method we use, following Istratescu [17], reduces the infinite-dimensional case to a sequence of finite-dimensional cases.

THEOREM 6.2 (Schauder's Fixed Point Theorem): Let C be a compact, convex subset of a Banach space X, and suppose T:C-->C is continuous. Then T has a fixed point in C.

PROOF [17]: Since C is compact, for every integer n we can find a finite set of points of C, $\{x_1^n, x_2^n, \ldots, x_m^n\}$, with the property that, for each $x \in X$, there exists an integer j such that $|x - x_j^n| < 1/n$.

Then we can define the following functions:

$$\alpha_i^n(x) = \begin{cases} (1/n) - |x - x_i^n| & \text{if } |x - x_i^n| < (1/n) \\ 0 & \text{otherwise} \end{cases}$$

These are clearly continuous, and by compactness,

$$\sum_{i=1}^{m} \alpha_i^n(x) \neq 0$$

for any x. Thus, we can define the following continuous functions:

$$T_n(x) = \sum_{i=1}^{m} \frac{\alpha_i^n(x) \cdot x_i^n}{\sum_1^m \alpha_i^n(x)}.$$

Let E_n be the closed convex hull of $\{x_1^n, \ldots, x_m^n\}$. E_n is compact since it is closed, and we observe that $T_n(C) \subseteq E_n$. Also, E_n is a subset of the vector space spanned by $\{x_1^n, \ldots, x_m^n\}$. Notice, too, that if $|x_i^n - x| \geq (1/n)$, then $\alpha_i^n(x) = 0$, so we have the following inequality:

$$|T_n x - x| = \left\| \frac{\sum \alpha_i^n(x) \cdot (x_i^n - x)}{\sum \alpha_i^n(x)} \right\|$$

$$< \left\| \frac{(1/n)\sum_{i=1}^{m}\alpha_i^n(x)}{\sum\alpha_i^n(x)} \right\|$$

$$= 1/n.$$

So $\{T_n\}$ converges uniformly to the identity. If we define $T_n'(x)=T_n(Tx)$, then $T_n' \to T$ uniformly.

By the remarks above, we see that $T_n'(E_n) \subseteq E_n$, so we can use Brouwer's fixed point theorem to find a fixed point x_n for each T_n'. Since C is compact, the sequence $\{x_n\}$ has a subsequence that converges, say to x_0. Assuming that $x_n \to x_0$, we easily obtain the result that

$$|Tx_0 - x_0| \le |Tx_0 - Tx_n| + |Tx_n - T_n'x_n| + |T_n'x_n - x_n| + |x_n - x_0|.$$

The right-hand side converges to 0, so x_0 must be a fixed point of T. □

VII. Some Miscellaneous Results

In this chapter, we touch briefly on two areas of fixed-point theory that we have not yet considered.

Common Fixed Points

One area of interest in fixed point theory is the idea of "common fixed points". When dealing with common fixed point questions, we generally have a metric space X and a family of mappings $\{T_\alpha : T_\alpha(X) \subseteq X\}$, which are usually considered to be commutative, and we want to know under what circumstances these mappings will have a common fixed point. The findings in this area tend to follow results for single mappings. See [10] or [11] for some examples of common fixed point theorems.

In this section, we present a theorem dealing with a sequence of mappings which, taken together, are similar to the Banach contractions, although not all the mappings will necessarily be contractions. This idea was inspired by a paper of Yun's [31]. The following has been presented in Shen & Sound [26].

DEFINITION 1: Let X be a metric space. A sequential mapping T on X is a sequence $\{T_n\}$ of self-mappings of X. By induction, we define $T^1 = T_1$, and $T^{n+1} = T_{n+1}T^n$. A point z is said to be a <u>common fixed point</u> of T if $T_n z=z$ for each T_n.

We call r_n the <u>contraction ratio of</u> T_n, in case

$$r_n = \sup\{d(T_n x, T_n y)/d(x,y):\ x,y\in X;\ x\neq y\}.$$

If this holds, clearly we have

$$d(T^n x, T^n y) \leq r_1 r_2 \ldots r_n d(x,y)$$

for each $x,y\in X$ and each n.

Then we say that a sequential mapping T is a <u>geometric mean contraction</u> if each T_n has a contraction ratio and there exists a constant G<1 such that for each n, the geometric mean satisfies

$$0\leq (r_1 r_2 \ldots r_n)^{1/n} < G < 1.$$

We say a sequential mapping T is <u>sequentially commutative</u> if, for any positive integers m,n, and each x in X, we have $T_m(T^n x)=T^n(T_m x)$.

<u>THEOREM</u> 7.1: Let X be a complete metric space and T a sequentially commutative geometric mean contraction on X. If there exists $x_0\in X$ such that the set

$$\{d(T_n x_0, x_0):\ n \text{ any positive integer}\}$$

is bounded, then T has a unique common fixed point z, and $T^n x \to z$, as $n\to\infty$, for each x in X.

<u>PROOF</u>: Let $D=\sup d(T_n x_0, x_0)$. Then consider $d(T^{n+1}x_0, T^n x_0)$.

$$\begin{aligned} d(T^{n+1}x_0, T^n x_0) &= d(T^n T_{n+1} x_0, T^n x_0) \\ &\leq G^n d(T_{n+1} x_0, x_0) \\ &\leq G^n D \end{aligned}$$

Then we have, similar to the argument in Theorem 2.1,

$$d(T^{m+k}x_0, T^m x_0) \leq G^m D/(1-G)$$

Then for $k \geq 0$, as $m \to \infty$, $d(T^{m+k}x_0, T^m x_0) \to 0$, so $\{T^m x_0\}$ is a Cauchy sequence, and thus converges to a point z in X. Moreover, if $y \in X$, then $d(T^n x_0, T^n y) \leq G^n d(x,y)$, so as $n \to \infty$, $d(T^n x_0, T^n y) \to 0$, and thus $T^n y \to z$ as well. Finally, z is also a fixed point of each T_n, for consider

$$\begin{aligned} z &= \lim T^m(T_n y) \\ &= \lim T_n(T^m y) \\ &= T_n z. \end{aligned}$$

Inductively, $T^n z = z$ also.

Furthermore, z is unique for each T^n, since all the T^n's are Banach contractions. In addition, z is the only fixed point shared by every T_n, although it may be possible for some T_n to have multiple fixed points.□

The Converse of Banach's Theorem

Another interesting question in fixed point theory is whether every function that has a unique fixed point is a contraction. Obviously, the answer is no, but the following theorem, from Bessaga [6], is interesting in this regard. It says, essentially that if T and each iteration of T have a unique fixed point, then we can always find a metric d that makes T a Banach contraction.

THEOREM 7.2: Let X be an abstract set, and T a self-mapping of X of which every iteration T^n has exactly one fixed point. Then for every real number K, $0<K<1$, there is a complete metric d on X with $d(Tx,Ty)<Kd(x,y)$.

PROOF: Since each iteration has only one fixed point, the fixed point must be the same for every iteration. We will call the fixed point z.

Now define two equivalence relations on X.

(1.) $x\sim y$ if and only if either $x=y$ or there is a positive integer n such that

$$T^{n-1}x\neq T^n x; \quad T^{n-1}y\neq T^n y; \text{and} \quad T^n x=T^n y.$$

(2.) $x\approx y$ if and only if there are positive integers m and n with

$$T^m x=T^n y.$$

We also define the equivalence class $[x]=\{y\in X: x\sim y\}$, for $x\in X$. Then, letting $[X]=\{[x]: x\in X\}$, define $[[x]]=\{[y]\in[X]: x\approx y\}$.

An interesting point here is that $x\in[z]$ (z being the fixed point) if and only if $x=z$. Otherwise, $x\sim z$ implies that $T^{n-1}z\neq T^n z$, for some positive integer n.

Now we can define an integer-valued function f on X that meets the following criteria:

i) If $x\sim y$, then $f(x)=f(y)$

ii) If $x\sim Ty$, then $f(x)=f(y)+1$.

We accomplish this by recourse to the Axiom of Choice. From each set $[[x]]$ where $[z]\notin[[x]]$, we can choose one element $[x]$. For every x' in $[x]$, set $f(x')=0$. On the other hand, if y is such that $[y]\neq[x]$, but $[y]$ is in $[[x]]$, then it must be that

$T^m x = T^n y$, for some positive integers m,n.

Of course, to satisfy ii, we would want $f(T^m x) = f(x) + m = m$, so set $f(T^m x) = m$. Also $T^m x \sim T^n y$, so set $f(T^n y) = m$ also. Then we merely set $f(y) = m-n$, and $f(y') = f(y)$ for all y' in [y]. In this way we can define f(x) for each x in X.

One might object, what happens if, for example, $x \sim y$ and $x \sim Ty$ at the same time? In that case, $y \sim Ty$, and it can be seen that $Ty = z$, so $x, y, Ty \in [z]$, which means $x = y = Ty = z$. In [[z]], then, we ought to choose $f(z) = 0$, and, for $y' \in [y] \in [[z]]$, set $f(y') = -n$ if n is the smallest number with $T^n y' = z$.

Finally, we are ready to define our metric. For a given pair $x, y \in X$, we find $m, n \in \{0, 1, 2, \ldots \infty\}$ as small as possible so that $T^m x = T^n y$. (Here we adopt the convention that $T^\infty x = z$, so this can be done unambiguously for every pair.) Then the metric d is given by

$$d(x,y) = \sum_{i=1}^{m} K^{f(x)+i} + \sum_{i=1}^{n} K^{f(y)+i}$$

where K, remember, is the Lipschitz constant chosen in the statement of the theorem. This can be be shown to be a valid metric, although the triangle inequality is a little complicated. It is also a complete metric, because, as we will show, any Cauchy sequence under this metric is either eventually constant or else convereges to z.

First note that, for any $x \in X$,

$$d(x,z) \leq \sum_{i}^{\infty} K^{f(x)+i} = (K^{f(x)+1})/(1-K).$$

Now let $\{x_n\}$ be a Cauchy sequence in X that is not eventually constant. Choose $\varepsilon>0$. If we take M such that $m,n \geq M$ implies $d(x_m,x_n)<\varepsilon(1-K)/2$, then for $n \geq M$, $d(x_n,z)<\varepsilon$. To show this, take any $x_m \neq x_n$, with $m \geq M$. Then it must be that

$$K^{f(x_m)+1} \leq d(x_m,x_n) < \varepsilon(1-K)/2.$$

(Note: It is possible that this may not actually be true for $f(x_m)$. But in that case, it must be true for $f(x_n)$, which is even better for us.) Then we also have

$$d(x_m,z) \leq K^{f(x_m)+1}/(1-K) < \varepsilon/2.$$

And in this case, of course, by the triangle inequality, we have $d(x_n,z)<\varepsilon$.

Lastly, we should show that $d(Tx,Ty) \leq Kd(x,y)$. If $T^m x=T^n y$ with m,n as small as possible, then $(m-1)$ and $(n-1)$ are the smallest numbers that make $T^{m-1}(Tx)=T^{n-1}(Ty)$ true also. Furthermore

$$\begin{aligned}
\sum_{i=1}^{m-1} K^{f(Tx)+i} &= \sum_{i=1}^{m-1} K^{f(x)+1+i} \\
&= K\sum_{i=1}^{m-1} K^{f(x)+i} \\
&\leq K\sum_{i=1}^{m} K^{f(x)+i} .
\end{aligned}$$

(The "$\leq$" comes in because it might be that $m=\infty$.) The same is true for y, so $d(Tx,Ty) \leq Kd(x,y)$.□

Also, we might mention that L. Janos [18] has given the following more topologically oriented converse:

THEOREM 7.3: Let X be a compact, metrizable topological space, and let T be a continuous self-mapping of X that satisfies

$$\bigcap_{1}^{\infty} T^{n}(X) = \{z\},$$

for z some element of X. Then if K is any number with $0<K<1$, there exists a metric d that generates the original topology on X, and that satisfies $d(Tx,Ty) \leq Kd(x,y)$, for all x, y in X.

REFERENCES

1. D.E. Alspach, A fixed point free nonexpansive map, Proc. Amer. Math. Soc. 82(1981), 423-424.

2. D.F. Bailey, Some theorems on contractive mappings, Journal London Math. Soc. 41(1966), 101-106.

3. J.-B. Baillon & R. Schoneberg, Asymptotic normal structure and fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 81(1981), 257-264.

4. S. Banach, Sur les operations dans les ensembles abstraites et leurs applications, Fund. Math. 3(1922), 133-181.

5. L.P. Belluce & W.A. Kirk, Fixed-point theorems for certain classes of nonexpansive mappings, Proc. Amer. Math. Soc. 20(1969), 141-146.

6. C. Bessaga, On the converse of the Banach "Fixed point principle", Colloq. Math. 7(1959), 41-43.

7. D.W. Boyd & J.S.W. Wong, Another proof of the contraction mapping principle, Canad. Math. Bull. 11(1968), 605-606.

8. F. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Nat. Acad. Sci. U.S.A. 53(1965), 1272-1276.

9. __________., Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A. 54(1965), 1041-1044.

10. R.E. Bruck, Jr., A common fixed point theorem for a commuting family of nonexpansive mappings, Pacific Journal of Math. 53(1974), 59-71.

11. R. DeMarr, Common fixed points for commuting contraction mappings, Pacific Journal of Math. 13(1963), 1139-1141.

12. J. Dugundji, Topology, Allyn and Bacon, Inc., Boston (1966).

13. M. Edelstein, On fixed and periodic points under contractive mappings, Journal London Math. Soc. 37(1962), 74-79.

14. __________., On nonexpansive mappings, Proc. Amer. Math. Soc. 15(1964), 689-695.

15. D. Gohde, Zum Princip der kontraktiven abbildung, Math

Nachr. 30(1965), 251-258.

16. R. Goldberg, Methods of Real Analysis, John Wiley & Sons, Inc, New York (1976).

17. V.I. Istratescu, Fixed Point Theory An Introduction, D. Reidel Publishing Co., Dordrecht, Holland (1981).

18. L. Janos, A converse of Banach's contraction theorem, Proc. Amer. Math. Soc. 18(1967), 287-289.

19. L.A. Karlovitz, Existence of fixed points of nonexpansive mappings in a space without normal structure, Pacific Journal of Math. 66(1976), 153-159.

20. W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72(1965), 1004-1006.

21. __________., On mappings with diminishing orbital diameters, Journal London Math. Soc. 44(1969), 107-111.

22. __________., Fixed point theory for nonexpansive mappings, Springer-Verlag Lec. Notes in Math. 886(1980), 484-505.

23. I.I. Kolodner, On the proof of the contractive mapping theorem, Amer. Math. Monthly 74(1967), 1212-1213.

24. H. Poincare, Sur les courbes definies par les equations differentielles, Journal de Math. 2(1886).

25. G. Schechtman, On commuting families of nonexpansive operators, Proc. Amer. Math. Soc. 84(1982), 373-376.

26. C.Y. Shen & A. Sound, Contractive mappings on a premetric space, preprint.

27. V. Smulian, On the principle of inclusion in the space of type(B), Mat. Sb.(N.S.) 5(1939), 327-328.

28. A. Taylor & D. Lay, Introduction to Functional Analysis, John Wiley & Sons, New York (1980).

29. S. Willard, General Topology, Addison Wesley Publishing Company, Reading, Mass. (1970).

30. J.S.W. Wong, On fixed points of distance diminishing transformations of metric spaces, preprint.

31. T. Yun, Fixed point theorems of sequential mappings with geometric mean contraction and applications, preprint.