RIGIDITY OF GRAPHS

by

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Rigidity of Graphs

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ABSTRACT

We define a graph G to be a set of vertices $V(G) = \{1,...,v\}$ together with a set of edges E(G) of unordered pairs from V. We represent G in R^n by points $p_1,...,p_v$, corresponding to the vertices of G, together with the line segments which join p_i and p_j when (i,j) is in E(G).

We say the representation of a graph G is rigid in R^n if every continuous movement of the representation which preserves edge lengths also preserves the distance between every pair of points in that representation. We say the representation of G is flexible if there is a continuous movement of that representation which preserves edge lengths but does not preserve the distance between every pair of points in the representation.

We use the Inverse Function Theorem to determine the rigidity or flexibility of a given representation of G. From this we show that if a representation of G is rigid in R^n and the affine hull of $p_1,...,p_v$ has dimension n then G must be n-connected, have at least $nv-n(n+1)/2$ edges and contain a subgraph which is minimally rigid in R^n. We demonstrate the existence of 2n-1 connected graphs which are flexible in R^n.

We apply the above results to the Structural Analysis of trusses and spaceframes. In particular, the determination of minimally rigid sub-graphs gives a new method for the automation of the flexibility method of structural analysis.

We describe some flexing panel structures including a quonset type shelter and a flexing tube.

The results from a computer program are used to determine the rigidity or flexibility of some specific examples.

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My wife, Tammy, drafted and constructed many of the diagrams and models which are referred to in this thesis. I owe a special debt of gratitude to Tammy for her patience and understanding during the preparation of this thesis.

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CHAPTER 1

INTRODUCTION

We see examples of rigid and flexible objects around us in many manifestations. These range from simple structures such as a tripod to sophisticated machinery such as jet aircraft and also include our own bodies.

When one studies a mechanism or designs a structure some sort of a model is used to define the relationships between the various components. One of the simplest models that is often used is a point and line diagram. Indeed one of the first things that a child learns to draw is a stick figure. This type of model arises naturally in many applications and corresponds directly to the mathematical concept of a graph as a set of points together with a set of relations between pairs of points. The intent of this thesis is to explore some of the relationships between the rigidity or flexibility of a structure and the properties of an abstract mathematical model of that structure.

The modern study of rigidity dates back to to 1812 when A. L. Cauchy [9] published his paper on the rigidity of convex polyhedra with rigid polygonal faces. There is no doubt that Cauchy had more than an academic interest in the theory of rigidity as his title is given as "Ingenieur des

Ponts et Chaussees " which literally translates to Engineer of Bridges and Roads. In the latter part of the nineteenth century and the early part of the twentieth century work was done by such people as James Maxwell, Sir Robert Ball and Raoul Bricard (see [6]) .

More recently a paper by G. Laman [9] in 1970 sparked renewed interest in rigidity theory with a graph theoretic approach. Laman outlines rigidity requirements for planar graphs and clearly distingushes between the concepts of *rigidity* and *infinitesimal rigidity.* Branko Grünbaum and G. C. Shephard [8] point out the ambiguity and lack of rigour in previous treatments of rigidity.

In 1974 H. Gluck [7] develops a theorem for determining rigidity in his treatment of closed simply connected surfaces. This theorem is expanded to deal with graphs in Euclidean n-space by L. Asimow and B. Roth [1] in 1978. L. Lovâsz and Y. Yemini [10] use combinatorial arguments and results from Matroid Theory to examine the rigidty of planar graphs in a paper published in 1982. Also published in 1982 is a paper by Henry Crapo and Walter Whitely [6] which deals with rigidity and statics of frameworks from the point of view of projective geometry.

This thesis is based primarily on the works of G. Laman and Asimow and Roth. In Chapter 2 the theory of rigidity in Euclidean n-space is examined.

Chapter 3 deals with minimally rigid graphs. These graphs have some interesting properties which have a special application in structural analysis. Also we show how the singular value decomposition can be used to apply rigidity theory to concrete examples.

In Chapter 4 we give a brief introduction to the methods employed in the structural analysis of bar and joint frameworks. We then show how the results of Chapters 2 and 3 can be applied to structural analysis. In particular we demonstrate how rigidity theory and the singular value decomposition can be used to automate the flexibility method of structural analysis.

Chapter 5 describes some flexing panel structures which can be created from a flat piece of material. There are a large number of different designs based on the relatively simple ideas outlined here. While this chapter may not be rigorous, it was the study of these structures that first interested the author in rigidity theory.

In Chapter 6 we examine some concrete examples using a

Fortran program written by the author. This program makes use of the singular value decompostion to apply the theory developed to the concrete examples. A listing of the program code is included in this thesis as an appendix.

The author plans further work in the application of these results to structural engineering and in the investigation of the properties of flexing panel structures.

CHAPTER 2

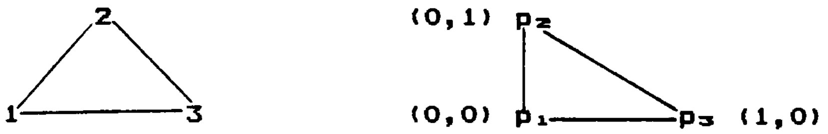
THE THEORY OF RIGIDITY OF GRAPHS

In this chapter we will develop mathematical tools for determining the rigidity (or flexibility) of almost all representations of a given graph in an n-dimensional Euclidean space. Also we will give some corollaries of the theory described.

2.1 PRELIMINARY DEFINITIONS AND MOTIVATION

For our purposes a graph $G=\{V,E\}$, where V is a set of vertices $V=(1,\dots,v)$ and E is a set of edges, where each edge is an unordered pair from V. We will restrict our attention to simple connected graphs. In other words we will allow no multiple edges, no edges containing only one vertex (a loop) and every vertex must be connected to every other vertex by a path. Throughout the rest of this thesis we will use v and e to denote the number of elements in V and E respectively.

We represent G in R^n by selecting v points $p_1,\dots,p_v$ in R^n such that p_i corresponds to vertex i of V. Note that $p_i=(p_{i,1},\dots,p_{i,n})$. These points in R^n then represent the vertices of our graph.



$G = \{V,E\} = \{(1,2,3),((1,2),(1,3),(2,3))\}$

Figure 2.1

The natural representation of the edges of our graph G in R^n is then the line segments connecting points p_i and p_j where (i,j) is an element of E.

If we consider the possible motions of one point in R^n we see that we require n coordinates to represent this. To represent the motion of v unconstrained points in R^n we require nv coordinates. Thus we can represent the points $p_1,\ldots,p_v$, p_i in R^n, by a single point p in R^{nv} such that

$$p = (p_{1,1},\ldots,p_{1,n},\ldots,p_{v,1},\ldots,p_{v,n}) .$$

That is the first n coordinates of p represent p_1, the second n coordinates of p represent p_2 , and so on. In the case of Figure 2.1 $p=(0,0,0,1,1,0)$. We will denote this *representation* of G in R^n at $p_1,\ldots$ $\ldots,p_v$ by $G(p)$.

The basic notion of rigidity requires that an object or structure be inflexible. This does not mean that the structure cannot move at all but that the structure cannot change shape. A triangle constructed by joining rigid bars together at the ends has this property. The triangle may be moved about or turned around but the shape does not change. However suppose four rigid bars are connected at the ends by flexible joints to form a rectangle, it is easy to see that this arrangement is not rigid.

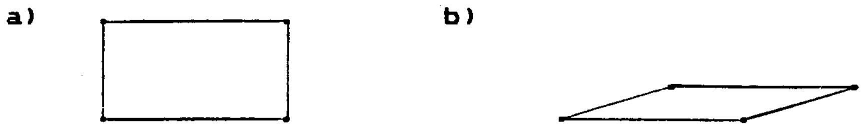


Figure 2.2

For our representation of the graph G in R^n, denoted by G(p), we will allow the points of G(p) to move in R^n but will require that the lengths of the line segments, corresponding to the edges of G, remain constant. To represent the edge lengths of G(p) we will define an *edge function* for G(p). We order the edges of G. Then the edge function is

$$f_G(p)=(\ldots,||p_i-p_j||,\ldots)$$

where $||p_i-p_j||$ is the k^{th} coordinate of $f_G(p)$ if (i,j) is the k^{th} edge of G. This gives us a function from R^{nv} into $R^{|E|}$. We note that the square of the edge lengths is used in [1].

Now that we have defined the edge function for some G(p) we will define rigidity and flexibility in terms of this edge function.

<u>Definition</u> We say that T is an *isometry* of R^n if

$$||Tx - Ty|| = ||x - y|| \quad \text{for all } x,y \text{ in } R^n .$$

We say that two representations of a graph are *congruent* if there is an isometry of R^n which maps one representation to the other .

Let G(q) be the representation of G at points $q_1,...,q_v$ in R^n. If

$$f_G(q) = f_G(p)$$

then the length of the corresponding edges of G(p) and G(q) are the same. This does not mean that G(p) is congruent to G(q). See Figure 2.3 for an example of two representations which are not congruent yet have the same edge lengths.

There are three cases. First, G(q) is congruent to G(p) . Second, G(q) may be a representation of G that can be reached by deforming G(p) while not changing the lengths of the edges (as in Figure 2.2 a and b). In this case we would call G(p) flexible. Third, G(q) is not congruent to G(p) and is not a flexing of G(p). Figure 2.3 gives an example of this in R^2 .

Figure 2.3

We are then interested in the set of points q in R^n such that $f_G(q)$ equals $f_G(p)$. This set is

$f_G^{-1}(f_G(p))$,

which we will call the *fibre* of G at p , and denote as fibre(G,p).

Let K_v be the complete graph on v vertices (every pair of vertices is joined by an edge) and K(p) be the representation of K_v on the points of G(p) . Then the distance between every pair of points of K(p) is fixed. Movements which preserve the distances between the points of K(p) correspond to rigid movements of G(p) . It is easy to see that the possible movements of K(p) must be contained by the possible movements of G(p) .

2.1.1 <u>Definition</u>: Let G be a graph on v vertices, K the complete graph on v vertices and p a point in R^{nv} . Then G(p) is *rigid* in R^n if there exists a neighborhood U of p in R^{nv} such that

$$\text{fibre}(G,p) \cap U = \text{fibre}(K,p) \cap U$$

G(p) is *flexible* in R^n if there exists a continuous path $x:[0,1] \to R^{nv}$ such that $x(0) = p$ and $x(t)$ is in

$$\text{fiber}(G,p) \cap U \; - \; \text{fibre}(K,p) \cap U$$

for all t in $(0,1]$ and some open neighborhood U of p.

In other words $G(p)$ is rigid only if any movement of the points of $G(p)$ which preserves edge lengths is a rigid movement of $G(p)$.

Note that if

$$\text{fiber}(G,p) \cap U \; - \; \text{fibre}(K,p) \cap U \neq \emptyset \; ,$$

we can construct a smooth path $x(t)$ with $x(0) = p$ by taking neighborhoods U_i such that U_i contains U_{i-1} and $x(t_i)$ is in $U_i - U_{i-1}$.

We will now develop this characterization of rigidity and flexibility into a useful tool.

2.2 APPLICATION OF THE INVERSE FUNCTION THEOREM

For a smooth map $f:X \to Y$ where X and Y are smooth manifolds, we denote the *Jacobian* of f at x in X by $df(x)$. Let $k = \max(\text{rank } df(x): x \text{ in } X)$. Then x is a *regular point* of f if rank $df(x) = k$ and a *singular point* otherwise.

2.3 <u>Proposition</u> [1] *Let* $f:R^n \to R^m$ *be a smooth map and* $k = \max\{\text{rank}(df(x)): x \text{ in } R^n\}$. *If* x_0 *is a regular point of* f *then the image under* f *of some neighborhood of* x_0 *is a* k*-dimensional manifold.*

<u>Proof</u> Let $f = (f_1, f_2)$ where f_1 consists of the first k coordinate functions of f and assume that rank $df_1(x_0) = k$. Since rank $df_1 = k$ the inverse function thereom [20,p34] yields local coordinates at x_0 such that $f_1(x_1, x_2) = x_1$.
Thus in local coordinates

$$df = \begin{vmatrix} I & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$$

Since rank$(df) = k$ near x_0, $\partial f_2/\partial x_2 = 0$ near x_0 . Then $f_2(x_1, x_2) = g(x_1)$ which gives

$$f(x_1, x_2) = (x_1, g(x_1))$$

near x_0 . Thus f maps some neighborhood of x_0 onto $\{(x,y): y = g(x)\}$, the graph of g, which is a k-dimensional manifold since g is differentiable. []

It follows that if p is a regular point of f_0 then fibre(G,x) is a manifold of co-dimension k near p .

A subset M of R^n is said to be an *affine set* if M contains the entire line through each pair of points in M . The *dimension* of an affine set is defined to be the dimension of the *subspace* $M-M = \{x-y : x,y \text{ in } M\}$ parallel to M . We will denote the dimension of an affine set M by dim(M) .

The *affine hull* of a set S in R^n is the smallest affine set containing S . Let P be the affine hull of $p_1,\ldots,p_v$.

2.3 DETERMINING RIGIDITY AND FLEXIBILITY

The following test for rigidity was introduced by Herman Gluck in [7] and expanded by Asimow and Roth to deal with higher dimensional cases in [1] .

2.3 Theorem [1] [7] *Let* G *be a graph with* v *vertices,* e *edges and edge function* f_G. *Suppose that* p *in* R^{nv} *is a regular point of* f_G *and let* dim(P) = m . *Then the graph* G(p) *is rigid in* R^n *if and only if*

$$\text{rank}(df_G(p)) = nv - (m+1)(2n-1)/2$$

and G(p) *is flexible in* R^n *if and only if*

$$\text{rank}(df_G(p)) < nv - (m+1)(2n-1)/2 .$$

Proof. Let $k = \max\{\text{rank } df_G(x) : x \text{ in } R^{nv}\}$. Then rank $df_G(p) = k$. By Proposition 2.3 there exists a neighborhood U of p in R^{nv} such that the intersection of fibre(G,p) and U is an (nv-k)-dimensional manifold.

Let J(n) be the n(n+1)/2-dimensional manifold of isometries of R^n and define $F: J(n) \to R^{nv}$ by

$$F(T) = (Tp_1, \ldots, Tp_v) \quad \text{for } T \text{ in } J(n) .$$

Note that F is smooth and that the image under F of $f_G(p)$ is fibre(K,p) (F corresponds to the rigid movements of G(p)). Then $F^{-1}(p)$ is the subgroup of J(n) consisting of isometries which yield the identity on P. Then $F^{-1}(p)$ can be identified with the (n-m)(n-m-1)/2 -dimensional manifold O(n) of orthogonal linear transformations of N where N is the (n-m)-dimensional subspace orthogonal to the m-dimensional subspace P-P .
Let

$$w: J(n) \to J(n)/F^{-1}(p)$$

be the natural projection and define

$$\underline{F}: J(n)/F^{-1}(p) \to R^{nv}$$

so that $F=\underline{F}\, w$. Then $\underline{F}$ is smooth and $\underline{F}: J(n)/F^{-1}(p) \to im(\underline{F})$ is a diffeomorphism. Since $J(n)/F^{-1}(p)$ is a manifold of dimension (m+1)(2n-m)/2 we conclude that $im(\underline{F})$ = im(F) = fibre(K,p) is an (m+1)(2n-m)/2-dimensional manifold. Note that this corresponds to the rigid movements of G(p).

Since all the rigid movements of G(p) are contained in the set of all possible edge length preserving movements of $G(p)$, then the intersection of fibre(K,p) and U is contained in the intersection of fibre(G,p) and U. This gives us

$$k \leq nv - (m+1)(2n-m)/2 .$$

Then $k = nv - (m+1)(2n-m)/2$ if and only if there exists a neighborhood W of p in R^{nv} such that

$$\text{fibre}(K,p) \cap W = \text{fibre}(G,p) \cap W .$$

Then the only possible edge length preserving movements of $G(p)$ are the rigid movements of $G(p)$. Since we have that $k \leq nv - (m+1)(2n-m)/2$, then $G(p)$ is flexible in R^n if and only if $k < nv - (m+1)(2n-m)/2$. []

2.4 COROLLARIES

In the first part of this section we will deal with representations of G in different dimensions. For a representation of G in R^m we will denote the edge function of the graph G by f_{Gm}.

2.4.1 <u>Lemma [1]</u> *Let* G *be a graph with* v *vertices. Suppose* p *in* R^{nv} *is a regular point of* f_{Gn} *and let* $m = \dim(P)$. *Then there exists* q *in* R^{mv} *such that* q *is a regular point of* f_{Gm}, $\dim(Q) = m$ *and* $\text{rank } df_{Gn}(p) = \text{rank } df_{Gm}(q)$. *If* $G(p)$ *is rigid in* R^n *then* $G(q)$ *is rigid in* R^m.

Proof: Define $C: R^m \to R^n$ by

$$C(x_1, \ldots x_m) = (x_1, \ldots x_m, 0, \ldots, 0) .$$

There exists an isometry T of R^n taking the m dimensional subspace im(C) onto the affine hull P of $p_1, \ldots, p_v$. Then (T o C) maps R^m onto P . Let $q_i = C^{-1}(T^{-1}p_i)$. Since T is nonsingular then dim(P) = dim(Q) where Q is the affine hull of $q_1, \ldots, q_v$. Since

$$\begin{aligned} \max(\text{rank } df_{om}) &\leq \max(\text{rank } df_{on}) \\ &= \text{rank } df_{on}(p) \\ &= \text{rank } df_{om}(q) \\ &\leq \max(\text{rank } df_{om}) , \end{aligned}$$

q is a regular point of f_{om}. []

Let G be a graph with v vertices. Then R^{nv} can be partitioned according to the rigidity or flexibility of G(p) , into the sets of regular and singular points of f_o , or according to whether dim(P) = min(v-1,n) or dim(P) < min(v-1,n) . The first few corollaries explore the relationships between these partitions of R^{nv}.

2.4.2 Corollary [1] *Let* G *be a graph with* v *vertices. If* G(p) *is rigid in* R^n *where* p *is a regular point of* f_o *then* dim(P) = min(v-1,n) .

Proof Let $m = \dim(P)$. By the lemma there exists a q in R^{mv} with q a regular point of f_{Gm}, $\dim(Q) = m$, rank $df_{Gm}(q)$ = rank $df_{Gn}(p)$ and $G(q)$ rigid in R^m. Then by the Theorem 2.3 we have

$$mv - (m+1)(2m-m)/2 = \text{rank } df_{Gm}(q) = \text{rank } df_{Gn}(p)$$
$$= nv - (m+1)(2n-m)/2 .$$

Since $g(x) = vx - (m+1)(2x-m)/2$ is affine and $g(m) = g(n)$, then $m=n$ or the coefficient of $v-(m+1)$ of x in $g(x)$ is zero. Therefore $m = \min(v-1,n)$. []

2.4.3 Corollary [1] *Let* G *be a graph with* v *vertices and edge function* f_G. *If* p,q *in* R^{nv} *are regular points of* f_G *and* $G(p)$ *is rigid in* R^n, *then* $G(q)$ *is rigid in* R^n *and* $\dim(P) = \dim(Q)$.

Proof Let $m = \dim(P)$ and $l = \dim(Q)$. Since p and q are both regular points of f_G then we have $df_G(p) = df_G(q)$ and by the lemma $G(q)$ is rigid. Applying Theorem 2.3 and Corollary 2.4.2 we have $m,l \geq v-1$ and

$$nv - (m+1)(2n-m)/2 = nv - (l+1)(2n-l)/2 .$$

This reduces to

$$(l-m)(l+m+1-2n) = 0 .$$

If $m \neq l$ then $m+l = 2n-1$ and either m or l is less than n . If we assume that $m < n$, then $\dim(P) = v-1$ and $\dim(Q) = v-1$ by Corollary 2.4.2 . []

2.4.4 <u>Corollary [1]</u> *Let* G *be a graph with* v *vertices and* e *edges. If* $e < nv - n(n+1)/2$ *and* $v > n$ *then* G(p) *is flexible in* R^n *for all regular points of* f_G.

<u>Proof</u> Let p in R^{nv} be a regular point of f_G and $\dim(P) = n$. Then

$$\text{rank } df_G \leq e < nv - n(n+1)/2$$

$$\leq nv - (m+1)(2n-m)/2$$

and thus G(p) is flexible by Theorem 2.3 .[]

2.4.5 <u>Corollary [1]</u> *Let* G *be a graph with* v *vertices and* e *edges. If* p *in* R^{nv} *is a regular point of* f_G , $\dim(P) = v-1$ *and* G(p) *is rigid in* R^n, *then* G *is the complete graph on* v *vertices.*

<u>Proof</u> Let $v-1 = m$. Then G(p) rigid implies that

$$e \geq nv - (m+1)(2n-m)/2 .$$

Substituting v-1 for m gives

$$e \geq v(v-1)/2 .$$

But $e \leq v(v-1)/2$, with equality holding only if G is the complete graph on v vertices. []

The next Corollary uses Euler's Formula (v-e+f=2) relating the numbers of vertices, faces and edges in a planar graph.

2.4.6 Corollary [1] *Let* G *be a planar graph such that* G(p) *is rigid for all regular points* p *, in* R^{2v} *of* f_G . *Then the average number* A *of edges on each face of* G *is less than* 4 *and if* $v > 2$ *then* G *contains a triangle.*

Proof: Since G(p) is rigid in R^2 at all regular points p in R^{2v} of f_G then $e \geq 2v - 3$ and

$$A = 2e/f = 2e/(2-v+e) \leq 4e/(e+1) < 4 \quad .$$

Suppose that $v > 2$ and that G has no faces with three edges. Then $A = 2e/f \geq 4$ since every face must have at least 4 edges. Since this can not hold at least one face has three edges. []

2.5 INFINITESIMAL RIGIDITY AND FLEXIBILITY

In the previous sections we have only dealt with rigidity and flexibility at regular points of the edge function. In this section we develop some theory to deal with singular points of the edge function.

Let G be a graph with edge function f_G and p a point in R^{nv} . Let x(t) be a smooth path in

R^{nv} with $x(0) = p$. Then $df(f_a \circ x)(0) = 0$ implies that at p the rate of change of the edge lengths is zero. This can be written as $df_a(p)\ dx(0) = 0$ Thus $dx(0)$ is an element of the kernel (or null space) of $df_a(p)$. Let X be the collection of all such paths. Note that if x is a smooth path in fibre(K,p) = $f^{-1}{}_K(f_K(p))$ with $x(0) = 0$ then x is in X . Thus the tangent space T_x to fibre(K,p) at p is a subspace of ker $df_a(p)$.

2.5.1 Definition G(p) is *infinitesimally rigid* in R^n if T_x = ker $df_a(p)$ and *infinitesimally flexible* otherwise.

Thus G(p) is infinitesimally flexible in R^n if and only if there is a path x in X which is not tangent at p to a smooth motion of K(p) in R^n. A simple example of this type of situation ocurrs when the the points of a triangle are co-linear.

From the previous section we have

$$\text{rank } df_a(p) \leq nv - (m+1)(2n-m)/2$$

where $m = \dim(P)$. Since T_x is contained in the kernel of $df_a(p)$ we have that G(p) is infinitesimally rigid in R^n if and only if

$$\text{rank } df_a(p) = nv - (m+1)(2n-m)/2$$

and $G(p)$ is infinitesimally flexible in R^n if and only if

$$\text{rank } df_G(p) < nv - (m+1)(2n-m)/2 \quad .$$

Thus at regular points of f_G rigidity and infinitesimal rigidity are the same, as are flexibility and infinitesimal flexibility. The following theorem deals with singular points of f_G .

2.5 <u>Theorem [2]</u> $G(p)$ *is infinitesimally rigid in* R^n *if and only if* p *is a regular point of* f_G *and* $G(p)$ *is rigid in* R^n .

<u>Proof</u>: If $G(p)$ is infinitesimally rigid then

$$\text{rank } df_G(p) = nv - (m+1)(2n-m)/2 \; .$$

Since this is maximal then p is a regular point of f_G and $G(p)$ is rigid. If $G(p)$ is rigid and p is a regular point of f_G then

$$\text{rank } df_G(p) = nv - (m+1)(2n-m)/2$$

and $T_x = \ker df_G(p)$ at p . Thus $G(p)$ is infinitesimally rigid. []

The proofs for the following corollaries are analogous to the proofs of the corresponding corollaries of the previous section.

2.5.1 <u>Corollary [2]</u> *If* G(p) *is infinitesimally rigid in* R^n *then* dim(P)=min(v-1,n) .

2.5.2 <u>Corollary [2]</u> *If* G(p) *is infinitesimally rigid for* p *a regular point of* f_G *then* G(q) *is infinitesimally rigid for all regular points* q *in* R^n .

2.5.3 <u>Corollary [2]</u> *If* G *is a graph with* v *vertices and* $e < nv - n(n+1)/2$ *edges then* G(p) *is infinitesimally flexible for all* p *in* R^n.

2.5.4 <u>Corollary [1]</u> *Let* G *be a graph with* v *vertices and* e *edges. If* dim(P) = v-1 *and* G(p) *is infinitesimally rigid in* R^n, *then* G *is the complete graph on* v *vertices.*

CHAPTER 3

MINIMALLY RIGID GRAPHS

In Chapter 2 we developed tools that can be used to determine the rigidity or flexibility of a given graph G for all regular points of the edge function. We will now use this theory to examine minimally rigid graphs. We say that a graph G is *minially rigid* in R^n if it is rigid in R^n and if the deletion of an edge of G results in a graph which is flexible in R^n. The simplest example of a minimally rigid graph is a triangle in R^2. Minimally rigid graphs have a special application in structural analysis. This will be explored in Chapter 4.

3.1 THEORY AND COROLLARIES

3.1 Theorem [9] *Let* G *be a graph with* v *vertices and edge function* f_G . *Suppose that* G *is rigid in* R^n *at a regular point* p *of the edge function and* dim(P) = m . *Then* G *contains a minimally rigid subgraph* G' *on* v *vertices with*

$$e' = nv - (m+1)(2n-m)/2 \quad \textit{edges.}$$

Proof: Since G(p) is rigid in R^n then

$$\text{rank } df_G(p) = nv - (m+1)(2n-m)/2$$

by Theorem 2.3 . Thus we can find nv − (m+1)(2n−m)/2 linearly independent rows of $df_{\sigma}(p)$. Each row corresponds to an edge of G. Let G′ be the graph induced by these rows. Then

$$\text{rank } df_{\sigma'}(p) = nv - (m+1)(2n-m)/2$$

and G′(p) is rigid in R^n . []

3.1.1 <u>Corollary [9]</u> *Let* G *be a graph on* $v > n$ *vertices. Suppose that* G *is minimally rigid in* R^n *at a regular point* p *of the edge function and* dim(P) = m . *Then* G *has exactly* $nv-(m+1)(2n-m)/2$ *edges.*

<u>Proof</u>: This follows immediately from Theorem 2.3 . []

The following Corollary is an expanded version of a theorem due to G. Laman concerning planar graphs. It gives a necessary conditions for a graph to be minimally rigid.

3.1.2 <u>Corollary [9]</u> *Let* G *be a graph on* v *vertices and* e *edges. Let* G′ *be a subgraph of* G *on* v′ *vertices with* e′ *edges and suppose that* dim(P) = n. *Then* G *is minimally rigid in* R^n *at a regular point* p *of the edge function only if*

$$e = nv - n(n+1)/2$$

and for every subgraph G′ *of* G

$$e' \leq nv' - n(n+1)/2 \quad .$$

<u>Proof</u>: Suppose that G is minimally rigid. Then from Theorem 2.3, rank $df_G(p) = nv - n(n+1)/2$. Since the rows of $df_G(p)$ correspond to the edges of G then

$$e = nv - n(n+1)/2 .$$

Suppose $e' > nv' - n(n+1)/2$ for some G' . Then

$$\text{rank } df_{G'}(p) = nv' - n(n+1)/2 < e'$$

and there is at least one linearly dependent row in $df_{G'}(p)$. Since the rows of $df_{G'}(p)$ are a subset of the rows of $df_G(p)$ this implies that $df_G(p)$ does not have full rank and thus $G(p)$ is not rigid. []

3.2 CONNECTIVITY AND RIGIDITY

We will use minimally rigid graphs to show a relationship between connectivity and rigidity. We say that a graph G is *k-vertex connected* if the deletion of any set of $k-1$ vertices does not disconnect G . A graph is *k-regular* if each vertex is incident with exactly k edges.

3.2.1 <u>Corollary</u> *Let* G *be a graph with* $v > n$ *vertices and edge function* f_G . *If* p *a is regular point of* f_G , $\dim(P) = n$ *and* $G(p)$ *is rigid in* R^n *then* G *is* *n-vertex connected.*

Proof: Assume that G is minimally rigid in R^n. Let C be a cut—set of vertices of G and assume that C contains $k = n-1$ vertices. Let $E(C)$ be the edges of G on C. Define G' and G'' so that the interestection of $V(G')$ and $V(G'')$ is C, G' and G'' have no edges in common and the union of G', G'' and $E(C)$ is G. Let v', v'' and e', e'' be the number of vertices and edges in G' and G'' respectively. Note that $v' + v'' = v + k$ and that there are at most $k(k-1)/2$ edges on C.

Since G is minimally rigid by Corollary 3.1.1

$$e' \leq nv' - n(n+1)/2 \quad ,$$

$$e'' \leq nv'' - n(n+1)/2 \quad \text{and}$$

$$e' + e'' \leq e = nv - n(n+1)/2 \quad .$$

Then

$$e' + e'' \leq n(v'+v'') - n(n+1)$$

$$= nv - n(n+1)/2 + nk - n(n+1)/2$$

$$\leq nv - n(n+1)/2 \quad .$$

Since G', G'' are contained in G we have must have

$$e' + e'' \leq nv - n(n+1)/2 - (nk - n(n+1)/2) \; .$$

Since G' and G'' have no edges in C we have

$$e' + e'' \geq nv - n(n+1)/2 - k(k-1)/2 \; .$$

Combining the last two equations gives us

$$k(k-1)/2 \geq nk - n(n+1)/2 \quad ,$$

which reduces to

$$(k-n)^2 \geq k-n \quad ,$$

which implies that $k \geq n$. This contradicts our assumption that $k = n-1$ and thus G must be n-vertex connected. []

3.2.2 <u>Corollary</u> *Let* G *be a k-regular graph such that* $k < 2n$ *and* $\dim(P) = n$. *If* $v > n(n+1)/(2n-k)$ *then* G(p) *is flexible in* R^n *for all regular points* p *of* f_G *such that* $\dim(P) = n$.

<u>Proof</u>: If G is k-regular then G has exactly vk/2 edges. Suppose that G(p) is rigid in R^n for p a regular point of f_G. Then

$$e = vk/2 \geq nv - n(n+1)/2 \quad \text{which reduces to}$$

$$v \leq n(n+1)/(2n-k) \ .$$

Thus for $v > n(n+1)/(2n-k)$ and $k < 2n$ G does not have enough edges to be rigid. []

These types of graphs do exist [14,p44] and a simple example which is 3-vertex connected with 8 vertices and 12 edges is given in Figure 3.1 . This graph flexes in R^2.

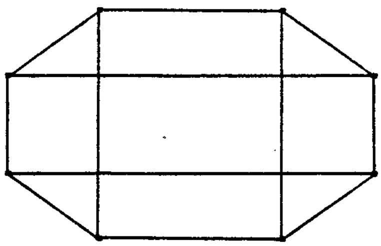


Figure 3.1

Since every rigid graph contains a minimally rigid subgraph these results hold for rigid graphs in general. Then if G(p) is rigid in R^n and dim(P)=n then G is n-vertex connected. However Corollary 3.2.2 shows that this is not sufficient.

3.3 APPLYING THE SINGULAR VALUE DECOMPOSITION

We can use the singular value decompostion to determine rigidity or flexibility for specfic examples. If G(p) is rigid then we can find a minimally rigid subgraph. If G(p) is flexible then we can identify the flexings of G(p).

We state the following theorem without proof. (This proof may be found in Golub and Van Loan [15, p 16-17] .)

<u>Theorem</u> *Let* A *be an* m by n *matrix. Then there exist orthogonal matrices* U *and* V *,where* U *is* m by m *and* V *is* n by n *such that*

$$U^T AV = \mathrm{diag}(s_1, \ldots, s_k)$$

where k = min(m,n) *and*

$$s_1 \geq s_2 \geq \ldots \geq s_k \geq 0 \quad .$$

The s_i's are the *singular values* of A and the index of the smallest non-zero singular value is the rank of A . Suppose rank A = r . Then the first r rows of U span the row space of A and the last $n-r$ rows of V span the null space of A .

We will assume that $\dim(P) = n$. Let $df_G(p) = A$. Then applying the singular value decomposition to A we get

$U^T AV = \mathrm{diag}(s_1,...,s_k)$.

Let s_i be the smallest non-zero singular value. If $i = nv - n(n+1)/2$ then $G(p)$ is infinitesimally rigid . and if $i < nv - n(n+1)/2$ then $G(p)$ is infinitesimally flexible . To determine if $G(p)$ is flexible one must insure that p is a regular point of the f_G .

If $G(p)$ is rigid we can find a minimally rigid subgraph $G'(p)$ by 'growing' a graph with successive applications of the singular value theorem. A more effective method of finding a set of linearly independent rows may be found in [15,p416] .

We can find an isometry T of R_n such that

$Tp_1 = (0,...,0)$

and for $i=2,...,n$

$Tp_i = (q_{i,1},...,q_{i,i-1},0,...,0)$

and for $i = n+1,...,v$

$Tp_i = q_i$.

Since T is an isometry the distances between every pair of points is preserved . This is a special application of the

QR decomposition. A treatment of this may be found in [15,p 164] . We say that G(p) is in *standard position* when $Tp_i = p_i$ for $i = 1,...,v$. Let us assume that G(p) is in standard position. Then by only considering movements of p_i in directions in which the coordinates of p_i are not identically zero we effectively fix G(p) in R^n.

If G(p) is flexible then we can determine the unconstrained points from the last $nv - n(n+1)/2 - r$ rows of V . These rows give us the tangents to the paths along which the points of G(p) can move while preserving the edge lengths of G(p). Since we know what types of paths the points must follow we can construct these paths from this information. For a given tangent vector we find a point which can move while preserving edge lengths but is connected to a fixed point. This point must then move on a spherical surface centered at the fixed point. Once this path is determined the paths of other points can be related to it. Note that each tangent vector corresponds to one degree of freedom for the flexing of the graph. Thus for k tangent vectors we need k independent variables to describe the flexings of the graph.

CHAPTER 4

APPLICATION OF RIGIDITY THEORY

TO

STRUCTURAL ANALYSIS

The determination of rigidity is an essential part of Structural Analysis. The natural model of a bar and joint framework structure (joints flexible) gives a representation of a graph in R^2 or R^3. The joints of the framework become the vertices of the graph and the bars of the framework become the edges of the graph. This natural correspondence suggests that results from rigidity theory can be applied in structural analysis .

4.1 STRUCTURAL ANALYSIS OF TRUSSES

The object of a structural analysis of a truss is to determine whether or not a truss of a given design can support the loads placed on it. This determination is made on the basis of the truss supporting the required load without being displaced more than a given amount. Thus the results of a structural analysis should give the displacement of a truss in terms of the applied load.

We will give a brief overview of the two basic methods employed in structural analysis. These two methods are the

Stiffness method and the Flexibility method. Both methods use systems of linear equations to relate the internal forces acting along the bars to the external (or loading) forces acting on the joints. Similarly the internal displacements (changes in the bars) are related to external displacements (movements of the joints). Implicit in these relations is a constraint on the displacements of the joints of the framework. These constraints correspond to the standard position (defined in Section 3.3 of Chapter 3) of a representation of a graph G in R^n . The constraints are necessary for a unique determination of the displacements of the joints. (see [18], [19])

These two systems of equations are linked together by a representation of the physical characteristics of the bars of the truss to give the final relation between the external forces and the external displacements.

The first method examined is the stiffness (or displacement) method. This method is the most widely used as it can be automated easily. However a solution of the stiffness method requires that a large matrix be inverted. The second method is the flexibility (or force) method which, at present, can not be automated easily. However the flexibility method can be solved by inverting a smaller matrix (in many cases much smaller) than the matrix inverted in the stiffness method.

4.1.1. The Stiffness Method of Structural Analysis

Let T , in R^2, be a truss with v joints and e bars. Let

$$x = (x_{2,1}, x_{3,1}, \ldots, x_{v,1}, x_{v,2})^T$$

represent the displacements of the joints of T under external forces

$$X = (X_{2,1}, X_{3,2}, \ldots, X_{v,1}, X_{v,2})^T$$

acting on those joints. Similarly let

$$s = (s_1, \ldots, s_e)^T$$

represent the changes of lengths of the bars of T (we only consider the bars as compressing or stretching) and

$$S = (S_1, \ldots, S_e)^T$$

be the corresponding forces along the bars. Now we construct the matrix B so that

(1) $$s = Bx \; .$$

This can be accomplished as follows. For each i select some k and set $x_{i,j} = 0$ for some i and all j not equal to k and then record the changes in the lengths of the bars affected by that displacement of that joint. By taking the sum of the changes of bar lengths over all possible i and k we have the changes in bar lengths in terms of the displacements of the joints (which is valid provided the relationship between displacement and force is linear).

Using a result based on the conservation of energy, which is called the "Principle of Virtual Work" [18,p25], we get the following relation between internal and external forces. This is

(2) $$X = B^T S \quad .$$

Now we require the relationship between the internal forces acting along the bars and the changes in lengths of the bars. This depends on the physical characteristics of the bars. Let $K' = \mathrm{diag}(k_1,\ldots,k_e)$ be an e by e matrix such that $S_i = k_i s_i$ (a unit force results in a compression of k_i units on bar b_i). Thus

(3) $$S = K's \ .$$

Combining expressions (1),(2) and (3) we get

$$X = B^T K' Bx \quad \text{or}$$

(4) $$X = Kx \ .$$

The final requirement is to compute K^{-1} to get

(5) $$x = K^{-1}X \quad .$$

It should be noted that the final computation is not trivial even for relatively small trusses.

4.1.2 The Flexibility Method

Let T be a truss with v joints and e bars and x, X, s and S be as above. First we determine the relationship between the forces acting along the bars and the external forces acting on the joints. This is done by treating the forces along redundant bars (a set of bars is redundant if they can be removed without making the structure flexible) as external forces so that the equilibrium equations have a unique solution. One of the aims of this thesis is to provide a method for automating the determination of a redundant set of bars in a given structure.

Let Y represent the forces along the redundant bars. Then we represent the new external force vector as $\begin{vmatrix} X \\ Y \end{vmatrix}$, and let B be the matrix such that

$$(1) \qquad S = B\begin{vmatrix} X \\ Y \end{vmatrix}$$

The relationship between the changes of length of the bars and the displacements of the joints is obtained from the Principle of Virtual Work. Since we are treating the forces along the redundant bars as external forces on the vertices the physical lengths of the redundant bars do not directly influence the displacements of the vertices. One might imagine the middle of the bar being replaced with a mechanism which maintains a constant force regardless of changes in the distance between the end points of the bar. After the flexibility equations are solved the engineer or designer must tailor the physical characteristics of the redundant bars so that they will be compatible with the structure. To represent this we extend x with the same number of zeros as there are redundant bars. We then get

(2) $$\begin{vmatrix} x \\ 0 \end{vmatrix} = B^T s \ .$$

We now deal with the relationship between the internal forces and the changes of lengths of the bars. Here we let $F' = \mathrm{diag}(f_1,...,f_e)$, where f_i is the flexibility of bar b_i and $s_i = f_i S_i$. Then

(3) $$s = F'S \ .$$

Combining (1),(2) and (3) we get

$$\begin{vmatrix} x \\ 0 \end{vmatrix} = B^T F' B \begin{vmatrix} X \\ Y \end{vmatrix} \quad \text{or}$$

(4) $$\begin{vmatrix} x \\ 0 \end{vmatrix} = F \begin{vmatrix} X \\ Y \end{vmatrix} \ .$$

We will rewrite F in (4) as

$$\begin{vmatrix} x \\ 0 \end{vmatrix} = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \begin{vmatrix} X \\ Y \end{vmatrix}$$

and then separate the equations to get

(5) $$x = F_{11}X + F_{12}Y \quad \text{and}$$

(6) $$0 = F_{21}X + F_{22}Y \ .$$

Solving (6) for Y and substituting into (5) we get

(7) $$x = [F_{11} - (F_{12}\ F_{22}^{-1}\ F_{21})]\ X \ .$$

Note that with the flexibility method we only have to invert an r by r matrix F_{22} , where r is the number of redundant members in T . Thus one would like a method for determining sets of redundant members that lends itself to automation.

4.2 RIGIDITY THEORY AND RESOLUTION OF FORCES

In this section we will develop the necessary conditions for a unique resolution of forces acting on a bar and joint framework structure. First we will assume that all forces act at the joints of the framework, the framework is infinitesimally rigid and the structure does not move in space.

4.2.1 Preliminary Definitions

Let G be a graph with v vertices (v>n), e edges and edge function f_G which represents the lengths of the edges of G in R^n. We represent G in R^n by identifying the vertex i of G with p_i in R^n. Suppose that G(p) is in standard position and that G(p) is rigid in R^n for some regular point p in R^{nv} of the edge function. (Hence G(p) is infinitesimally rigid.)

Let the external force acting on vertex i be denoted by X_i such that

$$X_1 = (0,\dots,0)^T$$

$$X_i = (X_{i,1},\dots,X_{i,i-1},0,\dots,0)^T$$

for $i = 2,\dots,n$ and

$$X_i = (X_{i,1},\dots,X_{i,n})^T$$

for $i = n+1,\dots,v$.

Also, let X be an (nv-n(n+1)/2)-vector constructed from the $X_{i,j}$'s which are not identically zero. Thus

$$X = (X_{2,1}, X_{3,1}, X_{3,2}, \ldots, X_{v,n})^T .$$

Similarly let x_i represent the displacement of vertex p_i . Then

$$x = (x_{2,1}, x_{3,1}, x_{3,2}, \ldots, x_{v,n})^T .$$

In this way we only apply forces on points in the direction(s) in which they may be displaced. This is necessary so that the displacements of the points can be uniquely determined.

For each edge (i,j) in E(G) we represent the force acting along the corresponding edge of G(p) by S_{ij} , where S_{ij} is a scalar representing the magnitude of the force on edge (i,j) . Then the force acting on point p_i due to (i,j) is given by

$$(1) \quad S_{ij} \frac{(p_i - p_j)}{||p_i - p_j||} .$$

Let

$$S = (S_1, \ldots, S_e)^T$$

be an e-vector with the k^{th} entry corresponding to the k^{th} edge of G . Similarly let the change in edge lengths be represented by

$$s = (s_1, \ldots, s_e)^T .$$

4.2.2 Equilibrium Conditions

Since we have assumed that G(p) is in standard position we only need to find the equilibrium conditions for the coordinates of the points of G(p) which are not identically zero.

We require that the sum of the forces at each vertex is zero. Using the notation defined above we have

$$(1) \quad X_i = \sum_{j \text{ in } a(i)} \left(S_{i,j} \frac{(p_i - p_j)}{||p_i - p_j||} \right)$$

where $a(i) = \{j : (i,j) \text{ in } E(G)\}$. This condition must hold for every vertex.

4.2.3 <u>The Edge Function and Resolution of Forces</u>

In Chapter 2 we dealt with the edge function of a graph, but we were only interested in the rank of the derivative of the edge function. Here we will determine the displacements of the points in terms of the applied forces, the physical characteristics of the edges and the derivative of the edge function. Again we assume that G(p) is rigid in R^n and in standard position. Recall that

$$(1) \quad f_G(p) = (\ldots, ||p_i - p_j||, \ldots)$$

where $||p_i - p_j||$ is the k^{th} entry of $f_G(p)$ if (i,j) is the k^{th} edge of G .

Then the derivative of the edge function, $df_G(p)$, is an e by nv matrix with $nv - n(n+1)/2$ non-zero columns. The entries of $df_G(p)$ are then

$$(2) \qquad df_{\sigma}(p)_{k,n(i-1)+m} = \begin{cases} \dfrac{(p_{i,m}-p_{j,m})}{\|p_i-p_j\|} \\ 0 \text{ otherwise} \end{cases}$$

where (i,j) is the k^{th} edge of G and $m=1,\ldots,n$. Thus each row of $df_{\sigma}(p)$ has at most 2n non-zero entries and the row sums are zero. This edge function is continuously differentiable provided the edge lengths are non-zero. If we assume that all edges have non-zero length then the results of Chapter 2 will hold.

Now we apply the equilibrium conditions to the Jacobian of the edge function to get a resolution of the external forces acting on the vertices in terms of the internal forces acting along the edges. Recall the equilibrium condition 4.2.2(1) which gives the requirements for the equilibrium of the forces at each vertex i ;

$$4.2.2(3) \quad X_i = \sum_{j \text{ in } a(i)} \left(S_{ij} \frac{(p_i-p_j)}{\|p_i-p_j\|} \right) .$$

Consider the columns of $df_{\sigma}(p)$ which correspond to vertex i . Then the k^{th} row entries in these columns are

$$\frac{(p_i-p_j)}{\|p_i-p_j\|} ,$$

where the k^{th} edge of G is (i,j) .

Let us denote $df_{\sigma}(p)$ by A . Then

$$(3) \qquad X = A^T S .$$

Since we have assumed that G(p) is rigid we know from Theorem 2.3 that the rank of $df_G(p)$ is $nv - n(n+1)/2$. Thus (3) has a unique solution only if $df_G(p)$ has full rank. From Chapter 3 this implies

$$e = nv - n(n+1)/2$$

and that G(p) must be minimally rigid.

4.3 APPLICATION TO STRUCTURAL ANALYSIS

The determination of theoretical rigidity in the evaluation of a proposed design is an immediate application of rigidity theory in the field of structural analysis. However rigidity theory can be applied to both the stiffness and flexibility methods of structural analysis in useful ways.

4.3.1 Application to The Stiffness Method

As in Section 4.2.1 let $S = (S_1,\ldots,S_e)^T$ be an e-vector representing the forces acting on the edges of G(p), $s = (s_1,\ldots,s_e)^T$ represent the changes of lengths of the edges of G(p), $x = (x_{2,1},\ldots,x_{v,n})^T$ represent the displacements of the vertices of G(p) and $X = (X_{2,1},\ldots,X_{v,n})^T$ be the corresponding forces on those vertices. Let A be the matrix created by taking columns of $df_G(p)$ which correspond to coordinates of the points of G(p) which are not identically zero.

We will rewrite the stiffness equations in terms of the A . From 4.2.3 (3) we have $X = A^T S$. This corresponds to the equilibrium equation for the stiffness method 4.1.1(2) . Thus we get the compatibility equation

4.1.1(1)

(2) $$s = Ax \quad .$$

And using 4.1.1 (3) we have

(3) $$S = K' s \quad ,$$

where K' is the edge stiffness matrix. Combining these equtions we get

(4) $$X = A^T K' A\, x \quad .$$

Thus we have the stiffness equation written in terms of the derivative of the edge function of G . However this still leaves us with a large matrix to invert.

4.3.2 Application to the Flexibility Method

By applying rigidity theory to the flexibility method of structural analysis we are able to express the flexibility equations in terms of the edge function defined above.We are also able to determine a set of redundant edges of $G(p)$ which are necessary to solve the flexibility equations.

Since the rows of $df_a(p)$ correspond to the edges of $G(p)$ and since the rank of $df_a(p)$ is $nv-n(n+1)/2$ we can find $nv - n(n+1)/2$ linearly independent rows of A .

(As noted in Chapter 3 this can be done using the singular value decompostion.) The graph $G'(p)$ induced by the edges corresponding to those rows is then rigid. We will call the edges of $G'(p)$ *basic* edges and the edges of $G(p)$ which are not in $G'(p)$ *redundant* edges. Note that this partitions the rows of $df_G(p)$ and hence the rows of A.

Let the matrix formed by the rows corresponding to the basic edges be denoted by A_b and the matrix formed by the rows corresponding to the redundant edges be denoted by A_r. Let S_b be the forces on the basic edges and S_r be the forces on the redundant edges. From 4.2.3(3) we have

(1) $$X = A^T S \quad .$$

Since A_b has full rank then $A_b A_b^T$ has an inverse which we will call M.
This gives

(2) $$S_b = M A_b X \quad .$$

And then MA_b corresponds to B in Section 4.1.2 and

(4) $$x = (MA_b)^T F' (MA_b) X \quad ,$$

where F' is the flexibility matrix of the edges .

Now we must represent the redundant edges of G(p) as forces acting on the points of G(p) . Again we use the derivative of the edge function to accomplish this. We can use (1) to find the forces the redundant edges exert on the points of G(p) . Let X_r represent these forces. Then

(5) $$A_r^T S_r = X_r \quad .$$

From (2) we can find the effect of these redundant edges on the basic edges.

(6) $$S_b = MA_b(X + X_r) \quad \text{or}$$

(7) $$S_b = MA_b(X + A_r^T S_r) \quad .$$

Since the stresses on the redundant edges must get mapped to themselves we get

(8) $$S = \begin{vmatrix} S_b \\ S_r \end{vmatrix} = \begin{vmatrix} MA_b & MA_b A_r^T \\ 0 & I \end{vmatrix} \begin{vmatrix} X \\ S_r \end{vmatrix}$$

which we will rewrite as

$$S = Q \begin{vmatrix} X \\ S_r \end{vmatrix} \quad .$$

Again using the Principle of Virtual Work we get

(9) $$\begin{vmatrix} x \\ 0 \end{vmatrix} = Q^T F' Q \begin{vmatrix} X \\ S_r \end{vmatrix} \quad \text{or}$$

$$\begin{vmatrix} x \\ 0 \end{vmatrix} = F \begin{vmatrix} X \\ S_r \end{vmatrix}$$

where F' is the edge flexibility matrix. Since we have assumed that the edges are only stretched or compressed we can split the flexibility matrix into a matrix for the basic edges and a matrix for the redundant edges denoted as F'_b and F'_r respectively. Separating the equations we find

(10) $$x = F_{11}X + F_{12}S_r \quad \text{and}$$

$$0 = F_{21}X + F_{22}S_r \quad ,$$

where

$$F_{11} = A_b^T M^T F'_b M A_b$$

$$F_{12} = A_b^T M^T F'_b M A_b A_r^T$$

$$F_{21} = A_r A_b^T M^T F'_b M A_b$$

$$F_{22} = A_r A_b^T M^T F'_b M A_b A_r^T + F'_r$$

Then from 4.1.2(7) we have the final solution

$$(11) \quad x = [F_{11} - (F_{12} F_{22}^{-1} F_{21})]X \; .$$

In Chapter 3 we described how the singular value deomposition can be used in conjunction with rigidity theory to determine whether or not the representation of a graph is rigid. If that representation is rigid then a linearly independent set of edges can be found. Since the singular value decomposition can be implemented [15,p293,p416] for both these applications we then have a method for automating the flexibility method of structural analysis.

As well we can use the Jacobian of the edge function of a representation of a graph to determine the relationship between the internal and external forces in a bar and joint structure and the relationship between the internal displacements (changes in lengths of the bars) and the external displacements (changes in the postions of the vertices).

CHAPTER 5

Flexing Panel Structures

In this chapter we will descibe some flexing panel structures. We use the term "flexing panel structure" to denote structures created by joining rigid polygonal panels together along their edges by flexible hinges. Some examples of this type of structure are of particular interest as they can fold up into very compact packages which can be deployed quickly and simply to provide strong lightweight shelters. The structures described below can be modelled using heavy paper which has been suitably folded.

5.1 PRELIMINARY DISCUSSION

The structures we are interested in can be broken down into similar strips of polygonal elements. These strips must then be able to flex so that each strip is compatible with its neighbors. Thus we will begin by examining the required behavior of these strips.

Consider a strip S of length L and width W lying on a flat surface. We want to consider the behavior of that strip when it has a single fold. Denote this fold by F and denote the acute angle between the line of F and the edge of the S by f. For now suppose that $0 < f < \pi/2$.

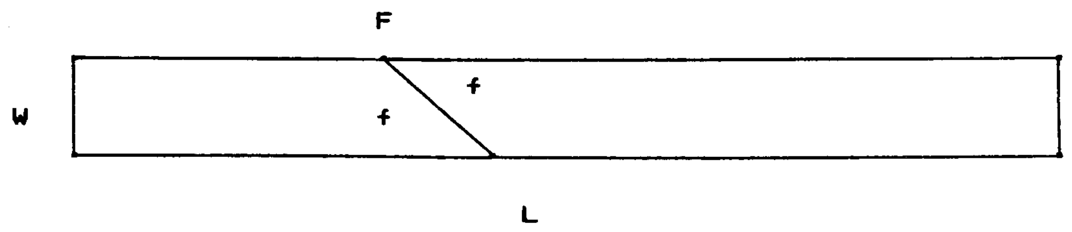


Figure 5.1

Let S be folded completely over at F as below.

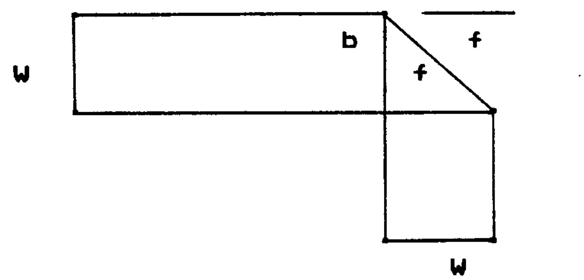


Figure 5.2

It can be seen immediately that the angle b formed on each edge of S is $b = \pi - 2f$. Now suppose that S is unfolded so that one edge of S remains on the surface. This requires that the other edge of S lift off the surface. Let the angle between S and the surface be a.

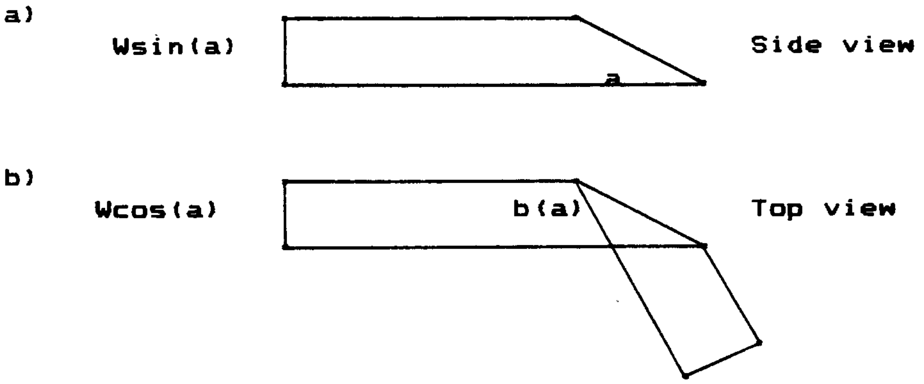


Figure 5.3

Then the projection of S directly down onto the surface has width $W\cos(a)$ and the top edge is lifted $W\sin(a)$ off the surface. Note that the plane induced by the top edge of S is parallel to the flat surface. Then the angle b(a) induced by F and a in the projection of S onto the surface is given by

(1) $$b(a) = \pi - 2\mathrm{Arctan}(\tan(f)\cos(a)) .$$

Now consider the case where S has more then one fold. We require that one edge of S remain on the flat surface at all times. This means that folds must be oriented so that a full twist is not induced in S (each fold induces a half twist in S). Also we require that no two folds cross on S.

We say that a fold is increasing if it is in the same direction as the first fold and decreasing if it is in the opposite direction to the first fold.

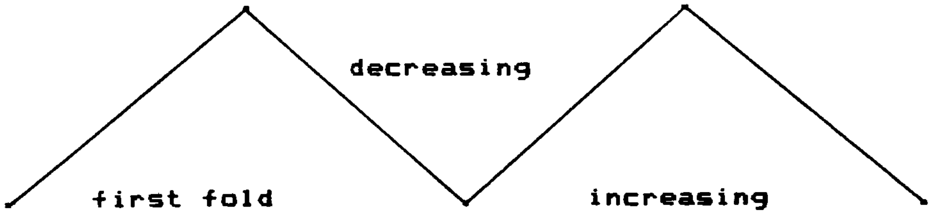


Figure 5.4

If F_i is an increasing fold then

(2) $$b_i(a) = \pi - 2\mathrm{Arctan}(\tan(f_i)\cos(a))$$

and if F_i is a decreasing fold then

(3) $$b_i(a) = \pi + 2\mathrm{Arctan}(\tan(f_i)\cos(a)) .$$

Now let S' be the mirror image of S (obtained by flipping S over). Then by joining S and S′ along a common edge we have a larger strip which is flexible. By repeating this process we can construct as wide a strip as desired. Also by picking appropriate folds we can create many different types of structures. One should be aware that certain constraints arise due to physical considerations.

5.2 A SIMPLE SHELTER

Here we will describe a design for a simple flexing structure that may be used for a shelter. This design folds up into a regular hexagon and can be deployed as a "quonset" type shelter. Designs based on other regular polygons can be constructed in a similar manner. We will base this description on the construction of a paper model.

Suppose that we have a heavy piece of paper of width 6 and length $6(3)^{1/2}$. Identify the lower left corner of the paper with (0,0) so that the width is measured along the vertical axis. Starting at 1 unit vertical rule 5 horizontal lines 1 unit apart on the paper.

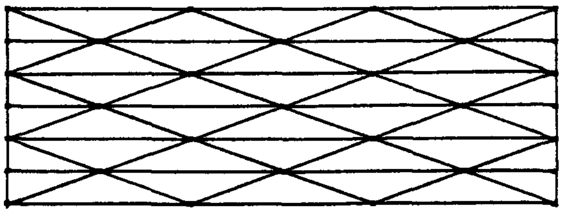


Figure 5.5

Rule a diagonal line from the lower left corner to the upper right corner and parallel lines that meet the edge of the paper at the ends of the horizontal lines. Then starting at the upper left corner and going to the lower right corner repeat this process.

This partitions the paper into triangles which correspond to our panels. Note that this design can be extended indefinitely in both directions. Also note that the horizontal lines partion the paper into strips similar to the strips described above. The lines that we have drawn will be the hinges of our structure. Since we are dealing with triangles this design might be constructed using a bar and joint framework where the line segments are the bars.

Now comes the tricky part. All the horizontal lines must be folded in one direction while the diagonal lines must all be folded in the other direction. Do not try and accomplish this in one pass. First make all the horizontal folds, flatten out the paper, make one set of diagonal folds and then the other. Now carefully compress the top and bottom edges together and correct any folds that are forming in the wrong direction. As this is done the paper will start to form an arch with the horizontal folds internal to the arch and the diagonal folds on the outside of the arch. By completing the folding process the model collapses into a regular hexagon.

5.3 FOLDING UP THE PLANE

This is a flexing panel design which can collapse a plane surface into a very compact package. One might think of this as a simultaneous double pleating of the plane. Again we base this description on the construction of a paper model.

We will start with a heavy piece of paper of width 6 and length 12. Identify the lower left corner of the paper with (0,0) so that the width is measured along the vertical axis. Rule horizontal lines at unit intervals and vertical lines at intervals of 3 (this interval is arbitrary but should be greater then 2). Now at points (i,3j) (i=0,2,4,6 j=1,3) draw diagonal lines at angles of $\pi/4$ and $3\pi/4$ through these points until the lines meet the adjacent horizontal lines. Repeat this process at points (i,6) (i=1,3,5). This will result in vertical strings of diamonds centered on these points.

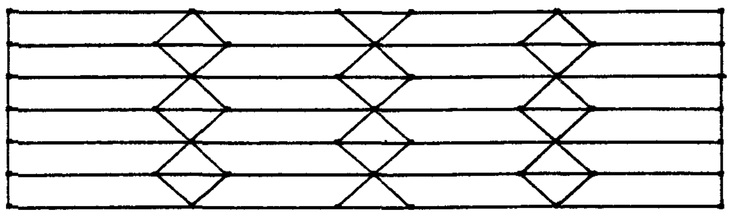


Figure 5.6

Again note that this design can be expanded indefintely in both directions.

We will concern ourselves only with the hoizontal lines and the diagonal lines. Consider the horizontal lines as line segments outside the diamonds and inside the diamonds. Going up the paper the line segments outside the diamonds must be folded alternately (pleated). Going across the paper the line segments in the diamonds must be folded opposite the line segments outside the diamonds. The diagonals must be folded opposite the line segments inside the diamond which contains that diagonal. Again crease all the folds in the proper direction individually. Then compress the top and bottom edges together correcting the folds as required. As the model takes shape it resembles a series of major peaks and valleys crossed by minor valleys and ridges. By completing the folding process the paper will collapse into a compact package 4 by 3 and 12 times the paper thickness deep. There are many possible varitations based on this basic design.

One interesting aspect of this design is that it does not trap any part of the paper "inside" the folds. The whole sheet unfolds in both directions simultaneously. This may be very useful for handling large flexible sheets of material.

5.4 A FLEXING TUBE

We can build a flexing cylinder if we allow perpendicular folds across the strips. The perpendicular fold does not open but allows one to double the strip back on itself. As we can see from 5.1(3) as a increases from 0 , the angle formed by the edges of the strip at a fold gets closer to π . If we allow a perpendicular fold next to a regular fold then as we increase a the angle formed by the edges of the strip still increases with a but as the fold opens it doubles the strip back onto itself. We will call these types of folds inverted folds. Using this technique of perpendicular folds we can construct a flexing tube. Note that due to the doubling back of the strip onto itself this tube is not isomorphic to a cylinder.

Let R = {i: F_i is a regular fold} and let S = {i:F_i is an inverted fold} . We require that

$$\sum_{i \text{ in } R} (f_i) + \sum_{i \text{ in } S} (f_i) = 4N\pi$$

where N is the total number of folds. This may seem to give a contradiction but using inverted folds we can create folds which have an angle of π and thus do not contribute to the angles of the polygon until the strip starts to unfold. Then the increase in the the sum of the angles of the regular folds is offset by the decrease in the angles of the inverted folds as the strp unfolds.

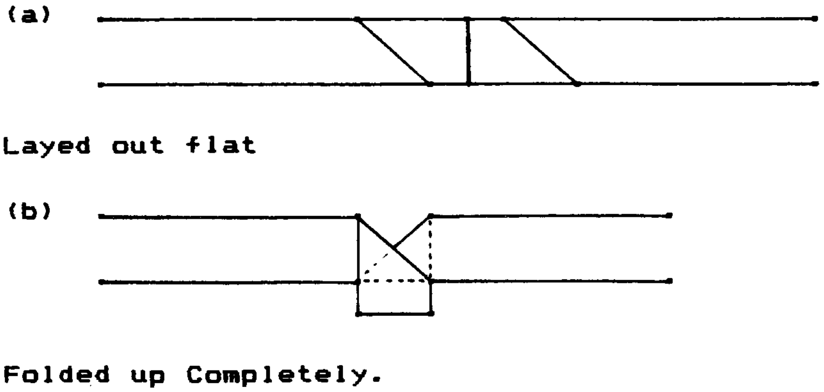


Figure 5.7 An inverted fold .

A simple example of such a flexing cylinder is a folded strip which which has one regular fold of angle $\pi/2$, an inverted fold of angle π, two more regular folds of angle $\pi/2$, another inverted fold of angle π and finally a regular fold of angle $\pi/2$. In this construction let the distances between the regular folds and the inverted folds be the same.

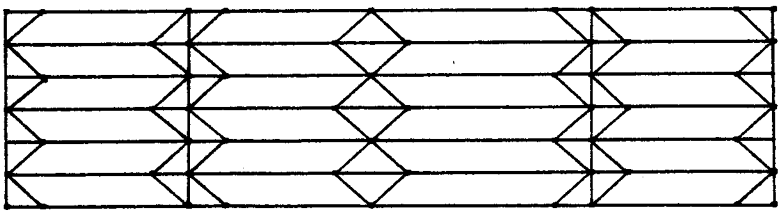


Figure 5.8 A diagram for a flexing tube.

The ends of the strips are then identified. There are many variations of this type of design. The basic idea is to make the changes of the sum of all the angles zero while the strip is flexing.

In Chapter 6 we give an example of a rigid tube constructed from triangular panels.

CHAPTER 6

CALCULATIONS

In this chapter we calculate the rank of the derivative of the edge function for some explicit examples of graphs represented in R^3 and R^4. The calculations are done using a Fortran program written by the author which uses the singular value decomposition to determine the rank of the derivative of the edge function. The actual determination of the singular value decomposition is carried out using the National Algorithm Group Fortran Library routine FO2WCF. This routine is available at Simon Fraser University in the public MTS file *NAGD.

6.1 THE PROGRAM

The Fortran program RIGID, written by the author, uses as input the number of vertices of graph G, the dimension of the space in which G is represented, the adjacency matrix of G and the positions of the points $p_1,...,p_v$. From this $df_G(p)$ is calculated and the NAG routine FO2WCF is then called to determine the singular values. The program then produces the number of vertices, edges and the singular values of $df_G(p)$. From this information we are able to determine whether or not

$$\text{rank } df_G(p) = nv - (m+1)(2n-m)/2 \quad ,$$

where m is the dimension of the affine hull of $p_1,...,p_v$.

6.2 THE EXAMPLES

Except in section 6.2.2 we work in R^3. Let C_n denote the n-cube.

6.2.1 The Cube

We represented C_3 (in 3-space) with edge lengths of 10 . The singular value decompostion produces exactly 12 identical singular values equal to 14.142 . Thus the rank of $df_C(p)$ is 12 . Since C_3 has 8 vertices and 12 edges this is maximal. But 3*8 - 6 = 18 so the cube must be flexible in R^3 .

We now examine the complete graph on the points of the cube. The program returns 18 non-zero singular values. Thus , as one would expect, this representation is rigid. Since K_8 has 28 edges we can find a subgraph with 18 edges which would be rigid.

6.2.2 The Hypercube

We represent C_4 in R^4 with edge lengths of 10. RIGID returns 32 identical singular values and again this is exactly the number of edges of C_4 . Since 4*16 - 10 is 46, C_4 is flexible in R^4.

We examine the complete graph on the points of the hypercube. RIGID returns 47 non-zero singular values. The 46^{th} singular value is 4.854 and the 47^{th} is 0.76 .
Since we expect at most 46 non-zero singular values this indicates that further work is necessary for the numerical computations. The unexpected singular value is probably due to rounding error in the computation. In any case we require 46 non-zero singular values and thus there are 74 redundant edges in this representation.

6.2.3 The Shelter

We represent a graph corresponding to a shelter similar to the one described in Chapter 5 . This graph has 22 vertices and 49 edges. The points are represented as follows;

p_{1+9i} = (-5.0,0.0 ,5.4*i) i=0,1,2

p_{2+9i} = (-2.5,4.33,5.4*i) i=0,1,2

p_{3+9i} = (2.5,4.33,5.4*i) i=0,1,2

p_{4+9i} = (5.0,0.0 ,5.4*i) i=0,1,2

p_{5+9i} = (-4.33,0.0 ,2.7+5.4*i) i=0,1

p_{6+9i} = (-4.33,2.5 ,2.7+5.4*i) i=0,1

p_{7+9i} = (0.0 ,5.0 ,2.7+5.4*i) i=0,1

p_{8+9i} = (4.33,2.5 ,2.7+5.4*i) i=0,1

p_{9+9i} = (4.33,0.0 ,2.7+5.4*i) i=0,1 .

The edges are as follows;

$E = \{(i,i+1): i=V\backslash 4,9,13,18,22\} \cup \{(i,i+5): i=V\backslash 9,18,19,20,21,22\}$

$\cup \{(i,i+4): i=V\backslash 5,14,19,20,21,22\}$.

RIGID return 49 non-zero singular values. Since 3*22-6=60 then the shelter is flexible and lacks at least 11 edges in order to be rigid in and of itself.

6.2.4 A Rigid Tube

This example is based on a tube constructed from identical triangular panels. The underlying graph has 12 vertices and 30 edges. The vertices are represented as follows;

$p_{1+6i} = (\ 0.0\ , 5.0\ , 5.4*i)\quad i=0,1$

$p_{2+6i} = (-4.33. -2.5, 5.4*i)\quad i=0,1$

$p_{3+6i} = (\ 4.33. -2.5, 5.4*i)\quad i=0,1$

$p_{4+6i} = (\ 4.33,\ 2.5, 2.7+5.4*i)\quad i=0,1$

$p_{5+6i} = (-4.33,\ 2.5, 2.7+5.4*i)\quad i=0,1$

$p_{6+6i} = (\ 0.0\ , -5.0, 2.7+5.4*i)\quad i=0,1$.

The edges of this graph are as follows;

$E = \{(i,i+1),(i,i+2),(i+1,i+2): i=1,4,7,10\} \cup \{(1,4),(1,5),$

$(2,5),(2,6),(3,4),(3,6),(4,7),(4,9),(5,7),(5,8),$

$(6,8),(6,9),(7,10),(7,11),(8,11),(8,12),(9,10),(9,12)\}$.

RIGID returns 30 non-zero singular values. Since 3*12-6 is 30 this representation is infinitesimally rigid. Also since this is exactly the number of edges the representation of this tube is an example of a minimally rigid graph.

APPENDIX A

The following is a listing of a Fortran program which computes the singular values of the derivative of the edge function for some representation of a graph G .

```
C     A-DERIVATIVE OF THE EDGE FUNCTION, Q-LEFT HAND
C     SINGULAR VECTORS, PT-RH SINGULAR VECTORS AS COLUMNS
C     P-ROW I OF P GIVES COORDINATES OF POINT I
C     SV-SINGULAR VALUES
C
      DOUBLE PRECISION A(200,200),Q(200,200),PT(200,200),
     *P(50,10),SV(50),WORK(40600)
C
C     ADJ-ADJACENCY MATRIX, E-# OF EDGES, V-# OF VERTICES
C     ND-DIM OF EUCLIDEAN SPACE, M-ROWS OF A, N-COLS OF A
C     EDGE-STORES EDGES OF GRAPH
C
      INTEGER ADJ(50,50),E,V,ND,M,N,EDGE(200,2)
C
C     INPUT # OF VERTICES AND DIMENSION OF E-SPACE
C
      READ(5,100) V,ND
  100 FORMAT(2I3)
C
C     INPUT ADJACENCY MATRIX
C
      DO 5 I=1,V
    5 READ(5,101) (ADJ(I,J),J=1,V)
  101 FORMAT(50I1)
C
C     COUNT EDGES
C
      E=0
      DO 10 I=1,V
      DO 20 I=1,I
      IF (ADJ(I,J).EQ.0) GOTO 20
      E=E+ADJ(I,J)
   20 CONTINUE
   10 CONTINUE
      WRITE(6,201) V,E
  201 FORMAT(1X,2I3)
C
C     SET DIMENSIONS OF A
C
      M=E
```

```
      N=V*ND
C
C     INPUT POSITIONS OF POINTS
C
      DO 30 I=1,V
   30 READ(5,102) (P(I,J),J=1,ND)
  102 FORMAT(10F6.3)
C
C     ZERO A,  PREPARE DERIVATIVE OF EDGE FUNCTION
C
      DO 40 I=1,E
      DO 50 J=I,N
   50 A(I,J)=0
   40 CONTINUE
C
C     ENTER ELEMENTS OF A , ORDER EGDES LEXICOGRAPHICALLY
C
      KE=0
      DO 60 I=1,V
      DO 70 J=I,V
      IF (ADJ(I,J).EQ.0) GOTO 70
      KE=KE+1
      I1=ND*(I-1)
      J1=ND*(J-1)
      DO 80 K=1,ND
      A(KE,I1+K)=P(I,K)-P(J,K)
      A(KE,J1+K)=P(J,K)-P(I,K)
      EDGE(KE,1)=I
      EDGE(KE,2)=J
 80   CONTINUE
 70   CONTINUE
 60   CONTINUE
C
C     FIND MIN M,N
C
      MINMN=MIN0(M,N)
C
C     CALL *NAGD ROUTINE F02WCF
C
      IFAIL=0
      LWORK=3*MINMN+MINMN*MINMN
      NRA=200
      NRQ=200
      NRPT=200
      CALL F02WCF(M,N,MINMN,A,NRA,Q,NRQ,SV,PT,NRPT,WORK,
     *LWORK,IFAIL)
C
C    OUTPUT SINGULAR VALUES
C
     WRITE(6,202) (SV(I),I=1,MINMN)
 202 FORMAT(1X,200F7.3)
     END
```

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