

ON EXACT SHEARING SOLUTIONS OF THE NONSTATIC SPHERICALLY
SYMMETRIC EINSTEIN FIELD EQUATIONS

by

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ABSTRACT

In this thesis we have sought solutions of the nonstatic spherically symmetric field equations which exhibit non-zero shear. The Lorentzian warped product construction is used to present the spherically symmetric metric tensor in double-null coordinates. The field equations, kinematical quantities, and Riemann invariants are computed for a perfect fluid stress-energy tensor. For a special observer, one of the field equations reduces to a form which admits wave-like solutions. Assuming a functional relationship between the metric coefficients, the remaining field equation becomes a second order nonlinear differential equation which may be reduced to a Bernoulli equation. Some special solutions are found which have shear and satisfy various weak energy conditions.

The double null coordinates are also used to study the existence of a timelike collineation vector parallel to the velocity of an anisotropic fluid. The resulting solutions are reducible spherically symmetric spaces.

DEDICATION

I dedicate this thesis to my parents.

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CHAPTER I

MOTIVATION AND HISTORICAL BACKGROUND OF THE PROBLEM

Introduction

In classical general relativity there are several major unresolved problems. Many of these problems involve the phenomena of gravitational collapse. Gravitational collapse is very important, since it simultaneously presents the greatest prediction of classical general relativity, and poses the most difficult problems of the theory. The most important unresolved problem is the cosmic censorship hypothesis which was first posed by Penrose [1]. There are several versions of this conjecture [2] but they all essentially assert that "realistic" gravitational collapse will result in a singular final state in which the singularity is enclosed in an event horizon. This conjecture is important for several reasons [3]:

- (a) singularities cannot appear at random in spacetime but are always enclosed in event horizons;
- (b) important results in general relativity, such as the Schoen-Yau Positive Mass Theorem [4,5], the Black-Hole Area Theorem [6], and several others, assume the validity of the cosmic censorship hypothesis;
- (c) in astrophysics, the final states of stars whose mass exceeds a certain threshold must be black holes and

not "naked" singularities.

There are many papers in the literature documenting the attempts to resolve this conjecture. Much of this work has been in the production of examples which test various aspects of this problem. A common feature of many of these examples is that they are physically unrealistic. An example is collapsing, radiating dust conjured in a manner so that, as the singularity forms, the matter becomes evanescent. This approach, which is entirely mathematical, has few advantages from a physical point of view. These examples also tend to neglect the effects that realistic matter might impose on the collapse. Seifert [2,7] has put forward the idea that the shear of a realistic collapsing fluid, and hence the viscosity due to shear-stresses, may be important for excluding naked singularities. Israel [8,9] has put forth a weaker conjecture, called the event horizon conjecture, for which there are no known counter-examples. The event horizon conjecture asserts that, when gravitational collapse has proceeded beyond a certain critical point, an event horizon forms. The nature of the critical point has been left vague. One could take Thorne's hoop conjecture [10] as an indicator of the critical point: "Horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $\mathcal{C} \leq 2\pi(2GM/c^2)$." One might also conjecture that the event horizon forms in order to preserve the generalized second law of black hole thermodynamics. In

any case a deeper study of the properties of matter and its motion during gravitational collapse will be necessary before these issues may be resolved.

Gravitational collapse has been studied in several contexts:

- (a) cosmological studies of galaxy formation;
- (b) astrophysical studies of supernovae;
- (c) study of formation of white dwarf stars and neutron stars;
- (d) study of the possible physical origin of singularities of solutions of Einstein's field equations.

C.W. Misner [11] has distinguished three types of gravitational collapse:

- (a) stabilized collapse when matter coagulates to form well-known astrophysical objects such as planets, stars, and galaxies;
- (b) catastrophic collapse when matter is in near free-fall with increasing density and remains in this state;
- (c) dynamical collapse when a catastrophic collapse is terminated by the formation of a quasi-static central mass such as a white dwarf star or a neutron star.

As gravitational collapse is associated with matter, we should expect that the mathematical properties of the models which we use for matter should appear prominently in the study of gravitational collapse. It is an unfortunate aspect of the

Einstein field equations that they are so difficult to solve in general. The specification of realistic models for matter makes these equations even more intractable. We are forced into using idealized matter models which lead to solvable systems of partial differential equations. These simpler models have their cost however.

Astrophysical objects are usually modelled in general relativity as perfect fluids. Perfect fluids are fluids whose internal resistance to flowing is zero. Realistic models of astrophysical objects should include viscosity, heat flows, and electromagnetic fields. For large classes of applications in astrophysics, static perfect fluid models are adequate, but recently anisotropic (the pressure is different in different directions) fluids and nonstatic perfect fluids motions have become topics of active research in general relativity. The objective in these lines of research is to produce models for the nonstatic interior of stars, compact ultradense objects, and to study the evolution of radiating spheres and gravitational collapse.

Viscous fluids are a specialization of anisotropic fluids to the case where the shear tensor of the fluid velocity is proportional to the anisotropic pressure. These fluids are of interest since they have been used to model relativistic quasi-static dissipative processes near thermodynamic equilibrium. Cutler and Lindblom [12] have studied the effect of fluid

viscosity on neutron star oscillations. Some of their interesting conclusions are that neutron star matter becomes more viscous in the superfluid state and that the dominant energy dissipation mechanism in neutron stars is the shear viscosity.

Exact solutions for viscous fluids are very difficult to find directly from the Einstein field equations. Nonstatic exact viscous solutions could be used as realistic models of gravitational collapse and might show some insight into the problem of cosmic censorship. Some of the technical difficulties of viscous gravitational collapse have recently been discussed by Coley and Tupper [13]. The problem we propose to study is related to, and was partially motivated by, the problems pointed out by Coley and Tupper. Coley and Tupper point out that that there are very few known viscous-fluid solutions of the Einstein field equations and that the problem of matching these solutions to a portion of the exterior Schwarzschild solution is not possible in general. Coley and Tupper also mention the difficulty in the matching problem related to the choice of interior and exterior coordinate systems. A problem which was not discussed by Coley and Tupper is the choice of the thermodynamic theory to be used in these studies. Usually the Eckart [14] model of relativistic thermodynamics is used for these studies. However, it suffers from a serious defect in that it is an acausal theory in which there may be certain signals which propagate with speed larger

than that of light. Israel [15], and Hiscock and Lindblom [16] have extended the Eckart theory in ways that make it causal. These new theories are much more complicated than that of Eckart. To our knowledge there have been no attempts to use these new thermodynamical theories to create realistic models of gravitational collapse.

A large collection of solutions of the viscous Einstein field equations is desirable so that the effects of viscosity may be better understood. Any method which may either reinterpret known metrics or which will help generate new solutions will be of interest. Through the work of Tupper [17,18,19] and others [20,21,22], it is well-known that a given metric tensor does not lead to a unique physical stress-energy tensor and thus may have various physical interpretations. Tupper has given conditions that allow certain perfect fluid solutions to be reinterpreted as viscous magnetohydrodynamic fluid with heat conduction. In [18] Tupper shows how under certain conditions that a perfect fluid spacetime may be reinterpreted as a magnetohydrodynamic fluid. In [19] Tupper shows how to find, under appropriate conditions, a viscous magnetohydrodynamic reinterpretation of a given perfect fluid solution. Tupper's method depends on finding the fluid velocity of the viscous magnetohydrodynamic fluid by equating the stress-energy tensor of the perfect fluid solution to the viscous magnetohydrodynamic stress-energy tensor. The

equations needed to find this velocity involve the shear of the viscous magnetohydrodynamic fluid. The viscosity introduces some freedom of choice for the viscous magnetohydrodynamic velocity. Resolution of this problem raises interesting questions about the inheritance of the symmetries of the perfect fluid solution by the magnetohydrodynamic solution. A mathematical description of the possible physical situations that may arise through Tupper's method can be made using the Raychaudhuri equation (see Chapter 2). Tupper's method has also been applied in studies of various cosmological models [23,24,25,26]. A large collection of perfect fluid solutions will be needed for effective use of Tupper's method in modelling realistic gravitational collapse.

As the collapse of matter is a nonstatic process, we should try to find nonstatic perfect fluid solutions. The role of shear has appeared repeatedly in the previous discussion hence we will look for nonstatic spherically symmetric perfect fluids with shear and possibly other nonzero kinematical quantities.

Statement of Problem

The Einstein field equations inside matter are

$$(1.1) \quad G_{ab} = T_{ab},$$

(we use units with $8\pi G = c = 1$) with reasonable physical side conditions on the stress-energy-momentum tensor T_{ab} . The

system of equations (1.1) is a quasi-linear second order coupled system of ten partial differential equations for the ten unknown functions of the metric tensor g_{ab} . The reasonable side conditions usually imposed on T_{ab} are that it should describe macroscopic matter with everywhere non-negative energy density, nonspacelike momentum flows, and pressures rather than tensions.

From (1.1) and the differential identity

$$(1.2) \quad G^{ab}{}_{;b} = 0,$$

we deduce that the stress-energy-momentum tensor must satisfy the equations

$$(1.3) \quad T^{ab}{}_{;b} = 0.$$

In conjunction with these side conditions we may impose conditions on the admissible matter motions. If we assume the matter is modelled by a fluid we may insist that the flow of the fluid has certain specified kinematical properties such as nonzero shear, acceleration, vorticity, or expansion.

From the preceding section we expect that shear will play a very important role in the study of realistic gravitational collapse. It would be useful to extend the collection of non-static spherically symmetric interior solutions with shear and other kinematical properties. There are many matter models one could choose for the interior of matter. We will concentrate on perfect fluids and later discuss some results of recent work on anisotropic fluids.

Kramer *et al.* [27] have reviewed the known (in 1980) non-static spherically symmetric perfect fluid solutions. Later in their plenary survey of exact solutions of Einstein's field equations Kramer and Stephani [28] note that only a few radially shearing solutions for perfect fluid interiors are known. This general class of solutions should be very large. Nonstatic perfect solutions can be classified according to properties possessed by the fluid flow *i.e.* shear, acceleration, vorticity, and expansion. All solutions in a spherically symmetric spacetime must have zero vorticity as the direction of vorticity would select a preferred direction.

We shall consider the problem of finding exact nonstatic spherically symmetric shearing solutions of the perfect fluid field equations and of the anisotropic fluid equations. In particular we want solutions that have flows which are shearing, expanding (or contracting), and which are accelerating. The known classes of these solutions with these properties are very sparse due to the inordinate difficulty in solving the field equations. We will not apply the causal thermodynamic theories to our work.

Methodology

The physical method for finding exact interior solutions in General Relativity has essentially three steps:

- (1) selection of allowable symmetries;

(2) selection of a matter model; and

(3) selection of initial (boundary) conditions.

The selection of allowable symmetries will usually constrain the mathematical form of the metric tensor. Often there is more than one system of coordinates which are "adapted" to the symmetries, thus a choice of coordinates may be involved in this step. The selection of a matter model not only may include the type of material (dust, radiation, perfect fluid, viscous fluid, anisotropic matter, etc.), but also may include an equation of state or a choice of thermodynamical model. We will neglect the contributions of other physical fields such as the electromagnetic fields and neutrino fields.

The selection of boundary conditions may involve initial conditions on a spacelike hypersurface, junction conditions on a timelike hypersurface (this also requires a choice of "exterior" spacetime model), or asymptotic conditions on the "boundary" of spacetime.

Even while employing very special choices in the physical method we will often encounter intractable systems of partial differential equations. The literature abounds with variants of the above strategy employed to overcome the practical difficulties encountered. In many cases numerical integration is used to find approximate solutions which yield useful information about the model.

An alternate method is to relax the "imposed" physics and

to adopt mathematical stratagems which will simplify the field equations to the extent that exact solutions may be found. We will call this strategy the method of mathematical simplicity. The physical interpretation of the solutions found in this way depends to a large degree on how much of the matter model has been retained. Often an explicit equation of state is not imposed which leads to the necessity of deriving it (if possible) from the field equations and the twice contracted Bianchi identities.

The mathematical conditions imposed usually fall into two classes:

- (a) special forms of the metric tensor, and
- (b) invariance of the physical fields under special motions or symmetries.

The two classes are not disjoint as special motions or symmetries will in general constrain the form of the metric tensor. Often some of the metric tensor components are assumed to be separable or have special functional forms of the coordinates. The invariance of the physical fields under various motions include Killing vector fields, homothetic motions, conformal Killing vector fields, affine conformal collineations, and others [29].

Treatment of perfect fluid models usually involve *ad hoc* simplifications to obtain analytical solutions for the metric coefficients, the pressure, and the density. These models are

only approximations of the real situation in nature. The degree of unrealism in a model is a cost of the modelling process that is hopefully compensated by simple analytic solutions. Often the unrealism appears in the form of unusual equations of state.

A vexing problem in these investigations is that a solution which has a very simple appearance in a certain coordinate system may have a very complicated appearance in another coordinate system. Many times equivalent solutions have been rediscovered by various researchers who were either unaware of the other solutions or could not see the equivalence. There is an algorithm for determining the equivalence of metrics but it is extremely difficult to carry out with hand calculations. Recently researchers have claimed that computer programs exist which can perform this difficult task.

The choice of a Lorentzian manifold as a model in which to put spacetime physics reflects our desire that spacetime should be a continuum (at least at the non-quantum level). A very useful tool, when it can be employed, is the concept of Lorentzian warped product manifolds. The notion of Lorentzian warped product manifold seems to have been first studied in General Relativity by Delsarte [30] in the 1930's. In the late 1960's Bishop and O'Niell [31] independently reintroduced the notion when studying manifolds of negative sectional curvature. Later O'Niell [32] and Beem and Ehrlich [33] published books

which have popularized the notion of Lorentzian warped product manifolds. Lorentzian warped product manifolds are extremely useful for the study of elementary causality. The Lorentzian warped product may be used to represent spherically symmetric metrics.

An important step in the solution of the field equations is the choice of coordinates. A suitably chosen system of coordinates may greatly assist in the solution of a difficult problem. On the other hand, a badly chosen system of coordinates may render the problem completely intractable. We shall employ double null coordinates for the reason that they make partial integration of the field equations easier. The idea for using double null coordinates comes from their use in the Kruskal-Szekeres completion of the Schwarzschild solution. Double null coordinates have been used before in the study of spherically symmetric interior solutions. Buchdahl [34] uses double-null coordinates in a search for the interior source of the Kruskal solution. He is motivated in his work by the earlier observation made by Synge [35] that by writing the spherically symmetric metric in double null coordinates, one is lead directly to the Kruskal solution for a vacuum. These two papers, particularly that of Synge, have greatly influenced our choice of double null coordinates for our investigation of perfect fluids. There may be disadvantages for physical interpretation of any solutions that we may find, but as Stephani

[36] has noted that there is no overabundance of exact spherically symmetric perfect fluid solutions.

Survey of Recent Research

A literature survey on exact interior solutions in general relativity, even when restricted to a particular form of the stress-energy-momentum tensor, is a large task. In the survey which follows we have tried to present as complete as possible listing of the work in the last 20 years which has as its primary objective the construction of exact nonstatic spherically symmetric perfect fluid solutions. In particular we have concentrated on those papers that have matter flows with nonzero shear and nonzero pressure. The zero pressure "dust" solutions are unrealistic in our opinion. Collins [37], Collins and Wainwright [38], and Ellis [39] have extensively discussed the role of shear in cosmological and stellar models in general relativity. We have included some of the more important studies on shear-free solutions as well. There are three groupings of papers depending upon whether the shear of the nonstatic perfect fluid solution has or has not been analyzed.

First we shall review some of the earlier work on nonstatic perfect fluid solutions whose flows were unclassified. Secondly we shall review some of the important recent work on exact nonstatic perfect fluid solutions with

shear-free flows. Finally we shall review the recent work done on exact nonstatic perfect fluid solutions with shear (and perhaps other kinematical properties).

There are many nonstatic solutions whose perfect fluid flows have been unanalyzed. A common technique of finding new solutions is to place restrictions on the form of the metric. McVittie [40] has found a class of nonstatic spherically symmetric perfect fluid solutions. McVittie assumes a particular form of the spherically symmetric metric tensor which allows the field equations to be written as a system of three ordinary second order differential equations. It has become common to call spherically symmetric metrics with the special form "McV-metrics". A property of the McV-metrics is that they depend on only one function of the timelike coordinate. This property is useful for seeing if a given metric belongs to the McV-metrics. In a later paper McVittie [41] elaborates on his method and relates the work of many researchers to his results. Dyer, McVittie, and Oates [42] have examined the connection between McVittie's method and the hypothesis that the metric tensor admit a conformal Killing vector. Dyer *et al.* find conditions that will force McV-metrics to admit conformal Killing vectors but they are not able to resolve the issue of whether this symmetry follows from the special form of the metric or the assumption of a similarity variable that McVittie uses in motivating his form of the metric. The investigations of Dyer

et al. were motivated by the results of Cahill and Taub [43] who found that the existence of a homothetic Killing vector implies the existence of a similarity variable. Cahill and Taub studied the problem of finding spherically symmetric perfect fluid similarity solutions.

Bonnor and Faulkes [44] found a class of nonstatic spherically symmetric perfect fluid solutions of uniform density but nonuniform pressure. McVittie [41] has shown that the class of solutions found by Bonnor and Faulkes is a special case of a McV-metric. The motion of the fluid was unanalyzed.

Thompson and Whitrow [45,46] studied nonstatic spherically symmetric perfect fluid bodies under the hypothesis that the density is uniform. Using a simple mathematical condition on the metric and imposing regularity on the solutions, Thompson and Whitrow were able to prove a theorem that showed the perfect fluid motion must be shear-free.

Nariai [47] studied gravitational collapse of a perfect fluid with a pressure gradient. The solutions found by Nariai are a special case of McV-solutions [41].

Faulkes [48] investigated nonstatic perfect fluid spheres with a pressure gradient. Faulkes solutions are special cases of those of Nariai [47] hence are McV-metrics as well. The kinematics of the fluid were unanalyzed.

Eisenstaedt [49] studied applications of perfect fluids to cosmology under the added hypotheses of a barotropic equation

of state and uniform density.

In a long series of recent papers Knutsen [50,51,52,53,54,55,56,57,58,59] has studied properties of nonstatic perfect fluid spheres. In most of his work he uses either the McV-form or a "generalized McV-form" of the spherically symmetric metric. Knutsen has found nonstatic analytic models of gaseous spheres (pressure and density are zero on the boundary). The kinematics of the fluid motion are unanalyzed for the solutions he finds.

Most of the known perfect fluid solutions have zero shear. The earliest nonstatic spherically symmetric shear-free perfect solutions were found by Wyman [60,61] but he did not recognize them as such [37]. Many nonstatic spherically symmetric shear-free perfect fluid solutions have been found by Kustaanheimo and Qvist [62,63]. A general class of solutions which are shear-free but expanding are contained in the class of Kustaanheimo and Qvist. Many special cases of this general class have been rediscovered by other researchers [27]. Stephani [36] has found a new class of shear-free perfect fluid nonstatic solutions. Stephani uses the method of Kustaanheimo and Qvist to find solutions that were overlooked in the earlier paper. Banerjee and Banerji [64] have studied perfect fluids with nonuniform density and pressure distributions under the assumption of shear-free radial motion. Glass [65] uses the method of Kustaanheimo and Qvist to study shear-free

gravitational collapse. Glass studied a particular second order nonlinear differential equation earlier considered by Faulkes [48]. Glass finds several shear-free collapsing solutions. McVittie [41] shows that some of the solutions of Glass may be represented as McV-metrics. Glass [65] does present solutions which are not McVittie metrics since the solutions depend on two arbitrary functions of time.

Recently, Sussman [66] has extensively surveyed the spherically symmetric shear-free perfect fluid solutions (both electrically neutral and electrically charged). Sussman finds a large class of "Charged Kustaanheimo-Qvist" solutions depending on two arbitrary functions and five parameters. The neutral members of this class with special values of the other parameters are identical with a class of metrics found by McVittie [67]. In a later paper [68], Sussman continues his work on spherically symmetric shear-free perfect fluid to examine the equations of state and singularities of these solutions.

Srivastava [69] has studied the methods of Kustaanheimo and Qvist [62,63], McVittie [41], Nariai [47], and Wyman [60] in an effort to achieve some degree of unification. Srivastava finds that the McV-metric can be viewed as special case of the Kustaanheimo and Qvist solution. Srivastava also shows that the "generalized McV-metrics" used by Nariai [47] and Knutsen [51] are equivalent to McV-metrics.

For the case when shear is nonzero there are only a few solutions known. Recently it has been shown that some of the previously known solutions with shear that were thought to be distinct are the same.

Misra and Srivastava [70] has proven a stronger version of the theorem of Thompson and Whitrow [45,46]. Misra and Srivastava showed that the mathematical condition of Thompson and Whitrow may be dropped: All regular, nonstatic, uniform density, perfect fluid solutions in comoving coordinates are shear-free.

Some solutions with nonzero shear, but with some of the other kinematical quantities (expansion, acceleration) equal to zero, have been found. McVittie and Wiltshire [71] found a special class of nonstatic shearing perfect fluid solutions which exhibit no acceleration. McVittie and Wiltshire study the use of non-comoving coordinates to find perfect fluid solutions. There are also solutions of McVittie and Wiltshire that have nonzero acceleration and expansion. Kramer *et al.* [27] assert that Skripkin [72] has found a special class of solutions with shear but zero expansion and constant density.

Under the hypothesis that the heat flux vanishes and separability of the metric, Lake [73] found a class of general fluid solutions with shear and with vanishing shear-viscosity. A special case of his solutions reduces to that of Gutman and Bespal'ko [74]. Shaver and Lake [75] have recently extended

the study of the solutions of Lake. Subject to the vanishing of the heat flux they find that all such solutions with shear and non-vanishing shear viscosity have a scalar polynomial singularity at the origin. They conclude that for their form of the metric the only fluid solutions of the field equations with vanishing heat flux which satisfies the energy conditions and are nonsingular at the origin are the Robertson-Walker solutions.

In Kramer *et al.* [27], a class of solutions with shear, acceleration, and expansion were credited to Gutman and Beshpal'ko [74]. (Note that this reference is in Russian and is difficult to obtain). This class of solutions was found in a comoving system of coordinates with the condition that one of the metric functions was separable.

Vaidya [76] found a class of nonstatic solutions for a perfect fluid whose streamlines are orthogonal to the isobaric surfaces. These solutions have nonzero shear, acceleration, and expansion.

Wesson [77] found a class of spherically symmetric nonstatic solutions with shear with a stiff equation of state. Kramer *et al.* [27] assert that Wesson's solution also has acceleration and shear.

Van Den Bergh and Wils [78] have found three new classes of exact spherically symmetric perfect solutions with shear and acceleration and an equation of state. Assuming the ansatz

that the metric coefficients were separable Van Den Bergh and Wils also found a generalization of the stiff equation of state solution of Wesson. This new solution has shear, acceleration, and expansion.

Hajj-Boutros [79] has obtained several classes of non-static, spherically symmetric perfect fluid solutions which exhibit shear, acceleration, and expansion. One of the classes is expressed in terms of Painlevé's third transcendent [80,81,82].

Collins and Lang [83] have recently found a class of spherically symmetric spacetimes which exhibit shear and acceleration. Collins and Lang imposed the condition of self-similarity on Lorentzian spaces with a perfect fluid. They discuss the work of Gutman and Bespal'ko [74], Wesson [77], Hajj-Boutros [79], Van den Bergh and Wils [78], and Herrera and Ponce de Leon [84]. Collins and Lang show that the metric of Gutman and Bespal'ko [74] is the same as that of Wesson [77]. Collins and Lang also show that the metrics of Van den Bergh and Wils [78] specialize to those of Wesson [77] and Gutman and Bespal'ko [74]. Collins and Lang point out that some of the work of Herrera and Ponce de Leon [84] with conformal Killing vectors may be specialized to self-similarity and hence arrive at the metric of Van den Bergh and Wils [78].

Nonstatic Spherically Symmetric Anisotropic Fluids and Conformal Symmetries

Spherically symmetric anisotropic fluids admitting conformal symmetries have recently been studied. The reason for this interest is that a connection between the existence of conformal Killing vectors and the equation of state has been found [85] by Herrera *et al.* The metrics considered were static and had a conformal Killing vector orthogonal to the fluid velocity. The existence of a conformal Killing vector was shown to constrain the pressure and density. It was also found that the orthogonality condition together with a special conformal Killing vector forced the fluid to have a stiff equation of state. In [85] it is asserted that recent realistic studies of stellar models indicate that an anisotropic fluid model may be more appropriate.

Herrera and Ponce de León [86,87] find families of expanding/contracting fluid anisotropic spheres, and confined non-static spheres whose total gravitational mass is zero. The problem of matching these spheres to exterior vacuum metrics is studied. These results have stimulated work on more general classes of symmetries such as collineations [29,88,89].

Duggal and Sharma [90] have recently investigated the similar dynamic restrictions imposed by a special conformal collineation in a class of anisotropic relativistic fluids without heat flux. In particular they showed that the stiff

equation of state is no longer singled out when the collineation vector is orthogonal or parallel to the velocity vector.

Shortly thereafter Maartens and Mason [89] extended and corrected some of the results of Duggal and Sharma. They showed that the assumption, by Duggal and Sharma, that a special conformal collineation vector preserves the fluid flow, is equivalent to assuming that the velocity vector is an eigenvector of the conformal collineation tensor (see Chapter 2 for the definition of the collineation tensor). Later Duggal [91,92] points out that, for fluid spacetimes, conformal symmetry plays the role of preserving the continuity of the matter flow at critical points of transition during a change of state. These ideas may have some use in cosmology. Duggal has found a connection between the shear of the fluid and the existence of a timelike conformal collineation vector. Duggal is able to generalize the theorem of Oliver and Davis [93] on the existence of timelike conformal Killing vectors to the case of timelike conformal collineation vectors. In his generalization of the Oliver-Davis theorem, Duggal shows that a fluid spacetime admits a timelike conformal Killing vector parallel to the fluid velocity if the shear tensor is proportional to the collineation tensor.

Duggal's theorem promises to be an important advance in the study of shearing fluids. Duggal points out that it is an

extremely difficult problem to characterize the collineation tensor so that physical examples of conformal collineation vectors may be found. In Chapter 5 we shall collect some partial results concerning this problem. In Chapter 5 and in an appendix we shall calculate the field equations in double null coordinates for an anisotropic fluid admitting various types of collineation vectors.

Results and Conclusions

In chapter 2 we present a summary of the mathematics which is used in general relativity. The usefulness of the Lorentzian warped product construction is noted, particularly in relation to the problem of discussing elementary causality. A specialized form of spherically symmetric double null coordinates is introduced. These coordinates have the advantage that they do not change their causal type when the metric coefficient change sign. However they do not cover as much of spacetime as the usual double null coordinates. A short presentation of the kinematics of an observer concludes the chapter.

In Chapter 3 we give a brief discussion of the classification of stress-energy tensors and the energy conditions which apply to them.

In Chapter 4 we use the double null coordinates to look for solutions of the perfect fluid field equations which

exhibit shear.

The field equations lead to a "wave-like" equation and the pressure isotropy equation. Although the wavelike equation admits a first integral, it cannot be completely integrated in general. The field equations were simplified by assuming a functional relationship between the metric coefficients. With a functional relationship of the form $f = r^\alpha$, was used for several cases. In particular $\alpha = -2, -1, -1/2, 0, 1/3$ lead to reasonable solutions which are nonstatic and have shear. Another case considered was $f = e^{ar}$. The functional relationship reduces the field equations to a Bernoulli equation. In six of the cases, the timelike weak energy conditions were satisfied. The dominant energy conditions were verified for four of the solutions and the strong energy conditions for four of the solutions. Unfortunately only two of the Bernoulli equations was integrable in closed form in terms of elementary functions. This fact greatly hindered analysis, particularly of the causality, which had been planned. In spite of this, some useful information was drawn from the mathematical form of the Bernoulli equations and the energy conditions. The kinematical quantities were computed when possible. Invariants derived from the Riemann tensor such as the Ricci scalar and the Kretschmann scalar were also computed. For all cases the acceleration of the fluid is zero. The shear tensor was always nonzero. An interesting observation about the solutions is

that they all satisfy the condition of being in a T-region [27]. There have been very few studies of T-models, until now only two or three solutions for dust, a stiff fluid, and a radiation solution. Three of our solutions are not among these solutions for the simple reason that they have much more complicated equations of state than previously known solutions. We found two solutions which have the equation of state $p = \mu/3$.

In Chapter 5 we present some recent results by Duggal connecting timelike collineations to the shear of a fluid. Calculations for timelike collineations in double null coordinates are presented. We state a theorem showing how a stress-energy tensor may be used to try to find a collineation. A theorem asserting the existence of a timelike collineation parallel to the velocity when given a collineation tensor and the velocity field. The equations for an anisotropic fluid admitting a timelike collineation vector parallel to the generalized comoving velocity are written in NN-coordinates. The condition that the collineation tensor is covariantly constant is investigated in detail for this case. Two solutions of the field equations result as a consequence of this study. Both of the solutions lead to reducible spaces in accordance with recent results of other researchers.

CHAPTER II

MANIFOLDS, TENSORS, AND LORENTZIAN WARPED PRODUCTS

Introduction

In this chapter we shall present several sections summarizing the mathematical tools and notation used in this work. This summary is a condensation of the treatments of similar material found in the texts of Sachs and Wu [94], Hawking and Ellis [95], Wald [96], Stephani [97], Frankel [98], O'Niell [32], Hicks [99], and Beem and Ehrlich [33]. Relevant comments have been added to aid in the assimilation of the large number of well-known definitions and concepts. More complete and detailed discussions of these topics may be found in the cited sources. In some cases there is still some disparity in the literature on the use of certain terms and definitions and the notation used to describe them. There is also a recognized profusion of sign disparities among relativists and differential geometers in the basic definitions of the classical tensor calculus. The uncertainty that these disparities impart leads inexorably to qualitative errors, particularly in expressions that involve the use of inequalities. We have tried to eliminate the possibility of such errors by systematically classifying the notation systems and conventions used by various authors. Appendix A is a summary of a system for translating tensorial expressions from one system of tensor

calculations to another.

Vector Spaces and Tensor Algebra

Let V denote a vector space over the real field \mathbb{R} . We shall almost always have $\dim_{\mathbb{R}} V = 4$ but most of the results are true as long as $\dim_{\mathbb{R}} V$ is finite. The dual space of V is denoted by V^* . The underlying set of V^* is the set of all \mathbb{R} -linear functionals on V . V^* is a real vector space and if $\dim_{\mathbb{R}} V$ is finite then $\dim_{\mathbb{R}} V^* = \dim_{\mathbb{R}} V$. If V and W are finite-dimensional real vector spaces then $V \oplus W$ denotes their direct sum. $V^* \otimes W^*$ will denote the vector space of \mathbb{R} -bilinear maps $V \times W \longrightarrow \mathbb{R}$. A standard theorem of linear algebra for finite dimensional vector spaces states

$$(2.1) \quad \dim_{\mathbb{R}}(V^* \otimes W^*) = (\dim_{\mathbb{R}} V)(\dim_{\mathbb{R}} W).$$

Since $\dim_{\mathbb{R}} V$ is always assumed to be finite we have

$$(2.2) \quad \begin{aligned} \dim_{\mathbb{R}} V &= \dim_{\mathbb{R}} V^* \\ &= \dim_{\mathbb{R}} V^{**}, \end{aligned}$$

where V^{**} is the dual space of V^* . Another well-known theorem of linear algebra [100] then states that $V \cong V^{**}$. This natural isomorphism is used repeatedly in tensor algebra to eliminate the unnecessary distinction of V from V^{**} .

Associated with V are its tensor spaces of type (r, s) , $T_{\circ}^r(V)$, where (r, s) is a pair of non-negative integers. A tensor of type (r, s) on V is defined to be a real-valued multilinear functional on $(V^*)^r \times (V)^s$. A standard theorem of tensor algebra asserts that $\dim_{\mathbb{R}}(T_{\circ}^r(V)) = [\dim_{\mathbb{R}}(V)]^{r+s}$.

Tensors of type $(r,0)$ are called contravariant tensors, while tensors of type $(0,s)$ are called covariant tensors. $T_s^r(V)$ may be viewed as the real vector space of multilinear functionals on $r+s$ factors, the first r factors being V^* , the remaining s factor being V . The tensor algebra on V is the direct sum over all pairs of non-negative integers (r,s) of the tensor spaces $T_s^r(V)$ i.e.

$$(2.3) \quad T(V) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} T_s^r(V)$$

If V is a vector space over \mathbb{R} , then an inner product on V , $g : V \times V \longrightarrow \mathbb{R}$, is a symmetric, bilinear functional. The inner product is the natural generalization of the dot product on \mathbb{R}^n . The index of g , $\text{Ind}(g)$, is defined to be the maximum dimension of the subspaces of V on which the restriction of g is negative definite i.e.

$$(2.4) \quad \text{Ind}(g) = \max\{\dim_{\mathbb{R}} W : W \text{ is a subspace of } V\}.$$

The nullity of g , $N(g)$, is defined to be the dimension of the subspace N of V on which the restriction of g is identically zero i.e.

$$(2.5) \quad N = \{X \in V : g(X,X) = 0\}$$

and

$$(2.6) \quad N(g) = \dim_{\mathbb{R}} N.$$

The signature of g , $\text{sig}(g)$, is defined as

$$(2.7) \quad \text{sig}(g) = \dim_{\mathbb{R}} V - 2\text{Ind}(g) - N(g).$$

The Lorentzian inner products to be introduced later will have

signature +2 since $\dim_{\mathbb{R}} V = 4$, $\text{Ind}(g) = 1$, and $N(g) = 0$.

Differential Manifolds and Smooth Maps

Differential manifolds are the setting for the appropriate generalization of calculus on vector spaces. Differential manifolds provide a setting in which we may have the notions of limit and derivative. Manifolds may be thought of as the abstraction of the idea of a surface in Euclidean space. We first introduce local objects (charts) on which limits and differentiation make sense, and then we patch these objects together in a smooth manner. Differential manifolds should not be simply viewed as parameterized surfaces since the notion of a parameterization involves the notion of an enveloping space.

The underlying set of a differential manifold is a topological manifold. An n-dimensional topological manifold is defined as a separable Hausdorff topological M space such that every point in M has an open neighbourhood which is homeomorphic to an open subset of \mathbb{R}^n . Note that M is σ -compact, paracompact, and has at most a denumerable number of components [101,102]. The paracompactness of M is essential for the theory of integration on manifolds. By considering simple examples such as the sphere S^n , $n \geq 1$, and the torus T^n , $n \geq 2$, in \mathbb{R}^{n+1} we quickly see that it is impossible to coordinatize these objects with a single coordinate chart. A similar situation holds in many of the models of spacetime that we use in general relativity.

A n-dimensional coordinate chart on M is a pair $(\mathcal{U}_\alpha, \varphi_\alpha)$ where \mathcal{U}_α is an open subset of M and φ_α is a homeomorphism of \mathcal{U}_α onto an open subset of \mathbb{R}^n (α is an arbitrary index in the set A; see Appendix B on symbols and notation conventions for the rules which indices obey). For each integer i , $0 \leq i \leq n-1$, and for each coordinate chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ we define the i-th coordinate function of φ_α , x_α^i , so that

$$(2.8) \quad x_\alpha^i \equiv \pi^i \circ \varphi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathbb{R}$$

where π^i is the i-th canonical projection on \mathbb{R}^n .

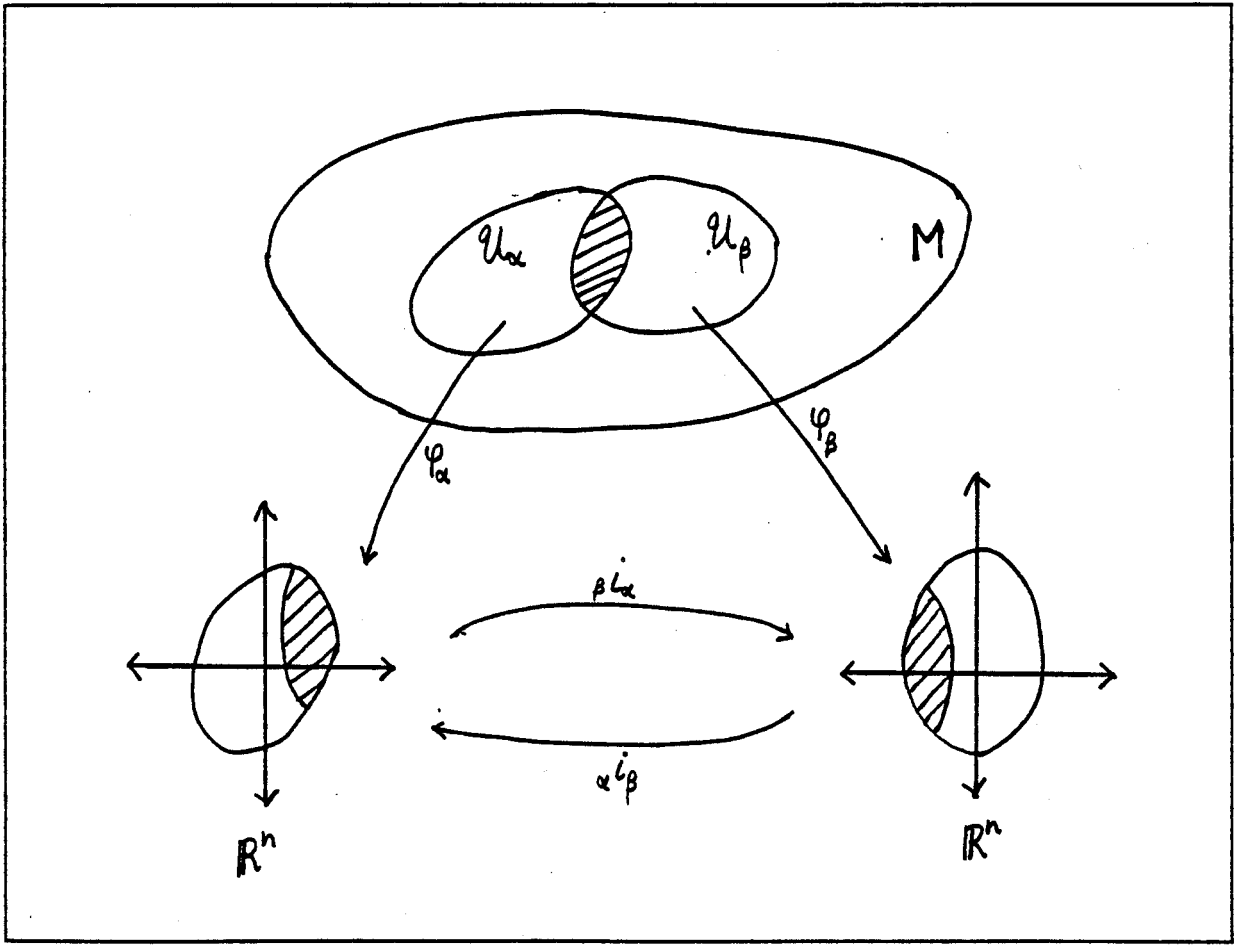


Figure 1 The Coordinate Transition Maps

Since more than one chart will usually be necessary to cover M , we must find a suitable compatibility criterion for the overlap of the chart domains. Two charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{U}_\beta, \varphi_\beta)$ are \mathcal{C}^r -compatible (see Figure 1) if the coordinate transition maps

$$(2.9) \quad \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha[\mathcal{U}_\alpha \cap \mathcal{U}_\beta] \longrightarrow \varphi_\beta[\mathcal{U}_\alpha \cap \mathcal{U}_\beta]$$

and

$$(2.10) \quad \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta[\mathcal{U}_\alpha \cap \mathcal{U}_\beta] \longrightarrow \varphi_\alpha[\mathcal{U}_\alpha \cap \mathcal{U}_\beta]$$

are \mathcal{C}^r -diffeomorphisms (coordinate diffeomorphisms) between their domains and ranges in \mathbb{R}^n . (Note that φ_α^{-1} denotes the inverse mapping of φ_α .)

An indexed collection $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha) : \alpha \in A\}$ of charts which cover M and satisfy the \mathcal{C}^r -compatibility criterion for all index pairs $(\alpha, \beta) \in A \times A$ will be called a \mathcal{C}^r -subatlas for M . Two \mathcal{C}^r -subatlases \mathcal{A} and \mathcal{B} of M are equivalent if $\mathcal{A} \cup \mathcal{B}$ is a \mathcal{C}^r -subatlas for M . An equivalence class of \mathcal{C}^r -subatlases for M will be called a \mathcal{C}^r -differentiable structure for M . Using inclusion as a partial order on a \mathcal{C}^r -differentiable structure we define a \mathcal{C}^r -atlas of M as the maximal element in the \mathcal{C}^r -differentiable structure. It is easily shown using Zorn's Lemma that a maximal atlas always exists. We shall always assume that $r \geq 3$. A \mathcal{C}^1 -differentiable structure always contains a \mathcal{C}^∞ -structure thus we will simply use the adjective "smooth" to denote the appropriate degree of differentiability.

Let M be a manifold endowed with a smooth differentiable

structure characterized by a subatlas $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha) : \alpha \in A\}$. For a coordinate chart $(\mathcal{U}_\alpha, \varphi_\alpha) \in \mathcal{A}$ and a function $f : M \rightarrow \mathbb{R}$, a coordinate expression for f in $(\mathcal{U}_\alpha, \varphi_\alpha)$, is the function

$$(2.11) \quad f_\alpha \equiv f \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n \rightarrow \mathbb{R}.$$

A function $f : M \rightarrow \mathbb{R}$ is smooth if f_α is smooth in the usual sense for all charts $(\mathcal{U}_\alpha, \varphi_\alpha) \in \mathcal{A}$. We will use $\mathcal{F}(M)$ to denote the commutative ring of smooth functions.

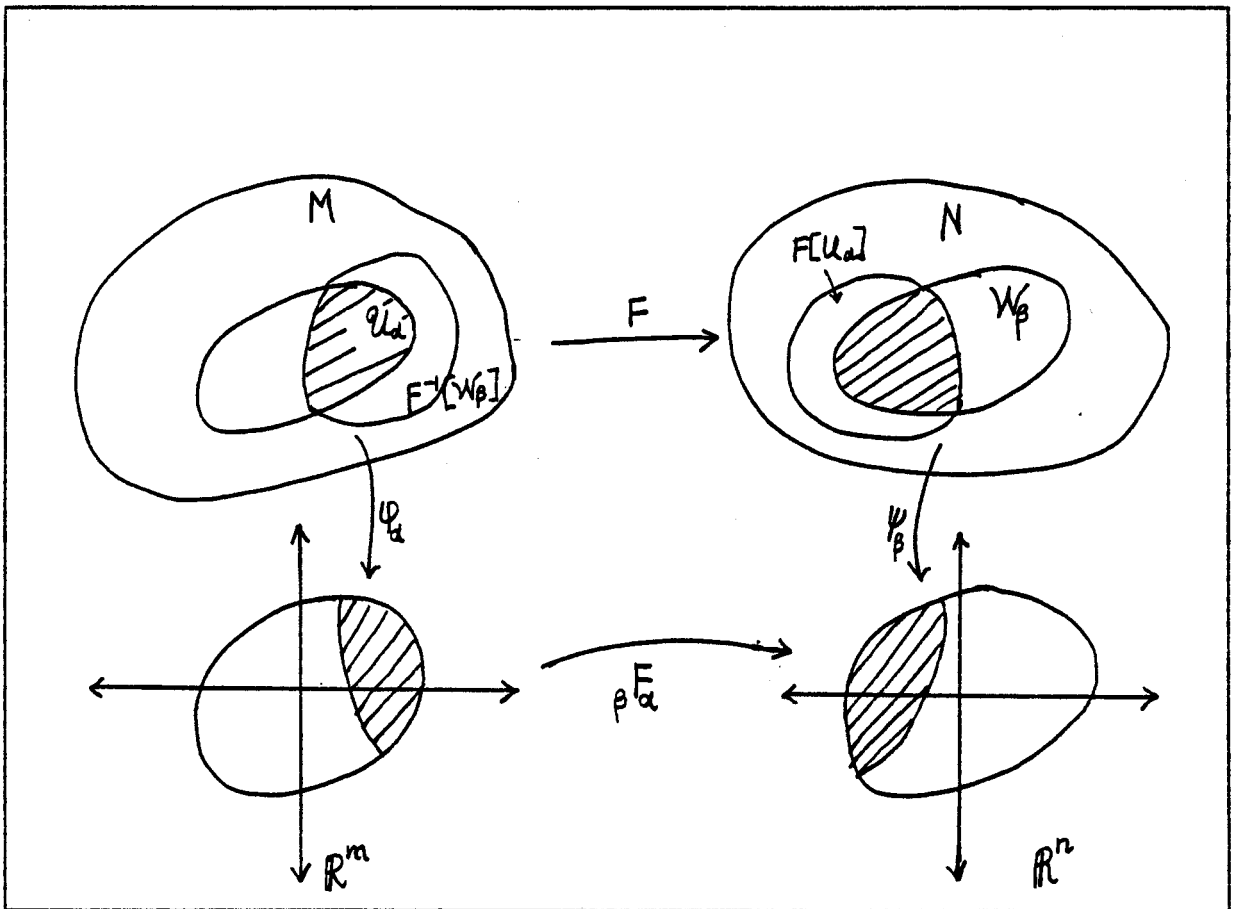


Figure 2 A Smooth Map between Manifolds M and N

Let N be another manifold with a smooth differentiable structure given by the subatlas $\mathcal{B} = \{(\mathcal{V}_\beta, \psi_\beta) : \beta \in B\}$. A

continuous map $F : M \longrightarrow N$ will be called smooth if for each index pair $(\alpha, \beta) \in A \times B$ the map

$$(2.12) \quad {}_{\beta}F_{\alpha} \equiv \psi_{\beta} \circ F \circ \phi_{\alpha}^{-1} : \phi_{\alpha}[\mathcal{U}_{\alpha} \cap F^{-1}(\mathcal{W}_{\beta})] \longrightarrow \psi_{\beta}[F(\mathcal{U}_{\alpha}) \cap \mathcal{W}_{\beta}]$$

(see Figure 2) is smooth in the usual sense. $\mathcal{E}^{\infty}(M;N)$ will denote the set of smooth maps from M to N .

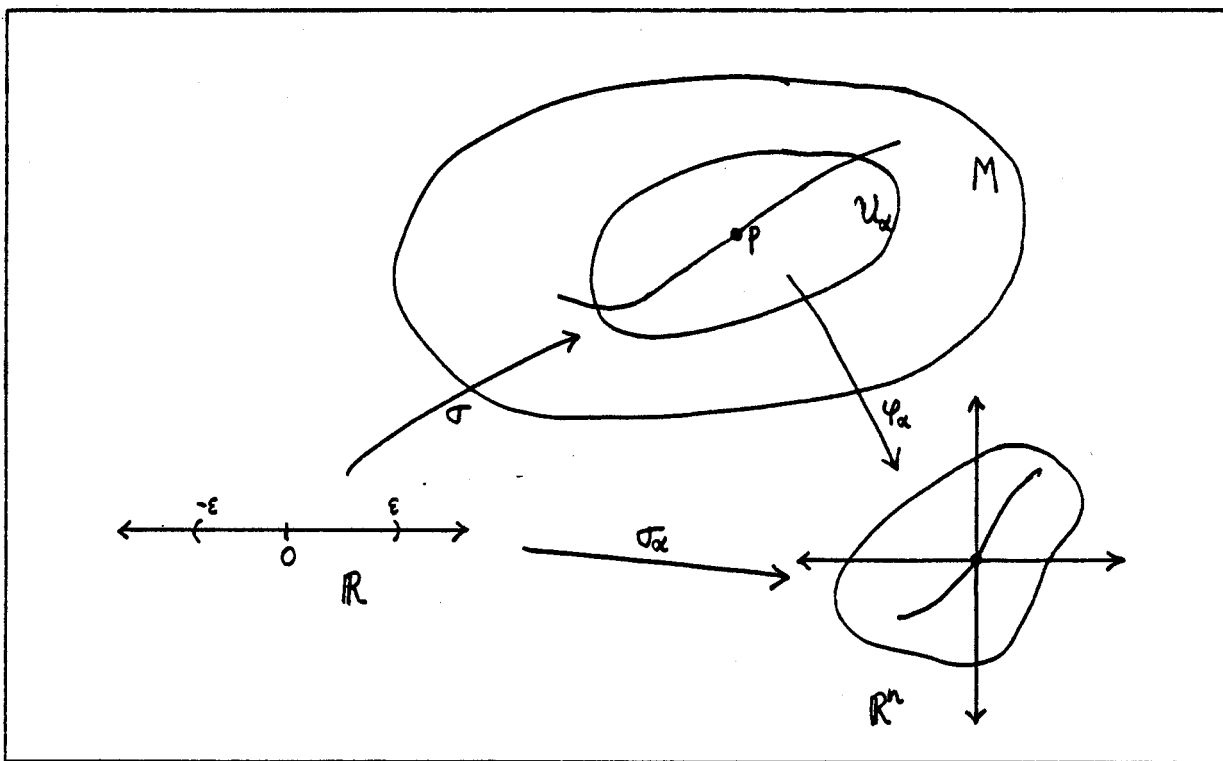


Figure 3 Curves in a Manifold

An important class of maps is $\mathcal{E}^{\infty}(I;M)$, where I is an open interval in \mathbb{R} . These maps are called curves in M . Let $\sigma \in \mathcal{E}^{\infty}(I;M)$ with $\sigma(0) = p$, then, for each $\alpha \in A$ such that $p \in \mathcal{U}_{\alpha}$, there is a local coordinate representation of σ ,

$$(2.13) \quad \sigma_{\alpha} \equiv \phi_{\alpha} \circ \sigma : (-\epsilon, \epsilon) \subset I \longrightarrow \phi_{\alpha}(\mathcal{U}_{\alpha}) \cap \mathbb{R}^n$$

and $\sigma_\alpha(0) = \varphi_\alpha(p)$ (see Figure 3). Note that in practice we almost never specify the curve σ directly. If a chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ is given, we just specify the functions σ_α directly.

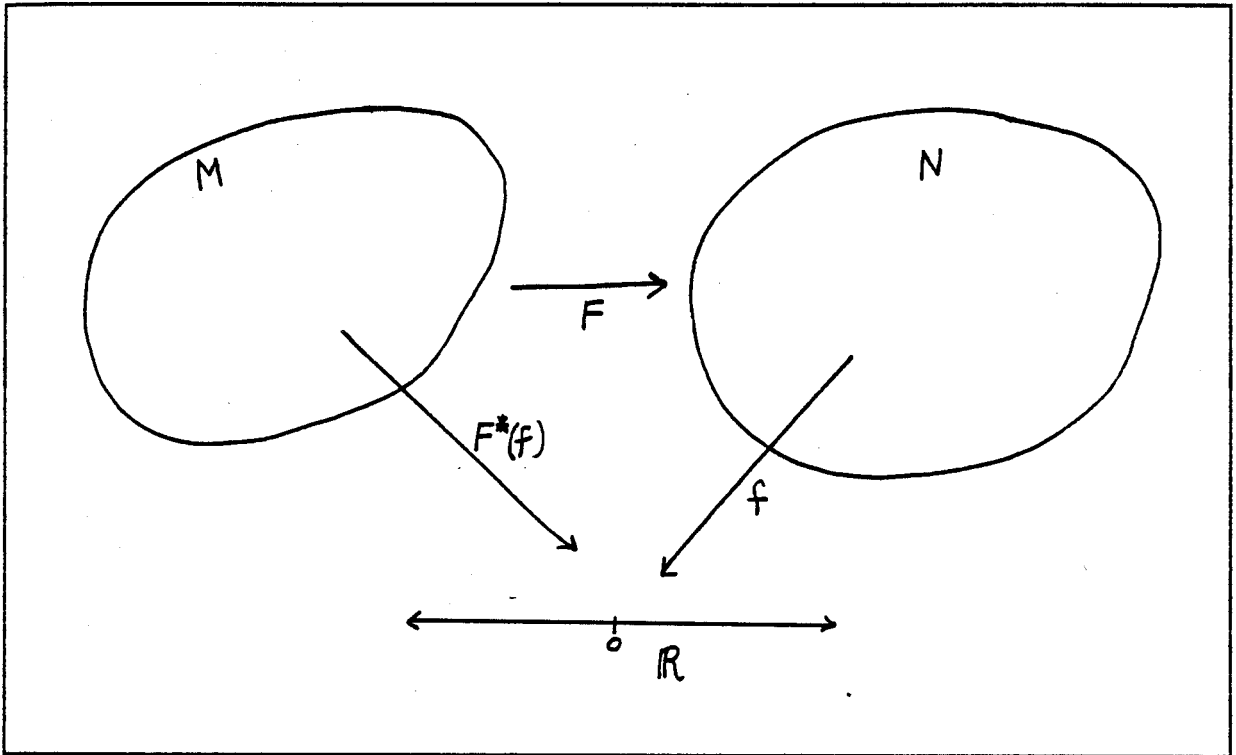


Figure 4 The Pullback of a Map

Each smooth map $F \in \mathcal{C}^\infty(M;N)$ determines an algebra homomorphism $F^* : \mathcal{F}(N) \longrightarrow \mathcal{F}(M)$ given by $F^*(f) = F \circ f$ where $f \in \mathcal{F}(N)$. The map F^* will be called the pullback by F (see Figure 4).

A diffeomorphism $F : M \longrightarrow N$ is a smooth mapping that has an inverse mapping $F^{-1} : N \longrightarrow M$ which is also smooth. The theory of differentiable manifolds can simply be described as the study of objects preserved by diffeomorphisms.

The main step in extending the calculus from vector spaces to differentiable manifolds is finding a generalization of directional derivatives. The appropriate concept for a differentiable manifold M is the tangent vector V_p at a point $p \in M$. There are four different ways of viewing the notion of tangent vectors on a manifold M [103]. Vectors at $p \in M$ may be viewed as equivalence classes of curves through p , as equivalence classes of n -tuples of real numbers at p , or as derivations on germs of functions defined on a neighbourhood of p . We will adopt the view that tangent vectors are derivations on $\mathcal{F}(M)$. A tangent vector of M at p is a linear map $V_p : \mathcal{F}(M) \longrightarrow \mathbb{R}$ such that

$$(2.14) \quad V_p[fg] = V_p[f]g(p) + f(p)V_p[g]$$

for arbitrary functions f, g in $\mathcal{F}(M)$. The collection of all tangent vectors at a point $p \in M$ is denoted $T_p M$, the tangent space to M at p . We can define addition of tangent vectors at p by

$$(2.15) \quad (V_p + W_p)[f] = V_p[f] + W_p[f],$$

and we can define scalar multiplication of tangent vectors at p by

$$(2.16) \quad (cV_p)[f] = cV_p[f].$$

It can easily be shown, using the operations above, that $T_p M$ is a real vector space and $\dim_{\mathbb{R}}(T_p M) = \dim_{\mathbb{R}}(M)$. For a given coordinate chart $(U_\alpha, \varphi_\alpha)$ we can find an induced coordinate (holonomic) basis for $T_p M$ i.e.

$\left\{ \partial_{\alpha i} \equiv \frac{\partial}{\partial x_{\alpha}^i} : 0 \leq i \leq 3 \right\}$. For a function $f \in \mathcal{F}(M)$ and a chart

containing p , $(\mathcal{U}_{\alpha}, \varphi_{\alpha})$, we define the action of $\partial_{\alpha i}|_p$ on $f \in \mathcal{F}(M)$ by

$$(2.17) \quad \begin{aligned} \partial_{\alpha i}|_p[f] &= \frac{\partial f(p)}{\partial x_{\alpha}^i} \\ &= \frac{\partial f^{\alpha}(\varphi_{\alpha}(p))}{\partial u^i} \end{aligned}$$

where the u^i are coordinates on \mathbb{R}^4 .

For a smooth mapping $F : M \longrightarrow N$ and for each $p \in M$ we define the map $F_{*p} : T_p M \longrightarrow T_{F(p)} N$ by

$$(2.18) \quad F_{*p}(V_p)[f] = V_p[F \circ f].$$

By choosing holonomic bases for both $T_p M$, $T_{F(p)} N$ associated with chart $(\mathcal{U}_{\alpha}, \varphi_{\alpha})$ and $(\mathcal{W}_{\beta}, \psi_{\beta})$ respectively i.e.

$$(2.19) \quad \left\{ \partial_{\alpha i} \equiv \frac{\partial}{\partial x_{\alpha}^i} : 0 \leq i \leq 3 \right\}$$

and

$$(2.20) \quad \left\{ \partial_{\beta i} \equiv \frac{\partial}{\partial x_{\beta}^i} : 0 \leq i \leq 3 \right\},$$

we can find a coordinate presentation of F_{*p} :

$$(2.21) \quad {}_{\beta}F_{\alpha * p}(\partial_{\alpha j}|_p) = \left[\frac{\partial(x_{\beta}^i \circ F)(p)}{\partial x_{\alpha}^j} \right] \partial_{\beta i}|_{F(p)}.$$

The term in the square brackets in (2.21) is the Jacobian matrix of ${}_{\beta}F_{\alpha * p}$. If ${}_{\beta}F_{\alpha * p}$ is injective then F is called an immersion. If F is an injective immersion, with N homeomorphic to $F(N)$, then F is an imbedding of N into M . A submersion $F : M \longrightarrow N$ is a smooth mapping onto N such that ${}_{\beta}F_{\alpha * p}$ is onto for all $p \in M$. When the charts $(\mathcal{U}_{\alpha}, \varphi_{\alpha})$ and $(\mathcal{W}_{\beta}, \psi_{\beta})$ are fixed

in an argument we will dispense with the more complicated notation ${}_{\beta}F_{\alpha \times p}$ and simply write F_{xp} . This same abuse of notation will hold in other similar notational situations.

A submanifold N of M is a topological subspace of M such that the inclusion map $j_N : N \longrightarrow M$ is smooth and for each $n \in N$ the induced map $j_{N \times n} : T_n N \longrightarrow T_n M$ is injective (one-to-one). The submanifold N inherits a differentiable structure \mathcal{A}_N related in a natural way to that of M . The notions of imbedding and submanifold are closely related: if N is a submanifold of M then the inclusion map j_N is an imbedding.

An abstract construction which appears frequently in theoretical physics and differential geometry is the fibre bundle [102] construction. Locally fibre bundles are product manifolds but the submersion Π may "twist" the product. Let a smooth submersion $\Pi : E \longrightarrow B$ be given for smooth manifolds E and B . The map Π has the local product property (see Figure 5) with respect to a smooth manifold F if there is

- (a) an open covering $\{\mathcal{U}_{\alpha} : \alpha \in A\}$ of B , and
- (b) a family of diffeomorphisms $\{\psi_{\alpha} : \mathcal{U}_{\alpha} \times F \longrightarrow \Pi^{-1}(\mathcal{U}_{\alpha})\}$ such that $\Pi \circ \psi_{\alpha}(b, f) = b$ for all $b \in \mathcal{U}_{\alpha}$, $f \in F$. (Π^{-1} is the usual inverse set function.)

The collection $\{(\mathcal{U}_{\alpha}, \psi_{\alpha}) : \alpha \in A\}$ is called a local decomposition of Π . A smooth fibre bundle is a four-tuple (E, Π, B, F) , where E is the bundle manifold, $\Pi : E \longrightarrow B$ is a smooth submersion, B is the base manifold, F is a smooth

manifold called the typical fibre.

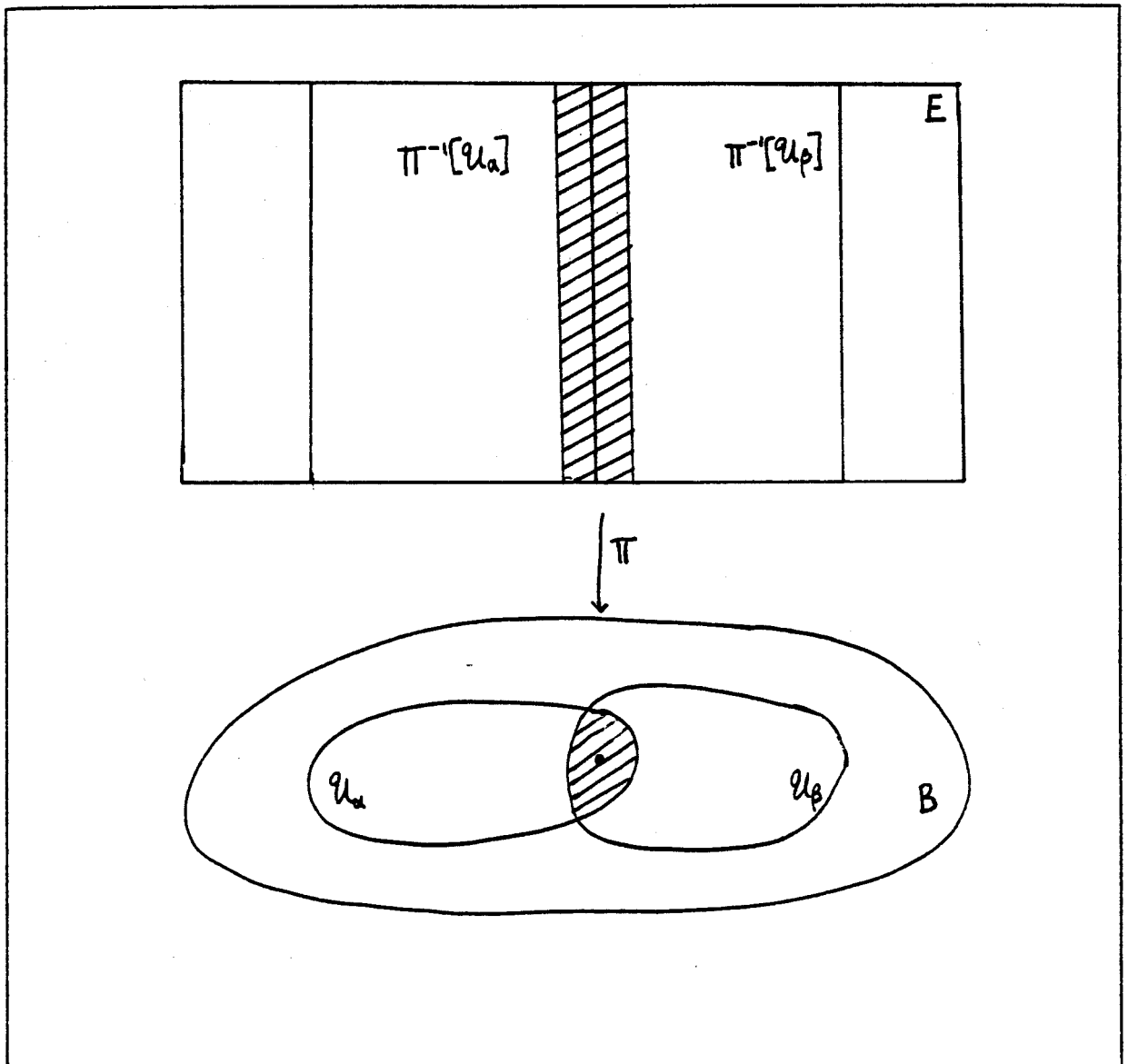


Figure 5 The Fibre Bundle Construction

There is a large variety of subtypes of the fibre bundle construction, depending on the type of fibre and its internal symmetry groups, but the one of interest is the vector bundle construction. A vector bundle is simply a fibre bundle that

attaches vector spaces to each point of the base manifold B . Many of the concepts and objects used in general relativity have a natural presentation in the setting of vector bundles.

Let $TM = \{(p, V) : p \in M, V \in T_p M\}$. The tangent bundle on M , TM , is a special case of the vector bundle construction on M [102]. TM is a smooth manifold with a smooth surjection

$\Pi : TM \longrightarrow M$ defined by

$$(2.22) \quad \Pi(p, V) = p.$$

Viewed as a fibre bundle, the base manifold of TM is M and the fibre manifold at $p \in M$ is the vector space $T_p M$. An atlas on M induces a differentiable structure on TM in a natural way. If

$(\mathcal{U}_\alpha, \varphi_\alpha)$ is a coordinate chart on M then there is a natural induced chart $(\mathcal{U}_\alpha \times \mathbb{R}^4, \varphi_\alpha \times \varphi_{\alpha x})$ on TM such that

$$(2.23) \quad \varphi_\alpha \times \varphi_{\alpha x}(p, V_p) = (x_\alpha^i(p), v_\alpha^i),$$

where

$$(2.24) \quad V_p = v_\alpha^i \partial_{\alpha i} \Big|_p$$

is the representation of V_p with respect to the holonomic basis induced on $T_p M$ by the coordinate chart $(\varphi_\alpha, \mathcal{U}_\alpha)$. The fibre

$\Pi^{-1}(p)$ is isomorphic to $T_p M$ and hence is identified with $T_p M$.

There are several other bundles of interest on M such as the cotangent, tensor, exterior algebra, and frame bundles [102].

Of these the most important is the tensor bundle, $T_s^r M$, of type (r, s) tensors over M .

A vector field V on a smooth map $F : N \longrightarrow M$ is a smooth mapping $V : N \longrightarrow TM$ such that $\Pi \circ V = F$, where Π is the

projection $TM \longrightarrow M$ (see Figure 6). There are three cases of F that occur repeatedly in general relativity:

- (a) $F = \sigma$, where $\sigma : \mathcal{U} \subset \mathbb{R} \longrightarrow M$ is a smooth curve in M ;
- (b) $F = j_N$, where $j_N : N \longrightarrow M$ is an imbedding of a smooth submanifold N into M ;
- (c) $F = i_M$, where i_M is the identity map on M .

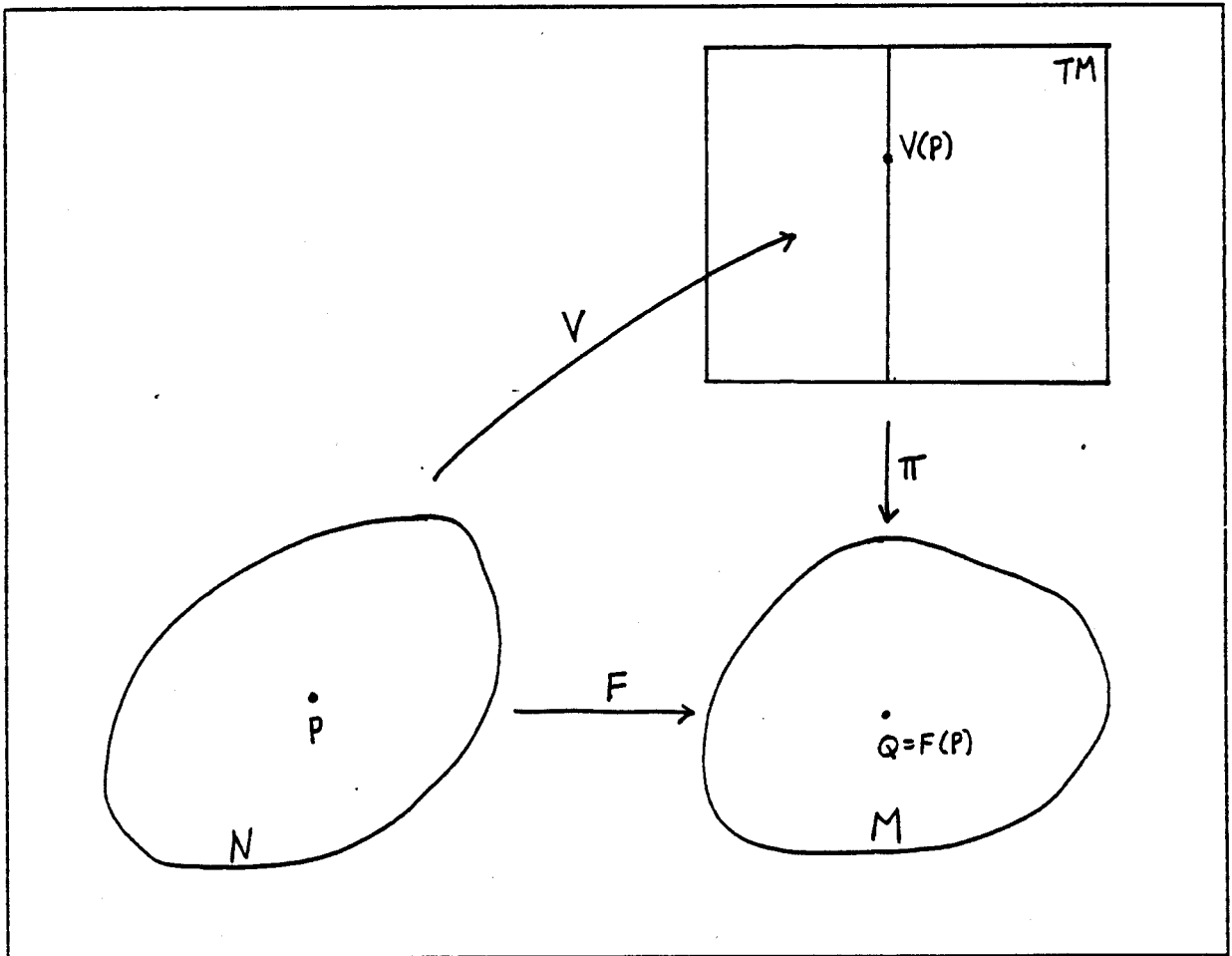


Figure 6 Vector Field over a Mapping

A vector field V over i_M on a differentiable manifold M may also be viewed as a function that assigns to each point

$p \in M$ a tangent vector $V_p \in T_p M$. For a smooth function $f \in \mathcal{F}(M)$, Vf is a function such that

$$(2.25) \quad (Vf)(p) = V_p[f].$$

The vector field V is smooth if Vf is smooth for all $f \in \mathcal{F}(M)$. It can be shown that the set of all smooth vector fields on M , $\mathfrak{X}(M)$, is an $\mathcal{F}(M)$ -module [32]. The set of all smooth vector fields over i_M may also be viewed as a vector bundle over i_M . Thus a smooth vector field is a section of a particular vector bundle. Let T be a tensor field over F . A derivation of tensor fields over F is an assignment of a tensor field DT over F such that

- (a) DT is the same type of tensor as T ;
- (b) $D(\alpha S + \beta T) = \alpha DS + \beta DT$ for all $\alpha, \beta \in \mathbb{R}$, and all tensor fields S, T over F ;
- (c) $D(S \otimes T) = DS \otimes T + S \otimes DT$ for all tensor fields S, T over F .

A connection D over F is an assignment to each vector field X over F of a derivation D_X of tensor fields over F so that

- (a) $D_X f = Xf$ if f is a function over F ;
- (b) $D_{fX+gY} T = fD_X T + gD_Y T$ for smooth functions f, g over F ;
- (c) D_X commutes with contraction.

If $F : N \longrightarrow M$ is a smooth map and D is a connection on M then F^*D is the unique connection over F such that

$$(2.26) \quad (F^*D)_X (V \circ F)(p) = D_{F_* X} V(F(p))$$

for all vectors $X \in N_p$ and for all $V \in \mathfrak{X}(M)$. The connection F^*D is called the induced connection.

The bracket of two vector fields V and W , denoted $[V,W]$, is the vector field such that for all $p \in M$ and all $f \in \mathcal{F}(M)$ we have

$$(2.27) \quad [V,W]_p[f] = V_p[Wf] - W_p[Vf].$$

The bracket operation on $\mathfrak{X}(M)$ has the following properties [32]

(a) $[aV + bW, X] = a[V, X] + b[W, X]$ for all $a, b \in \mathbb{R}$;

(b) $[V, W] = -[W, V]$;

(c) $[V, [W, X]] + [W, [X, V]] + [X, [V, W]] = 0$.

Vector fields are first order operators on functions in $\mathcal{F}(M)$.

A surprising fact is that the bracket of two vector fields is not a second order operator but is another vector field.

A smooth curve $\sigma : (-\varepsilon, \varepsilon) \subset I \longrightarrow M$ is a local integral curve of $V \in \mathfrak{X}(M)$ if $\sigma' = \alpha_{*x}(\partial_u) = (V \circ \alpha)(u)$ for all values $u \in (-\varepsilon, \varepsilon) \subset I$. Occasionally the domain of a local integral curve can be extended to the whole real line. If $\text{dom } \sigma = \mathbb{R}$ and $\sigma' = V$ then we say that V is complete. By considering the smooth vector field $V = (1, -y^2)$ on \mathbb{R}^2 it is easy to see that not all smooth vector fields are complete even when $\text{dom}(V) = M$. This arises since we have only a local existence and uniqueness theorem for differential equations. For each $p \in M$ and $V \in \mathfrak{X}(M)$ there is a unique maximal integral curve in the sense that the domain of σ cannot be extended. For a vector field $V \in \mathfrak{X}(M)$ we define a local flow as a map $\phi : \mathcal{U} \times I \longrightarrow M$ such that $\phi(p, t) = \phi_p(t)$ where $\phi_p(t)$ is the unique maximal integral

curve of V through p defined for all $p \in \mathcal{U}$ and all $t \in I$.

If $F : M \longrightarrow N$ is a smooth mapping of manifolds then F induces an associated mapping between each of the tensor bundles $T_s^r M$ and $T_s^r N$. For type $(1,0)$ we denote this map by TF and for type $(0,1)$ we denote this map by TF^* .

Differential Geometry

The setting for differential geometry is a manifold endowed with a notion of "inner product" called a metric. We wish to study the properties of manifolds and metrics which remain invariant under a group of diffeomorphisms which preserve some of the local algebraic properties (signature, index, nondegeneracy, nullity, symmetry) of the metric. Metrics are usually classified by their signature, nullity, and index. The most general class of metrics is the class of semi-riemannian metrics of arbitrary signature. The Riemannian metrics and the Lorentzian metrics may be viewed as subclasses of the semi-riemannian metrics.

A smooth assignment of a symmetric, nondegenerate bilinear form $g(p) : T_p M \times T_p M \longrightarrow \mathbb{R}$ such that the $\text{Ind}(g(p)) = 0$ and $N(g(p)) = 0$ for all $p \in M$ is called a Riemannian metric on M . A manifold (M, g) such that g is a Riemannian metric for each $p \in M$ is called a Riemannian manifold.

Similarly a smooth assignment of a symmetric, nondegenerate bilinear form $g(p) : T_p M \times T_p M \longrightarrow \mathbb{R}$ with

$\text{Ind}(g(p)) \neq 0$, $\dim_{\mathbb{R}} M$ for all $p \in M$ but constant over M , and arbitrary nullity is called a semi-riemannian metric on M . A manifold (M, g) such that g is a semi-riemannian metric for each $p \in M$ is called a semi-riemannian manifold.

As general relativity uses an inner product of index 1 and nullity 0, we shall restrict our description of differential geometry to this case. A smooth assignment of a symmetric, nondegenerate bilinear form $g(p) : T_p M \times T_p M \longrightarrow \mathbb{R}$ with $\text{Ind}(g) = 1$ at each point in M is called a Lorentzian metric on M . The pair $(T_p M, g(p))$ is a Lorentzian vector space for each $p \in M$. The corresponding manifold is called a Lorentzian manifold.

If (M, g) is a semi-riemannian manifold then there is a unique connection D over the identity i_M such that for all $V, W, X \in \mathfrak{X}(M)$

$$(2.28) \quad [V, W] = D_V W - D_W V;$$

$$(2.29) \quad X[g(V, W)] = g(D_X V, W) + g(V, D_X W).$$

D is called the Levi-Civita connection of (M, g) . The same properties hold for a Riemannian manifold. These properties define the interaction of the metric g and the connection D in such a way as to render D torsionless. Levi-Civita connections may be characterized by the Koszul formula [33]:

$$(2.30) \quad 2g(D_V W, X) = V[g(W, X)] - g(V, [W, X]) + W[g(X, V)] \\ - g(W, [X, V]) - X[g(V, W)] + g(X, [V, W]).$$

A vector field V is parallel with respect to D if $D_X V = 0$ for

all smooth vector fields X . If $V = \sigma_x(\partial_u)$ is the tangent vector field on a curve σ and $D_V V = fV$ for some smooth function f , then we say that σ is a pre-geodesic. If $D_V V = 0$ then σ is a geodesic. The distinction between a pre-geodesic and a geodesic lies in their parameterizations. A pre-geodesic may always be reparameterized to be a geodesic. The parameterization that makes a pre-geodesic a geodesic is not unique. Parameters of a geodesic are affinely related to each other.

A smooth mapping $F : (M, g) \longrightarrow (N, h)$ is called an isometry if F is a diffeomorphism and $g = F^*h$.

A submanifold N of (M, g) is nondegenerate if for each $p \in N$ and nonzero $V_p \in T_p N$ there is a $W_p \in T_p N$ so that $j_N^*g(V_p, W_p) \neq 0$, where j_N^*g is the pullback of the metric g via $j_N : N \longrightarrow M$. If j_N^*g is positive definite then N is a spacelike submanifold of M ; if j_N^*g has index 1 on $T_p N$ for all $p \in N$ then N is a timelike submanifold of M ; if j_N^*g is degenerate then N is a null submanifold.

A submanifold N of (M, g) is geodesic at $p \in N$ if each geodesic γ of (M, g) with $\gamma(0) = p$ and $\gamma'(0) \in T_p N$ is contained in N for some neighbourhood of P . N is totally geodesic if N is geodesic for each $p \in N$.

A submanifold P of (M, g) with dimension $\dim_{\mathbb{R}} P = \dim_{\mathbb{R}} M - 1$ has a number of distinguished forms defined on it. These forms are derived from the unit normal vector N by the following formulae:

$$(2.31) \quad L(X) = D_X N,$$

$$(2.32) \quad I(X, Y) = g(X, Y),$$

$$(2.33) \quad II(X, Y) = g(L(X), Y).$$

L is called the Weingarten map and is self-adjoint.

Tensor Analysis on Differentiable Manifolds

For any pair of nonnegative integers $(r, s) \neq (0, 0)$ and a ring K , a K -multilinear function $R : (V^*)^r \times V^s \longrightarrow K$ will be called a K -tensor of type (r, s) over V . In general relativity there are two cases of interest:

- (a) $K = \mathbb{R}$ and $V = T_p M$;
- (b) $K = \mathcal{F}(M)$ and $V = \mathfrak{X}(M)$.

If $U \subset M$ is an open set in a semi-riemannian manifold (M, g) and $R : U \longrightarrow T_s^r M$ is a map such that $\Pi \circ R = i_U$ then R is a type (r, s) tensor field on U . A tensor field of type $(1, 0)$ is called a vector field. A tensor field of type $(0, 1)$ is called a 1-form field. M has a Lorentzian metric g thus there is an associated isomorphism between $T_0^1 M$ and $T_1^0 M$ given by

$$b : T_0^1 M \longrightarrow T_1^0 M$$

so that

$$V^b = (v_\alpha^i \partial_{\alpha i})^b = v_{\alpha i} \omega_\alpha^i.$$

In terms of components with respect to $\left\{ \partial_{\alpha i} = \frac{\partial}{\partial x_\alpha^i} : 0 \leq i \leq 3 \right\}$

and its dual basis $\{\omega_\alpha^i = dx_\alpha^i : 0 \leq i \leq 3\}$ we have $g_{\alpha i j} v_\alpha^j = v_{\alpha i}$.

There is also an isomorphism between $T_1^0 M$ and $T_0^1 M$ given by

$$\# : T_1^0 M \longrightarrow T_0^1 M$$

so that

$$V^\# = (v_{\alpha i} \omega_\alpha^i)^\# = v_\alpha^i \partial_{\alpha i}.$$

In terms of components we have $g_\alpha^{ij} v_{\alpha j} = v_\alpha^i$. Thus $\#$ and \flat are just the usual index raising and index lowering isomorphisms of classical tensor analysis. We see that the index raising (lowering) isomorphisms define an equivalence between 1-forms and vector fields. A 1-form ω is metrically equivalent to a vector field W if $\omega^\# = W$ and $W^\flat = \omega$.

It is well-known that tensor analysis can be done with respect to arbitrary bases of $T_p M$ and $T_p^* M$. As all the tensor analysis that we shall use is done with respect to holonomic bases, we shall omit the extension of our notation to this case. In many cases there is only one coordinate chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ in use at one time so we will dispense with the subscript α indicating which chart we are using. In situations in which more than one chart is being used we will revert to the more complex, but mathematically unambiguous, notation.

The Christoffel symbols of the second kind associated with a metric tensor g and coordinates $(\mathcal{U}_\alpha, \varphi_\alpha)$ are written as

$$(2.34) \quad \Gamma_{ij}^k \equiv (g^{il}/2)(g_{jl,k} + g_{lk,j} - g_{jk,l}),$$

where g^{il} is found from the equation

$$(2.35) \quad g^{il} g_{lj} = \delta_{.j}^i.$$

Covariant differentiation in a chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ will simply be denoted by appending a covariant index preceded by a

semicolon. Thus the partial covariant derivative of a tensor

$$T^{i_1 \dots i_r}_{j_1 \dots j_s} \quad ;_k$$

(2.36)

$$T^{i_1 \dots i_r}_{j_1 \dots j_s ; k} = T^{i_1 \dots i_r}_{j_1 \dots j_s , k} + \sum_{\alpha=1}^r \Gamma^{i_\alpha}_{ak} T^{i_1 \dots i_{\alpha-1} a i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} - \sum_{\beta=1}^s \Gamma^b_{j_\beta k} T^{i_1 \dots i_r}_{j_1 \dots j_{\beta-1} b j_{\beta+1} \dots j_s}.$$

In the formula (2.36) above the ";k" denotes the partial covariant derivative with respect to $\partial_{\alpha k}$ and a comma denotes the partial coordinate derivative with respect to x_α^i .

The failure of partial covariant differentiation to commute leads to the Ricci identities which may be taken as the definition of the Riemann tensor:

(2.37)

$$T^{i_1 \dots i_r}_{j_1 \dots j_s ; kl} - T^{i_1 \dots i_r}_{j_1 \dots j_s ; lk} = - \sum_{\alpha=1}^r R^{i_\alpha}_{akl} T^{i_1 \dots i_{\alpha-1} a i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} + \sum_{\beta=1}^s R^b_{j_\beta kl} T^{i_1 \dots i_r}_{j_1 \dots j_{\beta-1} b j_{\beta+1} \dots j_s}.$$

The components of the Riemann tensor with respect to the coordinate basis and the dual basis of the chart $(\mathcal{U}_\alpha, \varphi_\alpha)$ may be written in terms of the Christoffel symbols of the second kind as:

$$(2.38) \quad R^i_{jkl} = \Gamma^i_{jk, l} - \Gamma^i_{jl, k} + \Gamma^i_{nl} \Gamma^n_{jk} - \Gamma^i_{nk} \Gamma^n_{jl}.$$

The Riemann tensor satisfies the following symmetry properties:

$$(2.39) \quad (a) \quad R_{jikl} = -R_{ijkl},$$

$$(2.40) \quad (b) \quad R_{ijkl} = -R_{ijlk},$$

$$(2.41) \quad (c) \quad R_{ijkl} = R_{klij},$$

$$(2.42) \quad (d) \quad R_{i[jkl]} = 0,$$

where indices inclosed in brackets indicate normalized anti-symmetrization. Contracting on the first and third indices leads us to the expression for the Ricci tensor in the same coordinates:

$$(2.43) \quad R_{ij} = R^a{}_{iaj}.$$

The Einstein tensor is defined by

$$(2.44) \quad G_{ij} = R_{ij} - (R/2)g_{ij},$$

where R is the curvature scalar given by

$$(2.45) \quad R = R^a{}_a.$$

The Riemann tensor also obeys the well-known Bianchi differential identities:

$$(2.46) \quad R_{ij[kl;m]} = 0.$$

A contraction on the Bianchi identities leads to the first-contracted Bianchi identities:

$$(2.47) \quad R^i{}_{jkl;i} + 2R_{j[k;l]} = 0.$$

Another contraction of the Bianchi identities implies the well-known differential identities for the Einstein tensor:

$$(2.48) \quad G^{ij}{}_{;j} = 0.$$

The Weyl conformal curvature tensor is defined by decomposing the Riemann tensor into products of the metric tensor, the Ricci tensor, and the curvature scalar. The Weyl conformal curvature tensor can be characterized by the fact that it has all the symmetries of the Riemann tensor and is

traceless with respect to all contractions. Thus

$$(2.49) \quad C^i{}^j{}_{kl} = R^i{}^j{}_{kl} - 2g^i{}_{[k} R^j]{}_{l]} + (R/3)g^i{}_{[k} g^j]{}_{l]},$$

and Weyl tensor satisfies the same symmetry properties as the Riemann tensor. The Weyl tensor also is traceless:

$$(2.50) \quad C^i{}_{jil} = 0.$$

In terms of the Weyl conformal curvature tensor it can be shown [39,104] that for a 4-dimensional Lorentzian manifold the Bianchi differential identities are equivalent to

$$(2.51) \quad C^{ijkl}{}_{;l} = R^{k[ii;jj]} - (1/6)g^{k[ii} R^{j]j]}.$$

The Weyl conformal curvature tensor has a decomposition for an observer with velocity U^i in terms of its "electric" part E_{ij} and its "magnetic" part H_{ij} given by [39]:

$$(2.52) \quad C_{ijkl} = (\eta_{ijpq} \eta_{klrs} + g_{ijpq} g_{klrs}) U^p U^r E^{qs} \\ - (\eta_{ijpq} g_{klrs} + g_{ijpq} \eta_{klrs}) U^p U^r H^{qs},$$

where

$$(2.53) \quad E_{ik} = C_{ijkl} U^j U^l,$$

$$(2.54) \quad H_{ik} = (1/2) \eta_{ip}{}^{rs} C_{rskq} U^p U^q,$$

and

$$(2.55) \quad g_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk}.$$

The tensors E_{ij} and H_{ij} can be shown to have the following properties which follow from the symmetries of the Weyl conformal curvature tensor:

$$(2.56) \quad E_{ij} = E_{ji}, \\ E^i{}_i = 0, \\ E_{ij} U^j = 0;$$

and

$$(2.57) \quad \begin{aligned} H_{ij} &= H_{ji}, \\ H^i_i &= 0, \\ H_{ij} U^j &= 0. \end{aligned}$$

Elementary Causality on Lorentzian Manifolds

A Lorentzian metric g for M is a smooth symmetric tensor field of type $(0,2)$ on M such that for each $p \in M$, the tensor $g(p)$ on $T_p M$ is a nondegenerate indefinite inner product of signature $(-,+,+,+)$. It is easily shown that a noncompact manifold admits a Lorentzian metric. On the other hand for compact manifolds it can be shown that a Lorentzian metric exists only if the Euler characteristic vanishes i.e. $\chi(M) = 0$.

A spacetime (M,g) is a connected, noncompact, smooth Hausdorff manifold of dimension 4 which has a countable basis, a Lorentzian metric of signature $(-,+,+,+)$ and a time orientation. We assume noncompactness in the definition of a spacetime since it can be shown that a compact spacetime admits closed timelike curves. We assume that M is connected since disconnected parts of a spacetime should not be able to interact. A spacetime (M,g) is time-oriented if M has a nowhere zero, continuous timelike vector field. There is a large number of inequivalent definitions of spacetime appearing in the literature. The definition we have adopted is the weakest one which is either consistent with or can easily be

adapted to the definition used in our major references [32,33,94,95]. As Hall [105] points out, there is still some debate among relativists on the nature of the topology of a spacetime. The usual manifold topology is a "homogeneous" topology *i.e.* for any two points $p, q \in M$ there is a homeomorphism which maps p to q . Thus the usual topology reflects the locally \mathbb{R}^4 nature of M rather than its Lorentz metric structure which in most cases imposes a local sense of time-orientation at each point. Göbel [106] has studied other choices of a topology for spacetime which are more compatible with its Lorentzian metric. As constructed above, the underlying manifold of spacetime is a manifold "modelled over \mathbb{R}^4 ". There are more abstract definitions of a manifold which allow other "model" spaces to be used [103,107]. Perhaps a "model" space can be found that intrinsically embraces the notions.

Two spacetimes (M, g) and (M', g') are equivalent, written as $(M, g) \approx (M', g')$, if there is a space-orientation and time-orientation preserving isometry between them. The spacetime equivalence class of (M, g) is the set

$$(2.58) \quad [(M, g)] = \{(M', g') : (M', g') \approx (M, g)\}.$$

In general relativity it is the classes $[(M, g)]$ that are of interest. In practical terms it is not a trivial task to establish that two spacetimes are equivalent. It is much easier to establish inequivalence by finding an isometric invariant that one spacetime has and which the other spacetime

does not have. For a fixed manifold M the set of all Lorentzian metrics on M is denoted by $\text{Lor}(M)$. We write $p \ll q$ if there is a future-directed piecewise smooth timelike curve in M from p to q . We write $p \leq q$ if $p = q$ or there is a future-directed piecewise smooth nonspacelike curve in M from p to q . If $p \in M$ then the chronological future of p is the set

$$(2.59) \quad I^+(p) = \{q \in M : p \ll q\}.$$

If $p \in M$ then the chronological past of p is the set

$$(2.60) \quad I^-(p) = \{q \in M : q \ll p\}.$$

If $p \in M$ then the causal future of p is the set

$$(2.61) \quad J^+(p) = \{q \in M : p \leq q\}.$$

If $p \in M$ then the causal past of p is the set

$$(2.62) \quad J^-(p) = \{q \in M : q \leq p\}.$$

A nonzero vector $V \in T_p M$ is called

$$(2.63) \quad (a) \text{ timelike if } g(p)(V, V) < 0;$$

$$(2.64) \quad (b) \text{ spacelike if } g(p)(V, V) > 0;$$

$$(2.65) \quad (c) \text{ null if } g(p)(V, V) = 0;$$

$$(2.66) \quad (d) \text{ nonspacelike if } g(p)(V, V) \leq 0.$$

The causal classification of vectors provides a partial classification of curves in spacetime. If $V = \sigma' = \sigma_x(\partial_u)$ then we say that the curve σ is

$$(2.67) \quad (a) \text{ timelike if } g(\sigma(u))(V, V) < 0 \text{ for all } u \in \text{dom}(\sigma);$$

$$(2.68) \quad (b) \text{ spacelike if } g(\sigma(u))(V, V) > 0 \text{ for all } u \in \text{dom}(\sigma);$$

$$(2.69) \quad (c) \text{ null if } g(\sigma(u))(V, V) = 0 \text{ for all } u \in \text{dom}(\sigma);$$

(2.70) (d) nonspacelike if $g(\sigma(u))(V,V) \leq 0$ for all
 $u \in \text{dom}(\sigma)$.

The classification is partial since we do not include curves of mixed causal nature *i.e.* a timelike curve continuously joined to a spacelike curve. The partial classification is sufficient since we do not observe curves which represent physical particles changing their causal type.

The elementary causality of a spacetime is defined as the collection of past and future causal sets and the properties they induce. A spacetime is chronological if it does not contain any closed timelike curves through $p \in M$ so that $p \notin I^+(p)$. The spacetimes of general relativity are usually assumed to be chronological on physical grounds. Since compact spacetimes are not chronological, most of the spacetimes in general relativity are non-compact. Note that the term compact is often abused in general relativity when it is used to refer to cosmological models whose spatial sections are compact *i.e.* Friedmann models.

A spacetime (M,g) is causal if it contains no closed nonspacelike curves. An open set $U \subset M$ is called causally convex if no nonspacelike curve intersects U in a disconnected set. A spacetime (M,g) with arbitrarily small causally convex neighbourhoods is called strongly causal. A spacetime (M,g) is globally hyperbolic if it is strongly causal and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

The next notion that we need is the idea of stability under perturbation of the metric g on M . In order to discuss this mathematically we need to topologize $\text{Lor}(M)$. Since M is paracompact we may find a fixed countable open covering of M by coordinate domains $\mathfrak{B} = \{B_\alpha : \alpha \in I\}$ with the property that only a finite number of the B_α intersect an given compact subset of M . Thus we have a locally finite subatlas of M . Let $\delta : M \longrightarrow \mathbb{R}^+$ be a continuous function. Two metrics g and $h \in \text{Lor}(M)$ are δ -close in the fine \mathcal{E}^r -topology if for each $p \in M$ all of the coefficients and all the derivatives up to the r -th order are $\delta(p)$ -close at p when calculated in each of the coordinate systems $(B_\alpha, \varphi_\alpha)$ which contain p . We write

$$|g - h|_{r, \mathfrak{B}} < \delta$$

to denote δ -closeness in the fine \mathcal{E}^r -topology on $\text{Lor}(M)$. The sets

$$(2.71) \quad N_g(\delta) \equiv \{h \in \text{Lor}(M) : |g - h|_{r, \mathfrak{B}} < \delta\}$$

with g an arbitrary element of $\text{Lor}(M)$ and $\delta : M \longrightarrow \mathbb{R}^+$ an arbitrary continuous function form a basis for the fine \mathcal{E}^r -topology on $\text{Lor}(M)$.

A spacetime (M, g) is stably causal if there is a fine \mathcal{E}^0 -neighbourhood $N_g(\delta)$ of g in $\text{Lor}(M)$ such that each Lorentzian metric $h \in N_g(\delta)$ is causal. Beem and Ehrlich [33] have shown that a Lorentzian manifold (M, g) with M homeomorphic to \mathbb{R}^2 is stably causal.

The fine \mathcal{E}^r -topologies on $\text{Lor}(M)$ for $r = 0, 1, 2$ may be

interpreted as follows:

- (a) $r = 0$: if h is δ -close in the fine \mathcal{C}^0 -topology then all the coefficients of h_{ij} are close to g_{ij} in the fixed covering \mathfrak{B} of M . Thus the lightcones of h and g are "close".
- (b) $r = 1$: if h is δ -close in the fine \mathcal{C}^1 -topology then all the coefficients of h_{ij} are close to g_{ij} and all the partial derivatives $h_{ij,k}$ are close to $g_{ij,k}$ in the fixed covering \mathfrak{B} of M . Thus the Levi-Civita connections of h and g are "close" hence the systems of geodesics of h and g are "close".
- (c) $r = 2$: if h is δ -close in the fine \mathcal{C}^2 -topology then all the coefficients of h_{ij} are close to g_{ij} and all the partial derivatives $h_{ij,k}$, $h_{ij,kl}$ are close to $g_{ij,k}$, $g_{ij,kl}$ respectively in the fixed covering \mathfrak{B} of M . Thus the curvature tensors of h and g are "close".

There are many more elementary causal properties [33,108] such as the causal simplicity, causal continuity, future and past distinguishing properties but the preceding ones are easily related to a global decomposition of the Lorentzian metric. The causal properties discussed above are suited to the Lorentzian warped product construction which we shall present in the next section. A large class of interesting spacetimes may be written as a Lorentzian warped product.

The causality relations can be related by a simple diagram:

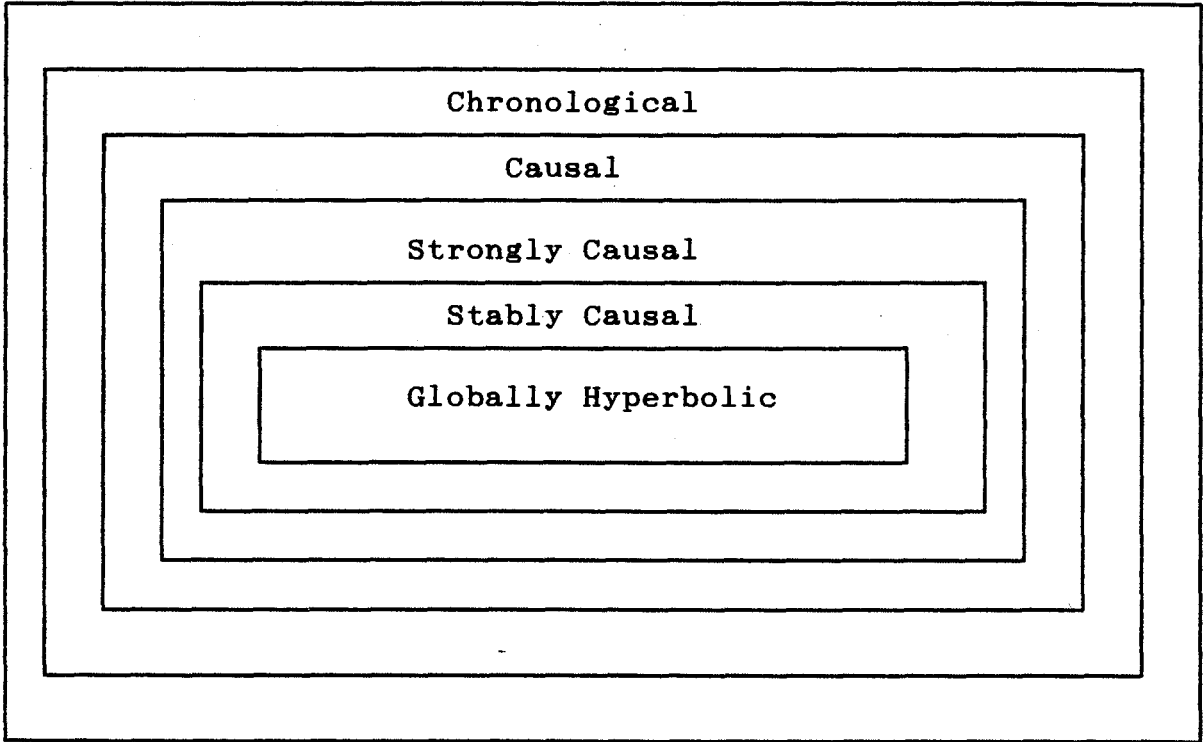


Figure 7 The Strengths of the Elementary Causality Conditions

Lorentzian Warped Products

Let (M, g) be a Lorentzian manifold of dimension m and (H, h) be a Riemannian manifold of dimension n . Suppose that

$f : M \longrightarrow \mathbb{R}^+$ be a smooth function. The Lorentzian warped product of the first type [33], $M \times_f H$ is the manifold

$$(\bar{M}, \bar{g}) = (M \times H, g \oplus fh).$$

If $\pi : \bar{M} \longrightarrow M$, and $\eta : \bar{M} \longrightarrow H$ are the projections onto M and N respectively, then we define the metric \bar{g} by

$$(2.72) \quad \bar{g}(v,w) = g(\pi_x v, \pi_x w) + f(\pi(p))h(\eta_x v, \eta_x w),$$

for $v, w \in T_p \bar{M}$. In a later section we will summarize the elementary causality of Lorentzian warped products of the first type.

The Lorentzian warped product of the second type on $M \times H$ is the manifold $(\bar{M}, \bar{g}) = (M \times H, fg \oplus h)$ where $f : H \longrightarrow \mathbb{R}^+$. We define \bar{g} by

$$(2.73) \quad \bar{g}(v,w) = f(\eta(p))g(\pi_x v, \pi_x w) + h(\eta_x v, \eta_x w)$$

for all $v, w \in T_p \bar{M}$. The elementary causality of warped products of the second type has been studied by Kemp [109]. All of the warped products in our work will be of the first type. As all spacetimes as we have defined them will be time-oriented we should find a criterion so that a warped product of the first type is time-oriented. Beem and Ehrlich [33] have proven the following fact. The warped product $M \times_f H$ of (M, g) and (H, h) may be time oriented if and only if either

- (a) $\dim M > 2$ and (M, g) is time-oriented; or
- (b) $\dim M = 1$ and $g = -dt^2$.

Let $M \times_f H$ be a Lorentzian warped product of the first type. The following is a list (a similar list may be found in Kemp [109] for warped products of the second type) of useful properties for warped products of the first type:

- (a) For each $b \in H$, the restriction $\pi|_{\eta^{-1}(b)} : \eta^{-1}(b) \longrightarrow M$ is an isometry of $\eta^{-1}(b)$ onto M .
- (b) For each $m \in M$, the restriction $\eta|_{\pi^{-1}(m)} : \pi^{-1}(m) \longrightarrow H$ is

a homothetic map of $\pi^{-1}(m)$ with homothetic factor $1/f(m)$.

(c) If $v \in T(M \times H)$, then $g(\pi_* v, \pi_* v) \leq \bar{g}(v, v)$.

Thus $\pi_* : T(M \times H) \longrightarrow T_{\pi(p)} M$ maps nonspacelike vectors to nonspacelike vectors and π maps nonspacelike curves of $M \times_f H$ to nonspacelike curves of M .

(d) For each $(m, b) \in M \times H$, the submanifolds $\pi^{-1}(m)$ and $\eta^{-1}(b)$ of $M \times_f H$ are nondegenerate when given their respective induced metrics.

(e) If $\phi : H \longrightarrow H$ is an isometry of H , then the map

$$\Phi \equiv \iota_M \times \phi : M \times_f H \longrightarrow M \times_f H$$

defined by

$$\Phi(m, b) = (m, \phi(b))$$

is an isometry of $M \times_f H$.

(f) If $\psi : M \longrightarrow M$ is an isometry of M such that $f \circ \psi = f$ then the map $\Psi = \psi \times \iota_H : M \times_f H \longrightarrow M \times_f H$ defined by

$$\Psi(m, b) = (\psi(m), b)$$

is an isometry of $M \times_f H$. If X is a Killing vector field on M ($\mathcal{L}_X g = 0$) with $X[f] = 0$ then the lift of X to $M \times_f H$, \bar{X} , such that $\bar{X}(p) = (X(\pi(p)), 0_{n(p)})$ is a Killing vector field on $M \times_f H$.

(g) For each $b \in H$, the leaf $\eta^{-1}(b)$ is a totally geodesic submanifold of $M \times_f H$.

Note that in (f) we need $f \circ \psi = f$ so that f is constant on the orbits of ψ otherwise the warping factor is changed.

Elementary Causality of Lorentzian Warped Product Manifolds of the First Type

Beem and Ehrlich [33] have proven the several propositions concerning the elementary causality of Lorentzian metrics with a warped product decomposition of the first type. Let (M,g) be a spacetime and let (H,h) be a Riemannian manifold. Then

- (a) $(M \times_f H, g \oplus fh)$ is chronological if and only if (M,g) is chronological;
- (b) $(M \times_f H, g \oplus fh)$ is causal if and only if (M,g) is causal;
- (c) $(M \times_f H, g \oplus fh)$ is strongly causal if and only if (M,g) is strongly causal;
- (d) $(M \times_f H, g \oplus fh)$ is stably causal if and only if (M,g) is stably causal and $\dim_{\mathbb{R}} M \geq 2$;
- (e) $(M \times_f H, g \oplus fh)$ is globally hyperbolic if and only if (M,g) is globally hyperbolic and (H,h) is a complete Riemannian manifold.

There are several more results available in Beem and Ehrlich [33]. Since the warped products we shall consider in Chapter 4 have Lorentzian factors which are two dimensional, we present some facts on two-dimensional Lorentzian manifolds. All these facts are found in [33].

Let (M,g) be a two dimensional spacetime. Then the following facts hold:

- (a) If M is homeomorphic to \mathbb{R}^2 , then (M,g) is stably causal;

- (b) If M is simply connected, then (M,g) is causal;
- (c) If M is simply connected then (M,g) is strongly causal;
- (d) The universal covering manifold of (M,g) is homeomorphic to \mathbb{R}^2 ;
- (e) If $M = \mathbb{R}^2$, then (M,g) is chronological.

The causal properties of a Lorentzian warped product of the first type will enable us to examine the elementary causality of spherically symmetric spacetimes with very little difficulty provided we can find adequate topological information on the factors. This feature of the warped product representation of metric tensors seems not to be widely appreciated in the literature. It should be noted that not all spherically symmetric spacetimes may be globally written as a Lorentzian warped product. Clarke [110] has recently produced an example of a spherically symmetric manifold which cannot be written as the direct product of two manifolds of lower dimension. Since the Lorentzian warped product construction depends on having a direct product decomposition we could not put a Lorentzian warped product on Clarke's example.

Spherically Symmetric Lorentzian Warped Product Manifolds of the First Type

A spacetime is spherically symmetric if its isometry group contains a subgroup isomorphic to the group $SO(3)$, and the

orbits of this group are two-dimensional spheres. The group action of $SO(3)$ on the orbits may be interpreted as a rotation. A metric tensor which is invariant under rotations is called spherically symmetric. A spherically symmetric metric tensor induces a metric on the orbits which is a multiple of the metric on a unit sphere. There are several types of coordinates in which it has become customary to represent spherically symmetric metrics. We will employ the Lorentzian warped product construction to examine them.

There are two subtypes of Lorentzian warped products of the first type so that the resulting Lorentzian manifold is four-dimensional. For $\dim_{\mathbb{R}} M = 1$ and $\dim_{\mathbb{R}} H = 3$ we find the class of Friedmann-Robertson-Walker-Lemaitre spacetimes if H is a space of constant curvature. (The naming of these metrics depends on whether the stress-energy tensor is that of a perfect fluid or the special case of "dust"). For $\dim_{\mathbb{R}} M = 2$ and $\dim_{\mathbb{R}} H = 2$ we find a class of Lorentzian warped product spacetimes which includes all of the spherically symmetric metrics (take $H = S^2$). Let (S^2, h) be the standard Riemannian differential geometry on the sphere. The coordinates we use on the sphere will be denoted (θ, ϕ) so that

$$(2.74) \quad h = d\theta d\theta + \sin^2 \theta d\phi d\phi.$$

Since (M, g) is a two-dimensional Lorentzian manifold we have four causally different ways of coordinatizing M . If we use coordinates (t, r) so that $\bar{g}(\partial_t, \partial_t) < 0$ and $\bar{g}(\partial_r, \partial_r) > 0$ then we

will call these coordinates TS-coordinates (since ∂_t and ∂_r are timelike and spacelike respectively). In TS-coordinates we can write the metric on M as

$$(2.75) \quad \bar{g} = -A^2(t,r)dt \otimes dt + B^2(t,r)dr \otimes dr.$$

If we use coordinates (t,u) so that

$$\bar{g}(\partial_t, \partial_t) < 0$$

$$\text{and } \bar{g}(\partial_u, \partial_u) = 0,$$

then we will call these coordinates TN-coordinates (since ∂_t and ∂_u are timelike and null respectively). In TN-coordinates we can write the metric on M as

$$(2.76) \quad \bar{g} = -A^2(t,u)dt \otimes dt + 2B^2(t,u)dt \otimes du.$$

If we use coordinates (u,r) so that

$$\bar{g}(\partial_u, \partial_u) = 0$$

$$\text{and } \bar{g}(\partial_r, \partial_r) > 0,$$

then we will call these coordinates NS-coordinates (since ∂_u and ∂_r are null and spacelike respectively). In NS-coordinates we can write the metric on M as

$$(2.77) \quad \bar{g} = 2A^2(u,r)du \otimes dr + B^2(u,r)dr \otimes dr.$$

If we use coordinates (u,v) so that

$$\bar{g}(\partial_u, \partial_u) = 0$$

$$\text{and } \bar{g}(\partial_v, \partial_v) = 0,$$

then we will call these coordinates NN-coordinates (since ∂_u and ∂_v are both null). In NN-coordinates we can write the metric on M as

$$(2.78) \quad g = -4A^2(u,v)du \otimes dv.$$

Synge [35] has shown that using NN-coordinates to formulate the vacuum field equations will directly produce the Kruskal-Szekeres vacuum solution. As the other coordinate systems have been extensively studied [111,112] we will confine our attention to NN-coordinate systems. Thus the metric for the spacetime which we use can be written in the form

$$(2.79) \quad \bar{g} = -4f^2(u,v)du \otimes dv + R^2(u,v)[d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi],$$

thus the metric is a type 1 warped product metric.

From a purely formal point of view there is no difference between the TN and NS types of coordinates. Goenner and Havas [113] point out that the direct integration of the field equations using a single type of coordinates and the corresponding form of the metric tensor may not produce the optimal set of solutions. Any spherically symmetric metric can be written in the NN-coordinates or the TS-coordinates [114]. A difficulty in working with different canonical forms of the spherically symmetric metric tensor lies in the fact that one must prove the inequivalence of the solutions found. It is well-known in general relativity that this equivalence problem is notoriously difficult; often more difficult than the process of finding the solutions themselves. There are various classification schemes of solutions by which one may prove solutions inequivalent. When these methods fail there is not much one can do in practical terms. For the NN-coordinates there has been little work so this problem is not so pressing

there.

The metric form (2.79) is preserved under coordinate transformations of the form

$$(2.80) \quad \begin{aligned} u &= f(u'), \\ v &= g(v'). \end{aligned}$$

Takeno [114] has shown that the coordinate transformation (2.80) is the most general that preserves the metric form (2.79).

Following Beem and Ehrlich [33], we set $\psi = \ln(R^2)$, and let D^1 and D^2 denote the Levi-Civita connections over the identity on (M, g) and (H, h) respectively. For vector fields $X_1, Y_1 \in \mathfrak{X}(M)$ and $X_2, Y_2 \in \mathfrak{X}(H)$, we may lift them to vector fields $X = (X_1, 0) + (0, X_2)$ and $Y = (Y_1, 0) + (0, Y_2)$ in $\mathfrak{X}(M \times H)$. Writing D for the Levi-Civita connection of \bar{g} and using the Koszul formula (2.30) we find the following formula for D :

$$(2.81) \quad \begin{aligned} D_X Y &= D_{X_1}^1 Y_1 + D_{X_2}^2 Y_2 + (1/2)[X_1(\psi)Y_2 + Y_1(\psi)X_2 \\ &\quad - \bar{g}(X_2, Y_2)\text{grad}_g \psi]. \end{aligned}$$

We use $\text{grad}_g \psi$ for the gradient of ψ on (M, g) . We identify the vector $D_{X_1}^1 Y_1|_m \in T_m M$ with the vector $(D_{X_1}^1 Y_1|_m, 0_b) \in T_{(m,b)}(M \times H)$. Similar identifications will be assumed.

Decompose tangent vectors $X \in T_p(M \times H)$ as $X = (X_1, X_2)$ and define the tensors H_ψ and h_ψ by

$$(2.82) \quad H_\psi(X_1) = D_{X_1}^1(\text{grad}_g \psi),$$

and

$$(2.83) \quad h_{\psi}(X_1, Y_1) = g(H_{\psi}(X_1), Y_1) \\ = g(D_{X_1}^1(\text{grad}_g \psi), Y_1).$$

Define the symbol $\|\text{grad}_g \psi\|_g^2 = g(\text{grad}_g \psi, \text{grad}_g \psi)$. Let R , R^1 , and R^2 be the curvature tensors on $(M \times_f H, g \oplus R^2 h)$, (M, g) , and (H, h) . For vector fields $X, Y, Z \in \mathfrak{X}(M \times H)$ we have

$$(2.84) \quad R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z.$$

Using (2.83), the decompositions of vector fields, and (2.84) we find the following formula for the Riemann tensor on the spherically warped product manifold of type 1:

$$(2.85) \quad R(X, Y)Z = R^1(X_1, Y_1)Z_1 + R^2(X_2, Y_2)Z_2 + (1/2)[h_{\psi}(X_1, Z_1)Y_2 \\ - h_{\psi}(Y_1, Z_1)X_2 + \bar{g}(X_2, Z_2)H_{\psi}(Y_1) - \bar{g}(Y_2, Z_2)H_{\psi}(X_1)] \\ + (1/4)[X_1(\psi)Z_1(\psi) + \bar{g}(X_2, Z_2)\|\text{grad}_g \psi\|_g^2]Y_2 \\ - (1/4)[Y_1(\psi)Z_1(\psi) + \bar{g}(Y_2, Z_2)\|\text{grad}_g \psi\|_g^2]X_2 \\ + (1/4)[Y_1(\psi)\bar{g}(X_2, Z_2) - X_1(\psi)\bar{g}(Y_2, Z_2)]\text{grad}_g \psi.$$

Similar formulae may be developed for the Ricci tensor [33] and the Einstein tensor, however these formulae depend explicitly on the nature of the basis chosen for the tangent spaces (orthonormal, non-null vectors). For pseudo-orthonormal bases which include null vectors, one must be careful when writing the above formulae.

Observers and the Kinematics of their Motion

In this section we formalize the notion of observer and congruences of observers. We also discuss the well known

Raychaudhuri decomposition of a timelike congruence of observers. The presentation we follow is a combination of that of Ellis [39,115], Frankel [98], Szekeres [116], and Greenberg [117].

An observer is a time-like curve $\gamma : I \subset \mathbb{R} \longrightarrow M$ such that $g(U,U) = -1$, where $U = \gamma_* (\partial_u)$ is future-pointing and u is the parameter of γ called proper time. The vector U determines a (3+1)-decomposition of $T_{\gamma(u)}M = R_\gamma \oplus \mathbb{R}$ along γ , where we call R_γ the rest space of γ at $\gamma(u)$. The image $\gamma(I)$ is called the world line of γ . The timelike unit vector $U = \gamma_* (\partial_u)$ is the 4-velocity of γ . An instantaneous observer is an ordered pair (p,U) where $p \in M$ and U is a future-pointing timelike unit vector in $T_p M$. One can always find a local observer whose tangent at p is U .

A reference frame Q on a spacetime (M,g) is a tetrad of orthonormal vector fields one of which is timelike and whose integral curves is an observer. A reference frame Q is geodesic if $D_T T = 0$ where T is the timelike unit future-pointing vector field in Q . Given an observer γ , we call a vector field X over γ a relative position vector if $X \circ \gamma(u) \in R_{\gamma(u)}$ for all $u \in \text{dom } \gamma$ and X is invariant under the flow induced by γ . The relative spatial velocity vector along γ is the vector

$$(2.86) \quad \begin{aligned} V(X) &= F_U X \\ &= D_U X - g(X, D_U U)U, \end{aligned}$$

where $U = \gamma_*(\partial_u)$.

Let (M, g) be a Lorentzian manifold. For a given open set $\mathcal{U} \subseteq M$, we define a congruence in \mathcal{U} as a family of curves such that for each point $p \in \mathcal{U}$ there is exactly one curve in the family which passes through p . Let U be a timelike unit vector field on (M, g) . The acceleration vector of a U -observer is the spacelike vector field $A = D_U U$ i.e. $A_i = U_{i;j} U^j$. For curves in the congruence of the U -observer the acceleration vector represents the non-gravitational forces acting on the observer. Computing the divergence of A we have

$$\begin{aligned} A^i{}_{;i} &= (U^i{}_{;j} U^j)_{;i} = U^i{}_{;ji} U^j + U^i{}_{;j} U^j{}_{;i} \\ &= U^i{}_{;ji} U^j + U_{i;j} U^{j;i}. \end{aligned}$$

By the Ricci identities

$$U_{i;kl} - U_{i;lk} = U_j R^j{}_{ikl}$$

and a contraction we find that

$$(2.87) \quad A^i{}_{;i} = U^i{}_{;ij} U^j + R_{ij} U^i U^j + U_{i;j} U^{j;i}.$$

We can decompose the "acceleration" tensor $U_{i;j}$ with respect to U^i and h_{ij} so that [39]

$$(2.88) \quad U_{i;j} = V_{ij} - A_i U_j,$$

where

$$(2.89) \quad V_{ij} = h^k{}_i h^l{}_j U_{k;l}$$

is called the relative velocity tensor. The tensor V_{ij} represents the relative velocities of particles in the rest 3-space of the observer U^k .

We now decompose V_{ij} into symmetric and antisymmetric parts as

follows:

$$(2.90) \quad V_{ij} = \theta_{ij} + \omega_{ij},$$

where

$$(2.91) \quad \theta_{ij} = V_{(ij)},$$

is called the strain-rate tensor, and where

$$(2.92) \quad \omega_{ij} = U_{[ij]},$$

is called the vorticity tensor. Since V_{ij} is the result of a projection into the rest-space of a U-observer we see that

$$(2.93) \quad \theta_{ij} U^j = 0,$$

and

$$(2.94) \quad \omega_{ij} U^j = 0.$$

The strain-rate tensor θ_{ij} can be further decomposed into its trace and its trace-free parts:

$$(2.95) \quad \theta_{ij} = \sigma_{ij} + (1/3)\theta h_{ij},$$

where σ_{ij} , the shear-rate tensor, satisfies

$$(2.96) \quad \begin{aligned} \sigma_{ij} &= \sigma_{ji}, \\ \sigma^i_i &= 0, \\ \sigma_{ij} U^j &= 0; \end{aligned}$$

and θ , the volume expansion scalar, is given by

$$(2.97) \quad \theta = U^i_{;i}.$$

The vorticity tensor ω_{ij} may be viewed as taking a sphere of particles in the rest-space of the U-observer into a rotated sphere of the same volume. A vector ω^i , called the vorticity vector, may be associated with ω_{ij} by the equation

$$(2.98) \quad \omega^i = (1/2)\eta^{ijkl} U_j \omega_{kl}.$$

From a knowledge of ω^i we can recover the vorticity tensor by

$$(2.99) \quad \omega_{ij} = \eta_{ijkl} \omega^k U^l.$$

The vorticity vector satisfies the following properties:

$$(2.100) \quad \omega_i U^i = 0,$$

and

$$(2.101) \quad \omega_{ij} \omega^i = 0.$$

The magnitude of the vorticity is defined by

$$(2.102) \quad \begin{aligned} \omega^2 &= \omega^i \omega_i \\ &= \omega^{ij} \omega_{ij} \\ &\geq 0. \end{aligned}$$

The shear tensor may be viewed as taking a sphere of particles in the rest-space of the U-observer into an ellipsoid of the same volume. The direction of any principal axis of the shear tensor is unchanged but all other directions are changed.

The magnitude σ of σ_{ij} is defined by

$$(2.103) \quad \begin{aligned} \sigma^2 &= (1/2) \sigma^{ij} \sigma_{ij} \\ &\geq 0. \end{aligned}$$

The effect of the volume expansion scalar θ is to change a sphere of particles in the rest-space of the U-observer into a larger sphere so that the logarithmic derivative of the radius of the sphere is $\theta/3$.

Using the Ricci identity $U_{i;k;l} - U_{i;l;k} = U_j R^j_{ikl}$ and multiplying by U^k we find

$$(2.104) \quad U_{i;k;l} U^k - U_{i;l;k} U^k - R^j_{ikl} U_j U^k = 0.$$

Using the definition of the relative velocity tensor V_{ij} and

using the projection h_{ij} we find the propagation equation for V_{ij} along the integral curve of U :

$$(2.105)$$

$$h^k{}_i h^l{}_j V_{kl;m} U^m - A_i A_j - h^k{}_i h^l{}_j A_{k;l} + V_{ik} V^k{}_j + R_{ikjl} U^k U^l = 0.$$

Since (2.105) is a propagation equation for V_{ij} it also contains the propagation equation of θ , σ_{ij} , and ω_{ij} . We will not find the propagation equation for ω_{ij} as we will deal exclusively with spherically symmetric spaces in which the vorticity tensor $\omega_{ij} = 0$.

The propagation equation for the expansion scalar θ is found from (2.105) by contracting on i and j :

$$(2.106) \quad \theta' - A^i{}_{;i} + (1/3)\theta^2 + 2\sigma^2 + (1/2)(\mu + 3p) = 0,$$

where we have used the field equations for a perfect fluid together with the definitions above. We have used the notation $\theta' = \frac{d\theta}{d\tau}$ where τ is the proper time along the integral curve of U .

Equation (2.106) can be written as

$$(2.107) \quad \theta' + (1/3)\theta^2 = -2\sigma^2 + A^i{}_{;i} - (1/2)(\mu + 3p).$$

In cosmology it is common practice to define a length scale L by the equation

$$(2.108) \quad \theta = 3(\ln L)'$$

In terms of the length scale L , which we may view as the distance from our fiducial timelike U -trajectory to a neighbouring U trajectory (L being measured in $R_{\gamma(\tau)}$), we have

$$(2.109) \quad 3L''/L = -2\sigma^2 + A^i{}_{;i} - (1/2)(\mu + 3p).$$

From this equation we see that shear induces a contraction of the flow; the divergence of the acceleration shows the tendency of pressure gradients to cause expansion; the terms from the trace of the stress-energy tensor show that pressure causes contraction.

The symmetric trace-free part of (2.105) is the shear propagation equation. From Ellis [39] we have (neglecting terms with vorticity)

$$(2.110) \quad h_i^k h_j^l (\sigma_{kl;m} U^m - A_{(k;l)}) - A_i A_j + \sigma_{ik} \sigma_j^k + (2/3)\theta \sigma_{ij} + h_{ij} (-(2/3)\sigma^2 + (1/3)A^i_{;i}) + E_{ij} = 0.$$

From this equation we see that the shear is controlled by the electric part E_{ij} of the Weyl tensor. Since E_{ij} represents the free gravitational field due to distant matter, these equations describe the "tidal forces" felt by a congruence of U-observers.

For spherically symmetric metrics and distributions of matter we should observe a distortion in the flow of U-observers as we move toward larger concentrations of matter. For a spherical fluid element in the rest space of a U-observer the cross-section of the fluid element orthogonal to the radial direction should decrease while the radial cross-section should elongate. The elongation in the radial direction is due to two effects - the acceleration of the congruence and the tidal forces.

Motions on a Lorentzian Spacetime

The use of symmetries to classify vacuum solutions of the field equations is well-known in general relativity. Symmetries may also be used to partially classify interior solutions. The stress-energy tensor is sometimes assumed to be invariant under the action of a symmetry *i.e.* isometric motion. The problem of how these symmetries effect the individual matter fields that contribute to the stress-energy tensor is probably difficult since a given stress-energy tensor may have several interpretations. We will briefly discuss some of the symmetries that have been used (a more complete list of symmetries is in Katzin *et al.* [29]. In Chapter 5 we will compute the field equations and other quantities for the case of nonstatic spherically symmetric anisotropic fluids in NN-coordinates.

A motion is generated by a Killing vector field X on (M, g) such that

$$(2.111) \quad \mathcal{L}_X g_{ij} = 0.$$

A conformal motion is generated by a vector field X (conformal Killing vector field) such that

$$(2.112) \quad \mathcal{L}_X g_{ij} = 2\psi g_{ij},$$

where ψ is the scalar conformal factor.

A homothetic motion is generated by an vector field X (homothetic Killing vector field) such that

$$(2.113) \quad \mathcal{L}_X g_{ij} = 2g_{ij}.$$

A special conformal motion is generated by an conformal vector field X such that

$$(2.114) \quad \begin{aligned} \mathcal{L}_X g_{ij} &= 2\psi g_{ij}, \\ \psi_{,ij} &= 0. \end{aligned}$$

where ψ is the scalar conformal factor. A conformal collineation is generated by an affine conformal vector field X such that

$$(2.115) \quad \mathcal{L}_X g_{ij} = 2\psi g_{ij} + H_{ij},$$

where the collineation tensor H_{ij} (one could also call H_{ij} a conformal Killing tensor) satisfies

$$(2.116) \quad H_{[ij]} = 0,$$

$$(2.117) \quad H_{ij;k} = 0,$$

and where ψ is the scalar conformal factor. It is still a difficult open problem to characterize the collineation tensor H_{ij} . Only a few conditions are known which will produce H_{ij} in such a way as to guarantee the existence of the vector X and the scalar ψ . It has proven difficult to find examples of proper affine conformal vectors (do not reduce of conformal Killing vectors) until recently when Sharma and Duggal [118] have provided an abstract example.

An affine conformal vector field is a generalization of a conformal Killing field. An affine conformal vector field reduces to a conformal Killing vector field if and only if $H_{ij} = \lambda g_{ij}$ where λ is a constant. We may view the symmetric tensor H_{ij} as a measure of how much X fails to be a conformal

Killing vector.

A special conformal collineation is generated by an affine conformal vector field X such that

$$(2.118) \quad \mathcal{L}_X g_{ij} = 2\psi g_{ij} + H_{ij},$$

i.e.

$$(2.119) \quad X_{i;j} + X_{j;i} = 2\psi g_{ij} + H_{ij},$$

where ψ is the scalar conformal factor and H_{ij} is the symmetric parallel tensor associated with X and obeys the following equations

$$(2.120) \quad H_{[ij]} = 0,$$

$$(2.121) \quad H_{ij;k} = 0,$$

$$(2.122) \quad \psi_{,ij} = 0.$$

The interest in these more general types of motions is a consequence of the following theorems. The first is due to Oliver and Davis [93].

Theorem: Let $X^i = \lambda U^i$, $U^i U_i = -1$ and $\lambda > 0$. A spacetime (M, g) admits a timelike conformal motion with symmetry vector X^i if and only if

$$(a) \quad \sigma_{ij} = 0,$$

$$(b) \quad A_i = (\log \lambda)_{,i} + (\theta/3)U_i,$$

where the conformal scalar

$$\psi = \lambda\theta/3$$

and σ_{ij} , θ , and A_i are, respectively, the shear tensor, the expansion scalar, and the acceleration vector of the timelike flow generated by U_i .

The second theorem is due to Duggal [91].

Theorem: Let $X^i = \lambda U^i$, $U^i U_i = -1$ and $\lambda > 0$. A spacetime (M, g) admits a timelike conformal collineation with symmetry vector X^i if and only if

$$(a) \sigma_{kl} = (2\lambda)^{-1} [h^i_k h^j_l H_{ij} - (2/3)\theta^* h_{kl}], \text{ and}$$

$$(b) A_i = \lambda^{-1} [\lambda_{,i} + \lambda_{,j} U^j U_i + H_{jk} U^k h^j_i],$$

where

$$\psi = (\lambda\theta - \theta^*)/3,$$

$$\theta^* = (1/2)[H^i_i + H_{ij} U^i U^j],$$

and σ_{ij} , θ , and A_i are, respectively, the shear tensor, the expansion scalar, and the acceleration vector of the timelike flow generated by U_i .

Duggal's theorem is very important for the study of shearing fluids since it relates the shear of the fluid to a symmetry *i.e.* the existence of a timelike affine collineation vector X and the "affine collineation tensor" H_{ij} . We conclude this section with some of the properties of collineations.

An affine conformal vector field is special if and only if it leaves the curvature tensor R^i_{jkl} invariant. A special affine conformal vector field X is a special case of a Ricci collineation vector field [29],

$$(2.123) \quad \mathcal{L}_X R_{ij} = 0.$$

The Lie derivative with respect to an affine conformal vector field X of a non-null unit vector Z is given by [89],

$$(2.124) \quad \mathcal{L}_X Z^i = -(\psi + (\epsilon/2)H_{jk} Z^j Z^k) Z^i + Y^i,$$

$$(2.125) \quad \mathcal{L}_X Z_i = (\psi - (\epsilon/2)H_{jk}Z^jZ^k)Z_i + H_{ij}Z^j + Y_i,$$

where Y^i is orthogonal to Z^i and ϵ is the indicator of Z^i .

If ψ is the flow generated by an affine conformal vector field X then [118]

- (a) a null vector field N will be transformed by ψ into a null vector field if and only if $H(N,N) = 0$;
- (b) a non-null vector field V retains its causal character under ψ ;
- (c) two orthogonal vector fields U , and V , will be transformed into orthogonal vector fields under ψ if and only if $H(U,V) = 0$.

There are many more properties of collineations which we shall omit. These generalized symmetries seem sure to play a very important role in future studies of realistic fluids. In Chapter 5 we shall compute the field equations with a conformal collineation in NN-coordinates for an anisotropic stress-energy tensor.

CHAPTER III

THE MATHEMATICS OF THE STRESS-ENERGY-MOMENTUM TENSOR

Classification of the Stress-Energy Tensor

In this section we present a brief summary of the algebraic classification of the stress-energy-momentum tensor. The techniques used will be applicable to any second order symmetric tensor. There are several classification schemes [27,119,120, 121] which have been developed in the last twenty years. The simplest of these is the Segré classification which uses the eigenvalues and eigenvectors of the R_{ij} with respect to g_{ij} . When working with indefinite metrics one must prescribe the field over which the Segré classification takes place. It is customary to use \mathbb{R} , the real field, for the Segré classification. In general the Segré class of R_{ij} over \mathbb{R} will be different from that over \mathbb{C} . From the field equations (1.1) and the definition of the Einstein tensor we see that any classification of R_{ij} is also a classification of T_{ij} . In fact an alternative way of writing the field equations is

$$(3.1) \quad R_{ij} = T_{ij} - (1/2)Tg_{ij},$$

where $T = T^i_i$. There is a shift of eigenvalues in passing from a classification of R_{ij} to that of T_{ij} but, as long as it is kept in mind, it poses no obstacle.

Hall [122,124] has proven that in a spacetime (M,g) if $p \in M$ so that $T_{ij} \neq 0$ at p then there always exists a real null

tetrad $\{L_i, N_i, X_i, Y_i\}$ such that T_{ij} assumes one of the following canonical forms:

$$(3.2) \quad T_{ij} = 2\rho_0 L_{(i} N_{j)} + \rho_1 (L_i L_j + N_i N_j) + \rho_2 X_i X_j + \rho_3 Y_i Y_j;$$

$$(3.3) \quad T_{ij} = 2\rho_1 L_{(i} N_{j)} \pm \rho_1 L_i L_j + \rho_2 X_i X_j + \rho_3 Y_i Y_j;$$

$$(3.4) \quad T_{ij} = 2\rho_1 L_{(i} N_{j)} + 2L_{(i} X_{j)} + \rho_1 X_i X_j + \rho_2 Y_i Y_j;$$

$$(3.5) \quad T_{ij} = 2\rho_0 L_{(i} N_{j)} + \rho_1 (L_i L_j - N_i N_j) + \rho_2 X_i X_j + \rho_3 Y_i Y_j;$$

where $\rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R}$, and in (3.5) $\rho_1 \neq 0$. The first form

(3.2) can be written with respect to a pseudo-orthonormal tetrad $\{T_i, Z_i, X_i, Y_i\}$ where $\sqrt{2}T^i = L^i - N^i$, $\sqrt{2}Z^i = L^i + N^i$ as

$$(3.6) \quad T_{ij} = -(\rho_0 - \rho_1)T_i T_j + (\rho_0 + \rho_1)Z_i Z_j + \rho_2 X_i X_j + \rho_3 Y_i Y_j.$$

The forms (3.1) and (3.5) correspond to Segré type $\{1;1,1,1\}$ where the numbers inside the braces refer to the degree of the elementary divisor corresponding to an eigenvalue. Each elementary divisor corresponds to at least one eigenvector. If an eigenvalue is algebraically degenerate (repeated eigenvalue) but the number of eigenvectors equals the multiplicity of the eigenvalue, then the elementary divisors for that eigenvalue are of degree 1. The Segré symbol indicates the algebraic degeneracy by enclosing the degrees of the elementary divisors corresponding to the degenerate eigenvalue in parentheses. If the degree of the elementary divisor exceeds the number of eigenvectors then the degree of the elementary divisors sum to the multiplicity of the eigenvalue. In this case at least one of the degrees of the elementary divisors of the degenerate eigenvalue is greater

than 1. The degree of the elementary divisor corresponding to a timelike eigenspace is separated from the other degrees by a semicolon, the others by a comma.

The form (3.3) corresponds to Segré class $\{2;1,1\}$ which has a unique null eigenvector in the L^i direction. The form (3.4) corresponds to Segré class $\{3;1\}$ which has a unique null eigenvector in the L^i direction. In (3.5) complex eigenvalues occur and X^i , and Y^i are the only real eigenvectors, thus the Segré class is written $\{z\bar{z};1,1\}$. For this T_{ij} is diagonalizable over \mathbb{C} but not over \mathbb{R} .

We see that T_{ij} always admits at least two eigenvectors. The Segré class $\{1;1,1,1\}$ and its algebraic degenerate subcases is the only class which admits a timelike eigenvector. If there is no timelike eigenvalue degeneracy (timelike eigenvalue is distinct from the spacelike eigenvalues) then the timelike eigenvector is unique.

Energy Conditions

The Einstein field equations (1.1) are usually solved subject to side conditions. These side conditions will reflect physical properties of the physical situation we are trying to model. As we are interested in exact interior solutions, the "energy conditions" will be applied. There are several types of distinct energy conditions mentioned in the literature [33,95, 96]. Unfortunately not all these energy conditions have

distinct names [96]. The role of all the energy conditions is to exclude unrealistic models of macroscopic matter.

There is a general consensus among researchers in general relativity that the energy density of classical macroscopic matter as measured by an observer with 4-velocity U^i is nonnegative *i.e.*,

$$(3.7) \quad T_{ij} U^i U^j \geq 0.$$

Since we do not admit the existence of privileged observers, this relation must hold for all timelike vectors U^i . These inequalities are called the timelike weak energy conditions. Tipler [123] has shown that the timelike weak energy conditions are the weakest energy conditions that can be locally defined which use the entire set of timelike vectors in $T_p M$.

If we write the stress-energy tensor T_{ij} as

$$(3.8) \quad T_{ij} = \sum_{a=0}^3 \lambda_{(a)} e_{(a)i} e_{(a)j}$$

where $\{e_{(a)}\}$ is an orthonormal eigenbasis with $e_{(0)}$ timelike, then the timelike weak energy conditions for a stress-energy tensor of Segré class $\{1,111\}$, and its algebraic degeneracies, are equivalent [94] to the following system of inequalities on the eigenvalues $\lambda_{(a)}$:

$$(3.9) \quad \begin{aligned} \lambda_0 &\geq 0, \\ \lambda_0 + \lambda_i &\geq 0, \quad i \in \{1,2,3\}. \end{aligned}$$

The null weak energy condition is

$$(3.10) \quad T_{ij} K^i K^j \geq 0$$

for all null vectors K^i .

The strong energy condition is

$$(3.11) \quad T_{ij} U^i U^j + (1/2)T \geq 0$$

for all unit timelike vectors U^i . This condition ensures that the matter stresses will not become so large that $R_{ij} U^i U^j \leq 0$.

The dominant energy condition assert that T_{ij} satisfies the the following conditions for each timelike future pointing vector U^i :

$$(3.12) \quad T_{ij} U^i U^j \geq 0, \text{ and}$$

$$(3.13) \quad T^i_j U^j \text{ is nonspacelike.}$$

This energy condition states that the speed of energy flow of matter is always less that the speed of light. The dominant energy condition implies the timelike weak energy condition.

The dominant energy condition excludes stress-energy-momentum tensors of Segré types $\{z\bar{z};1,1\}$ or $\{3;1\}$ and their algebraic degeneracies. It also severely restricts the possible eigenvalues of Segré classes $\{1;1,1,1\}$ and $\{2;1,1\}$ and their algebraic degeneracies. In particular for Segré class $\{1;1,1,1\}$ we must have the eigenvalues satisfying [124]

$$(3.14) \quad \lambda_0 \geq 0, \text{ and}$$

$$(3.15) \quad |\lambda_i| \leq \lambda_0 \text{ for } i \in \{1,2,3\}.$$

The strong energy condition implies the null weak energy conditions, but does not imply the timelike weak energy condition [96].

A stress-energy-momentum tensor is normal at $p \in M$ if $T^i_j X^j$ is timelike for all nonspacelike vectors $X \in T_p M$ [94].

T_{ij} is normal if it is normal for every $p \in M$. It can be shown that a normal stress-energy-momentum tensor has a unique unit future pointing timelike eigenvector [94].

Stress-Energy-Momentum Tensors

The material content of a spacetime is represented by the stress-energy-momentum tensor T_{ij} which appears on the right-hand side of the field equations (1.1). T_{ij} depends on the fields representing the matter, the covariant derivatives of these fields, and the metric tensor. Note that $T_{ij} = 0$ on an open set U in M means that there are no matter fields on U . By the field equations we see that the twice contracted Bianchi identities of the Einstein tensor

$$(3.16) \quad G^{ij}{}_{;j} = 0,$$

imply that the stress-energy-momentum tensor satisfies a set of differential identities

$$(3.17) \quad T^{ij}{}_{;j} = 0.$$

It has become common practice in general relativity to call these identities "the conservation equations" or, for the case of dust, "the equations of motion". As Wald [96] notes, the notion of (3.17) as conservation equations is only true in the differential sense. A better descriptive term for (3.17) is the "equations of hydrodynamical support".

For a fluid moving through spacetime with a unit-speed timelike tangent vector U , the flow lines are the integral

curves of the vector field U . We say that the fluid is a perfect fluid if the stress-energy-momentum tensor has the form

$$(3.18) \quad T_{ij} = (\mu+p)U_i U_j + p g_{ij},$$

where μ is the energy density measured by an observer with velocity U , p is the pressure common to all 2-planes in the rest-space of the observer.

A viscous fluid with coefficient of dynamic viscosity $\eta \geq 0$, bulk viscosity $\xi \geq 0$, flow vector U^i , energy density μ , isotropic pressure p , shear tensor σ_{ij} , expansion scalar θ , and heat flow vector Q^i , has a stress-energy-momentum tensor given by [125]

$$(3.19)$$

$$T_{ij} = (p - \xi\theta + \mu)U_i U_j + (p - \xi\theta)g_{ij} - 2\eta\sigma_{ij} + 2U_{(i}Q_{j)},$$

where

$$U_i U^i = -1,$$

$$U_i Q^i = 0,$$

$$\sigma_{ij} U^j = 0,$$

and

$$\sigma^i_i = 0.$$

Hall [119] and Hall and Negm [126] have shown that this form of the stress-energy-momentum tensor, without any energy condition imposed, does not restrict the Segré class. By using the projection tensor onto the 3-space orthogonal to U^i , *i.e.* $h_{ij} = g_{ij} + U_i U_j$, we can write the stress-energy-momentum tensor in the form:

(3.20)

$$T_{ij} = (T^{kl}U_kU_l + (1/3)T^{kl}h_k^eh_{el})U_iU_j + ((1/3)T^{kl}h_k^eh_{el})g_{ij} \\ + [T^{kl}h_{ik}h_{jl} - (1/3)(T^{kl}h_k^eh_{el})h_{ij}] - 2T^{kl}U_lh_{k(i}U_{j)}.$$

Equation (3.20) has the same form as (3.19) if we set

$$Q_i = T^{kl}U_lh_{ki}$$

and

$$-2\sigma_{ij} = [T^{kl}h_{ik}h_{jl} - (1/3)(T^{kl}h_k^eh_{el})h_{ij}]$$

with other obvious identifications. From (3.19) it is clear that setting $Q^i = 0$ forces T_{ij} to be Segré type $\{1;1,1,1\}$ or one of its degeneracies. A stress-energy-momentum tensor for a viscous fluid without heat flow is easily seen to be normal.

Setting ξ, η to zero and leaving $Q^i \neq 0$, we have a nonviscous fluid with heat flow. The vector U^i is not a timelike eigenvector in this case. Hall and Negm [126] have shown that the dominant energy conditions imply that two physical Segré classes arise from this specialization, namely $\{2,(11)\}$ and $\{1,1(11)\}$. Only in the last case is the stress-energy-momentum tensor normal.

Setting the viscosity coefficients ξ, η and the heat flow vector Q^i to zero leads to the Segré type $\{1;(1,1,1)\}$ of a perfect fluid. It is clear that the stress-energy-momentum tensor of a perfect fluid is normal.

Hall and Negm [124] give several combinations of matter fields and discuss their algebraic structure. The combinations include two non-zero interacting radiation fields, a perfect

fluid and a radiation field, and two noninteracting perfect fluids. All of these combinations have stress-energy-momentum tensors of Segré class $\{1,1(11)\}$ hence are all anisotropic.

For a perfect fluid the timelike weak energy conditions [96] become

$$(3.21) \quad \begin{aligned} \mu &\geq 0, \\ \mu + p &\geq 0. \end{aligned}$$

For a perfect fluid the strong energy conditions are

$$(3.22) \quad \begin{aligned} \mu + 3p &\geq 0, \\ \mu + p &\geq 0. \end{aligned}$$

For a perfect fluid the dominant energy conditions are

$$(3.23) \quad \mu \geq |p| \geq 0,$$

For anisotropic stress-energy tensors the situation is more complicated depending on the composition of the stress-energy tensor. Hall and Negm [124] have written the dominant energy conditions for several cases.

CHAPTER IV

SHEARING PERFECT FLUID SOLUTIONS IN SPHERICALLY WARPED PRODUCT MANIFOLDS OF THE FIRST TYPE

The Field Equations in NN-Coordinates

In this chapter we shall write the field equations for a perfect fluid in NN-coordinates and find some special solutions which exhibit nonzero shear. From Chapter 2 we know that a spherically symmetric metric may be written in a variety of ways when represented by a Lorentzian warped product of the first type. The advantage of using this representation is that one can immediately deduce certain aspects of elementary causality solely from this representation and an analysis of the causality of the two dimensional Lorentzian factor manifold.

The metric for the spacetime which we use can be written in the form

$$(4.1) \quad \bar{g} = -4f^2(u,v)du \otimes dv + r^2(u,v)[d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi].$$

This metric is a type 1 warped product metric. The coordinates u and v are null coordinates. There are few papers in the literature which use double null coordinates [35,127]. The papers [35,127] have similar calculations which were used to cross-check the calculations as far as possible.

The metric form (4.1) is not quite the same as that in [35,127]. In those papers the metric is presented with the

term $-4f^2(u,v)du \otimes dv$ replaced by $-2f(u,v)du \otimes dv$. The second choice allows the possibility that when f changes sign the coordinates u and v are interchanged.

Beem and Ehrlich [33] have proven the several propositions concerning the elementary causality of Lorentzian metrics with a warped product decomposition of the first type. Let (M,g) be a spacetime and let (H,h) be a Riemannian manifold. Then

- (a) $(M \times_f H, g \oplus fh)$ is chronological if and only if (M,g) is chronological;
- (b) $(M \times_f H, g \oplus fh)$ is causal if and only if (M,g) is causal;
- (c) $(M \times_f H, g \oplus fh)$ is strongly causal if and only if (M,g) is strongly causal;
- (d) $(M \times_f H, g \oplus fh)$ is stably causal if and only if (M,g) is stably causal and $\dim_{\mathbb{R}} M \geq 2$.

For the metric (4.1) we take M to be some open subset of \mathbb{R}^2 with metric $g = -4f^2(u,v)du \otimes dv$ where (u,v) are the coordinates on \mathbb{R}^2 ; for H we take S^2 , the standard sphere with coordinates (θ, ϕ) , with metric $h = d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi$. With these identifications the all four of the propositions hold when the appropriate conditions hold on the two-dimensional Lorentzian manifold (M,g) . Recall that S^2 is complete. The elementary causality of any perfect fluid solution determined from (4.1) (as far as the preceding four propositions are concerned) is determined by simply examining the properties of the (u,v) coordinates on an appropriate domain. Since the metrics

$g = -4f^2(u,v)du\otimes dv$ are all conformal to $g = -2du\otimes dv$ on \mathbb{R}^2 , the spaces (M,g) have similar causal properties as 2-dimensional Minkowski space. If global topological identifications are made, then great care must be used in trying to use the preceding theorems. Not all spherically symmetric metrics have these properties since they cannot all be written as a Lorentzian warped product.

We compute the standard quantities for the metric (4.1) next. All quantities are computed in the sign conventions of Chapter 2 (also see Appendix A for notation). The Christoffel symbols of the second kind are (with $(x^0, x^1, x^2, x^3) = (u, v, \theta, \phi)$)

$$(4.2) \quad \begin{array}{ll} \Gamma^0_{00} = 2f_u/f, & \Gamma^0_{22} = rr_v/(2f^2), \\ \Gamma^0_{33} = \Gamma^0_{22}\sin^2\theta, & \Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta, \\ & = rr_u\sin^2\theta/(2f^2), & = rr_v\sin^2\theta/(2f^2), \\ \Gamma^1_{22} = rr_u/(2f^2), & \Gamma^1_{11} = 2f_v/f, \\ \Gamma^2_{02} = r_u/r, & \Gamma^2_{12} = r_v/r, \\ \Gamma^2_{33} = -\cos\theta\sin\theta, & \Gamma^3_{03} = r_u/r, \\ \Gamma^3_{13} = r_v/r, & \Gamma^3_{23} = \cot\theta. \end{array}$$

Applying the Christoffel symbols we find the geodesic equations to be

$$(4.3) \quad \begin{aligned} rr_v(\phi')^2\sin^2\theta/(2f^2) + rr_v(\theta')^2/(2f^2) + 2(u')^2f_u/f + u'' &= 0, \\ rr_u(\phi')^2\sin^2\theta/(2f^2) + rr_u(\theta')^2/(2f^2) + 2(v')^2f_v/f + v'' &= 0, \\ 2r_u\theta'u'/r + 2r_v\theta'v'/r - \sin\theta\cos\theta(\phi')^2 + \theta'' &= 0, \\ 2r_u\phi'u'/r + 2r_v\phi'v'/r + 2\theta'\phi'\cot\theta + \phi'' &= 0, \end{aligned}$$

where the prime represents differentiation with respect to an affine parameter. In the paper of Synge [35], the Lagrangian defined from the metric (4.1) is used to discuss radial null and timelike geodesics for a vacuum in NN-coordinates.

The Killing equations corresponding to the metric (4.1) are

$$\begin{aligned}
(4.4) \quad & \xi_{0,0} - 2f_u \xi_0 / f = 0, \\
& \xi_{0,1} + \xi_{1,0} = 0, \\
& \xi_{0,2} + \xi_{2,0} - 2r_u \xi_2 / r = 0, \\
& \xi_{0,3} + \xi_{3,0} - 2r_u \xi_3 / r = 0, \\
& \xi_{1,1} - 2f_v \xi_1 / f = 0, \\
& \xi_{1,2} + \xi_{2,1} - 2r_v \xi_2 / r = 0, \\
& \xi_{1,3} + \xi_{3,1} - 2r_v \xi_3 / r = 0, \\
& 2\xi_{2,2} - rr_u \xi_1 / f^2 - rr_v \xi_0 / f^2 = 0, \\
& \xi_{2,3} + \xi_{3,2} - 2\cot\theta \xi_3 = 0, \\
& 2\xi_{3,3} - r\sin^2\theta(r_u \xi_1 + r_v \xi_0) / f^2 + \sin 2\theta \xi_2 = 0.
\end{aligned}$$

The spherical symmetry implies the existence of at least three

Killing vectors: $\xi_{(1)} = \frac{\partial}{\partial\phi}$, $\xi_{(2)} = \sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}$, and

$\xi_{(3)} = \cos\phi \frac{\partial}{\partial\theta} - \cot\theta \sin\phi \frac{\partial}{\partial\phi}$. Other Killing vectors may occur

if special assumptions are made on the metric functions f and r . In the following section we will see that assumptions that we make in order to render the field equations tractable will generate an "accidental" Killing vector.

The Riemann tensor has components given by

$$\begin{aligned}
 (4.5) \quad R_{0101} &= -4ff_{vu} + 4f_u f_v, \\
 R_{0202} &= 2rr_u f_u / f - rr_{uu}, \\
 R_{0212} &= -rr_{uv}, \\
 R_{0303} &= R_{0202} \sin^2 \theta, \\
 R_{0313} &= R_{0212} \sin^2 \theta, \\
 R_{1212} &= 2rr_v f_v / f - rr_{vv}, \\
 R_{1313} &= R_{1212} \sin^2 \theta, \\
 R_{2323} &= r^2 \sin^2 \theta (r_u r_v / f^2 + 1).
 \end{aligned}$$

We will write the tetrad components of the Riemann tensor for a pseudo-orthonormal tetrad $\{\lambda_A^i\}$ chosen as follows:

$$\begin{aligned}
 (4.6) \quad \lambda_{(0)}^i &= \delta_{(1)}^i / (\sqrt{2}f), \\
 \lambda_{(1)}^i &= \delta_{(0)}^i / (\sqrt{2}f), \\
 \lambda_{(2)}^i &= \delta_{(2)}^i / r, \\
 \lambda_{(3)}^i &= \delta_{(3)}^i / (r \sin \theta).
 \end{aligned}$$

Thus $g_{ij} \lambda_A^i \lambda_B^j = \eta_{AB}$ where

$$(4.7) \quad \eta_{AB} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The independent tetrad-components of the Riemann tensor are

$$\begin{aligned}
 (4.8) \quad R_{(0101)} &= R_{1010} / (4f^4) \\
 &= (-ff_{vu} + f_u f_v) / f^4, \\
 R_{(0202)} &= R_{1212} / (2f^2 r^2) \\
 &= (2r_v f_v / f - r_{vv}) / (2rf^2),
 \end{aligned}$$

$$\begin{aligned}
R_{(0212)} &= R_{1202}/(2f^2 r^2) \\
&= -r_{uv}/(2rf^2), \\
R_{(1212)} &= R_{0202}/(2f^2 r^2) \\
&= (2r_u f_u/f - r_{uu})/(2rf^2), \\
R_{(2323)} &= R_{2323}/(r^4 \sin^2 \theta) \\
&= (r_u r_v/f^2 + 1)/r^2.
\end{aligned}$$

The Ricci tensor components are

$$\begin{aligned}
(4.9) \quad R_{00} &= 4r_u f_u/(rf) - 2r_{uu}/r, \\
R_{01} &= -2r_{uv}/r + 2f_u f_v/f^2 - 2f_{vu}/f, \\
R_{11} &= 4r_v f_v/(rf) - 2r_{vv}/r, \\
R_{22} &= 1 + rr_{uv}/f^2 + r_u r_v/f^2, \\
R_{33} &= R_{22} \sin^2 \theta.
\end{aligned}$$

The tetrad components of the Ricci tensor are

$$\begin{aligned}
(4.10) \quad R_{(00)} &= 2r_v f_v/(rf^3) - r_{vv}/(rf^2), \\
R_{(01)} &= -r_{uv}/(rf^2) + f_u f_v/f^4 - f_{vu}/f^3, \\
R_{(11)} &= 2r_u f_u/(rf^3) - r_{uu}/(rf^2), \\
R_{(22)} &= 1/r^2 + r_{uv}/(rf^2) + r_u r_v/(r^2 f^2), \\
R_{(33)} &= 1/r^2 + r_{uv}/(rf^2) + r_u r_v/(r^2 f^2).
\end{aligned}$$

The Ricci scalar (curvature scalar) is

$$\begin{aligned}
(4.11) \quad R &= 2r_u r_v/(r^2 f^2) + 4r_{uv}/(rf^2) - 2f_u f_v/f^4 \\
&\quad + 2f_{vu}/f^3 + 2/r^2.
\end{aligned}$$

Other invariants constructed from the Riemann tensor are

$$\begin{aligned}
(4.12) \quad R_{ij} R^{ij} &= 2[2r_v f_v/(rf^3) - r_{vv}/(rf^2)][2r_u f_u/(rf^3) - r_{uu}/(rf^2)] \\
&\quad + 2[-r_{uv}/(rf^2) + f_u f_v/f^4 - f_{vu}/f^3]^2
\end{aligned}$$

$$+ 2[1/r^2 + r_{uv}/(rf^2) + r_u r_v/(r^2 f^2)]^2,$$

and the Kretschmann scalar

$$\begin{aligned}
(4.13) \quad R_{ijkl} R^{ijkl} &= 4(r_u r_v)^2/(r^4 f^4) + 8r_u r_v/(r^4 f^2) \\
&\quad + 16r_u r_v f_u f_v/(r^2 f^6) - 8r_u r_{vv} f_u/(r^2 f^5) \\
&\quad - 8r_{uu} r_v f_v/(r^2 f^5) + 4(r_{uv})^2/(r^2 f^4) \\
&\quad + 4r_{uu} r_{vv}/(r^2 f^4) + 4(f_u f_v)^2/f^8 \\
&\quad - 8f_u f_v f_{vu}/f^7 + 4(f_{vu})^2/f^6 + 4/r^4 \\
&= 4[2r_v f_v/f - r_{vv}][2r_u f_u/f - r_{uu}]/(r^2 f^4) \\
&\quad + 4(r_{uv})^2/(r^2 f^4) + 4[1+r_u r_v/f^2]^2/r^4 \\
&\quad + 4[f_u f_v - ff_{vu}]^2/f^8.
\end{aligned}$$

The components of the Einstein tensor are

$$\begin{aligned}
(4.14) \quad G_{00} &= 4r_u f_u/(rf) - 2r_{uu}/r, \\
G_{01} &= 2f^2/r^2 + 2r_u r_v/r^2 + 2r_{uv}/r, \\
G_{11} &= 4r_v f_v/(rf) - 2r_{vv}/r, \\
G_{22} &= -rr_{uv}/f^2 + r^2 f_u f_v/f^4 - r^2 f_{vu}/f^3, \\
G_{33} &= G_{22} \sin^2 \theta.
\end{aligned}$$

We are interested in finding shearing solutions for the case when the stress-energy tensor is that of a perfect fluid. Whenever interior solutions are considered one must decide on the reference frame of the observer. In NN-coordinates we shall use an observer who is "generalized comoving". This is a special choice of observer who shares some of the mathematical advantages of comoving observers in TS-coordinates. From a mathematical point of view, the choice of a comoving observer does not introduce new unknown functions into whatever problem

is being analyzed. This property is shared by the notion of a "generalized comoving" observer. A generalized comoving observer has a velocity vector orthogonal to the orbits of a symmetry group. For spherically symmetric metrics this means that the velocity is orthogonal to the two-dimensional orbits of $SO(3)$. In contrast, a comoving observer is defined to have velocity vector orthogonal to a hypersurface. In NN-coordinates for the metric (4.1), the velocity vector of a generalized comoving observer is given by

$$(4.15) \quad U_i = (f, f, 0, 0).$$

From now on all the problems to be analyzed will use this frame. In NN-coordinates an observer who is not comoving in the generalized sense, but still has velocity orthogonal to the orbits of $SO(3)$, has velocity given by $U_i = (a, b, 0, 0)$ where $ab = f^2$.

Associated with any velocity vector is a spatial projection tensor h_{ij} which completes the (3+1)-decomposition of spacetime. The components of h_{ij} for the generalized comoving velocity are

$$(4.16) \quad \begin{aligned} h_{00} &= f^2, & h_{01} &= -f^2, & h_{11} &= f^2, \\ h_{22} &= r^2, & h_{33} &= r^2 \sin^2 \theta. \end{aligned}$$

Kinematical quantities such as the acceleration vector, shear tensor, and expansion scalar are computed next. The nonzero components of the acceleration vector of the generalized comoving velocity are

$$(4.17) \quad \begin{aligned} A_0 &= (f_u - f_v)/(2f), \\ A_1 &= (-f_u + f_v)/(2f). \end{aligned}$$

The nonzero components of the shear tensor determined by U_i are

$$(4.18) \quad \begin{aligned} \sigma_{00} &= f(r_u + r_v)/(3r) - (f_u + f_v)/3, \\ \sigma_{01} &= -f(r_u + r_v)/(3r) + (f_u + f_v)/3, \\ \sigma_{11} &= f(r_u + r_v)/(3r) - (f_u + f_v)/3, \\ \sigma_{22} &= -r(r_u + r_v)/(6f) + r^2(f_u + f_v)/(6f^2), \\ \sigma_{33} &= \sigma_{22} \sin^2 \theta. \end{aligned}$$

The expansion scalar θ is found to be

$$(4.19) \quad \theta = -(r_u + r_v)/(rf) - (f_u + f_v)/(2f^2).$$

Using the stress-energy tensor for a perfect fluid with energy density μ and pressure p , $T_{ij} = (\mu+p)U_i U_j + pg_{ij}$, $\mu+p \neq 0$, we find the independent field equations to be

$$(4.20) \quad f^2(\mu + p) = 4r_u f_u/(rf) - 2r_{uu}/r,$$

$$(4.21) \quad f^2(\mu - p) = 2f^2/r^2 + 2r_u r_v/r^2 + 2r_{uv}/r,$$

$$(4.22) \quad f^2(\mu + p) = 4r_v f_v/(rf) - 2r_{vv}/r,$$

$$(4.23) \quad r^2 p = -rr_{uv}/f^2 + r^2 f_u f_v/f^4 - r^2 f_{uv}/f^3.$$

The conservation equations are written in the following form:

$$(4.24)$$

$$\begin{aligned} 2f^2(T^{0i}_{;i} + T^{1i}_{;i}) &= 2\mu(r_u + r_v)/r + 2p(r_u + r_v)/r \\ &\quad + \mu(f_u + f_v)/f + p(f_u + f_v)/f + \mu_u + \mu_v \\ &= 0, \end{aligned}$$

$$(4.25)$$

$$\begin{aligned} 2f^2(T^{0i}_{;i} - T^{1i}_{;i}) &= \mu(f_u - f_v)/f + p(f_u - f_v)/f + p_u - p_v \\ &= 0. \end{aligned}$$

Notice that the second conservation equation is identically satisfied if f and r (hence μ and p) are functions of $u+v$.

Solutions of the Field Equations in NN-Coordinates

In this section we seek solutions of the field equations (4.20) to (4.23) which exhibit shear. We rewrite the equations (4.20) to (4.23) as follows: to find μ we add (4.20),

(4.22), and twice (4.21) then divide by $4f^2$ to get

$$(4.26) \quad \mu = 1/r^2 + r_u r_v / (r^2 f^2) + r_{uv} / (r f^2) \\ + (r_u f_u + r_v f_v) / (r f^3) - (r_{uu} + r_{vv}) / (2r f^2).$$

To find p we add (4.20), (4.22), and subtract twice (4.21) then divide by $4f^2$ to get

$$(4.27) \quad p = -1/r^2 - r_u r_v / (r^2 f^2) - r_{uv} / (r f^2) \\ + (r_u f_u + r_v f_v) / (r f^3) - (r_{uu} + r_{vv}) / (2r f^2).$$

Taking the difference of (4.20) and (4.22) leads to

$$(4.28) \quad f(r_{uu} - r_{vv}) = 2r_u f_u - 2r_v f_v.$$

Equation (4.28) can also be written in the form

$$(4.29) \quad (r_u / f^2)_u = (r_v / f^2)_v.$$

Taking the difference of (4.23) and (4.27) leads to

$$(4.30) \quad (-f_u f_v + f f_{uv}) / f^4 - r_u r_v / (r^2 f^2) - (r_{uu} + r_{vv}) / (2r f^2) \\ + (r_u f_u + r_v f_v) / (r f^3) - 1/r^2 = 0.$$

Equation (4.30) is the pressure isotropy equation. The pressure isotropy equation may be written in the form

$$(4.31) \quad [\ln|f|]_{uv} / f^2 - [(r_u / f^2)_u + (r_v / f^2)_v] / (2r) \\ - r_u r_v / (r^2 f^2) - 1/r^2 = 0.$$

Using equation (4.29), equation (4.30) can be written as

$$(4.32) \quad (-f_u f_v + f f_{uv})/f^4 - (r_u/f^2)_u/r - r_u r_v/(r^2 f^2) - 1/r^2 = 0.$$

There are many ways which one may attempt to solve (4.28) and (4.30) for f and r . Equation (4.29) has a particularly interesting structure so we shall examine it first. The first integral of the equation (4.29) is $r_u/f^2 = \chi_v$, and $r_v/f^2 = \chi_u$, where χ is a \mathcal{E}^2 -function on a contractible domain. If r is a \mathcal{E}^2 -function then we can write (as a consequence of $r_{uv} = r_{vu}$)

$$(4.33) \quad \begin{aligned} (r_u + r_v)/f^2 &= \chi_u + \chi_v, \\ (r_u - r_v)/f^2 &= -(\chi_u - \chi_v). \end{aligned}$$

Defining new variables $\rho \equiv u-v$, and $t \equiv u+v$, we find (4.33)

becomes

$$(4.34) \quad \begin{aligned} r_t/f^2 &= \chi_t, \\ r_\rho/f^2 &= -\chi_\rho. \end{aligned}$$

We now make the *ad hoc* hypothesis that $f = F(t)$. The second of equations (4.34) can be integrated to find that

$$(4.35) \quad r(t, \rho) = g(t) - [F(t)]^2 \chi(t, \rho).$$

Using this in the first of (4.34) and integrating the resulting linear equation we find

$$(4.36) \quad r(t, \rho) = g(t) - F(t)[P(t) + h(\rho)]$$

where g and h are arbitrary smooth functions of one variable

and P is given by $P(t) \equiv (1/2) \int^t [g'(\xi)/F(\xi)] d\xi$. Using the

definitions of t and ρ we have

$$(4.37) \quad r(u, v) = g(u+v) - F(u+v)[P(u+v) + h(u-v)].$$

There is a dual case for the *ad hoc* hypothesis that $f = F(\rho)$ which leads to

$$(4.38) \quad r(u,v) = g(u-v) - F(u-v)[P(u-v) + h(u+v)].$$

In both of these cases, use of (4.37) or (4.38) in the pressure isotropy equation (4.30) leads to an *intractable equation* in general. Thus we have to look for another method of solving (4.28) and (4.30).

For a vacuum one can show that $2f^2 = (1 - 2m/r)U'(u)V'(v)$ which leads to Schwarzschild's solution. By a coordinate change the dependence on functions of u and v can be removed so that we have $f = F(r) = (1 - 2m/r)$. This motivates us to assume that for a class of solutions a *functional relationship exists between the metric coefficients f and r even in the case of a perfect fluid.*

The method that we shall use to study the field equations is to find a general solution of equation (4.28) under the hypothesis that a functional relationship $f = F(r)$ exists between the metric coefficients f and r . Once we have found a solution to equation (4.28) we shall use it in the pressure isotropy equation (4.30) to find a mathematical solution to the field equations. The mathematical solutions will then be tested to see that appropriate energy conditions are satisfied, that the shear tensor is nonzero, and that the resulting solution is nonstatic.

We will define several auxiliary functions which Figure 8

illustrates. For clarity in Figure 8 we have used distinct symbols for the value of a function and the function itself. It is a common abuse of notation in applied mathematics to use the same symbol for both. Thus $f = \mathcal{F}(u,v) = F \circ \mathcal{R}(u,v)$ and $r = \mathcal{R}(u,v)$ as a consequence of assuming that $f = F(r)$. When convenient we shall employ the usual abuse of notation.

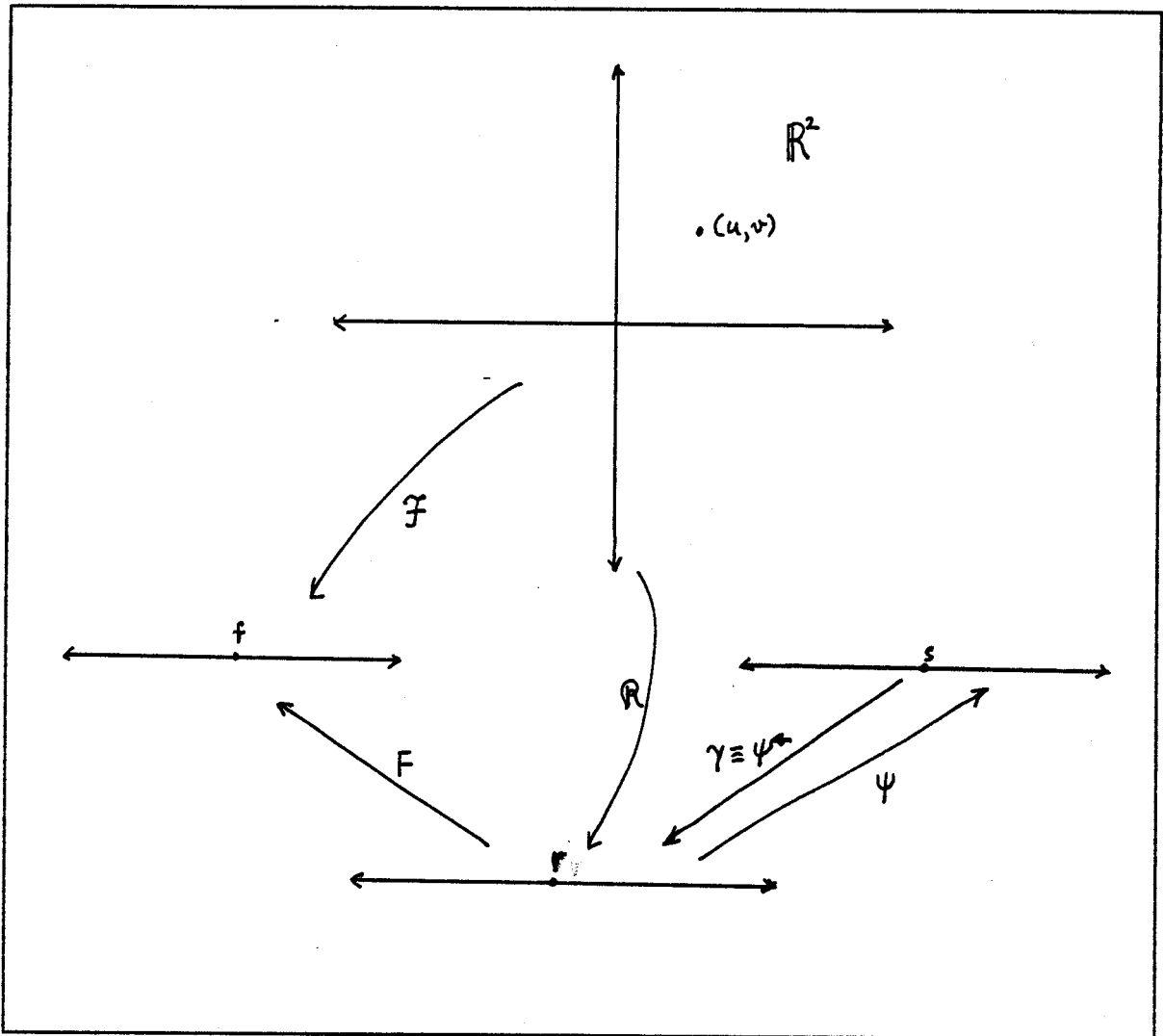


Figure 8 Functions related to $(r_u/f^2)_u = (r_v/f^2)_v$.

Let $\psi(r) \equiv \int^r [F(\xi)]^{-2} d\xi$. Then $\psi'(r) = [F(r)]^{-2} > 0$ hence ψ^{-1} exists and is smooth. The value of ψ is $s = \psi(r)$. We also define the function $\gamma(s) \equiv \psi^{-1}(s) = r$. By abuse of notation we have $s_u = r_u/f^2$ and $s_v = r_v/f^2$. Thus equation (4.29) leads to

$$(4.39) \quad s_{uu} - s_{vv} = 0.$$

Equation (4.39) has the general solution

$$(4.40) \quad s = g(u+v) + h(u-v)$$

where g and h are arbitrary C^2 -functions. We find r to be given by

$$(4.41) \quad \begin{aligned} r &= \gamma(s) \\ &= \gamma(g(u+v)+h(u-v)). \end{aligned}$$

Using (4.41) we find the following formulae for the partial

derivatives of r and f (we use $h' \equiv \frac{dh}{d(u-v)}$ and $g' \equiv \frac{dg}{d(u+v)}$)

$$(4.42) \quad \begin{aligned} r_u &= f^2(g' + h'), \\ r_v &= f^2(g' - h'), \\ r_{uu} &= f^2[(g'' + h'') + 2f\frac{df}{dr}(g' + h')^2], \\ r_{vv} &= f^2[(g'' + h'') + 2f\frac{df}{dr}(g' - h')^2], \\ f_u &= f^2\frac{df}{dr}(g' + h'), \\ f_v &= f^2\frac{df}{dr}(g' - h'), \\ f_{uv} &= f^2\left[\frac{df}{dr}(g'' - h'') + \frac{d}{dr}\left(f^2\frac{df}{dr}\right)(g'^2 - h'^2)\right]. \end{aligned}$$

Putting these into (4.30) we get

$$(4.43) \quad \begin{aligned} &\left[r^2\frac{d}{dr}\left(f^2\frac{df}{dr}\right) - r^2f\left(\frac{df}{dr}\right)^2 - f^3\right](g'^2 - h'^2) \\ &+ r\left(r\frac{df}{dr} - f\right)g'' - r\left(r\frac{df}{dr} + f\right)h'' - f = 0. \end{aligned}$$

Thus, if we find 3 functions f , g , and h so that (4.43) is satisfied, then we can use (4.41) to define r . This solution will be a mathematical solution of the field equations only-it still needs to have the nonstaticity, shear tensor, and energy conditions checked. Nonstaticity will follow automatically if $g' \neq 0$.

The first case that we shall study is the simple case when $f = 1$. By absorbing the constant of integration in (4.41), we can write

$$(4.44) \quad r = g(u+v) + h(u-v).$$

Equation (4.43) reduces to

$$(4.45) \quad g'^2 - h'^2 + (g + h)(g'' + h'') + 1 = 0.$$

Differentiating with respect to $t = u + v$, we find

$$(4.46) \quad g'(g'' + h'') + (g + h)g''' + 2g'g'' = 0.$$

Differentiating with respect to $\rho = u - v$, we find

$$(4.47) \quad h'(g'' + h'') + (g + h)h''' - 2h'h'' = 0.$$

Differentiating (4.47) with respect to t gives

$$(4.48) \quad g'h''' + g'''h' = 0.$$

If $g' \neq 0$ and $h' \neq 0$, then $h'''/h' = -g'''/g' = -\lambda$, where λ is a constant of separation. There are three subcases:

$$(i) \quad \lambda = 0,$$

$$(ii) \quad \lambda > 0,$$

$$(iii) \quad \lambda < 0.$$

For the case $\lambda = 0$, we have $g''' = 0$ and $h''' = 0$ so that we formally find

$$(4.49) \quad \begin{aligned} g(u+v) &= k_0 + k_1(u+v) + k_2(u+v)^2, \\ h(u-v) &= c_0 + c_1(u-v) + c_2(u-v)^2. \end{aligned}$$

Using these solutions in (4.46) and (4.47), we see that

(4.45) becomes

$$(4.50) \quad g'^2 - h'^2 + 1 = 0.$$

Thus the formal solutions (4.50) have the form

$$(4.51) \quad \begin{aligned} g(u+v) &= k_0 + (u+v)\sinh\alpha, \\ h(u-v) &= c_0 + (u-v)\cosh\alpha, \end{aligned}$$

where α is a real parameter. Using (4.51) we find that r is given by

$$(4.52) \quad r(u,v) = c_0 + k_0 + (u+v)\sinh\alpha + (u-v)\cosh\alpha.$$

Unfortunately, together with $f = 1$, (4.52) leads to the unreasonable result that $\mu = 0$ and $p = 0$. Thus case $\lambda = 0$ does not lead to a solution.

The case $\lambda > 0$ leads to the system of equations

$$(4.53) \quad \begin{aligned} g''' &= \alpha^2 g', \\ h''' &= -\alpha^2 h', \end{aligned}$$

where we have set $\lambda = \alpha^2$, $\alpha \neq 0$. This system of equations has the formal solution

$$(4.54) \quad \begin{aligned} g(u+v) &= k_0 + k_1 e^{\alpha(u+v)} - k_2 e^{-\alpha(u+v)}, \\ h(u-v) &= c_0 - c_1 \cos[\alpha(u-v)] + c_2 \sin[\alpha(u-v)]. \end{aligned}$$

Using (4.54) we can show $g'' + h'' = \alpha^2(g - h) + \alpha^2(c_0 - k_0)$.

Putting this into (4.46) and (4.47), noting that $g''' = \alpha^2 g'$

and $g' \neq 0$, we find that $g(u+v) = (3k_0 - c_0)/4$ for all values

of $u+v$. Since the set of functions $\{1, e^{\alpha(u+v)}, e^{-\alpha(u+v)}\}$ is

linearly independent when $\alpha \neq 0$ we see that $k_1 = k_2 = 0$ thus $g' = 0$ which contradicts the hypothesis $g' \neq 0$. Similarly we can show that $h' = 0$, contradicting the hypothesis that $h' \neq 0$. Thus there is no solution for the case $\lambda > 0$ when $g' \neq 0$ and $h' \neq 0$.

The case $\lambda < 0$ leads to the system of equations

$$(4.55) \quad \begin{aligned} g''' &= -\alpha^2 g', \\ h''' &= \alpha^2 h', \end{aligned}$$

where we have set $\lambda = -\alpha^2$, $\alpha \neq 0$. This system of equations has a formal solution similar to (4.54) and the same methods may be used to show that this case has no solution.

If we assume that $g' \neq 0$, $h' = 0$, then we find that

(4.45) reduces to

$$(4.56) \quad gg'' + g'^2 + 1 = 0$$

where we have absorbed the constant value of h into g .

We set $\beta = g'$ so that $g'' = \beta \frac{d\beta}{dg}$ and (4.56) becomes

$$(4.57) \quad g\beta\beta' + \beta^2 + 1 = 0,$$

where the prime means differentiation with respect to g . This equation reduces to the Bernoulli equation

$$(4.58) \quad \beta' = -\beta/g - \beta^{-1}/g.$$

Applying the method in [80] we arrive at the solution

$$(4.59) \quad \beta^2 = Cg^{-2} - 1,$$

where $C > 0$ is an integration constant. This equation leads to

$$(4.60) \quad g'^2 = Cg^{-2} - 1.$$

This equation can be completely integrated to give the

solution $g^2 = C - K^2 \mp 2K(u+v) - (u+v)^2$ where K is a constant of integration. The metric has the form

$$(4.61) \quad \bar{g} = -4du \otimes dv + [C - K^2 \mp 2K(u+v) - (u+v)^2] d\theta \otimes d\theta \\ + [C - K^2 \mp 2K(u+v) - (u+v)^2] \sin^2 \theta d\phi \otimes d\phi.$$

Since $C > 0$, there are two values of $u+v$ for which g would be zero. Constraining $u+v$ to lie strictly between these values guarantees that $g^2 > 0$. The region $g^2 > 0$ is an open strip in the (u,v) coordinate plane hence is homeomorphic to \mathbb{R}^2 and thus this solution is stably causal, hence strongly causal, causal, and chronological.

We now want to check the timelike weak energy conditions. For $f \equiv 1$, and $r = g(u+v)$ we find the energy conditions to be

$$(4.62) \quad \mu = (1 + g'^2)g^{-2} \geq 0,$$

which will hold for any $g \neq 0$, and

$$(4.63) \quad \mu + p = -2g''/g \geq 0.$$

Taking $g^2 = C - K^2 \mp 2K(u+v) - (u+v)^2$ and differentiating twice with respect to $u+v$ we find

$$(4.64) \quad 2g'^2 + 2gg'' = -2.$$

Thus

$$(4.65) \quad gg'' = -(1 + g'^2) < 0,$$

and the timelike weak energy conditions hold for the solution $g^2 = C - K^2 \mp 2K(u+v) - (u+v)^2$.

If we apply the dominant energy condition we just have to check the inequalities

$$(4.66) \quad \mu \geq |p| \geq 0,$$

which is equivalent to the three inequalities

$$(4.67) \quad \begin{aligned} \mu &= 1/r^2 + r_u r_v / (r^2 f^2) + r_{uv} / (r f^2) \\ &\quad + (r_u f_u + r_v f_v) / (r f^3) - (r_{uu} + r_{vv}) / (2 r f^2) \\ &\geq 0, \end{aligned}$$

$$(4.68) \quad \begin{aligned} \mu + p &= 2(r_u f_u + r_v f_v) / (r f^3) - (r_{uu} + r_{vv}) / (r f^2) \\ &\geq 0, \end{aligned}$$

$$(4.69) \quad \begin{aligned} \mu - p &= 2/r^2 + 2r_u r_v / (r^2 f^2) + 2r_{uv} / (r f^2) \\ &\geq 0. \end{aligned}$$

The first two are satisfied for the timelike weak energy conditions. All that remains is to check (4.69). We find that

$$(4.70) \quad \begin{aligned} \mu - p &= 2/g^2 + 2g \cdot^2 / g^2 + 2g'' / g^2 \\ &= 2(1 + g \cdot^2 + g'') / g^2 \\ &= 0 \end{aligned}$$

from (4.65). Thus the dominant energy conditions hold on the same region as the timelike weak energy conditions.

If we apply the strong energy condition we must check the inequalities

$$(4.71) \quad \begin{aligned} \mu + p &= 2(r_u f_u + r_v f_v) / (r f^3) - (r_{uu} + r_{vv}) / (r f^2) \\ &\geq 0, \end{aligned}$$

$$(4.72) \quad \begin{aligned} \mu + 3p &= -2/r^2 - 2r_u r_v / (r^2 f^2) - 2r_{uv} / (r f^2) \\ &\quad + 4(r_u f_u + r_v f_v) / (r f^3) - 2(r_{uu} + r_{vv}) / (r f^2) \\ &\geq 0. \end{aligned}$$

Using (4.65) we can show that $\mu + 3p \geq 0$ on the region where the solution is defined, thus the strong energy conditions hold

on this region as well.

The energy conditions, (4.67), and (4.70) show that the stiff equation of state $p = \mu$ holds. The expansion scalar is $\theta = -2g'/g$. The acceleration vector A is identically zero. The shear tensor σ_{ij} is nonzero since $\sigma_{00} = 2g'/(3g) \neq 0$. By a transformation to the coordinates t and ρ it is clear that this solution is nonstatic since $g' \neq 0$.

Since $f = 1$ and $r = g(u+v)$ we see that $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ is also a Killing vector. $\xi_{(4)}$ is orthogonal to the surfaces $u + v$ constant hence this solution, which is not a dust solution, has the same symmetry as the Kantowski-Sachs [27] dust metrics.

The radial ($\theta = \text{constant}$, $\phi = \text{constant}$) geodesic equations can be easily integrated for both the null and timelike cases. The radial null geodesics are

$$(4.73) \quad \begin{aligned} u &= b, \\ v &= d, \end{aligned}$$

where b and d are arbitrary constants. The radial timelike geodesics are given by

$$(4.74) \quad \begin{aligned} u(\lambda) &= a\lambda + b, \\ v(\lambda) &= c\lambda + d, \end{aligned}$$

where $4ac = 1$ and b and d are arbitrary constants.

The scalar invariants are found next. The curvature scalar is

$$(4.75) \quad R = -2(1 + g'^2)/g^2$$

where $g'^2 = C - K^2 \mp 2K(u+v) - (u+v)^2$. Other invariants constructed from the Riemann tensor are

$$\begin{aligned}
(4.76) \quad R_{ij}R^{ij} &= 4g''^2/g^2 \\
&= 4(1 + g'^2)^2/g^4 \\
&= R^2
\end{aligned}$$

and

$$\begin{aligned}
(4.77) \quad R_{ijkl}R^{ijkl} &= 12(1 + g'^2)^2/g^4 \\
&= 3R^2,
\end{aligned}$$

where $g^2 = C - K^2 \mp 2K(u+v) - (u+v)^2$, and we have used (4.65) repeatedly.

For the next example we take $f = r^{1/2}$. With this assumption we find that $s = \ln|r| + c$, where c is a constant of integration. Absorbing c into g where $s = g(u+v) + h(u-v)$ we find from (4.40) that r is given by

$$(4.78) \quad r(u,v) = Ce^{g(u+v) + h(u-v)},$$

where C is a nonzero constant. Using (4.78) in (4.43) we find after some simplification

$$(4.79) \quad g'^2 - h'^2 + (g'' + h'')/2 + e^{-(g+h)}/C = 0.$$

Consecutively differentiating with respect to $u+v$ and $u-v$ we find that either $g' = 0$ or $h' = 0$ for all values of $u+v$ and $u-v$. This shows that we cannot have both functions g and h to be nonconstant. If $g' = 0$ it is easily seen that any solution which might be derived from (4.79) will be static hence not of interest in our study. Taking $h' = 0$, we find the equation

$$(4.80) \quad g'' + 2g'^2 + 2e^{-g}/C = 0.$$

Setting $p(g) = g'$ and setting $U(p) = p^2$ we find the differential equation

$$(4.81) \quad \frac{dU}{dg} + 4U + 4e^{-g}/C = 0.$$

Thus $p^2 = -4e^{-g}/(3C) - 4Ke^{-4g}/C$ hence $C < 0$. There are three cases: $K < 0$, $K = 0$, $K > 0$.

If $K < 0$ we can perform a quadrature to implicitly determine g and hence r . Thus

$$(4.82) \quad D \pm (u+v) = (-3C/4)^{1/2} \int e^{g/2} (1+3Ke^{-3g})^{-1/2} dg,$$

where D is an integration constant. Since $p^2 = g'^2 \geq 0$, we see that $K < 0$ imposes a restriction on the domain of the solution.

Using $f = r^{1/2}$ and (4.80) in (4.67), we find that

$\mu \geq 0$ if $1 + Cg'^2 e^g \geq 0$. However the condition $\mu + p \geq 0$

cannot be satisfied when $1 + Cg'^2 e^g \geq 0$, hence the timelike weak energy conditions are not satisfied for the case for any

constant $C < 0$. Since this argument may be applied for the cases $K = 0$, and $K < 0$, we see that no reasonable solutions

arise from $f = r^{1/2}$.

Another case is given by $f = r^{1/3}$. We find that $s = 3r^{1/3} + c$, where c is a constant of integration. Absorbing c into $g(u+v)$ we find

$$(4.83) \quad r(u,v) = [g(u+v) + h(u-v)]^3/27.$$

Using this in (4.43) and simplifying gives

$$(4.84) \quad 10(g'^2 - h'^2)/9 + 2r^{4/3}g''/3 + 4r^{4/3}h'' + r^{1/3} = 0.$$

Let $\eta = h/3$, $\Gamma = g/3$, so that $f = \Gamma + \eta$. Then (4.84) will reduce to

$$(4.85) \quad 2(\Gamma + \eta)^3(\Gamma'' + 2\eta'') + 10(\Gamma + \eta)^2(\Gamma'^2 - \eta'^2) + 1 = 0.$$

This equation is difficult to solve, so it will be simplified

by putting $\eta = 0$ to get

$$(4.86) \quad 2\Gamma^3\Gamma'' + 10\Gamma^2\Gamma'^2 + 1 = 0,$$

where we use the dot to denote differentiation with respect

to $u+v$. Let $\alpha = \Gamma'$ so that $\Gamma'' = \alpha \frac{d\alpha}{d\Gamma}$, where $\alpha = \alpha(\Gamma)$. Thus

(4.86) is reduced to

$$(4.87) \quad 2\Gamma^3\alpha\alpha' + 10\Gamma^2\alpha^2 + 1 = 0,$$

where we use the prime to denote differentiation with

respect to Γ . Dividing by α we find

$$(4.88) \quad 2\Gamma^3\alpha' + 10\Gamma^2\alpha + \alpha^{-1} = 0,$$

which we recognize as a Bernoulli equation if we write it as

$$(4.89) \quad \alpha' = -5\alpha/\Gamma - \alpha^{-1}/(2\Gamma^3).$$

From [80] we find the solution

$$(4.90) \quad \alpha^2(\Gamma) = Y_1(\Gamma) + Y_2(\Gamma)$$

where

$$(4.91) \quad Y_1(\Gamma) = Ce^{\phi(\Gamma)},$$

$$(4.92) \quad Y_2(\Gamma) = -2e^{\phi(\Gamma)} \int [e^{-\phi(\Gamma)}/2\Gamma^3] d\Gamma,$$

$$(4.93) \quad \begin{aligned} \phi(\Gamma) &= 2 \int (-5/\Gamma) d\Gamma \\ &= -10 \ln |\Gamma| + K \end{aligned}$$

where K is an integration constant.

Thus we have

$$(4.94) \quad Y_1(\Gamma) = C\Gamma^{-10},$$

$$(4.95) \quad Y_2(\Gamma) = -1/(8\Gamma^2),$$

so that

$$(4.96) \quad \alpha^2(\Gamma) = C\Gamma^{-10} - 1/(8\Gamma^2),$$

where C is a constant of integration. In fact C must be

greater than zero so that $\alpha^2(\Gamma) \geq 0$. The definition of α shows that

$$(4.97) \quad \Gamma^2 = C\Gamma^{-10} - 1/(8\Gamma^2).$$

Let $H(\Gamma;C) \equiv C\Gamma^{-10} - 1/(8\Gamma^2)$ so that $H(\Gamma;C) \geq 0$. We can write a solution to (4.86) as

$$(4.98) \quad K \pm (u+v) = \int [H(\Gamma;C)]^{-1/2} d\Gamma \\ = \int [C\Gamma^{-10} - 1/(8\Gamma^2)]^{-1/2} d\Gamma.$$

Thus $f = \Gamma$ and $r = \Gamma^3$ give the solution of the field equations where Γ is determined by the quadrature (4.98). The integral in (4.98) is non-elementary when $C \neq 0$.

In terms of Γ the mass-energy density μ and the pressure p are given by

$$(4.99) \quad \mu = 1/\Gamma^6 + 15\Gamma^2/\Gamma^4,$$

and

$$(4.100) \quad p = -1/\Gamma^6 - 15\Gamma^2/\Gamma^4 - 6\Gamma''/\Gamma^3.$$

From (4.99) and (4.100) we see there is no simple equation of state for this solution.

The timelike weak energy condition $\mu \geq 0$ is satisfied by any solution $\Gamma \neq 0$ of (4.86). The energy condition $\mu + p \geq 0$ holds if $\Gamma\Gamma'' \leq 0$. Thus the timelike weak energy conditions are satisfied for a solution of (4.86) if $C > 0$ and $\Gamma\Gamma'' \leq 0$. From (4.86) we see that Γ satisfies $\Gamma\Gamma'' = -5\Gamma^2 - 1/(2\Gamma^2) < 0$ thus the timelike weak energy condition $\mu + p \geq 0$ holds.

For the dominant energy condition we have just to check the inequality (since the timelike weak energy conditions hold)

$$(4.101) \quad \mu - p = 2/r^2 + 2r_u r_v / (r^2 f^2) + 2r_{uv} / (rf^2) \\ \geq 0.$$

Using $f = \Gamma$, $r = \Gamma^3$, and (4.86) in (4.101) we find that

$$(4.101) \text{ cannot be satisfied since } \mu - p = -\Gamma^{-6} < 0.$$

For the strong energy condition we just have to check

$$(4.102) \quad \mu + 3p = -2/r^2 - 2r_u r_v / (r^2 f^2) - 2r_{uv} / (rf^2) \\ + 4(r_u f_u + r_v f_v) / (rf^3) - 2(r_{uu} + r_{vv}) / (rf^2) \\ \geq 0,$$

since $\mu + p \geq 0$ from the timelike weak energy conditions.

Using $f = \Gamma$, $r = \Gamma^3$, and (4.86) in (4.102), we find that

$$(4.103) \quad \mu + 3p = 7\Gamma^{-6} + 60\Gamma^{-2}\Gamma^{-4} \\ \geq 0,$$

thus the strong energy condition holds.

Any solution of (4.86) with $C > 0$ will satisfy the timelike weak energy conditions and the strong energy conditions. This solution has shear since $\sigma_{00} = (4/3)\Gamma$, nonzero expansion since the expansion scalar is $\theta = -7\Gamma/\Gamma^2$, and zero acceleration since Γ is a function of $u+v$. Writing the metric in terms of Γ we have

$$(4.104) \quad \bar{g} = -4\Gamma^2(u+v)du\odot dv + \Gamma^6(u+v)[d\theta\odot d\theta + \sin^2\theta d\phi\odot d\phi].$$

This metric is clearly nonstatic since $\Gamma \neq 0$. As the metric coefficients are functions of $u+v$ we see that $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ is an additional Killing vector. On each subregion of the (u,v) -plane where $\Gamma \neq 0$ the metric (4.104) is stably causal, hence strongly causal, causal, and chronological.

The Ricci scalar is found, using (4.11) and (4.86), to be

$$(4.105) \quad R = -30\Gamma^2\Gamma^{-4} - 5\Gamma^{-6}.$$

Other invariants constructed from the Riemann tensor are complicated nonzero expressions of Γ , Γ' , and Γ'' which we shall omit.

We next study the case when $f = F(r) = e^{ar}$ with $a \neq 0$.

Equation (4.30) becomes

$$(4.106) \quad -e^{2ar}/r^2 - r_u r_v / r^2 + ar_{uv} + a(r_u^2 + r_v^2)/r - (r_{uu} + r_{vv})/(2r) = 0.$$

Simplifying (4.106) we find (assuming that $r = R(u+v)$)

$$(4.107) \quad -e^{2ar} - r'^2 + ar^2 r'' + 2arr'^2 - rr'' = 0,$$

which we rewrite as

$$(4.108) \quad r(ar-1)r'' + (2ar-1)r'^2 - e^{2ar} = 0.$$

We set $\beta = r'$ so that $r'' = \beta \frac{d\beta}{dr}$ and (4.108) becomes

$$(4.109) \quad r(ar-1)\beta\beta' + (2ar-1)\beta^2 - e^{2ar} = 0,$$

where the prime means differentiation with respect to r .

Dividing by β and rewriting we get

$$(4.110) \quad \beta' = (1-2ar)\beta/r(ar-1) + e^{2ar}\beta^{-1}/[r(ar-1)]$$

which is a Bernoulli equation. Equation (4.110) can be written as

$$(4.111) \quad \beta' = g(r)\beta + h(r)\beta^k$$

where $g(r) = (1-2ar)/r(ar-1)$, $h(r) = e^{2ar}\beta^{-1}/[r(ar-1)]$, and

$k = -1$. If we set

$$(4.112) \quad \lambda(r) = (1 - k) \int g(r) dr,$$

then the solution of (4.108) is

$$(4.113) \quad \beta^{1-k} = De^{\lambda(r)} + (1-k)e^{\lambda(r)} \int e^{-\lambda(r)} h(r) dr.$$

Thus we are left with the first integral

$$(4.114) \quad r'^2 = e^{\lambda(r)} [D + 2 \int e^{-\lambda(r)} h(r) dr].$$

Evaluating the integral for $\lambda(R)$ we find the following first integral

$$(4.115) \quad r'^2 = Dr^{-2}(ar-1)^{-2} + e^{2ar}/(a^2 r^2).$$

Let $K(r;a,D) = Dr^{-2}(ar-1)^{-2} + e^{2ar}/(a^2 r^2)$. We can write implicitly a solution r to (4.115) as

$$(4.116) \quad E \pm (u+v) = \int [K(r;a,D)]^{-1/2} dr,$$

where E is a constant of integration. The timelike weak energy conditions become

$$(4.117) \quad \mu = Dr^{-4}(1+2ar)(ar-1)^2 e^{-2ar} + a^2 r^{-4}(1+ar)^2 \geq 0,$$

and, using (4.108) and (4.115)

$$(4.118) \quad \begin{aligned} \mu + p &= (4arr'^2 - 2rr'')/(r^2 e^{2ar}) \\ &= 2(ar+1)/(a^2 r^4) + 2D(2a^2 r^2 - 1)/[r^4 (ar-1)^3] \\ &\geq 0. \end{aligned}$$

These inequalities are very complicated depending on the signs of a , D , r . If we choose $D \geq 0$, then $0 < ar < 2^{-1/2}$ will satisfy both (4.117) and (4.118). Thus there is an open strip in the (u,v) -coordinate plane on which the timelike weak energy conditions are satisfied. The metric form of this solution is

$$(4.119) \quad \bar{g} = -4e^{2ar(u,v)} du dv + r^2(u,v) [d\theta^2 + \sin^2 \theta d\phi^2],$$

where $r(u,v)$ is the function implicitly defined by (4.116).

The mass-energy density is given by (4.117). The pressure is

given by

$$(4.120) \quad p = -1/r^2 - e^{-2ar}[r'^2 - 2rr'' - 2arr'^2]/r^2.$$

Since the open strip is homeomorphic to R^2 we see that the solution is stably causal hence strongly causal, causal, and chronological. The shear tensor σ_{ij} is nonzero since $\sigma_{00} = 2e^{ar}r'/(3r) - 2ae^{ar}r'/3$. The expansion scalar is found to be $\theta = -r'e^{-ar}/r - ae^{-ar}$. The metric (4.119) is nonstatic since both f and r depend on $u+v$. As the metric coefficients are functions of $u+v$ we see that $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ is an additional Killing vector.

The Ricci scalar is given by

$$(4.121) \quad R = 2e^{-2ar}[r(2+ar)r'' + r'^2 + r^2]/r^2.$$

Another scalar constructed from the Riemann tensor is

$$(4.122) \quad R_{ij}R^{ij} = 2e^{-4ar}[2ar'^2 - r'']^2/r^2 + 2e^{-4ar}r''^2[a-1/r]^2 + 2[1 + r'^2 + rr''e^{-2ar}]^2/r^4.$$

The scalar $R_{ijkl}R^{ijkl}$ is

$$(4.123) \quad R_{ijkl}R^{ijkl} = 4e^{-4ar}[(2arr'^2 - rr''^2)^2 + r^2r''^2 + 4a^2r^4r''^2 + (r'^2 + e^{2ar})^2]/r^4.$$

From the previous examples we have seen that setting $h = 0$ (or $g = 0$) will allow integration of the field equations for some specific cases. Now we investigate (4.30) more generally under this hypothesis. Using $h = 0$ in (4.43), as this will generate nonstatic solutions, we find the equation

$$(4.124) \quad \left[r^2 \frac{d}{dr} \left(f^2 \frac{df}{dr} \right) - r^2 f \left(\frac{df}{dr} \right)^2 - f^3 \right] g'^2 + r \left(r \frac{df}{dr} - f \right) g'' - f = 0.$$

Equation (4.41) yields $s = g(u+v)$ and $r = \gamma(s) = \gamma(g)$. From (4.41) and the Implicit Function Theorem we find $\frac{dy}{ds} = [F(r)]^2$. Putting this into (4.124), we get

(4.125)

$$\begin{aligned} & \left[[F(\gamma(g))]^3 + [\gamma(g)]^2 F(\gamma(g)) \left(\frac{dF}{dr} \right)^2 - [\gamma(g)] \frac{2d}{dr} \left(F \frac{2dF}{dr} \right) \right] g'^2 \\ & + \gamma(g) \left[F(\gamma(g)) - \gamma(g) \frac{dF}{dr} \right] g'' + F(\gamma(g)) = 0, \end{aligned}$$

where we evaluate $\frac{d}{dr}$ at $\gamma(g)$. If we prescribe F arbitrarily and use (4.41) to find r we see that solving the field equations reduces to solving the second order autonomous differential equation (4.125) for g . If we define

$$(4.126) \quad p \equiv g' = P(g),$$

$$(4.127) \quad A(g) \equiv [\gamma(g)/2] \left[F(\gamma(g)) - \gamma(g) \frac{dF}{dr} \right],$$

$$(4.128) \quad B(g) \equiv [F(\gamma(g))]^3 + [\gamma(g)]^2 F(\gamma(g)) \left(\frac{dF}{dr} \right)^2 - [\gamma(g)] \frac{2d}{dr} \left(F \frac{2dF}{dr} \right),$$

$$(4.129) \quad C(g) \equiv F(\gamma(g)),$$

(4.125) becomes the Bernoulli equation (for P):

$$(4.129) \quad A(g) \frac{d}{dg} [P(g)]^2 + B(g) [P(g)]^2 + C(g) = 0.$$

We shall analyze (4.129) in two cases: $A(g) \equiv 0$, and $A(g) \neq 0$.

If we assume that $A(g) \equiv 0$, then $F(r) - r \frac{dF}{dr} = 0$, thus we have $F(r) = cr$, where c is a constant of integration. Using this in (4.128) we find $B(g) \equiv 0$, thus (4.129) becomes

$$(4.130) \quad C(g) = 0.$$

From (4.130) we must have $c = 0$ which produces a singular metric. Thus $A(g) \equiv 0$ leads to no solutions.

If we assume that $A(g) \neq 0$, and let $U(p) = [P(g)]^2$, we have

$$(4.131) \quad \frac{dU}{dg} + B(g)U/A(g) + C(g)/A(g) = 0.$$

This equation is linear and can be integrated to find

$$(4.132) \quad U(g) = [K - \int^g C(\xi)\mu(\xi)/A(\xi)d\xi]/\mu(g),$$

where $\mu(g) = e^{\int^g B(\xi)/A(\xi)d\xi}$. Since $U(g) = [P(g)]^2 \geq 0$ we see that (4.131) imposes some conditions determined by $A(g)$ and $C(g)$ on the domain of g . Supposing that the domain of g is nonempty we can find g by a further quadrature. Thus

$$(4.133) \quad D \pm (u+v) = \int^g [U(\xi)]^{-1/2} d\xi,$$

where D is a constant of integration. We conclude that assuming that either g or h is identically zero will always lead to a differential equation of the Bernoulli type (neglecting the degenerate case when $A(g) \equiv 0$).

We will apply the above discussion to the study of the case $f = r^\alpha$, where α is a real number. Equation (4.125) becomes

$$(4.134) \quad (1 - \alpha)rg'' + (1 + \alpha - 2\alpha^2)r^{2\alpha}g'^2 + 1 = 0.$$

We find $s = r^{1-2\alpha}/(1-2\alpha)$ and thus

$$(4.135) \quad r(u, v) = [(1-2\alpha)g(u+v)]^{1/(1-2\alpha)}.$$

Putting r into (4.134), yields

$$(4.136) \quad (1-\alpha)[(1-2\alpha)g]^{1/(1-2\alpha)}g'' + \alpha^2[(1-2\alpha)g(u+v)]^{2\alpha/(1-2\alpha)}g'^2 + 1 = 0,$$

where $\alpha \neq 1, 1/2$. The exceptional case $\alpha = 1/2$ has already

been considered and has been shown to lead to no reasonable solutions. The exceptional case $\alpha = 1$ has no solutions since putting $f = r$ in (4.30) directly leads to $r^{-2} = 0$.

Now we consider the case when $\alpha = -1$. Using $f = r^{-1}$ leads to $s = r^3/3 + c$, where c is a constant of integration. Thus, absorbing c into the definition of g and letting $h = 0$, we find from (4.41) that

$$(4.137) \quad r(u,v) = [3g(u+v)]^{1/3}.$$

Putting this into (4.125) leads to

$$(4.138) \quad 2r^3 r'' + 5r^2 r'^2 + 1 = 0.$$

where we use the dot to denote differentiation with respect to $u+v$. Setting $\beta = r'$ so that $r'' = \beta \frac{d\beta}{dr}$, we get

$$(4.139) \quad 2r^3 \beta \beta' + 5r^2 \beta^2 + 1 = 0,$$

where now the prime represents differentiation with respect to r . Dividing by β and rearranging terms we find the Bernoulli equation

$$(4.140) \quad \beta' = -5\beta/2 - \beta^{-1}/(2r^2).$$

From [80] we find the solution of (4.140) to be

$$(4.141) \quad \beta^2(r) = Y_1(r) + Y_2(r)$$

where

$$(4.142) \quad Y_1(r) = De^{\lambda(r)},$$

$$(4.143) \quad Y_2(r) = -e^{\lambda(r)} \int [e^{-\lambda(r)}/r^2] dr,$$

and $\lambda(r)$ is given by

$$(4.144) \quad \begin{aligned} \lambda(r) &= 2 \int (-5/2) dr \\ &= -5r + C, \end{aligned}$$

with C a constant of integration. Thus we find

$$(4.145) \quad Y_1(r) = De^{-5r},$$

$$(4.146) \quad Y_2(r) = -e^{-5r} \int [e^{5r}/r^2] dr$$

Thus the solution of (4.140) is

$$(4.147) \quad \beta^2(r) = Y_1(r) + Y_2(r) \\ = De^{-5r} - e^{-5r} \int [e^{5r}/r^2] dr.$$

A first integral of (4.38) is

$$(4.148) \quad r'^2 = De^{-5r} - e^{-5r} \int [e^{5r}/r^2] dr,$$

where $D > 0$ so that (4.148) makes sense on some subregion of the (u,v) -plane. Let $K(r;D) \equiv De^{-5r} - e^{-5r} \int [e^{5r}/r^2] dr$ so that $K(r;D) \geq 0$. We can write a solution to (4.138) as

$$(4.149) \quad E \pm (u+v) = \int [K(r;D)]^{-1/2} dr.$$

This integral cannot be evaluated to find r explicitly in terms of $u+v$, E , and D . The metric form of this solution in terms of the implicit function r is

$$(4.150) \quad \bar{g} = -4r^{-2}(u,v) du \otimes dv + r^2(u,v) [d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi].$$

The mass-energy density μ is found to be

$$(4.151) \quad \mu = 3(2r'^2 - rr''),$$

and the pressure is

$$(4.152) \quad p = (2r'^2 - rr'')$$

so the equation of state $p = \mu/3$ holds.

Using (4.138) in (4.67) and (4.68) shows that the timelike weak energy conditions hold. The dominant energy condition holds since

$$(4.153) \quad \mu - p = 2/r^2 + 2r_{uv} r_v / (r^2 f^2) + 2r_{uv} / (rf^2)$$

$$= 2(2r'^2 - rr'')$$

$$\geq 0.$$

For the strong energy condition we just show that

$$(4.154) \quad \mu + 3p = -2/r^2 - 2r_u r_v / (r^2 f^2) - 2r_{uv} / (rf^2)$$

$$+ 4(r_u f_u + r_v f_v) / (rf^3) - 2(r_{uu} + r_{vv}) / (rf^2)$$

$$= 6(2r'^2 - rr'')$$

$$\geq 0$$

holds since we have shown that $\mu + p \geq 0$ for the timelike weak energy conditions. Thus the strong energy conditions hold on the region where the solution is defined.

The acceleration vector is zero since $r = R(u+v)$ and hence $f = F(u+v)$. Since $\sigma_{00} = 4r' / (3r^2)$ this solution has a nonzero shear tensor σ_{ij} . The expansion scalar is $\theta = -r'$. The solution is nonstatic since both f and r depend on $u+v$. As the metric coefficients are functions of $u+v$ we see $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ is an additional Killing vector.

The Ricci scalar is

$$(4.155) \quad R = 4r'^2 + 2rr'' + 2/r^2$$

$$= 1/r^2 - 5r'^2$$

where we have used (4.138).

Another invariant constructed from the Riemann tensor is

$$(4.156) \quad R_{ij} R^{ij} = 2[2r'^2 + rr'']^2 + 2r'^4 + 2[1/r^2 + rr'' + r'^2]^2$$

$$\geq 0.$$

The invariant $R_{ijkl} R^{ijkl}$ is found to be

$$(4.157) \quad R_{ijkl} R^{ijkl} = 20[2r'^2 + rr'']^2 + 4r^2 r''^2 + 4[rr'' - r'^2]^2$$

≥ 0 .

Since the solution is defined only where $r^{\cdot 2} > 0$ we see that the solution is stably causal. From the discussion of elementary causality in Chapter 2 we see that the solution is strongly causal, causal, and chronological.

Now we consider the case when $\alpha = -2$. Using $f = r^{-2}$ we find that

$$(4.158) \quad s = r^5/5 + c,$$

where c is a constant of integration. Solving (4.158) for r , setting $h \equiv 0$, and absorbing c into the definition of g we find

$$(4.159) \quad r(u,v) = [5g(u+v)]^{1/5}.$$

Instead of using (4.159) in (4.125), we shall put $f = r^{-2}$ directly into (4.30) to get

$$(4.160) \quad 3r^5 r'' + 3r^4 r'^2 + 1 = 0,$$

where we use the dot to denote differentiation with respect to $u+v$. Setting $\beta = r'$ so that $r'' = \beta \frac{d\beta}{dr}$, we get

$$(4.161) \quad 3r^5 \beta \beta' + 3r^4 \beta^2 + 1 = 0,$$

where now the prime represents differentiation with respect to r . Dividing by β and rearranging terms we find the Bernoulli equation

$$(4.162) \quad \beta' = -\beta/r - \beta^{-1}/(3r^5).$$

From [80] we find the solution of (4.162) to be

$$(4.163) \quad \beta^2(r) = Y_1(r) + Y_2(r)$$

where

$$(4.164) \quad Y_1(r) = De^{\lambda(r)},$$

$$(4.165) \quad Y_2(r) = -(2/3)e^{\lambda(r)} \int [e^{-\lambda(r)}/r^5] dr,$$

and $\lambda(r)$ is given by

$$(4.166) \quad \begin{aligned} \lambda(r) &= -2 \int r^{-1} dr \\ &= -2 \ln|r| + C, \end{aligned}$$

where C is a constant of integration. Thus we find

$$(4.167) \quad Y_1(r) = Dr^{-2},$$

$$(4.168) \quad Y_2(r) = (1/3)r^{-4}.$$

Thus the solution of (4.162) is

$$(4.169) \quad \begin{aligned} \beta^2(r) &= Y_1(r) + Y_2(r) \\ &= Dr^{-2} + (1/3)r^{-4}. \end{aligned}$$

A first integral of (4.160) is

$$(4.170) \quad r \cdot^2 = Dr^{-2} + (1/3)r^{-4}.$$

Let $K(r;D) \equiv Dr^{-2} + (1/3)r^{-4}$ so that $K(r;D) \geq 0$. We can write a solution to (4.160) as

$$(4.170) \quad C \pm (u+v) = \int [K(r;D)]^{-1/2} dr,$$

where C is an integration constant. This integral can be evaluated to find r implicitly in terms of $u+v$, C , and D . The metric form of this solution in terms of the implicit function r is

$$(4.172) \quad \bar{g} = -4r^{-4}(u,v) du \otimes dv + r^2(u,v) [d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi].$$

The mass-energy density μ is found to be

$$(4.173) \quad \mu = -3r^2(rr'' + 2r \cdot^2),$$

and the pressure is

$$(4.174) \quad p = r^2(rr'' - 2r \cdot^2).$$

The timelike weak energy inequalities (4.67) and (4.68) become

(using (4.160))

$$(4.175) \quad 1/r^2 \geq 3r^2 r'^2 \geq 0,$$

$$(4.176) \quad -8r^2 r'^2 - 2r^3 r'' \geq 0,$$

These two inequalities imply that

$$(4.177) \quad 1 \geq 3r^4 r'^2 \geq 0,$$

and

$$(4.178) \quad rr'' \leq -4r'^2 \leq 0.$$

The inequality (4.177) shows that as r grows in magnitude that r' tends to zero. The analysis of the timelike weak energy conditions may be broken into three cases depending on the sign of the integration constant D . For the case when the integration constant $D = 0$, the mathematical solution does not satisfy the weak energy conditions even for the special subcase when $C = 0$. Using $D > 0$ in (4.170) shows that (4.177) cannot be satisfied thus $D < 0$ is the only case left. Using $D < 0$ in (4.170) and (4.178) we find that the weak energy conditions will hold on the region defined by $0 < |r| \leq (-3D)^{-1/2}$. To check the dominant energy condition we have to check the inequality

$$(4.179) \quad \begin{aligned} \mu - p &= 2/r^2 + 2r_{uv} r' / (r^2 f^2) + 2r_{uv} / (rf^2) \\ &= 4/(3r^2) \\ &\geq 0, \end{aligned}$$

when $D < 0$ and we have used (4.173) and (4.174)..For the strong energy conditions it is straightforward to show that (4.173) and (4.174) imply $\mu + 3p = -6r^2 r'^2 < 0$ when $D < 0$.

This solution has nonzero shear tensor σ_{ij} on this region

since $\sigma_{00} = 2r \cdot r^{-3}$. The acceleration vector is zero since f is a function of $u+v$. The expansion scalar $\theta = 0$.

The Ricci scalar is found to be

$$(4.180) \quad R = 6D + 4/r^2.$$

The scalar $R_{ij}R^{ij}$ is given by

$$(4.181) \quad R_{ij}R^{ij} = 2[r^3 r'' + 4r^2 r \cdot^2]^2 + 2[r^3 r'' - 2r^2 r \cdot^2]^2 \\ + 2[1/r^2 + r^3 r'' + r^2 r \cdot^2]^2 \\ \geq 0.$$

The scalar $R_{ijkl}R^{ijkl}$ is found to be

$$(4.182) \quad R_{ijkl}R^{ijkl} = 4r^4[rr'' + 4r \cdot^2]^2 + 4r^4[3rr'' - 2r \cdot^2]^2 \\ + 16r^4[rr'' - r \cdot^2]^2 + 4r^6 r''^2 \\ \geq 0.$$

The final case we shall consider is the case when $\alpha = -1/2$.

Note that when $\alpha = -1/2$ that the second term in (4.134) is eliminated. Using $f = r^{-1/2}$ we find that

$$(4.183) \quad s = r^2/2 + c,$$

where c is a constant of integration. Solving (4.183) for r , setting $h \equiv 0$, and absorbing c into the definition of g we find

$$(4.184) \quad r(u,v) = [2g(u+v)]^{1/2}.$$

Since (4.184) implies that $r^2 = 2g$ we must have $g > 0$. When we find r we will have to consider the possible branches of this relation. From (4.134) we get

$$(4.185) \quad 3rg''/2 + 1 = 0,$$

where we use the dot to denote differentiation with respect to $u+v$. Since $r^2 = 2g$ we have

$$(4.186) \quad rr' = g',$$

$$g'' = r'^2 + rr'',$$

thus (4.185) becomes

$$(4.187) \quad 3r^2 r'' + 3rr'^2 + 2 = 0,$$

where we use the branch $r_+ = \sqrt{2g}$, thus $r_+ > 0$. Note that we shall only use the subscript "+" on r for clarity. Setting

$\beta = r'$ so that $r'' = \beta \frac{d\beta}{dr}$, we get

$$(4.188) \quad 3r^2 \beta \beta' + 3r\beta^2 + 2 = 0,$$

where now the prime represents differentiation with respect to r . Dividing by $3r^2\beta$ and rearranging terms we find the Bernoulli equation

$$(4.189) \quad \beta' = -\beta/r - 2\beta^{-1}/(3r^2).$$

From [80] we find the solution of (4.189) to be

$$(4.190) \quad \beta^2(r) = Y_1(r) + Y_2(r)$$

where

$$(4.191) \quad Y_1(r) = De^{\lambda(r)},$$

$$(4.192) \quad Y_2(r) = 2e^{\lambda(r)} \int [-2e^{-\lambda(r)}/(3r^2)] dr,$$

and $\lambda(r)$ is given by

$$(4.193) \quad \lambda(r) = -2 \int r^{-1} dr$$

$$= -2 \ln|r| + C,$$

where C is a constant of integration. Thus we find

$$(4.194) \quad Y_1(r) = Dr^{-2},$$

$$(4.195) \quad Y_2(r) = -4/(3r).$$

Thus the solution of (4.189) is

$$(4.196) \quad \beta^2(r) = Y_1(r) + Y_2(r)$$

$$= Dr^{-2} - (4/3r).$$

A first integral of (4.187) is

$$(4.197) \quad r'^2 = Dr^{-2} - (4/3r)$$

where we must choose $D > 0$. Let $K(r;D) = Dr^{-2} - (4/3r)$ so that $K(r;D) \geq 0$ defines a region of the (u,v) -plane.. We can write a solution to (4.187) as

$$(4.198) \quad C \pm (u+v) = \int [K(r;D)]^{-1/2} dr,$$

where C is an integration constant. This integral can be evaluated to find r_+ implicitly in terms of $u+v$, C , and D . The metric form of this solution in terms of the implicit function r_+ is

$$(4.199) \quad \bar{g} = -4r_+^{-1}(u,v)du \odot dv + r_+^2(u,v)[d\theta \odot d\theta + \sin^2 \theta d\phi \odot d\phi].$$

The timelike weak energy inequalities (4.67) and (4.68) become

$$(4.200) \quad \mu = 1/r^2 \\ \geq 0,$$

and

$$(4.201) \quad \mu + p = 4/(3r^2) \\ \geq 0,$$

using (4.185), (4.186), and the fact that $r > 0$. The pressure is

$$(4.202) \quad p = 1/(3r^2)$$

so we see that the equation of state $p = \mu/3$ holds. From (4.200) and (4.201) we see that the timelike weak energy conditions hold for the branch $r_+ = \sqrt{2g}$. For the dominant energy conditions we must check that $\mu - p \geq 0$.

Using (4.187) we find that

$$(4.203) \quad \mu - p = 2/(3r^2) \\ > 0.$$

Thus the dominant energy conditions hold for the $r = \sqrt{2g}$.

Checking the strong energy condition $\mu + 3p \geq 0$ we find

$$(4.204) \quad \mu + 3p = 2/r^2 \\ > 0.$$

The Ricci scalar for the branch $r = \sqrt{2g}$ is

$$(4.205) \quad R = 3r^{\cdot 2}/(2r) \\ > 0.$$

The invariant $R_{ij}R^{ij}$ for the branch $r = \sqrt{2g}$ is

$$(4.206) \quad R_{ij}R^{ij} = 4/(3r^4) \\ > 0.$$

The invariant $R_{ijkl}R^{ijkl}$ for the branch $r = \sqrt{2g}$ is

$$(4.207) \quad R_{ijkl}R^{ijkl} = 4r^{\cdot 2} + 4r^{\cdot\cdot 2} + 8(r^2 r^{\cdot\cdot} + 1/3)^2 \\ > 0.$$

The solution r_+ has nonzero shear tensor σ_{ij} on this region since $\sigma_{00} = r^{\cdot} r^{-3/2}$. The acceleration vector is zero since f is

a function of $u+v$. The expansion scalar $\theta = -(3/2)r^{\cdot} r^{-1/2}$.

Since the solution is defined only where $r^{\cdot 2} > 0$ we see

that the solution is stably causal. From the discussion of

elementary causality in Chapter 2 we see that the solution is

strongly causal, causal, and chronological. The metric (4.199)

is nonstatic since both f and r depend on $u+v$. As the metric

coefficients are functions of $u+v$ we see that $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ is

an additional Killing vector to those imposed by spherical symmetry.

Now we consider the other branch of $r^2 = 2g$ thus $r_- = -\sqrt{2g}$ and we see that $r_- < 0$. Using r_- in (4.134) we get

$$(4.208) \quad -3\sqrt{2g}g'' + 2 = 0.$$

Using (4.186) we find (4.208) becomes

$$(4.209) \quad -3r^2 r'' - 3r r'^2 + 2 = 0.$$

Setting $\beta(r) = r'$ so that $r'' = \beta \frac{d\beta}{dr}$, we get

$$(4.210) \quad -3r^2 \beta \beta' - 3r \beta^2 + 2 = 0,$$

where now the prime represents differentiation with respect to r . Dividing by $-3r^2 \beta$ and rearranging terms we find the Bernoulli equation

$$(4.211) \quad \beta' = -\beta/r + 2\beta^{-1}/(3r^2).$$

From [80] we find the solution of (4.211) to be

$$(4.212) \quad \beta^2(r) = Y_1(r) + Y_2(r)$$

where

$$(4.213) \quad Y_1(r) = De^{\lambda(r)},$$

$$(4.214) \quad Y_2(r) = 2e^{\lambda(r)} \int [2e^{-\lambda(r)}/(3r^2)] dr,$$

and $\lambda(r)$ is given by

$$(4.215) \quad \begin{aligned} \lambda(r) &= -2 \int r^{-1} dr \\ &= -2 \ln|r| + C, \end{aligned}$$

where C is a constant of integration. Thus we find

$$(4.216) \quad Y_1(r) = Dr^{-2},$$

$$(4.217) \quad Y_2(r) = 4/(3r).$$

Thus the solution of (4.211) is

$$(4.218) \quad \beta^2(r) = Y_1(r) + Y_2(r) \\ = Dr^{-2} + (4/3r).$$

A first integral of (4.209) is

$$(4.219) \quad r'^2 = Dr^{-2} + (4/3r)$$

where we must choose $D > 0$ since $r < 0$.

Let $K(r;D) \equiv Dr^{-2} - (4/3r)$ so that $K(r;D) \geq 0$ defines a region of the (u,v) -plane.. We can write a solution to (4.209) as

$$(4.220) \quad C \pm (u+v) = \int [K(r;D)]^{-1/2} dr,$$

where C is an integration constant. This integral can be evaluated to find r implicitly in terms of $u+v$, C , and D .

The timelike weak energy inequalities (4.67) and (4.68) become

$$(4.221) \quad \mu = 1/r^2 \\ \geq 0,$$

and

$$(4.222) \quad \mu + p = -4/(3r^2) \\ < 0,$$

using (4.185), (4.186), and the fact that $r < 0$. Thus the timelike weak energy conditions do not hold for $r_- = -\sqrt{2g}$.

The dominant energy conditions cannot hold since (4.222) is not positive. Similarly the strong energy conditions do not hold for $r_- = -\sqrt{2g}$. We will no longer consider $r_- = -\sqrt{2g}$.

We have looked at several cases of the field equations for a perfect fluid in NN-coordinates. In each case we see that the field equations may be reduced to a Bernoulli equation by

prescribing a functional relationship between f and r . If we make assumptions which force f and r to be functions of $u+v$, we are able in some cases to find reasonable solutions of the field equations which were nonstatic and which have nonzero shear tensors. In all of the acceptable cases the timelike weak energy conditions were satisfied. The dominant and strong energy conditions were applied where possible. In all cases the shear tensor σ_{ij} was nonzero and the acceleration was zero. The expansion scalar θ was computed for all solutions. For several of the cases some of the scalar invariants such as R , $R_{ij}R^{ij}$ and $R_{ijkl}R^{ijkl}$ derived from the Riemann tensor were computed.

An interesting observation about all the solutions is that they all satisfy the condition of being in a T-region [27] since $r_{,i}r'^i = -r_u r_v / f^2 < 0$ when $r = R(u+v)$. This is the result of insisting that f and r are functions of $u+v$ so that the solutions are nonstatic. The known perfect fluid solutions in a T-region are those of McVittie and Wiltshire [128] and Ruban [129,130]. The solution of [128] has an equation of state $p = (1/3)\mu$. The solution of [129] was a dust solution $p = 0$, and that of [130] had a stiff equation of state $p = \mu$. These solutions and the one found here share the property of having the same symmetries as the Kantowski-Sachs dust solutions. The solutions found here corresponding to $f = r^\alpha$, $\alpha = -2, 1/3$ do not have simple equations of state. Ruban has

made several interesting observations about T-regions.

The use of NN-coordinates seems to have some advantage in the reduction of the field equations. All the calculations were made with a "generalized comoving" observer. An interesting problem for future work will be to try to use the scheme of Tupper [18,19] in NN-coordinates to try to find viscous fluid solutions. A "tilting" observer in NN-coordinates could be taken to be $U_i = (a,b,0,0)$ where $ab = f^2$.

CHAPTER V

SPHERICALLY SYMMETRIC ANISOTROPIC FLUIDS IN NN-COORDINATES

Anisotropic Stress-Energy Tensors and Conformal Collineations

In this chapter we shall calculate the field equations for a spherically symmetric anisotropic fluid in NN-coordinates. In particular, we shall compute the equations for the case when the metric tensor admits a timelike conformal collineation vector parallel to a "generalized comoving" velocity vector.

From Chapter 2 we recall that the equations which determine a collineation vector X are

$$(5.1) \quad X_{i;j} + X_{j;i} = 2\psi g_{ij} + H_{ij},$$

where ψ is the scalar conformal factor. H_{ij} is the symmetric covariantly constant tensor associated with X and obeys the following conditions

$$(5.2) \quad H_{[ij]} = 0,$$

$$(5.3) \quad H_{ij;k} = 0.$$

If we have the additional condition on the conformal factor ψ

$$(5.4) \quad \psi_{,ij} = 0,$$

then the collineation vector is "special".

There are different methods in which conformal collineations may be used to study the problem of finding interior solutions. The first method is to prescribe the collineation tensor H_{ij} , assume that the metric admits a collineation vector X^i with collineation tensor H_{ij} , and then

applying this symmetry to the field equations. This is the method used in [89,90,91,92]. This method assumes that the matter fields constituting the stress-energy tensor partake in the symmetry and hence leads to difficult problems of "inheritance of symmetry". A second method is to follow the first method except for the last step of applying the symmetry to the field equations. In both of these methods the conformal collineation tensor is chosen in an *ad hoc* way. A third method is suggested by a theorem that we will prove later. Suppose one were able to construct a collineation tensor H_{ij} from the stress-energy tensor T_{ij} . Then applying the first or second methods above, we could seek solutions of the field equations as before. This third method seems to have the advantage of eliminating the *ad hoc* choice of the collineation tensor. It is clear that the third method will be available only for very special stress-energy tensors since $H_{ij;k} = 0$ is a very restrictive condition on the metric g_{ij} and T_{ij} .

Duggal [91] notes that the existence of a physical solution of the field equations which admits a proper conformal collineation vector (not a conformal motion) is a very difficult problem. Hall [131] asserts that a proper conformal collineation exists in a perfect fluid spacetime only for a stiff equation of state.

The existence of a conformal collineation vector depends on the existence of a covariantly constant symmetric tensor H_{ij}

other than g_{ij} . Katzin *et al.* [132] show that if a space admits a covariantly constant vector field which is the gradient of a nonconstant scalar field α , and $F \in \mathcal{E}^3$ so that $F^{(3)} = 0$, then we have

$$(5.5) \quad X_i = F'(\alpha)\alpha_{,i}$$

is a conformal collineation vector with collineation tensor

$$(5.6) \quad H_{ij} = 2F''(\alpha)\alpha_{,i}\alpha_{,j},$$

and conformal factor $\psi = 0$.

It is well-known [27] that reducible spacetimes may admit a covariantly constant second order symmetric tensor other than the metric tensor, but the spherical spacetimes we are using are not reducible. This follows from the fact that the Lorentzian warped product is not reducible unless the warping factor is trivial.

In the remainder of this section we present some facts concerning the third method of using collineations. In interior spacetimes one is motivated to look for a relationship between the stress-energy tensor and the collineation tensor. For a nonsingular anisotropic stress-energy tensor such a relationship may exist if the stress-energy tensor is recurrent ($T_{ij;k} = T_{ij}V_k$ for some vector V_k).

Walker [133] and Patterson [134] have studied the existence of covariantly constant second order symmetric tensors on a semi-riemannian manifold. (Eisenhart [135] studied the case of riemannian manifolds). Walker [133] proved the following

Theorem: If (M, g) is a semi-riemannian manifold and if T_{ij} is

- (a) symmetric,
- (b) not a constant multiple of g_{ij} ,
- (c) nonsingular ($\det[T^i_j] \neq 0$),
- (d) recurrent,

then $H_{ij} = \alpha(T_{ij} - \lambda g_{ij})$ is a symmetric covariantly constant second order tensor where α is a scalar and λ is one of the nonzero g -eigenvalues of T_{ij} .

If T_{ij} is an anisotropic stress-energy tensor we know that T_{ij} has Segré class $\{1, 1(11)\}$, hence has three distinct eigenvalues. Not all Segré class $\{1, 1(11)\}$ tensors satisfy the condition (c) in Walker's theorem. A radiating dust is of Segré class $\{1, 1(11)\}$ but has two zero eigenvalues, hence is singular. Furthermore, most stress-energy tensors are not recurrent, hence condition (d) is violated in Walker's theorem in general. For nonsingular recurrent anisotropic stress-energy tensors we are led to the following theorem.

Theorem: A nonsingular recurrent anisotropic stress-energy tensor T_{ij} in a Lorentzian manifold (M, g) may be used to construct three covariantly constant second order symmetric tensors of the form

$$(5.7) \quad H_{Aij} = \alpha_A (T_{ij} - \lambda_A g_{ij})$$

where the λ_A is one of the three nonzero g -eigenvalues of T_{ij} and α_A are certain scalars.

Proof: Following Walker [133] we define $\phi = \det[T_{ij}]/\det[g_{ij}]$.

Since spacetime is assumed to be four-dimensional we see that ϕ is a fourth degree homogeneous polynomial in the components T_{ij} with coefficients which are functions of the components of the metric tensor. Since T_{ij} and g_{ij} are nonsingular we have $\phi \neq 0$. Since T_{ij} is recurrent, we have $T_{ij;k} = T_{ij} V_k$ for some vector V_k . Taking the gradient of ϕ we find that $\phi_{;k} \equiv 4\phi V_k$, thus $V_k = \ln(\phi^{1/4})_{;k}$ and hence is a gradient.

Define $S_{ij} \equiv \phi^{-1/4} T_{ij}$. We claim $S_{ij;k} = 0$. Taking the covariant derivative of S_{ij} we have

$$\begin{aligned} S_{ij;k} &= -(1/4)\phi^{-3/4}\phi_{;k} T_{ij} + \phi^{-1/4} T_{ij} V_k \\ &= -(1/4)\phi^{-3/4}(4\phi V_k) T_{ij} + \phi^{-1/4} T_{ij} V_k \\ &= 0. \end{aligned}$$

Thus S_{ij} is nonsingular, S_{ij} is not proportional to g_{ij} , and $S_{ij;k} = 0$. Consider now the tensor $H_{ij} \equiv S_{ij} - \rho g_{ij}$. Each coefficient τ_A , $A = 1, 2, 3, 4$, of the polynomial

$$F(\rho) \equiv \det[H_{ij}]/\det[g_{ij}] = \rho^4 + \tau_1 \rho^3 + \tau_2 \rho^2 + \tau_3 \rho + \tau_4$$

is a sum of products of components of S_{ij} and g_{ij} . Since S_{ij} and g_{ij} are covariantly constant we see that each τ_A is constant. Thus every root of $F(\rho) = 0$ is constant and nonzero since S_{ij} is nonsingular. Now define $H_{Aij} \equiv S_{ij} - \rho_A g_{ij}$, where ρ_A is a root of $F(\rho)$. It is clear that $H_{Aij;k} = 0$ for each root ρ_A of $F(\rho) = 0$. T_{ij} is an anisotropic stress-energy tensor with three distinct nonzero eigenvalues λ_A so setting $\rho_A = \phi^{-1/4} \lambda_A$ and $\alpha_A = \phi^{-1/4}$ shows $H_{Aij} \equiv \alpha_A (T_{ij} - \lambda_A g_{ij})$.

We are interested in solutions with shear so we should

look for a relationship between the shear tensor and the existence of a conformal collineation. Duggal [91] has found such a relation between the collineation tensor of a collineation vector (parallel to the velocity) and the shear tensor of the velocity. Duggal's results are summarized in the following theorem.

Theorem: Let $X^i = \lambda U^i$, $U^i U_i = -1$ and $\lambda > 0$. A spacetime (M, g) admits a timelike conformal collineation with symmetry vector X^i , conformal scalar ψ , and collineation tensor H_{ij} if and only if

$$(a) \sigma_{kl} = (2\lambda)^{-1} [h^i_k h^j_l H_{ij} - (2/3)\theta^* h_{kl}]$$

$$(b) A_i = \lambda^{-1} [\lambda_{,i} + \lambda_{,j} U^j U_i + H_{jk} U^k h^j_i]$$

where

$$\psi = (\lambda\theta - \theta^*)/3,$$

$$\theta^* = (1/2)[H^i_i + H_{ij} U^i U^j],$$

and σ_{ij} , θ , and A_i are respectively, the shear tensor, the expansion scalar, and the acceleration vector of the timelike flow generated by U_i .

From Mason and Maartens [89] we find that condition (a) in Duggal's theorem is more transparently written as

$$(a') \sigma_{kl} = (2\lambda)^{-1} [h^i_k h^j_l - (1/3)h^{ij} h_{kl}] H_{ij}.$$

The condition (a') clearly shows the close relationship between the shear of U^i and the collineation tensor. Loosely speaking, (a') expresses the relationship that σ_{kl} is proportional to a projection of the collineation tensor H_{ij} . Thus to force a

solution of the field equations have have shear it is sufficient to assume the existence of a proper conformal collineation parallel to the velocity. Next we consider the problem of the existence of a timelike collineation vector X parallel to the velocity U when we are given H_{ij} , the collineation tensor.

Given a second order symmetric tensor H_{ij} which is covariantly constant, and a velocity field U^i , we can determine the expansion scalar θ from the kinematics of the congruence of U . If we seek a timelike collineation vector $X = \lambda U$ we must determine the scaling factor λ . The solution of this problem is the content of the following theorem.

Theorem: Let (M, g) be a spacetime and suppose that a second order symmetric covariantly constant tensor H_{ij} and a unit timelike vector field U^i is given. Then there is a scalar λ so that $X^i = \lambda U^i$ is an affine collineation vector with collineation tensor H_{ij} .

Proof: In [89] it is shown that under the hypotheses above that the conformal factor can be expressed as

$$\psi = \lambda_{;i} U^i + (1/2) H_{ij} U^i U^j,$$

and

$$\theta = 3\psi/\lambda + (1/2\lambda) h^{ij} H_{ij}.$$

Substituting the expression for ψ into the expression for θ we have

$$\lambda\theta = 3\psi + (1/2) h^{ij} H_{ij} = 3\lambda_{;i} U^i + (3/2) H_{ij} U^i U^j + (1/2) h^{ij} H_{ij}.$$

This last expression is a linear partial differential equation for λ : $-\lambda_{;i} U^i + \theta\lambda/3 = (1/2)H_{ij} U^i U^j + (1/6)h^{ij} H_{ij}$. Linear partial differential equations can, in principle, always be solved [136,137]. Thus we have shown that given a covariantly constant symmetric tensor H_{ij} and a velocity vector field, U^i , we can determine a timelike collineation vector X^i parallel to the velocity U^i with collineation tensor H_{ij} .

The two theorems just proved show that the third method of using collineations may be useful. A key question which will temper the application of this method is the characterization of recurrent stress-energy tensors.

The Anisotropic Field Equations in NN-Coordinates

In this section we shall write the field equations for a spherically symmetric anisotropic fluid in NN-coordinates. We shall assume that the velocity of the fluid has a special form in NN-coordinates is generalized comoving. We shall also compute the equations for a timelike collineation vector parallel to the velocity. We will assume that the stress-energy tensor is that of an anisotropic fluid whose anisotropy vector S is orthogonal to U and orthogonal to the two-dimensional pressure isotropy surfaces. Since $U_i = (f, f, 0, 0)$ and the metric is spherically symmetric we have $S_i = (f, -f, 0, 0)$. Thus the stress-energy tensor has the form

$$(5.8) \quad T_{ij} = (\mu+q)U_i U_j + qg_{ij} + (p-q)S_i S_j,$$

where $S_i S^i = 1$, $U^i U_i = -1$ and $S_i U^i = 0$. The eigenvalues of T_{ij} are $\lambda_0 = -\mu$, $\lambda_1 = p$, $\lambda_{2,3} = q$.

The form of the metric tensor we use is:

$$(5.9) \quad \bar{g} = -4f^2(u,v)du\odot dv + r^2(u,v)[d\theta\odot d\theta + \sin^2\theta d\phi\odot d\phi],$$

which we recognize as a spherically symmetric metric in NN-coordinates. Since the metric is a Type 1 Lorentzian warped product, the comments of Chapter 4 on elementary causality still hold. In these coordinates the anisotropic field equations become (independent equations only)

$$(5.10) \quad \mu - p = 2r_u r_v / (r^2 f^2) + 2r_{uv} / (r f^2) + 2/r^2,$$

$$(5.11) \quad \mu + p = 4r_v f_v / (r f^3) - 2r_{vv} / (r f^2),$$

$$(5.12) \quad \mu + p = 4r_u f_u / (r f^3) - 2r_{uu} / (r f^2),$$

$$(5.13) \quad q = -r_{uv} / (r f^2) + f_u f_v / f^4 - f_{vu} / f^3.$$

The conservation equations are

$$(5.14)$$

$$\begin{aligned} T^{0i}{}_{;i} &= \mu(r_u + r_v) / (2r f^2) + q r_v / (r f^2) + p(r_u - r_v) / (2r f^2) \\ &\quad + (\mu + p) f_u / (2f^3) + (\mu_u + \mu_v) / (4f^2) + (p_u - p_v) / (4f^2) \\ &= 0, \end{aligned}$$

$$(5.15)$$

$$\begin{aligned} T^{1i}{}_{;i} &= \mu(r_u + r_v) / (2r f^2) + q r_u / (r f^2) + p(r_v - r_u) / (2r f^2) \\ &\quad + (\mu + p) f_v / (2f^3) + (\mu_u + \mu_v) / (4f^2) + (p_v - p_u) / (4f^2). \\ &= 0. \end{aligned}$$

If we assume a timelike collineation vector $X = \lambda U$ with a collineation tensor H_{ij} , then we find the equations

$$(5.16) \quad H_{00} = 2f\lambda_u - 2\lambda f_u,$$

$$(5.17) \quad H_{01} = f\lambda_u + f\lambda_v + \lambda f_u + \lambda f_v + 4f^2\psi,$$

$$(5.17) \quad H_{11} = 2f\lambda_v - 2\lambda f_v,$$

$$(5.18) \quad H_{22} = -r\lambda r_u/f - r\lambda r_v/f - 2r^2\psi,$$

$$(5.19) \quad H_{33} = -r\lambda \sin^2\theta r_u/f - r\lambda \sin^2\theta r_v/f - 2r^2\psi \sin^2\theta \\ = H_{22} \sin^2\theta.$$

Note that we have not specified what H_{ij} is, but are just writing the right-hand side of $H_{ij} = X_{i;j} + X_{j;i} - 2\psi g_{ij}$.

From the condition $H_{ij;k} = 0$ in the definition of a conformal collineation vector we find the following equations

$$(5.20) \quad H_{00;0} = 2f\lambda_{uu} - 2\lambda f_{uu} + 8\lambda(f_u)^2/f - 8f_u\lambda_u,$$

$$(5.21) \quad H_{00;1} = 2f\lambda_{uv} - 2\lambda f_{uv} - 2f_u\lambda_v + 2f_v\lambda_u,$$

$$(5.22) \quad H_{01;0} = f\lambda_{uu} + f\lambda_{vu} + \lambda f_{uu} + \lambda f_{vu} - 2\lambda(f_u)^2/f \\ - 2\lambda f_u f_v/f + 4f^2\psi_u - f_u\lambda_v + f_v\lambda_u,$$

$$(5.23) \quad H_{01;1} = f\lambda_{uv} + f\lambda_{vv} + \lambda f_{uv} + \lambda f_{vv} - 2\lambda(f_v)^2/f \\ - 2\lambda f_u f_v/f + 4f^2\psi_v + f_u\lambda_v - f_v\lambda_u,$$

$$(5.24) \quad H_{02;2} = -r\lambda r_u f_u/(2f^2) - r\lambda r_u f_v/(2f^2) + r\lambda r_v f_u/f^2 \\ - r r_u \lambda_u/(2f) - r r_u \lambda_v/(2f) - r r_v \lambda_u/f \\ + \lambda(r_u)^2/f + \lambda r_u r_v/f,$$

$$(5.25) \quad H_{03;3} = \sin^2\theta H_{02;2},$$

$$(5.26) \quad H_{11;0} = 2f\lambda_{vu} - 2\lambda f_{vu} + 2f_u\lambda_v - 2f_v\lambda_u,$$

$$(5.27) \quad H_{11;1} = 2f\lambda_{vv} - 2\lambda f_{vv} + 8\lambda(f_v)^2/f - 8f_v\lambda_v,$$

$$(5.28) \quad H_{12;2} = r\lambda r_u f_v/f^2 - r\lambda r_v f_u/(2f^2) - r\lambda r_v f_v/(2f^2) \\ - r r_u \lambda_v/f - r r_v \lambda_u/(2f) - r r_v \lambda_v/(2f) \\ + \lambda(r_v)^2/f + \lambda r_u r_v/f,$$

$$(5.29) \quad H_{13;3} = \sin^2\theta H_{12;2},$$

$$(5.30) \quad H_{22;0} = r\lambda r_u f_u / f^2 + r\lambda r_v f_u / f^2 - r\lambda r_{uu} / f \\ - r\lambda r_{vu} / f - rr_u \lambda_u / f - rr_v \lambda_u / f \\ - 2r^2 \psi_u + \lambda(r_u)^2 / f + \lambda r_u r_v / f,$$

$$(5.31) \quad H_{22;1} = r\lambda r_u f_v / f^2 + r\lambda r_v f_v / f^2 - r\lambda r_{uv} / f \\ - r\lambda r_{vv} / f - rr_u \lambda_v / f - rr_v \lambda_v / f \\ - 2r^2 \psi_v + \lambda(r_v)^2 / f + \lambda r_u r_v / f,$$

$$(5.32) \quad H_{33;0} = \sin^2 \theta H_{22;0},$$

$$(5.33) \quad H_{33;1} = \sin^2 \theta H_{22;1}.$$

These equations, when combined with the field equations, form a formidable system of partial differential equations. The equations (5.20) to (5.33) show the severe constraints that the existence of a conformal collineation place on the metric.

The kinematic quantities of the timelike congruence determined by U are calculated next. The shear tensor of U is

$$(5.34) \quad \sigma_{00} = f(r_u + r_v) / (3r) - (f_u + f_v) / 3, \\ \sigma_{01} = -f(r_u + r_v) / (3r) + (f_u + f_v) / 3, \\ \sigma_{11} = f(r_u + r_v) / (3r) - (f_u + f_v) / 3, \\ \sigma_{22} = -r(r_u + r_v) / (6f) + r^2(f_u + f_v) / (6f^2), \\ \sigma_{33} = \sigma_{22} \sin^2 \theta,$$

The acceleration vector of U is

$$(5.35) \quad A_0 = (f_u - f_v) / (2f), \\ A_1 = (f_v - f_u) / (2f).$$

The expansion scalar of the congruence determined by U is

$$(5.36) \quad \theta = -(r_u + r_v) / (rf) - (f_u + f_v) / (2f^2).$$

It is customary to impose some energy conditions on an interior solution in order to give some physically plausible properties to the solution. As pointed out in Chapter 3, there are several alternatives available. We shall write the time-like weak energy conditions as they are the most straightforward to apply. The timelike weak energy conditions for the field equations (5.10) to (5.13) are

$$(5.37) \quad 0 \leq \mu = r_u r_v / (r^2 f^2) + (r_u f_u + r_v f_v) / (r f^3) \\ + r_{uv} / (r f^2) + 1/r^2 - (r_{uu} + r_{vv}) / (2r f^2),$$

$$(5.38) \quad 0 \leq \mu + p = 2r_u f_u / (r f^3) + 2r_v f_v / (r f^3) - r_{uu} / (r f^2) \\ - r_{vv} / (r f^2),$$

$$(5.39) \quad 0 \leq \mu + q = r_u r_v / (r^2 f^2) + r_u f_u / (r f^3) - r_{uu} / (2r f^2) \\ + 1/r^2 + r_v f_v / (r f^3) - r_{vv} / (2r f^2) + f_u f_v / f^4 \\ - f_{vu} / f^3.$$

The dominant energy conditions are given by the five inequalities

$$(5.40) \quad \mu \geq 0, \\ \mu + p \geq 0, \\ \mu - p \geq 0, \\ \mu + q \geq 0, \\ \mu - q \geq 0.$$

The strong energy conditions are given by the inequalities

$$(5.41) \quad \mu + p \geq 0, \\ \mu + q \geq 0, \\ \mu + 2p + q \geq 0.$$

A Study of the Conformal Collineation Equations

In the first section we defined a conformal collineation by equations (5.1) and wrote the condition $H_{ij;k} = 0$ in NN-coordinates in equations (5.20) to (5.33). In this section we will study the system (5.20) to (5.33) with a view to finding *nonstatic shearing* anisotropic solutions of the field equations (5.10) to (5.13) which admit a conformal collineation vector parallel to the generalized comoving velocity. Our approach is to study (5.20) to (5.33) and find candidate cases with nonzero shear tensor. These candidate cases will be used in the field equations and appropriate energy conditions.

If we subtract (5.26) from (5.21) we find that

$$(5.42) \quad f_u \lambda_v - f_v \lambda_u = 0.$$

This equation can have solutions in only five possible ways depending on the number of partial derivatives of f and λ which are zero. We will use (5.42) to systematically attempt to solve (5.20) to (5.33). With the understanding that c and k represent nonzero real numbers we define several cases for solving (5.42) as follows:

case A: $f = c \neq 0, \lambda = k \neq 0, c, k \in \mathbb{R};$

case B1: $f = c \neq 0, \text{ and } \lambda_u = 0, \lambda_v \neq 0;$

case B2: $f = c \neq 0, \text{ and } \lambda_u \neq 0, \lambda_v = 0;$

case C1: $\lambda = k \neq 0, f_u = 0, f_v \neq 0;$

case C2: $\lambda = k \neq 0, f_u \neq 0, f_v = 0;$

case D1: $\lambda = k \neq 0, f_u \neq 0, f_v \neq 0;$

case D2: $f = c \neq 0, \lambda_u \neq 0, \lambda_v \neq 0;$

case D3: $f_v = \lambda_v = 0, f_u \neq 0, \lambda_u \neq 0;$

case D4: $f_u = \lambda_u = 0, f_v \neq 0, \lambda_v \neq 0;$

case E1: $f_u \neq 0, \lambda_u \neq 0, f_v \neq 0, \lambda_v \neq 0, f\Lambda - \Lambda' = 0;$

case E2: $f_u \neq 0, \lambda_u \neq 0, f_v \neq 0, \lambda_v \neq 0, f\Lambda - \Lambda' \neq 0.$

We will consider Case A first. As a consequence of the hypothesis for case A, (5.20), (5.21), (5.26), and (5.27) are satisfied. Equations (5.22) and (5.23) imply that ψ is a constant. Equations (5.24) and (5.28) imply that r satisfies the following system of equations:

$$(5.43) \quad \begin{aligned} r_u(r_u + r_v) &= 0, \\ r_v(r_u + r_v) &= 0. \end{aligned}$$

At this point case A can be broken into four subcases depending on r :

subcase (1): $r_u = 0, r_v = 0;$

subcase (2): $r_u \neq 0, r_v = 0;$

subcase (3): $r_u = 0, r_v \neq 0;$

subcase (4): $r_u \neq 0, r_v \neq 0, r_u + r_v = 0.$

For subcase (1), r is a constant. From (5.34) we see that $\sigma_{ij} = 0$, thus this subcase is discarded.

For subcase (2), equation (5.43) produces $r_u = 0$ which contradicts the hypothesis of subcase (2). A similar conclusion holds for subcase (3).

The hypothesis of subcase (4), together with (5.43), implies that $r = \rho(u-v)$. Equations (5.30) and (5.31) are satisfied by

$r = \rho(u-v)$, but (5.34) shows that we have $\sigma_{ij} = 0$, hence we discard this subcase.

The preceding four subcases show that case A does not admit any solutions which lead to nonstatic shearing solutions of the field equations.

Next we consider Case B1. Since $\lambda_u = 0$ and $\lambda_v \neq 0$ we see that $\lambda = \Lambda(v)$. Equations (5.20), (5.21), and (5.26) are satisfied. Equations (5.22) and (5.23) imply

$$(5.44) \quad \psi(v) = -\Lambda'/(4f).$$

Equation (5.27) shows that $\Lambda(v) = av + b$, $a \neq 0$. Equation

(5.24) reduces to

$$(5.45) \quad r_u [-a/(av + b) + 2(r_u + r_v)/r] = 0.$$

Equation (5.28) becomes

$$(5.46) \quad (av + b)r_v [-a/(av + b) + 2(r_u + r_v)/r] = 2ar_u.$$

Now case B1 can be broken into four subcases depending on r :

subcase (1): $r_u = 0$, $r_v = 0$;

subcase (2): $r_u \neq 0$, $r_v = 0$;

subcase (3): $r_u = 0$, $r_v \neq 0$;

subcase (4): $r_u \neq 0$, $r_v \neq 0$.

For subcase (1), r is a constant (which we may assume is nonzero) and thus (5.45) and (5.46) are satisfied. Equation (5.31) implies that $\psi_v = 0$ thus $-\Lambda'/(4f) = 0$ and $a = 0$ which contradicts the hypothesis $\lambda_v \neq 0$ of case B1. Thus subcase (1) is discarded.

For subcase (2), using $r_u \neq 0$ and $r_v = 0$ in (5.45) and

(5.46), immediately leads to a contradiction hence this case is discarded.

For subcase (3), equation (5.45) is satisfied and (5.46) has the solution

$$(5.47) \quad r(u,v) = K(av + b)^{1/2},$$

where $K \neq 0$. Equation (5.31) is reduced to $2a/(av + b) + 1 = 0$ which cannot hold for arbitrary values of v . Thus subcase (3) is discarded.

For subcase (4), equation (5.45) reduces to

$$(5.48) \quad -a/(av + b) + 2(r_u + r_v)/r = 0.$$

If (5.48) holds, then from (5.46), $2ar_u = 0$ which either contradicts $a \neq 0$ or the hypothesis that $r_u \neq 0$. Thus we discard subcase (4). The case B1 does not lead to any solutions.

Case B2 is entirely dual to case B1 so we conclude that case B2 leads to no solutions.

We now consider case C1. Since $f_u = 0$, f is a function of v alone. By a coordinate transformation we can reduce this case to the case when f is a constant. Since λ is a constant, we see that case C1 reduces to the case A which has no acceptable solutions. We thus discard case C1. As case C2 is dual to C1 we discard it as well.

Now we consider case D1. Equations (5.20) implies that

$$(5.49) \quad (f^{-3})_u = G(v)$$

for some function G . Similarly (5.27) implies that

$$(5.50) \quad (f^{-3})_v = H(u),$$

for some function H . The integrability of (5.49) and (5.50) implies that

$$(5.51) \quad (f^{-3})_{uv} = K,$$

where K is a constant. Thus $G(v) = Kv + a$, $H(u) = Ku + b$ so (5.51) may be integrated to find

$$(5.52) \quad f(u,v) = [Kuv + au + bv + c]^{-1/3},$$

where a, b, c are constants of integration. Equation (5.21) implies that $f_{uv} = 0$. Using (5.52) we find $K = 0$ and $ab = 0$. If $a = 0$, then $f_u = 0$ contradicts the hypothesis that $f_u \neq 0$. A similar result holds for $b = 0$. Thus case D1 is discarded.

For case D2 we have (5.20), (5.21), (5.26), and (5.27)

leading to

$$(5.53) \quad \lambda(u,v) = au + bv + c,$$

where a, b, c are constants of integration and $a^2 + b^2 > 0$. Equations (5.22) and (5.23) imply that ψ is a constant.

Equation (5.24) reduces to

$$(5.54) \quad r_u[-(a + b)/2 + \lambda(r_u + r_v)/r] = ar_v.$$

Equation (5.28) reduces to

$$(5.55) \quad r_v[-(a + b)/2 + \lambda(r_u + r_v)/r] = br_u.$$

At this point case D2 can be broken into four subcases depending on r :

subcase (1): $r_u = 0, r_v = 0$;

subcase (2): $r_u \neq 0, r_v = 0$;

subcase (3): $r_u = 0, r_v \neq 0$;

subcase (4): $r_u \neq 0, r_v \neq 0$.

For subcase (1), r is a constant (which we take to be nonzero). Equations (5.30) and (5.31) imply that ψ is a constant. Equation (5.34) implies that the shear tensor σ_{ij} is zero so we discard this subcase.

For subcase (2), equation (5.55) implies $b = 0$ so that (5.53) gives $\lambda(u,v) = au + c$. Equation (5.54) can be integrated to find

$$(5.56) \quad r(u,v) = A(au + c)^{1/2},$$

where $A \neq 0$. Equation (5.31) is satisfied and (5.30) can be integrated to find

$$(5.57) \quad r(u,v) = B(au + c)^k,$$

where B is a nonzero constant of integration. Comparing (5.56) and (5.57) we see that if $A = B$ and $k = 1/2$ then we have a solution of (5.20) to (5.33). However we have seen that $\lambda(u,v) = au + c$, thus $\lambda_v = 0$, which contradicts the hypothesis of case D2. Since subcase (3) is dual to subcase (2), we will not discuss it further.

For subcase (4), equation (5.30) implies

$$(5.58) \quad (r_u + r_v)_u + a(r_u + r_v)/\lambda - r_u(r_u + r_v)/r = 0.$$

Similarly equation (5.31) implies

$$(5.59) \quad (r_u + r_v)_v + b(r_u + r_v)/\lambda - r_v(r_u + r_v)/r = 0.$$

Consider the case when $r_u + r_v = 0$. Then (5.58) and (5.59) are satisfied. Equation (5.54) becomes

$$(5.60) \quad -r_u(a + b)/2 = ar_v,$$

and equation (5.55) becomes

$$(5.61) \quad -r_v(a + b)/2 = br_u.$$

We can solve this linear system of equations for r_u and r_v only if $a = b$. This leads to $r(u,v) = \rho(u-v)$. From (5.34), we find the shear tensor σ_{ij} is zero since f is a constant.

Now consider the case when $r_u + r_v \neq 0$. Equation (5.58) implies

$$(5.62) \quad (r_u + r_v)_u / (r_u + r_v) + a/\lambda - r_u/r = 0,$$

which may be rewritten as

$$(5.63) \quad [\ln|r_u + r_v|]_u = [\ln|r/\lambda|]_u.$$

Similarly equation (5.59) reduces to

$$(5.64) \quad [\ln|r_u + r_v|]_v = [\ln|r/\lambda|]_v.$$

The integrability of (5.63) and (5.64) is clear so we obtain

$$(5.65) \quad r_u + r_v = Kr/\lambda,$$

where $K \neq 0$ is a constant of integration.

Using (5.65) in (5.54) and (5.55) gives

$$(5.66) \quad r_u[-(a + b)/2 + K] = ar_v.$$

and

$$(5.67) \quad r_v[-(a + b)/2 + K] = br_u.$$

Equations (5.66) and (5.67) are linear in r_u and r_v and can be solved for r_u and r_v only if

$$(5.68) \quad K = (a + b \pm \sqrt{ab})/2.$$

Using K in (5.65) leads to

$$(5.69) \quad r(u,v) = \rho(\alpha u + \alpha v),$$

where $\alpha \equiv K - (a + b)/2$. Thus (5.69) leads to a solution of

(5.20) to (5.33) which gives nonzero shear in general.

All that remains is to check the field equations and energy conditions.

Now we consider case D3. Since $f_u = 0$ we see that f is a function of v alone. By a coordinate transformation we can reduce this case to the case when f is a constant. Since λ is a constant we see that case D3 reduces to the case A which has no acceptable solutions. We thus discard case D3. Since case D4 is dual to D3 we discard it as well.

Consider case E1. From (5.42) that $f_u \lambda_v - f_v \lambda_u = 0$, and $f_u \neq 0$, $f_v \neq 0$, $\lambda_u \neq 0$, and $\lambda_v \neq 0$. The solution of (5.42) under these conditions is $\lambda = \Lambda(f)$. Equations (5.21) and (5.26) imply that

$$(5.70) \quad f \lambda_{uv} - \lambda f_{uv} = 0.$$

Using $\lambda = \Lambda(f)$ in (5.70) yields

$$(5.71) \quad (f\Lambda' - \Lambda)f_{uv} + \Lambda'' f f_u f_v = 0,$$

where the prime represents differentiation with respect to f .

The case E1 is characterized by

$$(5.72) \quad f\Lambda' - \Lambda = 0.$$

Solving (5.72) we find that $\Lambda(f) = af$, $a \neq 0$. Thus $\Lambda'' = 0$ so equations (5.21) and (5.26) are satisfied. Equations (5.20) and (5.27) are also satisfied. Equation (5.22) implies

$$(5.73) \quad \psi_u = -(a/2)[(f_u + f_v)/f]_u,$$

and equation (5.23) produces

$$(5.74) \quad \psi_v = -(a/2)[(f_u + f_v)/f]_v.$$

The integrability of (5.73) and (5.74) is satisfied and we find

$$(5.75) \quad \psi(u,v) = -(a/2)[(f_u + f_v)/f] + C,$$

where C is a constant of integration. Equation (5.24) reduces to

$$(5.76) \quad r_u[-(f_u + f_v)/f + (r_u + r_v)/r] = 0.$$

Equation (5.28) reduces to

$$(5.77) \quad r_v[-(f_u + f_v)/f + (r_u + r_v)/r] = 0.$$

Equation (5.30) becomes

$$(5.78) \quad \psi_u = -(a/2)[(r_u + r_v)/r]_u,$$

and equation (5.31) becomes

$$(5.79) \quad \psi_v = -(a/2)[(r_u + r_v)/r]_v.$$

The integrability of (5.78) and (5.79) is satisfied and the solution is

$$(5.80) \quad \psi(u,v) = -(a/2)[(r_u + r_v)/r] + K$$

where K is a constant of integration. Equations (5.75) and

(5.80) imply

$$(5.81) \quad (a/2)[(r_u + r_v)/r - (f_u + f_v)/f] = K - C.$$

At this point case E1 can be broken into four subcases depending on r:

subcase (1): $r_u = 0, r_v = 0$;

subcase (2): $r_u \neq 0, r_v = 0$;

subcase (3): $r_u = 0, r_v \neq 0$;

subcase (4): $r_u \neq 0, r_v \neq 0$.

For subcase (1), r is a nonzero constant, and equations (5.24) and (5.28) are satisfied. Equation (5.81) implies

$$(5.82) \quad f(u,v) = G(u-v)e^{(C-K)(u+v)/a},$$

where G is an arbitrary \mathcal{E}^2 -function. From (5.34) we find that $\sigma_{00} = -(f_u + f_v)/3 \neq 0$ in general. We still have to check the field equations and energy inequalities for this subcase.

For subcase (2), equation (5.76) implies that $K = C$ and

$$(5.83) \quad r_u/r - (f_u + f_v)/f = 0.$$

Since $r = \rho(u)$ we see (5.83) has the solution

$$(5.84) \quad f(u,v) = H(u-v)\rho(u),$$

where H is an arbitrary \mathcal{E}^2 -function. From (5.80) we see that

$$(5.85) \quad \psi(u,v) = -(a/2)\rho'(u)/\rho(u) + K.$$

From (5.34) we have $\sigma_{00} = 0$. Thus we discard this case.

Subcase (3) is dual to the subcase (2) so we shall consider it only in the sense that arbitrary functions of u are replaced with arbitrary functions of v etc.

For subcase (4), equations (5.76) and (5.77) imply that $K = C$. From (5.76), (5.77), and (5.34) we see that $\sigma_{ij} = 0$ hence we discard this case.

Finally we consider case E2. The condition $f\Lambda' - \Lambda \neq 0$ lead to three subcases:

$$\text{subcase (1): } \Lambda'' = 0;$$

$$\text{subcase (2): } \Lambda'' \neq 0, f^2\Lambda'' + 4(\Lambda - \Lambda'f) = 0;$$

$$\text{subcase (3): } \Lambda'' \neq 0, f^2\Lambda'' + 4(\Lambda - \Lambda'f) \neq 0.$$

For subcase (1) we find $\Lambda(f) = af + b$, where both a and b are nonzero. Equations (5.21) and (5.26) imply $f_{uv} = 0$ which has the solution

$$(5.86) \quad f(u,v) = g(u) + h(v).$$

Equation (5.20) and (5.86) lead to

$$(5.87) \quad g(u) = (c_1 u + d_1)^{-1/3},$$

where c_1 and d_1 are constants of integration.

Similarly, (5.27) and (5.86) lead to

$$(5.88) \quad h(v) = (c_2 v + d_2)^{-1/3},$$

where c_2 and d_2 are constants of integration. Now we define a coordinate transformation by

$$(5.89) \quad \begin{aligned} U &= c_1 u + d_1, \\ V &= c_2 v + d_2, \end{aligned}$$

and redefine f and r as functions of U and V . From (5.87) and (5.88) we have

$$(5.90) \quad f(U,V) = k[U^{-1/3} + V^{-1/3}],$$

where $k^2 = c_1 c_2 > 0$. From equation (5.22) we find

$$(5.91) \quad \psi_U = -(a/2)[f_U/f]_U - (b/4)[f_U/f^2]_U + (af + b)f_U f_V / f^3.$$

From equation (5.23) we find

$$(5.92) \quad \psi_V = -(a/2)[f_V/f]_V - (b/4)[f_V/f^2]_V + (af + b)f_U f_V / f^3.$$

The integrability of the (5.91) and (5.92) cannot be satisfied by a function of the form (5.90). Thus this subcase is discarded.

Subcase (2) assumes $\Lambda'' \neq 0$ and the equation

$$(5.93) \quad f^2 \Lambda'' + 4(\Lambda - \Lambda' f) = 0$$

which has the solution

$$(5.94) \quad \Lambda(f) = af + bf^4,$$

where $a \neq 0$ and $b \neq 0$ are constants of integration. Equations

(5.20) and (5.27) imply that

$$(5.95) \quad f(u,v) = cuv + du + k_1v + k_2,$$

where c, d, k_1, k_2 are constants of integration. Equations

(5.21) and (5.26) imply

$$(5.96) \quad f^2 f_{uv} + 4f_u f_v = 0.$$

Using (5.95) in (5.96), equation (5.96) has no solutions. Thus we discard this case.

Finally for subcase (3) we define a function χ so that

$$(5.97) \quad \chi''/\chi' \equiv f\Lambda''/(f\Lambda' - \Lambda) - 4/f.$$

Equation (5.97) has a solution given by

$$(5.98) \quad \chi(f) = K \int^f [(\xi\Lambda' - \Lambda)/\xi^4] d\xi.$$

Viewing χ as a function of u and v we see that by abuse of notation, and (5.20), we may write $\chi(u,v) = A(v)u + B(v)$.

Equation (5.27) implies

$$(5.99) \quad A''(v)u + B''(v) = 0.$$

At this point we consider two subcases:

subcase (1): $A''(v) \neq 0$;

subcase (2): $A''(v) = 0$.

For subcase (1) we see $u = -B''(v)/A''(v)$ which is impossible, so we discard this subcase.

For subcase (2) we have $A(v) = cv + d$, and $B(v) = k_1v + k_2$. Since χ is locally invertible we may write

$$(5.100) \quad \begin{aligned} f(u,v) &= \chi^{-1}(A(v)u + B(v)) \\ &= \chi^{-1}(cuv + du + k_1v + k_2). \end{aligned}$$

If $c = 0$ we can show using (5.21), (5.26), and (5.20) that

$f\Lambda' - \Lambda = 0$ which contradicts hypothesis. When $c \neq 0$, we define a new variable

$$(5.101) \quad \xi = cuv + du + k_1v + k_2.$$

Using (5.20), (5.21), and (5.26), we can show by a complicated argument that

$$(5.102) \quad f(u,v) = K\xi^{-1/4},$$

where K is a constant. From (5.102), f may be written as

$$(5.103) \quad f(u,v) = K[(\alpha u + \beta)(\gamma v + \delta)]^{-1/4},$$

where α , β , γ , and δ are constants. Defining new variables

(5.103) reduces to the case when f is a constant, which is a case already discussed.

Now we shall examine the field equations and the energy conditions for the few cases above which might admit a conformal collineation vector parallel to the comoving velocity. We shall rewrite the field equations in a more convenient fashion. Subtracting (5.11) from (5.12) we find after simplification

$$(5.104) \quad f(r_{uu} - r_{vv}) = 2r_u f_u - 2r_v f_v.$$

Equations (5.10), (5.11), and (5.12) determine μ ,

$$(5.105) \quad \mu = r_u r_v / (r^2 f^2) + (r_u f_u + r_v f_v) / (r f^3) + r_{uv} / (r f^2) + 1/r^2 - (r_{uu} + r_{vv}) / (2r f^2);$$

(5.11) plus (5.12) minus twice (5.10) determines p ,

$$(5.106) \quad p = -r_u r_v / (r^2 f^2) + r_u f_u / (r f^3) - r_{uu} / (2r f^2) - r_{uv} / (r f^2) - 1/r^2 + r_v f_v / (r f^3) - r_{vv} / (2r f^2);$$

and (5.13) determines q ,

$$(5.107) \quad q = -r_{uv} / (r f^2) + f_u f_v / f^4 - f_{vu} / f^3,$$

once f and r are chosen in a way to satisfy appropriate energy conditions.

We return to subcase (4) of case D2, we have

$$(5.106) \quad r(u,v) = \rho(au + \alpha v),$$

where $\alpha = K - (a + b)/2$. Using (5.106), and the fact that f is constant, in (5.104), we find that $a = \pm\alpha$. By a coordinate transformation we can take $a = 1$. If $\alpha = -1$, (5.104) is satisfied, but the solution is static and the shear tensor is zero. If $\alpha = 1$, equation (5.104) is satisfied and the shear tensor σ_{ij} will be nonzero. The solution is nonstatic since r is a function of $u+v$. The energy conditions will be examined later.

For subcase (1) of case E1, we have r is a nonzero constant, and from (5.82), $f(u,v) = G(u-v)e^{(C-K)(u+v)/a}$, where G is an arbitrary \mathcal{C}^2 -function. Equation (5.104) is satisfied and the shear tensor σ_{ij} is nonzero. The energy conditions will be examined later.

For subcase (2) of case E1, $r = \rho(u)$ and from equation (5.84), $f(u,v) = H(u-v)\rho(u)$, where H is an arbitrary \mathcal{C}^2 -function. Using r and f in (5.104), we find

$$(5.108) \quad 2\rho'/\rho - \rho''/\rho' = 2H'/H,$$

where the prime denotes differentiation with respect to $u-v$, and the dot denotes differentiation with respect to u . Equation (5.108) is separated in terms of the variables $\rho = u-v$ and u . Equation (5.108) may be integrated to find r and f such

that a transformation of variables leaves $r = \text{constant}$ and f as a function of ρ . Any such solutions will be static, so we discard this case. Since subcase (3) of case E1 is dual to subcase (2) of case E1, we also discard it.

The only cases which remain to apply the energy conditions to are:

(a) subcase (4) of case D2 with $a = \alpha = 1$;

(b) subcase (1) of case E1.

Consider subcase (4) of case D2 with $a = \alpha = 1$. We have f is a constant and from (5.69), $r(u,v) = \rho(u + v)$. We shall take $f = 1$ (we can always do this by a coordinate transformation). The timelike weak energy conditions with $f = 1$ and $r(u,v) = \rho(u+v)$ in (5.37), (5.38), and (5.39) produces

$$(5.109) \quad \begin{aligned} \mu &= (1 + \rho'^2)/\rho^2 \\ &\geq 0, \end{aligned}$$

$$(5.110) \quad \begin{aligned} \mu + p &= -2\rho''/\rho \\ &\geq 0, \end{aligned}$$

$$(5.111) \quad \begin{aligned} \mu + q &= (1 + \rho'^2)/\rho^2 - 2\rho''/\rho \\ &\geq 0. \end{aligned}$$

We see that any function which satisfies $\rho\rho'' \leq 0$ will satisfy the timelike weak energy conditions. For the dominant energy conditions we just have to satisfy the inequalities

$$(5.112) \quad \begin{aligned} \mu - p &= 2r_u r_v / (r^2 f^2) + 2r_{uv} / (rf^2) + 2/r^2 \\ &\geq 0, \end{aligned}$$

$$\begin{aligned}
(5.113) \quad \mu - q &= r_u r_v / (r^2 f^2) + (r_u f_u + r_v f_v) / (r f^3) \\
&\quad + 2r_{uv} / (r f^2) + 1/r^2 - (r_{uu} + r_{vv}) / (2r f^2) \\
&\quad - f_u f_v / f^4 + f_{vu} / f^3 \\
&\geq 0.
\end{aligned}$$

Using $f = 1$ and $r(u,v) = \rho(u+v)$ in (5.112) and (5.113) yields

$$\begin{aligned}
(5.114) \quad \mu - p &= 2(1 + \rho'^2) / \rho^2 + 2\rho'' / \rho \\
&\geq 0,
\end{aligned}$$

$$\begin{aligned}
(5.115) \quad \mu - q &= (1 + \rho'^2) / \rho^2 + \rho'' / \rho \\
&\geq 0.
\end{aligned}$$

The dominant energy conditions will hold if

$$(5.116) \quad -(1 + \rho'^2) \leq \rho\rho'' \leq 0.$$

For the strong energy conditions we just have to check the inequality

$$\begin{aligned}
(5.117) \quad \mu + 2p + q &= -r_u r_v / (r^2 f^2) - 1/r^2 - 3(r_{uu} + r_{vv}) / (2r f^2) \\
&\quad + 3(r_v f_v + r_u f_u) / (r f^3) + f_u f_v / f^4 - f_{vu} / f^3 \\
&\quad - 2r_{uv} / (r f^2) \\
&\geq 0.
\end{aligned}$$

Using $f = 1$ and $r(u,v) = \rho(u+v)$ in (5.117) yields

$$\begin{aligned}
(5.118) \quad \mu + 2p + q &= -(1 + \rho'^2 + 5\rho\rho'') / \rho^2 \\
&\geq 0.
\end{aligned}$$

The strong energy condition will hold if

$$(5.119) \quad \rho\rho'' \leq -(1 + \rho'^2) / 5.$$

The form of the metric for the solutions defined by the energy inequalities is

$$(5.120) \quad \bar{g} = -4du\odot dv + \rho^2(u+v)[d\theta\odot d\theta + \sin^2\theta d\phi\odot d\phi].$$

This metric is nonstatic and has an extra Killing vector given by $\xi_{(4)} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$. The Ricci scalar is given by

$$(5.121) \quad R = 2(1 + \rho'^2 + 2\rho\rho'')/\rho^2.$$

The scalar $R_{ij}R^{ij}$ is given by

$$(5.122) \quad R_{ij}R^{ij} = 4\rho''^2/\rho^2 + 2[1/\rho^2 + \rho''/\rho + \rho'^2/\rho^2]^2, \\ \geq 0.$$

The Kretschmann scalar is

$$(5.123) \quad R_{ijkl}R^{ijkl} = 8\rho''^2/\rho^2 + 4[1 + \rho'^2]^2/\rho^4 \\ \geq 0.$$

The nonzero components of the shear tensor σ_{ij} are

$$(5.124) \quad \begin{aligned} \sigma_{00} &= 2\rho'/(3\rho), \\ \sigma_{01} &= -2\rho'/(3\rho), \\ \sigma_{11} &= 2\rho'/(3\rho), \\ \sigma_{22} &= -\rho\rho'/3, \\ \sigma_{33} &= \sigma_{22}\sin^2\theta. \end{aligned}$$

The acceleration is identically zero. The expansion scalar is $\theta = -2\rho'/\rho$.

Now we consider subcase (1) of case E1. For this case we may take $r = 1$ as the simplest case of a constant r , and

$f(u,v) = G(u-v)e^{(C-K)(u+v)/a}$, where G is an arbitrary

e^2 -function and $a \neq 0$. If we define a constant

$\alpha = (C - K)/a$, then we have $f(u,v) = G(u-v)e^{\alpha(u+v)}$.

The field equation (5.102) is identically satisfied by $r = 1$

and $f(u,v) = G(u-v)e^{\alpha(u+v)}$. The timelike weak energy

conditions become

$$(5.125) \quad \mu = 1,$$

$$(5.126) \quad \mu + p = 0,$$

$$(5.127) \quad \mu + q = 1 + e^{-2\alpha(u+v)}(GG'' - G'^2)/G^4.$$

The pressures are

$$(5.128) \quad p = -1,$$

$$(5.129) \quad q = e^{-2\alpha(u+v)}(GG'' - G'^2)/G^4.$$

The dominant energy condition give

$$(5.130) \quad \mu - p = 2,$$

$$(5.131) \quad \mu - q = 1 - e^{-2\alpha(u+v)}(GG'' - G'^2)/G^4.$$

The class of functions G which satisfy dominant energy conditions is nonempty since $G(u-v) = e^{\alpha(u-v)}$ is in it. The

Ricci scalar is

$$(5.132) \quad R = 2[1 + e^{-2\alpha(u+v)}(GG'' - G'^2)/G^4] \\ = 2(\mu + q).$$

Other invariants constructed from the Riemann tensor are

$$(5.133) \quad R_{ij}R^{ij} = 2e^{-4\alpha(u+v)}(GG'' - G'^2)^2/G^8 \\ = 2q^2,$$

and

$$(5.134) \quad R_{ijkl}R^{ijkl} = 4 + 4e^{-4\alpha(u+v)}(GG'' - G'^2)^2/G^8 \\ = 4(1 + q^2).$$

The nonzero components of the shear tensor σ_{ij} are

$$(5.135) \quad \begin{aligned} \sigma_{00} &= -2\alpha Ge^{\alpha(u+v)}/3, \\ \sigma_{01} &= 2\alpha Ge^{\alpha(u+v)}/3, \\ \sigma_{11} &= -2\alpha Ge^{\alpha(u+v)}/3, \\ \sigma_{22} &= \alpha e^{-\alpha(u+v)}/(3G), \end{aligned}$$

$$\sigma_{33} = \sigma_{22} \sin^2 \theta.$$

The nonzero components of the acceleration are

$$(5.136) \quad \begin{aligned} A_0 &= G'/G, \\ A_1 &= -G'/G. \end{aligned}$$

The expansion scalar is

$$(5.137) \quad \theta = -\alpha e^{-\alpha(u+v)}/G.$$

The two cases, which lead to mathematically reasonable solutions of the field equations and which admit a timelike collineation vector parallel to the velocity, produce reducible spaces. This result is in agreement with recent unpublished (as yet) results of Hall and da Costa [144] who have shown that a collineation tensor H_{ij} with $H_{ij;k} = 0$ exists on a spacetime if and only if it is reducible. The result of Hall and da Costa shows that the usefulness of collineations will be fairly limited.

The recent advance made by Duggal in relating the existence of a timelike conformal collineation vector to the shear of the fluid velocity may be useful in future studies of shearing solutions in reducible spaces. The ability to make a mathematical hypothesis, about the symmetries to be imposed upon a problem, which leads to a nonzero shear tensor, should eliminate many *ad hoc* hypotheses which have been used to find solutions with shear.

Since the propagation of the shear tensor is closely connected with the electric part of the Weyl tensor E_{ij} ,

$$h_i^k h_j^l (\sigma_{kl;m} U^m - A_{(k;l)}) - A_i A_j + \sigma_{ik} \sigma_j^k \sigma_j^k + (2/3)\theta \sigma_{ij} + h_{ij} (-(2/3)\sigma^2 + (1/3)A^i_{;i}) + E_{ij} = 0,$$

Duggal's theorem shows that when a timelike collineation vector parallel to the velocity exists, the electric part of the Weyl tensor E_{ij} is partially determined by H_{ij} . Penrose [138] has also conjectured that there is a relation between gravitational entropy and the Weyl tensor. Since the magnitude of the Weyl tensor is a rough measure of the "clumping of matter", further studies of shearing solutions may lead to results related to the conjectures mentioned in Chapter 1.

APPENDIX A

TRANSFORMING SYSTEMS OF NOTATION IN GENERAL RELATIVITY

A frequently frustrating problem in general relativity is the nonstandardization of notation. There are hundreds of different notation systems available depending on the selection of a few parameters. It is a common piece of folklore in general relativity that it is "just a matter of fiddling to get the signs right". Any survey of the literature will reveal that many authors do not supply adequate information to decipher their results. In recent years, the influential text [125] has had a remarkable standardizing effect on this problem. The use of computers to perform calculations of evermore increasing complexity has also stimulated standardization. The system we have created here is not the most extensive possible - it would take 4 or 5 more "switches" to even approach a reasonable degree of completeness. The system presented in the following is a minimal system that will work for all the "standard calculations".

In this appendix the work of Ernst [139] and Misner *et al.* [125] is generalized and unified in a manner similar to that in Biech [140]. The advantage of this unified "translation formalism" is that computer programs can be written which are "independent" of notation conventions. In [140] it was noted that almost all systems of notation in current use in general

relativity can be classified by six parameters ϵ_i , $i = 1, \dots, 6$. These parameters take the values 1 or -1 depending on the notation convention. The five parameters ϵ_i , $i=1,2,3,5,6$ are independent and deal only with mathematical definitions. The parameter ϵ_4 is used as an auxiliary parameter in order to unify this classification with that of Ernst [139]. The three parameters defined in Misner *et al.* [125] are denoted by W_i , $i = 1,2,3$.

The sets of parameters $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ and $\{W_1, W_2, W_3\}$ are sufficient to transform any tensor expression which does not involve the Levi-Civita tensor density. Transformations of tensor expressions involving the Levi-Civita tensor density use the parameters ϵ_5, ϵ_6 . The definitions of the ϵ_i , $i = 1, \dots, 6$ are as follows:

$$\epsilon_1 \cdot \text{signature}(\mathbf{g}) = +2;$$

$$\epsilon_2 V_i R^i{}_{.jkl} = V_{j;lk} - V_{j;kl};$$

$$\epsilon_3 R_{ij} = R^m{}_{.imj};$$

$$\epsilon_4 G_{ij} = -\epsilon_1 \epsilon_2 \epsilon_3 \cdot 8\pi T_{ij};$$

$$\epsilon_5 = \begin{cases} +1 & \text{if spacetime indices run over } \{0,1,2,3\}, \\ -1 & \text{if spacetime indices run over } \{1,2,3,4\}; \end{cases}$$

$$\epsilon_6 \cdot \eta_{ijkl} = [-g]^{1/2} \text{sign}(ijkl),$$

where $g \equiv \det[g_{ij}]$ and $(ijkl)$ is a permutation in S_4 .

The parameters W_1, W_2, W_3 of [125] are defined by

$$W_1 \cdot \text{signature}(\mathbf{g}) = +2;$$

$$W_2 V_i R^i{}_{.jkl} = V_{j;kl} - V_{j;lk};$$

$$W_3 G_{ij} = 8\pi T_{ij};$$

$$W_2 W_3 R_{ij} = R^m{}_{.imj}.$$

From these definitions it is straightforward to find the relations relating these two sets of parameters:

$$\epsilon_1 = W_1;$$

$$\epsilon_2 = -W_2;$$

$$\epsilon_3 = W_2 W_3;$$

$$\epsilon_4 = W_1.$$

To transform tensor equations in one notation system (barred system) to another system (unbarred) it is necessary to calculate the six transformation parameters α_i , $i = 1, \dots, 6$ by the relation

$$\alpha_i = \epsilon_i \bar{\epsilon}_i \quad i = 1, \dots, 6.$$

With the parameters α_i we can convert tensor equations from one notation system to the other by the use of the following equations:

$$\bar{g}_{ij} = \alpha_1 g_{ij},$$

$$\bar{g}^{ij} = \alpha_1 g^{ij},$$

$$\bar{g} = g, \quad (g \text{ is the determinant of } g_{ij})$$

$$\bar{\eta}_{ij} = \alpha_1 \eta_{ij},$$

$$\bar{\eta}^{ij} = \alpha_1 \eta^{ij},$$

$$\overline{[ij,k]} = \alpha_1 [ij,k],$$

$$\overline{\Gamma_{ij}^k} = \Gamma_{ij}^k,$$

$$\overline{R_{ijkl}} = \alpha_1 \alpha_2 R_{ijkl}, \quad (\text{insert one factor } \alpha_1 \text{ for each index to be raised})$$

$$\overline{R_{ij}} = \alpha_2 \alpha_3 R_{ij}, \quad (\text{insert one factor } \alpha_1 \text{ for each index to be raised})$$

$$\bar{R} = \alpha_1 \alpha_2 \alpha_3 R, \text{ (R is the curvature scalar)}$$

$$\overline{G_{ij}} = \alpha_2 \alpha_3 G_{ij}, \text{ (insert one factor } \alpha_1 \text{ for each index to be raised)}$$

The conformal curvature tensor C_{ijkl} and the projective curvature tensor W_{ijkl} transform in the same way as the Riemann curvature tensor R_{ijkl} under changes in ϵ_1 and ϵ_2 but it misbehaves under changes in ϵ_3 .

For equations which involve the duals of tensor we use the transformation format

$$\overline{*(T \dots \dots)} = \alpha_5 \alpha_6 *(T \dots \dots).$$

All tensor expressions not involving the conformal curvature tensor or the projective curvature tensor should be transformable from one notation system to another by the above rules. The transformation scheme just outlined can also be extended to the tetrad equivalents of the above expressions.

For expressions employing the Cartan differential algebra we must add another indicator for the possible choices of the wedge product. Unfortunately since the different choices of wedge product are related by the "cocycle identity" [141] the indicator does not take just two values. We will omit further discussion on this and refer the reader to the literature [141,142].

APPENDIX B

SYMBOLS AND NOTATION

The general naming conventions and usage of symbols employed in this thesis are listed below. Lower case Greek indices take their values in a specified index set. They will always be used as labels, usually linking coordinate functions to their associated coordinate chart. They will never be used as tensorial indices. Lower case Roman indices have range $\{0,1,2,3\}$ and are used to indicate spacetime components in a holonomic basis. Upper case Roman indices have range $\{0,1,2,3\}$ and are used to indicate tetrad components for an anholonomic basis. We also follow the rules:

(a) The Einstein summation convention holds for the following index types:

(i) lower case Roman indices have range $\{0,1,2,3\}$;

(ii) upper case Roman indices have range $\{0,1,2,3\}$;

(b) covariant derivatives are indicated by a semi-colon: $X_{i;k}$;

(c) partial derivatives are indicated by a comma: $X_{i,k}$.

The following list describes most of the symbols used in this work:

A_i	acceleration 4-vector
β	velocity parameter (v/c)
\flat	abstract index lowering operator (vectors to forms)

#	abstract index raising operator (forms to vectors)
$\mathcal{C}^0(I;M)$	space of smooth curves in M
$\mathcal{C}^n(M;N)$	space of n-fold differentiable mappings from M to N
$\mathcal{C}^\infty(M;N)$	space of smooth mappings from M to N
C_{ijkl}	conformal curvature tensor
D_X	covariant derivative with respect to X
δ_{ij}	Kronecker tensor
∂_{α_i}	i-th coordinate basis vector for the chart $(U_\alpha, \varphi_\alpha)$
e_i	frame vectors $i = 0, 1, 2, 3$.
ϵ_i	indicator of e_i i.e. $\epsilon_i e_i \cdot e_i = 1$
f^*	pull-back by f
f_*	push-forward by f when f is a local diffeomorphism
g_{ij}	components of the metric tensor in a holonomic basis
g	$\det g_{ij}$, the determinant of g_{ij}
G_{ij}	components of \mathbf{G} in a holonomic basis
Γ^i_{jk}	Christoffel symbol of the second kind
h_{ij}	projection tensor associated with g_{ij} and u_i
H_{ij}	conformal collineation tensor
η	coefficient of shear viscosity
η_{ijkl}	Levi-Civita twisted 4-form
$i_{\alpha\beta}$	coordinate transition map from (U_β, φ_β) to $(U_\alpha, \varphi_\alpha)$
i_M	identity map on a manifold M
$I^+(p)$	chronological future set of p
$I^-(p)$	chronological past set of p
$\text{Ind}(g)$	index of the metric tensor g (1)

J_N	inclusion map for a submanifold N
$J^+(p)$	causal future set of p
$J^-(p)$	causal past set of p
$Lor(M)$	space of Lorentz metrics on M
\mathcal{L}_X	Lie derivative with respect to X
M	space-time manifold
$M \times_f H$	type 1 Lorentzian warped product of M, H
M_0	Minkowski space-time
μ	energy density
$N(g)$	the nullity of the metric tensor (0)
$N_g(\delta)$	δ -neighbourhood of g in fine \mathcal{E}^r -topology on $Lor(M)$
p	thermodynamic pressure
p_i	momentum 4-vector
R	scalar curvature
R_{ijkl}	components of Riemann curvature in a holonomic basis
R_{ij}	components of Ricci curvature in a holonomic basis
$sig(g)$	the signature of the metric tensor g (+2)
σ	shear scalar
σ_{ij}	shear tensor
T_{ij}	stress-energy-momentum tensor
$T_s^r(V)$	the space of type (r,s) tensors on the vector space V
TM	tangent bundle of M
T_p^*M	cotangent space of M at p
T_pM	tangent space of M at p
θ	spatial expansion rate

θ_{ij} rate of strain tensor
 U_i unit velocity 4-vector
 ζ coefficient of bulk viscosity
 ω_{ij} vorticity tensor
[X,Y] bracket of vector fields X, Y

APPENDIX C

CALCULATIONS IN NN-COORDINATES

This appendix contains some of the calculations which were not used directly in the text. They are all computed in NN-coordinates with metric given by (4.1). Unless otherwise noted all velocity dependent calculations used a generalized comoving velocity as defined in Chapter 4. The geodesic equations are

$$\begin{aligned} rr_v \phi'^2 \sin^2 \theta / (2f^2) + rr_v \theta'^2 / (2f^2) + 2u'^2 f_u / f + u'' &= 0, \\ rr_u \phi'^2 \sin^2 \theta / (2f^2) + rr_u \theta'^2 / (2f^2) + 2v'^2 f_v / f + v'' &= 0, \\ 2r_u \theta' u' / r + 2r_v \theta' v' / r - \phi'^2 \cos \theta \sin \theta + \theta'' &= 0, \\ 2r_u \phi' u' / r + 2r_v \phi' v' / r + 2\theta' \phi' \cot \theta + \phi'' &= 0, \end{aligned}$$

where the prime indicate differentiation with respect to an affine parameter. The conservation equations for a perfect fluid with a generalized comoving velocity in NN-coordinates may be written

$$\begin{aligned} 2f^2 (T^{0i}_{;i} + T^{1i}_{;i}) &= 2\mu(r_u + r_v) / r + 2p(r_u + r_v) / r \\ &\quad + \mu(f_u + f_v) / f + p(f_u + f_v) / f + \mu_u + \mu_v \\ &= 0, \\ 2f^2 (T^{0i}_{;i} - T^{1i}_{;i}) &= \mu(f_u - f_v) / f + p(f_u - f_v) / f + p_u - p_v \\ &= 0. \end{aligned}$$

We now calculate equations for a spacelike collineation vector in NN-coordinates. We assume that the velocity is generalized comoving and the collineation vector is parallel to

the spacelike anisotropy vector S . The direction of anisotropy is assumed orthogonal to the comoving velocity thus has nonzero components $S_0 = f$, $S_1 = -f$. The spacelike conformal collineation vector ξ has nonzero components $\xi_0 = f\lambda$, $\xi_1 = -f\lambda$, where λ is the scaling factor. The equations for a conformal collineation with H_{ij} as collineation tensor are

$$H_{00} \equiv 2f\lambda_u - 2\lambda f_u,$$

$$H_{01} \equiv -f\lambda_u + f\lambda_v - \lambda f_u + \lambda f_v + 4f^2\psi,$$

$$H_{11} \equiv -2f\lambda_v + 2\lambda f_v,$$

$$H_{22} \equiv r\lambda r_u/f - r\lambda r_v/f - 2r^2\psi,$$

$$H_{33} \equiv r\lambda \sin^2\theta r_u/f - r\lambda \sin^2\theta r_v/f - 2r^2\psi \sin^2\theta,$$

where ψ is the conformal factor. Note we set $H_{ij;k} = 0$ as the condition for the existence of a conformal collineation vector.

$$H_{00;0} = 2f\lambda_{uu} - 2\lambda f_{uu} + 8\lambda(f_u)^2/f - 8f_u\lambda_u,$$

$$H_{00;1} = 2f\lambda_{uv} - 2\lambda f_{uv} - 2f_u\lambda_v + 2f_v\lambda_u,$$

$$H_{01;0} = -f\lambda_{uu} + f\lambda_{vu} - \lambda f_{uu} + \lambda f_{vu} + 2\lambda(f_u)^2/f - 2\lambda f_u f_v/f \\ + 4f^2\psi_u - f_u\lambda_v + f_v\lambda_u,$$

$$H_{01;1} = -f\lambda_{uv} + f\lambda_{vv} - \lambda f_{uv} + \lambda f_{vv} - 2\lambda(f_v)^2/f + 2\lambda f_u f_v/f \\ + 4f^2\psi_v - f_u\lambda_v + f_v\lambda_u,$$

$$H_{02;2} = r\lambda r_u f_u/(2f^2) - r\lambda r_u f_v/(2f^2) + r\lambda r_v f_u/f^2 + r r_u \lambda_u/(2f) \\ - r r_u \lambda_v/(2f) - r r_v \lambda_u/f - \lambda(r_u)^2/f + \lambda r_u r_v/f,$$

$$H_{03;3} = r\lambda \sin^2\theta r_u f_u/(2f^2) - r\lambda \sin^2\theta r_u f_v/(2f^2) + r\lambda \sin^2\theta r_v f_u/f^2 \\ + r \sin^2\theta r_u \lambda_u/(2f) - r \sin^2\theta r_u \lambda_v/2f - r \sin^2\theta r_v \lambda_u/f \\ - \lambda \sin^2\theta (r_u)^2/f + \lambda \sin^2\theta r_u r_v/f,$$

$$H_{11;0} = -2f\lambda_{vu} + 2\lambda f_{vu} - 2f_u\lambda_v + 2f_v\lambda_u,$$

$$H_{11;1} = -2f\lambda_{vv} + 2\lambda f_{vv} - 8\lambda(f_v)^2/f + 8f_v\lambda_v,$$

$$H_{12;2} = -r\lambda r_u f_v / f^2 + r\lambda r_v f_u / (2f^2) - r\lambda r_v f_v / (2f^2) + r r_u \lambda_v / f \\ + r r_v \lambda_u / (2f) - r r_v \lambda_v / (2f) + \lambda (r_v)^2 / f - \lambda r_u r_v / f,$$

$$H_{13;3} = -r\lambda \sin^2 \theta r_u f_v / f^2 + r\lambda \sin^2 \theta r_v f_u / (2f^2) - r\lambda \sin^2 \theta r_v f_v / (2f^2) \\ + r \sin^2 \theta r_u \lambda_v / f + r \sin^2 \theta r_v \lambda_u / (2f) - r \sin^2 \theta r_v \lambda_v / (2f) \\ + \lambda \sin^2 \theta (r_v)^2 / f - \lambda \sin^2 \theta r_u r_v / f,$$

$$H_{22;0} = -r\lambda r_u f_u / f^2 + r\lambda r_v f_u / f^2 + r\lambda r_{uu} / f - r\lambda r_{vu} / f + r r_u \lambda_u / f \\ - r r_v \lambda_u / f - 2r^2 \psi_u - \lambda (r_u)^2 / f + \lambda r_u r_v / f,$$

$$H_{22;1} = -r\lambda r_u f_v / f^2 + r\lambda r_v f_v / f^2 + r\lambda r_{uv} / f - r\lambda r_{vv} / f + r r_u \lambda_v / f \\ - r r_v \lambda_v / f - 2r^2 \psi_v + \lambda (r_v)^2 / f - \lambda r_u r_v / f,$$

$$H_{33;0} = -r\lambda \sin^2 \theta r_u f_u / f^2 + r\lambda \sin^2 \theta r_v f_u / f^2 + r\lambda \sin^2 \theta r_{uu} / f \\ - r\lambda \sin^2 \theta r_{vu} / f + r \sin^2 \theta r_u \lambda_u / f - r \sin^2 \theta r_v \lambda_u / f \\ - 2r^2 \sin^2 \theta \psi_u - \lambda \sin^2 \theta (r_u)^2 / f + \lambda \sin^2 \theta r_u r_v / f,$$

$$H_{33;1} = -r\lambda \sin^2 \theta r_u f_v / f^2 + r\lambda \sin^2 \theta r_v f_v / f^2 + r\lambda \sin^2 \theta r_{uv} / f \\ - r\lambda \sin^2 \theta r_{vv} / f + r \sin^2 \theta r_u \lambda_v / f - r \sin^2 \theta r_v \lambda_v / f \\ - 2r^2 \sin^2 \theta \psi_v + \lambda \sin^2 \theta (r_v)^2 / f - \lambda \sin^2 \theta r_u r_v / f.$$

We set $\psi_{;ij} = 0$ as the condition for a special conformal collineation vector.

$$\psi_{;00} = -2f_u \psi_u / f + \psi_{uu},$$

$$\psi_{;01} = \psi_{uv},$$

$$\psi_{;10} = \psi_{vu},$$

$$\psi_{;11} = -2f_v \psi_v / f + \psi_{vv},$$

$$\psi_{;22} = -r r_u \psi_v / (2f^2) - r r_v \psi_u / (2f^2),$$

$$\psi_{;33} = -r \sin^2 \theta r_u \psi_v / (2f^2) - r \sin^2 \theta r_v \psi_u / (2f^2),$$

Now we write the equations for a "tilting" timelike

collineation vector. We assume ξ has nonzero components given by $\zeta_0 = a$, $\xi_1 = b$.

The tilting conformal collineation tensor equations are

$$H_{00} = -4af_u/f + 2a_u,$$

$$H_{01} = 4f^2\psi + a_v + b_u,$$

$$H_{11} = -4bf_v/f + 2b_v,$$

$$H_{22} = -arr_v/f^2 - brr_u/f^2 - 2r^2\psi,$$

$$H_{33} = -arsin^2\theta r_v/f^2 - brsin^2\theta r_u/f^2 - 2r^2\psi sin^2\theta,$$

We set $H_{ij;k} = 0$ as the condition for the existence of a conformal timelike tilting collineation vector.

$$H_{00;0} = 20 a(f_u)^2/f^2 - 4af_{uu}/f - 12a_u f_u/f + 2a_{uu},$$

$$H_{00;1} = 4 a f_u f_v/f^2 - 4af_{uv}/f - 4a_v f_u/f + 2a_{uv},$$

$$H_{01;0} = -2a_v f_u/f - 2b_u f_u/f + 4f^2\psi_u + a_{vu} + b_{uu},$$

$$H_{01;1} = -2a_v f_v/f - 2b_u f_v/f + 4f^2\psi_v + a_{vv} + b_{uv},$$

$$H_{02;2} = 2arr_v f_u/f^3 + ar_u r_v/f^2 + b(r_u)^2/f^2 - ra_u r_v/f^2 \\ - ra_v r_u/(2f^2) - rb_u r_u/(2f^2),$$

$$H_{03;3} = 2arsin^2\theta r_v f_u/f^3 + asin^2\theta r_u r_v/f^2 + bsin^2\theta (r_u)^2/f^2 \\ - rsin^2\theta a_u r_v/f^2 - rsin^2\theta a_v r_u/(2f^2) \\ - rsin^2\theta b_u r_u/(2f^2),$$

$$H_{11;0} = 4bf_u f_v/f^2 - 4bf_{vu}/f - 4b_u f_v/f + 2b_{vu},$$

$$H_{11;1} = 20 b(f_v)^2/f^2 - 4bf_{vv}/f - 12b_v f_v/f + 2b_{vv},$$

$$H_{12;2} = a(r_v)^2/f^2 + 2brr_u f_v/f^3 + br_u r_v/f^2 - ra_v r_v/(2f^2) \\ - rb_u r_v/(2f^2) - rb_v r_u/f^2,$$

$$H_{13;3} = asin^2\theta (r_v)^2/f^2 + 2brsin^2\theta r_u f_v/f^3 + bsin^2\theta r_u r_v/f^2 \\ - rsin^2\theta a_v r_v/(2f^2) - rsin^2\theta b_u r_v/(2f^2)$$

$$\begin{aligned}
& - r \sin^2 \theta b_v r_u / f^2, \\
H_{22;0} &= 2arr_v f_u / f^3 - arr_{vu} / f^2 + ar_u r_v / f^2 + 2brr_u f_u / f^3 \\
& - brr_{uu} / f^2 + b(r_u)^2 / f^2 - ra_u r_v / f^2 - rb_u r_u / f^2 \\
& - 2r^2 \psi_u, \\
H_{22;1} &= 2arr_v f_v / f^3 - arr_{vv} / f^2 + a(r_v)^2 / f^2 + 2brr_u f_v / f^3 \\
& - brr_{uv} / f^2 + br_u r_v / f^2 - ra_v r_v / f^2 - rb_v r_u / f^2 - 2r^2 \psi_v, \\
H_{33;0} &= 2arsin^2 \theta r_v f_u / f^3 - arsin^2 \theta r_{vu} / f^2 + asin^2 \theta r_u r_v / f^2 \\
& + 2brsin^2 \theta r_u f_u / f^3 - brsin^2 \theta r_{uu} / f^2 + bsin^2 \theta (r_u)^2 / f^2 \\
& - r \sin^2 \theta a_u r_v / f^2 - r \sin^2 \theta b_u r_u / f^2 - 2r^2 \sin^2 \theta \psi_u, \\
H_{33;1} &= 2arsin^2 \theta r_v f_v / f^3 - arsin^2 \theta r_{vv} / f^2 + asin^2 \theta (r_v)^2 / f^2 \\
& + 2brsin^2 \theta r_u f_v / f^3 - brsin^2 \theta r_{uv} / f^2 + bsin^2 \theta r_u r_v / f^2 \\
& - r \sin^2 \theta a_v r_v / f^2 - r \sin^2 \theta b_v r_u / f^2 - 2r^2 \sin^2 \theta \psi_v.
\end{aligned}$$

The nonzero components of a timelike conformal collineation vector parallel to the generalized comoving velocity are $\xi_0 = f\lambda$, $\xi_1 = f\lambda$. The equations for a timelike conformal collineation vector parallel to the comoving velocity are

$$\begin{aligned}
H_{00} &= 2f\lambda_u - 2\lambda f_u, \\
H_{01} &= f\lambda_u + f\lambda_v + \lambda f_u + \lambda f_v + 4f^2 \psi, \\
H_{11} &= 2f\lambda_v - 2\lambda f_v, \\
H_{22} &= -r\lambda r_u / f - r\lambda r_v / f - 2r^2 \psi, \\
H_{33} &= -r\lambda \sin^2 \theta r_u / f - r\lambda \sin^2 \theta r_v / f - 2r^2 \psi \sin^2 \theta,
\end{aligned}$$

We set $H_{ij;k} = 0$ as the condition for the existence of a conformal collineation vector.

$$H_{00;0} = 2f\lambda_{uu} - 2\lambda f_{uu} + 8\lambda(f_u)^2 / f - 8f_u \lambda_u,$$

$$\begin{aligned}
H_{00;1} &= 2f\lambda_{uv} - 2\lambda f_{uv} - 2f_u\lambda_v + 2f_v\lambda_u, \\
H_{01;0} &= f\lambda_{uu} + f\lambda_{vu} + \lambda f_{uu} + \lambda f_{vu} - 2\lambda(f_u)^2/f - 2\lambda f_u f_v/f \\
&\quad + 4f^2\psi_u - f_u\lambda_v + f_v\lambda_u, \\
H_{01;1} &= f\lambda_{uv} + f\lambda_{vv} + \lambda f_{uv} + \lambda f_{vv} - 2\lambda(f_v)^2/f - 2\lambda f_u f_v/f \\
&\quad + 4f^2\psi_v + f_u\lambda_v - f_v\lambda_u, \\
H_{02;2} &= -r\lambda r_u f_u/(2f^2) - r\lambda r_u f_v/(2f^2) + r\lambda r_v f_u/f^2 - r r_u\lambda_u/(2f) \\
&\quad - r r_u\lambda_v/(2f) - r r_v\lambda_u/f + \lambda(r_u)^2/f + \lambda r_u r_v/f, \\
H_{03;3} &= -r\lambda \sin^2\theta r_u f_u/(2f^2) - r\lambda \sin^2\theta r_u f_v/(2f^2) \\
&\quad + r\lambda \sin^2\theta r_v f_u/f^2 - r \sin^2\theta r_u\lambda_u/(2f) - r \sin^2\theta r_u\lambda_v/(2f) \\
&\quad - r \sin^2\theta r_v\lambda_u/f + \lambda \sin^2\theta (r_u)^2/f + \lambda \sin^2\theta r_u r_v/f, \\
H_{11;0} &= 2f\lambda_{vu} - 2\lambda f_{vu} + 2f_u\lambda_v - 2f_v\lambda_u, \\
H_{11;1} &= 2f\lambda_{vv} - 2\lambda f_{vv} + 8\lambda(f_v)^2/f - 8f_v\lambda_v, \\
H_{12;2} &= r\lambda r_u f_v/f^2 - r\lambda r_v f_u/(2f^2) - r\lambda r_v f_v/(2f^2) - r r_u\lambda_v/f \\
&\quad - r r_v\lambda_u/(2f) - r r_v\lambda_v/(2f) + \lambda(r_v)^2/f + \lambda r_u r_v/f, \\
H_{13;3} &= r\lambda \sin^2\theta r_u f_v/f^2 - r\lambda \sin^2\theta r_v f_u/(2f^2) - r\lambda \sin^2\theta r_v f_v/(2f^2) \\
&\quad - r \sin^2\theta r_u\lambda_v/f - r \sin^2\theta r_v\lambda_u/(2f) - r \sin^2\theta r_v\lambda_v/(2f) \\
&\quad + \lambda \sin^2\theta (r_v)^2/f + \lambda \sin^2\theta r_u r_v/f, \\
H_{22;0} &= r\lambda r_u f_u/f^2 + r\lambda r_v f_u/f^2 - r\lambda r_{uu}/f - r\lambda r_{vu}/f - r r_u\lambda_u/f \\
&\quad - r r_v\lambda_u/f - 2r^2\psi_u + \lambda(r_u)^2/f + \lambda r_u r_v/f, \\
H_{22;1} &= r\lambda r_u f_v/f^2 + r\lambda r_v f_v/f^2 - r\lambda r_{uv}/f - r\lambda r_{vv}/f - r r_u\lambda_v/f \\
&\quad - r r_v\lambda_v/f - 2r^2\psi_v + \lambda(r_v)^2/f + \lambda r_u r_v/f, \\
H_{33;0} &= r\lambda \sin^2\theta r_u f_u/f^2 + r\lambda \sin^2\theta r_v f_u/f^2 - r\lambda \sin^2\theta r_{uu}/f \\
&\quad - r\lambda \sin^2\theta r_{vu}/f - r \sin^2\theta r_u\lambda_u/f - r \sin^2\theta r_v\lambda_u/f \\
&\quad - 2r^2 \sin^2\theta \psi_u + \lambda \sin^2\theta (r_u)^2/f + \lambda \sin^2\theta r_u r_v/f, \\
H_{33;1} &= r\lambda \sin^2\theta r_u f_v/f^2 + r\lambda \sin^2\theta r_v f_v/f^2 - r\lambda \sin^2\theta r_{uv}/f
\end{aligned}$$

$$\begin{aligned}
& - r\lambda \sin^2 \theta_{vv} / f - r \sin^2 \theta_{uv} \lambda / f - r \sin^2 \theta_v \lambda / f \\
& - 2r^2 \sin^2 \theta_{\psi} + \lambda \sin^2 \theta (r_v)^2 / f + \lambda \sin^2 \theta_{ur} r_v / f.
\end{aligned}$$

APPENDIX D

A MUTENSOR PROGRAM FOR A PERFECT FLUID IN NN-COORDINATES

The following is a MuTensor script file listing which computes many of the quantities used for General Relativity. The following program listing was used, together with variants, to produce many of the calculations used in this thesis. The programs were developed for the MuTensor computer algebra system version 3.75 [143].

%COMMENT

TITLE: PERFSTAN.NNC

STATUS: working LAST UPDATE: 16:50/21/3/88

NOTATION: W=(+,+,+) E=(+,-,+,+,+,+)

This muTensor script computes the field equations in double null coordinates for a spherically symmetric metric with a perfect fluid stress-energy tensor the velocity vector is assumed to be "COMOVING."

A = acceleration vector

G = Einstein tensor (from RIC) W(3)=-1

M = conservation vector from twice contracted Bianchi identity

NG = Einstein tensor (from NRIC) W(3)=+1

NRIC = Ricci tensor (R contracted on 1-3 positions)

P = projection tensor onto 3-space orthogonal to U

R = Riemann tensor E(2)=-1 W(2)=+1

RIC = Ricci tensor (R contracted on 1-4 positions) E(3)=-1

T = stress-energy tensor of type (0,2)
 TUP = stress-energy tensor of type (2,0)
 U = 4-velocity of fluid
 V = relative acceleration tensor for 3-space orthogonal to U
 W = vorticity tensor of the relative velocity in 3-space
 orthogonal to U

VFE = field equation tensor (identically zero) (W(3)=+1)
 E(4)=+1

f = metric component function

m = density of rest mass

p = pressure

r = metric component function

Th = expansion tensor

Units are chosen so that $c=h=kappa*G=1$.

%

RECLAIM();

DENNUM:NUMNUM:DENDEN:EXPBAS:BASEXP:30\$

NUMDEN:0\$

TRGSQ:4\$

COORDS : '(u v th ph)\$

ds: $-4*f^2*d(u)*d(v) + r^2*d(th)^2 + r^2*SIN(th)^2*d(ph)^2$;

DEPENDS (f(u,v))\$

DEPENDS (r(u,v))\$

METRIC(ds);

%COMMENT

We compute the Christoffel symbols and read in the velocity

vector so that we can later compute the stress-energy tensor after we compute the shear tensor etc.

%

CHRISTOFFEL1()\$

CHRISTOFFEL2()\$

MKTNSR ('U, '(-1), '())\$

U[0] :: f\$

U[1] :: f\$

U[2] :: 0\$

U[3] :: 0\$

%COMMENT

Here we are going to compute the acceleration 4-vector

%

MKTNSR ('A, '(-1), '())\$

A[a] :: U[a,!!b]*U[^b]\$

%COMMENT Here we are going to define the projection tensor related to metric and the 4-velocity.

%

MKTNSR ('P, '(-1 -1), '((1 1 2)))\$

P[a,b] :: g[a,b] + U[a]*U[b]\$

%COMMENT Here we define the relative acceleration tensor which is defined to be the covariant derivative of the 4-velocity corrected for acceleration orthogonal to the 3-space defined by the 4-velocity.

%

MKTNSR ('V, '(-1 -1), '((1 1 2)))\$

```
V[a,b] :: U[a,!!b] + A[a]*U[b]$
```

```
%COMMENT
```

```
Here we decompose the covariant derivative of the 4-velocity  
into its symmetric and anti-symmetric parts.
```

```
%
```

```
MKTNSR ('Th, '(-1 -1), '((1 1 2)))$
```

```
MKTNSR ('W, '(-1 -1), '((-1 1 2)))$
```

```
Th[a,b] :: V[[a,b]]$
```

```
W[a,b] :: V[{a,b}]$
```

```
%COMMENT
```

```
Here we are going to compute the shear rate tensor
```

```
We also compute the expansion "e".
```

```
%
```

```
MKTNSR ('E, '(-1 -1), '((1 1 2)))$
```

```
e :: Th[^a,a];
```

```
DEPENDS (e(u,v))$
```

```
E[a,b] :: Th[a,b] - (e/3)*P[a,b];
```

```
%COMMENT
```

```
Here we compute the stress-energy momentum tensor.
```

```
%
```

```
MKTNSR ('T, '(-1 -1), '((1 1 2)))$
```

```
MKTNSR ('TUP, '(1 1), '((1 1 2)))$
```

```
T[a,b] :: (m + p)*U[a]*U[b] + p*g[a,b]$
```

```
DEPENDS (p(u,v))$
```

```
DEPENDS (m(u,v))$
```

```
SHIFT (T[^a,b]);
```



```

MKTNSR ('M, '(1), '())$
TUP[a,b] :: T[^a,^b]$
M[a] :: TUP[a,b,!!b]$
RIEMANN();
WEYL();
RICCI();
MKTNSR ('NRIC, '(-1 -1), '((1 1 2)))$
NRIC[a,b] :: -RIC[a,b]$
%COMMENT
Here we program around a bug in the MuTensor program. It does
not compute the Ricci scalar or the Einstein tensor properly.
%
RR :: NRIC[^a,a];
DEPENDS (RR(u,v))$
MKTNSR ('NG, '(-1 -1), '((1 1 2)))$
NG[a,b] :: NRIC[a,b] - RR*g[a,b]/2$
MKTNSR ('N, '(1), '())$
MKTNSR ('NUP, '(1 1), '((1 1 2)))$
NUP[a,b] :: NG[^a,^b]$
N[a] :: NUP[a,b,!!b]$
%COMMENT
The vector N should be identically 0 so that  $G^{ij}_{;j} = 0$ . For the
code above this identity is true.
%
N[];
MKTNSR ('VFE, '(-1 -1), '((1 1 2)))$

```

VFE[a,b] :: NG[a,b] - T[a,b]\$

SHIFT (VFE[^a,b]);

RECLAIM ();

Several lines for printing output omitted here.

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