HYPERIDENTITY AND HYPERVARIETY RESULTS

FOR VARIETIES OF SEMIGROUPS

by

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Hyperidentity and Hypervariety Results for Varieties of Semigroups

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ABSTRACT

A hypervariety is a class of varieties closed under the formation of equivalent, product, reduct, and sub-varieties. Hyperidentities are identities which define hypervarieties, in the same way that ordinary identities define varieties. This thesis explores the concepts of hypervariety and hyperidentity in relation to varieties of semigroups.

Chapters 1 and 2 provide an introduction and background, describing the relationships between hyperidentities, hypervarieties, and varieties of clones. Chapter 3 gives the semigroup-theoretic results needed for later chapters; these are mainly of the form of an equational description of the joins of various equationally-defined varieties of semigroups.

Chapter 4 begins the study of hyperidentities satisfied by various varieties of semigroups, and the properties of two operators, the hypervariety and the closure operators, on varieties of semigroups. For the lattice of all varieties of bands, we identify which varieties are closed, and produce hyperidentities to distinguish the corresponding hypervarieties, giving a countably infinite chain of hypervarieties. Similar results are obtained for the varieties of k-nilpotent semigroups, and for joins of these varieties with varieties of bands.

The final two chapters consider the commutative varieties satisfying identities of the form $x^n = x^{n+m}$, and other related varieties. We produce several families of hyperidentities satisfied by such varieties; in particular, we introduce a technique

for producing a hyperidentity related to any given identity for semigroups. We also investigate length restrictions on what types of hyperidentities such varieties can satisfy. These results combine with the join results of Chapter 3 to provide information about the action of the hypervariety and closure operators on such varieties.

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Chapter 1

Introduction

Hypervarieties and hyperidentities have been defined by Taylor in [15]. A hypervariety is a class of varieties closed under the formation of equivalent, product, reduct, and sub-varieties. Hyperidentities are used to define hypervarieties, in the same way that ordinary identities define varieties. In this thesis we explore the concepts of hypervariety and hyperidentity as they relate to varieties of semigroups.

We begin in Chapter 2 with a general study of hypervarieties and hyperidentities. Section 1 introduces the concept of a clone, a particular type of heterogeneous algebra associated with any variety of algebras. Several theorems then set up a correspondence between varieties of clones and hypervarieties. This makes it possible to describe hypervarieties equationally, in terms of identities for varieties of clones. However such identities are generally very complicated, and we turn to hyperidentities as an alternate approach. Section 2 presents a theorem proved by Taylor in [15] that relates hypervarieties and hyperidentities, along with some terminology for and examples of hyperidentities.

In Section 3 of Chapter 2, we introduce the operator $\mathcal{H}$, which takes any variety V to the hypervariety $\mathcal{H}$(V) it generates, and look at the properties of $\mathcal{H}$ when it is restricted to the lattice of varieties of semigroups. A Galois correspondence is set up between varieties of semigroups and sets of hyperidentities, leading to the definition of a closed variety of semigroups and of a closure operator on varieties of semigroups. This begins the main work of the thesis, the study of hyperidentities satisfied by various varieties of semigroups, and of the closure and hypervariety operators.

Chapter 3 presents the semigroup-theoretic results to be used in identifying the closures of various varieties. The first section gives a brief overview of the structure of the lattice of all varieties of semigroups, establishing the notation to be used later. It concludes with a description of the technique to be used in obtaining closure results. This technique depends heavily on obtaining an equational description of the join U $\vee$ V for various equationally-defined varieties U and V. Thus the remainder of this chapter presents the necessary join results: in Sections 2, 3, and 4 we look at joins of various varieties V with the varieties of rectangular bands, of zero semigroups, and of nilpotent semigroups respectively. Both syntactic and structural proofs are used, and in many cases the identities obtained for the joins have been suggested by the hyperidentities to be described in subsequent chapters.

In Chapter 4 we begin looking at hyperidentities satisfied by various varieties of semigroups. The lattice of varieties of bands is examined in Section 1. Because its structure is completely known (see [2], [6], or [7]), and because the properties of idempotence and duality are so strong, we are able to obtain complete hyperidentity and closure results for the varieties in this lattice. We obtain a countably infinite chain of hypervarieties of the form $\mathcal{H}(V)$ where V is a self-dual variety of bands, with corresponding bases of hyperidentities to define them. In Section 2 we consider the varieties of nilpotent semigroups, again with complete results; and in the final section we combine the band and nilpotent results to describe the closure of any variety of the form $U \vee V$, for U a variety of bands and V a variety of nilpotent semigroups.

In Chapters 5 and 6 we consider the varieties A_m and $A_{n,m}$, consisting of commutative semigroups satisfying $x^m y = y$ and $x^n = x^{n+m}$ respectively. We begin with some general remarks about the construction of hyperidentities satisfied by such varieties. We then present in Section 5.2 a technique which allows us to take any identity and construct a hyperidentity in some sense based on the given identity. As well as some hyperidentities for the varieties A_m and $A_{n,m}$, this technique yields several interesting results about the closure and hypervariety operators as they apply to varieties of commutative semigroups and monoids. Section 5.3 explores a different type of hyperidentity for A_m and $A_{n,m}$. Here we are led to consider two length parameters depending on m which seem to determine "how long" a hyperidentity satisfied by one of these varieties has to be. The final section of Chapter 5 explores further this idea of length restriction, with several lemmas giving conditions which a hyperidentity must meet in order to be satisfied by certain varieties $A_{n,m}$.

In Chapter 6 we combine the hyperidentity information of Chapter 5 with the join results of Section 3.4 to discuss the closure of A_m, $A_{n,m}$, and several related varieties. The results are divided into two cases, in terms of a length parameter $t(m)$ discussed in Chapter 5. In the first case, when n is greater than $t(m)$, we obtain complete closure results for $A_{n,m}$ and related varieties. In the second case, when n is less than or equal to $t(m)$, we obtain results only for certain values of m, determined again in terms of restrictions on the value of $t(m)$. We conclude with a conjecture about the closure of $A_{n,m}$ for the remaining values of n and m.

Chapter 2

Hypervarieties and Hyperidentities

This chapter introduces the concepts of hypervariety and hyperidentity, which will be studied throughout this thesis. The first section defines and explores the connections between varieties of clones and hypervarieties. Since the equational description of these classes quickly becomes very complex, we turn in Section 2 to an equivalent approach, that of hyperidentities. We then define a closure operator on varieties of semigroups, in terms of the hyperidentities they satisfy.

2.1. Clones and Hypervarieties

In this section we introduce the concepts of clone and hypervariety. These structures are considered by W. Taylor in [14] and W.D.Neumann in [9], and the reader is referred to these papers for a more detailed discussion, including proofs of the results stated here.

We begin with the definition of a heterogeneous algebra, assuming that the reader is familiar with ordinary (homogeneous) algebras and varieties, as discussed in [3], for instance.

In a heterogeneous algebra we allow more than one sort of object, so that our "universe" is a family of non-empty sets $(A_i: i \in I)$, indexed by some set I. Each fundamental operation then includes not only an arity, but also information about which sets it acts on. For example, we might have I the set of natural numbers, and a ternary operation $F: A_1 \times A_2 \times A_3 \rightarrow A_2$. A zero-ary operation corresponds to a distinguished element of one of the sets A_i. Formally then a heterogeneous algebra of a given type is a system $<A_i; F_t: i \in I, t \in T>$, where each A_i is a non-empty set and the type includes the necessary information about each F_t.

Most of the standard results of universal algebra carry over easily to heterogeneous algebras. In particular, we may define heterogeneous subalgebras, products, and homomorphic images in the obvious way, as well as free algebras and equational classes; and a Birkhoff-type theorem then relates the two approaches.

The heterogeneous algebras we will be considering are called clones. For any homogeneous variety V, we define a heterogeneous algebra C(V) in the following way. For the underlying sets, we use $F_n(V)$, $n \geq 1$, the V - free algebra on the n generators $x_1, \ldots, x_n$. Thus the members of the nth universe are the n-ary terms of V. For each n, m $\geq$ 1, we have a fundamental operation C^n_m which is (n + 1)-ary, and maps $F_n(V) \times (F_m(V))^n \rightarrow F_m(V)$. This is defined by the rule that if t is in $F_n(V)$ and $t_1, \ldots, t_n$ are in $F_m(V)$, then $C^n_m(t, t_1, \ldots, t_n)$ is the m-ary term obtained from t by simultaneous substitution of the t_j for the variables x_j, $1 \leq j \leq n$. We also have a set of zero-ary operations or distinguished objects: for each $n \geq 1$ and each $1 \leq i \leq n$, we distinguish e^n_i, the ith n-ary projection. Thus

$$C(V) = < F_n(V); C^n_m; e^n_i: n,m, i \in N, 1 \leq i \leq n >.$$

This algebra is called the (concrete) clone of V.

It is easily verified that for any variety V, C(V) satisfies the following three identities:

1. $C^p_m(z, C^n_m(y_1, x_1, \ldots, x_n), \ldots, C^n_m(y_p, x_1, \ldots, x_n))$

   $= C^n_m(C^p_n(z, y_1, \ldots, y_p), x_1, \ldots, x_n), \qquad m, n, p \geq 1.$
2. $C^n_m(e^n_i, x_1, \ldots, x_n) = x_i, \qquad m \geq 1, \quad 1 \leq i \leq n.$
3. $C^n_n(y, e^n_1, \ldots, e^n_n) = y, \qquad n \geq 1.$

The last two of these say that the e^n_i's act as projections, while the first is a super-associativity law for the C^n_m operators. More generally, the family of all heterogeneous algebras of the type $<A_n; C^n_m; e^n_i: n,m,i \in N, 1 \leq i \leq n>$ with $C^n_m : A_n \times (A_m)^n \rightarrow A_m$ and $e^n_i \in A_n$, which satisfy the three identities above forms a (heterogeneous) variety, K_0. The algebras in K_0 are called abstract clones. Any concrete clone C(V) is of course an abstract clone; the next theorem shows that the converse is also true.

<u>Theorem 2.1.1</u> (Taylor, [14]): For any abstract clone C in K_0, there is a homogeneous variety V such that C is isomorphic to C(V).

The familiar operations which may be performed on (heterogeneous) algebras such as clones are related closely to constructions on (homogeneous) varieties. Two varieties are said to be equivalent if they have isomorphic clones. If $(V_i\colon i \in I)$ is a class of varieties of the same type, their product $\bigotimes_{i \in I} V_i$ is defined to be the variety whose clone is $\prod_{i \in I} C(V_i)$. A more complicated construction is that of a reduct variety. A reduct of a homogeneous algebra $\langle A; F_t\colon t \in T\rangle$ is any algebra $\langle A; F_s\colon s \in S\rangle$ for $S \subseteq T$. Let V be a variety of type T, and fix $S \subseteq T$. The reduct variety of V determined by S is the variety W of type S generated by the class of all algebras $\langle A; F_s\colon s \in S\rangle$ for which $\langle A; F_t\colon t \in T\rangle$ is in V. The next proposition relates reduct varieties and subvarieties to clones.

<u>Proposition</u> <u>2.1.2</u> (Taylor, [14]):

i) If W is a reduct variety of a variety V, then C(W) is a subclone of C(V). Conversely, if a clone C is a subclone of C(V), then there is a variety V_0 equivalent to V and a reduct variety W of V_0 such that C is isomorphic to C(W).

ii) If W is a subvariety of a variety V, then C(W) is a homomorphic image of C(V). Conversely, if a clone C is a homomorphic image of C(V), then C is isomorphic to C(W) for some subvariety W of V.

We now define a hypervariety to be any collection of varieties closed under the formation of equivalent, product, sub- and reduct varieties. The preceding comments and Proposition set up a correspondence between hypervarieties and varieties of clones: equivalent varieties correspond to isomorphic clones, products of varieties to products of clones, subvarieties to homomorphic image clones, and reduct varieties to subclones. The next Proposition, whose proof follows easily from the previous Proposition, expands on this correspondence.

<u>Proposition</u> <u>2.1.3:</u>

i) Let $\mathcal{H}$ be a hypervariety. Let $\mathcal{C}(\mathcal{H})$ be the class of all clones C isomorphic to C(V) for some variety V in $\mathcal{H}$. Then $\mathcal{C}(\mathcal{H})$ is a variety of clones.

ii) Let $\mathcal{C}$ be a variety of clones. Let $\mathcal{H}(\mathcal{C})$ be the class of all varieties V whose clones C(V) are in $\mathcal{C}$. Then $\mathcal{H}(\mathcal{C})$ is a hypervariety.

iii) Let $\mathcal{C}$ be a variety of clones, and let $\mathcal{H}$ be a hypervariety. Then

$$\mathcal{C}(\mathcal{H}(\mathcal{C})) = \mathcal{C} \qquad \text{and } \mathcal{H}(\mathcal{C}(\mathcal{H})) = \mathcal{H}.$$

Since the class of all hypervarieties forms a class which is ordered under inclusion, has a largest member (the hypervariety containing all varieties), and is closed under intersection, this class forms a complete lattice under inclusion, and for any variety V there is a smallest hypervariety $\mathcal{H}(V)$ containing V. This sets up an operator $\mathcal{H}$ from the class of all varieties to the lattice of hypervarieties. The properties of $\mathcal{H}$ as it acts on varieties of semigroups will be studied in subsequent sections. For the remainder of this section we give some general results about $\mathcal{H}$ and a related operator $\mathcal{C}$. $\mathcal{C}$ is also defined on the class of all varieties; for any variety V, $\mathcal{C}(V)$ is the variety of clones generated by the clone C(V).

Proposition 2.1.4: $\mathcal{C}$ and $\mathcal{H}$ are both monotonic operators. Moreover, for any two varieties V and W, $\mathcal{C}(V) = \mathcal{C}(W)$ iff $\mathcal{H}(V) = \mathcal{H}(W)$.

More information about $\mathcal{H}$ is gained from an alternate approach to the definition of the product of varieties. Let $(A_i\colon i \in I)$ be an indexed family of algebras, possibly of different types. The non-indexed product of the A_i's is the algebra $\bigotimes_{i \in I} A_i$ whose universe is the Cartesian product of the universes of the A_i's, and which has an n-ary fundamental operation p corresponding to each indexed family $(p_i : i \in I)$ of n-ary term functions p_i of A_i; p is defined in the obvious coordinate-wise way. Now let $(V_i : i \in I)$ be an indexed family of varieties, possibly of different types. The product of the varieties V_i is the variety $\bigotimes_{i \in I} V_i$ generated by all non-indexed products $\bigotimes_{i \in I} A_i$ for A_i in V_i. Since Taylor proved in [14] that the clone of this product of the V_i's is isomorphic to the product of the clones $C(V_i)$, it follows that this definition of product of varieties is equivalent to the one given earlier. We make use of this new definition in the following proposition.

Proposition 2.1.5: Let $(V_i : i \in I)$ be a class of varieties all of the same type. Then the join of these varieties is a subvariety of a reduct variety of their product.

Proof: Let the fundamental operations of the common type of the varieties be indexed by the set T. Let $(A_i : i \in I)$ be any collection of algebras with A_i in V_i, for all i in I. Then for each t in T, A_i has an operation f_t^i, so that $\bigotimes_{i \in I} A_i$ has an operation $F_t = (f_t^i : i \in I)$. The algebra B with universe the Cartesian product of the universes of the A_i's and fundamental operations F_t for t in T is then a reduct of $\bigotimes_{i \in I} A_i$ to the type T. In fact, B is just the (ordinary) product $\prod_{i \in I} A_i$ of the A_i's.

Let W be the reduct variety of $\bigotimes_{i \in I} V_i$ determined by T. By definition, W is generated by the class of all reducts of algebras in $\bigotimes_{i \in I} V_i$ to type T, which includes the class of all reducts of algebras $\bigotimes_{i \in I} A_i$, $A_i \in V_i$, to type T. In particular, W contains all of the products $\prod_{i \in I} A_i$ for A_i in V_i. From this it follows that the join of the V_i's is a subvariety of W, a reduct variety of $\bigotimes_{i \in I} V_i$.

Two important corollaries of this result will be used extensively in our later study of $\mathcal{H}(V)$ for V a variety of semigroups.

Corollary 2.1.6: Any hypervariety is closed under the operation of taking joins of varieties of the same type.

Corollary 2.1.7: The operators $\mathcal{C}$ and $\mathcal{H}$ both preserve joins of varieties of the same type.

Proof: Let $(V_i : i \in I)$ be a class of varieties all of the same type. By monotonicity of $\mathcal{C}$, we have

$$\bigvee_{i \in I} \mathcal{C}(V_i) \quad \subseteq \quad \mathcal{C}\Big(\bigvee_{i \in I} V_i\Big).$$

Conversely, the variety $\bigvee_{i \in I} \mathcal{C}(V_i)$ contains each $C(V_i)$, hence contains $\coprod_{i \in I} C(V_i)$, which is isomorphic to $C(\bigotimes_{i \in I} V_i)$. By the preceding Proposition, any variety of clones which contains $C(\bigotimes_{i \in I} V_i)$ must contain $C(\bigvee_{i \in I} V_i)$. Therefore

$$\mathcal{C}\Big(\bigvee_{i \in I} V_i\Big) \quad \subseteq \quad \bigvee_{i \in I} \mathcal{C}(V_i).$$

For $\mathcal{H}$, again one inclusion follows by monotonicity. Since $\bigvee_{i \in I} \mathcal{H}(V_i)$ contains all the V_i's, it contains their join too, by the previous Corollary. Hence the opposite inclusion also holds.

Note that although $\mathcal{H}$ preserves joins, we will show by example later that $\mathcal{H}$ does not preserve meets.

2.2. Hyperidentities

We have seen that hypervarieties correspond in a very precise way to varieties of clones. This correspondence allows us to describe hypervarieties equationally, by the clone equations which define the corresponding varieties of clones. However, the clone equations are generally very complicated and unwieldy, so a different approach is needed. In this section we introduce hyperidentities, and show how they provide this alternate description of hypervarieties.

Hyperidentities are used by Taylor in [15], a paper which presents the thereoms stated without proof below and provides many interesting examples. A hyperidentity is defined to be formally the same as an identity. We use the letters $x,y,z,w,x_1,x_2, \ldots$, for variables, and $F,G,J,K,F_1,F_2, \ldots$ for operation symbols. A variety V is said to satisfy a hyperidentity H if whenever the operation symbols of H are replaced by terms of V of the appropriate arity, then the identity which results holds true in V. The identities produced in this way, by a choice of V-terms for the operation symbols of H, are called (V-)instances of H. For example, the hyperidentity $F(x,x) = x$ will be satisfied by a variety V iff every binary term of V is idempotent. So for instance the varieties of bands and of lattices satisfy this hyperidentity, which is called the idempotent hyperidentity. Another hyperidentity we will frequently encounter is

$$F(G(x,y),\ G(z,w)) \quad = \quad G(F(x,z),\ F(y,w));$$

it is easily verified that any variety of commutative groups or semigroups satisfies this hyperidentity. We will say that a variety V satisfies a set Σ of hyperidentities if it satisfies every hyperidentity in Σ.

Saying that a variety V satisfies a given hyperidentity H says something about all the terms of V of certain arities; that is, about the free algebras $F_n(V)$ for certain $n \geq 1$. Thus such a statement corresponds to a statement about the clone C(V) of V, which in turn corresponds to a statement about the hypervariety $\mathcal{H}(V)$. In fact, there is a Birkhoff-type theorem relating hyperidentities and hypervarieties.

Theorem 2.2.1 (Taylor, [15]): Every hyperidentity defines a hypervariety, and conversely every hypervariety is definable by a set of hyperidentities.

Combining this theorem with Proposition 2.1.4 and Corollary 2.1.6, we get the following Corollaries:

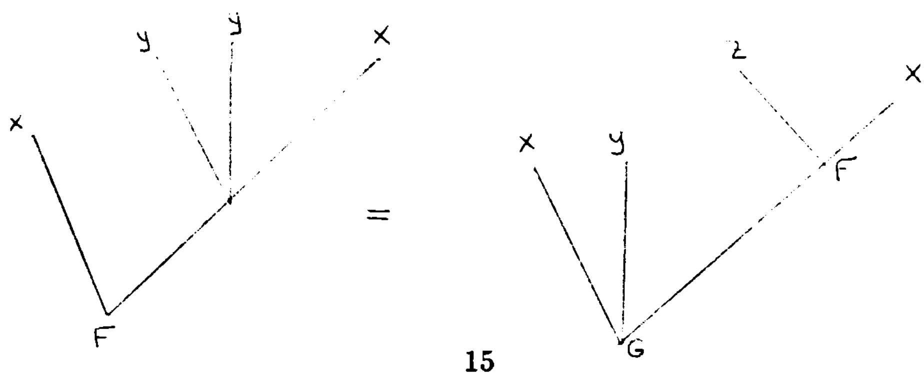
Corollary 2.2.2: Let V and W be varieties. Then $\mathcal{C}(V) = \mathcal{C}(W)$ iff $\mathcal{H}(V) = \mathcal{H}(W)$ iff V and W satisfy precisely the same hyperidentities.

Corollary 2.2.3: Let V and W be varieties of the same type, and let H be any hyperidentity. Then V and W both satisfy H iff $V \vee W$ satisfies H.

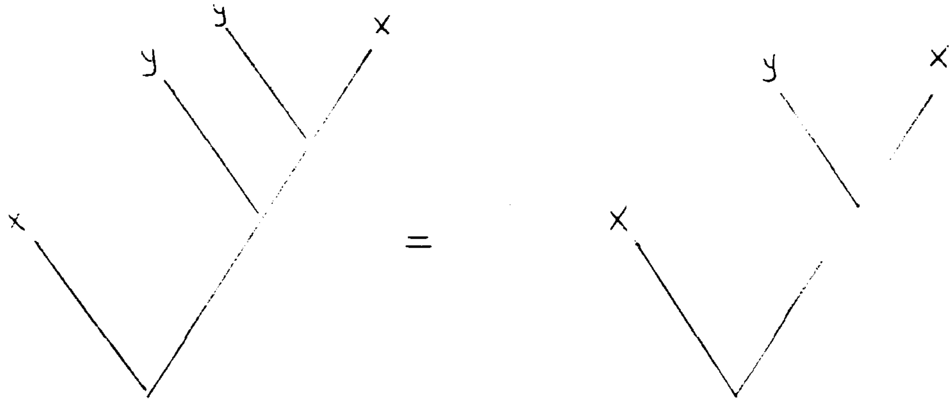
These results are crucial to our study of hyperidentities and hypervarieties. Corollary 2.2.2. provides a general technique for showing that two varieties generate different hypervarieties, namely producing a hyperidentity satisfied by one of the varieties but not by the other. Taylor used this method in [15] to produce several examples of $2^{\aleph_0}$ different hypervarieties, and asked whether distinct varieties of groups always generate distinct hypervarieties. This question has since been answered in the negative by Bergman [1], who showed that the variety of all groups satisfies precisely the same hyperidentities as the variety of metabelian groups. We discuss a similar question for varieties of semigroups in later Chapters, using the closure operator defined in the next section to examine in more detail which varieties generate the same hypervariety.

One feature of hyperidentities has important repercussions. For any non-trivial variety V, and any $n \geq 1$, the n-ary terms of V include the n "projections", $x_1, \ . \ . \ . \ , x_n$. If V satisfies a hyperidentity H, it means in particular that the identity we obtain from H by replacing every operation symbol F in H by the projection term x_1 of the appropriate arity must hold in V. It is easily verified that no matter what form H has, the identity thus obtained is just $x = y$, where x is the first variable to appear on the left-hand-side of H, and y is the first variable to appear on the right-hand-side. For this to hold in a non-trivial variety V, we must have x and y actually the same variable. Therefore, any hyperidentity H satisfied by a non-trivial variety must have the same first variable on either side. By a dual argument, such an H must also have the same last variable on each side. The significance of these facts will be seen later, in Section 1 of Chapter 3.

By analogy with the terminology for terms and identities, we will use the name "hyperterm" for the two expressions equated in a hyperidentity. Following Taylor [15] , we will frequently present hyperterms by means of tree diagrams. Each non-leaf node of a tree will correspond to an operation symbol, starting with the outermost operation symbol of the hyperterm on the root of the tree; and each leaf will correspond to a variable. For instance,



represents the hyperidentity F(x,G(y,y,x)) = G(x,y,F(z,x)). By convention, any unlabelled non-leaf nodes of the same arity in such a tree stand for the same operation symbol. Thus



represents F(x,F(y,F(y,x))) = F(x,F(y,x)). A hyperidentity all of whose operation symbols are n-ary (for some n $\geq$ 1) will be called an n-ary hyperidentity.

2.3. The Closure Operator

In this section we begin the study of the operator $\mathcal{H}$ as it applies to varieties of semigroups. We introduce a closure operator on varieties of semigroups, and use this to get information about hypervarieties generated by such varieties. We note that although our discussion is carried out in terms of varieties of semigroups, we could in fact consider varieties of any fixed type of algebra. No particular knowledge of semigroups is assumed in this section; we use only the fact that the collection of all varieties of semigroups forms a complete lattice under inclusion.

For any variety W of semigroups, we define HI(W) to be the set of all hyperidentities satisfied by W. Conversely, for any set Σ of hyperidentities, we define V(Σ) to be the largest variety of semigroups to satisfy Σ. By Corollary 2.2.2, V(Σ) is equal to the join of all the varieties of semigroups which satisfy Σ.

Lemma 2.3.1: Let U and W be any varieties of semigroups, and let Σ and Γ be any sets of hyperidentities. Then

1. If $U \subseteq W$ then $HI(W) \subseteq HI(U)$.
2. If $\Sigma \subseteq \Gamma$ then $V(\Gamma) \subseteq V(\Sigma)$.
3. $\Sigma \subseteq HI(V(\Sigma))$ and $W \subseteq V(HI(W))$.
4. $V(HI(V(\Sigma))) = V(\Sigma)$ and $HI(V(HI(W))) = HI(W)$.

Proof: These claims all follow easily from the definitions of $V(\Sigma)$ and $HI(W)$.

From this Lemma we see that there is a Galois correspondence

$$W \rightarrow HI(W)$$
$$V(\Sigma) \leftarrow \Sigma,$$

between varieties of semigroups and sets of hyperidentities. A variety W of semigroups will be called closed if $V(HI(W)) = W$; that is, if W is the largest variety of semigroups to satisfy all the hyperidentities satisfied by W. It follows from Lemma 2.3.1 that varieties of the form $V(\Sigma)$ and $V(HI(W))$ are always closed.

Proposition 2.3.2: The intersection of closed varieties is closed.

Proof: Let $(W_i: i \in I)$ be any collection of closed varieties. Since

$$\bigcap_{i \in I} W_i \subseteq W_j$$

for all $j \in I$, we have

$$V(HI(\bigcap_{i \in I} W_i)) \subseteq V(HI(W_j)) = W_j,$$

for all j in I. Therefore

$$V(HI(\bigcap_{i \in I} W_i)) \subseteq \bigcap_{i \in I} W_i.$$

The opposite inclusion also holds, by Lemma 2.3.1(3), showing that $\bigcap_{i \in I} W_i$ is closed.

The variety S of all semigroups is a closed variety, since S is the largest variety of semigroups to satisfy the trivial hyperidentity $x = x$. This combined with Proposition 2.3.2 means that for any variety W of semigroups there is a smallest closed variety containing it, namely the intersection of all the closed varieties containing W. We call this variety the closure of W, and denote it by $\overline{W}$.

Corollary 2.3.3: For any variety W of semigroups, $\overline{W} = V(HI(W))$.

Proof: $V(HI(W))$ is a closed variety containing W, so $\overline{W} \subseteq V(HI(W))$. For the opposite direction, suppose that U is any closed variety containing W: then $V(HI(W)) \subseteq V(HI(U)) = U$. Therefore $V(HI(W)) \subseteq \overline{W}$.

Corollary 2.3.4: Let U and W be any varieties of semigroups. If $U \subseteq W$, then $\overline{U} \subseteq \overline{W}$. Also $\overline{U \cap W} \subseteq \overline{U} \cap \overline{W}$ and $\overline{U} \vee \overline{W} \subseteq \overline{U \vee W}$.

Thus the closure operator is a monotone one. We will show later that $\overline{U \cap W}$ may be a proper subvariety of $\overline{U} \cap \overline{W}$, so that the closure operator does not preserve intersections. It is not known whether it preserves joins.

A set Σ of hyperidentities will be called closed if $HI(V(\Sigma)) = \Sigma$. By dualizing Lemma 2.3.2 and its Corollaries, we may show that any intersection of closed sets of hyperidentities is closed. This allows us to define for any set Σ of hyperidentities its closure $\overline{\Sigma}$ as the smallest closed set of hyperidentities containing Σ, and it follows that $\overline{\Sigma} = HI(V(\Sigma))$.

Let Σ and Γ be any sets of hyperidentities, and let H be any hyperidentity. H is said to be a consequence of Σ if H is in $\overline{\Sigma} = HI(V(\Sigma))$; that is, if any variety of semigroups which satisfies Σ must satisfy H. If every hyperidentity in Γ is a consequence of Σ, we say that Γ is a consequence of Σ, or equivalently that Σ yields Γ.

Now let W be any variety of semigroups. A set Σ of hyperidentities is called a basis for HI(W), or a hyperidentity-basis for W, if Σ yields HI(W). In this case every hyperidentity satisfied by W is a consequence of Σ. We emphasize that this definition of a basis is for varieties of semigroups only; there would be a similar notion of basis for varieties of other types, and the most general definition of basis would encompass varieties of all types of algebras.

<u>Lemma</u> <u>2.3.5:</u> Let W be a closed variety of semigroups, and let Σ be a set of hyperidentities. Then Σ is a basis for HI(W) iff the varieties of semigroups satisfying Σ are precisely W and its subvarieties; that is, iff $V(\Sigma) = W$.

<u>Proof:</u> Let Σ be a basis for HI(W), so that $W \subseteq V(\Sigma)$ and also $HI(W) \subseteq$

HI(V(Σ)). Applying V to the second of these inclusions gives $V(HI(V(\Sigma))) \subseteq V(HI(W))$, which since W and V(Σ) are both closed reduces to $V(\Sigma) \subseteq W$. Therefore $W = V(\Sigma)$, as required.

Conversely, suppose that $V(\Sigma) = W$, so that W is the largest variety of semigroups to satisfy Σ. Then certainly $HI(W) \subseteq HI(V(\Sigma)) = \overline{\Sigma}$, so that Σ is a basis for HI(W).

Since by Theorem 2.2.1 hyperidentities precisely define hypervarieties, our correspondence $W \rightarrow HI(W)$, $\Sigma \rightarrow V(\Sigma)$ between varieties of semigroups and sets of hyperidentities can be extended to a correspondence between varieties of semigroups and hypervarieties. But for any variety W, the hypervariety determined by HI(W) is just the smallest hypervariety to contain W, which we have been denoting $\mathcal{H}(W)$. Thus we are led once again to consider the operator $\mathcal{H}$, this time restricted to varieties of semigroups.

We will use the notation L(S) for the lattice of varieties of semigroups, and $\mathcal{HV}$ for the collection of all hypervarieties, which we saw in Section 2.1 is also a complete lattice under inclusion. The closed varieties of semigroups include S itself, and are closed under intersection, so they too form a complete lattice, which we will denote by L(CS). It is not known if this lattice is a sublattice of L(S): the problem of whether or not the join of two closed varieties is closed is equivalent to the problem mentioned earlier of whether the closure operator preserves joins.

We have shown in Corollary 2.1.7 that $\mathcal{H}$ preserves joins on the two lattices L(S) and L(CS). Since by definition $\mathcal{H}(V) = \mathcal{H}(\overline{V})$ for any V in L(S), the images $\mathcal{H}(L(S))$ and $\mathcal{H}(L(CS))$ are the same. This image is then a join-subsemilattice of the lattice $\mathcal{H}\mathcal{V}$ of all hypervarieties. It is not known if it is in fact a sublattice, although it does form a complete lattice. Note also that $\mathcal{H}$ is one-to-one on the lattice L(CS), since $\mathcal{H}(V) = \mathcal{H}(W)$ for V,W closed varieties implies that V and W satisfy precisely the same hyperidentities, so that $V = \overline{V} = \overline{W} = W$.

Chapter 3

Semigroup Results

Having introduced the closure operator in Chapter 2, our goal now is to identify the closures of various varieties of semigroups. In this chapter we present some of the semigroup-theoretic results necessary for that goal. We start by giving some notation and background on the varieties of semigroups to be considered, and describing the general technique used to find the closure $\overline{V}$ of a variety V. One of the three stages of this technique involves showing that a given set of identities defines the join of given varieties, motivating us to consider joins of varieties. In the last three sections of the chapter then we focus on this question of joins, looking in particular at some of the joins to be encountered in later work on hyperidentities.

3.1. Varieties of Semigroups

This section presents some information about the lattice L(S) of varieties of semigroups, especially about some of the varieties whose closures will be investigated later. Only enough background for this later investigation is given here, and all results are stated without proof. A more detailed survey of the lattice of varieties of semigroups is given in [5].

Our interest in varieties of semigroups will be in terms of the identities they satisfy; that is, we will view them as equational classes. We fix a countably infinite set of variables, including $x,y,z,w,x_1,x_2, \ldots ,y_1,y_2, \ldots$, and use words from the free semigroup on this set. For any word u, $|u|$ denotes the length of u. An identity is then an equation $u = v$ where u and v are words. A trivial identity is one in which u and v are identical words. For any set I of identities, we use V(I) to denote the class of all semigroups satisfying I. If I contains only one identity $u = v$, we simplify this to $V(u = v)$. For example, $V(xy = yx)$ is the variety A of abelian semigroups. A set I of identities is a basis for a variety V if $V(I) = V$, and hence all the identities satisfied by V are consequences of the identities in I. The terms of a variety V are equivalence classes of terms; we will identify words with the equivalence classes they represent, and refer to terms such as xy, xyx, and so on.

The collection of all varieties of semigroups is a complete lattice under inclusion. We use L(S) to denote this lattice, and in general for any variety V in L(S), we use L(V) for the lattice of all subvarieties of V. The largest element of L(S) is the variety S of all semigroups, and the smallest element is the trivial variety T of one-element semigroups. Evans [5] has shown that L(S) is uncountably infinite, and that there are varieties of semigroups which do not have finite bases of identities. However, Perkins [10] showed that any variety of commutative semigroups is finitely based, so that L(A) is a countably infinite lattice.

For convenience we list below the varieties of semigroups to be referred to in the rest of this thesis:

$A = V(xy = yx)$, the variety of abelian semigroups.

$M = V(xyzw = xzyw)$, the variety of medial semigroups.

$B = V(x^2 = x)$, the variety of bands (idempotent semigroups).

$SL = V(xy = yx,\ x^2 = x) = A \cap B$, the variety of semilattices.

$LZ = V(xy = x)$, the variety of left-zero semigroups.

$RZ = V(xy = y)$, the variety of right-zero semigroups.

$RB = V(xyz = xz)$, the variety of rectangular bands.

$LNB = V(xyz = xzy,\ x^2 = x)$, the variety of left normal bands.

$RNB = V(xyz = yxz,\ x^2 = x)$, the variety of right normal bands.

$NB = V(xyzw = xzyw,\ x^2 = x) = M \cap B$, the variety of normal bands.

$Z = V(xy = zw)$, the variety of zero semigroups.

$N_k = V(x_1 \cdots x_k = y_1 \cdots y_k)$, the variety of k-nilpotent semigroups.
Note that $N_2 = Z$.

$AN_k = V(xy = yx,\ x_1 \cdots x_k = y_1 \cdots y_k)$,
the variety of abelian k-nilpotent semigroups.

$MN_k = V(xyzw = xzyw,\ x_1 \cdots x_k = y_1 \cdots y_k)$,
the variety of medial k-nilpotent semigroups.

$A_m = V(xy = yx,\ xy^m = x)$, the variety of abelian groups of exponent m.

$B_m = V(xy^m = x,\ y^m x = x)$, the variety of groups of exponent m.

$A_{n,m} = V(xy=yx,\ x^n = x^{n+m})$, the variety of commutative semigroups
satisfying $x^n = x^{n+m}$. Note that $A_{1,1} = SL$,
and $A_{1,m} = SL \vee A_m$.

$B_{n,m} = V(x^n = x^{n+m})$, the Burnside variety of semigroups satisfying

$x^n = x^{n+m}$. Note that $B_{1,1} = B$.

$M_{n,m} = V(xyzw = xzyw, x^n = x^{n+m})$, the variety of medial semigroups satisfying $x^n = x^{n+m}$. Note that $M_{1,1} = NB$.

The atoms of L(S) are the varieties LZ, RZ, SL, Z, and the A_p's for p prime. Any variety in L(S) contains one of these atoms as a subvariety. The four non-group atoms generate a sixteen-element Boolean algebra, with $NB \vee Z$ as their join. The join of all the group atoms is A, the variety of commutative semigroups. The join of all the atoms is the medial variety M (see [5]). In particular, this implies that

$$M = A \vee LZ \vee RZ \vee SL \vee Z = A \vee RB \vee SL \vee Z = A \vee RB.$$

The varieties A_m, with m square-free, form a distributive lattice. For any l and m both square-free, their greatest common divisor gcd(l,m) and their least common multiple lcm(l,m) are also square-free, and we have $A_l \cap A_m = A_{gcd(l,m)}$ and $A_l \vee A_m = A_{lcm(l,m)}$. Similarly, the varieties $A_{n,m}$, with n and m ≥ 1, form a lattice, with $A_{k,l} \cap A_{n,m} = A_{min(k,n),gcd(l,m)}$ and $A_{k,l} \vee A_{n,m} = A_{max(k,n),lcm(l,m)}$.

The variety RB of rectangular bands plays a significant role in what follows. It is the join of the two atoms LZ and RZ. Both LZ and RZ have the special property that their terms are all words of length one; that is, for any $n \geq 1$, the n-ary terms are just the n projections $x_1, \ldots, x_n$. This means that the clones of LZ and RZ are the same, and contain only the projections e_i^n, and so they are subclones of C(V) for any other non-trivial variety V (of any type, in fact). Translating this into the language of hypervarieties, we have $\mathcal{H}(LZ) = \mathcal{H}(RZ) \subseteq \mathcal{H}(V)$ for any non-trivial variety V. Hence also $RB = LZ \vee RZ \in \mathcal{H}(V)$ for any non-trivial V. This proves the following important result.

<u>Proposition 3.1.1:</u> For any non-trivial variety V of semigroups, $\mathcal{H}(RB) \subseteq \mathcal{H}(V)$, and $RB \subseteq \overline{V}$.

Although our proof started with clones, we could equally well have established this result by arguing about hyperidentities instead. Then our observations in Section 2.2 that any hyperidentity satisfied by a non-trivial variety V has the same variable appearing first in each of its hyperterms, and the same variable appearing last, tell us that any such hyperidentity is satisfied by both LZ and RZ, and hence by their join RB. This shows again that $RB \subseteq \overline{V}$.

Proposition 3.1.1 is our first step towards concrete information about closures of varieties. In fact, it allows us to prove our first non-trivial closure result.

<u>Proposition 3.1.2:</u> $\overline{A} = M$.

<u>Proof:</u> We know from Proposition 3.1.1. that $A \vee RB \subseteq \overline{A}$. We have also observed in Section 2.2 that the variety A satisfies the hyperidentity

$$F(G(x,y),G(z,w)) = G(F(x,z),F(y,w)).$$

By substituting for both F and G the binary term x_1x_2, we get as an instance of this hyperidentity the medial identity, $xyzw = xzyw$. Therefore any variety which satisfies all the hyperidentities that A does must be contained in the medial variety M, and $\overline{A} \subseteq M$. But also, as we noted earlier in this section, $M = A \vee RB$. Thus we have $A \vee RB \subseteq \overline{A} \subseteq M = A \vee RB$, and our conclusion follows.

The hyperidentity used in this proof has the property that the varieties of semigroups which satisfy it are precisely the medial variety M and its subvarieties. Therefore by Lemma 2.3.5, the set consisting of this one hyperidentity forms a basis for the family HI(M) of hyperidentities satisfied by M. We will refer to this hyperidentity as the medial hyperidentity.

The proof of Proposition 3.1.2 serves to illustrate, in a simple case, the method to be used in identifying $\overline{V}$ for various varieties V. This method involves three stages. First, we prove that certain varieties are contained in $\overline{V}$; these include at least RB and V itself, perhaps more. Then we produce some hyperidentities satisfied by V. From these we get some instances, which we use as defining identities for a variety V_0, with $\overline{V} \subseteq V_0$. Finally, we show that V_0 is in fact the join of the varieties found at the first stage. Then we may conclude that $\overline{V} = V_0$.

The first two of these stages involve hyperidentities and hypervarieties, and are discussed more fully in Chapters 4 and 5. The third stage however involves proofs of a purely semigroup-theoretic nature, and the rest of this chapter is devoted to it.

The problem of finding the join of two or more semigroup varieties is not in general an easy one. Specifically, we would like when given equational descriptions of two varieties to produce a set of identities which will define their join. For most of the joins to be found in the next three sections, we first obtain such a set of identities by guess-work (often stimulated by knowledge of what hyperidentity instances we have been able to obtain for the relevant varieties). We then have to prove that the conjectured set of identities does indeed define the required join. For this we have two basic methods, one using a structural approach and one a syntactic approach. We conclude this section with a discussion of these two methods, both of which will be illustrated in the next sections.

Let $V = V(I)$, $W = V(J)$, and $U = V(K)$, where I, J, and K are sets of identities, and $V \vee W \subseteq U$. The syntactic approach is to consider the identities satisfied by $V \vee W$. If we can show that any non-trivial identity satisfied by $V \vee W$ is a consequence of the identities in K, then any such identity is also satisfied by U. From this it follows that $U \subseteq V \vee W$, giving us $U = V \vee W$.

For the structural approach, we will show that $U \subseteq V \vee W$ by showing that every semigroup in U is a subdirect product of a semigroup in V and a semigroup in W. The following Lemma sets up the machinery to be used in proofs of this type.

<u>Lemma</u> <u>3.1.3:</u> Let U, V, and W be varieties of semigroups. Let C be a semigroup in U. Suppose that there is a map $\Theta: C \rightarrow C$ which satisfies the following conditions:

1. Θ is a homomorphism;

2. Θ is a retraction; that is, $\Theta^2 = \Theta$;

3. the image $\Theta(C)$ is an ideal of C;

4. $\Theta(C)$ is in V;

5. the Rees quotient $C/\Theta(C)$ is in W.

Then C is a subdirect product of $\Theta(C)$ and $C/\Theta(C)$, so that C is in $V \vee W$.

<u>Proof:</u> Let ρ be the canonical homomorphism from C to its Rees quotient $C/\Theta(C)$. The condition that Θ is a retraction ensures that the intersection of the kernels of ρ and Θ is trivial. From this it follows (see [2]) that C is a subdirect product of the images $\Theta(C)$ of Θ and $C/\Theta(C)$ of ρ. Conditions 4 and 5 then imply that C is in $V \vee W$.

3.2. Joins with RB

The first type of joins we consider are those of the form $V \vee RB$, for certain varieties V of semigroups. We have seen that the variety RB is contained in $\overline{V}$ for any non-trivial variety V, so that $V \vee RB \subseteq \overline{V}$, and hence it will be necessary in identifying closures to have equational descriptions of $V \vee RB$. In particular, we will consider for V the varieties A_m and $A_{n,m}$, and N_k, MN_k, and AN_k, for n, m ≥ 1 and $k \geq 2$. A useful observation is that since $A_{1,m} = SL \vee A_m$, and $NB = RB \vee SL$, we have $A_{n,m} \vee RB$ equal to $A_{n,m} \vee NB$ for all n and m ≥ 1.

<u>Proposition</u> <u>3.2.1:</u> Let $m \geq 2$. Then $A_{1,m} \vee RB = M_{1,m}$.

Proof: By definition $A_{1,m} \vee RB \subseteq M_{1,m}$. For the converse, let C be any semigroup in $M_{1,m}$. By [11, IV.4.6] it suffices to show that C is completely regular and that its idempotents form a normal band, for then C is in LNB $\vee$ RNB $\vee$ $A_m \vee SL = NB \vee A_{1,m} = A_{1,m} \vee RB$. It follows easily from the medial identity of $M_{1,m}$ that the idempotents of C form a normal band. Moreover, for any c in C, we have $c = c^{m+1} = c^{m-1}c^2 \in c^2C \cap Cc^2$, and hence by [11, IV.1.2] C is also completely regular.

Corollary 3.2.2: Any semigroup in $M_{1,m}$ is a strong semilattice of rectangular groups of exponent m.

Proof: This follows from the previous proof, again using [11, IV.4.6].

A slight variation of the preceding proof lets us describe $A_m \vee RB$.

Proposition 3.2.3: Let $m \geq 2$. Then

$$A_m \vee RB = \quad V(xyzw = xzyw,\ x = x^{m+1},\ xy^mz = xz).$$

Proof: Inclusion in one direction is clear. If C is a semigroup in the second variety, then as in the previous proof C is completely regular and its idempotents form a normal band. Hence C is a subdirect product of some C_1 in LZ or LNB, C_2 in RZ or RNB, and C_3 in A_m or $A_{1,m}$. But SL does not satisfy the identity $xy^mz = xz$, so neither do LNB, RNB, or $A_{1,m}$. Therefore we must have C_1 in LZ,

C_2 in RZ, and C_3 in A_m. From this we get $C \in LZ \vee RZ \vee A_m = A_m \vee RB$.

Note that in this proof, the role of the identity $xy^mz = xz$ is to force the semigroup C to be in $A_m \vee RB$ rather than $A_{1,m} \vee RB$. For this purpose any identity which is satisfied by A_m and RB but not by SL will suffice. One such identity which we will encounter later in investigating hyperidentities is

$$wxy^mw^{2^{m+1}-m-2} = wxw^{2^{m+1}-2}.$$

This gives

Corollary 3.2.4:

$$A_m \vee RB = V(xyzw = xzyw,\ x = x^{m+1},\ wxy^mw^{2^{m+1}-m-2} = wxw^{2^{m+1}-2}).$$

For $A_{n,m} \vee RB$ when $n \geq 2$ the structural approach does not work, and we turn instead to a syntactic method. Also unlike the $n = 1$ case, $A_{n,m} \vee RB$ is a proper subvariety of $M_{n,m}$ when $n \geq 2$, as indicated by the identity $x^{n-1}yx = x^{n-1+m}yx$ used in the next result.

Proposition 3.2.5: Let $n \geq 2$ and $m \geq 1$. Then the variety $A_{n,m} \vee RB$ is defined by the identities

$$xyzw = xzyw,\ x^n = x^{n+m}, \text{ and } x^{n-1}yx = x^{n-1+m}yx.$$

Proof: Let the variety defined by the three given identities be called W. Clearly $A_{n,m} \vee RB \subseteq W$. For the opposite inclusion, suppose that $u = v$ is any non-trivial identity satisfied by $A_{n,m} \vee RB$. We show that W also satisfies $u = v$.

Since RB satisfies $u = v$, we know that u and v start with the same letter, x say, and end with the same letter, y say (with x and y possibly the same). Since $A_{n,m}$ satisfies $u = v$, u and v have the same content, and for each letter z in this content, either the number of occurrences of z in u is equal to the number of occurrences of z in v, or these two quantities are both $\geq$ n and are congruent modulo m. Using this information we show how to deduce $u = v$ from the identities defining W.

We first transform u and v into a "standard form" $\bar{u}$ and $\bar{v}$ as follows. Write

$$\bar{u} = x^a z_1^{a_1} \ . \ . \ . \ z_p^{a_p} y^b \qquad \text{and} \qquad \bar{v} = x^c z_1^{c_1} \ . \ . \ . \ z_p^{c_p} y^e.$$

where $x, z_1, \ldots z_p, y$ are the distinct (except possibly $x = y$) letters appearing in u and v, and if $x = y$, then $b = e = 1$. The identities $u = \bar{u}$ and $v = \bar{v}$ hold in W, just by use of the medial identity.

If $x \neq y$, then from the above information we may deduce $\bar{u} = \bar{v}$ simply by using the identity $x^n = x^{n+m}$. Hence in this case, W satisfies $u = v$.

If $x = y$, we have $b = e = 1$ by construction. Again we may deal with the "interior" letters $z_1, \ldots z_p$ using only $x^n = x^{n+m}$, so we may reduce this case to considering words $\bar{u}' = x^a wx$ and $\bar{v}' = x^c wx$, for some word w. From the comments above, either $a+1 = c+1$, or $a+1$ and $c+1$ are both $\geq$ n and are congruent modulo m. If $a = c$, we are done; otherwise, both a and c are $\geq$ n-1 and a and c are congruent modulo m, and we have two cases to consider.

If a and c are both $\geq$ n, with a and c congruent modulo m, then $x^a = x^c$ holds in W, and so does $\bar{u} = \bar{v}$. Finally, suppose that a = n-1 and c $\geq$ n (or dually). Then c is congruent to n-1 modulo m, and c may be be written as km+n-1 for some k $\geq$ 1. But then $\bar{u}' = x^{n-1}wx$ and $\bar{v}' = x^{km+n-1}wx$, and $\bar{u}' = \bar{v}'$ holds in W by repeated use of the identity $x^{n-1}yx = x^{n-1+m}yx$. Hence in either case W satisfies $\bar{u} = \bar{v}$ and therefore also u = v.

For the remainder of this section we focus on the joins of some nilpotent varieties with RB.

Proposition 3.2.6: Let k $\geq$ 3. Then

$$N_k \vee RB \quad = \quad V(x_1 \cdots x_k = x_1 y_2 \cdots y_{k-1} x_k).$$

Proof: We will call the right-hand-side variety W_k. Clearly it contains $N_k \vee RB$. To prove the opposite inclusion, let C be any semigroup in W_k. Define a map $\Theta : C \to C$ by $\Theta(c) = c^k$, for all c in C. Then

1. Θ is a homomorphism, since $x^k y^k = (xy)^k$ holds in W_k;
2. Θ is a retraction, since $(x^k)^k = x^k$ holds in W_k;
3. $\Theta(C)$ is an ideal of C, since for any c and d in C we have

   $c^k d = c^k d^k = (cd)^k \in \Theta(C)$, and similarly $dc^k \in \Theta(C)$;
4. $\Theta(C)$ is in RB, since $(x^k)^2 = x^{2k} = x^k$ and $x^k y^k z^k = (xyz)^k = x^k y^k$ both hold in W_k;
5. $C/\Theta(C)$ is in N_k, since for any $c_1, \ldots, c_k$ in C, we have

$$c_1 \cdots c_k = c_1^k \cdots c_k^k = (c_1 \cdots c_k)^k \in \Theta(C).$$

Therefore by Lemma 3.1.3 we have $C \in N_k \vee RB$.

<u>Proposition</u> <u>3.2.7:</u> Let $k \geq 3$. Then

$$MN_k \vee RB = V(xyzw = xzyw,\ x_1 \cdots x_k = x_1y_2 \cdots y_{k-1}x_k).$$

<u>Proof:</u> The proof is very similar to the previous one, with the map Θ: $C \rightarrow C$ as before. But now $\Theta(C)$ is in RB and $C/\Theta(C)$ is in MN_k, so that C is in $MN_k \vee RB$.

<u>Proposition</u> <u>3.2.8:</u> Let $k \geq 2$. Then $AN_k \vee RB = MN_k \vee RB$.

<u>Proof:</u> If $k = 2$, $AN_k = MN_k = N_k$, and the result is obvious, so we assume that $k \geq 3$. Since $AN_k \vee RB \subseteq MN_k \vee RB$, it suffices to prove that every non-trivial identity satisfied by $AN_k \vee RB$ is also satisfied by $MN_k \vee RB$. So suppose that $AN_k \vee RB$ satisfies $u = v$. Since RB satisfies $u = v$, u and v have the same first letters and the same last letters. Thus if both $|u|$ and $|v|$ are $\geq k$, then we are done: we use the identities defining $MN_k \vee RB$ (from the previous Proposition) to deduce $u = v$. Otherwise, consider the case where $|u| < k$ or $|v| < k$ (or both). Since AN_k satisfies $u = v$, there exists a chain $u = u_0 = u_1 = \ldots = u_l = v$, with each step $u_i = u_{i+1}$ a consequence of either $xy = yx$ or $x_1 \cdots x_k = y_1 \cdots y_k$. But steps which are consequences of the first of these identities do not change the length of words involved, while steps which are

consequences of the second identity can only be used on words u_i of length $\geq$ k. Thus if $|u| < k$ or $|v| < k$, then $|u_i| < k$ for all $0 \leq i \leq l$, and in fact the abelian variety A satisfies $u = v$. Then $M = A \vee RB$ also satisfies $u = v$, so $MN_k \vee RB$ does too.

3.3. Joins with Z

In this section we consider joins of the form $V \vee Z$, where V is a variety of semigroups and $Z = N_2$ is the variety of zero semigroups. In this special case, the concept of an inflation of a semigroup proves crucial to the investigation of such joins. A semigroup C is called an inflation of a semigroup D, where $D \subseteq C$, if $C^2 \subseteq D$ and there is a homomorphism Φ of C onto D with the property that $\Phi^2 = \Phi$. Notice that this definition implies that D is then an ideal of C, since $CD \cup DC \subseteq C^2 \subseteq D$. The importance of inflations is seen in the following result of Clarke's:

<u>Proposition</u> <u>3.3.1</u>(Clarke, [4]): Let V be any variety of semigroups. Then the class of all inflations of semigroups in V is a variety, which is in fact the variety $V \vee Z$.

The next lemma is very simple to prove, but is surprisingly useful when combined with the fact that Z is an atom of L(S).

<u>Lemma</u> <u>3.3.2:</u> Let C be a semigroup in a variety V, and let C be an inflation of a semigroup D. Then both D and C/D are also in V; moreover, C/D is in $Z \cap V$.

Proposition 3.3.3: Let V be any variety of semigroups which does not contain the variety **Z**. Then V $\vee$ **Z** covers V in the lattice L(S).

Proof: Suppose there is a variety U with V $\subseteq$ U $\subseteq$ V $\vee$ **Z**. Since U $\cap$ **Z** $\subseteq$ **Z** and **Z** is an atom of L(S), U $\cap$ **Z** is either the trivial variety T or **Z** itself. If U $\cap$ **Z** = **Z**, then **Z** $\subseteq$ U so that V $\vee$ **Z** $\subseteq$ U $\subseteq$ V $\vee$ **Z** and U = V $\vee$ **Z**.

Otherwise, U $\cap$ **Z** = T, and we have V $\subseteq$ U but U properly contained in V $\vee$ **Z**. If also V is properly contained in U, then there would be a semigroup C in U - V. Then C is is V $\vee$ **Z** so that C is an inflation of some D $\in$ V; but by Lemma 3.3.2, C/D is in **Z** $\cap$ U = T, so that C = D. This contradicts the fact that C is in U - V. Thus in this case we must have V = U.

Proposition 3.3.4: Let V and W be varieties of semigroups such that V covers W in L(S). Then either V $\vee$ **Z** = W $\vee$ **Z**, or V $\vee$ **Z** covers W $\vee$ **Z**.

Proof: Suppose there is a variety U with W $\vee$ **Z** $\subsetneqq$ U $\subseteq$ V $\vee$ **Z**. Then there exists a semigroup C in U -(W $\vee$ **Z**). Then C must be an inflation of some D in V - W. Let Y be the variety generated by W $\cup$ {D}. Clearly W $\subsetneqq$ Y $\subseteq$ V. Since V covers W, we get Y = V; that is, W $\cup$ {D} generates V. Now we have W $\subseteq$ U, and by Lemma 3.3.2, D is in U. Thus V $\subseteq$ U. Since also **Z** $\subseteq$ W $\vee$ **Z** $\subseteq$ U, we get V $\vee$ **Z** $\subseteq$ U $\subseteq$ V $\vee$ **Z**, forcing U = V $\vee$ **Z**.

The next result allows us to describe all the subvarieties of $V \vee Z$ in terms of the subvarieties of V, for any $V \in L(S)$.

Proposition 3.3.5: Let V be any variety of semigroups. For any variety $U \subseteq V \vee Z$, either $U \subseteq V$ or $U = (U \cap V) \vee Z$ with $U \cap V \subseteq V$.

Proof: If $Z \subseteq V$ the result is trivial, so we assume that $Z \nsubseteq V$. Suppose that $U \subseteq V \vee Z$, but $U \nsubseteq V$. If $V \subseteq U$, then $V \subseteq U \subseteq V \vee Z$, so by the previous Proposition either $U = V$ or $U = V \vee Z = (U \cap V) \vee Z$, as required.

Otherwise, neither of V or U contains the other. In this case, $U \cap V$ is a proper subvariety of both U and V. Since $U \subseteq V \vee Z$, any C in U is an inflation of some D in V, and by Lemma 3.3.2, D is also in U. Thus any such C in U is in $(U \cap V) \vee Z$, so that $U \subseteq (U \cap V) \vee Z$. Then $U \cap V \subsetneqq U \subseteq (U \cap V) \vee Z$, and $Z \nsubseteq V$ means $Z \nsubseteq U \cap V$, so by Proposition 3.3.3 we have $U = (U \cap V) \vee Z$, as required.

We next consider the map α on $L(S)$ which takes any variety V to its join $V \vee Z$ with Z. The next results describe some properties of α. It is obvious that α preserves joins on $L(S)$. In general, α is not one-to-one; for instance, we may have $U \vee Z \neq U$, yet of course $(U \vee Z) \vee Z = U \vee Z$.

Lemma 3.3.6: Let U and V be varieties, neither of which contain Z. If $U \neq V$, then $U \vee Z \neq V \vee Z$.

Proof: Since $U \neq V$, there exists a semigroup C in $U - V$ say. Suppose that $U \vee Z = V \vee Z$. Then $C \in U \subseteq U \vee Z = V \vee Z$, so that C is an inflation of some semigroup D in V. Since $C \notin V$, $C \neq D$. However, C/D is in $U \cap Z$ by Lemma 3.3.2; and when $Z \nsubseteq U$ we have $U \cap Z = T$, forcing $C = D$. This contradiction shows that $U \vee Z \neq V \vee Z$ must hold.

Corollary 3.3.7: Let V be a variety of semigroups which does not contain Z. Then the map α is one-to-one on $L(V)$.

The proof that α preserves meets on $L(S)$, making it a lattice homomorphism, is broken into two parts. We first consider the action of α on varieties not containing Z, showing that under certain restrictions α becomes a lattice isomorphism.

Proposition 3.3.8: Let W and Y be varieties of semigroups, neither of which contain Z. Then $(W \cap Y) \vee Z = (W \vee Z) \cap (Y \vee Z)$. In particular, if V is a variety which does not contain Z, then α preserves meets on $L(V)$.

Proof: By definition, $(W \cap Y) \vee Z \subseteq (W \vee Z) \cap (Y \vee Z)$. For the reverse inclusion, note that $U = (W \vee Z) \cap (Y \vee Z)$ is a subvariety of both $W \vee Z$ and $Y \vee Z$. From Proposition 3.3.5, we know the form of such subvarieties:

$$U = U' \text{ or } U' \vee Z, \qquad \text{for } U' \subseteq W,$$

and $\quad U = U'' \text{ or } U'' \vee Z$, $\quad$ for $U'' \subseteq Y$.

This gives us four cases to consider:

1. If $U = U' = U''$, then $U \subseteq W \cap Y \subseteq (W \cap Y) \vee Z$.
2. If $U = U' \subseteq W$, and also $U = U'' \vee Z$ for $U'' \subseteq Y$, then we would have $Z \subseteq U'' \vee Z = U \subseteq U' \subseteq W$. This contradicts our assumption that W does not contain Z.
3. The case $U = U' \vee Z = U'' \subseteq Y$, where $U' \subseteq W$, is the dual of case 2, and similarly leads to a contradiction.
4. If $U = U' \vee Z = U'' \vee Z$, for $U' \subseteq W$ and $U'' \subseteq Y$, then by Lemma 3.3.6, $U' = U''$. Hence $U = U' \vee Z \subseteq (W \cap Y) \vee Z$.

Therefore in all possible cases we get $(W \vee Z) \cap (Y \vee Z) \subseteq (W \cap Y) \vee Z$, as required.

<u>Corollary</u> <u>3.3.9:</u> Let V be any variety of semigroups which does not contain Z. Let $L(V) \vee Z$ be the collection of varieties of the form $U \vee Z$, $U \in L(V)$. Then the map α is a lattice isomorphism of $L(V)$ onto $L(V) \vee Z$, which is therefore a sublattice of $L(S)$.

<u>Proposition</u> <u>3.3.10:</u> Let V be a variety of semigroups which does not contain Z. Then $L(V \vee Z)$ is isomorphic to $L(V) \times L(Z)$.

<u>Proof:</u> Since Z is an atom of $L(S)$, $L(Z)$ is just the two-element lattice $T \subseteq Z$. Define a map γ from $L(V) \times L(Z)$ to $L(V \vee Z)$ by $\gamma(U,W) = U \vee W$, for U in $L(V)$ and W in $L(Z)$.

By Proposition 3.3.5, γ maps onto $L(V \vee Z)$. It is also very easy to show that γ is one-to-one. Since γ clearly preserves joins, it suffices to check that γ preserves meets. Let U and U′ be in L(V), and W, W′ be in L(Z): so W and W′ are either T or Z. We need to show that

$$(U \cap U') \vee (W \cap W') \quad = \quad (U \vee W) \cap (U' \vee W').$$

If $W = W' = T$, this equation reduces to a trivial one. If $W = W' = Z$, it reduces to $(U \cap U') \vee Z = (U \vee Z) \cap (U' \vee Z)$, which by Proposition 3.3.8 holds for U and U′ in L(V). Finally, suppose that $W = T$ while $W' = Z$ (or dually). In this case our equation reduces to

$$U \cap U' = U \cap (U' \vee Z).$$

Here the containment from left to right is obvious. For the opposite direction, we know that $U \cap (U' \vee Z)$ is a subvariety of $U' \vee Z$, so it is either a subvariety of U′ or of the form $U'' \vee Z$ for some $U'' \subseteq U'$. The latter would imply that $Z \subseteq U'' \vee Z = U \cap (U' \vee Z) \subseteq U$, which is impossible since by assumption Z is not contained in U. Therefore we must have $U \cap (U' \vee Z) \subseteq U' \cap U$, as required. This shows that γ preserves intersections, and finishes the proof.

We now return to the study of α as a map applied to the entire lattice L(S). On this domain α is still a lattice homomorphism.

Proposition 3.3.11: The map α taking V to $V \vee Z$ is a lattice homomorphism on the lattice L(S) of varieties of semigroups.

Proof: Since α clearly preserves joins, it suffices to prove that it also preserves meets. For this we need to show that for any V and W in L(S), $(V \cap W) \vee Z = (V \vee Z) \cap (W \vee Z)$. Inclusion from left to right is always true, so we consider only the reverse inclusion. We examine three cases:

1. If V and W both contain Z, then we get $(V \vee Z) \cap (W \vee Z) = V \cap W = (V \cap W) \vee Z$, as required.

2. If neither V nor W contains Z, the desired inclusion holds by Proposition 3.3.8.

3. If V contains Z but W does not, (or dually), we let $Y = (V \vee Z) \cap (W \vee Z)$. Then Y is contained in both V and $W \vee Z$; by Proposition 3.3.5, it is either contained in W, or we have $Y = (Y \cap W) \vee Z$. If $Y \subseteq W$, then $Y \subseteq V \cap W \subseteq (V \cap W) \vee Z$, as required. Otherwise, we have $Y = (Y \cap W) \vee Z$ and $Y \subseteq V$, so that $Y \cap W \subseteq V \cap W$, and therefore $Y = (Y \cap W) \vee Z \subseteq (V \cap W) \vee Z$ again.

The fact that the map $V \to V \vee Z$ is a homomorphism on L(S) suggests that Z is a special type of element in the lattice. In fact we shall show that Z satisfies the conditions needed to make it a neutral element of L(S).

Definition 3.3.12: An element a in a lattice L is called neutral if

1) if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ then $x = y$, for all x and y in L,

2) the map $\alpha: x \to x \vee a$ is a lattice homomorphism,

and 3) the map $\beta: x \to x \wedge a$ is a lattice homomorphism.

The importance of neutral elements is reflected in the following proposition.

Proposition 3.3.13: Let a be an element of a lattice L. Then a is a neutral element of L iff the map $\delta : x \to (a \wedge x, a \vee x)$ is an isomorphism of L onto a subdirect product of $\{x \in L: x \leq a\}$ and $\{x \in L: x \geq a\}$.

For the remainder of this section, we verify that Z is a neutral element of L(S).

Proposition 3.3.14: Let V and W be varieties of semigroups. If $V \vee Z = W \vee Z$ and $V \cap Z = W \cap Z$, then $V = W$.

Proof: We know that $V \cap Z$ is either T, if Z is not contained in V, or Z, if Z is contained in V, and similarly for $W \cap Z$. Thus the assumption that $V \cap Z = W \cap Z$ implies that either both V and W contain Z, or neither do. In the first case, we get $V = V \vee Z = W \vee Z = W$ immediately. In the second case, we get $V = W$ from $V \vee Z = W \vee Z$ by Proposition 3.3.6.

Proposition 3.3.15: The map $\beta : V \to V \cap Z$ is a lattice homomorphism on L(S).

Proof: Since β obviously preserves meets, we need only check that $(V \vee W) \cap Z = (V \cap Z) \vee (W \cap Z)$, for any V and W in L(S). If either V or W contains Z, both sides of this equation become Z. If neither of V or W contain Z, then it is easily verified (by syntactic arguments) that $V \vee W$ does not contain Z either; and then both sides of the equation become T, the trivial variety.

Corollary 3.3.16: The variety Z of zero semigroups is a neutral element of L(S). Thus L(S) is isomorphic to a subdirect product of L(Z) and the interval sublattice [Z,S].

Remark 3.3.17: All of the results in this section depend heavily on the concept of inflation, as expressed in Proposition 3.3.1 and Lemmma 3.3.2. Unfortunately this concept does not seem to extend to higher nilpotency indices: no corresponding construction has been found for describing the varieties $V \vee N_k$ when $k \geq 3$. For this reason the results described in the next section for such varieties are not nearly so complete.

3.4. Joins with N_k

Volkov [16] has proved that if V is a finitely based variety then so is $V \vee N_k$, for $k \geq 2$, but without giving a method to explicitly produce a finite basis for $V \vee N_k$ from one for V. Clarke [4] has given a method for converting a basis of identities for a variety V into a basis of identities for $V \vee Z$. In this section we try to extend both these results. For certain varieties V we are able to produce identities which define $V \vee N_k$, for all $k \geq 2$. We begin with the variety $B = B_{1,1}$ of all bands.

Proposition 3.4.1: Let $k \geq 2$. The variety $B \vee N_k$ is defined by the identities $x^k = x^{2k}$, $x^k y = (xy)^k = xy^k$, and $x_1 \cdots x_k = (x_1 \cdots x_k)^k$.

Proof: Let W be the variety determined by the given identities. It is clear that $B \vee N_k \subseteq W$, and also that W satisfies the additional identity $x^k y^k = (xy)^k$.

Now let C be any semigroup in W. We define a map $\Theta : C \rightarrow C$ by $\Theta(c) = c^k$, for all c in C. Then it follows from the identities given that

1. Θ is a homomorphism;
2. $\Theta^2 = \Theta$, since W satisfies $x^k = x^{2k}$ and hence $x^k = x^{k^2}$;
3. $\Theta(C)$ is an ideal of C;
4. $\Theta(C)$ is in B, since for any c in C, $(c^k)^2 = c^k$ holds;
5. $C/\Theta(C)$ is in N_k, since the product of any k elements of C is in $\Theta(C)$.

Hence by Lemma 3.1.3, C is in $B \vee N_k$.

Corollary 3.4.2: Any semigroup in $B \vee N_k$ is a subdirect product of a band and a k-nilpotent semigroup.

The identities exhibited in Proposition 3.4.1 will be used later in showing that the varieties $B \vee N_k$, for $k \geq 2$, are all closed. For now we proceed to generalize this result, producing identities for $B_{1,m} \vee N_k$, for all $m \geq 1$ and all $k \geq 2$, and then considering $W \vee N_k$ for $W \subseteq B_{1,m}$.

Proposition 3.4.3: Let $m \geq 1$ and $k \geq 2$. Let a be the first number $\geq k$ which is congruent to 1 modulo m. Then the variety $B_{1,m} \vee N_k$ is defined by the identities

$$x^a y = xy^a = (xy)^a, \quad x^a = x^{(m+1)a}, \quad \text{and } x_1 \cdots x_k = (x_1 \cdots x_k)^a.$$

Proof: Let W be the variety defined by the given identities. First, since a is congruent to 1 modulo m, $x^a = x$ holds in $B_{1,m}$, and so clearly $B_{1,m} \vee N_k \subseteq W$. Conversely, let C be any semigoup in W. Define $\Theta : C \to C$ by $\Theta(c) = c^a$ for all c in C. Then

1. Θ is a homomorphism, since $x^a y^a = (xy)^{a^2} = (xy)^a$ holds in W because a^2 and a are congruent modulo m;
2. Θ is a retraction, since $x^{a^2} = x^a$ holds in W;
3. $\Theta(C)$ is an ideal of C, since for any c and d in C we have $c^a d = (c^a d)^a \in \Theta(C)$, and similarly dc^a is in $\Theta(C)$;
4. $\Theta(C)$ is in $B_{1,m}$, since $x^a = (x^a)^{m+1}$ holds in W;
5. $C/\Theta(C)$ is in N_k, since for any $c_1, \ldots, c_k$ in C, $c_1 \cdots c_k = (c_1 \cdots c_k)^a \in \Theta(C)$;

Therefore by Lemma 3.1.3 C is in $B_{1,m} \vee N_k$.

Proposition 3.4.4: Let $W \subseteq B_{1,m}$ for $m \geq 1$. Let $k \geq 2$, and let a be the first number $\geq$ k which is congruent to 1 modulo m. Let Σ be a basis for W, with $x = x^{m+1}$ in Σ. Write $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where $\Sigma_1 = \{x = x^{m+1}\}$, $\Sigma_2 = \{u = v \in \Sigma: |u|, |v| \geq k\}$, and $\Sigma_3 = \Sigma - (\Sigma_1 \cup \Sigma_2)$. Let $\Sigma_3^* = \{u^* = v^*: u = v \in \Sigma_3\}$, where u^* is obtained from u by replacing each letter x in u by x^a, each time it occurs. Then $W \vee N_k$ is defined by the identities in $\Sigma_2 \cup \Sigma_3^*$ plus the additional identities $x^a y = xy^a = (xy)^a$, $x_1 \cdots x_k = (x_1 \cdots x_k)^a$, and $x_1 \cdots x_k = (x_1 \cdots x_k)^a$.

Proof: Note that since $x^a = x$ holds in W, $W \vee N_k$ satisfies all of these identities. The proof then follows exactly that of the previous proposition, using Θ, up to part 4. This time we have $\Theta(C)$ in W, since by construction $\Theta(C)$ satisfies all the identities in Σ. The conclusion follows.

Corollary 3.4.5: Let $k \geq 2$, and let W be a subvariety of $B_{1,m}$ for some $m \geq 1$. Then any semigroup in $W \vee N_k$ is a subdirect product of a semigroup in W and a k-nilpotent semigroup.

In investigating closures of varieties later we will be interested in the following special cases:

Corollary 3.4.6:

i) Let $W = V(x = x^2, u = v)$ be a variety of bands. For $k \geq 2$, $W \vee N_k$ is defined by the identities $x^k y = xy^k = (xy)^k$, $x_1 \cdots x_k = (x_1 \cdots x_k)^k$, $x^k = x^{2k}$, and either $u = v$, if both $|u|$ and $|v|$ are $\geq k$, or $u^* = v^*$, otherwise, where u^* and v^* are formed from u and v respectively by replacing each letter x by x^k.

ii) For $m \geq 1$ and $k \geq 2$, it follows from the proof of Proposition 3.4.3 that

$M_{1,m} \vee MN_k = (B_{1,m} \vee N_k) \cap M$, so this variety is defined by the identities $x^a y = xy^a = (xy)^a$, $x^a = x^{(m+1)a}$, $x_1 \cdots x_k = (x_1 \cdots x_k)^a$, and $xyzw = xzyw$, where a is the first integer $\geq$ k and congruent to 1 modulo m.

The variety $M_{1,m} \vee MN_k$ turns out to be significant, since under certain restrictions on m and k it corresponds to the closure of $A_{1,m}$. Thus we now give a different set of identities for $M_{1,m} \vee MN_k$. These identities will correspond to hyperidentities; and they have the additional advantage that they can be generalized to deal with $M_{n,m} \vee MN_k$ for $n \geq 2$ as well.

<u>Notation 3.4.7:</u> Let $m \geq 1$ and $k \geq 2$. We use $\Sigma_{1,m,k}$ for the set of identities

$$x_1 \cdots x_k = x_1^{m+1} x_2 \cdots x_k,$$

$$x_1 \cdots x_k = x_1 x_2^{m+1} \cdots x_k,$$

$$x_1 \cdots x_k = x_1 \cdots x_{k-1} x_k^{m+1}, \text{ and } xyzw = xzyw.$$

<u>Proposition 3.4.8:</u> For any $m \geq 1$ and any $k \geq 2$, $M_{1,m} \vee MN_k = V(\Sigma_{1,m,k})$.

<u>Proof:</u> Clearly $M_{1,m} \vee MN_k \subseteq V(\Sigma_{1,m,k})$. Conversely, let C be any semigroup in $V(\Sigma_{1,m,k})$. Note that C satisfies the additional identities $x^k = x^{k+m}$ and $x_1 \cdots x_k = x_1^{m+1} x_2^{m+1} \cdots x_k^{m+1} = (x_1 \cdots x_k)^{m+1}$. Define a map $\Theta : C \to C$ by $\Theta(c) = c^{mk+1}$, for all c in C. Then

1. Θ is a homomorphism, because of the medial identity;

2. Θ is a retraction, since C satisfies $(x^{mk+1})^{mk+1} = x^{mk+1+mk(mk+1)} = x^{mk+1}$;

3. $\Theta(C)$ is an ideal of C, since C satisfies $xy^{mk+1} = x^{mk+1}y^{mk+1} = (xy)^{mk+1}$, and dually $x^{mk+1}y = (xy)^{mk+1}$;

4. $\Theta(C)$ is in $M_{1,m}$, since C satisfies both the medial identity and $(x^{mk+1})^{m+1} = x^{mk+1+m(mk+1)} = x^{mk+1}$;

5. $C/\Theta(C)$ is in MN_k, since C satisfies $x_1 \cdots x_k = (x_1 \cdots x_k)^{m+1} = (x_1 \cdots x_k)^{mk+1}$;

Thus C is in $M_{1,m} \vee MN_k$, as required, by Lemma 3.1.3.

Corollary 3.4.9: Any semigroup in $M_{1,m} \vee MN_k$ is a subdirect product of a k-nilpotent semigroup and a strong semilattice of rectangular groups of exponent m.

Proof: This follows from the proof of Proposition 3.4.8 and Corollary 3.2.2.

The argument in the proof of Proposition 3.4.8 can be modified slightly to deal with the join $A_m \vee RB \vee MN_k$, rather than $A_{1,m} \vee RB \vee MN_k$. Let $\Sigma^*_{1,m,k}$ be the set of identities formed from $\Sigma_{1,m,k}$ by adding the identity

$$wxy^m w^{2^{m+1}-m-2} = wxw^{2^{m+1}-2}.$$

(This new identity will appear later in our investigation of hyperidentities.)

Proposition 3.4.10: For any $m \geq 2$ and any $2 \leq k \leq 2^{m+1}$, $A_m \vee RB \vee MN_k = V(\Sigma^*_{1,m,k})$.

Proof: Since $k \leq 2^{m+1}$, it is clear that $A_m \vee RB \vee MN_k$ is contained in $V(\Sigma^*_{1,m,k})$. For the opposite inclusion, we repeat the proof of Proposition 3.4.8 exactly, except for part 4. There we still have $\Theta(C)$ in $M_{1,m}$, but now by Corollary 3.2.4, $\Theta(C)$ is also in $A_m \vee RB$, as required.

The identities in $\Sigma_{1,m,k}$ can be extended to deal with $M_{n,m} \vee MN_k$ for $n \geq 2$. Note that when $n \geq k$ we have $MN_k \subseteq M_{n,m}$, and hence $M_{n,m} \vee MN_k = M_{n,m}$, so we now consider the case where $n < k$.

Notation 3.4.11: Let $k \geq 2$ and $n, m \geq 1$, with $n < k$. Set $s = k-n+1$. We use $\Sigma_{n,m,k}$ for the set of identities

$$x_1^n x_2 \cdots x_s = x_1^{n+m} x_2 \cdots x_s,$$

$$x_1 x_2^n x_3 \cdots x_s = x_1 x_2^{n+m} x_3 \cdots x_s,$$

$$x_1 \cdots x_{s-1} x_s^n = x_1 \cdots x_{s-1} x_s^{n+m},$$

and

$$xyzw = xzyw.$$

Note that this includes the previous definition of $\Sigma_{1,m,k}$. However when $n > 1$ we can no longer give a structural proof that the identities in $\Sigma_{n,m,k}$ define the variety $M_{n,m} \vee MN_k$: our previous proof method breaks down since $\Theta(C)$ will no longer be an ideal of C. Instead we give a syntactic proof.

Proposition 3.4.12: Let $m \geq 1$, $k \geq 2$, and $1 \leq n < k$. Then $M_{n,m} \vee MN_k = V(\Sigma_{n,m,k})$.

Proof: Since $M_{n,m} \vee MN_k \subseteq V(\Sigma_{n,m,k})$, it will suffice to prove that any identity satisfied by $M_{n,m} \vee MN_k$ is also satisfied by $V(\Sigma_{n,m,k})$.

Let $u = v$ be a non-trivial identity satisfied by $M_{n,m} \vee MN_k$, and hence by both $M_{n,m}$ and MN_k. Then either $u = v$ is a consequence of medial, and so is certainly satisfied by $V(\Sigma_{n,m,k})$, or both $|u|$ and $|v|$ are $\geq k$. So we will now assume that $k \leq |u| \leq |v|$. We will prove that there is a sequence $u = u_0 = u_1 = \ldots = u_r = v$ such that each move $u_i = u_{i+1}$ is a consequence of the medial identity or the identity $x^n = x^{n+m}$, and such that $|u_i| \geq k$ for all $0 \leq i \leq r$. From this it will follow that $V(\Sigma_{n,m,k})$ also satisfies $u = v$.

We now describe how to produce such a sequence. First, by repeated use of the medial identity, we may write any word w in a "standard form" $\bar{w}$ as follows. Rewrite any string $(w_1 \cdots w_l)^c$ in w as $w_1^c w_2^c \cdots w_l^c$. Then as in the proof of Proposition 3.2.5, express the rewritten string as

$$x^a y_1^{a_1} \cdots y_p^{a_p} y^b,$$

where $x, y_1, \ldots y_p, y$ are the distinct (except possibly $x = y$) letters occurring in the word w; y_t occurs a_t times in w for $1 \leq t \leq p$; and x and y occur a and b times respectively in w, except that if $x = y$ then $b = 1$ and x occurs $a+1$ times in w.

Now by construction $M_{n,m}$ satisfies $u = \bar{u}$ and $v = \bar{v}$, and in fact there are deductions of these two identities involving only words of length $\geq k$. Since $M_{n,m}$ satisfies $u = v$, it also satisfies $\bar{u} = \bar{v}$. Also, $|u| = |\bar{u}|$ and $|v| = |\bar{v}|$. Thus it will suffice to produce a deduction of $\bar{u} = \bar{v}$ in $M_{n,m}$ in which the length of any intermediate word is $\geq k$.

Consider first the case where x and y are distinct letters. Then we can write

$$\bar{u} = x^a y_1^{a_1} \cdots y_p^{a_p} y^b,$$

and $$\bar{v} = x^c y_1^{c_1} \cdots y_p^{c_p} y^d.$$

Since $M_{n,m}$ satisfies $\bar{u} = \bar{v}$, we must have $a_l = c_l$ or a_l and c_l both $\geq n$ and congruent modulo m, for each $1 \leq l \leq p$, and similarly for a and c and b and d. For any variable z in $\bar{u}$, the net change in power on z as we go from $\bar{u}$ to $\bar{v}$ can then be accomplished as a series of moves of the form $z^e = z^{e+m}$ (an increase) or $z^{e+m} = z^e$ (a decrease), for some $e \geq n$. It is clear that having grouped together all occurrences of each such variable z, the moves done to one variable are independent of those done to another, and such moves can be done in any order. Therefore we can arrange to move from $\bar{u}$ to $\bar{v}$ in such a way that all increases are done first, and then any decreases. Since $|u|$ and $|v|$ are $\geq k$, this guarantees that any intermediate word in the sequence of moves also has length $\geq k$, as required.

The case $x = y$ is handled in much the same way. This time we have

$$\bar{u} = x^a y_1^{a_1} \cdots y_p^{a_p} x \quad \text{and} \quad \bar{v} = x^c y_1^{c_1} \cdots y_p^{c_p} x,$$

which we will simplify a bit and write as $\bar{u} = x^a wx$ and $\bar{v} = x^c w'x$, where w and w′ are words not involving the letter x. As in the $x \neq y$ case, we can always change w to w′ using only $M_{n,m}$ identities by doing all the necessary increases first, then all the decreases, so that the greatest possible length is maintained. So we concentrate now on the letter x. If $a = c$, we are done. Otherwise, we must have a and c both $\geq$ n, and congruent modulo m. So again the net change in power on x is either an increase or a decrease, by a multiple of m. If this net change is an increase, we do it first, then change w to w′ as previously described; if the net change is a decrease, we do it after the change from w to w′ is made. In either case we move from $\bar{u} = x^a wx$ to $\bar{v} = x^c w'x$, maintaining at each stage a word-length $\geq$ k. This completes the proof of the proposition.

We will conclude this chapter with the join $A_{n,m} \vee RB \vee MN_k$, which will appear later in Chapter 6 as the closure of $A_{n,m}$ for certain n, m, and k combinations. The syntactic proof given below combines the arguments used for $A_{n,m} \vee RB$ in Proposition 3.2.5 and for $M_{n,m} \vee MN_k$ in Proposition 3.4.12.

<u>Proposition</u> <u>3.4.13:</u> Let $k \geq 2$, $2 \leq n < k$, and $m \geq 1$. The variety $A_{n,m} \vee RB \vee MN_k$ is defined by the following identities:

$$xyzw = xzyw$$

$$x_1^n \cdots x_{k-n+1} = x_1^{n+m} \cdots x_{k-n+1},$$

$$x_1 x_2^n \cdots x_{k-n+1} = x_1 x_2^{n+m} \cdots x_{k-n+1},$$

$$x_1 \cdots x_{k-n+1}^n = x_1 \cdots x_{k-n+1}^{n+m},$$

and

$$x^{n-1}y_1 \cdots y_{k-n}x = x^{n-1+m}y_1 \cdots y_{k-n}x.$$

Proof: Let U be the variety defined by the given identities. Then certainly $A_{n,m} \vee RB \vee MN_k \subseteq U$. Conversely, we show that any non-trivial identity $u = v$ satisfied by $A_{n,m}$, RB, and MN_k is also satisfied by U.

When MN_k satisfies $u = v$, either M and hence U satisfies $u = v$, and we are done, or $|u|$ and $|v|$ are both $\geq k$. Since RB satisfies $u = v$, u and v have the same first letter, x say, and the same last letter, y say, with x and y possibly equal. Since $A_{n,m}$ satisfies $u = v$, u and v contain exactly the same letters, and for any letter z in u or v, either the number of occurrences of z in u is equal to the number of occurrences of z in v, or these two quantities are $\geq$ n and are congruent modulo m. Therefore we will transform u and v into the standard form $\bar{u}$ and $\bar{v}$ of Propositions 3.2.5 and 3.4.12. As before, $A_{n,m}$ and $M_{n,m}$ still satisfy $u = \bar{u}$ and $v = \bar{v}$, and $|u| = |\bar{u}|$ and $|v| = |\bar{v}|$.

The case where x and y are distinct variables is dealt with exactly as in the proof of Proposition 3.4.12: we ensure a sequence of moves from $\bar{u}$ to $\bar{v}$ in which all intermediate words have length $\geq$ k, by performing all necessary increases first, and then any necessary decreases. The first four of the five defining identities for U are sufficient for this.

In the case where $x = y$ we consider $\bar{u} = x^a wx$ and $\bar{v} = x^c w'x$, where w and w' are words not containing the letter x. As before, we are able to change w to

w' in such a way as to maintain maximum length of words. If $a = c$, therefore, we are done. Otherwise, we have a and $c \geq n\text{-}1$, and a congruent to c modulo m. In this case we use the identity $x^{n-1}y_1 \cdots y_{k-n}x = x^{n-1+m}y_1 \cdots y_{k-n}x$ from U to make the change from x^a to x^c. As in the $x = y$ case in Proposition 3.4.12, there are two possibilities: if $a > c$, we first transform w to w', then x^a to x^c, to get a deduction of $\bar{u} = \bar{v}$; while if $a < c$ we change x^a to x^c first, then change w to w'. In either situation we produce a deduction of $\bar{u} = \bar{v}$ in which all words have length $\geq k$, as required.

Chapter 4

Some Closure Results

In this chapter we investigate closure properties for some interesting classes of varieties. The varieties of bands studied in Section 4.1 have the special features of idempotence and duality, and by exploiting these features we produce a complete description of how the closure operator and the hypervariety operator $\mathcal{H}$ act on such varieties. The nilpotent varieties are also examined, and finally varieties obtained by taking the joins of varieties of bands with nilpotent varieties. The results obtained are incorporated into a picture of part of the lattice $\mathcal{H}(L(CS))$.

4.1. Varieties of Bands

The variety B of all bands is easily seen to be closed. It satisfies the idempotent hyperidentity $F(x,x) = x$, since all the instances of this hyperidentity are of the form $x^a = x$ for some $a \geq 1$. In particular, the binary term x_1x_2 gives the instance $x^2 = x$, so $B \subseteq \overline{B} \subseteq V(x^2 = x)$, and B is closed. Moreover any variety which satisfies the idempotent hyperidentity is a variety of bands, so by Lemma 2.3.5 we have a basis of size one for the set HI(B) of hyperidentities satisfied by B.

In this section we present closure and hypervariety results for subvarieties of B. The completeness of these results is due to two reasons: the structure of the lattice L(B) of all varieties of bands is known, with equational descriptions of the varieties; and the special properties of idempotence and duality make possible the construction of useful hyperidentities.

The structure of the lattice L(B) has been described independently by Birjukov [2], Fennemore [6], and Gerhard [7]. We will use the notation of Fennemore, whose diagram of the lattice is shown in Figure 4.1. Each variety of bands is defined by the idempotent identity $x^2 = x$ and one additional identity; it is these additional identities which label the various varieties in Figure 4.1. The words R_n, Q_n, and S_n, for $n \geq 3$, are defined inductively on the alphabet $\{x_1, \ldots, x_n\}$. For our purposes, it will suffice to know that for any $n \geq 3$, R_n, Q_n, and S_n all have length $\geq n$, all begin with the same variable, and also all end with the same variable. For any word w, we use w^d for the dual word. Duality is an important feature of L(B): the lattice is symmetric about its centre column, with mirror-image varieties $V = V(x^2 = x, u = v)$ and $V^d = V(x^2 = x, u^d = v^d)$, the dual variety of V. The self-dual varieties are those in the centre column, which are equal to their own duals. The next proposition shows how duality enters into the study of hyperidentities and hypervarieties.

<u>Proposition</u> <u>4.1.1:</u> Let V be any variety of bands which is not self-dual. Then the clones C(V) and C(V^d) are isomorphic, so that $C(V) = C(V^d)$ and $\mathcal{H}(V) = \mathcal{H}(V^d)$.

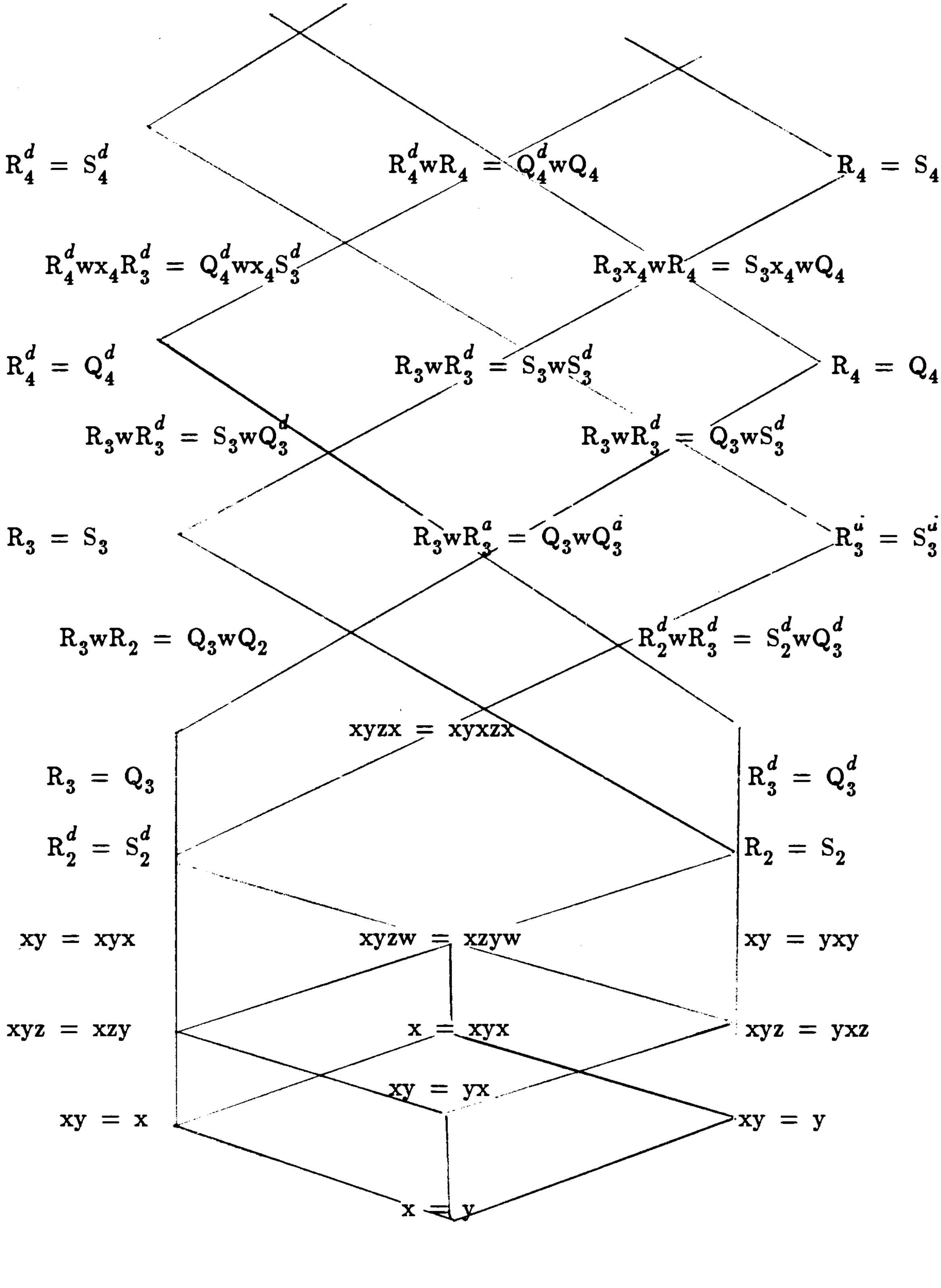


Figure 4-1: The Lattice L(B) of Varieties of Bands

Proof: For any n-ary term $t = t(x_1, \ldots, x_n)$ in the clone of V, let t^d be the term defined by the dual word $(t(x_1, \ldots x_n))^d$. Then t^d is in the clone of V^d. This sets up a mapping δ: $C(V) \rightarrow C(V^d)$. Clearly δ is a bijection which maps the n-ary projection terms x_i, $1 \leq i \leq n$, of V to the n-ary projection terms x_i of V^d, and it is easily verified that δ is compatible with the composition of terms. The claims then follow from this and Proposition 2.1.4.

Corollary 4.1.2: For any variety V of bands which is not self-dual, $\mathcal{H}(V \vee V^d) = \mathcal{H}(V) \vee \mathcal{H}(V^d) = \mathcal{H}(V) = \mathcal{H}(V^d)$, and $V \vee V^d \subseteq \overline{V}$.

This tells us that the closure operator and the operator $\mathcal{H}$ both induce a certain amount of collapsing on L(B). By applying $\mathcal{H}$ to the known structure of L(B) we produce a chain of hypervarieties, as shown in Figure 4.2. In particular, note that by Proposition 3.1.1, $\mathcal{H}(NB) = \mathcal{H}(RB \vee SL) = \mathcal{H}(RB) \vee \mathcal{H}(SL) = \mathcal{H}(SL)$.

Our goal now is to show that there is no further collapsing of L(B) under $\mathcal{H}$, so that all the hypervarieties $\mathcal{H}(V)$ shown in Figure 4.2 are distinct. We do this by producing, for each such self-dual V, a hyperidentity satisfied by V but not by the next variety in the chain. The hyperidentities produced will also allow us to identify the closure $\overline{V}$ of any variety V of bands.

All the hyperidentities to be used in this section involve only a single binary operation symbol. The advantage of this is that any variety of bands has at most six binary terms, namely x_1, x_2, x_1x_2, x_2x_1, $x_1x_2x_1$, and $x_2x_1x_2$. Thus it will be

$\mathcal{H}(B)$

$\vdots$

$\mathcal{H}(V(x^2 = x,\ R_4^d w R_4 = S_4^d w S_4))$

$\mathcal{H}(V(x^2 = x,\ R_4^d w R_4 = Q_4^d w Q_4))$

$\mathcal{H}(V(x^2 = x,\ R_3 w R_3^d = S_3 w S_3^d))$

$\mathcal{H}(V(x^2 = x,\ R_3 w R_3^d = Q_3 w Q_3^d))$

$\mathcal{H}(V(x^2 = x,\ xyzx = xyxzx))$

$\mathcal{H}(NB) = \mathcal{H}(LN) = \mathcal{H}(RN) = \mathcal{H}(SL)$

$\mathcal{H}(RB) = \mathcal{H}(LZ) = \mathcal{H}(RZ)$

$\mathcal{H}(T)$

Figure 4-2: The Lattice $\mathcal{H}(L(B))$

easy to verify that a given variety of bands does indeed satisfy a given hyperidentity: we produce the corresponding six instances, and check that each one is an identity of the given variety.

When dealing with a hyperidentity involving only one binary operation symbol, we often focus on the particular instance we get from substitution of the term x_1x_2. We will say that the hyperidentity is based on this instance. Note that this instance can be easily obtained by reading off the list of variables in the hyperidentity, in order of occurrence from left to right.

Proposition 4.1.3:

i) The variety T of trivial semigroups satisfies the hyperidentity $F(x,y) = x$, while no other variety of semigroups does.

ii) The variety RB satisfies the hyperidentity $F(x,F(y,x)) = x$, while the variety NB does not.

iii) The variety NB satisfies the hyperidentity

$$F(x,F(F(y,z),x)) = F(x,F(F(z,y),x))$$

while the variety $V(x^2 = x,\ xyzx = xyxzx)$ does not.

iv) The variety $V(x^2 = x,\ xyzx = xyxzx)$ satisfies the hyperidentity

$$F(x,F(F(y,z),x)) = F(x,F(F(y,F(x,z)),x)),$$

while the variety $V(x^2 = x,\ R_3wR_3^d = Q_3wQ_3^d)$ does not.

Proof: In each case it is easy to verify that the given variety V does satisfy the given hyperidentity: we omit the details. In i), use of the projection term x_2 yields the identity $y = x$, so that only the trivial variety satisfies the given hyperidentity. In the remaining cases, note that we have given for each variety $V = V(x^2 = x,\ u = v)$ a hyperidentity based on the identity $u = v$. Thus no variety above V in the lattice L(B) satisfies the hyperidentity.

Corollary 4.1.4: $\mathcal{H}(T) \subsetneqq \mathcal{H}(RB) \subsetneqq \mathcal{H}(NB) \subsetneqq \mathcal{H}(V(x^2 = x,\ xyzx = xyxzx))$ $\subsetneqq \mathcal{H}(V(x^2 = x,\ R_3wR_3^d = Q_3wQ_3^d))$.

Corollary 4.1.5: Each of the varieties T, RB, NB, and $V(x^2 = x,\ xyzx = xyxzx)$ is closed.

Proof: From the proof of Proposition 4.1.3 we have $\overline{T} \subseteq V(x = y) = T$, so that $T = \overline{T}$. Each of the remaining varieties mentioned may be expressed as $V = V(x^2,\ u = v)$ for some self-dual identity $u = v$. Again from the proof of Proposition 4.1.3, we have $\overline{V} \subseteq V(u = v)$. But also any variety of bands satisfies the idempotent hyperidentity based on $x^2 = x$, so that in fact we have $\overline{V} \subseteq V(x^2 = x,\ u = v) = V$. This proves that $V = \overline{V}$ and V is closed.

Corollary 4.1.4 allows us to prove some remarks made in Sections 2.1 and 2.3 about the closure operator and the operator $\mathcal{H}$. From this Corollary and Proposition 4.1.1, we get

$$\mathcal{H}(T) = \mathcal{H}(LZ \cap RZ) \subsetneqq \mathcal{H}(LZ) \cap \mathcal{H}(RZ) = \mathcal{H}(RB) \cap \mathcal{H}(RB) = \mathcal{H}(RB).$$

Thus we know that $\mathcal{H}$ does not preserve intersections, so is not a homomorphism on the lattice L(S). Similarly,

$$T = \overline{T} = \overline{LZ \cap RZ} \subsetneqq \overline{LZ} \cap \overline{RZ} = RB \cap RB = RB,$$

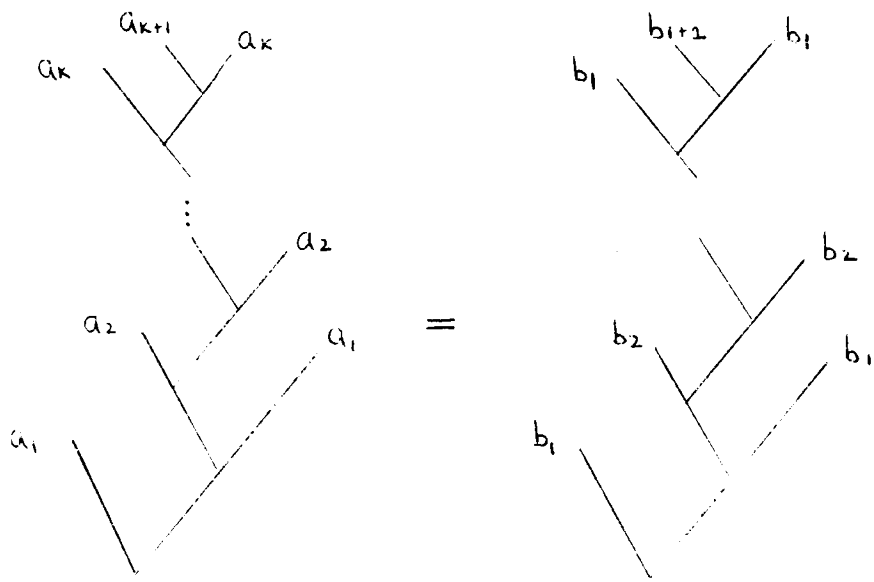
so the closure operator also does not preserve intersections.

We have now reached the inductively-defined part of the chain in Figure 4.2. Here we must consider the varieties $V(x^2 = x, R_n w R_n^d = Q_n w Q_n^d)$ and $V(x^2 = x, R_n w R_n^d = S_n w S_n^d)$ for $n \geq 3$ and n odd, and their duals for $n \geq 4$ and n even. Each of these varieties may be written as $V(x^2 = x, u = v)$, where $u = v$ is a self-dual identity, and u and v begin with the same variable and end with the same variable. Our technique is the one used in the previous proposition: we produce a hyperidentity HB(u =v) which is based on u = v and satisfied by the variety $V(x^2 = x, u = v)$.

From our assumptions about each such identity u = v, we may write it in the form

$$a_1 a_2 \cdots a_k a_{k+1} a_k \cdots a_2 a_1 = b_1 b_2 \cdots b_l b_{l+1} b_l \cdots b_2 b_1,$$

where k and l are ≥ 3, the a_i's and b_i's are variables from our standard alphabet, and $a_1 = b_1$. We define the hyperidentity HB(u = v) to be



The left-hand tree diagram represents the hyperterm

$$F(a_1,F(F(a_2,F(F(a_3, \ . \ . \ . \ F(F(a_k,F(a_{k+1},a_k)),a_{k-1})), \ . \ . \ . \ a_2)),a_1)),$$

and similarly for the right-hand tree.

Proposition 4.1.6: Let $V = V(x^2 = x, u = v)$ be a self-dual variety of bands properly containing the variety $V(x^2 = x, xyzx = xyxzx)$. Then V satisfies the hyperidentity $HB(u = v)$ as constructed above; and the only varieties of bands to do so are subvarieties of V.

Proof: Substitution of either of the projection terms x_1 or x_2 into $HB(u = v)$ yields the identity $a_1 = b_1$, which is known to be trivial. Use of the term x_1x_2 gives precisely the identity $u = v$. This of course holds in V and any of its subvarieties, but not in any other variety of bands, which establishes the second claim. Use of the term x_2x_1 also gives $u = v$, since $u = v$ is self-dual.

For the term $x_1x_2x_1$, we will show by induction on k that the hyperterm just before the statement of Proposition 4.1.6 when evaluated using $x_1x_2x_1$ produces exactly the word $a_1 \cdots a_k a_{k+1} a_k \cdots a_1$. From this it will follow that under this choice of term, the hyperidentity $HB(u = v)$ again yields the identity $u = v$.

For the induction base $k=3$, it is easy to show that $F(a_1,F(F(a_2,F(F(a_3,F(a_4,a_3)),a_2)),a_1))$ evaluates under $x_1x_2x_1$ to $a_1a_2a_3a_4a_3a_2a_1$ (making repeated use of the band identity $x^2 = x$). Then for $k \geq 3$, evaluating with $x_1x_2x_1$ yields

$$F(a_1,F(F(a_2,F(F(a_3, \ \ldots \ F(F(a_{k+1},F(a_{k+2},a_{k+1})),a_k)), \ \ldots \ a_3)),a_2)),a_1))$$
$$= \ F(a_1,F(a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_3a_2, \ a_1))$$

$= F(a_1, a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_2a_1a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_2)$

$= a_1a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_2a_1a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_2a_1$

$= a_1a_2a_3 \cdots a_{k+1}a_{k+2}a_{k+1}a_k \cdots a_2a_1.$

The remaining term, $x_2x_1x_2$, may be shown in a very similar way to also produce only $u = v$ as an instance.

Corollary 4.1.7: The hypervarieties $\mathcal{H}(V)$, for V a self-dual variety of bands other than SL, form a countably infinite chain, as shown in Figure 4.2.

Proof: From Propositions 4.1.3 and 4.1.6 it follows that the hypervarieties shown in Figure 4.2 are all distinct.

Propositions 4.1.3 and 4.1.6 also allow us to completely describe the closure operator as it acts on varieties of bands. For any self-dual variety $V = V(x^2 = x, u = v)$ except SL, we have shown that V satisfies hyperidentities based on $x^2 = x$ and $u = v$. Thus $V \subseteq \overline{V} \subseteq V(x^2 = x, u = v)$, and so $V = \overline{V}$ and V is closed. If V is not self-dual, then by Corollary 4.1.2, $V \vee V^d \subseteq \overline{V} \subseteq \overline{(V \vee V^d)} = V \vee V^d$, since $V \vee V^d$ is self-dual and hence closed; therefore $\overline{V} = V \vee V^d$ in this case. In the special case of SL, we saw in the comments following Corollary 4.1.2 that $\mathcal{H}(SL) = \mathcal{H}(NB)$, so that $NB = SL \vee RB \subseteq \overline{SL} \subseteq NB$, and $\overline{SL} = NB$. We summarize this in:

Corollary 4.1.8: For any variety V of bands except SL, $\overline{V} = V \vee V^d$; and $\overline{SL} = NB$.

Our hyperidentity results also give us information about bases for HI(V), when V is a non-trivial closed variety of bands. Let V = V(x^2 = x, u = v) be closed. We have seen that only subvarieties of V satisfy both HB(u = v) and the idempotent hyperidentity F(x,x) = x. Therefore by Lemma 2.3.5, the set Σ containing these two hyperidentities is a basis for HI(V). In fact, it is an irredundant basis, in the sense that neither hyperidentity is a consequence of the other. We show this by finding for each of the two hyperidentities a variety which satisfies it, but not the other one. The variety B of all bands certainly satisfies the idempotent hyperidentity, but does not satisfy HB(u = v) when u = v is non-trivial. Conversely, since all instances of HB(u = v) involve words of length $\geq$ 2, the variety Z of zero semigroups satisfies HB(u = v) too, and hence by Corollary 2.2.3 so does V $\vee$ Z. But V $\vee$ Z does not satisfy F(x,x) = x, since its subvariety Z does not satisfy the base instance x^2 = x. This proves the following result.

Proposition 4.1.9: For any non-trivial closed variety V of bands, there is an irredundant basis of size two for HI(V).

4.2. Nilpotent Varieties

The next varieties whose closures we investigate are the nilpotent varieties N_k = V($x_1 \cdots x_k = y_1 \cdots y_k$), for k $\geq$ 2. We have seen that RB $\subseteq \overline{N_k}$, so $\overline{N_k}$ is at least $N_k \vee$ RB, and we know immediately that N_k is not closed. In hyperidentity terms, the RB condition translates into a requirement that any hyperidentity satisfied by N_k must have the same first variable in each hyperterm, and the same last variable. This suggests some hyperidentities we might try.

As in the previous section, all hyperidentities to be used here involve only a single binary operation. We note that for any $k \geq 2$, the binary terms of N_k are all possible words of length $< k$ on the variables x_1 and x_2, plus the word x_1^k; all words of length $\geq k$ are identified in N_k. This also means that a non-trivial identity $u = v$ holds in N_k iff both $|u|$ and $|v|$ are $\geq k$. We begin with the variety $N_2 = Z$.

Proposition 4.2.1: The variety Z satisfies the following hyperidentities:

$$F(x,F(y,x)) = F(x,x), \qquad F(x,F(x,y)) = F(x,y),$$
$$\text{and} \qquad F(x,F(y,y)) = F(x,y).$$

Proof: In each case the projection terms x_1 and x_2 lead to trivial identities $x = x$ or $y = y$. For any other binary terms all three hyperidentities yield only instances $u = v$ in which $|u|$ and $|v|$ are both ≥ 2.

Corollary 4.2.2: $\overline{Z} = Z \vee RB$.

Proof: From the hyperidentities in the previous Proposition, we have

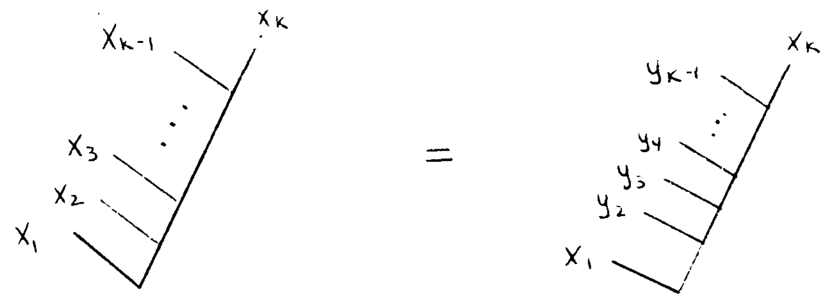
$$Z \vee RB \subseteq \overline{Z} \subseteq V(xyx = x^2,\ x^2y = xy,\ xy^2 = xy).$$

This latter variety is clearly contained in the variety

$$V(x^2y = xy^2 = xy,\ xy = (xy)^2,\ x^2 = x^4,\ x^2y^2x^2 = x^2),$$

which by Corollary 3.4.6 is precisely $Z \vee RB$.

Proposition 4.2.3: Let $k \geq 3$. The variety N_k satisfies the hyperidentity



based on $x_1x_2 \cdots x_k = x_1y_2 \cdots y_{k-1}x_k$.

Proof: Substituting terms of the form x_1^i, for $i \geq 1$, into this hyperidentity produces only trivial identities $x_1^i = x_1^i$. Using terms of the form x_2^j, for $j \geq 1$, also produces only trivial instances. But from any term described by a term involving both x_1 and x_2 we get an identity $u = v$ where both $|u|$ and $|v|$ are $\geq k$. Hence for any choice of term, the resulting identity is satisfied by N_k, as required.

We are now in a position to use some of the join results from Section 3.2. We saw in Proposition 3.2.5 that for $k \geq 3$, the identity $x_1x_2 \cdots x_k = x_1y_2 \cdots y_{k-1}x_k$ defines the variety $N_k \vee RB$, which together with the previous proposition proves that $\overline{N_k} = N_k \vee RB$ for $k \geq 3$. With only slight modifications we can also identify the closures of MN_k and AN_k, the varieties of medial and abelian k-nilpotent semigroups. Both varieties satisfy the hyperidentity given in Proposition 4.2.3, and also the medial hyperidentity. Hence we have

$$MN_k \vee RB \subseteq \overline{MN_k} \subseteq V(xyzw = xzyw,\ x_1x_2 \cdots x_k = x_1y_2 \cdots y_{k-1}x_k).$$

But by Proposition 3.2.7, this latter variety is in fact $MN_k \vee RB$, and we have $\overline{MN_k} = MN_k \vee RB$ for $k \geq 3$. Furthermore, we have

$$AN_k \vee RB \subseteq \overline{AN_k} \subseteq \overline{MN_k} = MN_k \vee RB = AN_k \vee RB,$$

the last equality from Proposition 3.2.8, so that $\overline{AN_k} = AN_k \vee RB = \overline{MN_k}$ for $k \geq 3$. This proves the following result.

Proposition 4.2.4: For $k \geq 2$, $\overline{N_k} = N_k \vee RB$. For $k \geq 3$, $\overline{MN_k} = MN_k \vee RB = \overline{AN_k}$.

Proposition 4.2.5: For $k \geq 2$, the hypervarieties $\mathcal{H}(N_k \vee RB)$ form a countably infinite chain in $\mathcal{HV}$.

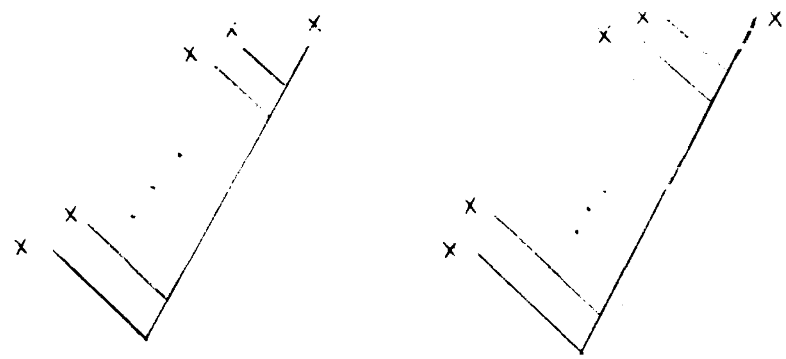
Proof: The hyperidentities in Proposition 4.2.1 are satisfied by $Z \vee RB$, but not by N_3, showing that $\mathcal{H}(Z \vee RB)$ is properly contained in $\mathcal{H}(N_3 \vee RB)$. For $k \geq 3$, $N_k \vee RB$ satisfies the hyperidentity in Proposition 4.2.3, based on an identity which is not satisfied by N_{k+1}, so that $\mathcal{H}(N_k \vee RB)$ is properly contained in $\mathcal{H}(N_{k+1} \vee RB)$. This gives a countably infinite chain of hypervarieties.

Note that the hyperidentity exhibited in Proposition 4.2.3 for N_k forms a basis of size one for the family of hyperidentities satisfied by N_k (or $N_k \vee RB$), since its base instance defines the closure $N_k \vee RB$.

4.3. Combining Band and Nilpotent Results

In this section we show that varieties of the form $V \vee N_k$, for V a non-trivial closed variety of bands, are also closed. We begin with $B \vee N_k$, and then extend to subvarieties of B. Recall from Proposition 3.4.1 that for $k \geq 2$, the variety $B \vee N_k$ is defined by the identities $x^k = x^{2k}$, $x^k y = (xy)^k = xy^k$, and $x_1 \cdots x_k = (x_1 \cdots x_k)^k$. Our first goal is to produce hyperidentities based on these identities. Note that they all at least are satisfied by RB.

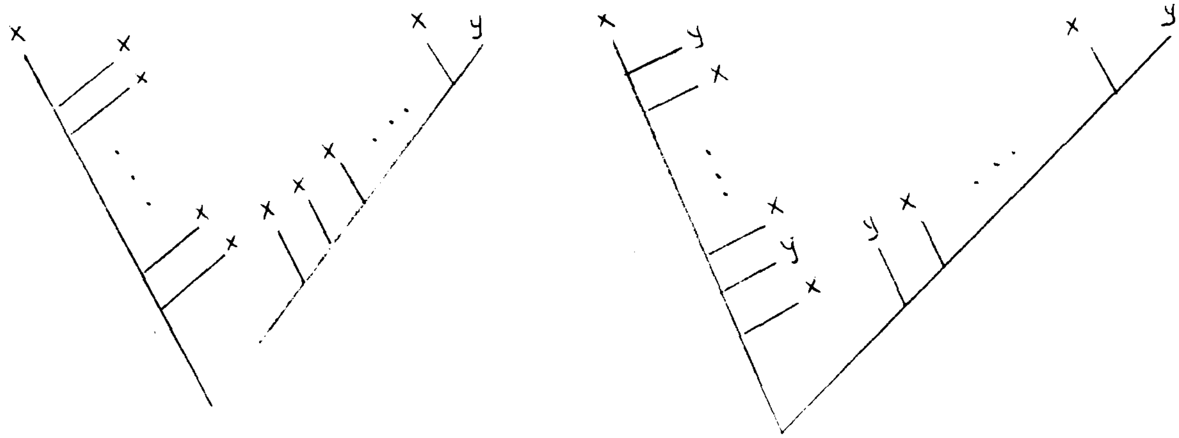
Proposition 4.3.1: Let $k \geq 2$. The variety $B \vee N_k$ satisfies the hyperidentity



based on $x^k = x^{2k}$.

Proof: Since any choice of semigroup term to be substituted for the binary operation symbol of this hyperidentity leads to an identity of the form $x^a = x^b$ for some a and $b \geq 1$, it is clear that B satisfies the hyperidentity. Moreover, it is also clear that either $a = b$ (terms x_1^i for $i \geq 1$), or a and b are both $\geq k$ (terms x_2^j for $j \geq 1$ or $x_1^i x_2^j$ for $i+j \geq 1$). Hence N_k satisfies the hyperidentity too, and so does the join $B \vee N_k$ by Corollary 2.2.3.

Proposition 4.3.2: Let $k \geq 2$. The variety $B \vee N_k$ satisfies the hyperidentity

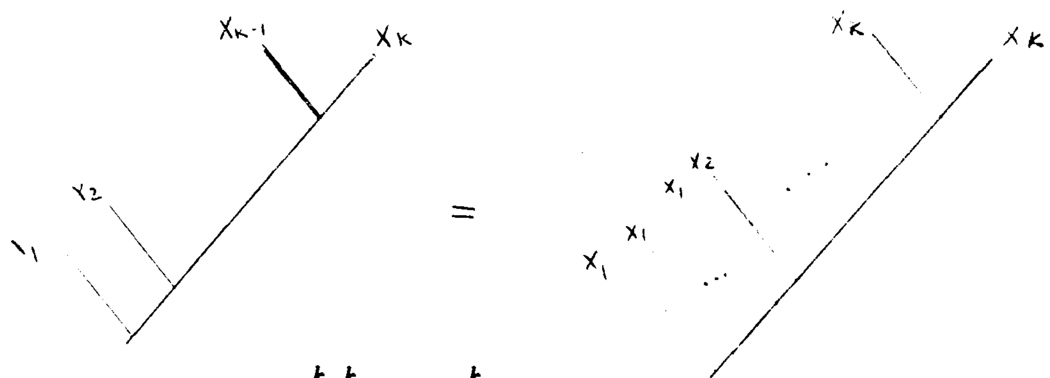


based on $x^{2k}y = (xy)^k$, and its dual.

Proof: By construction, using a projection term x_1 or x_2 in this hyperidentity yields an identity of the form $x^a = x^b$, with either $a = b$ or both a and b $\geq$ k, or of the form $y^c = y^c$. Using any other semigroup term gives an identity $u = v$ with both $|u|$ and $|v| \geq k$. Hence N_k satisfies the given hyperidentity.

To see that B also satisfies the hyperidentity, it suffices to check the four remaining binary band terms. Clearly the term x_1x_2 gives the base instance $x^{2k}y = (xy)^k$, which holds in B, while x_2x_1 gives a dual result. For the term $x_1x_2x_1$, it is easily verified that from either hyperterm we obtain a word which begins and ends with the letter x, and contains only the letters x and y. In B any two such words are equal, since they both reduce to the word xyx. Thus the identity obtained by using the term $x_1x_2x_1$ holds in B. A dual argument holds for the term $x_2x_1x_2$.

<u>Proposition</u> <u>4.3.3:</u> Let $k \geq 2$. The variety $B \vee N_k$ satisfies the hyperidentity



based on $x_1 \cdots x_k = x_1^k x_2^k \cdots x_k^k$.

<u>Proof:</u> As in the previous two proofs, any non-trivial instances $u = v$ of this hyperidentity have both $|u|$ and $|v| \geq k$, so that N_k satisfies the hyperidentity. For B, the two projection terms give trivial instances $x_1 = x_1$ or $x_k = x_k$. The terms x_1x_2 and x_2x_1 yield the base instance and its dual, both of which hold in B.

For the term $x_1x_2x_1$, a simple induction argument based on the shape of the two hyperterms shows that this choice of term leads to the identity

$$x_1 \cdots x_k x_{k-1} \cdots x_2 x_1 = x_1^k x_2^k \cdots x_k^k x_{k-1}^k \cdots x_2^k x_1^k.$$

This too holds in B.

Finally, we consider what effect the term $x_2x_1x_2$ has in the evaluation of the hyperidentity. In the right-hand hyperterm, evaluation starts at the top of the tree diagram, with k occurrences of x_k: using $x_2x_1x_2$ produces some power of x_k, which in B is just x_k again. At subsequent stages, evaluation involves expressions of the form $F(z,F(z,y))$. But such an expression gives

$$F(z,yzy) = yzyzyzy = yzy = F(z,y),$$

so by induction an expression

$$F(z,F(z, \ldots ,F(z,F(z,y)), \ldots ,),$$

with k occurrences of z, gives the same result as $F(z,y)$ alone. From this it follows that the given hyperidentity yields under the substitution of the term $x_2x_1x_2$ an identity which holds in B.

<u>Proposition 4.3.4:</u> For $k \geq 2$, the variety $B \vee N_k$ is closed.

<u>Proof:</u> Always $B \vee N_k \subseteq \overline{B \vee N_k}$. Starting with the base instances of the hyperidentities of the three previous propositions, we get

$$\begin{aligned}
\overline{B \vee N_k} &\subseteq V(x^k = x^{2k},\ x^{2k}y = (xy)^k = xy^{2k},\ x_1 \cdots x_k = x_1^k \cdots x_k^k) \\
&\subseteq V(x^k = x^{2k},\ x^ky = (xy)^k = xy^k,\ x_1 \cdots x_k = x_1^k \cdots x_k^k) \\
&\subseteq V(x^k = x^{2k},\ x^ky = (xy)^k = xy^k,\ x_1 \cdots x_k = x_1^k \cdots x_k^k,\ (xy)^k = x^ky^k) \\
&\subseteq V(x^k = x^{2k},\ x^ky = (xy)^k = xy^k,\ x_1 \cdots x_k = (x_1 \cdots x_k)^k\ ) \\
&= B \vee N_k
\end{aligned}$$

by manipulation of identities and then using Proposition 3.4.1. Therefore $B \vee N_k = \overline{B \vee N_k}$, and this variety is closed.

The hyperidentities given in Propositions 4.3.1 and 4.3.3 are based on instances which involve words of length k. Thus these hyperidentities, although satisfied by $B \vee N_k$, are not satisfied by N_{k+1} or by $B \vee N_{k+1}$. This establishes that these closed varieties generate distinct hypervarieties.

<u>Corollary</u> <u>4.3.5:</u> The hypervarieties $\mathcal{H}(B)$ and $\mathcal{H}(B \vee N_k)$, for $k \geq 2$, form a countably infinite chain of hypervarieties.

The preceding results for $B \vee N_k$ extend easily to subvarieties of B. Let $V = V(x^2 = x, u = v)$ be a non-trivial closed (hence self-dual) variety of bands. For the join $V \vee N_k$, we must distinguish whether or not $|u|$ and $|v|$ are $\geq k$.

Suppose that $k \leq |u| \leq |v|$. From Corollary 3.4.6 we know that $V \vee N_k$ is then defined by the four identities $x^k = x^{2k}$, $x^k y = (xy)^k = xy^k$, $x_1 \cdots x_k = (x_1 \cdots x_k)^k$, and $u = v$. Since $V \vee N_k \subseteq B \vee N_k$, $V \vee N_k$ also satisfies the hyperidentities of Propositions 4.3.1 - 4.3.3, and so we may obtain the first three of these identities as hyperidentity (base) instances. Now recall from Section 4.1 that V satisfies the hyperidentity HB($u = v$), based on $u = v$. For that hyperidentity, terms such as x_1^i or x_2^j, for i and $j \geq 1$, yield only trivial instances, while any term represented by a word involving both x_1 and x_2 leads to an identity in which both words have length $\geq |u| \geq k$. Hence N_k also satisfies HB($u = v$), and so does $V \vee N_k$. Combining this information, we get $\overline{V \vee N_k} \subseteq V \vee N_k$, so that $V \vee N_k$ is closed.

Now suppose that $|u|$ or $|v|$ is $< k$. Again Corollary 3.4.6 provides us with defining identities for $V \vee N_k$: besides the three usual ones for $B \vee N_k$, we need

$u^* = v^*$, where u^* and v^* are formed from u and v by replacing each variable x by x^k. As in the first case we have hyperidentity instances for the first three identities. However, the hyperidentity HB(u = v) is no longer satisfied by N_k, and our problem now is to find a hyperidentity based on $u^* = v^*$ instead. We do this by suitably modifying HB(u = v). Starting with HB(u = v) as described in Section 4.1, replace each variable x occurring in the hyperidentity by $F(x,F(x, \ldots F(x,x)) \ldots)$, an expression with k occurrences of x. This ensures that the new hyperidentity is based on $u^* = v^*$. It is obvious that this hyperidentity is satisfied by V: under any choice of term, wherever we previously had x we now have x^a for some $a \geq 1$, but such a change makes no difference within B. It is also clear that N_k satisfies the hyperidentity, since x_1^i or x_2^j terms give trivial instances and all other terms give instances of sufficient length. Therefore $V \vee N_k$ satisfies the hyperidentity, and we are able to obtain $u^* = v^*$ as a hyperidentity instance. Putting all this together in the usual way, we obtain $\overline{V \vee N_k} \subseteq V \vee N_k$ in this case too.

Finally, if V is a variety of bands which is not closed, we have $V \vee V^d = \overline{V}$ by Corollary 4.1.8, and it is easy to see that $\overline{V \vee N_k} = V \vee V^d \vee N_k$. All of this proves the following result.

<u>Proposition</u> <u>4.3.6:</u> If V is a non-trivial closed variety of bands, then $V \vee N_k$ is also closed. If V is a variety of bands which is not closed, then $\overline{V \vee N_k} = V \vee V^d \vee N_k$.

Figure 4.3 shows a part of the image $\mathcal{H}(L(CS))$ of hypervarieties generated by closed varieties of semigroups.

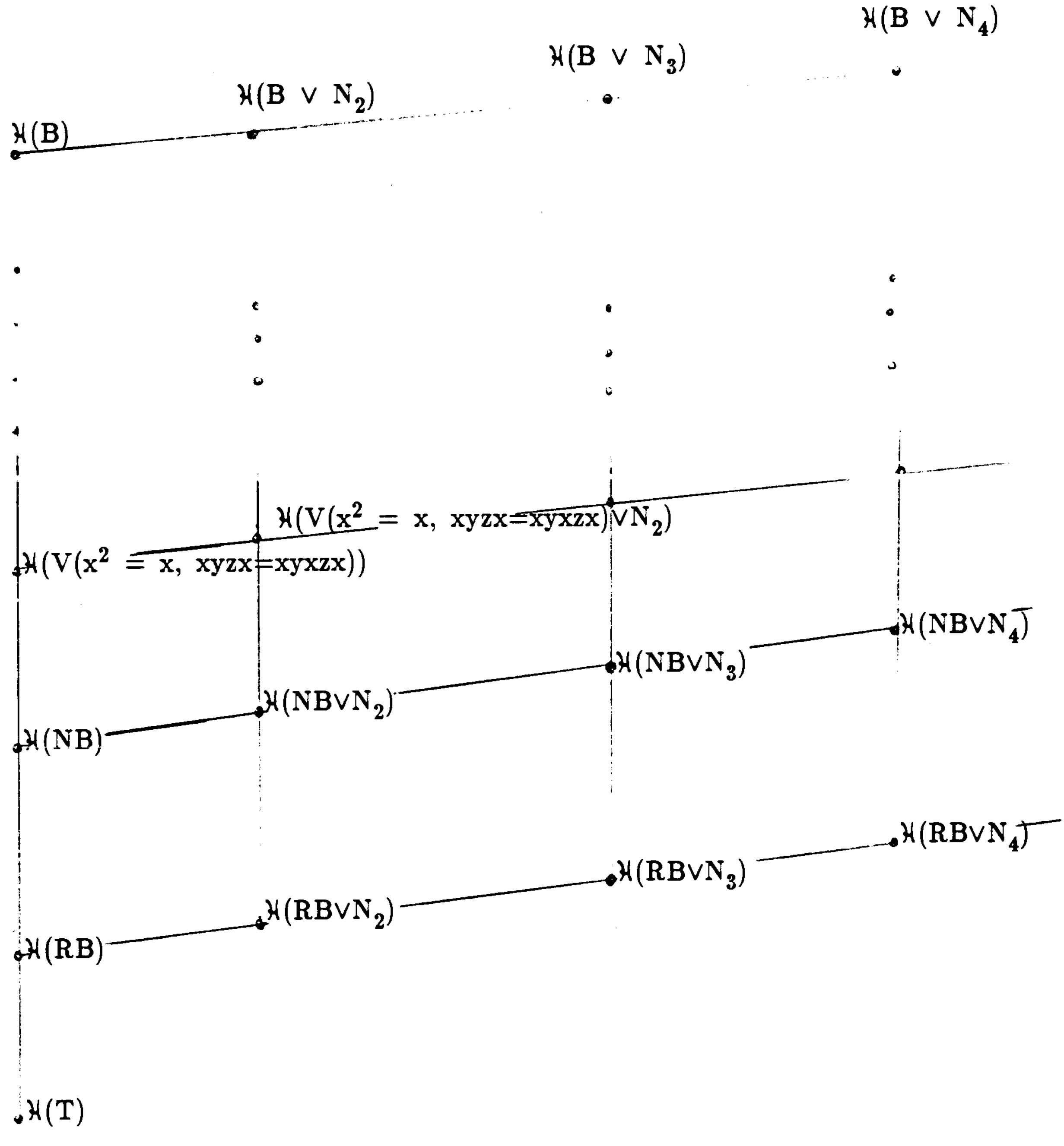


Figure 4-3: A Portion of the Lattice $\mathcal{H}(L(CS))$

Our previous results prove that the hypervarieties shown in Figure 4.3 are all distinct. For convenience of notation, we will use N_1 for the trivial variety, with

$V \vee N_1 = V$ for any variety V. Then any two hypervarieties in the diagram can be represented as $\mathcal{H}(V \vee N_k)$ and $\mathcal{H}(W \vee N_l)$ for some V and W in L(B) and some k and $l \geq 1$, with either $V \neq W$ or $k \neq l$. If $V \neq W$, we may use a hyperidentity HB(u = v) to distinguish the two hypervarieties; if $k \neq l$, we use the hyperidentity of Proposition 4.3.1.

We know from Proposition 3.1.1 that $\mathcal{H}(RB)$ is an atom of the lattice $\mathcal{H}\mathcal{V}$. Other results from Chapter 3 allow us to show that between the hypervarieties shown in the first two columns of Figure 4.3, no other intermediate hypervarieties of the form $\mathcal{H}(V)$, for V in L(S), are possible. First, since $\mathcal{H}$ preserves joins, the join of any two hypervarieties in the first two columns is again one of these hypervarieties. Now suppose that $\mathcal{H}(V \vee N_k)$ is one of the hypervarieties in the diagram, with V a closed variety of bands and k = 1 or 2, and that $\mathcal{H}(W) \subseteq \mathcal{H}(V \vee N_k)$ for some variety W of semigroups. Then $\overline{W} \subseteq \overline{V \vee N_k} = V \vee N_k \subseteq V \vee N_2 = V \vee Z$. By Proposition 3.3.5 then, either $\overline{W} \subseteq V$ or $\overline{W} = (\overline{W} \cap V) \vee Z$. If $\overline{W} \subseteq V$, then $\overline{W}$ is a closed variety of bands, and $\mathcal{H}(W) = \mathcal{H}(\overline{W})$ is one of the hypervarieties below $\mathcal{H}(V)$ in the first column of the diagram. Otherwise, if $\overline{W} = (\overline{W} \cap V) \vee Z$, then $\mathcal{H}(W) = \mathcal{H}(\overline{W}) = \mathcal{H}((\overline{W} \cap V) \vee Z)$, with $\overline{W} \cap V$ a closed subvariety of V, so it is one of the hypervarieties below $\mathcal{H}(V \vee N_2)$ in the second column of the diagram. We conclude therefore that no other hypervarieties of the form $\mathcal{H}(V)$, for V in L(S), are possible in the first two columns of Figure 4.3.

Chapter 5

Hyperidentities for the $A_{n,m}$'s

In order to obtain any results about closures of commutative varieties, we must first produce some hyperidentities which they satisfy. The problem of constructing hyperidentities which are satisfied by a given variety and in some sense "define" that variety is not in general an easy one. For varieties of bands, for instance, the task was made much simpler by the presence of idempotence and duality, both very strong properties. In this chapter we construct some families of hyperidentities satisfied by the commutative varieties A_m and $A_{n,m}$, n, m $\geq$ 1. These hyperidentities provide us with information about the hypervariety operator $\mathcal{H}$, as well as with instances which will be used in Chapter 6 to determine the closures of some of these varieties.

In Section 1 we illustrate some general discussion about the construction of hyperidentities with a specific example, the K_p family. Section 2 gives a new construction technique for building a hyperidentity corresponding to a given semigroup identity, subject to certain restrictions. This technique is then used to obtain several results about the behaviour of $\mathcal{H}$ on L(A) and other lattices; for instance, we show that $\mathcal{H}$ is injective on the lattice of varieties of commutative monoids. A different family of hyperidentities is examined in Section 3, this time depending on a "length parameter" t. In the final section this length parameter is used to give restrictions on what types of hyperidentities some of the $A_{n,m}$'s can satisfy.

5.1. Construction of Hyperidentities

The hyperidentities to be considered in this Chapter will all involve only one operation symbol F, which is binary. This, plus the fact that we consider only commutative or medial varieties, will be strongly exploited. In particular, we note that for commutative varieties, any binary term $t(x_1,x_2)$ can be expressed as $x_1^i x_2^j$ for some i, j $\geq$ 0, i+j $\geq$ 1.

Once again using Taylor's method of representing hyperidentities by trees, we now want to consider (not necessarily complete) binary trees. In a binary tree of height l, we may associate with each leaf of the tree an l-tuple of i's and j's, corresponding to the path taken from the root of the tree to that leaf: an i in the kth position indicates a left branch taken at node k, while a j indicates a right branch ($1 \leq k \leq l$). Since we will be considering commutative varieties, we may abbreviate such an l-tuple as $i^t j^{l-t}$, where $1 \leq t \leq l$. In a complete binary tree, for example, there are $\binom{l}{t}$ leaves with associated l-tuple $i^t j^{l-t}$, for $0 \leq t \leq l$.

The importance of this method of associating to each leaf in a binary tree an index $i^t j^{l-t}$ comes when we consider the tree as representing a hyperterm. Suppose we label the leaves of the tree with the r distinct variables $x_1, \ldots, x_r$, (r $\geq$

1). If we now replace the binary operation symbol of the tree by the binary term $x_1^i x_2^j$ for i, j $\geq$ 0, and evaluate, we get an expression of the form

$$x_1^{a_1} x_2^{a_2} \ldots x_r^{a_r},$$

where each exponent a_k is precisely the sum of all the l-tuples $i^t j^{l-t}$ associated with the leaves labelled x_k. Here we are using a commutativity assumption to group together all occurrences of each variable x_k. In this chapter such an assumption will usually be justified; if it is not, we assume at least the presence of mediality, and then keep track of the first and last letters in our expressions.

Our first hyperidentity construction illustrates these ideas.

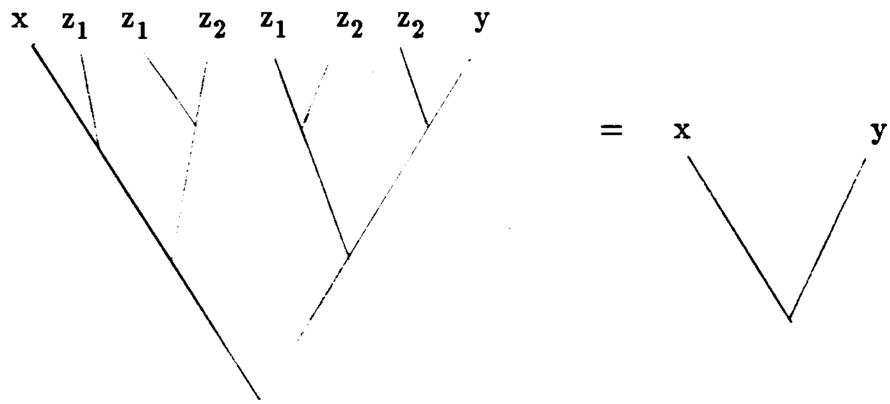
Construction:

Let p $\geq$ 2 be a prime. Form a complete binary tree of height p, and to each of its 2^p leaves associate an index $i^k j^{p-k}$, as described above. We will label two leaves with the same variable name if their associated indices are the same. In particular, there is exactly one leaf (the left-most one) with index $i^p j^0$: label this leaf x. There is exactly one leaf (the right-most one) with index $i^0 j^p$: label it y. In between, for $1 \leq k \leq p-1$, there are $\binom{p}{k}$ leaves corresponding to $i^k j^{p-k}$: label each such leaf z_k. Note that for each such k, the number $\binom{p}{k}$ of leaves labelled x_k is divisible by p. Call this labelled tree S_p. Finally, let K_p be the hyperidentity formed by equating S_p and the binary tree representing F(x,y).

Example: For p = 3, we get the hyperidentity K_3

$$F(F(F(x,z_1,),\ F(z_1,z_2)),\ F(F(z_1,z_2),\ F(z_2,y))) = F(x,y)$$

or



The hyperidentity K_3 is given in [15] as a hyperidentity satisfied by the variety of rings of characteristic 3, with the suggestion that it can be generalized to rings of characteristic p.

<u>Proposition</u> <u>5.1.1:</u> Let $p \geq 2$ be a prime. Then the variety A_p satisfies the hyperidentity K_p, while the varieties $A_{1,p}$, A, and A_q do not, for any prime $q \neq p$.

<u>Proof:</u> Any binary term of the variety A_p is of the form $x_1^i x_2^j$ for some $0 \leq i, j \leq p$, $i + j \geq 1$. We must examine the identity which is obtained from evaluation of K_p under the substitution of such a term. Making use of the comments above, we obtain an identity involving the letters $x, z_1, \ldots, z_{p-1}$, and y; x appears to the power i^p on the left-hand-side and to the power i on the right-hand-side; y appears to the power j^p on one side and to the power j on the other; and any other variable z_k appears to the power $\binom{p}{k} i^{p-k} j^k$ on one side and

not at all on the other. Since i^p is congruent to i modulo p, j^p is congruent to j modulo p, and $\binom{p}{k}$ is congruent to 0 modulo p for any $1 \leq k \leq p\text{-}1$, it follows that this identity does hold in A_p. Thus A_p satisfies K_p.

Since K_p is based on the identity

$$xz_1^{\binom{p}{1}} z_2^{\binom{p}{2}} \cdots z_{p-1}^{\binom{p}{p-1}} y = xy,$$

which does not hold in A or $A_{1,p}$, nor in any A_q, for q a prime different from p, the remaining claims of the proposition hold.

Corollary 5.1.2: The hypervarieties $\mathcal{H}(A_p)$, for p prime, are all distinct, and none of them are equal to $\mathcal{H}(A)$.

The K_p hyperidentity can also be used to answer a question posed by Taylor in [15] about varieties of groups. For any prime p, the variety G_p of abelian groups which satisfy $x^p = 1$ satisfies the hyperidentity K_p: the proof of Proposition 5.1.1 still holds if we allow binary terms $x_1^i x_2^j$ where now i and j may be < 0. Since the base identity (described in the proof above) does not hold in G_q for any prime $q \neq p$, we are able to distinguish by hyperidentities the various varieties G_p, $p \geq 2$, p prime.

The basic hyperidentity K_p can be modified in several ways. For instance, if we identify all those variables z_k in K_p whose associated indices $i^k j^{p-k}$ have $k \leq p/2$ with x, and all remaining z_k's with y, we get a hyperidentity based on the instance

$$x^{2^{p-1}} y^{2^{p-1}} \quad = \quad xy.$$

This hyperidentity is satisfied by $A_{1,p}$ but not by $A_{2,p}$, showing that these varieties generate different hypervarieties when p is prime. However, the general construction introduced in the next section will imply this result, along with many others.

5.2. The H(u = v) Construction

In order to distinguish between the hypervarieties determined by two distinct varieties of semigroups, we need to produce a hyperidentity satisfied by one which has as an instance an identity not satisfied by the other. Ideally, we would like a method which given an identity u = v produces a hyperidentity, preferably one based on u = v, which is satisfied by V(u = v) or at least by V(xy = yx, u = v). With varieties of bands we were able to do this, but for the $A_{n,m}$'s we usually cannot do so. For one thing, since $\mathcal{H}(RB) \subseteq \mathcal{H}(V)$ for any variety V, we must at least modify u = v into something that is satisfied by RB. Other "length" factors are also involved, as we will see in later sections. In this section we present a modified version of the ideal method described above: given any u = v, we produce a hyperidentity based on a rectangularized and padded version of u = v, which is satisfied by V(xy = yx, u = v). We then examine some uses of this construction method for particular identities u = v.

<u>Construction:</u> Let u = v be a semigroup identity, with $k = |u| \leq |v| = l$. Let w be any variable not occurring in either u or v. The hyperidentity H(u = v) will consist of two complete binary trees, each of height l, labelled as follows. Each

tree will have l leaves with associated index $i^{l-1}j$. On the first tree, label k of these leaves with the k letters of the word u,in order, and label all remaining leaves with the letter w. On the second tree, label the l leaves with this index with the letters of the word v, and all remaining letters with w. Note that the left-most and right-most letter on each tree is w, and that these four occurrences of w are all at height l; this ensures that projection terms of the form x_1^i or x_2^j always give only trivial identities.

Proposition 5.2.1: Let u = v be any non-trivial semigroup identity. Then both the varieties V(xy = yx, u = v) and V(xyzw = xzyw, u = v) satisfy the hyperidentity H(u = v).

Proof: Upon substitution of the term $x_1^i x_2^j$ in H(u = v), we obtain the identity

$$w^{i^l} u^{i^{l-1}j} w^a \quad = \quad w^{i^l} v^{i^{l-1}j} w^b,$$

where $b = \sum_{t=2}^{l} \binom{l}{t} i^{l-t} j^t$ and $a = b + (l - k) i^{l-1} j$. (This assumes of course that we use the medial identity to collect all the letters in the words u and v together, yet leaving the first and last letters (w) alone.)

If k = l, then it is clear that V(xyzw = xzyw, u = v) satisfies this identity for any choice of i and j $\geq$ 0. If k < l, then V(xyzw = xzyw, u = v) also satisfies the identity $x^k = x^l$. Hence $w^a = w^b$ holds, since $b \geq k$ and a - b is divisible by l - k. So in this case too V(xyzw = xzyw, u = v) satisfies the required identity. Therefore this variety and its subvariety V(xy = yx, u = v) both satisfy the given hyperidentity.

Using $i = j = 1$ in the identity of the proof above, we see that $H(u = v)$ is based on the identity

$$wuw^{2^l-k-1} \quad = \quad wvw^{2^l-l-1}. \qquad (*)$$

Thus we have rectangularized and "padded out" the original identity $u = v$. But although this falls short of our ideal, it nevertheless has many interesting consequences.

One of these involves looking at varieties of commutative monoids. When an identity element 1 is available, it follows that a variety satisfies the padded version (*) of $u = v$ iff it satisfies $u = v$. Thus for any two distinct varieties of commutative monoids, we can use an identity satisfied by one and not the other to produce a hyperidentity satisfied by one and not the other.

<u>Corollary</u> <u>5.2.2:</u> The operator $\mathcal{H}$ is injective on the lattice of varieties of commutative monoids.

$\mathcal{H}$ is not injective on the lattice of varieties of commutative semigroups, since we will show in Section 5.4 that $\mathcal{H}(A_m) = \mathcal{H}(A_m \vee Z)$.

An important special case of the $H(u = v)$ construction is obtained when we take $u = v$ to be $xy^m = x$ or $x^n = x^{n+m}$. This gives us hyperidentities satisfied by the varieties A_m and $A_{n,m}$. Instances of these hyperidentities will be valuable when we consider $\overline{A}_m$ and $\overline{A}_{n,m}$ in Chapter 6. For now they also allow us to show that $\mathcal{H}$ distinguishes various subvarieties of the A_m's and $A_{n,m}$'s.

The relationships between the varieties A_m and $A_{n,m}$ for m, n $\geq$ 1 are discussed in [5]. The subvarieties of A_m are precisely the varieties A_l, for l a divisor of m. For m $\geq$ 2, the subvarieties of $A_{1,m}$ are the A_l and $A_{1,l}$, for l a divisor of m; the lattice of such subvarieties is in fact isomorphic to the product of the lattice of subvarieties of $A_{1,1}$ = SL and the lattice (under divisibility) of divisors of m. Thus for any m $\geq$ 2, $A_m \subseteq A_{1,m}$, and if l divides m then $A_l \subseteq A_m$ and $A_{1,l} \subseteq A_{1,m}$. Also $A_{k,l} \subseteq A_{n,m}$ if k $\leq$ n and l divides m, for k, l, n, and m all $\geq$ 1. Each such inclusion gives us a corresponding one when we apply $\mathcal{H}$. Using the H(u = v) construction we get the following results.

Proposition 5.2.3:

i) For m $\geq$ 1, $\mathcal{H}(A_{1,m}) \subsetneqq \mathcal{H}(A_{2,m}) \subsetneqq \mathcal{H}(A_{3,m}) \subsetneqq \ldots$

ii) For m $\geq$ 1, $\mathcal{H}(A_m) \subsetneqq \mathcal{H}(A_{1,m})$.

iii) For m $\geq$ 2 and l a proper divisor of m,

$$\mathcal{H}(A_l) \subsetneqq \mathcal{H}(A_m) \text{ and } \mathcal{H}(A_{1,l}) \subsetneqq \mathcal{H}(A_{1,m}).$$

Proof: All of these results are handled by applying the H(u = v) technique to the relevant identity. We illustrate only one case, $\mathcal{H}(A_{n,m}) \subseteq \mathcal{H}(A_{n+1,m})$, for n and m both $\geq$ 1. By Proposition 5.2.1, $A_{n,m}$ satisfies the hyperidentity $H(x^n = x^{n+m})$. By the remarks following the proof of that Proposition, this hyperidentity is based on the identity

$$wx^n w^a \quad = \quad wx^{n+m} w^b,$$

where a - b = m and b $\geq$ n. Since this identity is not satisfied by $A_{n+1,m}$, we have produced a hyperidentity satisfied by $A_{n,m}$ but not by $A_{n+1,m}$.

It is known (see [5]) that if V is a variety of commutative monoids, then either $V = A_m$ for some $m \geq 1$, or V is the variety of monoids defined by the identities $xy = yx$ and $x^n = x^{n+m}$ for some n and $m \geq 1$. In fact the natural map between the A_m's and $A_{n,m}$'s and the lattice of varieties of commutative monoids is a lattice isomorphism. Thus Proposition 5.2.3 extends to a second proof that $\mathcal{H}$ is injective on the lattice of varieties of commutative monoids.

A different approach to the study of the varieties of commutative semigroups has been used by Nelson [8], who defined the varieties $A^r_{n,m}$. For $r \geq 0$ and n and $m \geq 1$, let

$$A^r_{n,m} = V(xy = yx,\ x^n = x^{n+m},\ x^r y^n = x^{r+m} y^n).$$

It is clear from this definition that $A^r_{n,m} \subseteq A_{n,m}$, for any $r \geq 0$, and that $A^r_{n,m} = A_{n,m}$ for $r \geq n$. Nelson has shown that the interval $[A^r_{n,1}, A^r_{n,m}]$ consists of all the varieties $A^r_{n,l}$ where l divides m; and that the interval $[A^0_{n,m}, A^n_{n,m}]$ consists of all the varieties $A^r_{n,m}$ where $0 \leq r \leq n$. Our hyperidentity construction technique then shows that all the corresponding $\mathcal{H}$-inclusions are proper ones.

Proposition 5.2.4: For any n and $m \geq 1$, any $1 \leq r \leq n$, and $l > 1$ a proper divisor of m,

i) $\mathcal{H}(A^r_{n,m}) \subsetneqq \mathcal{H}(A^r_{n+1,m})$

ii) $\mathcal{H}(A^r_{n,1}) \subsetneqq \mathcal{H}(A^r_{n,l}) \subsetneqq \mathcal{H}(A^r_{n,m})$;

iii) $\mathcal{H}(A^0_{n,m}) \subsetneqq \mathcal{H}(A^r_{n,m}) \subsetneqq \mathcal{H}(A^n_{n,m})$, when $r < n$.

We may also use the $H(u = v)$ hyperidentity technique to show that the lattice $\mathcal{H}(L(A))$ is not modular. It is known that the lattice $L(A)$ of varieties of commutative semigroups is not modular (although the A_m's and $A_{n,m}$'s form a large distributive sublattice). Schwabauer [13] gives the following example. Let

$$V_1 = V(xy = yx,\ xy^9 = x^2y^8,\ x^3y^7 = x^4y^6),$$

$$V_2 = V(xy = yx,\ x^2y^8 = x^3y^7,\ x^4y^6 = x^5y^5),$$

and $$V_3 = V(xy = yx,\ xy^9 = x^2y^8,\ x^3y^7 = x^4y^6,\ xy^9 = x^5y^5).$$

Then $V_3 \subseteq V_1$, but $(V_1 \cap V_2) \vee V_3$ satisfies $xy^9 = x^5y^5$ while $V_1 \cap (V_2 \vee V_3)$ does not. So $(V_1 \cap V_2) \vee V_3$ is properly contained in $V_1 \cap (V_2 \vee V_3)$, and $L(A)$ is not modular. Since $V_1 \cap V_2 \subseteq V_3$, in fact $(V_1 \cap V_2) \vee V_3 = V_3$. When we apply $\mathcal{H}$ to the various varieties involved, we get the sublattice of $\mathcal{H}(L(A))$ shown in Figure 5.1 below.

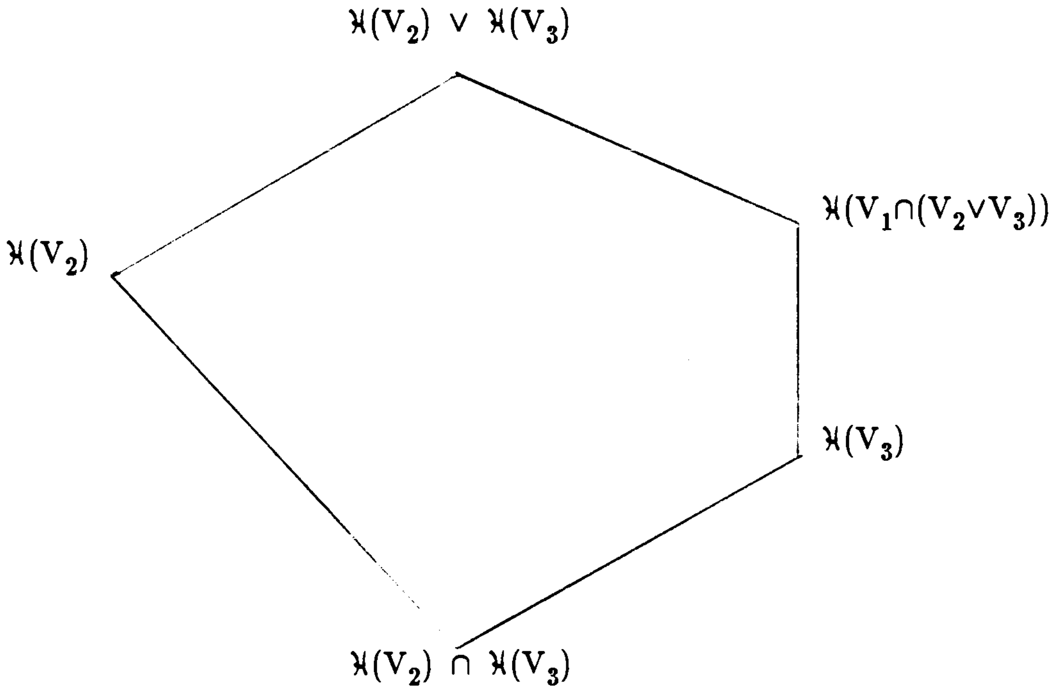


Figure 5-1: A Non-Modular Sublattice of $\mathcal{H}(L(A))$.

We now show that these five hypervarieties are all distinct. First, consider $\mathcal{H}(V_2)$ and $\mathcal{H}(V_3)$. The hyperidentity $H(x^2y^8 = x^3y^7)$ is satisfied by V_2; but the padded version of $x^2y^8 = x^3y^7$ is not satisfied by V_3, so neither is this hyperidentity. This shows that $\mathcal{H}(V_3)$ is not contained in $\mathcal{H}(V_2)$. Analogously, we use $H(xy^9 = x^2y^8)$ to show that $\mathcal{H}(V_2)$ is not contained in $\mathcal{H}(V_3)$. From this it follows that $\mathcal{H}(V_2) \cap \mathcal{H}(V_3)$ is properly contained in each of $\mathcal{H}(V_2)$ and $\mathcal{H}(V_3)$. It also follows that $\mathcal{H}(V_2)$ is properly contained in the join $\mathcal{H}(V_2) \vee \mathcal{H}(V_3)$. Similarly, since V_3 satisfies $H(xy^9 = x^5y^5)$ while the variety $V_1 \cap (V_2 \vee V_3)$ does not, we have $\mathcal{H}(V_3)$ properly contained in $\mathcal{H}(V_1 \cap (V_2 \vee V_3))$. Finally, suppose that $\mathcal{H}(V_1 \cap (V_2 \vee V_3)) = \mathcal{H}(V_2 \vee V_3)$. Then we would have

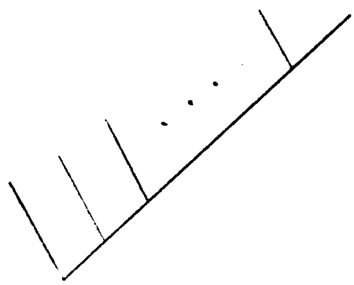
$$\begin{aligned} \mathcal{H}(V_2) \vee \mathcal{H}(V_3) &= \mathcal{H}(V_2 \vee V_3) \\ &= \mathcal{H}(V_1 \cap (V_2 \vee V_3)) \\ &\subseteq \mathcal{H}(V_1) \cap \mathcal{H}(V_2 \vee V_3) \\ &\subseteq \mathcal{H}(V_1), \end{aligned}$$

so that $\mathcal{H}(V_2) \subseteq \mathcal{H}(V_1)$. But this is false, since V_1 satisfies $H(x^3y^7 = x^4y^6)$ while V_2 does not. So we must after all have $\mathcal{H}(V_1 \cap (V_2 \vee V_3))$ properly contained in $\mathcal{H}(V_2 \vee V_3)$. This proves the following:

Proposition 5.2.5: The lattice $\mathcal{H}(L(A))$ is not modular.

5.3. The Parameters t(m) and d(m)

The hyperidentities discussed in the previous section involved a process of padding identities. The padded identities will be used in determining closure results, but they alone are not sufficient. We now consider a new family of hyperidentities, while examining the question of how much, if any, padding is unavoidable. The hyperidentities to be considered here all involve binary trees of the same shape, which we we will refer to throughout this section as the basic (binary) shape:



We are also still interested mainly in abelian or medial varieties, so we will make use again of the comments of Section 5.1.) One of the simplest hyperidentities of this shape which we might try is $H_{n,k}$, based on the identity $x^n = x^{n+k}$, Our first investigations of this hyperidentity reveal the following:

<u>Proposition 5.3.1:</u> Let $n \geq 2$. The variety $A_{n,k}$ satisfies the hyperidentity $H_{n,k}$ for $k = 1$ and $k = 2$, but not for $k = 3$.

<u>Proof:</u> Any identity produced from $H_{n,k}$ has the form $x^a = x^b$ for some a and $b \geq 1$. For $k = 1$ and $k = 2$, it is a matter of routine verification that for any such identity, we have either $a = b$, or $b \geq a \geq n$ and b-a congruent to 0 modulo k. For $k = 3$ and $n = 2$, however, evaluation under the substitution of the term x_2^2 yields the identity $x^2 = x^{16}$, which does not hold in $A_{2,3}$.

Note that this proves again, in the limited case $k = 1$ or 2, the result of Proposition 5.2.3(i). But since the hyperidentity here involves only one variable x, we may also extend to the non-commutative case, with

$$\mathcal{H}(B_{2,k}) \subsetneqq \mathcal{H}(B_{3,k}) \subsetneqq \mathcal{H}(B_{4,k}) \subsetneqq \ \ldots, \qquad \text{for } k = 1 \text{ or } 2.$$

But for $k > 2$, even in the commutative case, we must look further.

When we examine one-variable hyperidentities of this shape further, an interesting pattern emerges. We discover two parameters associated with each natural number m, whose relationship with n seems to determine which of these hyperidentities are satisfied by which $A_{n,m}$ varieties. For any $m \geq 2$, write

$$m \quad = \quad p_1^{\alpha_1} \cdots p_v^{\alpha_v}$$

as a product of distinct primes. We define

$$t(m) \quad = \quad \max\{\alpha_1, \ \ldots, \ \alpha_v\},$$

the highest power of a prime to divide m. This parameter t(m) seems to measure "how long" hyperidentities have to be, in a sense to be made more precise later. We also define

$$d(m) \quad = \quad \phi(p_1^{\alpha_1+1} \cdots p_v^{\alpha_v+1}),$$

where ϕ is the Euler-ϕ-function. By common properties of this function, it follows that for each prime divisor p of m, if p^α divides m but $p^{\alpha+1}$ does not, then d(m) is divisible by $p^\alpha(p - 1)$. This also implies that m divides d(m). For the special case $m = 1$, we set $t(m) = d(m) = 1$.

Before proceeding with an examination of hyperidentities, we give two technical lemmas. These embody some congruence properties which will be used repeatedly in our hyperidentity proofs. We will also make use of Euler's Theorem, which states that if j is a number relatively prime to m, then $j^{\phi(m)}$ is congruent to 1 modulo m.

Lemma 5.3.2: Let p be a prime, $\alpha \geq 1$, and $2 \leq j \leq p^{\alpha}-1$, and let j and p^{α} be relatively prime. Let $d \geq 2$ be any number divisible by $p^{\alpha}(p - 1)$. Then $(j^d - 1)/(j - 1)$ is congruent to 0 modulo p^{α}.

Proof: By Euler's Theorem, $j^{\phi(p^{\alpha+1})}$ is congruent to 1 modulo $p^{\alpha+1}$, and since $\phi(p^{\alpha+1}) = p^{\alpha}(p - 1)$ divides d, we have $j^d - 1$ congruent to 0 modulo $p^{\alpha+1}$. Thus if j is not congruent to 1 modulo p, or if j is congruent to 1 modulo p but is not congruent to 1 modulo p^2, the desired congruence will hold. So we now assume that j is congruent to 1 modulo p^{β}, but j is not congruent to 1 modulo $p^{\beta+1}$, where $2 \leq \beta \leq \alpha - 1$. In this case, we need to show that $j^d - 1$ is divisible by $p^{\alpha+\beta}$. By assumption we may write $j = kp^{\beta}+1$ for some $k \geq 1$, and therefore

$$\begin{aligned} j^d - 1 &= (kp^{\beta} + 1)^d - 1 \\ &= \sum_{l=1}^{d} \binom{d}{l} (kp^{\beta})^l. \end{aligned}$$

Any terms of this sum for which $\beta l \geq \alpha + \beta$ are divisible by $p^{\alpha + \beta}$. Thus it suffices to prove that for those l for which $\beta l < \alpha + \beta$, the coefficient $\binom{d}{l}$ is divisible by $p^{\alpha+\beta-\beta l}$. For $l = 1$ this is easy, since $\binom{d}{1} = d$ is divisible by p^{α} by assumption. For $l \geq 2$, we write

$$\binom{d}{l} = \frac{d\ (d\text{-}1)\ (d\text{-}2)\ \cdots\ (d\text{-}(l\text{-}1))}{1\ (1)\ \ (2)\ \ \cdots\ \ (l\text{-}1)}$$

$$= \frac{d}{l}\ \frac{(d\text{-}1)}{1}\ \frac{(d\text{-}2)}{2}\ \cdots\ \frac{(d\text{-}(l\text{-}1))}{(l\text{-}1)}$$

We first examine quotients of the form (d-w)/w, where $1 \leq w \leq l\text{-}1$. There are three cases. If w is not divisible by p, such quotients may be ignored. If w is divisible by p^γ for some $\gamma > \alpha$, then $w \geq p^\gamma > p^\alpha$, so $l > p^\alpha$; therefore $\beta l \geq \beta p^\alpha \geq \beta\alpha \geq \beta + \alpha$, so in this case $\binom{d}{l}(kp^\beta)^l$ is already divisible by $p^{\beta+\alpha}$ anyway. If however w is divisible by p^γ for some $1 < \gamma \leq \alpha$, then d - w is divisible by p^γ too. Hence in any case the quotients (d-w)/w remove no powers of p from our total.

Finally, consider the quotient d/l. Again if p does not divide l, we have no problem, since p^α divides d. Suppose then that $l = bp^\gamma$ for some $\gamma > 1$ and some b relatively prime to p. As before we need only consider $\gamma \leq \alpha$. Then $\gamma < l$, so $\gamma \leq l\text{-}1 \leq \beta(l\text{-}1)$; therefore $\alpha\text{-}\gamma \geq \alpha\text{-}\beta(l\text{-}1)$. Since d/l is divisible by $p^{\alpha\text{-}\gamma}$, it is thus divisible by at least $p^{\alpha\text{-}\beta(l\text{-}1)}$. This means that $\binom{d}{l}(kp^\beta)^l$ is divisible by $p^{\alpha\text{-}\beta(l\text{-}1)+\beta l} = p^{\alpha+\beta}$, as required.

<u>Lemma</u> <u>5.3.3:</u> Let p be a prime, and α, β, j, and d be natural numbers such that $\beta \geq \alpha$, $2 \leq j \leq p^\alpha\text{-}1$, and $p^\alpha(p\text{-}1)$ divides d. Then both $j^\beta(j^d - 1)$ and $j^\beta(j^d - 1)/(j - 1)$ are congruent to 0 modulo p^α.

Proof: If j is not relatively prime to p, then p divides j, and j^β is certainly divisible by p^α, proving both claims. If j is relatively prime to p, then by Euler's Theorem plus the fact that $\phi(p^\alpha)$ divides $p^\alpha(p-1)$ which in turn divides d, we have j^d-1 congruent to 0 modulo p^α. This proves the first claim in this case. The second claim, in the case where j is relatively prime to p, is proved by the preceding Lemma.

Corollary 5.3.4: Let n and m be ≥ 1, $2 \leq j \leq n+m-1$, and $1 \leq i \leq n+m-1$. Let $d = d(m)$ and let $\beta \geq t(m)$. Then $j^\beta(j^d-1)$ and $ij^\beta(j^d-1)/(j-1)$ are congruent to 0 modulo m.

Proof: The congruence modulo m reduces to a series of congruences modulo p^α, one for each prime p for which p^α divides m but $p^{\alpha+1}$ does not. We have seen that for each such p, $p^\alpha(p-1)$ divides the parameter d(m). The result then follows from the previous Lemma.

We are now ready to consider some hyperidentities for the variety $A_{n,m}$. First we use the shape referred to earlier as $H_{n,d}$, where $d = d(m)$. With the right conditions on n, this is successful. Note that this next result includes the two cases already seen in Proposition 5.3.1

Proposition 5.3.5: Let $m \geq 1$, and let $t = t(m)$ and $d = d(m)$. Let $n \geq t+1$.

Then the variety $A_{n,m}$ satisfies the basic shape hyperidentity based on the identity $x^n = x^{n+d}$.

Proof: The case $m = d = 1$ was already handled in Proposition 5.3.1, so we may assume that m and d are ≥ 2. Since the hyperidentity involves only the one variable x, checking whether $A_{n,m}$ satisfies the hyperidentity reduces to comparing two exponents on x. From any binary term $x_1^i x_2^j$, we obtain the two exponents

$$i + ij + \ \ldots \ + ij^{n-2} + j^{n-1}$$

and $$i + ij + \ \ldots \ + ij^{n+d-2} + j^{n+d-1},$$

for $0 \leq i, j \leq n+m-1$.

If $j = 0$, or if $i = 0$ and $j = 1$, these exponents are equal; otherwise we must show that they are both $\geq n$ and congruent modulo m. It is clear that any other choice of i and j values does make both exponents $\geq n$. If $i = 0$ and $j > 1$, the exponents become j^{n-1} and j^{n+d-1}, and since $n-1 \geq t$ the difference $j^{n-1}(j^d - 1)$ is congruent to 0 modulo m by Corollary 5.3.4.

Now assume that $i > 0$. If $j = 1$, we get only $(n-1)i+1$ and $(n+d-1)i+1$, which are congruent modulo m since m divides d. For $j > 1$, the difference in the exponents becomes

$$\begin{aligned} & i + ij + \ \ldots \ + ij^{n+d-2} + j^{n+d-1} \\ & \qquad - (i + ij + \ \ldots \ + ij^{n-2} + j^{n-1}) \\ = \ & ij^{n-1}(1 + j + \ \cdots \ + j^{d-1}) + j^{n-1}(j^d - 1) \\ = \ & ij^{n-1} \ (j^d - 1)/(j - 1) \quad + \quad j^{n-1}(j^d - 1). \end{aligned}$$

Again by Corollary 5.3.4 this last quantity is congruent to 0 modulo m, since $n-1 \geq t$.

A close examination of this last proof, along with the proofs of the preceding technical lemmas, reveals some of the significance of the two parameters t and d. Each plays a role in assuring that our congruences work when terms $x_1^i x_2^j$ are substituted into the hyperidentity formula. When j is relatively prime to some prime p dividing m, we need to use d(m) rather than just m to ensure the presence of enough powers of p, as in Lemma 5.3.2. When j is not relatively prime to such a p, we need $n-1 \geq t$, or $n \geq t+1$, again to provide enough powers of p.

In the situation where $1 \leq n \leq t$, we would like a hyperidentity based on $x^n = x^{n+m}$, or even on $x^n = x^{n+d}$, which is satisfied by $A_{n,m}$. Using the $K_{n,d}$ shape this is impossible. (We will discuss in the next section the question of whether it is possible using a different shape of hyperidentity.) However, we can use the same shape if we again pad out our base identity to a suitable length, namely length t+1. Notice in the following proof how once again we have contrived a factor of j^t.

<u>Proposition</u> <u>5.3.6:</u> Let $m \geq 2$, $t = t(m)$ and $d = d(m)$, and $1 \leq n \leq t+1$. The variety $A_{n,m}$ satisfies the basic shape hyperidentity based on the identity $y_1 \cdots y_s x^n = y_1 \cdots y_s x^{n+d}$, where $s = t+1-n$.

Proof: Once again we reduce to a comparison of exponents, this time for each of the variables $y_1, \ldots, y_s$ and x in turn. For each variable y_l, $1 \leq l \leq s$, we clearly have the same exponent on each side of the hyperidentity. So we need only examine the exponents on the variable x. For any $0 \leq i, j \leq n+m-1$, $i+j \geq 1$, these exponents are

$$ij^s + ij^{s+1} + \cdots + ij^{s+n-2} + j^{s+n-1}$$

$$\text{and} \quad ij^s + ij^{s+1} + \cdots + ij^{s+n+d-2} + j^{s+n+d-1}.$$

If $j = 0$, or if $j = 1$ and $i = 0$, these are equal; otherwise, (even if $i = 0$,) they are both $\geq n$. We check whether their difference is congruent to 0 modulo m. This difference is

$$\begin{aligned} & ij^{s+n-1} + ij^{s+n} + \cdots + ij^{s+n+d-2} + j^{s+n+d-1} - j^{s+n-1} \\ = \; & ij^{s+n-1}(1 + j + \cdots + j^{d-1}) + j^{s+n-1}(j^d-1) \\ = \; & ij^{s+n-1}(j^d-1)/(j-1) \quad + \quad j^{s+n-1}(j^d-1) \\ = \; & ij^t(j^d-1)/(j-1) \quad + \quad j^t(j^d-1) \end{aligned}$$

using the fact that $t = s+n-1$. By Corollary 5.3.4, this difference is congruent to 0 modulo m for any $1 \leq j \leq m+n-1$.

By dualizing the preceding arguments (including the technical lemmas,) we obtain the following similar results.

Proposition 5.3.7: Let $m \geq 2$, and $t = t(m)$ and $d = d(m)$. Let $1 \leq n \leq t+1$, and $s = t+1-n$. Then the variety $A_{n,m}$ satisfies hyperidentities based on the following instances:

1. $x^n y_1 \cdots y_s = x^{n+d} y_1 \cdots y_s$;

2. $y_1 \cdots y_s^{\,n} x = y_1 \cdots y_s^{\,n+d} x$;

3. $x y_1^{\,n} y_2 \cdots y_s = x y_1^{\,n+d} y_2 \cdots y_s$.

<u>Proof:</u>

1. This is just the dual of the hyperidentity seen in Proposition 5.3.6. The proof is therefore obtained by interchanging the roles of i and j in the previous proof.

2. The exponents obtained for the variable $y_1, \ldots, y_{s-1}$ are the same on either side of the hyperidentity. For x we have exponents j^{s+n-1} and $j^{s+n+d-1}$, whose difference $j^{s+n-1}(j^d-1) = j^t(j^d-1)$ is congruent to 0 modulo m. Finally for the variable y_s the difference in exponents is easily shown (as in the proof of Proposition 5.3.6,) to be $ij^t(j^d-1)/(j-1)$, which by Corollary 5.3.4 again is also congruent to 0 modulo m.

3. This is proved similarly to 2.

The next two propositions involve a similar construction of a hyperidentity corresponding to the identity $x^{n-1}yx = x^{n-1+m}yx$. As before, we cannot use such an identity itself as a base instance for a hyperidentity: to begin with we must use d(m) instead of m. When $n \geq t(m)+1$ we are able to use $x^{n-1}yx = x^{n-1+d}yx$ as a base; but for $1 \leq n \leq t(m)$ we once again resort to padding the desired instance out to length t+1.

<u>Proposition 5.3.8:</u> Let $m \geq 1$, $t = t(m)$, and $d = d(m)$, and let $n \geq t+1$. Then

the variety $A_{n,m}$ satisfies the basic shape hyperidentity based on the identity $x^{n-1}yx = x^{n-1+d}yx$.

<u>Proof:</u> Let $0 \leq i, j \leq n+m-1$, and $i+j \geq 1$. For the exponents on the variable y, we need ij^{n-1} and ij^{n+d-1} to be either equal, or both $\geq$ n and congruent modulo m. If $i = 0$, $j = 0$, or $j = 1$, these quantities are equal. For all other i and j values, both are $\geq$ n, and their difference is $ij^{n-1}(j^d-1)$. Since we have $n-1 \geq t(m)$, this difference is congruent to 0 modulo m by Corollary 5.3.4, for $d \geq 2$. (The case $d = 1 = m$ is trivial.)

For the exponents on x, we must consider the quantities

$$i + ij + \cdots + ij^{n-2} + j^n$$

and

$$i + ij + \cdots + ij^{n+d-2} + j^{n+d}.$$

If $i = 0$ we have $j^{n+d}-j^n = j^n(j^d - 1)$, which as usual is congruent to 0 modulo m for all possible values of j. For $i > 0$ and $j = 1$, we have $(n-1)i + 1$ and $(n+d-1)i + 1$; again these are congruent modulo m since m divides d. So assume that $i > 0$ and $j > 1$, and consider the difference

$$ij^{n-1} + \cdots + ij^{n+d-2} + j^{n+d} - j^n.$$

As usual, this simplifies to $ij^{n-1}(j^d - 1)/(j - 1) + j^n(j^d - 1)$, which we know by Corollary 5.3.4 is congruent to 0 modulo m when $n-1 \geq t$.

<u>Proposition</u> <u>5.3.9:</u> Let $m \geq 2$, $t = t(m)$, and $d = d(m)$, and let $2 \leq n \leq t$. Set $s = t+1-n$. Then the variety $A_{n,m}$ satisfies the basic shape hyperidentity based on $xy_1 \cdots y_s x^{n-1} = xy_1 \cdots y_s x^{n+d-1}$, and its dual.

Proof: Clearly the exponents on any variable y_l, for $1 \leq l \leq s$, are the same on either side of the hyperidentity. For exponents on x we have

$$i + ij^{s+1} + \cdots + ij^{s+n-2} + j^{s+n-1}$$

and
$$i + ij^{s+1} + \cdots + ij^{s+n+d-2} + j^{s+n+d-1}.$$

As in the previous proofs, the difference of these exponents reduces to $ij^t(j^d - 1)/(j - 1) + j^t(j^d - 1)$, which we know is congruent to 0 modulo m, as required.

It should be apparent from the construction of the hyperidentities in the last four propositions that we did not need the full strength of commutativity. In fact in each case the hyperidentity is also satisfied by the medial variety $M_{n,m}$, for the appropriate n and m values. This will be useful in determining $\overline{M}_{n,m}$ in Chapter 6.

5.4. The Length Restriction Lemmas

It appeared in Section 5.3 that, at least for the particular shape of hyperidentity being considered there, we could not find a hyperidentity satisfied by $A_{n,m}$ which is based on a non-trivial identity $u = v$ with $|u|$ and $|v| < t(m)+1$. Thus the parameter t(m) seemed to measure "how long" a hyperidentity had to be in order to be satisfied by $A_{n,m}$. We now examine whether this is true about all possible hyperidentities satisfied by $A_{n,m}$. The results obtained are quite limited, and we conclude the section with a conjecture involving the parameter t.

Lemma 5.4.1: Let $m \geq 2$. The variety A_m cannot satisfy any hyperidentity H for which a choice of A_m-terms gives a non-trivial instance u = v with $|u| = 1$. In particular, A_m satisfies no non-trivial hyperidentities of the form

$$x \quad = \quad F(\quad \ldots \quad),$$

where x is a single variable and F( ) is any hyperterm.

<u>Proof:</u> Suppose that A_m satisfies a non-trivial hyperidentity H, and there is a choice of A_m-terms giving as an instance the identity u = v, with $|u| = 1$. If $|v| = 1$ also, we must have u = v trivial. So we suppose that $|v| > 1$. This means that while only projection terms are involved on the left-hand-side of H in the evaluation to get u, at least one non-projection term must have been used on the right-hand-side to produce v. Form a new hyperidentity H′ from H by substituting into H the projection terms in our initial choice of A_m-terms, and identifying all variables as x. Then H′ is still satisfied by A_m, and it has the form

$$x \quad = \quad G(\quad \ldots \quad),$$

for some operation symbol G, where G is not just a variable. Now make the following choice of terms to use in H′: each k-ary operation symbol will be replaced by the k-ary term x_1^m. This yields an identity of the form $x = x^{m^e}$ (where $e \geq 1$ corresponds to the number of operation symbols encountered before the first x on the right-hand-side of H′). But such an identity cannot hold in A_m, since m^e is not congruent to 1 modulo m. This contradiction establishes the claim.

Note that exactly the same proof can be used for the same statement about $A_{1,m}$, for $m \geq 2$. The significance of this result is that it tells us that any instances of a hyperidentity satisified by A_m or $A_{1,m}$ are either trivial or of length ≥ 2. This means that the variety Z of zero semigroups satisfies all the hyperidentities satisified by A_m. Hence we have:

<u>Corollary</u> <u>5.4.2:</u> For any $m \geq 2$ and $n \geq 2$, $\quad Z \subseteq \overline{A}_m \subseteq \overline{A}_{1,m} \subseteq \overline{A}_{n,m}$.

This is our first indication of any varieties other than RB entering into a closure: we now have $A_m \vee RB \vee Z \subset \overline{A}_m$ for $m > 2$, and similarly for $\overline{A}_{1,m}$. These ideas will be expanded upon in the next chapter. For now, we continue with general results about what kind of hyperidentities A_m can satisfy.

<u>Lemma</u> <u>5.4.3:</u> Let $m = p^2a$, where p is prime, so that $t(m) \geq 2$. Suppose that A_m satisfies a hyperidentity of the form

$$F(\_\_, \;...\;, \_\_, x, \_\_, \;...\;, \_\_) \quad = \quad G(\;\cdots\;),$$

with the ith component of F consisting of a single variable x, $1 \leq i \leq$ arity of F. Then $F = G$, and the ith entry of G is also x.

<u>Proof:</u> If F and G were different operation symbols, then we could substitute the ith projection term for F in the hyperidentity, to obtain a new hyperidentity of the form

$$x \quad = \quad G(\;...\;)$$

still satisfied by A_m. But this contradicts Lemma 5.4.1, so we must have $F = G$.

Again replacing F by the ith projection term, we obtain x from the left-hand-side of the hyperidentity. On the right-hand-side, we project down the ith component until we reach either a variable (which must be x), or another operation symbol $K \neq F$. This latter situation leads to a hyperidentity $x = K(\ldots)$, again contradicting Lemma 5.4.1. Thus we must eventually reach an x, having encountered only F's as we project down the ith component. Let e be the number of such F's encountered; that is, the depth at which the x is nested in its hyperterm. Now consider evaluating the hyperidentity with F replaced by the term x_i^p. The identity which results is precisely $x^p = x^{p^e}$. Unless $e = 1$, this identity will not hold in A_m, since p^2 divides m. Thus the hyperidentity must have the specified form.

Lemma 5.4.4: Let $m \geq 2$, with $t(m) \geq 2$. If A_m satisfies a hyperidentity H and some choice of A_m-terms for H yields an instance $u = v$ from H where $|u| = 2$, then $u = v$ is trivial.

Proof: Suppose that A_m satisfies H as described. Since projection operations do not change the lengths of words, we may assume that H has been modified by carrying out any projections specified in our choice of terms. Thus we are assuming that only non-projection terms are used to produce $u = v$ from H. The only way to produce a word of length two without using projections is from a hyperterm such as F(___,x,___,y,___), using the term $x_i x_j$ (of appropriate arity) for F, or such as F(___,x,___), using x_i^2 for F, in each case with appropriate values for i and j. In either case, Lemma 5.4.3 guarantees that the instance $u = v$ will be trivial.

Corollary 5.4.5: Let $m \geq 2$, $t(m) \geq 2$, and $n \geq 2$. Then $N_3 \subsetneq \overline{A}_m \subsetneq \overline{A}_{1,m} \subsetneq \overline{A}_{n,m}$.

Lemma 5.4.6: Let $m = p^3a$ where p is prime, so that $t(m) \geq 3$. If A_m satisfies a hyperidentity H and some choice of A_m-terms yields an instance $u = v$ from H where $|u| = 3$, then $u = v$ is trivial.

Proof: As in the proof of Lemma 5.4.4, we may assume that no projection terms are used in obtaining the instance $u = v$ from H. We consider how a word of length three may be obtained from the hyperterm, say F( ), on the right-hand-side of H. There are only four possibilities:

1. F is replaced by the term $x_ix_jx_k$ for some indices $1 \leq i \neq j \neq k \leq$ arity of F, and in the hyperterm F has single variables only in its ith, jth, and kth components.

2. F is replaced by the term x_i^3, for some $1 \leq i \leq$ arity of F, and in the hyperterm F has a single variable in its ith component.

3. F is replaced by the term $x_i^2x_j$, for some $1 \leq i \neq j \leq$ arity of F, and F has single variables only in its ith and jth components.

4. F is replaced by the term x_ix_j, for some $1 \leq i \neq j \leq$ arity of F, and F has a single variable x say in its ith component, and a hyperterm G( ) in its jth component; and under our choice of terms, G( ) also

gives a word of length two.

Now Lemma 5.4.3 tells us immediately that the first three of these cases lead to trivial identities. In the fourth case the argument is more complicated. Here again Lemma 5.4.3 shows that we may represent H by

$$F(\_\_,x,\_\_,G(\_\_\_),\_\_\_) \quad = \quad F(\_\_,x,\_\_,J(\_\_\_),\_\_\_),$$

where the x's are in the ith component on either side, G and J are in the jth components, and G and J are operation symbols, possibly equal to F or to each other.

If F $\neq$ G, then any variable x at depth one in G is accessible by projections, that is, by replacing F by x_i and G by x_k for the appropriate index k. But in order to produce the word u of length three, any variable x inside G which enters into u must indeed be at depth one in G. Thus this particular choice of terms for F and G will result in an identity of the form x = J(. . .) which by Lemma 5.4.1 is impossible, unless J is equal to one of F or G. The case J = F is quickly ruled out, since it allows us to produce an identity of the form $x^p = x^{p^2}$ by replacing F by x_j^p and G by x_k, for the appropriate k; such an identity cannot hold in A_m when p^3 divides m. So we must have J = G.

Thus we now consider

$$F(\_\_,x,\_\_,G(\_\_\_),\_\_\_) \quad = \quad F(\_\_,x,\_\_,G(\_\_\_),\_\_\_).$$

On the left-hand-side the components of G which are used to form the word u are single variables, each accessible at depth two by a choice of projection terms for F and G. By Lemma 5.4.1, these projections must reach the same variable on the right-hand-side of H, and the variable must be at the same depth to avoid an identity of the form $x^{p^b} = x^{p^c}$ for $b \neq c$ and $b \leq 2$. This establishes that only trivial identities $u = v$ can be obtained from H in the case $F \neq G$.

When $F = G$, we consider

$$F(\_\_,x,\_\_,F(\_\_,y,\_\_,z,\_\_),\_\_\_) \quad = \quad F(\_\_,x,\_\_,J(\_\_\_),\_\_\_),$$

where x occurs in the ith component of the first F on each side, the second F on the left-hand-side and the J on the right-hand-side occur in the jth components of the outermost F's, and y and z occur in the ith and jth components respectively of the inner F on the left-hand-side. Now the variable z is accessible on the left-hand-side by the choice of the projection term x_j for F, so it must be accessible, using only this term, on the right-hand-side too. This forces $J = F$. Also we must have z at the same depth, depth two, on both sides: replacing F by x_j^p produces $z^{p^2} = z^{p^e}$ where e is the depth of nesting of z on the right-hand-side. So the right-hand-side of H must look like

$$F(\_\_,x,\_\_,F(\_\_,K(\_\_\_),\_\_,z),\_\_\_),$$

for some hyperterm $K(\_\_\_)$ in the ith place in the second F. We will show that in fact this hyperterm must be a single variable w. Then since $u = v$ is produced by the choice of x_ix_j for F, it has the form $xyz = xwz$, and for this to hold in A_m we must have $y = w$, so that $u = v$ is indeed trivial.

Suppose that $K(\_\_\_)$ is not just a single variable, but involves an operation symbol K of arity ≥ 1. Make the following choice of terms: $x_i^p x_j^p$ for

F, and $x_1^p \cdots x_n^p$ for any other n-ary operation symbol, $n \geq 1$. We will also simplify by identifying all the variables in H as x. Note that A_m must still satisfy the resulting simplified identity. Under this evaluation, K(____) will produce x^r for some $r > 1$, with r divisible by p. With further calculations we get the identity

$$x^{p+2p^2} \quad = \quad x^{p+rp^2+p^2}.$$

Thus we need to have $p + 2p^2$ congruent to $p + rp^2 + p^2$ modulo m. This reduces to the requirement that $p^2(r - 1)$ be congruent to 0 modulo m. Since p^3 divides m, this is only possible if r - 1 is congruent to 0 modulo p. But p divides r, and $r > 1$, so this is impossible. This contradiction shows that K(____) must after all be a single variable, and finishes the proof.

Corollary 5.4.7: Let $m \geq 2$, $t(m) \geq 3$, and $n \geq 2$. Then $N_4 \subseteq \overline{A}_m \subseteq \overline{A}_{1,m} \subseteq \overline{A}_{n,m}$.

These results about the significance of the parameter t(m) for length of instances of hyperidentities satisfied by the variety A_m are as good as possible for t(m) = 1, 2, or 3. At length four however the situation becomes more complicated. There are hyperidentities satisfied by A_m, for <u>any</u> m, which have non-trivial instances u = v with |u| = 4: the medial hyperidentity is an obvious example, with its base instance xyzw = xzyw. Thus we cannot hope to produce lemmas such as 5.4.4 and 5.4.6 for higher values of t(m), without at least modifying our statements to include the medial identity and its consequences. Since the only "too short" hyperidentities we have found are ones based on consequences of the medial identity, we present the following conjecture.

Conjecture 5.4.8: Let $m = p^r a$, with p a prime and a and $r \geq 1$, so that $t(m) \geq r$. If A_m satisfies a hyperidentity H, and some choice of A_m-terms yields an instance $u = v$ from H where $|u| = r$, then $u = v$ is either trivial or a consequence of the medial identity. Therefore for $t(m) \geq r$, $MN_{r+1} \subseteq \overline{A_m} \subseteq \overline{A_{1,m}}$.

Note that since $MN_r = N_r$ for $1 \leq r \leq 3$, Corollaries 5.4.2, 5.4.5, and 5.4.7 say precisely that this conjecture is true for $r = 1$, 2, or 3. We have been unable to obtain a proof for $r \geq 4$. It seems difficult to prove anything about all the hyperidentities satisfied by A_m for $t(m) \geq 4$. Even for $t(m) = 4$, if we attempt a proof using the approach of the previous lemmas, analyzing all possible ways to produce an instance $u = v$ with $|u| = 4$, we find that a large number of cases and sub-cases is needed. We have managed, by a very lengthy argument, to eliminate all but one possible case, which can only arise when the prime p in question is 2. Obviously, this method is not fruitful, and a different approach is needed.

Chapter 6

Closure Results for the $A_{n,m}$'s

The previous chapter presented some hyperidentities satisfied by the varieties $A_{n,m}$ under appropriate conditions on n and m. These hyperidentities give us some instances which $\overline{A}_{n,m}$ must satisfy. In this chapter we combine this information with the join results from Chapter 3, to identify $\overline{A}_{n,m}$ in some cases. Since the hyperidentity results depended so strongly on the interaction of n and the parameter t(m), we distinguish two cases: $n \geq t(m) + 1$, and $1 \leq n \leq t(m)$. Throughout this chapter we let $t = t(m)$ and $d = d(m)$, and when $t + 1 \geq n$, $s = s(n,m) = t(m)+1-n$. We conclude with closure results for the related varieties $M_{n,m}$ and $B_{n,m}$, and a brief discussion of how the hypervarieties $\mathcal{H}(A_p)$ and $\mathcal{H}(A_{1,p})$ for m square-free fit into the lattice $\mathcal{H}(L(CS))$.

6.1. The Case $n \geq t(m)+1$

In this section we consider varieties $A_{n,m}$ for which n is larger than the length index t(m). We begin with the special case $m = 1$, requiring then that $n \geq 2 = t(m)+1$. Although this is the simplest case, the same method will be used throughout this and the next section.

From Propositions 5.3.5 and 5.3.8, we know that $A_{n,1}$ satisfies hyperidentities based on $x^n = x^{n+1}$ and $x^{n-1}yx = x^nyx$. We also obtain the medial identity as a hyperidentity instance for $A_{n,1}$, since any commutative variety satisfies the medial hyperidentity. Therefore we have

$$\overline{A}_{n,1} \subseteq V(xyzw = xzyw,\ x^n = x^{n+1},\ x^{n-1}yx = x^nyx).$$

By Proposition 3.2.5, this latter variety is precisely $A_{n,1} \vee RB$. Since by Proposition 3.1.1 $RB \subseteq \overline{A}_{n,1}$, we have proved the following result.

<u>Proposition</u> <u>6.1.1:</u> Let $n \geq 2$. Then $\overline{A}_{n,1} = A_{n,1} \vee RB$.

We note that in this case, we were able to produce a hyperidentity based on one of the two defining identities of the variety, $x^n = x^{n+1}$; it is the abelian identity $xy = yx$ which must be "rectangularized" into the medial identity to include RB in the closure. However, if we consider for the moment non-commutative varieties, we can easily determine $\overline{B}_{n,1}$ and $\overline{M}_{n,1}$ for $n \geq 2$ as well. The hyperidentity based on $x^n = x^{n+1}$ involved only the single variable x, so it is of course also satisfied by $B_{n,1}$ and $M_{n,1}$. From this it is immediate that $B_{n,1}$ is closed for $n \geq 2$: $B_{n,1} \subseteq \overline{B}_{n,1} \subseteq V(x^n = x^{n+1}) = B_{n,1}$. Similarly for $M_{n,1}$ we obtain both $x^n = x^{n+1}$ and the medial identity as hyperidentity instances, so that $M_{n,1}$ too is closed for $n \geq 2$. (The cases $B_{1,1} = B$ and $M_{1,1} = NB$ are already known.)

<u>Proposition</u> <u>6.1.2:</u> For $n \geq 1$, $B_{n,1}$ and $M_{n,1}$ are closed varieties.

The situation for $m \geq 2$, still with $n \geq t(m)+1$, is only slightly more complicated than for $m = 1$. As we saw in Section 5.3, we are unable in general to produce a hyperidentity based on $x^n = x^{n+m}$, but must use $x^n = x^{n+d}$. By Propositions 5.3.5 and 5.3.8 again, we obtain instances $x^n = x^{n+d}$ and $x^{n-1}yx = x^{n-1+d}yx$. But also, by the construction method of Section 5.2 (see equation (*) after the proof of Proposition 5.2.1), we know that $A_{n,m}$ satisfies a hyperidentity based on

$$wx^n w^{2^{n+m}-n-1} = wx^{n+m} w^{2^{n+m}-n-m-1}.$$

We will show that this additional instance enables us to get $x^n = x^{n+m}$ after all (although not directly from a hyperidentity). In particular, let $W_{n,m}$ be the variety defined by the identities $xyzw = xzyw$, $x^n = x^{n+d}$, $x^{n-1}yx = x^{n-1+d}yx$, and $wx^n w^{2^{n+m}-n-1} = wx^{n+m} w^{2^{n+m}-n-m-1}$. We have just shown that $A_{n,m} \subseteq W_{n,m}$ when $n \geq t+1$.

<u>Lemma 6.1.3:</u> Let $m \geq 2$ and $n \geq t+1$. Then $W_{n,m}$ satisfies the identities $x^n = x^{n+m}$ and $x^{n-1}yx = x^{n-1+m}yx$.

<u>Proof:</u> If $m = d$ the claim is trivial, so we assume that $m < d$. Since $W_{n,m}$ satisfies $x^n = x^{n+d}$, it satisfies $x^a = x^{a+b}$ for some minimal $a \leq n$ and b dividing d. Then for any semigroup C in $W_{n,m}$ and any c in C, the subsemigroup $\{c^a, c^{a+1}, \ldots, c^{a+b-1}\}$ is a subgroup of C of order b. But from the fourth of the defining identities for $W_{n,m}$ we know that any group in $W_{n,m}$ satisfies $x^n = x^{n+m}$ (by taking $w = 1$ in the identity) and hence also $x^m = 1$; that is, the order of any such group divides m. But now it follows that $W_{n,m}$ satisfies $x^a = x^{a+m}$ and hence also $x^n = x^{n+m}$.

Using this, and using the fact that m divides d to write $d = km$ for some $k > 1$, we get

$$\begin{aligned} x^{n-1}yx &= x^{n-1+d}yx = x^{n-1+km}yx \\ &= x^{n-1+(k-1)m+m}yx = x^{n-1+m+(k-1)m}yx \\ &= x^{n-1+m}yx, \end{aligned}$$

giving the required identity for $W_{n,m}$.

Corollary 6.1.4: For $m \geq 2$ and $n \geq t+1$, $\overline{A}_{n,m} \subseteq W_{n,m} \subseteq A_{n,m} \vee RB$.

Proof: By Proposition 3.2.5, the variety $A_{n,m} \vee RB$ is defined by the identities $xyzw = xzyw$, $x^n = x^{n+m}$, and $x^{n-1}yx = x^{n-1+m}yx$.

Corollary 6.1.5: For $m \geq 2$ and $n \geq t+1$, $\overline{A}_{n,m} = A_{n,m} \vee RB$.

A very similar argument can be used for $\overline{M}_{n,m}$. As we saw in Proposition 5.2.1, the hyperidentity $H(x^n = x^{n+m})$ is also satisfied by $M_{n,m}$, as is the hyperidentity based on $x^n = x^{n+d}$. Thus $\overline{M}_{n,m}$ satisfies the identities $xyzw = xzyw$, $x^n = x^{n+d}$, and $wx^nw^{2^{n+m}-n-1} = wx^{n+m}w^{2^{n+m}-n-m-1}$. As in Lemma 6.1.3, these last two identities imply $x^n = x^{n+m}$ as well. Then $M_{n,m} \subseteq \overline{M}_{n,m} \subseteq M_{n,m}$, and $M_{n,m}$ is closed when $m \geq 2$ and $n \geq t+1$. For $B_{n,m}$, however, we can only say that $B_{n,m} \subseteq \overline{B}_{n,m} \subseteq V(x^n = x^{n+d}) = B_{n,d}$.

Proposition 6.1.6: Let $m \geq 2$ and $n \geq t(m)+1$. Then $M_{n,m}$ is closed.

6.2. The Case $1 \leq n \leq t(m)$

In this section we consider varieties $A_{n,m}$ for which $1 \leq n \leq t(m)$. This includes the special case $m = 1$: then $t(m) = 1 = n$, and we have only the variety $A_{1,1} = SL$, whose closure is already known to be NB. Hence we will assume that $m \geq 2$.

When $1 \leq n \leq t(m)$ the length considerations of Sections 5.3 and 5.4 are involved. In this situation we have the hyperidentities for $A_{n,m}$ given in Propositions 5.3.6 and 5.3.7, giving us the base instances

$$x^n y_1 \cdots y_s = x^{n+d} y_1 \cdots y_s,$$

$$xy_1^n \cdots y_s = xy_1^{n+d} \cdots y_s,$$

and

$$xy_1 \cdots y_s^n = xy_1 \cdots y_s^{n+d},$$

where $s = t+1-n$. We also know that $A_{n,m}$ satisfies the hyperidentity $H(x^n = x^{n+m})$ as constructed in Section 5.2, with the base instance $wx^n w^{2^{n+m}-n-1} = wx^{n+m} w^{2^{n+m}-n-m-1}$. Let the variety defined by these four instances and the medial identity be called $U_{n,m}$, so that $\overline{A}_{n,m} \subseteq U_{n,m}$ in this case.

<u>Lemma</u> <u>6.2.1:</u> Let $m \geq 2$ and $1 \leq n \leq t$. Then $U_{n,m}$ satisfies the identities

$$x^{t+1} = x^{t+1+m},$$

$$x^n y_1 \cdots y_s = x^{n+m} y_1 \cdots y_s,$$

$$xy_1^n \cdots y_s = xy_1^{n+m} \cdots y_s,$$

and

$$xy_1 \cdots y_s^n = xy_1 \cdots y_s^{n+m}.$$

Proof: The claim is trivial if $m = d$, so we assume that $m < d$. We show first that $U_{n,m}$ satisfies $x^{t+1} = x^{t+1+m}$. From the defining identities for $U_{n,m}$ we know that it satisfies $x^{t+1} = x^{t+1+d}$. Hence $U_{n,m}$ satisfies $x^a = x^{a+b}$ for some minimal a and b, with $a \leq t+1$ and b dividing d. Now as in the proof of Lemma 6.1.3, we can produce a semigroup in $U_{n,m}$ which is a group of order b, and use the identity $wx^n w^{2^{n+m}-n-1} = wx^{n+m}w^{2^{n+m}-n-m-1}$ to show that the order of any such group must divide m. Therefore b divides m, so that $U_{n,m}$ satisfies $x^a = x^{a+m}$ and $x^{t+1} = x^{t+1+m}$.

Next we show that $x^{t+1} = x^{t+1+m}$ and the given identity $x^n y_1 \cdots y_s = x^{n+d} y_1 \cdots y_s$ imply $x^n y_1 \cdots y_s = x^{n+m} y_1 \cdots y_s$. The other identities required may be deduced similarly. Since by construction $m > t$ and $n \geq 1$, we have $n+m \geq t+1$; and we may write $d = cm$ for some $c > 1$. Then

$$\begin{aligned} x^{n+m} y_1 \cdots y_s &= x^{n-1+m-t+t+1} y_1 \cdots y_s \\ &= x^{n-1+m-t} x^{t+1} y_1 \cdots y_s \\ &= x^{n-1+m-t} x^{t+1+(c-1)m} y_1 \cdots y_s \\ &= x^{n-1+m-t+t+1+(c-1)m} y_1 \cdots y_s \\ &= x^{n+m+(c-1)m} y_1 \cdots y_s \\ &= x^{n+cm} y_1 \cdots y_s \\ &= x^{n+d} y_1 \cdots y_s \\ &= x^n y_1 \cdots y_s. \end{aligned}$$

Corollary 6.2.2: If $m \geq 2$, then $\overline{A}_{1,m} \subseteq A_{1,m} \vee RB \vee MN_{t+1}$.

Proof: From Lemma 6.2.1 and Proposition 3.4.8, we know that $\overline{A}_{1,m} \subseteq U_{1,m} \subseteq M_{1,m} \vee MN_{t+1}$. Since by Proposition 3.2.1 $M_{1,m} = A_{1,m} \vee RB$, the claim follows.

The case $n > 1$ is slightly more complicated, since we must introduce another identity. For $1 \leq n \leq t$ and $m \geq 2$, $A_{n,m}$ also satisfies the hyperidentity given in Proposition 5.3.9, and based on the instance $xy_1 \cdots y_s x^{n-1} = xy_1 \cdots y_s x^{n-1+d}$. We will use $Y_{n,m}$ for the subvariety of $U_{n,m}$ which satisfies this additional identity.

Lemma 6.2.3: Let $m \geq 2$ and $1 \leq n \leq t$. Then the variety $Y_{n,m}$ satisfies the identity $xy_1 \cdots y_s x^{n-1} = xy_1 \cdots y_s x^{n-1+m}$, and its left-right dual.

Proof: By Lemma 6.2.1, both $U_{n,m}$ and its subvariety $Y_{n,m}$ satisfy $x^n = x^{n+m}$. As before, we assume that $d > m$, and write $d = mc$ for some $c > 1$. Then we have

$$\begin{aligned} xy_1 \cdots y_s x^{n-1} &= xy_1 \cdots y_s x^{n-1+d} \\ &= xy_1 \cdots y_s x^{n-1+cm} \\ &= xy_1 \cdots y_s x^{n+m-1+(c-1)m} \\ &= xy_1 \cdots y_s x^{n-1+m}. \end{aligned}$$

The dual case is handled similarly.

Corollary 6.2.4: Let $m \geq 1$ and $1 \leq n \leq t$. Then $\overline{A}_{n,m} \subseteq A_{n,m} \vee RB \vee MN_{t+1}$.

Proof: The claim follows from Lemmas 6.2.1 and 6.2.3 and Proposition 3.4.13.

We know of course that $A_{n,m} \vee RB$ is always contained in $\overline{A}_{n,m}$. For the MN_{t+1} factor, we turn to the length restrictions lemmas of Section 5.4. For $1 \leq t(m) \leq 3$, we know from Corollaries 5.4.2, 5.4.5, and 5.4.7 that $MN_{t+1} \subseteq \overline{A}_{n,m}$; for larger values of $t(m)$, we have only the conjecture that $MN_{t+1} \subseteq \overline{A}_{n,m}$. Thus we have the following results.

Proposition 6.2.5: Let $m \geq 2$. If $1 \leq n \leq t(m) \leq 3$, then

$$\overline{A}_{n,m} = A_{n,m} \vee RB \vee MN_{t+1}.$$

If $1 \leq n \leq t(m)$ and $t(m) \geq 4$, then

$$A_{n,m} \vee RB \vee MN_4 \subseteq \overline{A}_{n,m} \subseteq A_{n,m} \vee RB \vee MN_{t+1}.$$

Conjecture 6.2.6: Let $m \geq 2$, with $t(m) \geq 4$, and let $1 \leq n \leq t(m)$. Then $\overline{A}_{n,m} = A_{n,m} \vee RB \vee MN_{t+1}$.

Some slight variations on the proofs of this section allow us to identify $\overline{V}$ for some related varieties V. We consider first the varieties A_m, for $m \geq 2$. Since $\overline{A}_m \subseteq \overline{A}_{1,m}$, we have the same instances for A_m as we had for $A_{1,m}$. We also have the instance

$$wxy^m w^{2^{m+1}-m-2} \quad = \quad wxw^{2^{m+1}-2},$$

upon which $H(xy^m = x)$ is based. Then as before we argue that

$$\begin{aligned} \overline{A_m} \quad &\subseteq \quad U_{1,m} \ \cap \ V(wxy^m w^{2^{m+1}-m-2} wxw^{2^{m+1}-2}) \\ &= \quad A_m \vee RB \vee MN_{t+1}, \end{aligned}$$

where we have used Proposition 3.4.10 to verify that the identities we have do indeed define $A_m \vee RB \vee MN_{t+1}$. Combining this with the length restriction information gives the following.

Proposition 6.2.7: Let $m \geq 2$. If $t(m) \leq 3$, then

$$\overline{A_m} = A_m \vee RB \vee MN_{t+1}.$$

If $t(m) \geq 4$, then

$$A_m \vee RB \vee MN_4 \quad \subseteq \overline{A_m} \subseteq A_m \vee RB \vee MN_{t+1}.$$

The case for $M_{n,m}$, when $1 \leq n \leq t(m)$, is much the same. We remarked at the end of Section 5.3 that $M_{n,m}$ also satisfies the hyperidentities of Propositions 5.3.6 and 5.3.7; and in Section 5.2 that $M_{n,m}$ satisfies the hyperidentity $H(x^n = x^{n+m})$. From Proposition 3.4.12, the base instances of these three hyperidentities plus the medial hyperidentity are precisely the identities which define $M_{n,m} \vee MN_{t+1}$, which proves the following.

Proposition 6.2.8: Let $m \geq 2$ and $1 \leq n \leq t(m)$. If $t(m) \leq 3$, then $\overline{M_{n,m}} = M_{n,m} \vee MN_{t+1}$. If $t(m) \geq 4$, then $M_{n,m} \vee MN_4 \subseteq \overline{M_{n,m}} \subseteq M_{n,m} \vee MN_{t+1}$.

The situation for $B_{n,m}$ is not so straightforward. Without mediality, $B_{n,m}$ does not satisfy any of the hyperidentities of Chapter 5, except those involving only one variable. Thus when $1 \le n \le t(m)$, all we can say is that $\overline{B}_{n,m} \subseteq V(x^{t+1} = x^{t+1+d})$.

For example, when $n = 1$ and $m = 2$, we have $t(m) = 1$ and $d(m) = m = 2$, so $B_{1,2}$ satisfies the hyperidentity based on $x^2 = x^4$. In fact, we have been unable to produce any hyperidentities satisfied by $B_{1,2}$ which are not also satisfied by $B_{2,2}$, leading to the conjecture that $\overline{B}_{1,2} = B_{2,2}$. Since $B_{1,2} \vee Z$ is a proper subvariety of $B_{2,2}$, this suggests that in the non-commutative case there is more involved in the closure operation than just rectangularizing and lengthening appropriately.

When m is square-free, the parameter t(m) is 1, and it follows from Propositions 6.1.5, 6.2.6 and 6.2.8 that

$$\overline{A}_m = A_m \vee RB \vee MN_2 = A_m \vee RB \vee Z,$$

and $$\overline{A}_{1,m} = A_{1,m} \vee RB \vee MN_2 = A_{1,m} \vee RB \vee Z.$$

We combine this information with a result of Petrich's to study how χ acts on the varieties A_m and $A_{1,m}$ when m is square-free.

<u>Proposition</u> <u>6.2.9</u> (Petrich [12]:) Let $m \ge 2$. Any subvariety V of the variety $A_{1,m} \vee RB \vee Z$ can be uniquely expressed as

$$V = V_1 \vee V_2 \vee V_3,$$

where $V_1 \subseteq A_m$, $V_2 \subseteq SL$, and $V_3 \subseteq RB \vee Z$.

Proposition 6.2.10: Let p be prime. The closed variety $\overline{A}_{1,p}$ has thirty-two subvarieties, of which seven are closed, and the inclusions shown in Figure 6.1 are all coverings in the image lattice $\mathcal{H}(L(CS))$.

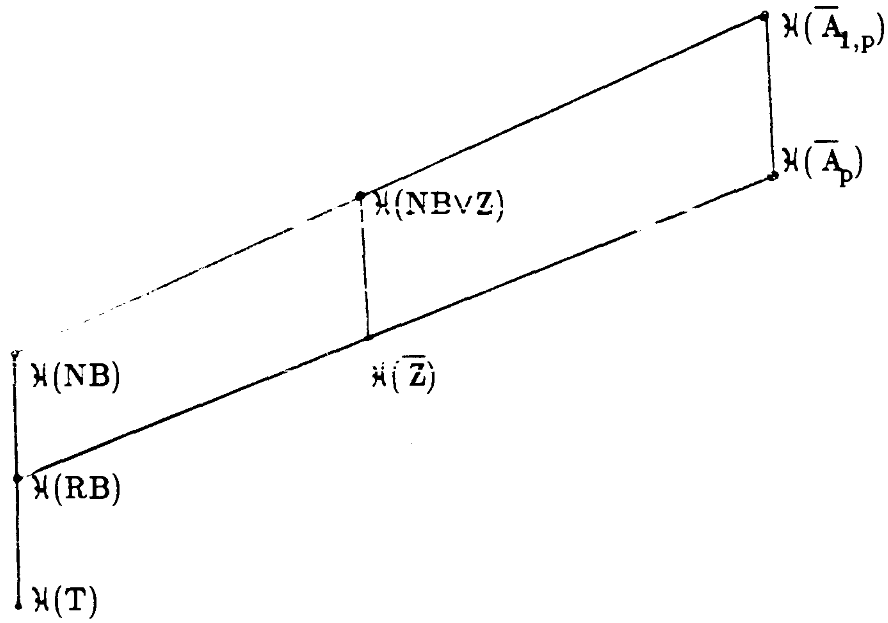


Figure 6-1: A Portion of the Lattice $\mathcal{H}(L(CS))$

Proof: Suppose that $\mathcal{H}(V) \subseteq \mathcal{H}(\overline{A}_{1,p})$, and $\overline{V} \subseteq \overline{A}_{1,p} = A_{1,p} \vee RB \vee Z$. By Proposition 6.3.1, $\overline{V}$ can be expressed as the join of some $V_1 \subseteq A_p$, $V_2 \subseteq SL$, and $V_3 \subseteq RB \vee Z$. Since A_p and SL are atoms of L(S), there are only two possibilities each for V_1 and V_2; by Proposition 3.3.10, $L(RB \vee Z)$ is isomorphic to the product of the lattices L(RB) and L(Z), and consists of the eight subvarieties T, LZ, RZ, RB, Z, LZ $\vee$ Z, RZ $\vee$ Z, and RB $\vee$ Z. Altogether we have thirty-two different subvarieties of $\overline{A}_{1,p}$. Examining each of these in turn, we find that under the closure operator these thirty-two varieties collapse to seven closed varieties: T, RB, $\overline{Z}$, NB, NB $\vee$ Z, $\overline{A}_p$, and $\overline{A}_{1,p}$. Then $\overline{V}$ is one of these seven, and $\mathcal{H}(V) = \mathcal{H}(\overline{V})$ is one of the seven hypervarieties shown in Figure 6.1.

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