EQUILIBRIA AND CHAOTIC SOLUTIONS OF A THREE DIMENSIONAL DYNAMICAL SYSTEM

by

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ABSTRACT

The interaction of three nonlinearly coupled differential equations modelling the co-evolution of three natural electrostatic waves in a plasma is studied. The equilibria of the dynamical system are investigated qualitatively in the phase space. The dynamics is governed by certain parameters related to the growth and decay rates of the interacting waves and the phase mismatch. It is shown that the saturation of the interacting waves is possible for certain values of the parameters. The bifurcations of the equilibrium points to periodic orbits,period doubling bifurcations and chaotic solutions characteristic of a strange attractor are established for different values of the same parameters.

It is shown that the tangent bifurcations from chaotic to periodic solutions can be explained on the one-dimensional map derivable from the original set of ordinary differential equations. An analytical derivation of the map substantiating the numerical study is presented.

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INTRODUCTION

Deterministic equations for systems with few degrees of freedom whose solutions may exhibit chaotic behaviour have been the subject of considerable attention in recent years.Such equations are not only intrinsically interesting from a mathematical viewpoint but also because of the striking similarity between the evolution of their solutions and turbulent events in nature. In other words, the randomness of solutions is reminiscent of the behaviour of turbulence in fluids and atmosphere.

Here we refer to a set of equations which gives the evolution of the amplitudes and phases of three nonlinearly coupled waves. It was introduced by Wersinger, Finn and Ott [1] who analysed a model of nonlinear system consisting of resonant three-wave coupling equations. The modelling equations have the form

$$\begin{aligned}
\frac{dC_1}{dt} &= \gamma_1 C_1 + MC_2C_3\, e^{-i\delta t}, \\
\frac{dC_2}{dt} &= \gamma_2 C_2 + MC_1C_3^{*} e^{-i\delta t}, \\
\frac{dC_3}{dt} &= \gamma_3 C_3 + MC_1C_2^{*} e^{-i\delta t},
\end{aligned} \tag{1}$$

where $\delta=\omega_1-\omega_2-\omega_3$ is the phase mismatch in frequencies ω_k, k=1,2,3 ; M is the coupling coefficient, and the constants γ_k give the temporal behavior of the waves on the long time scale (instability or damping depending on the sign of γ_k). The model (1) was applied to the case where the high frequency wave (k=1) is unstable ($\gamma_1>0$) and the low frequency waves (k=2,3) are linearly damped ($\gamma_{2,3} > 0$). This case represents a physical state for the nonlinear saturation where the temporal increase of energy of wave 1 (due to instability) can be diverted by nonlinear coupling to the damped low frequency waves (waves 2 and 3) and captured in their vicinity.

Using the study of Wersinger, Finn and Ott [1] as a base we analyse the interaction of two Langmuir [2] (longitudinal) waves with an ion-acoustic wave (longitudinal). One of the Langmuir wave here is considered as the growing wave and both, the other Langmuir and the ion-acoustic wave as damped waves.

The use of physical model makes the study more natural and substantiates the earlier work done on (1).The main objective of this thesis is to investigate the equilibria of the model (1) as pertaining to a real world situation and the emergence of chaotic behavior in the light of developments of nonlinear dynamics.

In chapter 1 we present a survey of the earlier work done on model (1) and relevant to the thesis recent work on nonlinear dynamics.

In chapter 2 we introduce the basic equations which govern the phenomenon of the interaction of the two Langmuir waves with the ion-acoustic wave and show that under appropriate conditions , these can be reduced to a set of three nonlinearly coupled first-order ordinary differential equations. We also discuss the assumptions under which the derived set of equations represent the actual physical phenomenon.

Chapter 3 deals with qualitative study of the model. Conditions for the existence of equilibrium points are found and their nature and stability property discussed. Also phase portraits in R^3 are given.

In chapter 4 we study numerically the bifurcation of the equilibrium points to periodic orbits, period doubling bifurcations and chaotic behavior characteristic of a strange attractor. We show that the tangent bifurcations from chaotic to periodic solutions can be explained on the one dimensional map derivable from the original set of ordinary differential equations. In addition we give the analytic expression for the one-dimensional map.

In the conclusion a short discussion of the results obtained is presented.

CHAPTER 1

PRELIMINARIES

In this chapter we summarize in brief some results on the three wave interaction.We also give a short survey of the research done on this topic in various fields. We introduce the basic assumptions of the model (1).

1.1 GENERAL DERIVATION OF THE MODEL EQUATIONS

In this section using the work of Sagdeev and Galeev [3] we derive the model (1) from the following equations representing the interaction between three harmonic oscillators,

$$\frac{d^2X}{dt^2} + 2\nu_1\frac{dX}{dt} + \omega_1^2 X = MYZ,$$

$$\frac{d^2Y}{dt^2} + 2\nu_2\frac{dY}{dt} + \omega_2^2 Y = MZX, \qquad (1.1)$$

$$\frac{d^2Z}{dt^2} + 2\nu_3\frac{dZ}{dt} + \omega_3^2 Z = MXY,$$

where X , Y and Z are the amplitudes of the three coupled oscillators, ν_k , k=1,2,3 are the damping coefficients,ω_k , k=1,2,3 the natural frequencies of the oscillators and M is the coupling constant.To derive the model (1) we use the averaging principle which is described briefly (see Stix [4]). Suppose we have two vector quantities A and B and we are interested in the product

$$AB = \mathrm{Re}(A_0 e^{-i\phi})\mathrm{Re}(B_0 e^{-i\phi})$$

$$= \frac{1}{4}[A_0B_0e^{-2i\phi} + A_0^*B_0^*e^{-2i\phi^*} + A_0B_0^*e^{-i(\phi-\phi^*)} + A_0^*B_0e^{i(\phi^*-\phi)}] , \; i = \sqrt{-1} \; .$$

The asterisk indicates the complex conjugate. If $e^{\pm i\phi_r}$ is periodic in space or time where $\phi = \phi_r + i\phi_i$, we may average over a single period to obtain

$$\overline{AB} = \frac{1}{4}(A_0B_0^* + A_0^*B_0)e^{2\phi i} , \qquad (1.2)$$

provided ϕ_i is constant over the same interval . If ϕ_r is almost periodic in space or time, or if A_0,B_0, or ϕ_i are slowly varying functions of space or time, eq.(1.2) is still correct in some mean value sense.With this physical interpretation in mind, we write for the amplitudes of each of the oscillators

$$X = C_1(t)e^{i\omega_1 t} + \text{c.c.},$$

$$Y = C_2(t)e^{i\omega_2 t} + \text{c.c.}, \tag{1.3}$$

$$Z = C_3(t)e^{i\omega_3 t} + \text{c.c.},$$

where $C_j(t)$, $j = 1,2,3$, are the slowly varying parts of the amplitudes and c.c. represents the complex conjugate part . In the following formulation we will also assume that the coupling is small, which justifies the use of the oscillator co-ordinates as the product of a slowly varying amplitude and rapidly oscillating exponential as in (1.3).

The boundary conditions for the physical problem are based on the fact that the first oscillator has already been excited to a much larger amplitude than the other two oscillators.

Substituting the expression (1.3) into (1.1) yields the equations

$$\left[\frac{d^2C_1(t)}{dt^2} + 2i\omega_1\frac{dC_1(t)}{dt} - \omega_1^2C_1(t) + 2\nu_1\frac{dC_1(t)}{dt} + 2i\nu_1\omega_1C_1(t) + \omega_1^2C_1(t)\right]e^{i\omega_1 t} + \text{c.c.}$$

$$= M\left[C_2e^{i\omega_2 t} + \text{c.c.}\right]\left[C_3e^{i\omega_3 t} + \text{c.c.}\right],$$

$$\left[\frac{d^2C_2(t)}{dt^2} + 2i\omega_2\frac{dC_2(t)}{dt} - \omega_2^2C_2(t) + 2\nu_2\frac{dC_2(t)}{dt} + 2i\nu_2\omega_2C_2(t) + \omega_2^2C_2(t)\right]e^{i\omega_2(t)} + \text{c.c.}$$

$$= M\left[C_1(t)e^{i\omega_1(t)} + \text{c.c.}\right]\left[C_3(t)e^{i\omega_3 t} + \text{c.c.}\right], \tag{1.4}$$

$$\left[\frac{d^2C_3(t)}{dt^2} + 2i\omega_3\frac{dC_3(t)}{dt} - \omega_3^2C_3(t) + 2\nu_3\frac{dC_3(t)}{dt} + 2i\nu_3\omega_3C_3(t) + \omega_3^2C_3(t)\right]e^{i\omega_3 t} + \text{c.c.}$$

$$= M\left[C_1(t)e^{i\omega_1 t} + \text{c.c.}\right]\left[C_2(t)e^{i\omega_2 t} + \text{c.c.}\right].$$

Since $C_i(t)$ is a slowly varying function of time,we may neglect the second derivative terms (i.e. $\frac{d^2C_j(t)}{dt^2}$) compared with $\omega_j\frac{dC_j(t)}{dt}$, being of order $\omega_j\tau$ which is <<1.

Hence we have the equations,

$$\frac{dC_1}{dt} + \nu_1 C_1 + \frac{(c.c.)e^{-2i\omega_1 t}}{2i\omega_1} = \frac{M}{2i\omega_1}\Big[C_2C_3e^{i(\omega_2+\omega_3)t-i\omega_1 t} + C_2C_3^*e^{i(\omega_2-\omega_3)t-i\omega_1 t} + C_2^*C_3e^{i(-\omega_2+\omega_3)t-i\omega_1 t} + C_2^*C_3^*e^{i(-\omega_2-\omega_3)t-i\omega_1 t}\Big],$$

$$\frac{dC_2}{dt} + \nu_2 C_2 + \frac{(c.c.)e^{-2i\omega_2 t}}{2i\omega_2} = \frac{M}{2i\omega_2}\Big[C_1C_3e^{i(\omega_1+\omega_3)t-i\omega_2 t} + C_1C_3^*e^{i(\omega_1-\omega_3)t-i\omega_2 t} + C_1^*C_3e^{i(-\omega_1+\omega_3)t-i\omega_2 t} + C_1^*C_3^*e^{i(-\omega_1-\omega_3)t-i\omega_2 t}\Big], \tag{1.5}$$

$$\frac{dC_3}{dt} + \nu_3 C_3 + \frac{(c.c.)e^{-2i\omega_3 t}}{2i\omega_3} = \frac{M}{2i\omega_3}\Big[C_1C_2e^{i(\omega_1+\omega_2)-i\omega_3 t} + C_1C_2^*e^{i(\omega_1-\omega_2)t-i\omega_3 t} + C_1^*C_2e^{i(-\omega_1+\omega_2)t-i\omega_3 t} + C_1^*C_2^*e^{i(-\omega_1-\omega_2)t-i\omega_3 t}\Big].$$

Equations (1.5) have a complicated structure and can be simplified using the averaging principle which allows one to average over the fast time scale associated with the oscillator frequencies since the amplitudes C_j may be considered constant . In general the averaging procedure will make all the terms on the right hand side of (1.5) vanish and will yield C_1,C_2,C_3 =constant.

However, when the oscillator frequencies ω_j , j=1,2,3, satisfy a resonance condition that reduces the oscillation rate of one of the exponentials on the right hand side of (1.5) to the time scale associated with the amplitudes, this particular exponential will remain unaffected by the averaging process.This can be seen when the frequencies satisfy the resonance condition $\omega_1=\omega_2+\omega_3$,then averaging of (1.5) yields

$$2i\omega_2\frac{dC_2}{dt} = MC_1C_3 ,$$

$$2i\omega_3\frac{dC_3}{dt} = MC_1^*C_2 , \tag{1.6}$$

where C_1 is considered to be a constant.

We seek a solution of (1.6) in the form of $C_j = c_j\, e^{i\nu t}$, $j=2,3$. Working only to lowest order by making a small parameter expansion around ν/ω_j, we get the dispersion relation

$$\nu = \pm\frac{|C_1||M|}{\sqrt{4\omega_2\omega_3}} \tag{1.7}$$

From the above result we see that C_2 and C_3 may have growing solutions only when the product $\omega_2\omega_3$ is negative. The resonance condition $\omega_1=\omega_2+\omega_3$ combined with the condition that $\omega_2\omega_3$ be negative gives an equivalent condition that $|\omega_1| > |\omega_2|,|\omega_3|$ and also (1.5) reduces to the model (1).

This analysis leads to the conclusion that if one of the oscillators is initially excited to a much larger energy than the other two oscillators, then it can transfer energy to the other two oscillators if and only if it has a higher frequency than the other two oscillators. We may also conclude that the three-wave interaction is the lowest order nonlinear effect (expanding in the wave amplitudes). The interaction is coherent if the spectral widths in k and ω of the interacting wave packets are small,respectively,compared to the inverse spatial scale length and the inverse of the interaction time (see Tsytovich [5]).

1.2 REVIEW OF THE PREVIOUS WORK

Nonlinear three-wave interactions have been studied in various areas: in the context of parametric amplifiers (Cullen [6]; Louisell [7]), nonlinear optics (Armstrong et.al.[8]), interactions of water waves (Bretherton [9]; McGoldrick [10]; Benney and Newell [11]), and interactions of bulk acoustic waves (Shiren [12]) and surface acoustic waves (Svaasand [13]; Newhouse et.al.[14]; Davis and Newhouse [15]).We are usually concerned with three wave interactions in which one of the waves is externally excited, for example, in laser-plasma interactions or in lower hybrid heating of a tokamak plasma. The other two waves initially have their amplitudes at the noise level. Hence,the wave externally excited has a much larger amplitude than that of the other two waves. The externally excited wave is called the "pump" and the other two interacting waves,the "daughter"waves. In order to describe the initial development of the interaction we discard terms involving products of the amplitudes(assumed initially small). We obtain a pair of linear relations describing the evolution of the initially small amplitude waves and the amplitude of the pump is taken to be time independent. In the course of the interaction , if the amplitude of the low-frequency waves becomes comparable to the amplitude of the pump,then the linear approximation breaks down and the subsequent evolution must be described by the full set of nonlinear equations.

In reality,an externally excited wave will interact with the whole spectrum of pairs of plasma waves. Up to the time the linear approximation remains valid,each such pair evolves independently of the others. The assumption is that the unstable spectrum is sufficiently narrow for the subsequent nonlinear interaction to be described by the coherent three wave equations. Essentially, the nonlinear evolution of the pump is determined by its interaction with the first pair of waves to grow to a large amplitude.

The nonlinear coupling of three waves is typically encountered in the description of any conservative nonlinear medium where:(a) The nonlinear dynamics can be considered as a perturbation of the linear wave solutions;(b) The lowest order nonlinearity is quadratic in the field amplitudes;(c) The three-wave resonance conditions (1.2) can be satisfied. Benney and Newell [11] showed that the nonlinearly coupled three-wave equations can be obtained from an appropriate nonlinear model.

1.3 ELECTRODYNAMICS OF THE THREE-WAVE INTERACTION

Here we discuss the derivation of the basic equations in the context of the electrodynamics of weakly nonlinear media. We follow Bers,Reiman and Kaup [16].

The linear dynamics of a medium can be represented in general by a linear(space-time integral) dependence of the electric current density $\mathbf{J}(\mathbf{r},t)$ upon the electric field $\mathbf{E}(\mathbf{r},t)$. For a homogeneous medium using complex Fourier transforms for the fields $\left\{ e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \text{ dependence} \right\}$, this relationship can be written as

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega)E_j(\mathbf{k},\omega)\ , \quad k = (k_1,k_2,k_3)^T. \tag{1.8}$$

Substituting (1.8) into Maxwell's equations gives the homogeneous set of equations

$$D_{ij}(\mathbf{k},\omega)E_j(\mathbf{k},\omega) = 0\ , \tag{1.9}$$

where the dispersion tensor (MKS units) is

$$D_{ij} = (1 - \frac{c^2k^2}{\omega^2})\delta_{ij} + \frac{c^2k^2}{\omega^2}k_ik_j + i\frac{\sigma_{ij}(\mathbf{k},\omega)}{\omega\varepsilon_0}. \tag{1.10}$$

The field equations are thus constrained by the dispersion relation

$$D(\mathbf{k},\omega) = \det D_{ij}(\mathbf{k},\omega) = 0\,, \tag{1.11}$$

giving $\omega(\mathbf{k}) \equiv \omega_k$. The linear field solutions can therefore be written in general as

$$E(r,t) = \sum_{k} \mathbf{E}_k e^{i(\mathbf{k}.\mathbf{r}-\omega_k t)} \tag{1.12}$$

For weakly dissipative media ($|\sigma_{ij}^h| << |\sigma_{ij}^a|$,where the superscript h stands for the Hermitian part

and the superscript a stands for the anti-Hermitian part) we take these fields to be the weakly damped (or growing) propagating waves,i.e.,for **k** real $|\nu| \equiv |\mathrm{Im}\omega_k| << |\mathrm{Re}\omega_k| \equiv |\omega|$ and thus from (1.11) with $|D_i| \equiv \mathrm{Im}|D(k,\omega)| << |\mathrm{Re}D(k,\omega)| \equiv |D_r|$ one obtains

$$D_r(k,\omega) = 0 \text{ giving } \omega(k) \tag{1.13}$$

and

$$\nu(\mathbf{k}) = - \left.\frac{D_i}{(\partial D_r/\partial\omega)}\right|_{D_r=0} \tag{1.14}$$

Consider now the nonlinear electrodynamics of the medium as a perturbation. The nonlinear electric current density to second order in the electric field will be given by

$$J_i^{(2)} = \sigma_{ijk}^{mn} E_j^m E_k^n\,, \tag{1.15}$$

where for brevity the superscripts m and n stand for the dependence upon $(\mathbf{k}_m,\omega_m)$ and $(\mathbf{k}_n,\omega_n)$,respectively,of the field variables and the third rank tensor. We now assume that this second-rank current will produce a slowly varying space-time amplitude variation in the linear field solutions, so that (1.12) now becomes

$$E(\mathbf{r},t) = \sum_{k} E_k u_k(r,t) e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \tag{1.16}$$

where the $\mathbf{r} \equiv x_i$ and t variation in u_k is slow compared to,respectively,k_i^{-1} and ω^{-1}. In general,the perturbation will produce a slowly varying amplitude and polarization(i.e.,orientation) of the electric field vector. It can,however,be shown to second-order that the dynamic equations for the amplitude are decoupled from those for the polarization. Here we consider only the amplitude equations. When (1.15) and (1.16) are substituted in Maxwell equations and the nonlinearity is considered with slow variation,we obtain an infinite set of coupled partial differential equations for the slowly varying amplitudes. From the structure of (1.15) it is clear that the simplest coupling will consist of a resonant triplet of linear waves satisfying

$$k_1 = k_2 + k_3 \,, \tag{1.17}$$

and

$$\omega_1 = \omega_2 + \omega_3 \,. \tag{1.18}$$

We shall write down the resulting coupled equations for the slowly varying amplitudes of the three waves which one obtains when the nonlinearity is conservative. We will normalize the slowly varying amplitude $u(\mathbf{r},t)$ so that its magnitude square is the action density of the wave. Let this normalized,slowly varying amplitude be $a_k(\mathbf{r},t) = a_{k0}u_k(\mathbf{r},t)$,

$$|a_{k0}|^2 = \left|\frac{\omega k0}{\omega}\right| = \frac{\varepsilon E_{k0}^2}{4}\left|\frac{\partial D_k}{\partial \omega}\right|_{Dr=0} , \tag{1.19}$$

where we have set $E_k = eE_{k0}$ and $\left(\frac{\partial D_k}{\partial \omega}\right) = e_i^* \left(\frac{\partial D_{ij}^n}{\partial \omega}\right) e_j$.

Then the three coupled equations are:

$$\left(\frac{\partial}{\partial t} + v_1.\nabla + \nu_1\right) a_1 = p_1 K a_2 a_3 \,,$$

$$\left(\frac{\partial}{\partial t} + v_2.\nabla + \nu_2\right) a_2 = p_2 K^* a_1 a_3^* \,, \tag{1.20}$$

$$\left(\frac{\partial}{\partial t} + v_3.\nabla .+ \nu_3\right) a_3 = p_3 K^* a_1 a_2^* \,,$$

where,assuming $\omega>0$, $p_k = \mathrm{sgn}(\omega_{k0}) = \pm 1$ is the energy parity ,

$$v_k = \frac{\partial \omega}{\partial k} \tag{1.21}$$

is the group velocity , and K is given by

$$\sqrt{|a_{10}a_{20}a_{30}|}\, K = -\frac{E_1^*.J_{23}^{(2)}}{4\omega_1} = -\frac{E_2.J_{1\text{-}3}^{(2)*}}{4\omega_2} = -\frac{E_3.J_{1\text{-}2}^{(2)*}}{4\omega_3}. \tag{1.22}$$

In eq(1.22) the subscripts on the second-order current indicate the wave amplitudes on which it depends (i.e.,superscripts on the right-hand side of eq(1.15)). Eq.(1.20) with all $v=0$ and each $v.\nabla = v_x \frac{\partial}{\partial x}$ are just the equations

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x}\right) a_i = p_i K a_j a_k ,$$

$$\left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x}\right) a_j = - p_j K^* a_i a_k^* , \tag{1.23}$$

$$\left(\frac{\partial}{\partial t} + v_k \frac{\partial}{dx}\right) a_k = - p_k K^* a_i a_j^* ,$$

for the one-dimensional space and time evolution of the three-wave resonant interaction. Also,with all $v=0$,all $\left(\frac{\partial}{\partial t}\right) = 0$, and each $v.\nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{dy}$, eq.(1.20) describe the two dimensional steady state resonant interaction of three waves.

1.4 RELEVANCE OF THE PROBLEM TO OUR PRESENT WORK

In plasma physics, and especially for high-temperature plasmas in a magnetic field, the linear dynamics of plasmas involves a very rich variety of waves. Nonlinear dynamics of plasmas can involve both wave-particle interactions and wave-wave interactions. The latter, in its simplest form, is described by (1.20). Such interactions are of importance in ionospheric propagation,in the evolution of various plasma instabilities, and more recently in problems of plasma heating with high-power electromagnetic sources,eg., with lasers for pellet fusion, and with radio-frequency to microwave and millimeter sources for magnetically confined fusion plasmas. In all cases where the nonlinear interactions were solved for pump depletion, only their evolution in time was considered .

Until recently, in discussing the problem of the onset of turbulence, the question of specific methods of eliminating an instability developing in a viscous nonlinear medium and the question of the mechanisms leading to the appearance of disordered random motion were, as a rule, considered independently. The elimination of an instability was generally attributed to systematic intraspectral transfer of the energy of diversely-scaled perturbations into the region of strongly damped small scales. On the other hand, the onset of chaos was associated either with the excitation of a large number of independent perturbations, or with some "conservative" mechanisms. It has become clear in recent years as a result of investigations of stochastic auto-oscillations in a dynamical system with a small number of modes that, in principle , the possibility of the onset of chaos in dissipative systems can itself be related to the mechanism underlying the limitation of the instability. In particular, the onset of stationary disordered motions in systems, in which the stabilization of linearly amplifiable modes is affected by the decay or parametric mechanism of energy transfer to damped perturbations of multiple scales, has now been investigated in detail.

Natural interest attaches to the observation of stochasticity within the framework of models in which stabilization is affected through the transfer of energy to neighbouring(nearly unstable) scales. The simplest model of this sort can be obtained by generalizing the well-known Landau model by allowing for the broadening of the spectrum in the course of the self-modulation(or self-focussing) of the wave packet, i.e.,for the excitation of close modes.

CHAPTER 2

Derivation of the model equations from the coupling equations of the three plasma waves

In this chapter we present the actual equations representing the coupling of the two high-frequency Langmuir (longitudinal) waves with the low-frequency ion-acoustic (longitudinal) wave in the presence of collisions which constitute the damping of the waves. We also present the conditions under which these equations can be reduced to the model (1).

2.1 MOTIVATION OF THE STUDY

Until quite recently, it was thought that turbulence,i.e.,stochastic self-oscillations of a continuous medium,was related exclusively to the excitation of an exceedingly large number of degrees of freedom. Although fully developed turbulence may be a problem with infinitely many relevant degrees of freedom, it is now widely believed that near the onset, a finite number of modes should be enough for a qualitative description of the system.

When a system is controlled by one heat reservoir it is known that the symmetry of the system may be lowered in a co-operative way through a definite critical temperature. This is known as the phase transition in thermodynamic equilibrium. Any ordered state which emerges as a result of such a transition is generally restricted to a regular spatial structure or pattern. In other words it always belongs to the fixed points of the motion. This restriction is due to the time reversal symmetry which is characteristic of thermodynamic equilibrium.

When the system is controlled by more than one independent external reservoir the state of the system generally deviates from thermal equilibrium (G. Nicolis and I.Prigogine [17]). Namely, in addition to the fixed points (of lower symmetry) which are familiar in thermal equilibrium, there may appear recurring orbital motions which are structurally stable, thus forming new phases. In particular there appear periodic orbits which are known as limit cycles. This kind of nonlinear oscillation exhibits much less amplitude response than a linear oscillator, and the dynamic stability of various kind of biorhythms may be understood from this point of view.

Recently, however, a third phase, different either from fixed points or from periodic orbits, has come under serious consideration . This is associated with the appearance of a recurrent aperiodic orbit(Rabinovich [18]). Although the existence of such solutions is known for some time, the recognition of the structural stability of such a "phase' belongs to relatively recent years. In fact the turbulence phenomenon in hydrodynamic has been reinterpreted from this point of view. This third variety is generally called a "chaotic phase" in contrast to "spatial pattern" or "temporal rhythm" and is one of the topical subjects of nonlinear dynamics.

Investigations of turbulence in dissipative media are called for in many problems of physics, such as Langmuir turbulence in a collision dominated plasma, acoustic turbulence in solids, investigations of thermal conduction, of the boundary layer,etc. In the majority of cases, turbulence in dissipative media is not similar to the most thoroughly investigated turbulence in media characterized by the presence of a wide inertia interval in which both the instability and damping can be neglected. The turbulence that is established within the inertia interval usually has a universal spectrum k_s (Tsytovich [3]), is fully described by the energy flux over the spectrum, and is essentially independent of the cause of its excitation be it the intrinsic instability outside the inertia interval, the action of electric fields, etc. In the case of a weak nonlinearity, such a turbulence is described by the kinetic equation for the waves (Tsytovich [3]), which can be obtained by using the random phase approximation. On the other hand, if the inertia interval is either small or nonexistent, then the phases of the individual modes can no longer be regarded as independent even in the case of a weak nonlinearity, and the random-phase approximation cannot be used. This situation, wherein the phases of the individual waves are interconnected in the case of multiwave interaction, is customarily called strong wave turbulence.

2.2 Model of the turbulence with spectrally narrow excitation region

If an unstable mode exists in a nonlinear dissipative medium with dispersion, it is neccesary in the general case to take into account the following nonlinear processes:

1) Generalization of harmonics of the growing mode and the resultant synchronization of the phases of the nonlinear periodic waves whose waveform depends on the magnitude for the given type of waves.

2) Decay of the growing wave (ω_0,k_0) into pairs of low-frequency waves (ω_1,k_1) and (ω_0-ω_1,k_0-k_1) with synchronized phases , where the waves in each pair can be of different types.

3) The interaction between the different nonlinear waves and the decay pairs.

If the medium has a sufficiently strong dispersion for all the wave types that participate in the interaction, then no harmonics can be produced and the principal elementary process in the general picture of the turbulence is the decay of the growing mode into pairs that are parametrically coupled with it.

2.3 Coupled mode equations

In the study of the dynamics of plasma systems, two descriptions are generally used. These are the hydrodynamical fluid description and the more detailed kinetic description. In the first case the Navier-Stokes equation is used in conjunction with the Lorentz force term and in the second case the Boltzmann equations are used. Both these sets of equations are nonlinear. In order to avoid complexities of the nonlinear treatment one considers only small variations in time and space of the quantities studied. In the linear approximation products of the oscillating quantities are neglected and therefore, the disturbance of the medium due to one wave does not influence other waves and thus the waves become uncoupled in this approximation. In the nonlinear theory we also take into account the influence of the waves upon each other and thus the waves become coupled. To be able to treat this situation mathematically we must assume that the interaction is in some sense weak so that the influence of the contributions will decrease rapidly with increasing the order. To first nonlinear order, called the second order approximation, we only include contributions to the interaction which are proportional to products between two oscillating quantities. If the second order oscillating quantities satisfy the dispersion relation of the medium they may produce a third wave, and we thus obtain a system of three interacting waves .

We follow Sjolund and Stenflo[19] to derive the coupled mode equations for the interaction between two Langmuir waves and one ion-acoustic wave.

We consider a plasma of electrons and ions, and the unperturbed plasma is assumed to be quasi-neutral and homogeneous in space. The following notation are used:

m = electron mass

M = ion mass

-e = electron charge

N_0 = unperturbed electron and ion density

n_e = perturbation in electron density

n_i = perturbation in the ion density

v_e = electron fluid velocity

v_i = ion fluid velocity

E and B = electric field and magnetic induction

v_{the} = electron fluid velocity

v_{thi} = ion thermal velocity

c_s = ion acoustic velocity.

The dynamic equations for the plasma will then be:

$$\frac{\partial n}{\partial t} + N_0\, \nabla.v = -\, \nabla(nv),$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0,$$

$$\frac{\partial v_e}{\partial t} + \frac{eE}{m} + \frac{v_{the}^2}{N_0}\nabla n_e = -(v_e.\nabla)v_e - \frac{e}{m} v_e{\times}B + \frac{v_{the}^2}{N_0{}^2} n_e \nabla n_e - \nu_e v_e\,, \qquad (2.1)$$

$$\frac{\partial v_i}{\partial t} - \frac{eE}{M} + \frac{v_{thi}^2}{N_0} dn_i = -(v_i.\nabla)v_i - \frac{e}{M} v_i{\times}B + \frac{v_{thi}^2}{N_0{}^2} n_i \nabla n_i - \nu_i v_i\,,$$

$$\frac{\partial E}{\partial t} - \nabla{\times}B - eN_0(v_e - v_i) = en\,(v_e - v_i)\,,$$

where the nonlinear terms have been written on the right hand side.

We will now consider the interaction between waves. In the linear approximation the dynamic quantities of one wave may be presented in the form

$$A_j e^{i(\omega_j t\, -\, k_j x)}\,, \qquad (2.2)$$

where A_j is a constant vector and the propagation is taken to be in the x-direction.

If we take all the perturbed quantities in eq.(2.1) to vary in the form (2.2) and we neglect the nonlinear terms, we obtain the condition for a nontrivial solution

$$(\omega_j^2 - \omega_p^2 - k_j^2 v_{the}^2)(\omega_j^2 - k_j^2 c_s^2) = 0,$$

where $\omega_p = \sqrt{\frac{N_0 e^2}{m\varepsilon_0}}$ is the plasma frequency . The frequency ω and the wave number k are thus related by one of the dispersion relations

$$\omega_j^2 = \omega_p^2 + k_j^2 v_{the}^2 , \qquad (2.3)$$

representing a Langmuir wave and

$$\omega_j^2 = k^2 c_s^2, \qquad (2.4)$$

representing an ion-acoustic wave.

In the presence of the nonlinear terms in (2.1), the solution will no longer be in the form (2.2). We will choose a Fourier component in space and study its evolution in time. We will now use the coupled-mode method. Instead of studying a particular oscillating component a linear combination is chosen which is the normal mode a. It may be found by substitution of a linear combination, a, with unknown coefficients, of the dynamic variables n,E,..... into the relation

$$\frac{\partial a}{\partial t} = - i \, \omega a . \qquad (2.5)$$

The time derivatives are eliminated with the help of the linearized hydrodynamic equations where a harmonic variation in space ($\frac{\partial}{\partial x} \equiv - ik$) is assumed. By making use of the dispersion relation and equating the coefficient for each of the dynamic variables to zero the unknown coefficients may be found. This implies that linearly the relation(2.5) is valid for arbitrary and independent variations in time of the dynamic variables. In (2.5) ω is the solution to the dispersion relation corresponding to the k-th Fourier component which we study. The advantage of introducing the normal mode a is that we obtain the relation(2.5) without using any linear relation between the oscillating quantities which implies a harmonic time-dependence.

First we study a Langmuir wave. Assuming the propagation in the x-direction the equations governing the Langmuir (longitudinal) mode are obtained from

$$\frac{\partial n}{\partial t} = -N_0\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}(Nv_0),$$

$$\frac{\partial v}{\partial t} = -\nu_e v - \frac{e}{m}E - \frac{v_{the}^2}{N_0}\frac{\partial n}{\partial x}, \tag{2.6}$$

$$\frac{\partial E}{\partial t} = 4\pi e N_0 v + 4\pi e N v_0 .$$

The linear equations for this mode are given by

$$\frac{\overline{n}}{N_0} = \frac{k_1\overline{v}}{\omega_1},$$

$$\overline{E} = \frac{4\pi i N_0 e\,\overline{v}}{\omega_1}. \tag{2.7}$$

Let us now define the longitudinal normal mode as

$$a_1 = \frac{\overline{v}}{v_{the}} + \frac{k_1}{\omega_1}v_{the}\frac{\overline{n_e}}{N_0} - \frac{ie}{mv_{the}}\frac{1}{\omega_1}\overline{E}, \tag{2.8}$$

where

$$\omega_1^2 = \omega_p^2 + k_1^2 v_{the}^2,$$

and

$$v = \overline{v}\,e^{-i\omega_1 t} + \overline{v}^*\,e^{i\omega_1 t}.$$

Substituting (2.7) in (2.8) gives

$$a_1 = \frac{\overline{v}}{v_{the}}\left[1 + \frac{k_1^2 v_{the}^2}{\omega_1^2} + \frac{\omega_p^2}{\omega_1^2}\right] = \frac{2\,\overline{v}}{v_{the}},$$

or

$$\frac{a_1}{2} = \frac{\overline{v}}{v_{the}} . \tag{2.9}$$

Substituting $\overline{v}$ from (2.9) into (2.7) gives

$$\frac{\overline{n}}{N_0} = \frac{k_1 v_{the}}{\omega_1}\frac{a_1}{2} ,$$

and

$$\overline{E} = \frac{4\pi i N_0 e}{\omega_1} v_{the}\frac{a_1}{2} .$$

Now, including the nonlinear terms in (2.6) we obtain

$$\frac{\partial a_1}{\partial t} = \frac{1}{v_{the}}\frac{\partial \overline{v}}{\partial t} + \frac{k_1 v_{the}}{\omega_1}\frac{1}{N_0}\frac{\partial \overline{n}}{\partial t} - \frac{ie}{m v_{the}\omega_1}\frac{1}{}\frac{\partial \overline{E}}{\partial t}$$

$$= -\frac{1}{v_{the}}(\nu_e \overline{v} + \frac{e}{m}\overline{E} + \frac{v_{the}^2}{N_0} ik_1 \overline{n}) - \frac{k_1 v_{the}}{\omega_1}(ik_1 \overline{v} + ik_1 \frac{\overline{Nv_0}}{N_0})$$

$$-\frac{i\omega_p^2}{v_{the}}\frac{1}{\omega_1}\overline{v} - \frac{i\omega_p^2}{v_{the}\omega_1}\frac{\overline{Nv_0}}{N_0} ,$$

or with (2.7) and (2.8) we have

$$\frac{\partial a_1}{\partial t} = -\frac{a_1}{2}[\nu_e + \frac{i\omega_p^2}{\omega_1} + \frac{ik_1^2 v_{the}^2}{\omega_1} + \frac{ik_1^2 v_{the}^2}{\omega_1} + \frac{i\omega_p^2}{\omega_1} - \frac{i}{\omega_1}(\omega_p^2 + k_1^2 v_{the}^2)\frac{\overline{Nv_o}}{N_0 v_{the}}$$

$$= \frac{-\nu_e a_1}{2} - i\omega_1 a_1 - i\omega_1 \frac{\overline{Nv_0}}{N_0 v_{the}}$$

We use the averaging principle to calculate the value of Nv_0, according to which

$$Nv_0 = [\,\overline{N}e^{i\omega_2 t} + \overline{N^*}e^{i\omega_2 t}\,][\,\overline{v_0}e^{-i\omega_0 t} + \overline{v_0^*}e^{i\omega_0 t}\,]$$

$$= \overline{N}\,\overline{v_0}e^{-i(\omega_0+\omega_2)t} + \overline{N}\,\overline{v_0^*}e^{i(\omega_0-\omega_2)t} + \overline{N^*}\,\overline{v_0}e^{-i(\omega_0-\omega_2)t}$$

$$+ \overline{N^*}\,\overline{v_0^*}e^{i(\omega_0+\omega_2)t} ,$$

or

$$\overline{Nv_0} = \overline{N^*}\ \overline{v_0}$$

since $\omega_0=\omega_1+\omega_2$, an assumption made for the interaction not to be averaged out due to rapid harmonic oscillator of the driving terms. However, due to the nonlinear interaction the Fourier components in space are not Fourier components in time and thus the relation $\omega_0=\omega_1+\omega_2$ need not be exactly fulfilled. In fact the dynamic quantities $\overline{v}$, $\overline{n}$, $\overline{E}$ may be written in the form of (here we present only $\overline{v}$)

$$\overline{v_y^{(j)}} = \overline{v_{y0}^{(j)}}\ e^{i\omega_j t},$$

where the condition for the weak interaction is

$$\left|\frac{\partial}{\partial t}\left(\ln \overline{v_{y0}^{(j)}}\ (t)\right)\right| << \omega_j .$$

This means that $\overline{v_y^{(j)}}$ consists only of frequencies close to ω_j .

Therefore, we have

$$\frac{\partial a_1}{dt} = \frac{-\nu_e}{2} a_1 - i\omega_1 a_1 - i\omega_1 \frac{\overline{N^*}\ \overline{v_0}}{N_0 v_{the}}$$

$$= \frac{-\nu_e}{2} - i\omega_1 a_1 - \frac{i\omega_1}{4} a_2^* a_0 \ ,$$

since

$$\frac{\overline{N^*}}{N_0} = \frac{a_2^*}{2}\ , \qquad \frac{v_0}{v_{the}} = \frac{a_0}{2}\ .$$

Now we do a similar analysis for the Langmuir pump wave. The governing equations can again be written as

$$\frac{\partial n_0}{\partial t} = -N_0 \frac{\partial v_0}{\partial x} - \frac{\partial}{\partial x}(N\overline{v})\ ,$$

$$\frac{\partial v_0}{\partial t} = \frac{-e}{m} E_0 - \frac{v_{the}^2}{N_0}\frac{\partial n_0}{\partial x}\ , \qquad (2.10)$$

$$\frac{\partial E_0}{\partial t} = 4\pi e N_0 v_0 + 4\pi e (Nv) \ .$$

We define the pump wave normal mode as

$$a_0 = \frac{\overline{v_0}}{v_{the}} + \frac{k_0 v_{the}}{\omega_0}\frac{\overline{n_0}}{N_0} - \frac{ie}{mv_{the}}\frac{1}{\omega_0}\overline{E_0}$$

and differentiating, we get

$$\frac{\partial a_0}{\partial t} = \frac{1}{v_{the}}\frac{\partial \overline{v_0}}{\partial t} + \frac{k_0 v_{the}}{\omega_0}\frac{1}{N_0}\frac{\partial \overline{n_0}}{\partial t} - \frac{ie}{mv_{the}}\frac{1}{\omega_0}\frac{\partial \overline{E_0}}{\partial t}$$

$$= \frac{1}{v_{the}}\left(-\frac{e}{m}E_0 - \frac{v_{the}^2 \partial n_0}{n_0\ \partial x}\right) - \frac{k_0 v_{the}}{\omega_0}\left(ik_0\overline{v_0} + ik_0\frac{\overline{Nv}}{N_0}\right)$$

$$-\frac{i\omega_p^2}{v_{the}}\frac{1}{\omega_0}\overline{v_0} - \frac{i\omega_p^2}{v_{the}\omega_0}\frac{\overline{Nv}}{N_0}$$

or

$$\frac{\partial a_0}{\partial t} = -\frac{a_0}{2}\left[\frac{i\omega_p^2}{\omega_0} + \frac{ik_0^2 v_{the}^2}{\omega_0} + \frac{ik_0^2 v_{the}^2}{\omega_0} + \frac{i\omega_p^2}{\omega_0}\right] - \frac{i}{\omega_0}\left(k_0^2 v_{the}^2 + \omega_p^2\right)\frac{\overline{Nv}}{N_0 v_{the}} ,$$

which can be written as

$$\frac{\partial a_0}{\partial t} = -i\omega_0 a_0 - i\omega_0\frac{\overline{Nv}}{N_0 v_{the}} \tag{2.11}$$

Again we obtain,

$$Nv = [\overline{N}\, e^{-i\omega_2 t} + \overline{N^*}\ e^{i\omega_2 t}][\overline{v}\, e^{-i\omega_1 t} + \overline{v^*}\, e^{i\omega_1 t}]$$

$$= \overline{N}\ \overline{v}\ e^{-i(\omega_1+\omega_2)t} + \overline{N}\ \overline{v^*}\ e^{i(\omega_1-\omega_2)t} + \overline{N^*}\ \overline{v}\ e^{-i(\omega_1-\omega_2)t}$$

$$+ \overline{N^*}\ \overline{v^*}\ e^{i(\omega_1+\omega_2)t}$$

Averaging out under the assumption $\omega_0=\omega_1+\omega_2$, gives

$$Nv = \overline{N}\ \overline{v}\ e^{-i\omega_0 t}$$

$$\overline{Nv} = \overline{N}\ \overline{v}\ .$$

with which (2.11) becomes

$$\frac{\partial a_0}{\partial t} = -i\omega_0 a_0 - i\omega_0 \frac{\overline{N}}{N_0}\frac{\overline{v}}{v_{the}}$$

$$= -i\omega_0 a_0 - \frac{i\omega_0}{4} a_2\, a_1 .$$

Introducing the growth term the above can be written as

$$\frac{\partial a_0}{\partial t} = \gamma\, a_0 - i\omega_0 a_0 - i\frac{\omega_0}{4} a_2\, a_1 ,$$

where the first term on the right hand side is the growth term.

Now, we derive the equation for the ion-acoustic mode. This is again a longitudinal mode.

In this case the nonlinearity comes through the beating of the Langmuir mode, but the ions itself are linear. Therefore the equations for the linear ions are

$$\frac{\partial N}{\partial t} = - N_0 \frac{\partial V}{\partial t} ,$$

$$\frac{\partial V}{\partial t} = - \frac{e}{M} ik^2\, \phi - \nu_i V . \tag{2.12}$$

The contribution of the nonlinear electrons comes through

$$\frac{\partial}{\partial x}(v_0.v) = \frac{e}{m}\frac{\partial \phi}{\partial x} - \frac{v_{the}^2}{N_0}\frac{\partial n}{\partial x}$$

$$\Rightarrow v_0.v = \frac{e}{m}\phi - \frac{v_{the}^2}{N_0} n$$

$$\Rightarrow \frac{e\phi}{M} = c_s^2 \frac{n}{N_0} + \frac{m}{M}(v_0.v) . \tag{2.13}$$

Using (2.13), we obtain for (2.12) the equations

$$\frac{\partial N}{\partial t} = - N_0 \frac{\partial V}{\partial x} ,$$

$$\frac{\partial V}{\partial t} = - \frac{c_s^2}{N_0} \frac{\partial N}{\partial x} - \frac{m}{M} \frac{\partial}{\partial x} (v_0 . v) - \nu_i V .$$

Let us define the ion acoustical normal mode as

$$a_2 = \frac{\overline{N}}{N_0} + \frac{\overline{V}}{c_s} , \tag{2.14}$$

where

$$N = \overline{N} e^{-i\omega_2 t} + \overline{N^*} e^{i\omega_2 t} , \quad V = \overline{V} e^{-i\omega_2 t} + \overline{V^*} e^{i\omega_2 t} .$$

Now, we write

$$\frac{\overline{N}}{N_0} = \frac{\overline{V}}{c_s} = \frac{1}{2} a_2 ,$$

$$\frac{\omega_2}{k_2} = c_s . \tag{2.15}$$

Differentiating (2.14), we have

$$\frac{\partial a_2}{\partial t} = \frac{1}{N_0} \frac{\partial \overline{N}}{\partial t} + \frac{1}{c_s} \frac{\partial \overline{V}}{\partial t}$$

$$= - \frac{\partial \overline{V}}{\partial x} - \frac{c_s}{N_0} \frac{\partial \overline{N}}{\partial x} - \frac{m}{M c_s} \frac{\partial}{\partial x} v_0 . v - \frac{\nu_i}{c_s} \overline{V}$$

$$= -ik_2 \overline{V} - ik_2 c_s \frac{\overline{N}}{N_0} - \frac{\nu_i}{c_s} \overline{V} - \frac{ik_2 m}{M c_s} \overline{v_0 . v}$$

$$= -(i\omega_2 - \nu_i) \frac{\overline{V}}{c_s} - i\omega_2 \frac{\overline{N}}{N_0} - \frac{i\omega_2}{v_{the}^2} \overline{v_0 . v}$$

or

$$\frac{\partial a_2}{\partial t} = -i\omega_2 a_2 - \frac{\nu_i}{2} a_2 - \frac{i\omega_2}{v_{the}^2} \overline{v_0 . v} .$$

Using the averaging principle we get

$$v_o.v = (\overline{v_0}\ e^{-i\omega_0 t} + \overline{v_o^*}\ e^{i\omega_0 t})(\overline{v}\ e^{-i\omega_1 t} + \overline{v^*}\ e^{i\omega_1 t})$$

$$= \overline{v_0}\ \overline{v}\ e^{-i(\omega_0+\omega_1)t} + \overline{v_0}\ \overline{v^*}\ e^{-i(\omega_0-\omega_1)t} + \overline{v_o^*}\ \overline{v}\ e^{i(\omega_0-\omega_1)t})$$

$$+ \overline{v_o^*}\ \overline{v^*}\ e^{i(\omega_0+\omega_1)t}.$$

Again, averaging out, using the relation $\omega_0=\omega_1+\omega_2$,we have

$$\overline{v_0.v} = \overline{v_0}\ .\ \overline{v^*}\ .$$

Therefore,

$$\frac{\partial a_2}{\partial t} = -i\omega_2 a_2 - \frac{\nu_i}{2} a_2 - \frac{i\omega_2}{v_{the}^2}\ \overline{v_0}\ \overline{v^*}\ ,$$

and since $\frac{v_0}{v_{the}} = \frac{a_0}{2}$, with (2.9) we obtain

$$\frac{\partial a_2}{\partial t} = -\nu_2 a_2 - i\omega_2 a_2 - \frac{i\omega_2}{4} a_0\, a_1^*\ . \tag{2.16}$$

Hence, we have the following three coupled nonlinear ordinary differential equations representing the three coupled harmonic oscillators :

$$\frac{\partial a_0}{\partial t} = \gamma a_0 - i\omega_0 a_0 - \frac{i\omega_0}{4} a_1\, a_2\ ,$$

$$\frac{\partial a_1}{\partial t} = -\nu_1\, a_1 - i\omega_1 a_1 - \frac{i\omega_1}{4} a_0\, a_2^*\ , \tag{2.17}$$

$$\frac{\partial a_2}{\partial t} = -\nu_2\, a_2 - i\omega_2 a_2 - \frac{i\omega_2}{4} a_0\, a_1^*\ .$$

Further, let us define

$$a_s = A_s(t)\ e^{-i\omega_s t}, \qquad s=0,1,2\ . \tag{2.18}$$

Substituting (2.18) into (2.17) gives

$$\frac{\partial A_0}{\partial t} = \gamma A_0 - \frac{i\omega_0}{4} A_1 A_2 e^{i\delta t} , \qquad (2.19)$$

$$\frac{\partial A_1}{\partial t} = - \nu_1 A_1 - \frac{i\omega_1}{4} A_0 A_2^* e^{-i\delta t} , \qquad (2.20)$$

$$\frac{\partial A_2}{\partial t} = - \nu_2 A_2 - \frac{i\omega_2}{4} A_0 A_1^* e^{-i\delta t} , \qquad (2.21)$$

where $\delta = \omega_0 - \omega_1 - \omega_2$.

Substituting

$$A_s = u_s e^{i\phi_s} \quad , s = 0,1,2 \qquad (2.22)$$

into (2.19),(2.20) and (2.21), and separating the real and imaginary part of the equations, gives

$$\begin{aligned} \frac{\partial u_0}{\partial t} + iu_0\frac{\partial \phi_0}{\partial t} &= \gamma u_0 - \frac{i\omega_0}{4} u_1 u_2 e^{i(\phi_1+\phi_2+\delta t-\phi_0)} , \\ \frac{\partial u_1}{\partial t} + iu_1\frac{\partial \phi_1}{\partial t} &= - \nu_1 u_1 - \frac{i\omega_1}{4} u_0 u_2 e^{i(\phi_0-\phi_2-\delta t-\phi_1)} , \qquad (2.23) \\ \frac{\partial u_2}{\partial t} + iu_2\frac{\partial \phi_2}{\partial t} &= - \nu_2 u_2 - \frac{i\omega_2}{4} u_0 u_1 e^{i(\phi_0-\phi_1-\delta t-\phi_2)}. \end{aligned}$$

We define $\phi = \phi_0 - \phi_1 - \phi_2 - \delta t$ and separate the real and imaginary parts of eqs.(2.23). Then the real parts are

$$\begin{aligned} \frac{\partial u_0}{\partial t} &= \gamma u_0 - \frac{\omega_0}{4} u_1 u_2 \sin\phi , \\ \frac{\partial u_1}{\partial t} &= - \nu_1 u_1 + \frac{\omega_1}{4} u_0 u_2 \sin\phi , \qquad (2.24) \\ \frac{\partial u_2}{\partial t} &= - \nu_2 u_2 + \frac{\omega_2}{4} u_0 u_1 \sin\phi , \end{aligned}$$

and the imaginary parts are

$$\frac{\partial\phi_0}{\partial t} = -\frac{\omega_0}{4}\frac{u_1u_2}{u_0}\cos\phi ,$$

$$\frac{\partial\phi_1}{\partial t} = -\frac{\omega_1}{4}\frac{u_0u_2}{u_1}\cos\phi , \qquad (2.25)$$

$$\frac{\partial\phi_2}{\partial t} = -\frac{\omega_2}{4}\frac{u_0u_1}{u_2}\cos\phi .$$

Differentiating ϕ we have

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi_0}{\partial t} - \frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_2}{\partial t} - \delta$$

$$= -\delta - \backslash\phi(1,4)\ (\omega_0\frac{u_1u_2}{u_0} - \omega_1\frac{u_0u_2}{u_1} - \omega_2\frac{u_0u_1}{u_2})\cos\phi . \qquad (2.26)$$

Consider the set of four coupled ordinary differential equations given by (2.24) and (2.26). We normalize by $\frac{4}{\omega_0}$ and change $\phi \rightarrow \phi - \frac{\pi}{2}$.

Then we have for (2.25) and (2.26)

$$\frac{\partial u_0}{\partial\tau} = \gamma u_0 + u_1u_2\cos\phi ,$$

$$\frac{\partial u_1}{\partial\tau} = -\nu_1u_1 - \frac{\omega_1}{\omega_0}u_0u_2\cos\phi , \qquad (2.27)$$

$$\frac{\partial u_2}{\partial\tau} = -\nu_2u_2 - \frac{\omega_2}{\omega_0}u_0u_1\cos\phi ,$$

$$\frac{\partial\phi}{\partial\tau} = -\delta - (\frac{u_1u_2}{u_0} - \frac{\omega_1}{\omega_0}\frac{u_0u_2}{u_1} - \frac{\omega_2}{\omega_0}\frac{u_0u_1}{u_2})\sin\phi .$$

We define

$$U_0 = \frac{\sqrt{\omega_1\omega_2}}{\omega_0}u_0 , \quad U_1 = \sqrt{\frac{\omega_2}{\omega_0}}u_1 , \quad U_2 = \sqrt{\frac{\omega_1}{\omega_0}}u_2 ,$$

which transforms (2.27) to

$$\frac{\partial U_0}{\partial \tau} = \gamma U_0 + U_1 U_2 \cos\phi ,$$

$$\frac{\partial U_1}{\partial \tau} = - \nu_1 U_1 - U_0 U_2 \cos\phi ,$$

$$\frac{\partial U_2}{\partial \tau} = - \nu_2 U_2 - U_0 U_1 \cos\phi , \tag{2.28}$$

$$\frac{\partial \phi}{\partial \tau} = - \delta - (\frac{U_1 U_2}{U_0} - \frac{U_0 U_2}{U_1} - \frac{U_0 U_1}{U_2}) \sin\phi .$$

Further, assuming that $\nu=\nu_1=\nu_2$ and $U_1=U_2=U$, which is a special case of (2.28), we get

$$\frac{\partial U_0}{\partial \tau} = \gamma U_0 + U^2 \cos\phi ,$$

$$\frac{\partial U}{\partial \tau} = - \nu U - U_0 U \cos\phi , \tag{2.29}$$

$$\frac{\partial \phi}{\partial \tau} = - \delta - (\frac{U^2}{U_0} - 2U_0)\sin\phi.$$

Finally, the substitution

$$X=U_0\cos\phi , \quad Y=U_0\sin\phi , \quad Z=U^2 , \tag{2.30}$$

transforms (2.29) into

$$\frac{\partial X}{\partial \tau} = \frac{\partial U_0}{\partial \tau}\cos\phi - U_0 \sin\phi \frac{\partial \phi}{\partial \tau}$$

$$= \gamma U_0 \cos\phi + U^2\cos^2\phi + \delta U_0 \sin\phi + U^0(\frac{U^2}{U_0} - 2U_0)\sin^2\phi ,$$

$$\frac{\partial Y}{\partial \tau} = \frac{\partial U_0}{\partial \tau}\sin\phi + U \cos\phi \frac{\partial \phi}{\partial \tau}$$

$$= \gamma U_0 \sin\phi + U^2 \sin\phi\cos\phi - \delta U_0 \cos\phi - U_0(\frac{U^2}{U_0} - 2U_0)\sin\phi\cos\phi ,$$

$$\frac{\partial Z}{\partial \tau} = 2U\frac{\partial U}{\partial \tau}$$

$$= - 2\nu U^2 - 2U_0U^2\cos\phi ,$$

or

$$\frac{\partial X}{\partial \tau} = \gamma X + Z + \delta Y - 2Y^2 ,$$

$$\frac{\partial Y}{\partial \tau} = \gamma Y - \delta X + 2XY , \tag{2.31}$$

$$\frac{\partial Z}{\partial \tau} = -2Z(\nu + X) .$$

Note the substitution

$$X=\gamma\xi , \qquad Y=\gamma\eta , \qquad Z=\gamma^2\zeta , \qquad \tau=\tau_1\gamma \tag{2.32}$$

scales the system (2.31) to

$$\frac{d\xi}{\delta\tau_1} = \xi + \zeta + \frac{\delta}{\gamma}\eta - 2\eta^2 ,$$

$$\frac{d\eta}{d\tau_1} = \eta - \frac{\delta}{\gamma}\xi + 2\xi\eta , \tag{2.33}$$

$$\frac{d\zeta}{d\tau_1} = - 2\zeta(\frac{\nu_1}{\gamma} + \xi) ,$$

which is given in the paper [1] by Wersinger ,Finn and Ott (according to their notation ξ=X , η=Y , ζ=Z , $\frac{\delta}{\gamma} = \Delta$, $\frac{\nu_1}{\gamma} = \Gamma$). They investigated the system (2.34) numerically .In the thesis we study the equivalent system (2.32) and besides results similar to [1] we show numerically the presence of a strange attractor.

CHAPTER 3

Equilibria of the three-wave interaction model and its stability propertities

In this chapter we study the equlibria of the three wave interaction model as given by the three coupled nonlinear ordinary differential equations (2.32). We show by qualitative reasoning the existence of chaotic solutions.

3.1 Nature and stability of the equlibria of the system

Consider the system of ordinary differential equations given by (2.32). We find the equilibrium points of the system (2.32) by equating each of the equations to zero. The equilibrium points are

$$E_1(0\,,0\,,0\,) \tag{3.1}$$

and

$$E_2\left(-\nu\,,\frac{\nu\delta}{2\nu-\gamma}\,,\gamma\nu\left[\,1+\frac{\delta^2}{(2\nu-\gamma)^2}\,\right]\right), \tag{3.2}$$

where it is assumed that $2\nu - \gamma > 0$. We investigate the nature and stability of each of the equilibria by computing the Jacobian matrix J of (2.32) at an equlibrium E (x_0 , y_0 , z_0) :

$$J^0\,(x_0,y_0,z_0) = \begin{bmatrix} \gamma & -\delta+2y_0 & -2z_o \\ \delta-4y_0 & \gamma+2x_0 & 0 \\ 1 & 0 & -2(\nu+x_0) \end{bmatrix}. \tag{3.3}$$

The determinant of J^0 is

$$\det\ J^0(x_0,y_0,z_0) = -2\gamma(\gamma+2x_0)(\nu+x_0) + 2(\delta-4y_0)(\nu+x_0)(2y_0-\delta)+2z_0(\gamma+2x_0) \tag{3.4}$$

It is known (Hirsh and Smale [20]) that if $\det J^0(x_0,y_0,z_0) \neq 0$ then $E(x_0,y_0,z_0)$ is a simple equilibrium and there is no other equilibrium state in its neighbourhood.

For the equilibrium point $E_1(0\,,0\,,0)$, the value of the determinant (3.4) is

$$\det J^0(0\,,0\,,0) = -\,2\nu(\gamma^2+\delta^2) \neq 0$$

which shows that the equlibrium is a simple one. For the other equilibrium point E_2(see (3.2)), the value of the determinant is

$$\det J^0(E_2) = 2\gamma(\gamma - 2\nu)\left(1 + \frac{\delta^2}{(2\nu - \gamma)^2}\right) \neq 0 ,$$

and, hence, this is also a simple equilibrium point.

The characteristic equation for the Jacobian matrix (3.3) is

$$- \det |J^0 - \lambda I| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 , \tag{3.5}$$

where I is the unit matrix,

$$\begin{aligned} a_1 &= 2\gamma - 2\nu , \\ a_2 &= (\delta - 4y_0)(2y_0 - \delta) + \gamma(- \gamma+4\nu+2x_0) + 4x_0(\nu+x_0) - 2z_0 , \\ a_3 &= 2(\nu+x_0)(\delta - 4y_0)(2y_0 - \delta) - 2\gamma(\nu+x_0)(\gamma+2x_0) + 2z_0(\gamma+2x_0). \end{aligned} \tag{3.6}$$

It is well known that the stability property of an equilibrium point $E(x_0,y_0,z_0)$ is determined by the signs of the real parts of the eigenvalues of the Jacobian matrix (3.3), i.e., the roots of (3.5). From (3.4) and the third eq.(3.6) it follows that $a_3 \neq 0$ for both equilibrium points, and hence the characteristic eq.(3.5) has no zero root. The roots of (3.5) can be distinct or repeated. An equilibrium of a system of autonomous ordinary differential equations is called hyperbolic if its characteristic equation has no roots with zero real parts,i.e.,no zero roots and purely imaginary roots; otherwise, the equilibrium is nonhyperbolic (see for example Hirsh and Smale [20]). A hyperbolic equilibrium is structurally stable.Here it is possible for both cases to exist, a hyperbolic equilibrium (if (3.5) has only nonzero real roots or a nonzero real root and a pair of complex roots with nonzero real part) or a nonhyperbolic equilibrium (if (3.5) has a pair of purely imaginary roots). Reyn [21] presented a detailed classification of the nature and stability of the equilibrium points of a three dimensional linear differential system. For a nonlinear system in R^3 see Bojadziev and Sattar [22].

Let us study the nature of the equilibrium $E_1(0 , 0 , 0)$. This equilibrium implies (using (2.30)) that

$$U^2 = 0 , \qquad U_0\cos\phi = 0 , \qquad U_0\sin\phi = 0 . \tag{3.7}$$

The first eq.(3.7) shows that there are no decay waves and the second and the third equation (3.7) imply that there is an absence of the pump wave. This is a typical physical situation in which the absence of the pump wave implies that the daughter waves do not exist.In our case this means that in the absence of the growing Langmuir wave (the pump wave), the damped Langmuir wave and the damped ion-acoustic wave do not exist. Therefore, it is natural that $E_1(0, 0, 0)$ is a fixed point. The eigenvalues of the Jacobian matrix (3.3) i.e.,the roots of (3.5) corresponding to E_1 are

$$\lambda_1 = -2\nu , \qquad \lambda_2 = \gamma + i\delta \qquad \text{and} \qquad \lambda_3 = \gamma - i\delta \tag{3.8}$$

This equilibrium point is a saddle-focus and hence, unstable. Let us interpret this physically.

We can find the eigenfunctions corresponding to the above eigenvalues by solving the system of equations

$$\begin{aligned} (\lambda - \gamma)p &= r + \delta q , \\ (\lambda - \gamma)q &= -\delta p , \\ (\lambda + 2\nu)r &= 0 . \end{aligned} \tag{3.9}$$

Let $r = r_0$ implying that $r \neq 0$ or, $\lambda = -2\nu$, which conforms with our result (3.8). Then solving the first equations (3.9) gives the values of p and q as,

$$p = (\lambda - \gamma)\{ \delta^2 + (\lambda - \gamma)^2\} r_0 ,$$

and

$$q = -\frac{\delta}{\delta^2 + (\lambda - \gamma)^2} r_0 .$$

The above analysis shows that U_0, or the pump Langmuir wave grows linearly and hence, the equilibrium point is unstable.

With regard to the second equilibrium point, E_2 using (2.30) and (3.2), we have

$$X = -\nu = U_0\cos\phi_0 ,$$

$$Y = \frac{\nu}{2\nu - \gamma}\delta = U_0\sin\phi_0 . \qquad (3.10)$$

From equations (3.10) we get the relations

$$U_0 = -\frac{\nu}{\cos\phi_0} ,$$

$$\tan\phi_0 = -\frac{\delta}{2\nu - \gamma} . \qquad (3.11)$$

Therefore, we have

$$\sin\phi_0 = \frac{\delta}{\sqrt{\delta^2 + (2\nu - \gamma)^2}} ,$$

and

$$\cos\phi_0 = -\frac{(2\nu - \gamma)}{\sqrt{\delta^2 + (2\nu - \gamma)^2}} . \qquad (3.12)$$

Using (2.30), (3.11) and (3.12)

we get

$$U^2 = \frac{\nu\gamma}{\cos 2\phi_0}$$

or,

$$U = -\frac{\sqrt{\nu\gamma}}{\cos\phi_0}$$

which implies that at the equilibrium E_2, $U > 0$ and hence, saturation of the instability is possible.

Following closely [1] we get equivalent results which are here expressed in our notations. The divergence of the flow of the system (2.32) in the phase space is calculated as

$$\frac{\partial}{\partial X}\left(\frac{\partial X}{\partial \tau}\right) + \frac{\partial}{\partial Y}\left(\frac{\partial Y}{\partial \tau}\right) + \frac{\partial}{\partial Z}\left(\frac{\partial Z}{\partial \tau}\right) = 2(\gamma - \nu)$$

From the above the time evolution of the phase-space volume turns out to be

$$V(\tau) = V(0)\, e^{2(\gamma - \nu)\tau}.$$

For $\gamma > \nu$ the volume expands and for the other limit it contracts.The former case implies that when the growth rate of the high-frequency wave is high and the decay rates of the low-frequency waves are too low the phase space trajectory is unbounded and saturation fails. Interest lies in the study of the latter case for which there is a possibility of saturation.

Also, if we add the second eq.(2.32) multiplied by Z and the third by Y, we get

$$\frac{d}{d\tau}(YZ) = -(2\nu - \gamma)YZ - \delta XZ$$

For $\delta=0$, we obtain

$$\frac{d}{d\tau}(YZ) = -(2\nu - \gamma)YZ. \tag{3.13}$$

The above result shows that the trajectories approach the planes Y=0 or Z=0 as $\tau \to \infty$ if $\nu > \gamma/2$.

The stability of E_2 can be determined by finding the roots of the characteristic equation (3.5) corresponding to E_2. The coefficients a_1, a_2, a_3 are

$$a_1 = 2(\nu - \gamma),$$

$$a_2 = \frac{\gamma\nu}{(2\nu - \gamma)^2}\left(\frac{\gamma}{\nu}(2\nu - \gamma)^2 + 4\delta^2\nu + 2\nu\gamma\delta^2\right), \tag{3.14}$$

$$a_3 = 2\gamma\nu(2\nu - \gamma) + \frac{2\gamma\nu\delta^2}{2\nu - \gamma}$$

Since we are only interested in the case $\nu > \gamma$ (contraction of volume in phase space), always $a_1,a_2,a_3 > 0$. For $a_3 > a_1a_2$, there are two roots with positive real part which implies that E_2 is unstable. For $a_3 < a_1a_2$ all roots of (3.5) have negative real parts, hence the equilibrium E_2 is asymptotically stable general node.For more detailed study concerning the stability (instability) nature of E_2 one can use the paper [22] by Bojadziev and Sattar .

At the critical case $a_3=a_1a_2$, which according to (3.14) can be written as

$$\frac{2\gamma\nu(\nu - \gamma)}{(2\nu - \gamma)^2}\{\frac{\gamma}{\nu}(2\nu - \gamma)^2 + \delta^2(4\nu+2\nu\gamma)\} = 2\gamma\nu(2\nu - \gamma) + \frac{2\gamma\nu\delta^2}{2\nu - \gamma}.$$

or,

$$\delta^2 = \delta_0^2 \equiv 4 (\nu - \frac{\gamma}{2})^2[4(\nu - \frac{\gamma}{2})^2 + 1][4(\nu - \frac{\gamma}{2})^2 - 3]^{-1} , \tag{3.15}$$

the characteristic equation (3.5) has a pair of imaginary roots $\lambda_{1,2} = \pm i\beta$, where $\beta=\sqrt{a_2}$. The third root is $\lambda_3= - a_1 < 0$. Then the equilibrium E_2 is nonhyperbolic and stable, a divergent vortex focus (see [22]), but structurally unstable.It is of interest to investigate the roots of (3.5) near the critical case. They are of the type $\alpha \pm i\beta$ and with some algebraic calculations and using (3.15) we find that $\alpha > 0$ for $\delta < \delta_0$ and $\alpha < 0$ for $\delta > \delta_0$; of course $\alpha = 0$ for $\delta = \delta_0$. Therefore E_2 is asymptotically stable if $\delta > \delta_0$; we require $4(\nu - \frac{\gamma}{2})^2 > 3$ (see (3.15)) in order δ to be real. For $\delta<\delta_0$ the equilibrium E_2 is unstable. For $4(\nu - \frac{\gamma}{2})^2 > 3$, whenever δ decreases through δ_0, two complex conjugate roots λ_1 and λ_2 have a real part going through zero from negative to positive. The third root λ_3 is real and negative at that time. (This is characteristic of a Hopf bifurcation).

3.2 Phase Plane Analysis

Here we investigate the behavior of the trajectories in phase space of the dynamical system (2.32) assuming γ to be small. The phase space is devided into two regions "fast" and "slow". The regions of slow motions, which correspond to the amplitudes of the low-frequency modes staying near zero, are located near the line $Y=\frac{\delta}{2}$ and Z=0, and are defined by the inequalities

$$|Z| \leq \gamma |X|, \qquad |2Y^2 - \delta Y| \leq \gamma |X|.$$

3.2.1 Exact synchronism (δ=0)

In this case the equilibrium point at the origin is a saddle node corresponding to the eigenvalues given by (3.8) as $\lambda_1 = -2\nu$ and $\lambda_{2,3} = \gamma$; the equilibrium point is unstable.

We already saw in (3.13) that the phase space of the dynamical system contains two integral surfaces Y=0 and Z=0 which are approached closely by the trajectories as $\tau \to \infty$ but cannot be intersected.

The cones of slow motions are located at the intersection of the two planes . Consider the two planes separately. On the plane Z=0, the system (2.32) reduces to

$$\frac{\partial X}{\partial \tau} = \gamma X - 2Y^2,$$

$$\frac{\partial Y}{\partial \tau} = \gamma Y + 2XY \qquad (3.16)$$

Its equilibrium point is (0,0). A stability analysis shows that (0,0) is an unstable node. A phase portrait for the plane Z=0 is given in fig.3.1. The curve on which $\frac{\partial X}{\partial \tau}=0$ is given by the parabola $2Y^2=\gamma X$ and the curve on which $\frac{\partial Y}{\partial \tau}=0$ is given by Y=0 and $X=-\frac{\gamma}{2}$. Therefore we can say that on the plane Z=0 the region of slow motions is bounded by the parabola $2Y^2=\gamma X$ and at X>0 it is the isocline of the horizontal tangents. Outside this region the trajectories resemble the circles X^2+Y^2=const,on which the energy is conserved implying that the influence of the small growth rate is negligible. The motions on this plane is asymptotically stable with respect to Z in the region $X > -\nu$.

We will, further, investigate the qualitative features of the motion, such as the existence of periodic regimes and the onset of stochasticity with the help of point transformation (Rabinovich [18]).We will ascertain in which manner the points of the narrow vertical strip $0<Y_1<Y<Y_2<<1$ on the plane X=0 are mapped by the phase trajectories at X>0 into the points of the horizontal strip $0<Z_1<Z<Z_2<<1$ of the same plane, and then, how they are mapped for the trajectories at X<0 again into the points of the vertical strip. This mapping of the motion co-ordinates Z and Y at the end of the period on the values of Z and Y at the start of the period makes the study clear.

In fig.3.3, we show the mapping of the vertical plane into the horizontal one, and again into the vertical one, qualitatively. This illustrates the successive transformations of a certain line in the vertical strip for two different cases, depending on the closeness of this to the z-axis.

Let us consider the fig.(3.3b). Considering the system (2.32) we find that the larger the value of Z at which the phase trajectory crosses the vertical strip, the farther from the X-axis it drops on the plane Z=0. The trajectory that begins on the plane X=0 at sufficiently large Z,instead of falling in the region of slow motions, moves immediately on the circle $X^2+Y^2=R^2$, after dropping to the plane Z=0.Close to this circle there is another trajectory, which, after passing through the vertical strip at small Z, drops on the plane Z=0 in the region of slow motions and emerges from that region, intersecting the parabola, at $X^2 \neq R^2$. Thus, trajectories cross the vertical strip far from each other.After crossing the horizontal strip they are along-side each other.This behavior explains the appearance of "horseshoes" in the mapping picture.

A similar analysis applies to the X<0 trajectories starting from the plane Z=0 and in the vicinity of the plane Y=0.The parabola of the earlier case is replaced by the straight line $Z= - \gamma X$. The characteristics of the mapping of the horizontal strip into a vertical one stays the same, thus explaining the possibility of the appearance of a "double horseshoe".

In order to carry out further investigation, we use a model in which the relation between points having equal Z in the vertical strip with points having equal Y in the horizontal strip.We map the line Z into itself establishing the dependence $\overline{Z}$(Z) of the consecutive points $\overline{Z}$ on the initial points Z (fig.3.4).There are either a single stable single-period regime (fig.3.4a), or a set of regimes with modulation-multiperiod regimes at different positions of $\overline{Z}$(Z) relative to the straihgt line $\overline{Z}$=Z corresponding to the closed cycles on the ($\overline{Z}$,Z).

From Sharkovskii [23] we know that if in a system described by the mapping $\overline{Z}$(Z) there are periodic motion with an odd number of periods then regardless of its stability there exists in the system one more denumerable set of unstable multiperiodic motions and a finite or denumerable number of stable motions.From fig.3.4b we see that such odd (say three-period) motions exists in our robust model described by the mapping $\overline{Z}$(Z), and hence,a denumerable set of unstable periodic motions exists in this system.

For sufficiently small γ the existence of this set in a bounded region of phase space is expected in the initial system. If the stable motions did not exist in this bounded phase-space region,to which the neighbouring trajectories tend, then it would imply the onset of chaos(Kiyashko and Rabinovich [24])). For the case δ=0 these stable motions do exist and correspond to cycles supported on one of the vertices at the "bottom" of the function $\overline{Z}$(Z).

3.2.2 Influence of the detuning on the dynamics of the triplet

For finite values of δ the integral surface Y=0 is destroyed and the equilibrium point E_1(0 , 0 , 0) is transformed from a saddle node to a saddle focus (fig.3.5).At δ>0 the stable separatrix of this equilibrium state enters the half space $Y<\frac{\delta}{2}$, which becomes twisted around the point X= - ν, Z=γ in the case δ=0 as t ->∞. We assume δ to be small since the qualitative analysis is impossible due to the increased complicated behavior of the trajectories. We disregard the destruction of the integral surface Y=0 and therefore the finiteness of δ only hinders the exchange of energy between the waves or to a decrease of the maximum attainable values of Z at $X = -\nu$. This leads to the the division of the mapping under consideration into two classes of motion by a separatrix that enters into the equilibrium point $E_1(0, 0, 0)$.The trajectories that enter in the vicinity of the plane Y=0, outside the unwinding separatrix (fig.3.1) cross the plane X=0 and move farther in a direction of increasing X. The trajectories falling inside the separatrix turn towards the decreased values of X before they cross the plane X=0, after several revolutions in the region of X<0 and at larger Z. Fig.3.4c gives the "robust" mapping of a line into a line and the discontinuties near small Z explains the existence of two classes of motion.

Thus, in the presence of phase-mismatch ,the stability regime of the unstable high-frequency wave due to the decay into the low-frequency damped waves can be chaotic.

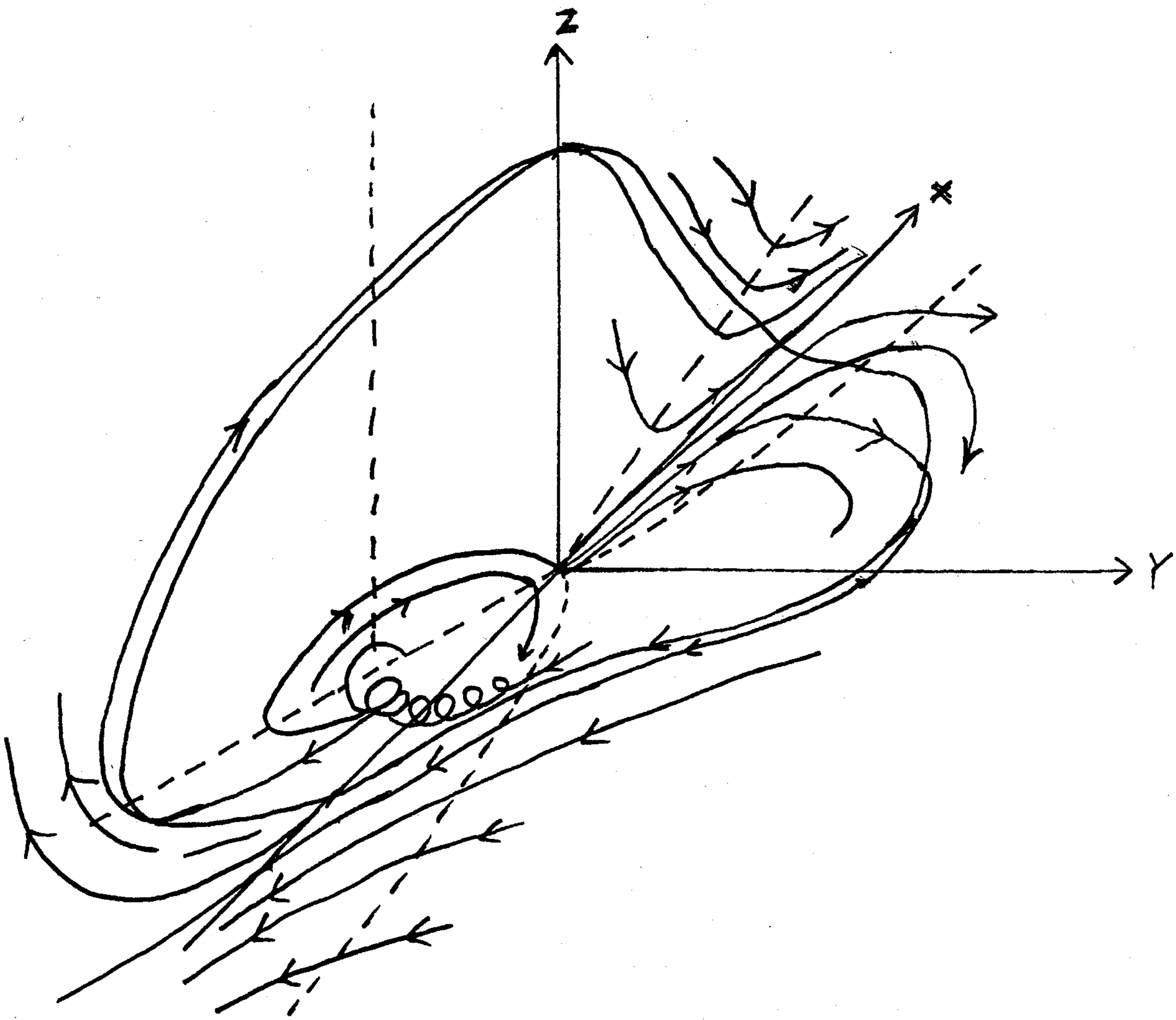


Fig. 3·1

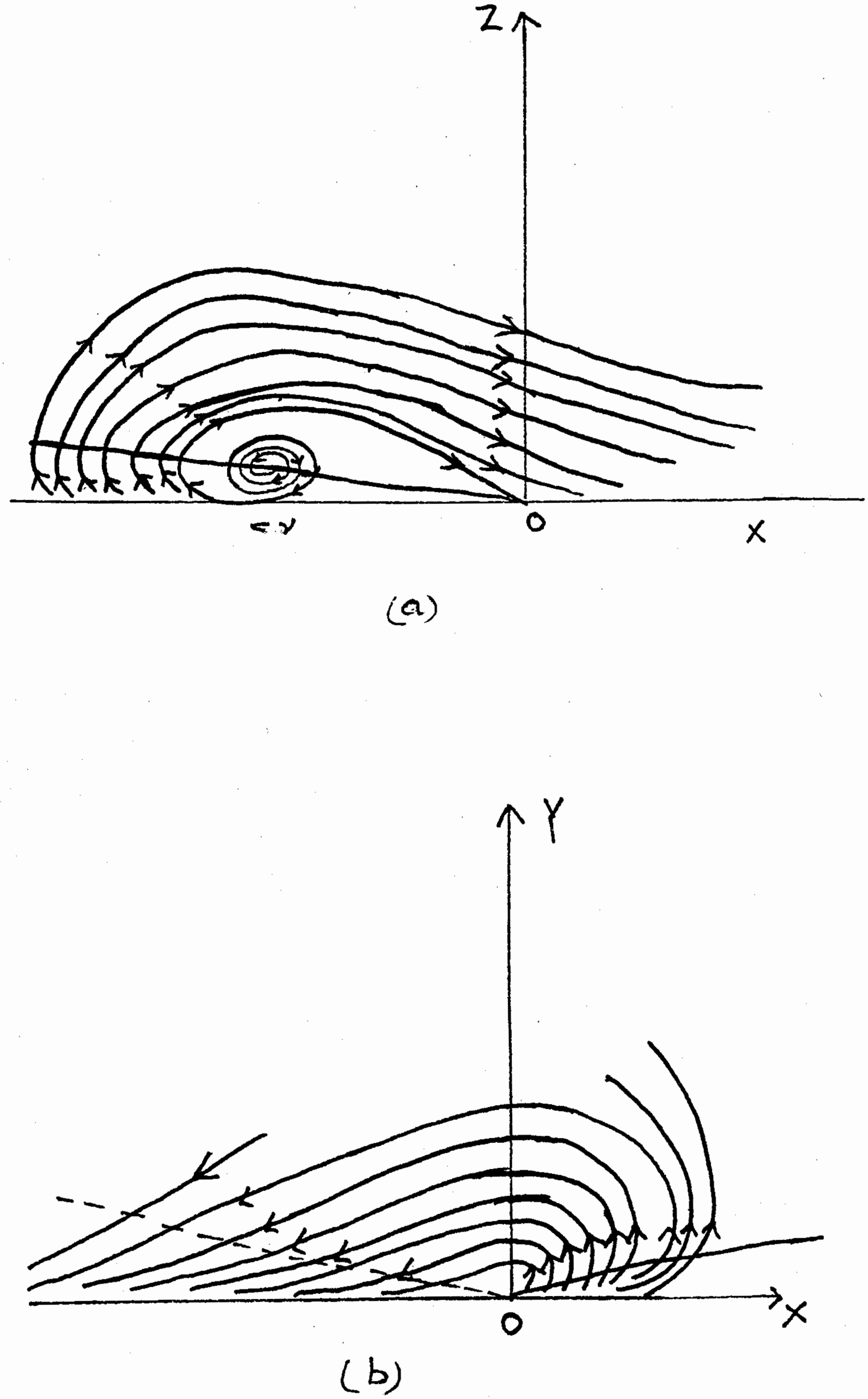


Fig 3.2

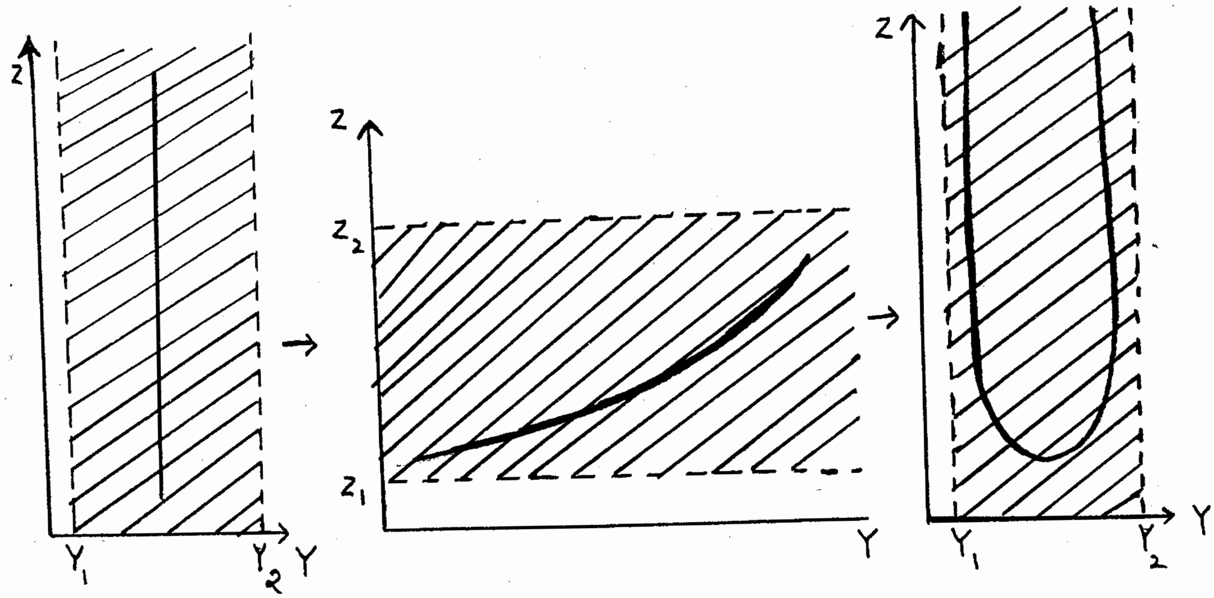


Fig 3·3a

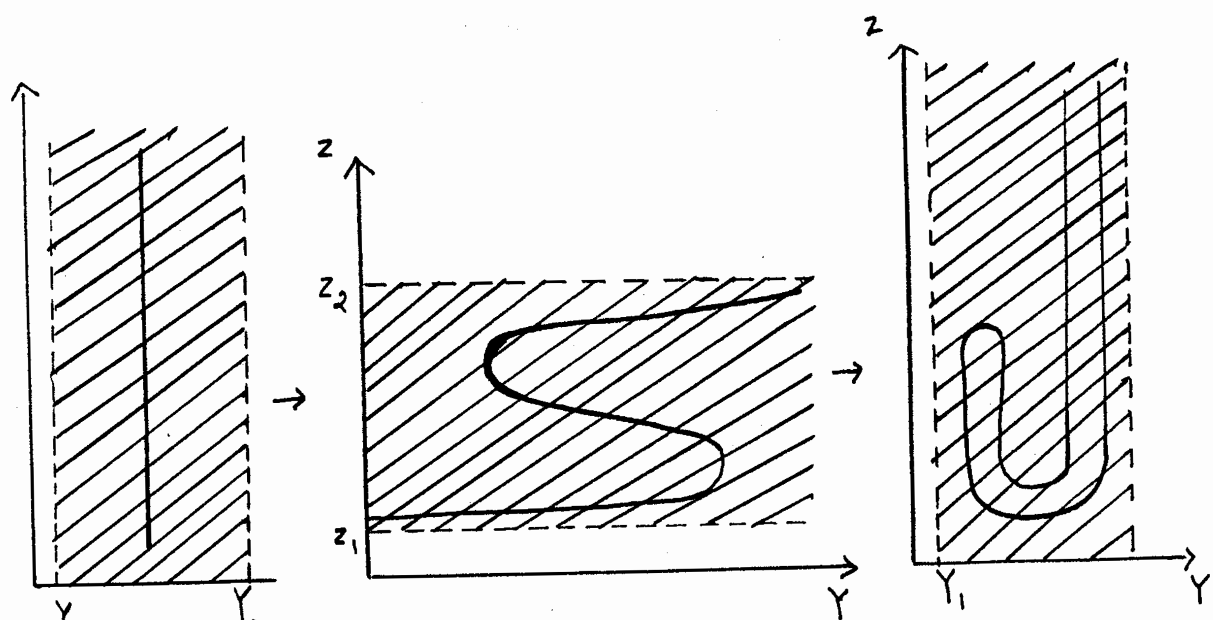


Fig 3·3b

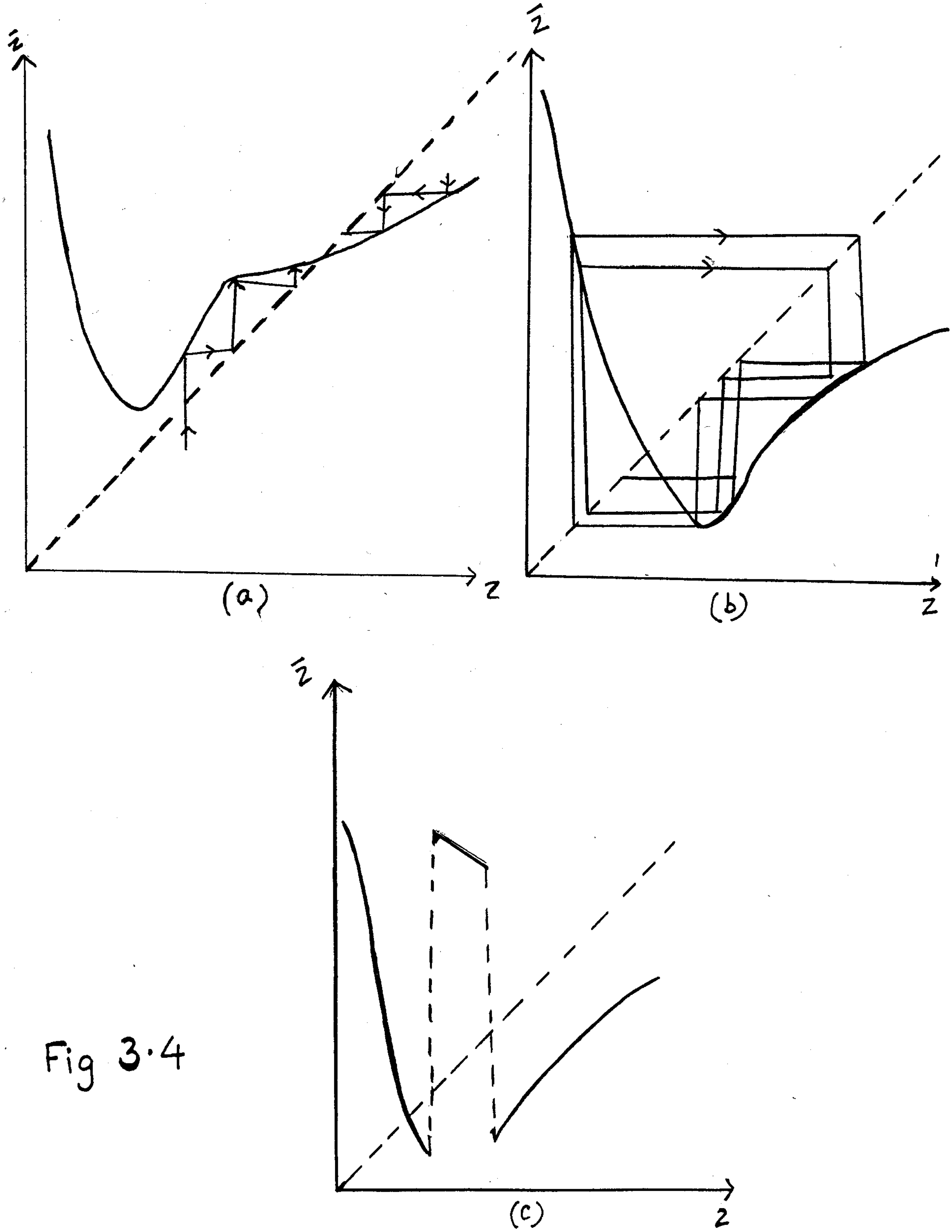


Fig 3·4

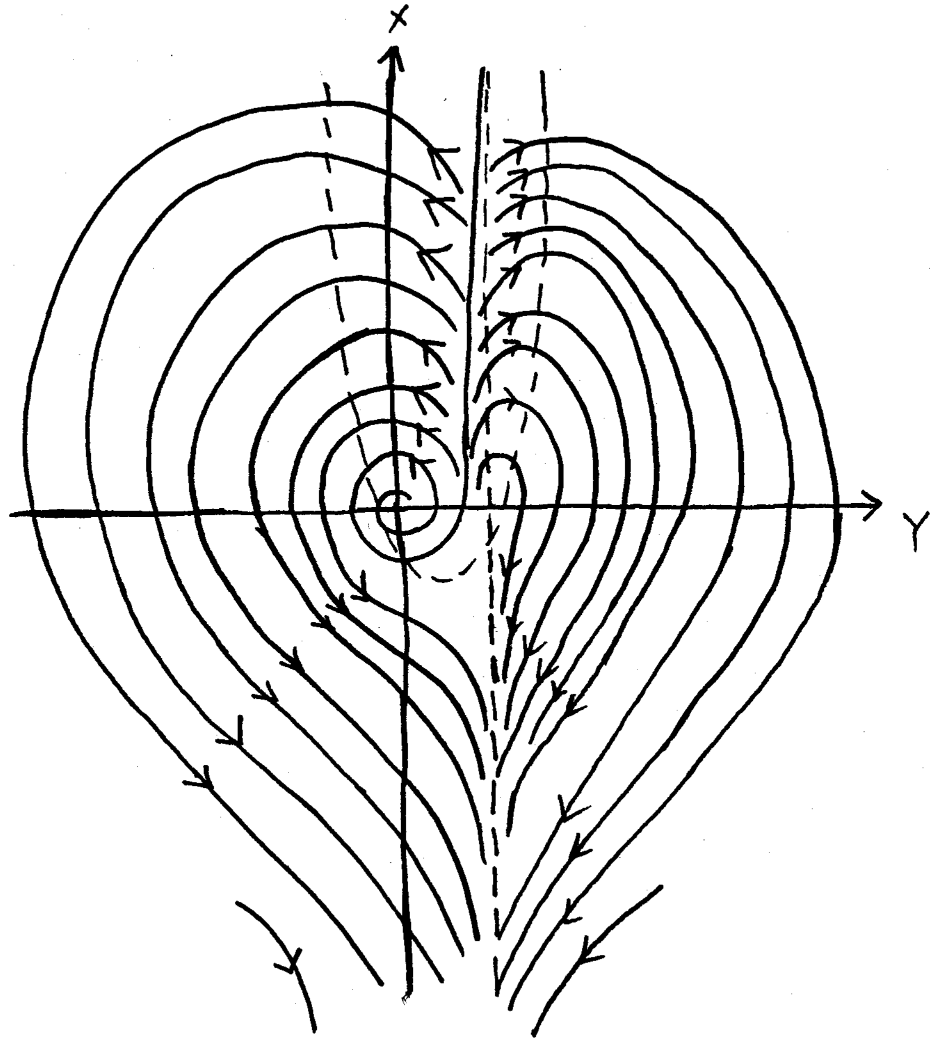


Fig 3.5

CHAPTER 4

Numerical solutions and analytical justification

In this chapter we present the results of the numerical investigations carried out on the system (2.29) and the system (2.31). In both cases we show the existence of periodic regimes,bifurcation points and the chaotic transitions.

Subsequently we present the analytical derivation of the one-dimensional mapping and show its similarity with the numerical solutions.

4.1 Numerical integrations

Now, we turn to the numerical integration of the system of eqs.(2.31) and also (2.29).Let us discuss the evolution of phase-space volumes as governed by the system(2.31). That is, we consider the volume(V) enclosed by some closed surface S in the X,Y,Z phase space, and let the surface evolve by having each point on the surface follow an orbit generated by (2.31).

We already saw in section (3.1) that the system (2.31) yields,

$$V(\tau) = V(0)\, e^{2(\gamma - \nu)\tau} \,.$$

For the case $\nu > \gamma$, the phase-space volume contracts exponentially in time. The special case of three ordinary autonomous differential equations with negative phase-space flow divergence presents a very clear case for the possible existence of a strange attractor. Since phase-space volume contracts to zero in the limit of large time, it follows that any attractor must have zero volume. A natural assumption might then be that the attractor would have to be a surface (two dimensional), a curve(one dimensional), or a point (zero dimensional). However, none of these allows chaotic motion. In particular, not even the highest dimension (two) of the above three possibilities allows chaos. For example, for orbits within a finite section of a plane, the Poincare-Bendixson theorem shows that the only possible attractor for the orbit must be either a point, a simple closed curve, or a self-intersecting closed curve, which looks like a character eight (see Hirsch and Smale [20]). Thus if one observes chaotic motion in the system(2.31), which has negative phase-space flow divergence as we already saw, then one is faced with something like a paradox. One way out is to realize that attractors with zero volume may have not only zero, one, or two, but can, in fact, have noninteger dimension. In particular, chaotic motion is possible if (2.31) has an attractor of dimension greater than two but less than three (so that the volume of the attractor is zero),i.e., a strange attractor.

In the numerical integration of our system (2.31) and also, (2.29) we observe the existence of periodic oscillations, limit cycles, bifurcations and chaotic solutions. We also demonstrate the existence of a strange attractor. We have not determined the dimension of the strange attractor, which will be the topic of further work.

We have already seen in the previous chapter that the system (2.31) has two fixed points. Features of interest have been numerically investigated for the second fixed point E_2 given by (3.2).This point remains asymptotically stable for $|\delta| > \delta_0$ where $\delta_0^2 = 4\,(\nu-\frac{\gamma}{2})^2\left[4(\nu-\frac{\gamma}{2})^2+1\right]\left[4(\nu-\frac{\gamma}{2})^2-3\right]^{-1}$ for all values of $\nu > \frac{1+\sqrt{3}}{2}$. E_2 loses stability by either subcritical or supercritical Hopf bifurcation [1], when the representative point of the system in the parameter space crosses the critical curve $\delta=\delta_0(\nu)$.

We follow the work of Meunier,Bussac and Laval [25],closely,to carry out the numerical investigation of (2.31).Hence, three regions in the (γ,δ) plane are distinguished which have been extensively studied by numerical methods for the parameter values $0<\delta<10, \frac{(1+\sqrt{3})}{2} < \nu < 50$. According to fig.4.1 we discuss the following regions :

4.1.1 - region 1 : the locally stable region

E_2 is locally stable, which implies that the linearly growing high-frequency wave saturates at a constant amplitude.

4.1.2 - region 2 : the adiabatic regime

In this region, the system loses stability by subcritical bifurcation. The variables $X(\tau)$,$Y(\tau)$ and $Z(\tau)$ oscillate with exponentially growing amplitudes and frequencies and hence, bounded solutions do not exist.The linear instability is not efficiently saturated by the decay process.

4.1.3 - region 3 : periodic and aperiodic regimes :

The equilibrium point E_2 loses stability by supercritical bifurcation. There is a complicated bifurcation sequence with alternating periodic (multilooped cycles) and non-periodic attractors (fig 4.2). These attracting sets do not achieve global stability and they have large attraction basins .

Depending on parameter values the transition between limit cycles and strange attractors follow two different schemes :

(i) In the first one, the cycles are generated by consecutive subharmonic bifurcations. Fiegenbaum [25], May [26] and Coullet and Tresser [27] showed similar bifurcation sequences in iterations of single-hump functions,e.g.,logistic equation.

(ii) In the second, a simple periodic attractor appears instead of an aperiodic one. Intermittency occurs at that transition in the case of period 2 and period 3 cycles. It is found that transitions with intermittency effectively occur at certain tangent bifurcations. It was shown by Tritton [29] that such a phenomenon occurs in fluid experiments, where turbulence appears as random bursts. The Lorentz system shows such a behavior. Our dynamical system is different from the Lorentz system. Namely, it does not have either symmetry or Liapounov functions ensuring the boundedness of solutions.

From the time series $X(\tau)$,$Y(\tau)$ and $Z(\tau)$, we generate a "surface of section" which is used as a diagnostic tool.The plane $X= \nu$ is chosen as the section surface, together with the condition $\frac{dX}{d\tau} < 0$.The section of the attractor by that plane consists of a finite set of points, which shows the presence of limit cycles or of arcs of curves(strange attractors,fig.4.4).Noting that Z takes its largest values in the plane $X=\nu$ and the reduced Poincare map f_z yields the recurrence scheme for Z maxima,i.e.,the maximal values of the low-frequency wave energy, we parametrize the position of a point by its Z-co-ordinate,since the curves exhibit no singularity.

The representative curve f_z, depends on the values of the parameters(fig 4.5). For low ν values, we obtain a smooth parabolic curve (fig 4.5a) for higher ν values , the curve presents a hump(fig.4.5b). The system displays the transition from a strange attractor to a double looped cycle in the latter case. We observe an intermittent behavior near the bifurcation point.

4.2 Intermittency

For $\delta=5$, the transition occurs at $\nu=21.685$(fig.4.3). Since the limit cycle has two loops, we consider the second-return map,i.e.,we plot Z_{n+2} vs Z_n [1].

The representative curve intersects the first bisectrix four times, giving rise to two pairs of fixed points for $\nu > 21.6$. Each pair corresponds to a stable fixed point (section of one loop of the cycle) and an unstable one. Each pair merges to a single point at $\nu_c = 21.685$. The curve is then tangent to the bisectrix in two points. For ν slightly smaller than 21.685, the curve does not cross the bisectrix but remains close to it (fig.4.6). The change in the behavior of the curve resembles the translation of parabolic curve, accompanied by a change of its second derivative.

Pomeau and Manneville [30] have studied these bifurcations naming such phenomenon as type 1 intermittency. We do the same for our study.

We have observed the expected behavior in the variation of successive values of Z with time. For ν slightly smaller than ν_c, laminar periods (no loss of correlations) are interrupted by random turbulent bursts(loss of correlations). There is an upper bound to the duration of laminar periods.

The upper bound depends on the distance to the bifurcation threshold. We have accordingly plotted the maximal duration as a function of $\varepsilon = \frac{\nu_c - \nu}{\nu_c}$. As predicted by Pomeau and Manneville [30], we have then obtained a $\varepsilon^{-0.5}$ law with a good accuracy (fig.4.7). Similar features are obtained at the intermittent onset of turbulence following a period 3 cycle.

4.3 Numerical investigation in the case $\delta=2$ of (2.29)

Following [1] , we choose $\delta=2$ in describing the properties of numerical solutions for a range of ν, $2 \leq \nu \leq 18$.

For $\nu < 3$ the evolution of the system is unbounded for almost all initial conditions.

For $3 \leq \nu \leq 8.5$ the instability saturates and a periodic oscillation is observed for wave amplitudes and phase ϕ (fig 4.8). In the phase space the asymptotic motion takes place on a simple limit cycle.

For $\nu \approx 8.5$ the simple period 1 limit cycle bifurcates to a period 2 limit cycle. The graphs of the wave amplitudes versus time become a periodic function with two alternating maxima (fig.4.9).

For $8.5 < \nu < 13.2$, as ν is increased the period 2 cycle splits in a period 4 cycle ($\nu = 12$), which splits in a period 8 cycle ($\nu = 13$) and so on till $\nu = 13.2$ where a non-periodic solution is observed.

For $13.2 < \nu < 16.85$, increasing ν furthermore, one observes chaotic regimes interrupted by stable periodic regime for some narrow windows in the parameter ν.

For $\nu = 16.85$ the widest such window is for the stable period 3 cycle. It sets in at $\nu = 16.85$ and then undergoes a sequence of bifurcations 3×2^n, creating stable period 6 cycle at $\nu = 17.4$ and so on till $\nu = 18.5$ where chaotic behavior again resumes.

The transition from a stable limit cycle ($\nu > 16.85$) to an aperiodic attractor $\nu < 16.85$ is an intermittent transition to turbulence. At $\nu \leq 16.85$ stable period 3 oscillations of wave amplitudes are randomly interrupted by random bursts(fig.4.10). Decreasing ν further, the duration of the laminar periods decreases more and more until the system reaches chaotic regime. If the intermittent signal is recorded long enough, it appears that the number of oscillations between two bursts varies as $\frac{1}{\sqrt{\nu_c - \nu}}$, where $\nu_c = 16.85$. This feature is characteristic of type 1 intermittency (Pomeau and Manneville [30]).

4.4 Correspondence with the quadratic map of the interval

For the δ value, δ=2, the behavior is the same as depicted by Feigenbaum [26] for the quadratic map, $P_r(x)$, defined as

$$x_{n+1} = r\, x_n\,(1 - x_n)\,, \quad 0 < r \leq 4\,, \qquad 0 \leq x \leq 1 \tag{4.1}$$

The map $P_r(x)$ shows an infinite sequence of period doubling bifurcations of stable periodic orbits at definite values of the parameter r,such as r_1 (period 1 to period 2), r_2(period 2 -> period 4),r_3---r_c as r is increased from r=0 to r=3.57. An aperiodic attractor appears at $r=r_c$. Beyond the critical bifurcation point r_c,upto which there is a cascade of period doublings, an inverse cascade of "noisy cycles" of periodicity $2^n \to 2^{n-1} \to \cdots 2 \to 1$.At r = 3.678 (Eckman [31]) the two sequences merge into one band.It is found that the parameter values r_n scale as $|r_c - r_n| \propto \alpha^{-n}$, where $\alpha = 4.699$ is a universal scaling factor for both period-doubling and band merging and depends on the quadratic nature of the maximum of the map. We have found that there exist periodic solutions for certain narrow windows in r. The widest such window of stability beginning at $r = r_T = 3.838$ and ends at r = 3.849, is for 3×2^n cycles. This corresponds to a tangent bifurcation, whereby a periodic orbit occurs ($r > r_T$) after a region of chaotic motion ($r < r_T$) and the mapping $x_{n+3} = f*f*f(x_n)$ has three stable and three unstable fixed points which merge into one as r is decreased. With further decrement in the value of the parameter r, intermittent turbulent behavior of random duration is observed, with the appearance of laminar phase in between of mean duration $\sim \dfrac{1}{\sqrt{r_T - r}}$.

If we consider the results from numerically generated section of the attractor on which the motion takes place for ν=15 (fig.4.11),the points generated in the surface of section appear to lie on an arc. The plot of $y_{n+1} = \sqrt{U_0^2 + U^2(t_{n+1})}$ vs. $y_n = \sqrt{U_0^2 + U^2(t_n)}$ where t_n is the nth time when the phase space trajectory pierces the surface of section, is given by the curve $y_{n+1} = f(y_n)$ in fig.4.12. The rounded maximum makes it resemble the simple quadratic map very closely and hence, yield a good qualitative model for behavior in ordinary differential equations.

4.5 Analytical investigation of nonlinear states

We give a qualitative analysis of the asymptotic solutions of the three wave system in the section.The method uses techniques originally due to Melnikov [32] and the results are similar to that of Bussac [33].In order to clarify the effects of dissipative terms,we first examine the solutions for the conservative system.

4.5.1 When there is no dissipation or mismatch in frequencies,eq.(2.29) reduces to

$$\frac{\partial U_0}{\partial \tau} = \gamma U_0 + U^2 \cos\phi$$

$$\frac{\partial U}{\partial \tau} = - U_0 U \cos\phi \qquad (4.2)$$

$$\frac{\partial \phi}{\partial \tau} = \left\{ 2U_0 - \frac{U^2}{U_0} \right\} \sin\phi$$

The system (4.2) is an example of a hamiltonian system and has two integrals of motion. Introducing the following variables: $M = \gamma U_0^2 + U^2$, the total wave energy ; $z = \frac{U^2}{M}$, the normalized energy of the low-frequency wave; and the "time" $u = \int_0^t \sqrt{M(t)}\, dt$, we try to obtain an analytical solution in terms of elliptic integrals.

The quantities M,z are positive and $0 \leq z < 1$. Then (4.2) writes,

$$\frac{dz}{du} = 2z\sqrt{1-z}\cos\phi = \frac{\partial H}{\partial \phi},$$

$$\frac{d\phi}{du} = (2\sqrt{1-z} - \frac{z}{\sqrt{1-z}})\sin\phi = \frac{\partial H}{\partial z}, \qquad (4.3)$$

$$\frac{d\sqrt{M}}{du} = 0,$$

where $h = 2z\sqrt{1-z}\sin\phi$ is the time - independent hamiltonian and (z,ϕ) evolve independently of the energy M(u) = constant which enters in the solutions $z(u),\phi(u)$ only as a time scale. Therefore, the phase space reduces to the plane (ϕ,z), $\sqrt{M}=\sqrt{M(u_0)}$ = constant. The trajectories in phase space for $\phi \in [0,2\pi]$ are plotted in fig.4.13 showing the periodicity in ϕ. Stable fixed points are located at $\phi=0,\pi,z=\frac{2}{3}$ and the hyperbolic fixed points are located at $z=0,\phi=\frac{\pi}{2},\frac{3\pi}{2}$. The separatrix, h=0, consists of z=0, z=1, $\phi=\frac{\pi}{2}$ $(0\leq z\leq 1)$, $\phi=\frac{3\pi}{2}$, the hamiltonian h is negative, $-\frac{4}{3\sqrt{3}}\leq h\leq 0$and the solutions $z(u),\phi(u)$ are periodic. The solution of (4.2) is (Whittaker and Watson [34]),

$$z = a - (a-b)\,\mathrm{sn}^2[K - \sqrt{a} - c(u-u_0)] \tag{4.4}$$

where $a\geq b\geq c$ are the roots of $(1-z)z^2 - \frac{h^2}{4}=0$; $K = K(\sqrt{(a-b)(a-c)})$ is the complete elliptic integral of first kind; $\mathrm{sn}(\theta)$ is the sine-amplitude Jacobian elliptic function, and the time origin $u=u_0$ has been taken at z=b. The solutions $z(u),\phi(u)$ are periodic in u, with a period $TU=\frac{2K}{\sqrt{a-c}}$ .We will refer to this solution as $z_c(u,h)$, $\theta_c(u,h)$ and have assumed that $\gamma<<1$.

4.5.2 Analytical investigation of the dissipative system

Introduction of dissipation and the frequency mismatch destroys the constancy of the energy M(u), and the hamiltonian h(u). The equations for $(z,\phi,\sqrt{M})$ are then given as

$$\frac{dz}{du} = -\frac{2z(1-z)(1+\nu)}{\sqrt{M}} + 2z(\sqrt{1-z}\,)\cos\phi \tag{4.5a}$$

$$\frac{d\phi}{du} = -\frac{\delta}{\sqrt{M}} + (\,2\sqrt{1-z}\; - \frac{z}{\sqrt{1-z}})\sin\phi \tag{4.5b}$$

$$\frac{d\sqrt{M}}{du} = 1 - (1+\nu)\, z \tag{4.5c}$$

where $u = \int_0^t \sqrt{M(t)}\, dt$ .When only dissipative terms are present , one recovers the linear instability,$z \to 0,\sqrt{M(u)} \to +\infty$, as $u\to\infty$. As long as the wave energy ($\sqrt{M}$) is small, the dissipative terms in eqs.(4.4a,4.4b) are in average larger than the nonlinear terms, making the average energy to grow. Consequently the dissipative terms in eqs.(4.4a,4.4b) decrease, till they become of same order or smaller than the nonlinear one. If instability does not saturate, or saturate at an energy level such as $\frac{\nu}{\sqrt{M}}, \frac{\delta}{\sqrt{M}} \sim \varepsilon << 1$, dissipative and frequency mismatch terms in eqs.(4.5a,b) remain small in the asymptotic regime. The equation for the hamiltonian is given as

$$\frac{dh}{du} = \frac{(1+\nu)(3z-2)h}{\sqrt{M}} + \frac{2\delta}{\sqrt{M}} z\sqrt{1-z}\cos\phi . \tag{4.6}$$

From(4.6), the right hand side of which is of order $\frac{\nu}{\sqrt{M}}, \frac{\delta}{\sqrt{M}} \sim \varepsilon$,we see that at the lowest order in ε the motion is given by (4.2). On the other hand, the variation of $\sqrt{M(u)}$ is of order 0. Therefore, in the asymptotic regime, the trajectory $(\phi, z, \sqrt{M})$ departs from the surface h=constant, only by terms of order $\frac{\nu}{\sqrt{M}}, \frac{\delta}{\sqrt{M}}$, but on this surface the variation of the energy $\sqrt{M}$ is finite. At saturation eq.(4.5c) gives the average value of z(u) as

$$<z> = \lim_{u \to \infty} \int_0^u z(u)du = \frac{1}{1+\nu} . \tag{4.7}$$

At lowest order $z(u) = z_c(u,h)$, and $<z>$ is approximately given by $<z> = c + (a-c)\frac{E}{K} = <z(h)>$. Then, the surface h=constant around which the asymptotic motion takes place is determined by the dissipation.If $2(\nu+1) >> 1$, $\mathrm{Log}\frac{h}{16} = -2(1+\nu)$, and the trajectory remains close to the separatrix surface of the conservative system. For $\delta > 0$, $h \leq 0$ in the asymptotic regime.

In order to determine the average wave energy , and to describe the dynamical behavior of the system, we now compute how far the trajectory in phase space, departed from the surface $h= -16\, e^{-2(1+\nu)}$.

The state of the system completely described by the variables $h, \sqrt{M}$ is recorded at moments crossing the surface $\phi=\pi, \frac{d\phi}{du} < 0$. The equations of motion determine the relation between values $\sqrt{M(0)}$, h(0) at one crossing of the plane $\theta=\pi$, and $\sqrt{M'(0)}$,h'(0) at the next crossing, or a mapping of the plane $\sqrt{M(0)}$,h(0) onto itself used in discussing the numerical investigation. However for a theoretical analysis,the variable h(0) is not very convenient. We found in numerical and analytical investigation of the system (2.29) , the main change in h happens just near the plane $\theta=\pi$, for which z is close to its maximum value $z \leq 1$, while over the rest of the trajectory h~const<<1(fig.4.13). The trajectory spends a very long "time" u near the hyperbolic points $z=0,\phi=\frac{\pi}{2},\frac{3\pi}{2}$ since it remains close to the separatrix surface. The time duration between two successive crossings is essentially determined by the values of h, corresponding to $z=z_{min}\sim 0$, and this value of h between successive crossings of the plane $\theta=\pi$, $z\leq 1$ is chosen as the dynamical variable. In the following we shall denote by h and $x=-\frac{1}{2}\log(-\frac{h}{16})$, the values of h(u) and x(u) before the crossing of the plane $\phi=\pi(z=z_{max})$ with a value y of the energy $\sqrt{M(u)}$, and by h',x' for that after this crossing or before the next one with y' value of $\sqrt{M}$. Notice that 2x' is the time duration between two successive crossings of the plane $\theta=\pi$, y -> y' and that y acts as an initial phase for $\sqrt{M(u)}$ and h(u).

We turn now to the evaluation of an explicit form for the mapping (x,y) -> (x',y') and note that

$$z_c(u) = 1 - \tanh^2 u + o(h)\,, \tag{4.7}$$

for small values of h(u),where u=o is the time at which z_c is maximum ($\theta=\pi$). Then eq.(4.5c) yields

$$\sqrt{M(u)} = y + u - (\nu+1)\tanh u\,, \tag{4.8}$$

where y is the energy for u=0. The energy change between two crossings is approximately equal to ,

$$y' = y + 2x' - 2(\nu+1)\,. \tag{4.9}$$

This change depends on the value of the time duration between crossings $2x' = -\log\frac{h'}{16}$. To get the change in h(or x), we integrate eq.(4.5) from u=-x to u=+x', having substituted in its right side , the expressions (4.7) and (4.8) for z(u) and $\sqrt{M(u)}$ deduced for h=const~0. For analytical evaluation of the integral it is convenient to introduce the quantity

$$P(u) = (\sqrt{M})^3 h - \delta M\left(z - \frac{1}{2(1+\nu)}\right), \tag{4.10}$$

obtaining

$$P'(x') = \left\{P(-x) + \frac{1+2\nu}{2(1+\nu)}\delta\int_{-x}^{x'}\sqrt{M(u)}du\, e^{(2\nu-1)\int_{-x}^{u}\frac{du'}{\sqrt{M(u')}}}\right\} e^{\left((1-2\nu)\int_{-x}^{x'}\frac{du}{\sqrt{M(u)}}\right)} . \quad (4.11)$$

In order to evaluate the integrals, we approximate $\sqrt{M(u)}$ as

$$\sqrt{M(u)} = y + u + (\nu + 1) \qquad -x<u<-1.5 ,$$

$$\sqrt{M(u)} = y + \frac{1-2\nu}{3}u \qquad -1.5<u<1.5 , \qquad (4.12)$$

$$\sqrt{M(u)} = y + u -(\nu + 1) \qquad 1.5<u<x'.$$

Then we get

$$h' = \left\{ h \left(\frac{y+\nu-0.5}{y+\nu+1-x}\right)^{-2(1+\nu)} + \frac{2\delta}{y^2-(\nu-0.5)^2}\right\}\left(\frac{y+x'-\nu-1}{y-\nu+0.5}\right)^{-2(1+\nu)} , \qquad (4.13)$$

$$x' = (1+\nu)\log\left(\frac{y+x'-\nu-1}{y-\nu+0.5}\right) - 0.5 \log\left\{e^{-2x}\left(\frac{y-0.5+\nu}{y-x+\nu+1}\right)^{-2(1+\nu)} - \frac{\delta}{8(y^2-(\nu-0.5)^2)}\right\} , \qquad (4.14)$$

$$y' = y + 2x' - 2(\nu+1) \qquad (y>\nu-0.5, x>1) . \qquad (4.15)$$

In the case of no mismatch in frequencies, $\delta=0$, eq.(4.15) yields $x'>x$. Therefore,y, and consequently the average wave energy, increase indefinitely as iteration proceeds. This result is in agreement with the numerical solution of the three wave system and the qualitative analysis . The instability does not saturate for $\delta=0$.

To compare the analytical analysis to the numerical solution , we discuss now the case $\delta=2$, $5 < \nu < 18$.

From eq.(4.15), we deduce

$$< x > = \lim_{N\to\infty} \frac{1}{N}\sum_{0}^{N} x_n .$$

Then for finite δ values, we expect the quantity e^{-2x} to be much smaller than $\frac{\delta}{8(y^2 - (\nu - 0.5)^2)}$ providing that y does not increase too much. Hence, we neglect the term e^{-2x} in eq.(4.14) and retain only the δ term. This assumption has been cheked for the parameter range $5 < \nu \leq 20$. Then for δ=2, the mapping (4.14) reduces to a one-dimensional map. Substituting y in terms of (y',x') in eq.(4.14), we get the explicit equation of the attractor,

$$x' = (1+\nu) \log\left\{1+\frac{x'-0.5}{Y'-2x'+2\nu+2}\right\} + 0.5\log\frac{8}{\delta}(Y'-2x'+2\nu+2) + 0.5(Y'-2X'+4\nu+1) \quad , (4.16)$$

where the variable $Y' = y'+1.5-(\nu+1) = \sqrt{M}$ (u=1.5) has been introduced in order to simplify the expression of x'(Y). The dynamic of the motion on the attractor is described by the one dimensional map $Y_{n+1} = f(Y_n)$, or

$$Y' = y + 2x'(Y) - 2(\nu+1) \; , \tag{4.17}$$

where x'(Y) is the solution of

$$x' = (\nu+1) \log\left[1+\frac{x'-0.5}{Y}\right] + 0.5\log\left[\frac{8}{\delta} Y (Y+2\nu-1)\right] .$$

The map (4.17) has two fixed points of which one is always unstable, given by $x=\nu+1$, $y \sim e^{2(\nu+1)}$. For this value of Y, the approximation $e^{-2x} << \frac{\delta}{8Y^2}$ fails.The other one $x_0=\nu+1$, $Y_0=Y_0(\nu)$ is stable if $\nu<8.6$. For $\nu=8.6, Y_0=8.92$ this simple stable fixed point corresponds to a stable limit cycle for the three wave system(2.29). In the asymptotic regime, the average energy $M= \gamma U_0^2+U^2$ oscillates with a period

$$T = \int_{-(\nu+1)}^{\nu+1} \frac{du}{\sqrt{M(u)}} \; ,$$

where $\sqrt{M(u)} = Y_o + u - (\nu+1)\tanh u - 0.5 + \nu$; $-(\nu+1)<u<(\nu+1)$. From the definition of $t = \int^u \frac{du}{\sqrt{M(u)}}$ one may deduce the expression of $M(t) = \gamma U_0^2+U^2$ and $\frac{U^2(t)}{M(t)} = z(t)$ as

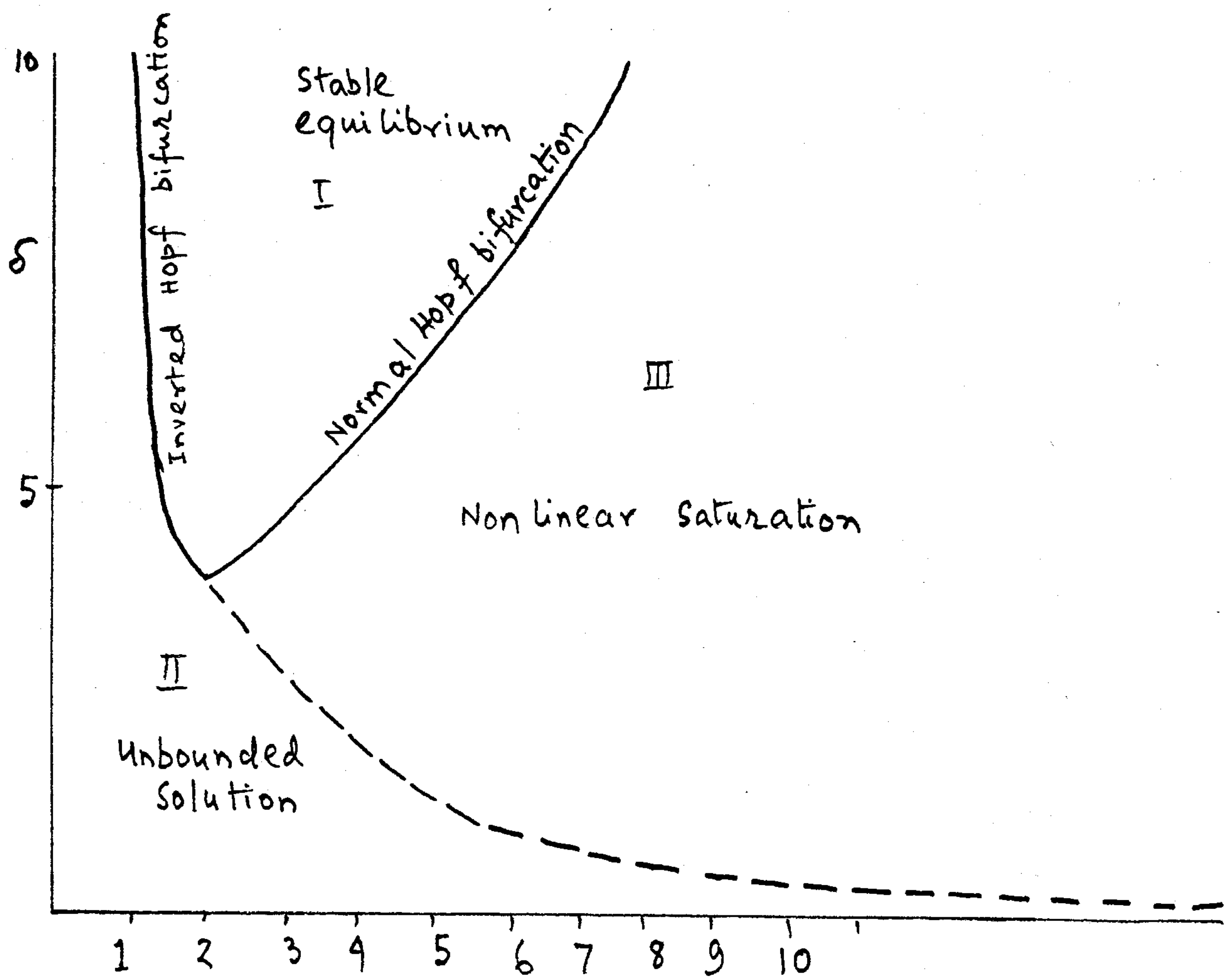
$z(t) \sim 1-\tanh^2 u(t)$.

(i) For ν=8.6, the fixed point loses its stability through a pitchfork bifurcation as $\frac{\partial f(Y)}{\partial Y}|_{Y_0}=-1$. A stable period two cycle is observed.

(ii) For 8.6 <ν<13, as ν is increased the cascade of period doubling is observed till ν=13.

(iii) For 13<ν<16.5 chaotic motion is observed. In fig.4.14, the one-dimensional attractor on which the motion takes place, has been plotted. In order to compare numerical and analytical constructed maps, we have plotted in fig.4.15 the minimum value of $\sqrt{M}$ versus the "time interval" $2x=\int_0^t\sqrt{M(t)}\,dt$ between two minimas for ν=14.8,δ=2. In the same way , we have recorded the successive minima of $\sqrt{M(t)}$, $\sqrt{M_n}$,$\sqrt{M_{n+1}}$ .... Fig. 4.16 shows the numerical computed map $\sqrt{M_{n+1}}$ vs $\sqrt{M_n}$. On the other hand we have plotted in fig.4.17, the graph of Y_{n+1} vs Y_n as generated by the mapping 4.17 for ν=14.8,δ=2. Within the approximation for $\sqrt{M(u)}$, 4.12, Y= $\sqrt{M(1.5)}$ represents the minimum value of $\sqrt{M(u)}$.

From this analysis, we conclude that the critical ν-values for which the attractor changes its topology are in good agreement with those deduced from the numerical investigation of the system (2.29).



10
5
δ
Stable
equilibrium
I
Inverted Hopf bifurcation
Normal Hopf bifurcation
III
Non linear saturation
II
Unbounded
Solution
1
2
3
4
5
6
7
8
9
10

Fig 4.1

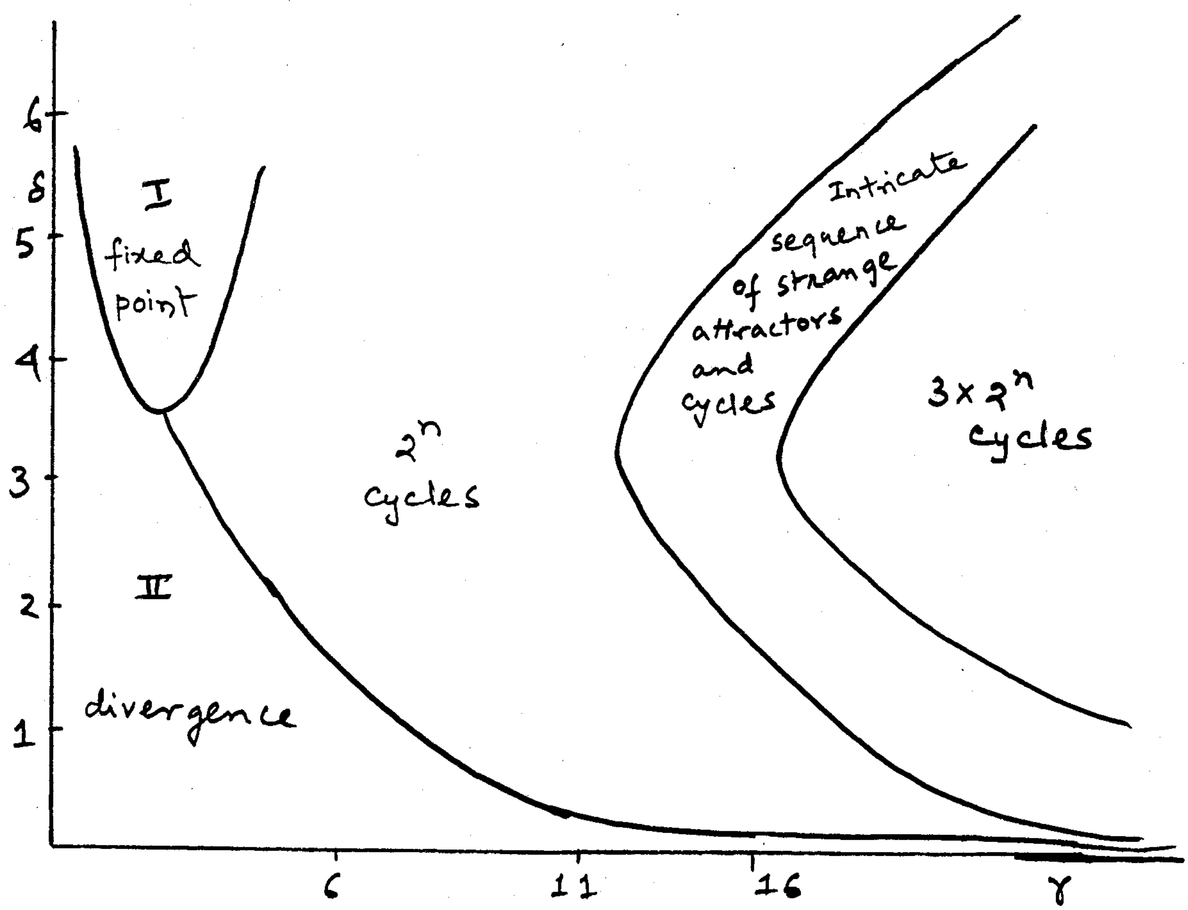

6
δ
5
4
3
2
1
I
fixed
point
II
divergence
2^n
cycles
Intricate
sequence
of strange
attractors
and
cycles
3×2^n
cycles
6
11
16
γ

Fig 4·2

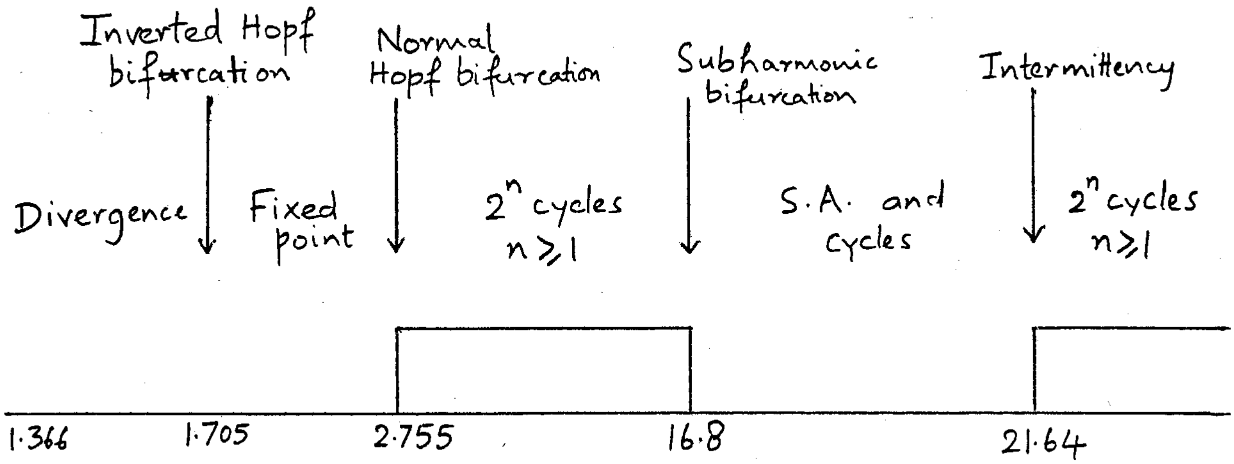
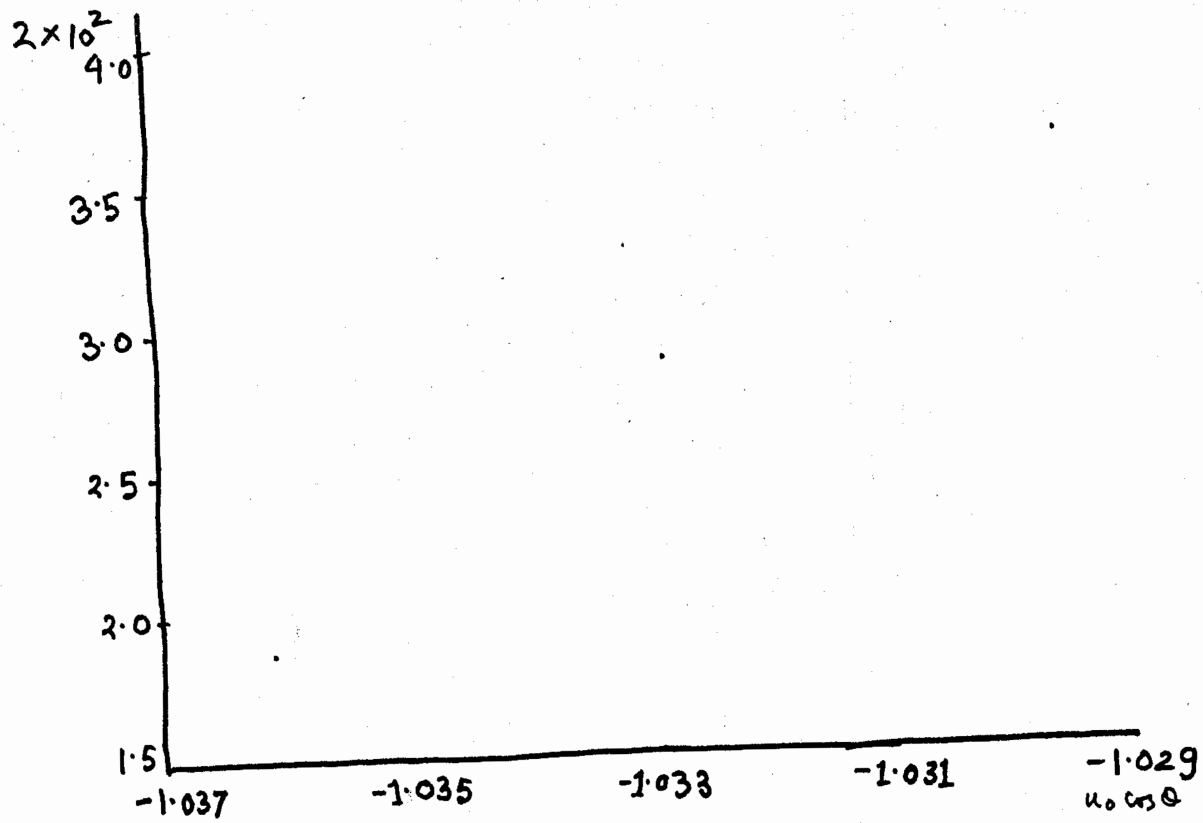

Inverted Hopf bifurcation
Normal Hopf bifurcation
Subharmonic bifurcation
Intermittency
Divergence
Fixed point
2^n cycles $n \geq 1$
S.A. and cycles
2^n cycles $n \geq 1$
1·366
1·705
2·755
16·8
21·64

Fig. 4·3


2×10²
4·0
3·5
3·0
2·5
2·0
1·5
-1·037
-1·035
-1·033
-1·031
-1·029
u₀ cos θ

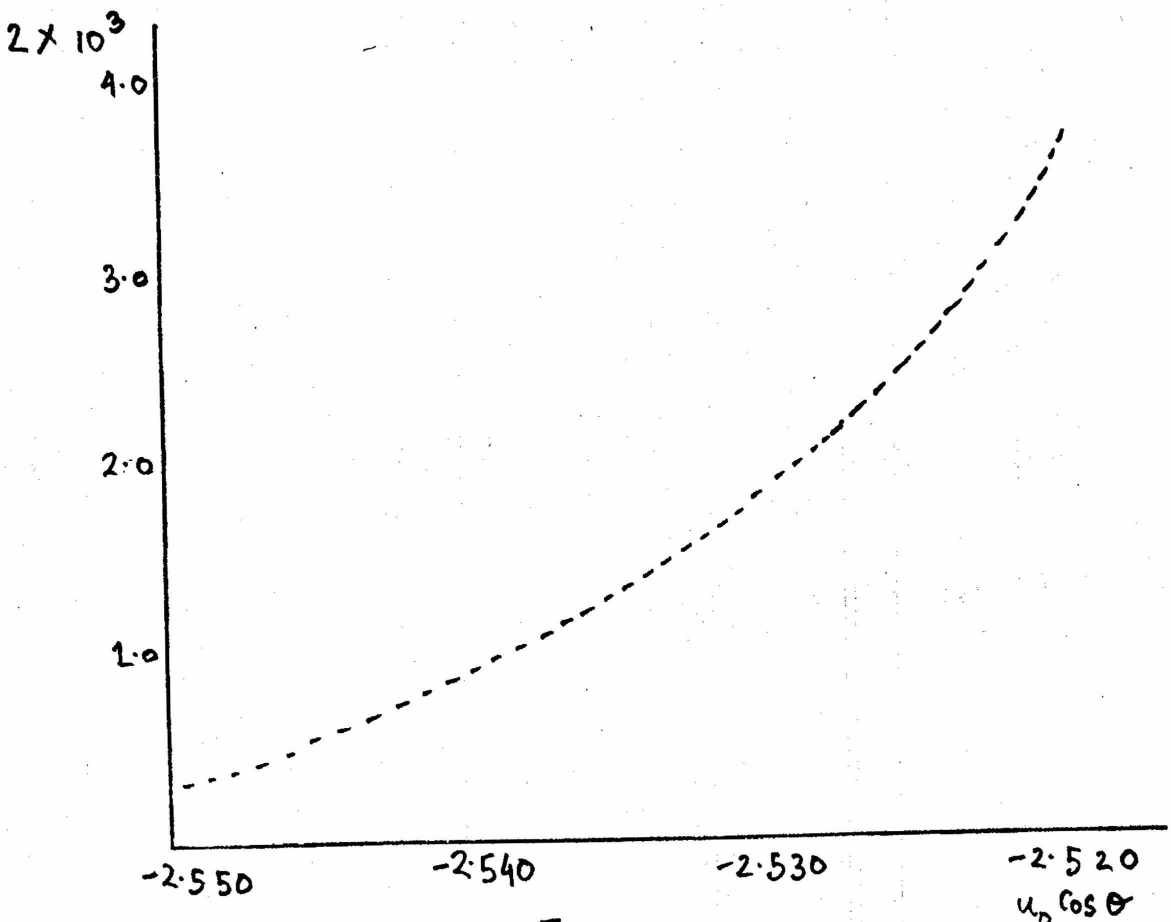
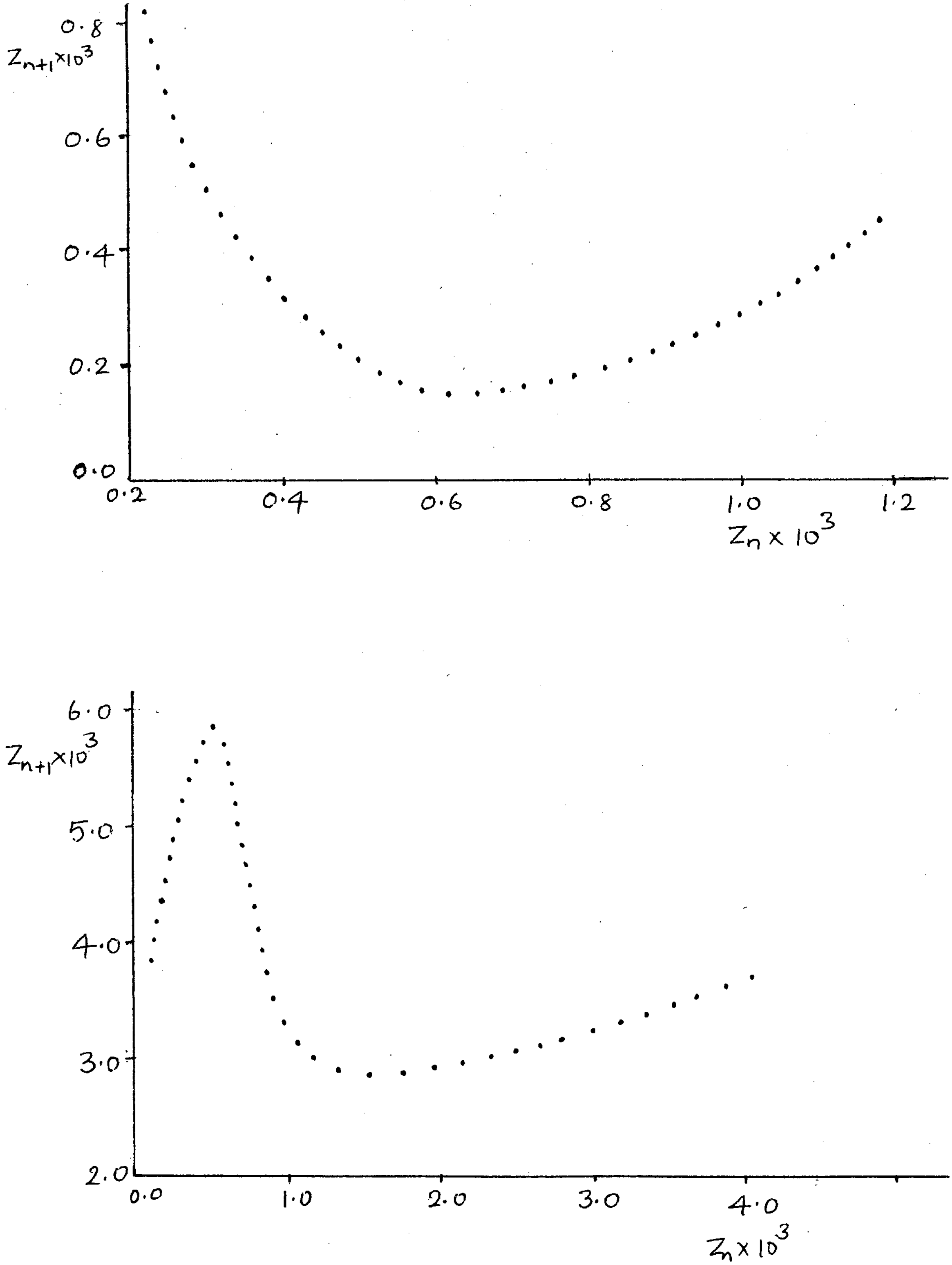

2× 10³
4·0
3·0
2·0
1·0
-2·550
-2·540
-2·530
-2·520
u₀ cos θ

Fig. 4·4



0.8
$Z_{n+1} \times 10^3$
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$Z_n \times 10^3$
6.0
$Z_{n+1} \times 10^3$
5.0
4.0
3.0
2.0
0.0
1.0
2.0
3.0
4.0
$Z_n \times 10^3$

Fig 4.5

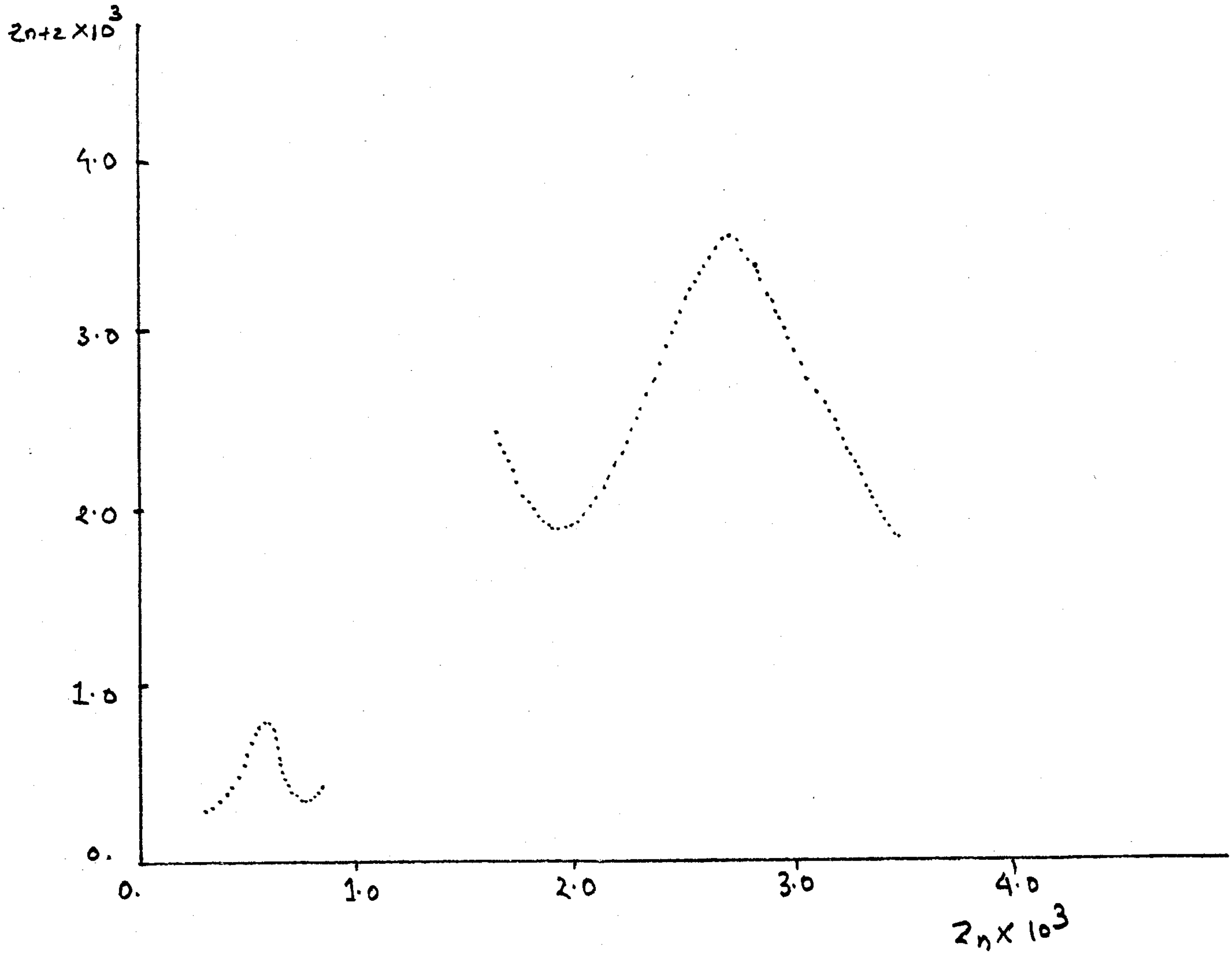


Fig 4.6

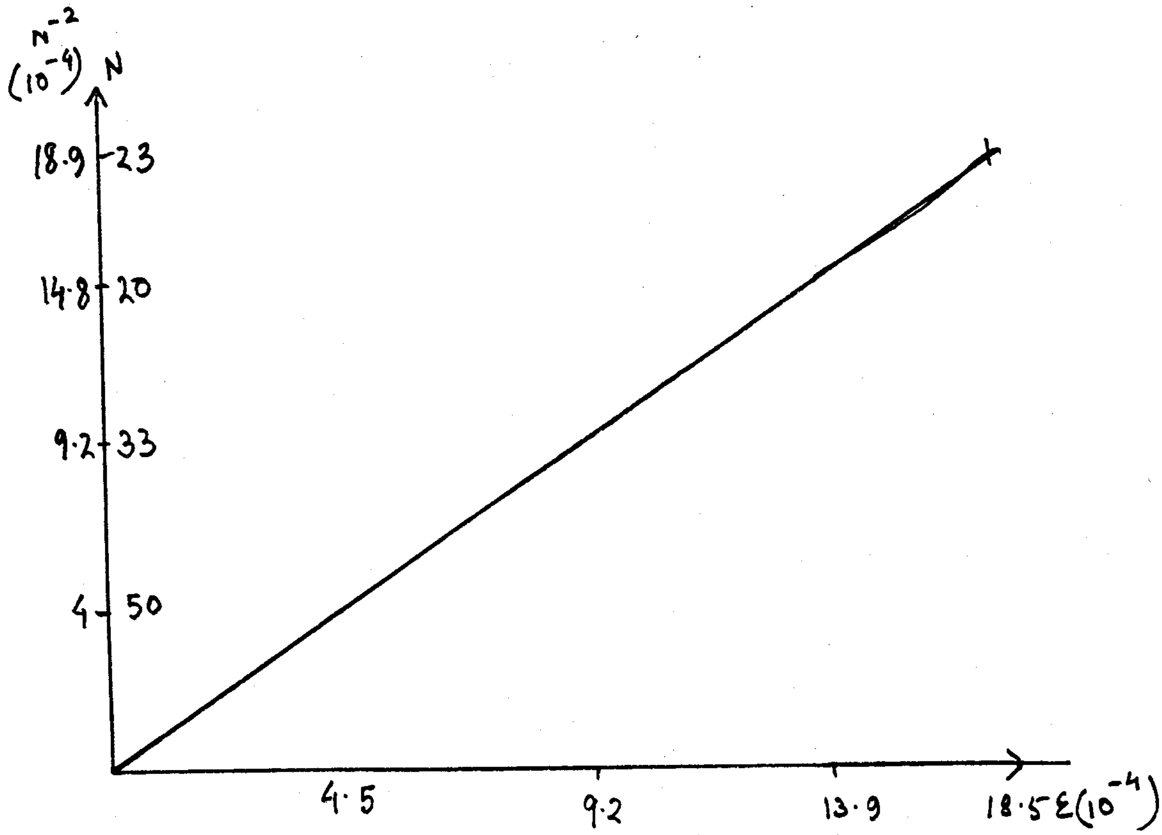


Fig 4.7

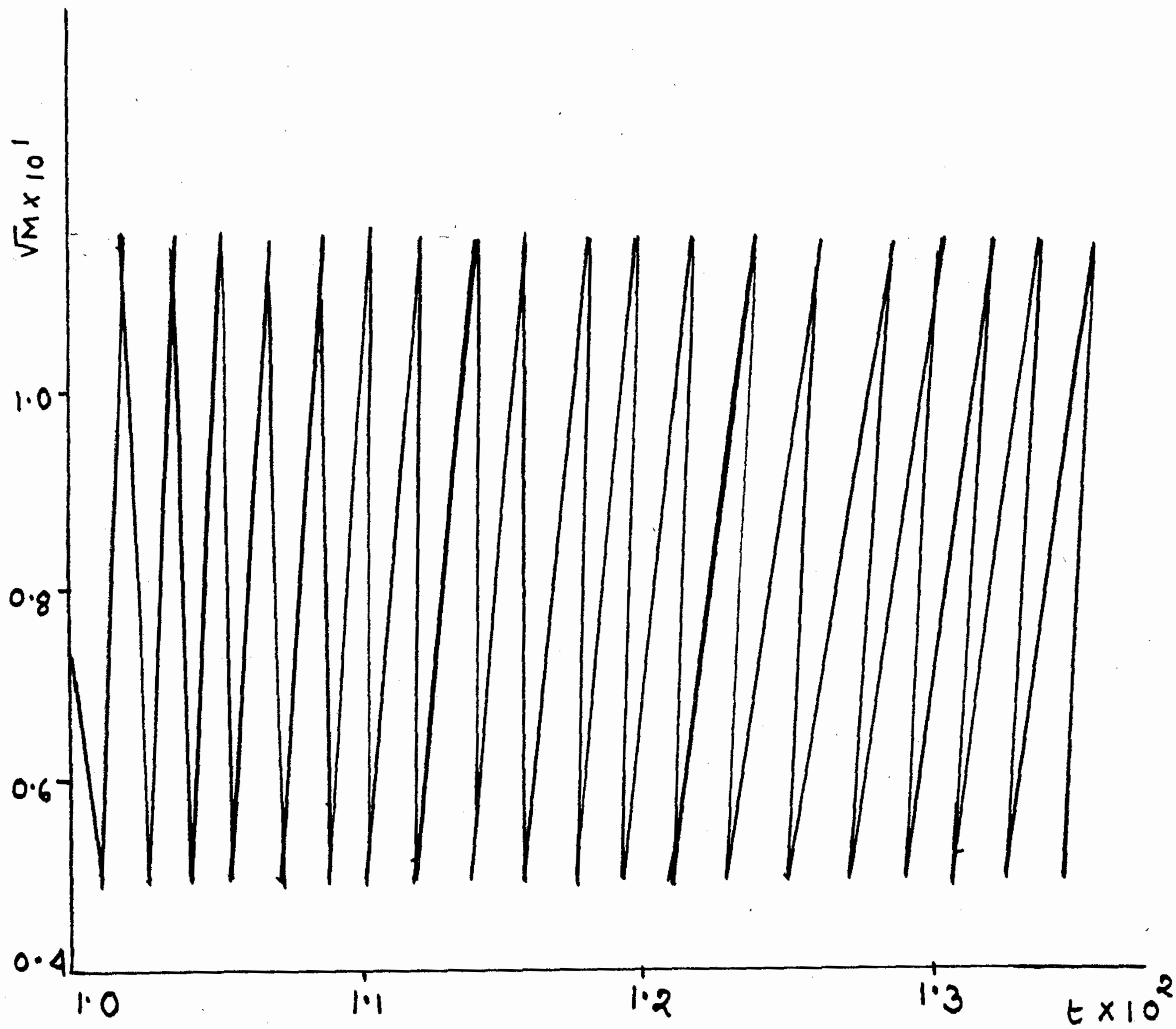


Fig 4.8

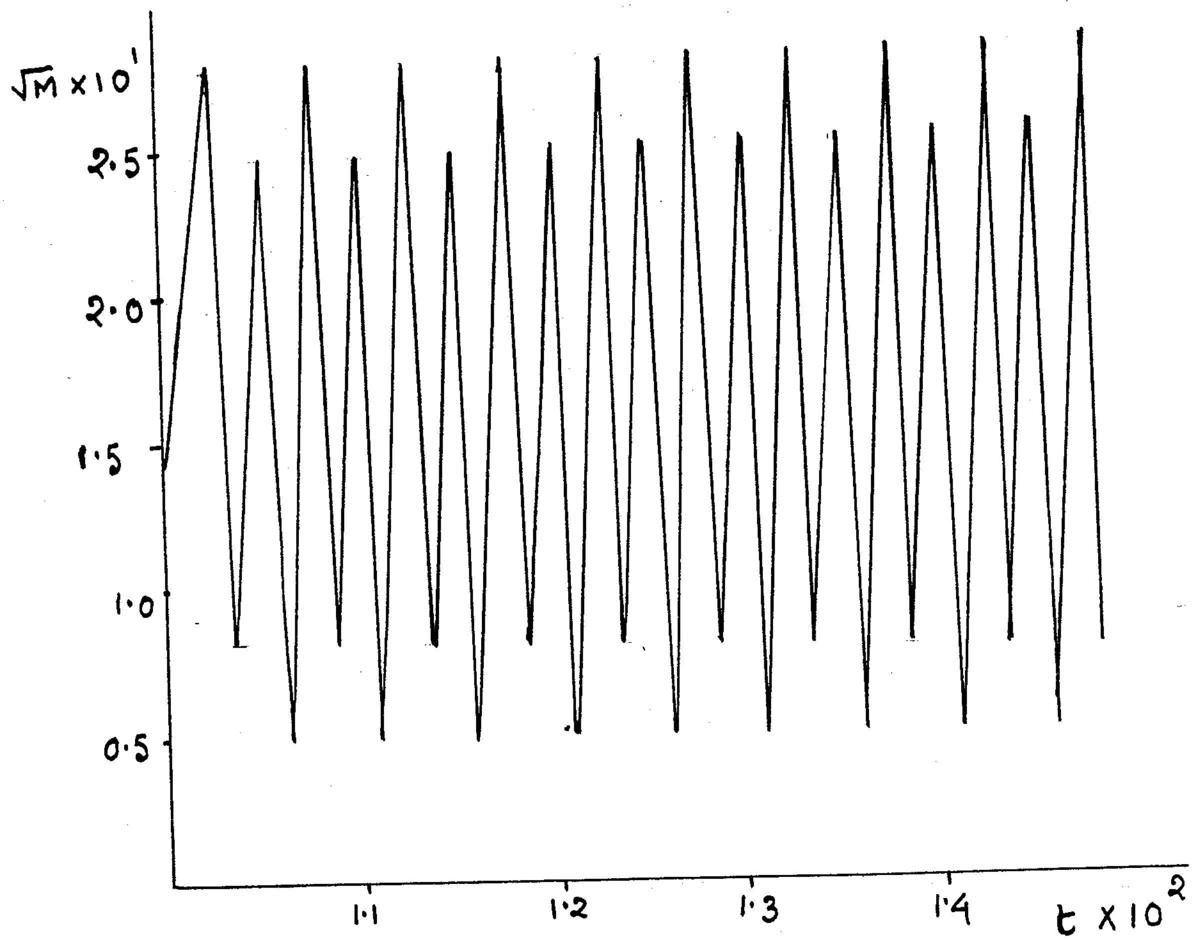


Fig 4·9

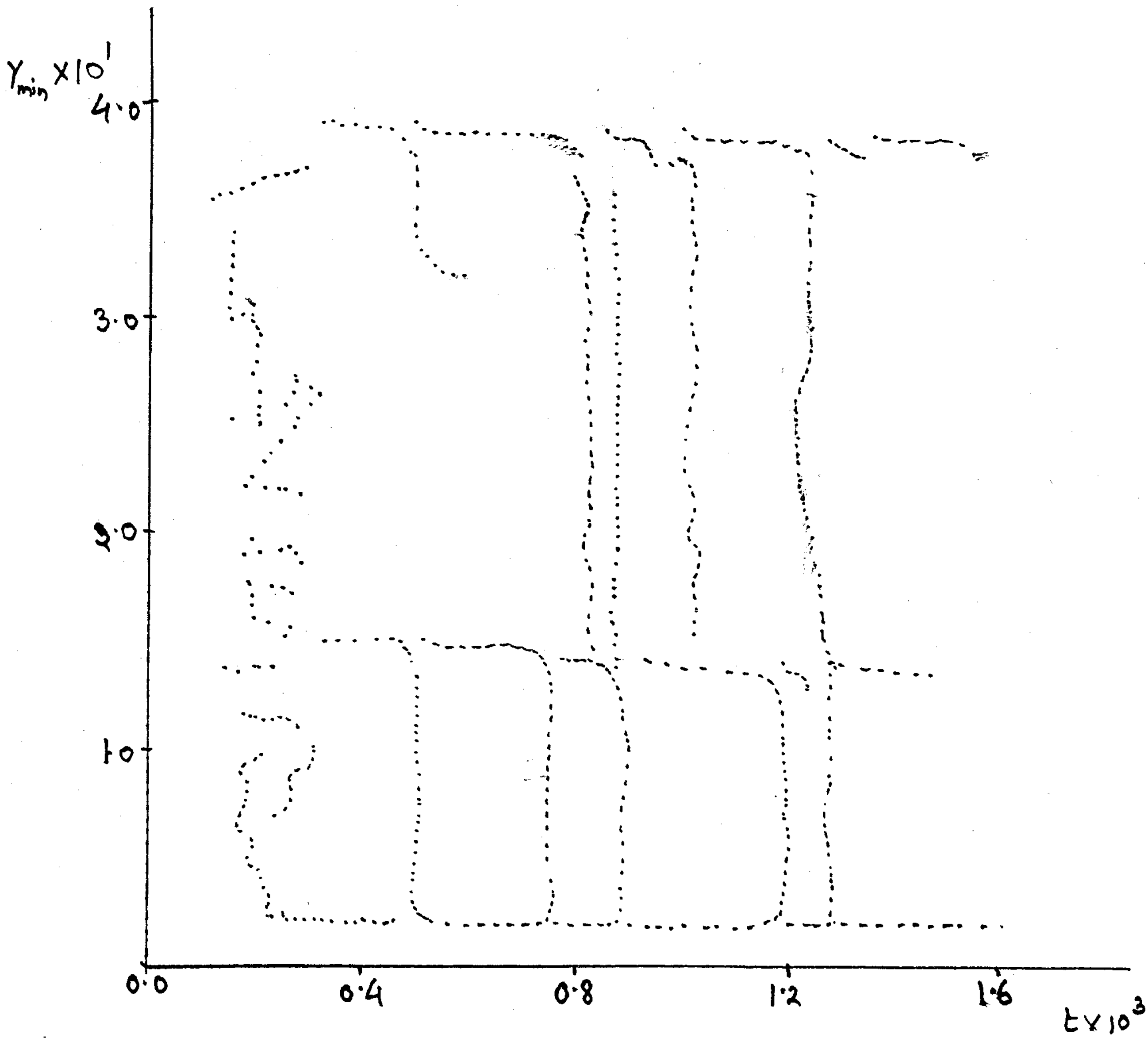


Fig 4·10

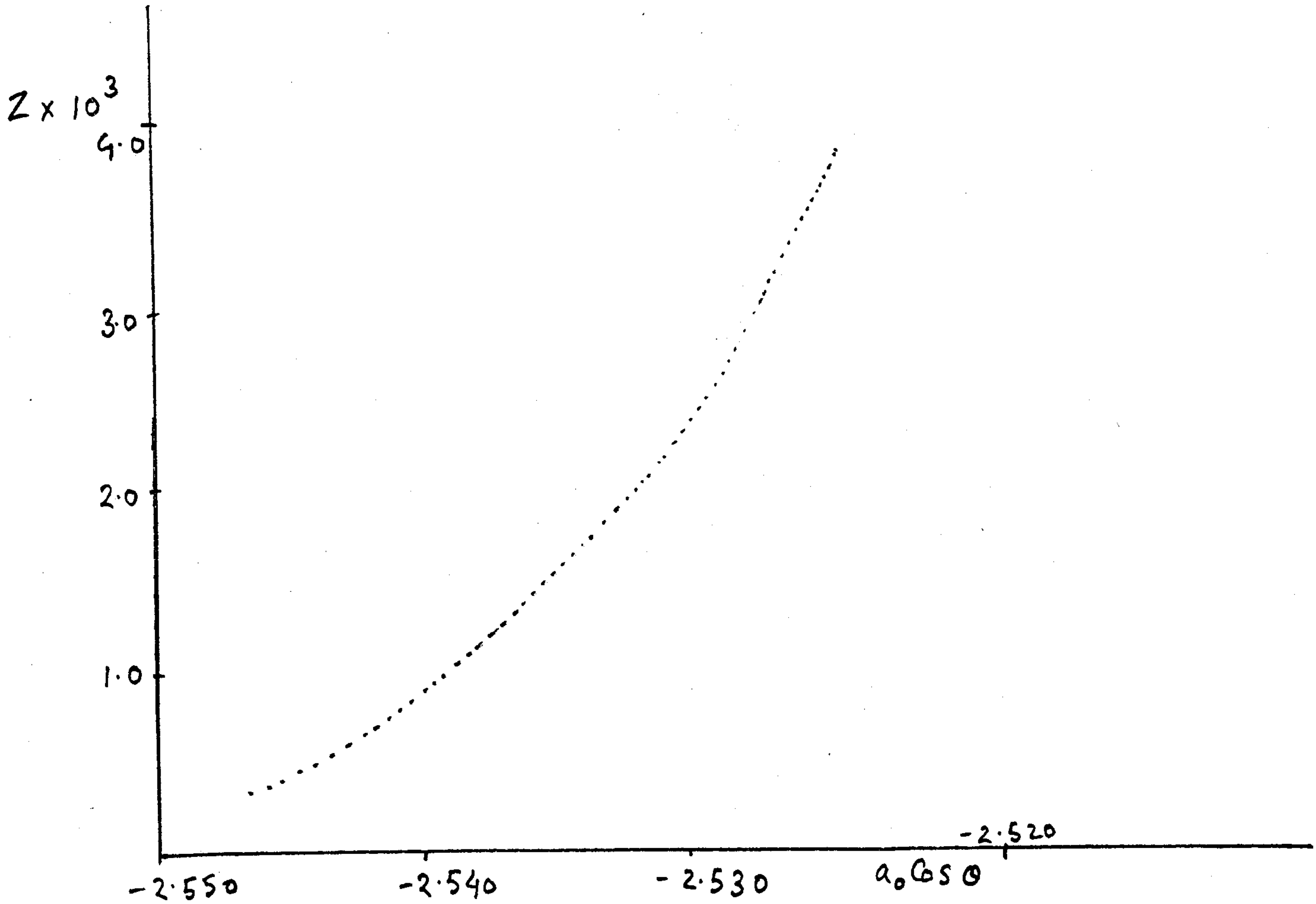


Fig 4·11

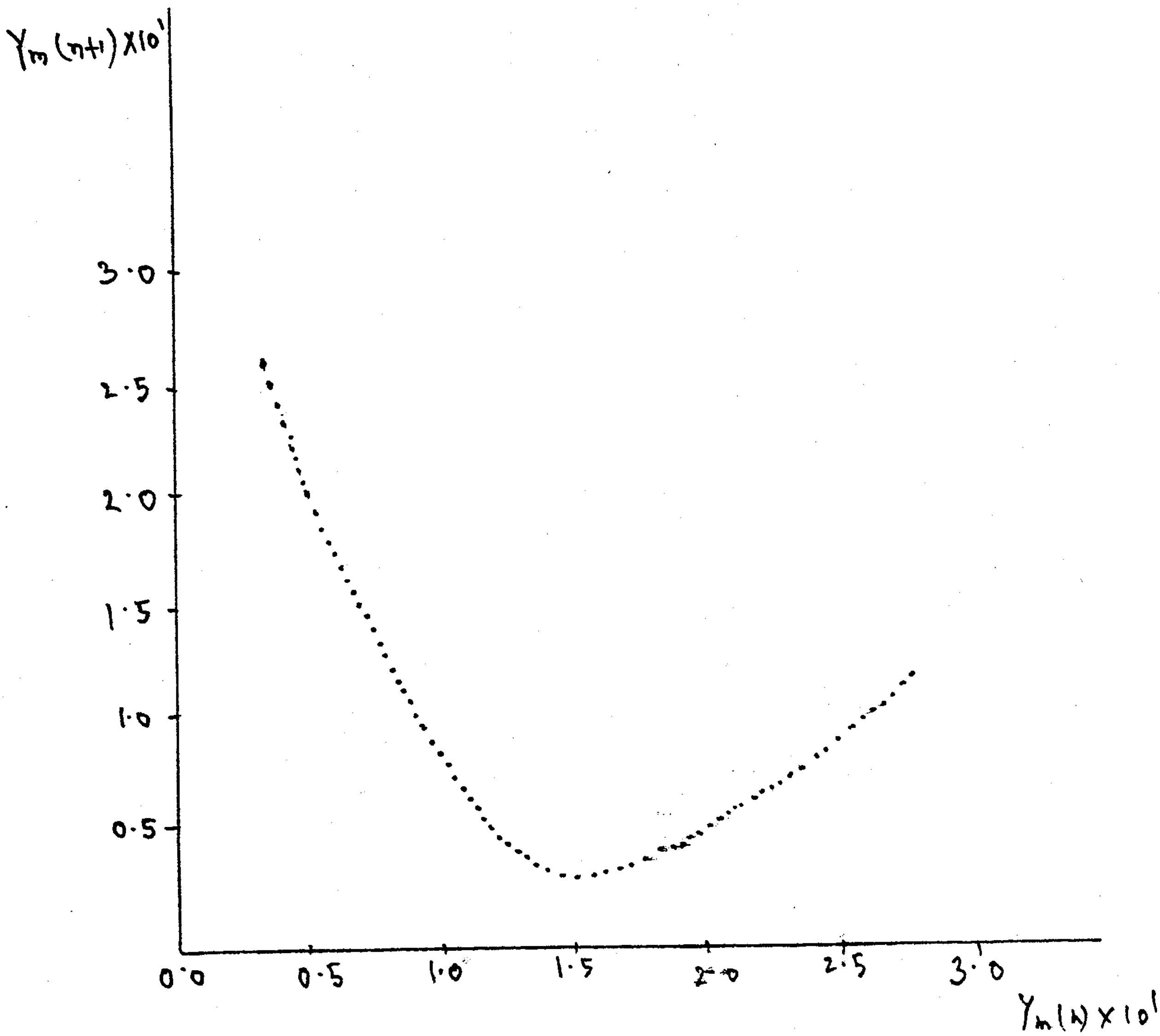


Fig 4·12

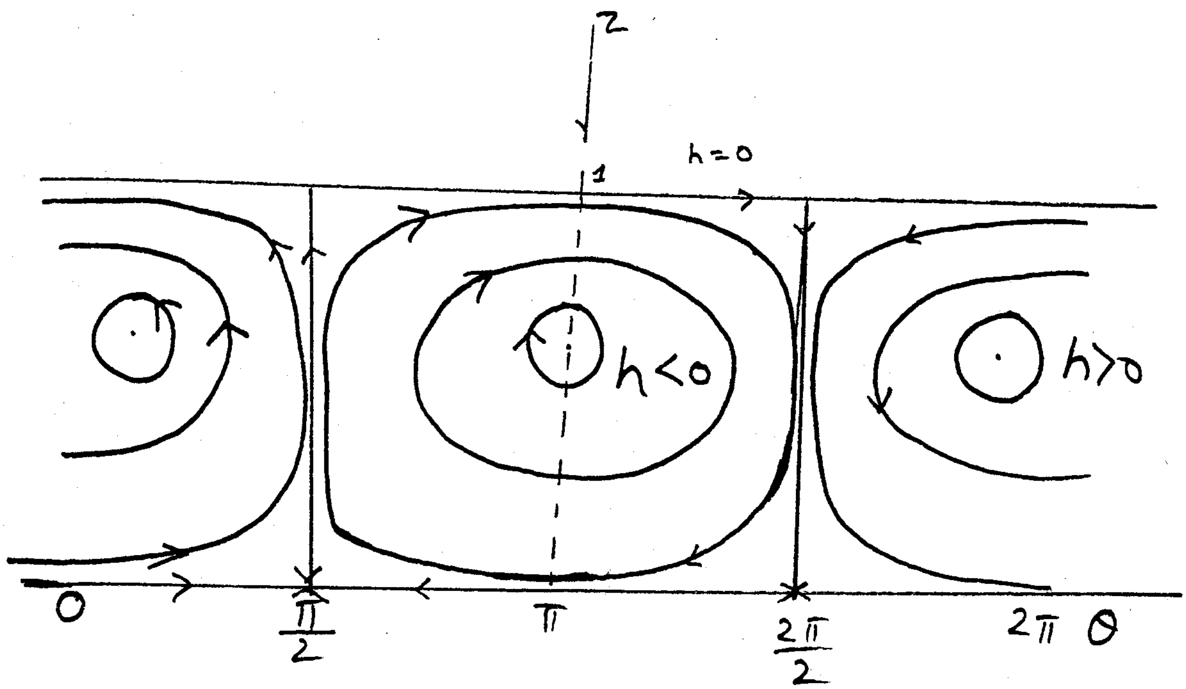


Fig 4·13

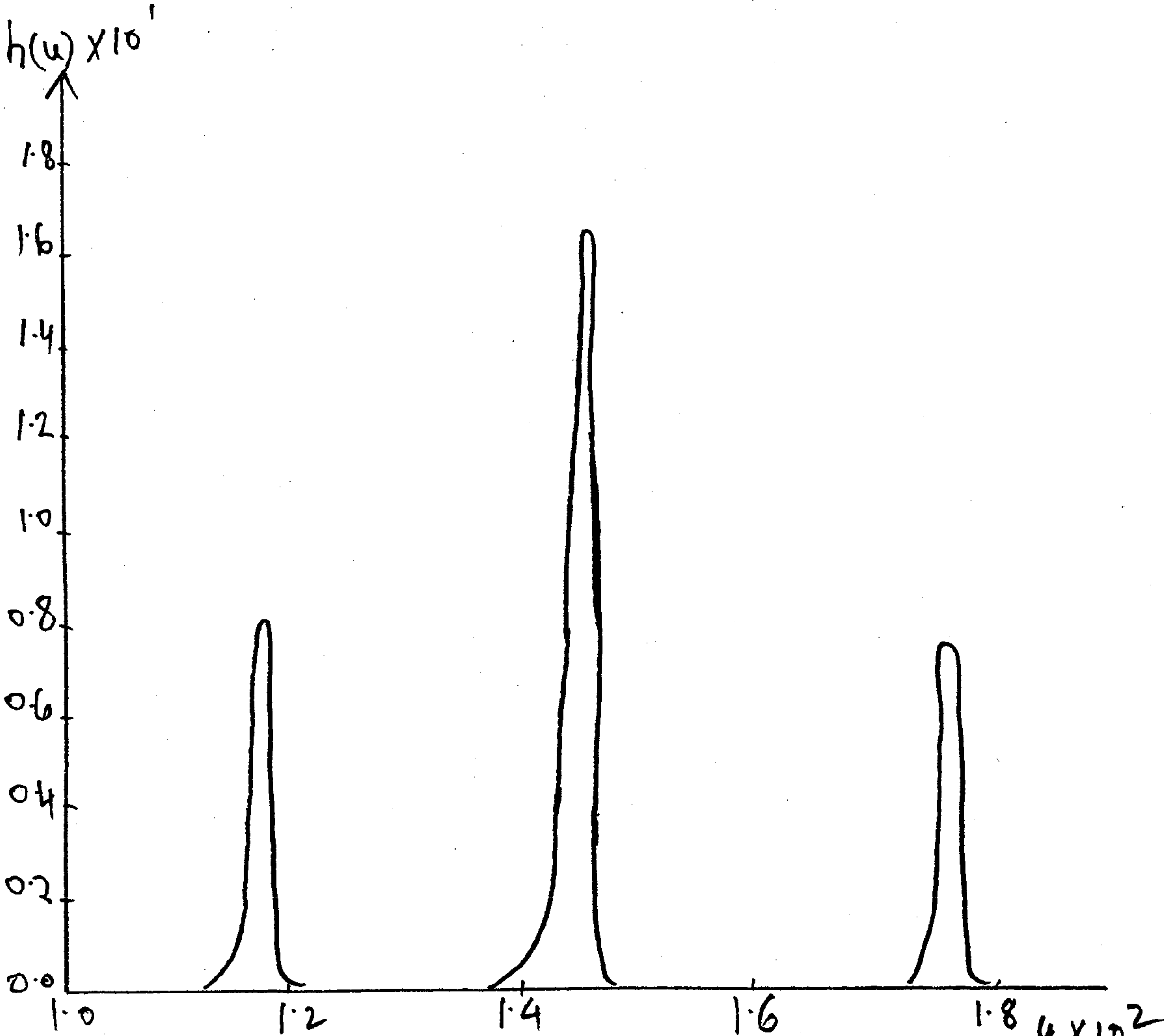


Fig 4·14

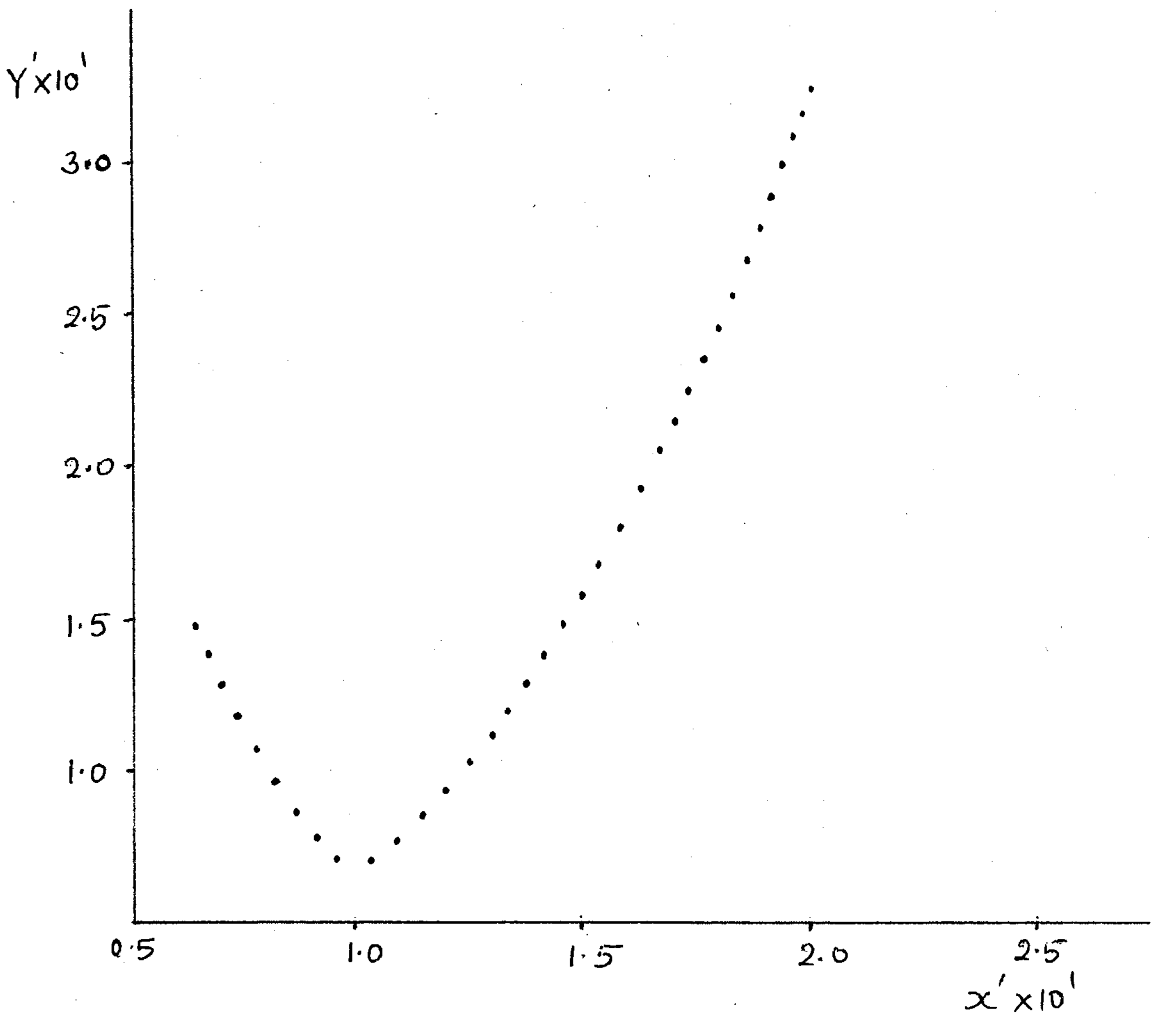


Fig 4·15

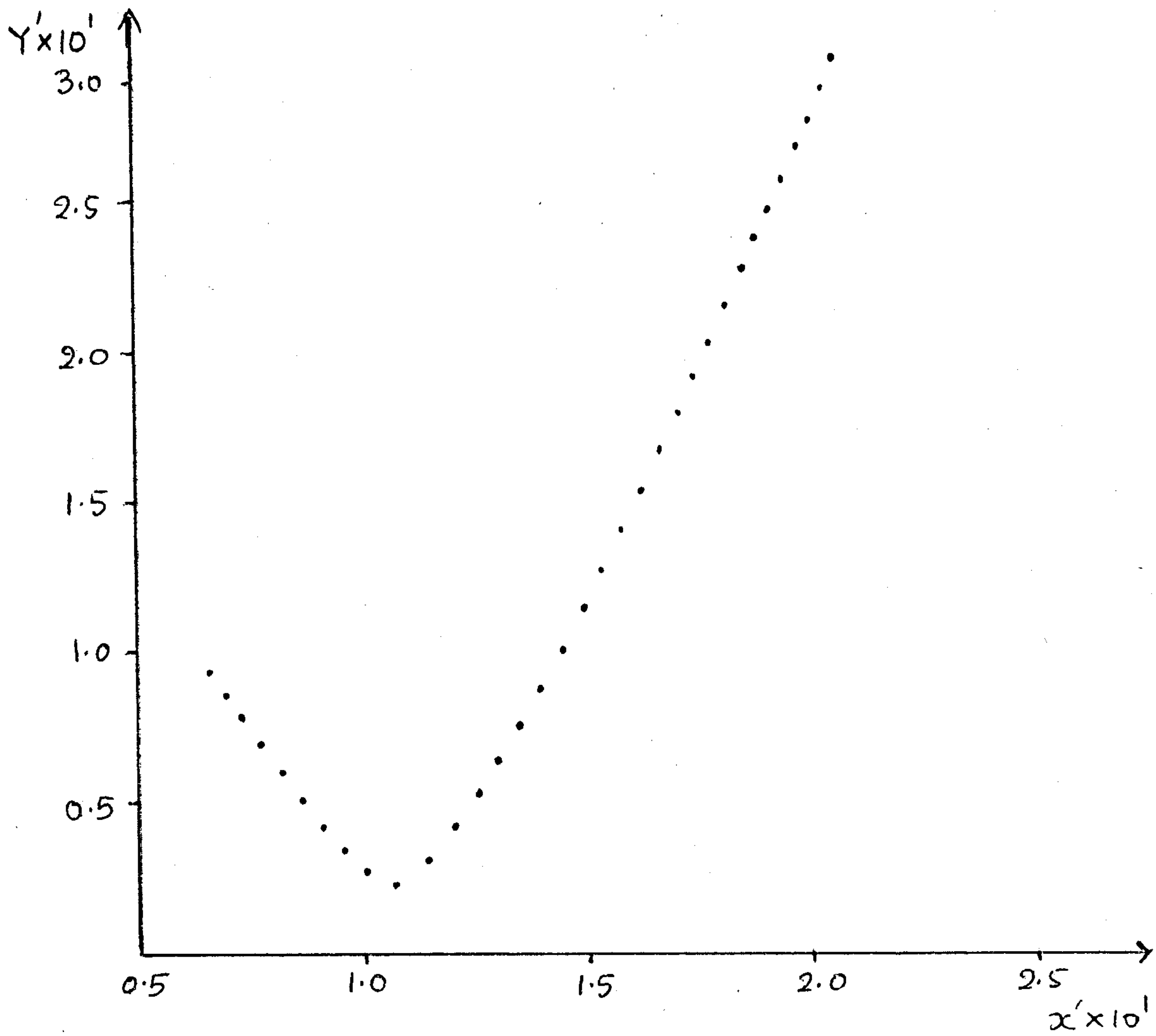


Fig 4.16

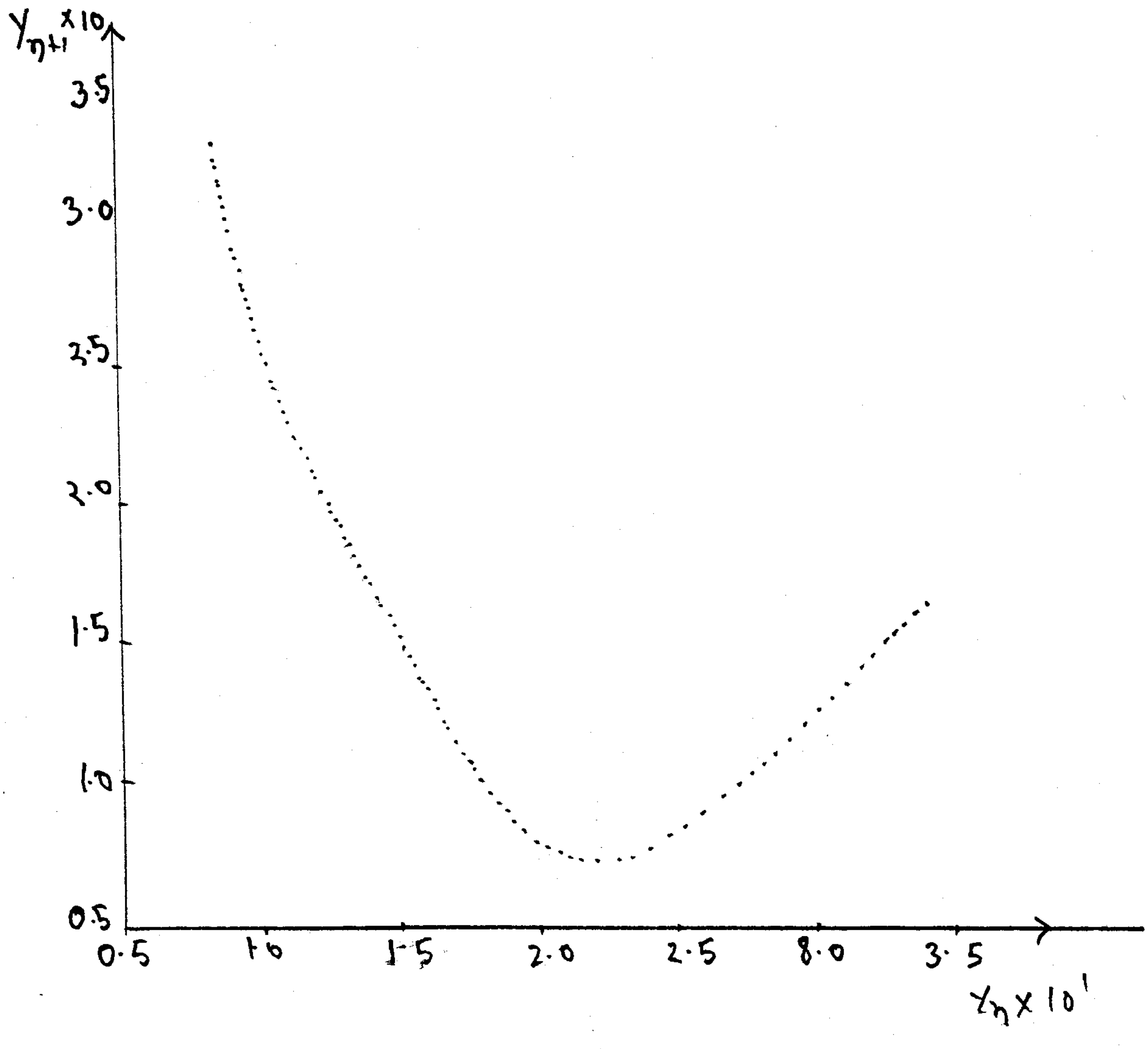


Fig 4·17

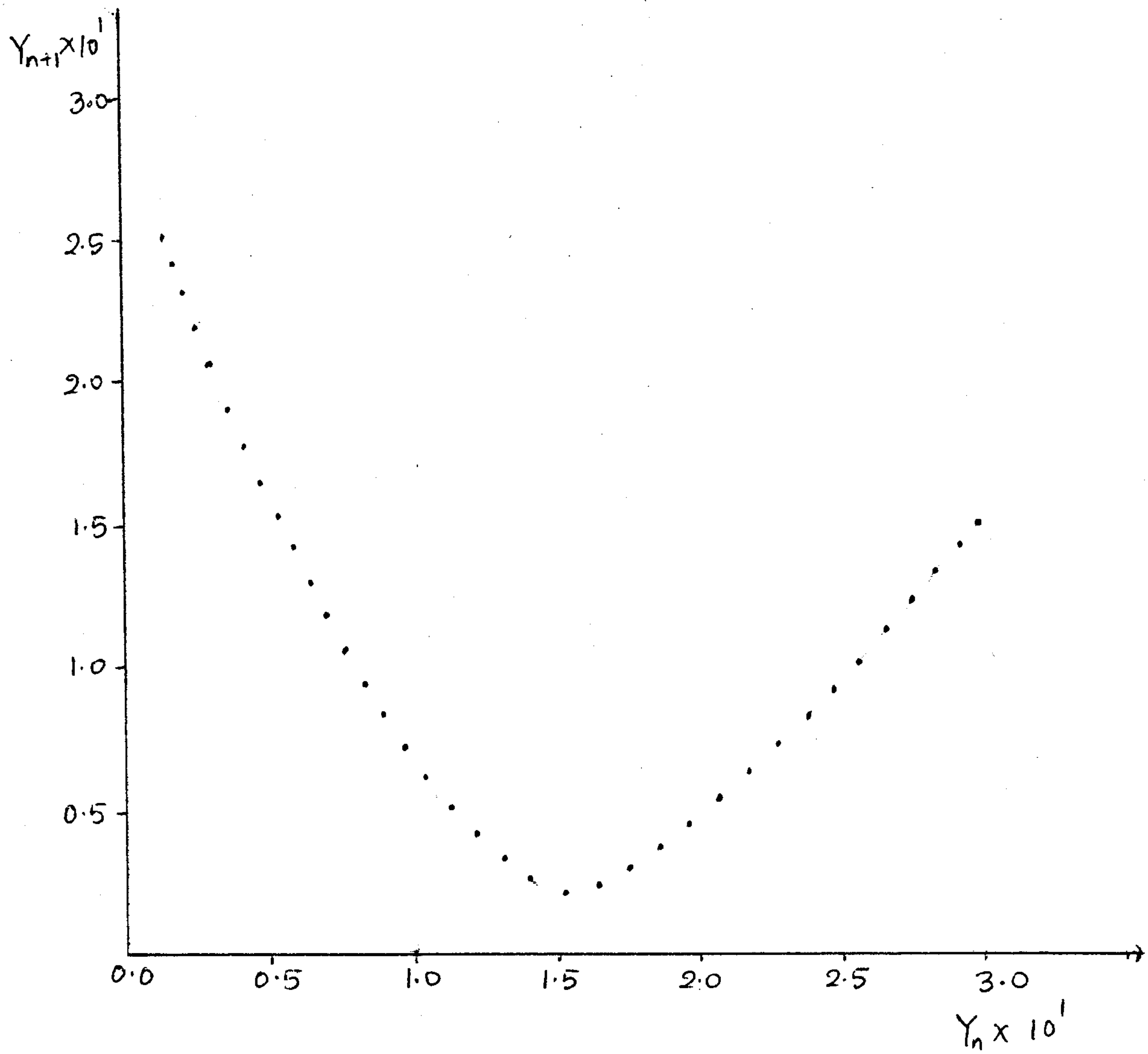


Fig 4.18

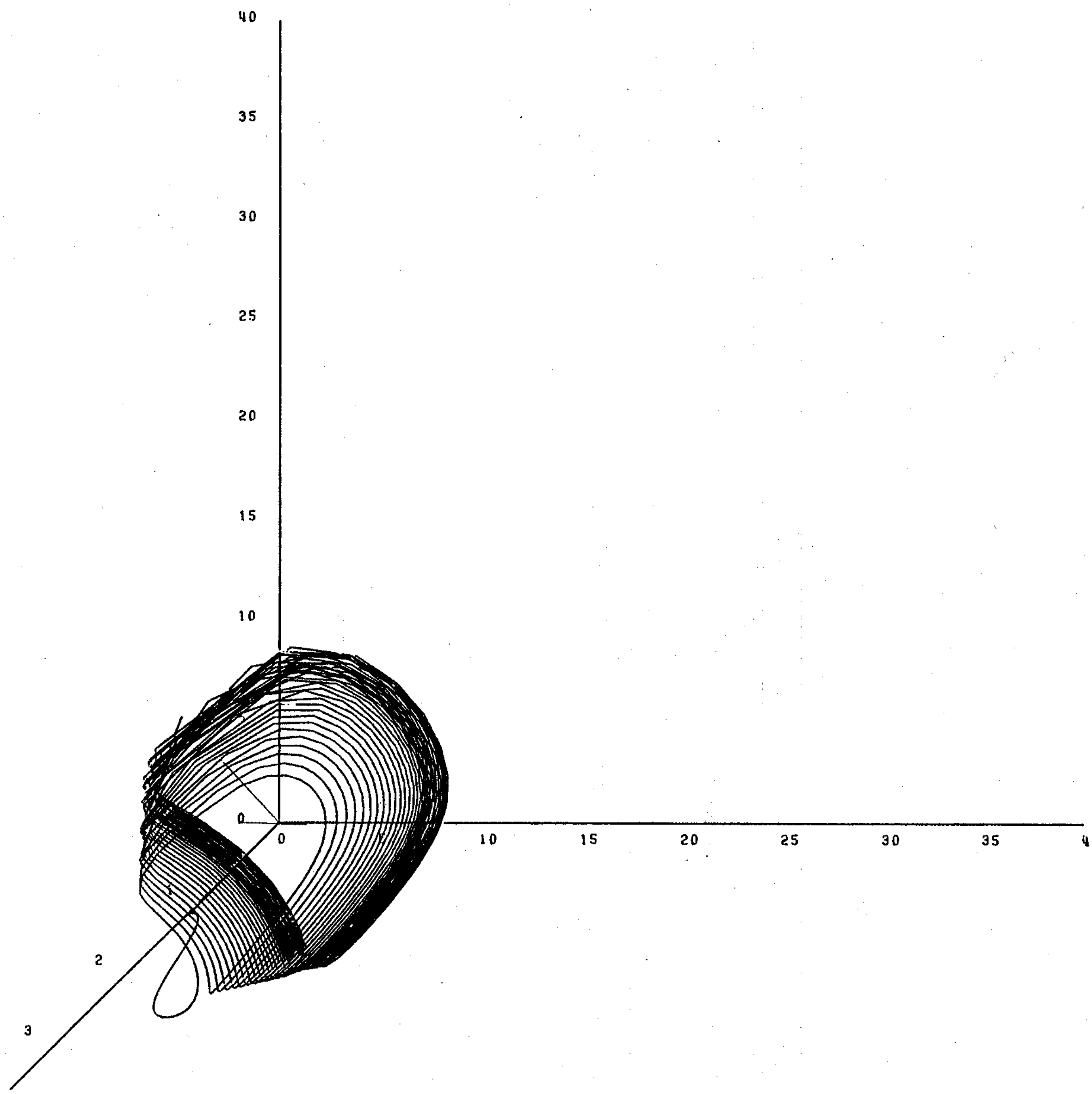


Fig. 4.19

CONCLUSION

We have studied the interaction of three electrostatic waves in a plasma which is represented by three first order nonlinear ordinary differential equations. The system has two equilibrium points both being unstable. They involve critical parameters which depend on the growth rate of the high-frequency wave and the decay rates of the two low-frequency waves. The second nontrivial critical point changes from an unstable saddle-node to a saddle focus as the parameter δ changes its value from zero to a finite quantity.

We have studied the system (2.31) numerically and found that there is a presence of intermittency which causes turbulence. We have explained the results of the numerical work with the help of a one-dimensional map. We have reproduced the results of the previous work of Wersinger,Finn and Ott [1] and also,found an analytical expression for the map from the original set of equations(2.31).The analytical results are in accordance the numerical results.

Further we establish numerically the presence of a strange attractor(fig.4.18).

BIBLIOGRAPHY

[1] Wersinger,J.-M.,Finn,J.M. and Ott,Edward,Bifurcation and "strange" behavior in instability saturation by nonlinear three-wave mode coupling, Phys. Fluids ,23: 1142-1154,(1980).

[2] Sagdeev,R.Z. and Galeev,A.A.,"Nonlinear Plasma Theory", W.A.Benjamin,Inc.,New York, 1969.

[3] Tsytovich,V.N.,"Nonlinear Effects in Plasma",Plenum Press,New York , 1970.

[4] Stix,T.H.,"The Theory of Plasma Waves",McGraw Hill , New York,1962.

[5] Tsytovich,V.N.,The Theory of Plasma Turbulence",Pergamon Press,Great Britain,1972.

[6] Cullen,A.L.,Proc.I.E.F.,107b:101,(1960).

[7] Louisell,W.H.,"Coupled Mode and Parametric Electronics",J.Wiley & Sons,New York,1960.

[8] Armstrong,J.A.,Bloembergen,N.,Ducuing,J. and Pershan,P.S.,Phys. Rev.,127:1918, (1962).

[9] Bretherton,F.P.,J.Fluid Mech.,20:457,(1964).

[10] McGoldrick,L.,J.Fluid Mech.,21:305,(1965).

[11] Benney,D.J., and Newell,A.C.,J.Math.Phys.,46:133,(1967).

[12] Shiren,N.S.,Proc.I.E.E.E.,53:1540,(1965).

[13] Svaasand,L.O.,Appl. Phys.Lett.,15:300,(1969).

[14] Newhouse,V.L.,Chen,C.L., and Davis,K.L.,J.Appl.Phys.,43:2603,(1972).

[15] Davis,K.L.,and Newhouse,V.L.,I.E.E.E. Trans.Sonics Ultrason.,Sv-22:33,(1975).

[16] Kaup,D.J.,Reiman,A., and Bers,a.,Reviews Mod. Phys.,51:275,(1979).

[17] Nicolis,G.,and Prigogine,I.,"Self-organization in Non-equilibrium Systems", J.Wiley & Sons,New York,1977.

[18] Rabinovich,M.I.,Stochastic self-oscillations and turbulence, Sov.Phys.Usp.,21:443-469,(1978).

[19] Sjolund,A.,and Stenflo,L.,Physica,35:499,(1967).

[20] Hirsch.,M.W.,and Smale,S.,"Differential Equation,Dynamical Systems,and Linear Algebra",Academic Press,New York,1974.

[21] Reyn,J.W.,Classification and description of singular points of a system of three linear differential equation,Z.Agnew,Math.Phys.,15:540-557,(1964).

[22] Bojadziev,G.,and Sattar,M.A.,Perturbations in the three dimensional Kolmogorov Model ,Math.Biosc.,78:293-305,(1986).

[23] Sharkovskii,A.N.,Proc.Fifth Intern.Conf. on Nonlinear Oscillations,2:541(1970).

[24] Kiyashko,S.V.,and Rabinovich,M.I.,Sov.Phys.JETP,39:798,(1974).

[25] Meunier,C., Bussac,M.N., and Laval,G., Physica D,2:236,(1982).

[26] Feigenbaum,M.J.,J.Stat.Phys.,19:25,(1978).

[27] May,R.M.,Simple Mathematical with very complicated dynamics, Nature,261: 456,(1976).

[28] Coullet,P.andTresser,C.,Iterations d'endomorphismes et groupe de renormalization, J.de Phys.Colloq.,C5:5-25,(1978).

[29] Tritton,D.J.,"Physical Fluid Dynamics",Van Norstrand-Reinhold",New York,1977.

[30] Pomeau,Y.and Manneville,P.,Intermittent transition to turbulence in dissipative dynamical systems,Commun.Math.Phys.,74:189-197,(1980).

[31] Eckmann,J.P.,Rev.Mod.Phys.,53:643-654,(1981).

[32] Melnikov,V.K.,Trans.Mosc.Soc.,12:1-56,(1963).

[33] Bussac,M.N.,Physica Scripta , T2:110-118,(1982).

[34] Whittaker,E.T. and Watson,G.N.,"A Course of Modern Analysis",Cambridge University Press,London,(1978).