DYNAMICAL CHIRAL SYMMETRY BREAKING IN 2+1 DIMENSIONAL QED

by

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B.Sc., University of Science and Technology of China, 1984

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Physics

SIMON FRASER UNIVERSITY

March 1988

ISBN 0-315-66197-6

APPROVAL

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Degree: Master of Science

Title of thesis: Dynamical Chiral Symmetry Breaking in 2+1 Dimensional QED

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Dynamical Chiral Symmetry Breaking in 2+1

Dimensional QED

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April 14, 1988

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ABSTRACT

Dynamical chiral symmetry breaking in 2+1 dimensional quantum electrodynamics with N fermion flavors is studied by using a modified effective potential proposed recently by Haymaker, Matsuki and Cooper. This effective potential contains the same physics as the original effective potential formulation due to Cornwall, Jackiw and Tomboulis as far as the Schwinger-Dyson equation is concerned and remedies the defects of the CJT effective potential.

The chiral symmetry breaking solutions are found by solving the Schwinger-Dyson equation numerically for $N = 0.5, 1.0, 1.2, ..., 2.8$. It is suggested that the chiral symmetry breaking solutions exist for any finite value of N.

The local stability of the vacuum configurations corresponding to the chiral symmetry breaking solutions is analyzed both analytically and numerically. It is shown that the chirally symmetric vacuum may be unstable and that the symmetry-breaking solutions correspond to the locally stable vacuum configurations and are then preferred energetically by the vacuum of this model. For comparison it is also shown that the same symmetry-breaking solutions are the locally unstable points, namely, saddle points in the CJT formalism.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to Dr. K.S. Viswanathan for suggesting this topic and helping me go through the study to its completion. He spent a lot of time helping me to understand the subject and the theories related to it. He read through my thesis and made helpful suggestions and corrections.

I am very grateful to Dr. T. Matsuki for having taken time to patiently explain to me things that I didn't understand. In addition, I wish to thank my supervisory committee for its help in my graduate studies.

The financial support provided by Simon Fraser University and the Natural Sciences and Engineering Research Council is also gratefully acknowledged.

DEDICATION

To My Parents

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CHAPTER 1

INTRODUCTION

Systems of fermions coupled by gauge forces have a very rich structure of global symmetries called "chiral symmetries". The realization of chiral symmetries and the causes and the consequences of their spontaneous breaking have always been a very important and interesting research area in high-energy particle physics.

In the theories of fundamental interactions, the quark model is widely accepted. The basic ideas of the quark model are that quarks are the fundamental constituents of all hadrons and that all the baryons consist of three quarks while all the mesons are formed by quark and antiquark. Quarks carry colour "charges" as well as electric charges and experience both strong and electromagnetic forces, as well as the more feeble, weak and gravitational interactions. As far as the strong interaction is concerned, the colour "charges" of the quarks act as the source of the strong force between quarks just as electric charge acts as source of the electromagnetic force between electrically charged particles.

There is considerable evidence that the underlying theory of the strong interaction possesses a near chiral symmetry, $SU(2)\times SU(2)\times U(1)$ because of the approximate masslessness of the up and down quarks. This symmetry must then break spontaneously in order to explain the effective 300 MeV masses these quarks possess as the constituents of hadrons. The Goldstone theorem requires that the spontaneous breaking of any continuous symmetry necessarily leads to the existence of massless Goldstone bosons. In the case of chiral symmetry it is believed that the pions play the role of Goldstone bosons. It is a major goal of particle physicists to understand why this chiral symmetry should be spontaneously broken and in what pattern.

The dynamical treatment of the chiral symmetry of the strong interaction was first brought up by Nambu and Jona-Lasinio ([1]). They suggested that the nucleon mass arises largely as self-energy of some primary fermion field through the same mechanism as the appearance of the energy gap in the theory of superconductivity. Much of the progress of theoretical particle physics in the 1960's occured through exploration of the phenomenological consequences of this spontaneous chiral symmetry breaking ([2] [3]). Our understanding of the underlying mechanism of chiral symmetry breaking, however, has not advanced very much.

In the 1970's, the great success of electromagnetic and weak interaction gauge theories proved that gauge theories (Yang and Mills [4]) are powerful theories in describing the fundamental interactions. The gauge theory of strong interactions is Quantum Chromodynamics (QCD) which is based on three basic ideas:

(1). All hadrons consist of fundamental constituents called quarks;

(2). There exists a quantum number for the quarks called "colour". Each quark can have three different colours, called red, green, and blue and the corresponding symmetry is the exact symmetry of nature (in other words, it is an unbroken symmetry.). The statistics of the hadrons come out right with these three colours;

(3). The transformations of the "colour" symmetry may be position dependent. This fundamental concept of Yang-Mills gauge theories introduces a set of massless vector fields which couple to fermions (quarks). In fact, the only choice of the colour symmetry group is SU(3)---special unitary group formed by 3×3 unitary matrices with determinant 1.

Reflecting the success of gauge theories, a renewed interest in the study of the fermionic symmetries has been seen during these past few years for three reasons. First, numerical treatment of the strong interaction gauge theories, especially in their lattice formulation, has been approaching a quantitative calculation of the hadron spectrum. There is, then, a need for physical ideas about quark dynamics to match these numerical calculations. Secondly, the gauge theoretic descriptions of the weak interactions have focused attention on the problem of explaining the quark and lepton spectra. From the perspective of the gauge theories, the quark and lepton masses are simply the parameters of chiral symmetry breaking in the interactions which determine the structures of these particles. Dynamical theories of the fermion mass matrix thus require an understanding of chiral symmetries in systems different from the usual strong interactions; such theories often require that chiral symmetry is realized in an unfamiliar way. Finally, the viewpoint provided by gauge theories has led to some striking qualitative conclusions about chiral symmetry which might form the basis of a more detailed theory.

To get a good understanding of the problem of this fermionic symmetry, it is necessary to describe some basic elements of the physics concerning this problem.

Chiral symmetries are normally introduced as formal symmetries of the massless theory of Dirac particles (fermions). The Lagrangian of this theory in four dimensional space-time including the interaction between fermions and an Abelian gauge field (i.e the group transformations related to the gauge field commute with each other) takes the form

$$L = \overline{\Psi}\, i\gamma^{\mu}(\partial_{\mu} - i g A_{\mu})\Psi \quad ; \qquad \overline{\Psi} = \Psi^{+}\gamma^{0} \tag{1.1}$$

where the Dirac field $\Psi(x)$ represents the spin 1/2 fermions and has four components and γ^{μ} ($\mu=0, 1, 2, 3$) is a set of 4×4 matrices satisfying the following algebra

$$\{\gamma^{\mu}\ ,\ \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\, g^{\mu\nu} \tag{1.2}$$

The metric tensor of the four dimensional space-time $g^{\mu\nu}$ is, in our convention, $(g^{\mu\nu}) = \mathrm{diag}\,(+1, -1, -1, -1)$.

There is an obvious symmetry of the Lagrangian

$$\Psi \rightarrow \Psi' = \exp[\,i\,\alpha\,]\,\Psi \qquad\qquad \overline{\Psi} \rightarrow \overline{\Psi}' = \overline{\Psi}\exp[\,i\,\alpha\,] \tag{1.3}$$

which corresponds to fermion-number conservation.

But the massless particle theory has another symmetry, using γ^5:

$$\Psi \to \Psi' = \exp[i\alpha\gamma^5]\Psi \quad , \quad \overline{\Psi} \to \overline{\Psi}' = \overline{\Psi}\exp[i\alpha\gamma^5] \quad , \tag{1.4}$$

The exponentials cancel because the matrix γ^5 anticommutes with the other four gamma matrices, i.e.

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{1.5}$$

To understand these symmetries physically, let us choose the following representation of the Dirac matrices:

$$\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad , \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad (i = 1, 2, 3)$$
$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \quad , \quad \alpha^i = \gamma^0\gamma^i = \begin{bmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix} \tag{1.6}$$

where σ^i (i=1,2,3) are 2×2 Pauli matrices.

In this representation, the Dirac Hamiltonian is

$$H = \int d^3x\, \Psi^+(x)\,[\vec{\alpha}\cdot(\vec{p} + g\vec{A}) - gA^0]\,\Psi(x) \tag{1.7}$$

If we write

$$\Psi = \begin{bmatrix} \Psi_L \\ \Psi_R \end{bmatrix} \tag{1.8}$$

Then,

$$H = \int d^3x\, \{\Psi_R^+[\vec{\sigma}\cdot(\vec{p} + g\vec{A}) - gA^0]\,\Psi_R + \Psi_L^+[(-\vec{\sigma})\cdot(\vec{p} + g\vec{A}) - gA^0]\,\Psi_L\} \tag{1.9}$$

where Ψ_R and Ψ_L describe, respectively, right and left-handed massless fermions because of the fact that

$$\gamma^5 \Psi_R = \Psi_R$$

$$\gamma^5 \Psi_L = -\Psi_L \tag{1.10}$$

It is easily seen then that the fermion numbers of Ψ_R and Ψ_L are (formally) separately conserved. In fact, this is the origin of the extra γ^5 symmetry.

The two pieces of (1.9) are not actually of different form. We can write Ψ_R as a second form of Ψ_L by applying charge conjugation

$$\Psi_{L2}(x) = \sigma^2 \Psi^*_R(x) \tag{1.11}$$

the first term of (1.9) becomes

$$\int d^3x\, \Psi^+_R [\vec{\sigma} \cdot (\vec{p} + g\vec{A}) - gA^0]\, \Psi_R = \int d^3x\, \Psi^+_{L2} [(-\vec{\sigma}) \cdot (\vec{p} - g\vec{A}) + gA^0]\, \Psi_{L2} \tag{1.12}$$

This is nothing else but a Ψ_L Hamiltonian with the opposite sign of charge g.

This construction is readily generalized to non-Abelian gauge theories. In the non-Abelian case, the Lagrangian is built as

$$L = \bar{\Psi} i \gamma^\mu D_\mu \Psi = \bar{\Psi}^a i\gamma^\mu (D_\mu)_{ab} \Psi^b \tag{1.13}$$

where a and b are "colour" indices.

The covariant derivative D_μ is defined as

$$D_\mu = \partial_\mu - i g A^a_\mu t^a_r \tag{1.14}$$

where the index a runs over the generators of the gauge group and the matrices t^a_r represent these generators in the representation r of the gauge group to which the fermions are assigned.

Representation matrices for the complex conjugate representation $\bar{r}$ are related by

$$t^a_{\bar{r}} = -(t^a_r)^* = (t^a_r)^T \tag{1.15}$$

where T stands for the transpose operation.

This notation allows us to recast the Hamiltonian for a Ψ_R as that of Ψ_L in the complex conjugate representation $\bar{r}$:

$$\int d^3x\, \Psi_R^+ [\vec{\sigma}\cdot(\vec{p} + g\vec{A}\cdot t) - gA^0\cdot t]\, \Psi_R = \int d^3x\, \Psi_{L2}^+ [(-\vec{\sigma})\cdot(\vec{p} + g\vec{A}\cdot(-t^*) + gA^0\cdot t^*]\, \Psi_{L2} \tag{1.16}$$

In this notation, the most general Hamiltonian coupling to gauge fields may be written compactly in the following form

$$H = \sum_{\text{rep.}r}\sum_{i=1}^{n_r} \int d^3x\, \Psi_{Lri}^+ [(-\vec{\sigma})\cdot(\vec{p} + g\vec{A}\cdot t_r) - gA^0\cdot t_r]\, \Psi_{Lri} \tag{1.17}$$

where the index i refers to "flavour" and corresponds to observed degrees of freedom of existing hadrons. At present six flavours are known. The gauge theory of strong interaction is diagonal in flavour index, i.e. the flavour index plays no dynamical role here.

Once H has been cast into this form, it is easy to read off the global symmetries of this system concerning "flavours": for each representation r, this Hamiltonian is (formally) invariant under general unitary transformations

$$\Psi_{Lri} \to U_{ij}\Psi_{Lrj} \tag{1.18}$$

The full global symmetry group is, therefore,

$$G = \left[\prod_r U(n_r)\right] / U(1) \qquad (1.19)$$

G is called the group of chiral symmtries of such a theory.

As an example of this notation, consider the case of the strong interactions which are described by a set of two almost massless Dirac fermions (quarks) coupled in the triplet (corresponding to three colours) representation to an non-Abelian gauge group SU(3). These almost massless fermions may be written as left-handed fermions, two in the 3 and two in the 3 representations of the colour group SU(3). In the limit of zero quark masses, the chiral symmetry of this theory is $SU(2)_L \times SU(2)_R \times U(1)$.

It is believed that the full group $G=SU(2)_L \times SU(2)_R \times U(1)$ is a symmetry of the strong interactions; however, hadrons do not form the multiplets classified by G in the real world but only by $SU(2) \times U(1)$ (isospin and baryon number). A part of G must, then, be spontaneously broken. Although the elucidation of this mechanism is, to a great extent, still an open problem in the theory of strong interactions, we present a rather simple intuitive argument due to Nambu and Jona-Lasinio ([1]). Its basic idea is that the condensation of fermion-antifermion pairs in the vacuum state of the theory causes this chiral symmetry to break down. The gauge coupling of the colour group SU(3) becomes arbitrarily large in the infrared regime. Let us observe the change in the structure of the vacuum state of this theory as the coupling g is increased from zero. Imagine that we can integrate over the quantum fluctuations of the gauge field; then H takes the form

$$H = H_d + H_{o\text{-}d} \tag{1.20}$$

where H_d is diagonal in the number of quark-antiquark pairs and $H_{0\text{-}d}$ changes the number of such pairs. $H_{0\text{-}d}$ is of order of g^2 and is a small perturbation when g is small. In this regime it makes sense to approximate H by H_d. Diagonalizing H_d yields a ground state close to the free field vacuum. Now, slowly increase g. If the fermions have zero mass and experience attractive interactions, H_d decreases as g increases. $H_{0\text{-}d}$, of course, increases. At some value of g it becomes appropriate to treat $H_{0\text{-}d}$ as our zeroth order problem and H_d as a perturbation. But $H_{0\text{-}d}$ changes the number of quark-antiquark pairs, so its ground state has an indefinite number of fermion pairs. We would still expect the ground state to be invariant under Lorentz transformations; hence these pairs must have vacuum quantum numbers---zero total momentum and angular momentum. The only pairs one can form from 3 and 3 left-handed fermions and their (right-handed) anti-particles which satisfy this condition are those of the form of Fig.(1) and the corresponding pairs of anti-fermions.

Figure 1. Fermion bilinear with vacuum quantum numbers.

The pair shown in Fig.(1) carries a net charge under the transformations:

$$\Psi_{L3i} \rightarrow \exp[i\alpha]\Psi_{L3i} \qquad \Psi_{L\bar{3}i} \rightarrow \exp[-i\alpha]\Psi_{L\bar{3}i}$$

$$\Psi_{L3i} \rightarrow U_{ij}\Psi_{L3j} \qquad \Psi_{L\bar{3}i} \rightarrow V_{ij}\Psi_{L\bar{3}j} \tag{1.21}$$

(The indicies i,j=1,2,3, are isospin labels.)

The presence of an indefinite number of such pairs in the vacuum breaks these symmetries. More formally, we have found that the ground state $|\Omega>$ of H has the property that an operator which destroys a fermion pair has a non-zero vacuum expectation value.

Let us assume that $|\Omega>$ gives the pair annihilation operator the rather simple expectation value:

$$< \Omega \,|\Psi_{L3i}\Psi_{L\bar{3}j}\,|\Omega > = \Delta\,\delta_{ij} \tag{1.22}$$

(where $\Delta \neq 0$ corresponds to equal condensation of pairs of each isospin). This expression is preserved by the transformations:

$$\Psi_{L3i} \rightarrow \exp[i\alpha]\Psi_{L3i} \qquad \Psi_{L\bar{3}i} \rightarrow \exp[-i\alpha]\Psi_{L\bar{3}i}$$

$$\Psi_{L3i} \rightarrow U_{ij}\Psi_{L3j} \qquad \Psi_{L\bar{3}i} \rightarrow \Psi_{L\bar{3}j}U^{-1}_{ij} \tag{1.23}$$

These transformations are those of an $SU(2) \times U(1)$ group of unbroken symmetries which corresponds precisely to isospin and baryon number. The remaining three symmetry directions of $SU(2) \times SU(2) \times U(1)$ must be spontaneously broken symmetries.

This chiral symmetry breaking is easily realized by adding a quark mass term to the Lagrangian. This mass term takes the form

$$L_M = m\,\overline{\Psi}(x)\,\Psi(x) \qquad (1.24)$$

Obviously this term is not γ^5 invariant. It can be put in by hand or generated dynamically, and the latter is what we are interested in.

The mass generation can be thought of as result of the chiral symmetry breaking. An intuitive argument could give us a simple picture of that ([15]). We have noted that there is a condensation of fermion and antifermion pairs in the vacuum. Now let us consider a bound state a massless quark and antiquark pair. Because of the uncertainty principle, the energy of the ground state will be given by $E_\pi \approx p - g^2/r \approx p\,(1-g^2)$ where p and r denote the relative momentum and coordinate, respectively. In a fully relativistic formulation, this relation may be replaced by $E_\pi{}^2 \approx p^2 - g^2/r^2 \approx p^2(1-g^2)$. When the gauge coupling g exceeds order one, there will be a tachyon bound state in the vacuum, indicating instability of the vacuum configuration. In order to cure this instability, the vacuum rearranges itself and gives mass to quarks so as to eliminate the tachyons and keep the bound state massless.

The spontaneous chiral symmetry breaking and mass generation are purely non-perturbative phenomena which cannot be easily seen in the usual perturbation series. What is needed is an approximation scheme that preserves some of the non-linear features of the field theory, which presumably leads to these cooperative and coherent effects. The effective potential formalism due to Cornwall, Jackiw and Tomboulis ([5]) has been developed to serve this role. We will do a self-consistent study of dynamical chiral symmetry breaking using an improved effective potential which is a variant of the Cornwall, Jackiw and Tomboulis (CJT) effective potential.

In this thesis, we are concerned with a gauge field theory in which the problem of dynamical chiral symmetry breaking can be systematically analyzed. This model is Quantum Electrodynamics in three dimensional space-time. In this model, N (fermion flavour number) fermions couple to a simple Abelian gauge field.

We have good reasons for choosing this model even though it is not QCD and not even four dimensional. This theory is actually a genuine gauge field theory and the first quantum field theory we know of, above two space-time dimensions that permits a systematic treatment of chiral symmetry breaking. This model, furthermore, has properties reminisent of four-dimensional theories, which we will see in the following chapters. Another reason is that four-dimensional realistic physical theories and three-dimensional theories are not unrelated. In fact, when one examines a four-dimensional field theory at finite temperature, one will find that at a temperature near the critical temperature a three-dimensional field theory becomes effective in the description of the dynamics near the phase transition. Moreover, at high temperatures---even away from the critical point, the infrared behaviour of any theory is described by the same theory in one less dimension ([6]). We certainly hope that the study of the chiral symmetry breaking in this model will shed some light on the more complicated problem of the chiral symmetry breaking in the four-dimensional theories like QCD.

The major tasks in studying the chiral symmetry breaking are to establish the Schwinger-Dyson equation for the fermion self-energy, which takes into account the non-perturbative features of the theory and then to investigate whether this equation admits a non-zero fermion mass as a solution. Along this line some work has been done in the past few years.

Pisarski ([7]) studied this model in the limit of large N (fermion flavour number). He derived the Schwinger-Dyson integral equation (S-D equation) of the fermion self-energy $\Sigma(p)$ and computed the ratio $\Sigma(0)/\alpha$ (α is a intrinsic energy scale of this particular model) by assuming that $\Sigma(p)$ was a constant (w.r.t. p) and cutting the integral off at a momentum of order α. And for large N, he obtained a solution for the fermion mass, $\Sigma(0)/\alpha = c \exp[-\pi^2 N/8]$, where c is of order 1. He then concluded that chiral symmetry breaking solutions exist.

T. Appelquist, M. Bowick, E. Cohler and L.C.R. Wijewardhana also did detailed analysis of chiral symmetry breaking within the framework of the 1/N expansion. They set up the S-D equation of the fermion self-energy by effectively summing over a selective set of terms in the 1/N expansion. They analyzed this model and their results show that chiral symmetry in this model can be broken for any value of N. They also suggested that it should be preferable for the theory to dynamically generate masses for fermions. The magnitude of the generated mass is roughly exponentially suppressed in N from the fundamental dimensionful scale $\alpha=Ne^2$ (e is the gauge coupling of this theory whose square has the dimension of mass).

We will start from the effective potentials and adopt the same 1/N expansion as in Ref.[7] to derive the S-D equation for the fermion self-energy. We will solve this equation numerically and show that the dynamical chiral symmetry breaking may be possible for any value of the fermion flavour number.

Following the study of the possibility of the chiral symmetry breaking, an important question of whether the vacuum configurations corresponding to the non-zero solutions are stable or not would naturally arise. We will do the stability analysis which has not yet been done so far.

The CJT effective potential is, however, unbounded from below, and hence the study of the global stability of a chiral symmetry breaking solution of the S-D equation in this formulation becomes meaningless. Recently an improved effective potential ([9]) was proposed to remedy the defect of the CJT effective potential. This modified effective potential gives us the same S-D equation as the CJT effective potential does.

However, its second derivative can be shown to be a mass squared of the composite fermion-antifermion bound state. This explanation helps us understand the stability problem. That is, if the mass squared is positive, symmetry is already broken and the corresponding vacuum configuration is stable; if negative, this symmetry broken vacuum then is unstable. Hence checking the second derivative of this improved effective potential answers the question of stability.

We will evaluate the improved effective potential for this particular model to examine the stability of the chiral symmetry breaking vacuum, namely, the eigenvalue equations for the second derivative operator of this effective potential will be solved at the stationary points, e.g. chiral symmetry breaking solutions. Both the analytical analysis and numerical evidence will show that the chiral symmetry breaking solutions are at locally stable points. For comparison, similar calculations will be done for the CJT potential and will show that the chiral symmetry breaking solutions correspond to the saddle points of this potential.

CHAPTER 2

CHIRAL SYMMETRY IN QED_3 AND THE EFFECTIVE POTENTIALS

§ 2.1 The Model and its Chiral Symmetries:

In this section we will define the model to be studied and discuss its chiral symmetries and other properties and the use of the 1/N expansion in analyzing chiral symmetry breaking.

The Lagrangian for massless quantum electrodynamics in three space-time dimensions (QED_3) is

$$L = -\frac{1}{4}F_{\mu\nu}^2 + \sum_{i=1}^{N} \overline{\Psi}_i i\gamma^\mu (\partial_\mu - ieA_\mu)\Psi_i \qquad (2.1)$$

where Ψ_i represents massless fermions with flavour index i, A_μ is the electromagnetic field (Abelian gauge field), e is the gauge coupling describing the interaction strength between fermions and the gauge field and

$$-\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad (2.2)$$

is the Lagrangian of the Abelian gauge field.

Chiral symmetries of this model are a bit unusual. A spinorial representation of the Lorentz group SO(2,1) in three dimensions is provided by two-component Dirac spinors Ψ_i, with the corresponding 2×2 representation of the Dirac algebra being given by the Pauli matrices

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^2 \qquad (2.3)$$

Obviously, this theory which we refer to as a massless theory has the flavour symmetry U(N) because the Lagrangian is invariant under the unitary transformations:

$$\Psi_i \rightarrow U_{ij}\Psi_j \qquad (2.4)$$

Considering the change of $\Psi(x)$ under the unitary transformation

$$\overline{\Psi}_i \rightarrow \overline{\Psi}_i' = (\Psi_i^+)'\gamma^0 = (U_{ij}\Psi_j)^+\gamma^0 = \Psi_j^+\gamma^0 U_{ji}^+ = \overline{\Psi}_j U_{ji}^+ \qquad (2.5)$$

we see that

$$L \rightarrow L' = -\frac{1}{4}F_{\mu\nu}^2 + \sum_{i=1}^{N}\overline{\Psi}_i' i\gamma^\mu(\partial_\mu - ieA_\mu)\Psi_i'$$

$$= -\frac{1}{4}F_{\mu\nu}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}\overline{\Psi}_i U_{ij}^+ i\gamma^\mu(\partial_\mu - ieA_\mu)U_{jj'}\Psi_{j'}$$

$$= -\frac{1}{4}F_{\mu\nu}^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}\overline{\Psi}_i i\gamma^\mu(\partial_\mu - ieA_\mu)U_{ij}^{-1}U_{jj'}\Psi_{j'}$$

$$= L$$

For two-component spinors, however, the flavour symmetry is the same whether the fermions are massless or not, and so it is not the chrial symmetry. In fact, there is no other 2×2 matrix anticommuting with all of the γ^μ. There is, therefore, nothing to generate a chiral symmetry that would be broken by a mass term $m\overline{\Psi}\Psi$, whether it is explicitly or dynamically generated.

Consider therefore the basic fermion field to be a four-component spinor. The three 4×4 γ-matrices can be taken to be

$$\gamma^0 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \qquad \gamma^1 = \begin{bmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{bmatrix} \qquad \gamma^2 = \begin{bmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{bmatrix} \qquad (2.6)$$

In contrast to two-component spinors, four-component spinors have symmetries which are chiral. Massless fermions then have a greater symmetry than massive ones. Simply put, there are two 4×4 matrices γ^3 and γ^5 that anticommute with γ^0, γ^1 and γ^2. The massless theory will be invariant under the "chiral" transformations:

$$\Psi \rightarrow \exp[\, i\alpha\gamma_3\,]\,\Psi \tag{2.7a}$$

$$\Psi \rightarrow \exp[\, i\beta\gamma_5\,]\,\Psi \tag{2.7b}$$

where

$$\gamma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \gamma^5 = i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.8}$$

For each four-component spinor, these will be a global symmetry U(2) with generators

$$1, \quad \gamma^3, \quad \gamma^5, \quad \text{and} \quad [\gamma^3, \gamma^5] \tag{2.9}$$

and the full symmetry is then U(2N). The algebra of U(2N) is the direct product of the algebra of U(N) and that of U(2). A mass term $m\overline{\Psi}\Psi$ would break this symmetry to the subgroup $SU(N)\times SU(N)\times U(1)\times U(1)$.

This would be understood more easily if we discuss this symmetry using two-component spinors.

Choosing four-component spinors actually doubles the fermion species. Essentially, we have 2N varieties of two-component spinors. Without fermion mass, we would have a flavour symmetry U(2N). A four-component spinor mass term $m\overline{\Psi}\Psi$ can be written in terms of two-component spinors. Writing

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \qquad (2.10)$$

(Ψ_1 and Ψ_2 are two-component spinors.), $m\bar{\Psi}\Psi$ becomes

$$m\bar{\Psi}\Psi = m\Psi_1^+\sigma_3\Psi_1 - m\Psi_2^+\sigma_3\Psi_2 = m\bar{\Psi}_1\Psi_1 + (-m)\bar{\Psi}_2\Psi_2 \qquad (2.11)$$

It is then easily seen that this is just the situation in which among 2N fermions N fermions have equal positive mass and the other N fermions have equal negative mass. The flavour symmetry with this mass term present is, therefore, $U(N)\times U(N)$, or more specifically $SU(N)\times SU(N)\times U(1)\times U(1)$.

Although the four-component mass term violates the chiral symmetries, it is parity conserving. The two-component mass term is, however, odd under the parity transformation. In 2+1 dimensions, the parity transformation corresponds to inverting one axis, since inversion of both axes could be undone by a π-rotation. Thus

$$P: \quad (x,y) \rightarrow (x,y)_p = (-x,y) \qquad (2.12)$$

The corresponding operation on the two-component spinor is

$$P\Psi(\vec{x},t)P^{-1} = \sigma^1\Psi(\vec{x}_p,t) \qquad (2.13)$$

The parity operation on the two-component mass term is then,

$$P[\,m\overline{\Psi}(\vec{x},t)\,\Psi(\vec{x},t)\,]\,P^{-1} = m\,P\,\overline{\Psi}(\vec{x},t)\,P^{-1}\;P\Psi(\vec{x},t)P^{-1}$$

$$= m\,P\Psi^{+}(\vec{x},t)P^{-1}\,\sigma^3\,P\Psi(\vec{x},t)P^{-1} = m\,\Psi^{+}(\vec{x}_p,t)\,\sigma^1\,\sigma^3\,\sigma^1\,\Psi(\vec{x}_p,t)$$

$$= -\,m\,\Psi^{+}(\vec{x}_p,t)\,\sigma^3\,\Psi(\vec{x}_p,t) = -\,m\,\overline{\Psi}(\vec{x}_p,t)\,\Psi(\vec{x}_p,t) \qquad (2.14)$$

This shows that the two-component mass term is parity violating. In the four-component formalism,

$$\Psi(\vec{x},t) \equiv \begin{bmatrix} \Psi_1(\vec{x},t) \\ \Psi_2(\vec{x},t) \end{bmatrix} \qquad (2.15)$$

The parity transformation becomes

$$\begin{aligned} P: \qquad & \Psi_1(\vec{x},t) \rightarrow \sigma^1\,\Psi_2(\vec{x}_p,t) \\ & \Psi_2(\vec{x},t) \rightarrow \sigma^1\,\Psi_1(\vec{x}_p,t) \end{aligned} \qquad (2.16)$$

The four-component mass term

$$m\,\overline{\Psi}\,\Psi = m\,\Psi_1^{+}\,\sigma^3\,\Psi_1 - m\,\Psi_2^{+}\,\sigma^3\,\Psi_2 \qquad (2.17)$$

transforms in the following way under the parity transformation:

$$P[\,m\overline{\Psi}(\vec{x},t)\Psi(\vec{x},t)\,]\,P^{-1} = P[\,m\Psi_1^{+}(\vec{x},t)\sigma^3\,\Psi_1(\vec{x},t) - m\Psi_2^{+}(\vec{x},t)\sigma^3\,\Psi_2(\vec{x},t)\,]\,P^{-1}$$

$$= m\Psi_2^{+}(\vec{x}_p,t)\sigma^1\,\sigma^3\,\sigma^1\Psi_2(\vec{x}_p,t) - m\Psi_1^{+}(\vec{x}_p,t)\,\sigma^1\,\sigma^3\,\sigma^1\,\Psi_1(\vec{x}_p,t)$$

$$= m\overline{\Psi}_1(\vec{x}_p,t)\Psi_1(\vec{x}_p,t) - m\overline{\Psi}_2(\vec{x}_p,t)\Psi_2(\vec{x}_p,t) = m\overline{\Psi}(\vec{x}_p,t)\Psi(\vec{x}_p,t) \qquad (2.18)$$

Therefore, this mass term is even under the parity transformation.

This point is being stressed because there is an alternative possibility in three-dimensions. Another acceptable candidate for a mass term is

$$m\bar{\Psi}\frac{1}{2}[\gamma^3, \gamma^5]\Psi = m\Psi_1^+\sigma^3\Psi_1 + m\Psi_2^+\sigma^3\Psi_2 \tag{2.19}$$

This term is invariant under the chiral transformations (2.7) but not invariant under the parity transformation (2.16). Such a parity-violating mass is in fact the only possibility in the two-component formalism. It is known that it will induce a Chern-Simons mass term for the gauge field via one-loop vacuum polarization ([9]). That such a fermion mass and the corresponding Chern-Simons mass could arise spontaneously, leading to the spontaneous violation of parity in QED_3, is an interesting and important possibility ([10]). In this thesis, however, attention will be restricted to the possible spontaneous appearance of a parity-conserving chiral-symmetry violating mass.

We now turn to the perturbative properties of this theory and introduce the 1/N expansion. From the Lagrangian (2.1), it is easily seen that the gauge coupling constant e is dimensionful and its square has the dimension of mass. This theory then is completely ultraviolet (large-momentum regime) finite.

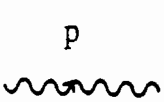
Since the coupling constant ($\alpha=Ne^2$) has dimension of mass this massless theory is plagued with infrared divergences. The effective loop expansion parameter is α/k, leading to infrared divergent Green's functions already at the two-loop level ([5],[11]). One scheme which leads to infrared finite results is an expansion in the dimensionless parameter 1/N with the coupling constant α fixed---1/N expansion. In this non-perturbative scheme it can be shown that this theory stays infrared finite to any order in 1/N ([11]). Whether this finite theory admits the spontaneous chiral symmetry breaking is the central problem of this thesis.

Now we pay some attention to the 1/N expansion. Each order in the 1/N expansion sums an infinite class of Feynman graphs which in turn leads to infrared finite amplitudes at the next level of approximation. For example, to leading (zeroth) order in 1/N, only those graphs are included which contain one closed fermion loop for every additional coupling factor of α/N. The only possibility is the correction to the gauge boson propagator shown in Fig.2.

Figure.2: The leading correction to the photon propagator in the 1/N expansion.

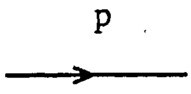
The Feynman rules can be read off from the Lagrangian (2.1) and all computations are performed in the Landau gauge.

Bare gauge boson propagator:



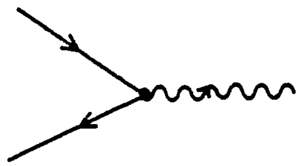
$$= i D_{\mu\nu}(p) = -i \frac{g_{\mu\nu} - p_\mu p_\nu / p^2}{p^2} \tag{2.20a}$$

Bare fermion propagator:



$$= i S_f(p) = \frac{i}{\not p} \qquad (\not p = p_\mu \gamma^\mu) \tag{2.20b}$$

Bare vertex:



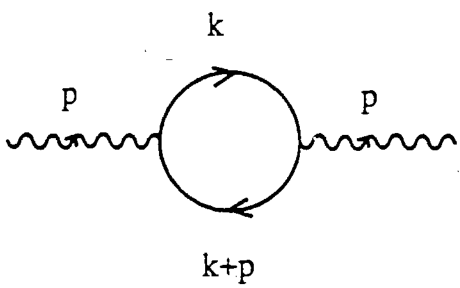
$$= i e \gamma^\mu \tag{2.20C}$$

Let us denote the corrected gauge boson propagator by $iD'_{\mu\nu}(p)$. The series expansion shown in Fig.2 can then be written down in a compact form:

$$i D'_{\mu\nu}(p) = i D_{\mu\nu}(p) + i D_{\mu\nu}(p)\,(i \Pi_{\mu\nu}(p))\,(i D'_{\mu\nu}(p)) \tag{2.21}$$

where $i\Pi_{\mu\nu}(p)$ is given by the closed fermion loop

$$i \Pi_{\mu\nu}(p) =$$



$$= -(-ie)^2 N \int \frac{d^3 k}{(2\pi)^3} \mathrm{Tr}\left[\frac{1}{\not k} \gamma^\mu \frac{1}{\not k + \not p} \gamma^\nu \right] \tag{2.22}$$

A straightforward computation gives the corrected gauge boson propagator (see appendix:A)

$$D'_{\mu\nu}(p) = -\frac{g_{\mu\nu} - p_\mu p_\nu}{p^2 [1 + \pi(p)]} \tag{2.23}$$

where $\pi(p)$ is given by

$$\pi(p) = \alpha / 8p \tag{2.24}$$

The Euclidean momentum is represented by p. To the leading order, it is not necessary to consider the fermion self-energy and vertex corrections since these enter only at next level in 1/N expansion.

The propeties of QED_3 make it clear that spontaneous chiral symmetry breakdown will not take place to any finite order in the 1/N expansion. To investigate this non-perturbative phenomenon, therefore, we have to go beyond the finite orders. In doing this, we will derive the effective potentials within the framework of 1/N expansion, to give the Schwinger-Dyson equation which sums a selective set of terms in the 1/N expansion. This is the major task of the following section.

§2.2 The Effective Potential Formalism

The dynamical generation of fermion masses in gauge theories can be studied in continuum space-time using the effective potential formalisms. These techniques lead to systematic resummation of graphs which is capable of describing non-perturbative phenomena such as chiral symmetry breaking in a systematic sequence of approximations.

The effective potential $V(\varphi)$ for an elementary field $\Phi(x)$---a possible vacuum expectation value of the quantum field $\Phi(x)$---has a simple interpretation as the energy density subject to the constraint that $\Phi(x)$ has some definite vacuum expectation value $\varphi(x)$. One can compute this effective potential and minimize it with respect to $\varphi(x)$ to determine the vacuum value of the field $\Phi(x)$ ([12]). In our study of chiral symmetry breaking, we need to know how to test whether the energy of the vacuum is lowered if the fermion bilinear $\Psi(x)\overline{\Psi}(y)$ acquires a nonzero vacuum expectation value. An effective potential formalism similar to the effective potential for single elementary field was proposed by Cornwall, Jackow and Tombulis ([4]). This effective potential depends on $S(x,y)$---a possible vacuum expectation value of the composite field operators such as fermion bilinear. Physical solutions require

$$\frac{\delta V(S)}{\delta S} = 0 \qquad (2.25)$$

Hence this formalism is especially appropriate for the study of dynamical chiral-symmetry breaking, which is characterized by the fact that non-zero solutions could exist for (2.25). This object has a simple interpretation only at the stationary points (referred to (2.25)) where it equals to the vacuum energy density.

Now we go through the sketch of the CJT (referring to Cornwall, Jackiw, Tomboulis) effective potential for fermi fields. To produce a vacuum expectation value of a fermion bilinear operator, we must, in principle, turn on some external field (analogous to a magnetic field orienting a potentially ferromagnetic system), construct the ordered vacuum in the presence of this external field, and then see if the order in this vacuum survives when we turn off the external field.

In doing this, one introduces sources coupled to the fermion bilinear $\Psi(x)\overline{\Psi}(y)$, Legendre transform the vacuum energy functional to an effective action which can be given as a two-particle irreducible loop expansion in terms of the full fermion propagator $S(x,y)$---a vacuum expectation value of the fermion bilinear. A graph is said to be two-particle irreducible if it doesn't become disconnected upon opening two lines.

Consider the generating functional of the theory governed by a Lagrangian $L(x)$ such as (2.1)

$$e^{iW[J]} = \int [d\Psi\, d\overline{\Psi}\, dA_\mu]\, e^{i(L - \overline{\Psi} J \Psi)} \tag{2.26}$$

where the exponential is a supressed form

$$L - \overline{\Psi} J \Psi = \int (dx)\, L(x) - \int (dx)\,(dy)\, \overline{\Psi}_\alpha(x)\, J_{\alpha\beta}(x,y)\, \Psi_\beta(y) \tag{2.27a}$$

and in three-dimensional space-time

$$\int (dx) = \int d^3x \tag{2.27b}$$

and $J_{\alpha\beta}(x,y)$ is an arbitrary bilocal matrix function. We will omit the spinor indices α and β later on.

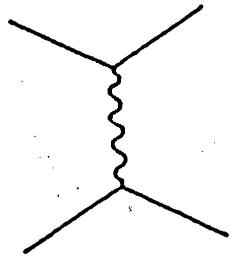
For the case of QED_3 in which the Lagrangian is given by (2.1), the path integral over the gauge field can actually be done as indicated giving a non-local four-point interaction term I_{int} (Fig.3)

$$e^{iW[J]} = \int [d\Psi\, d\overline{\Psi}]\, e^{i\overline{\Psi}(iS_0^{-1} - J)\Psi + iI_{int}} \tag{2.28}$$

where S_0^{-1} is the inverse of the free fermion propagator and

$$I_{int} = \int (dx)\,(dy)\,(\overline{\Psi}\gamma^{\mu}\Psi)_x\, D_{\mu\nu}(x-y)\,(\overline{\Psi}\gamma^{\nu}\Psi)_y \qquad (2.29)$$

here $D^{\mu\nu}(x-y)$ represents the gauge boson propagator.



$$= (\overline{\Psi}\gamma_{\mu}\Psi)_x\, D^{\mu\nu}(x-y)\,(\overline{\Psi}\gamma_{\nu}\Psi)_y$$

Figure 3. Non-local four-point interaction term.

Defining the Legendre-transform in the usual way gives:

$$\frac{\delta W[J]}{\delta J} = S \qquad (2.30a)$$

$$\Gamma[S] = W[J] - \mathrm{Tr}\, JS \qquad (2.30b)$$

$$\frac{\delta \Gamma[S]}{\delta S} = -J \qquad (2.30c)$$

here the trace operation denoted by Tr is taken both in the matrix sense and functional sense and the Legendre variable S conjugate to J is the full fermion propagator $<\overline{\Psi}(x)\,\Psi(y)>$ as can be seen from Eq.(2.28).

Since physical processes correspond to vanishing sources $J(x,y)$, equation (2.30c) provides a derivation of the stationary requirement

$$\frac{\delta \Gamma[S]}{\delta S} = 0 \qquad (2.31)$$

To compute the effective actions, CJT proposed a loop expansion for Γ. To exhibit this, it is useful to work backwards from the answers, defining a quantity Γ_2 which has a more transparent functional integral representation and actually as we will see, is the sum of two-particle irreducible vacuum graphs of the theory with fermion propagator S.

Decompose Γ in the following way

$$\Gamma \equiv -i\,\mathrm{Tr}\,(\mathrm{Ln}\,S^{-1} + S_0^{-1} S) + \Gamma_2 \tag{2.32}$$

Taking a variation of Γ with respect to S gives:

$$\frac{\delta \Gamma}{\delta S} = -J = i\,(S^{-1} - S_0^{-1}) + \frac{\delta \Gamma_2}{\delta S} \tag{2.33}$$

Express the path integral (the generating functional) (2.28) in terms of Γ_2, eliminating W with the help of Eqs.(2.30b) and (2.32)

$$e^{i[\Gamma_2 - \mathrm{Tr}\,S(iS_0^{-1} - J)]} = \frac{\int [d\Psi\, d\bar{\Psi}]\, e^{i\bar{\Psi}(iS_0^{-1} - J)\Psi + i I_{int}}}{e^{\mathrm{Tr}\,(\mathrm{Ln}\,S^{-1})}} \tag{2.34}$$

Finally,

$$\Gamma_2 - \mathrm{Tr}\,(S \frac{\delta \Gamma_2}{\delta S})$$

$$= -i\ \mathrm{Ln}\,\left(\frac{\int [d\Psi\, d\bar{\Psi}]\, e^{i[\bar{\Psi}(iS^{-1})\Psi + I_{int} + \bar{\Psi}\frac{\delta \Gamma_2}{\delta S}\Psi]}}{\int [d\Psi\, d\bar{\Psi}]\, e^{i\bar{\Psi}(iS^{-1})\Psi}} \right) \tag{2.35}$$

where Eq.(2.33) is used to eliminate J.

Eq.(2.35) is an expression for Γ_2. The right hand side clearly generates the sum of all connected vacuum graphs with lines representing the full fermion propagator S and interactions given by

$$\bar{I}_{int} = I_{int} + \bar{\Psi}\,\frac{\delta\Gamma_2}{\delta S}\,\Psi \tag{2.36}$$

It is not obvious, however, that the second term on the right hand side of (2.36) will, on expansion itself, cancel all two-particle reducible graphs, leaving all two-particle irreducible (2PI) graphs. Indirect arguments are given for this in Ref.[4],[13]. To further clarify this, I will give in Appendix B an illustration of how the two-particle-reducible graphs are cancelled. Hence,

$\Gamma_2[S]$ = sum of all 2PI vacuum graphs with propagator denoting S and interactions given by I_{int} .

In the lowest non-trivial order (two-loop) approximation, the following diagram contributes to Γ_2, where all internal lines represent full propagators.

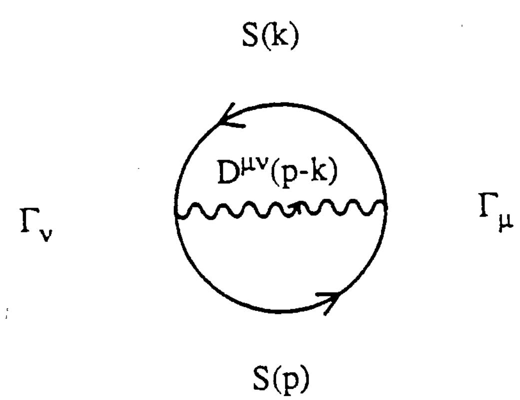


Figure.4: Two-loop approximation in deriving effective action. The solid line represents the exact fermion propagator S(p), the wavy line represents the full gauge boson propagator $D^{\mu\nu}(p-k)$.

We then evaluate the effective action in the lowest order and the first two derivatives for later reference.

$$\Gamma_{CJT}[S] = -i\,\mathrm{Tr}\,(\mathrm{Ln}S^{-1} + S_0^{-1}S) + \frac{i}{2}\mathrm{Tr}\,(SDS) \tag{2.37a}$$

$$\frac{\delta\,\Gamma_{CJT}[S]}{\delta S} = -J = i(S^{-1} - S_0^{-1}) + i\,D\,S \tag{2.37b}$$

$$\frac{\delta^2\Gamma_{CJT}[S]}{\delta S\,\delta S} = -i\,S^{-2} + i\,D \tag{2.37c}$$

$$W_{CJT}[J] = -i\,\mathrm{Tr}\,(\mathrm{Ln}\,S^{-1}) - \mathrm{Tr}\,S\,(i\,S_0^{-1} - J) + \frac{i}{2}\mathrm{Tr}\,(SDS) \tag{2.37d}$$

where $D(p-k)$ has the definition

$$D(p-k) = i\,e^2\,\Gamma^{\mu}\,\Gamma^{\nu}\,D_{\mu\nu}(p-k) \tag{2.38}$$

As I mentioned in the first chapter (Introduction), the CJT effective potential has a defect that it is unbounded from below. This makes the study of the stability of chiral symmetry breaking vacuum meaningless. An alternative was then proposed to remedy this defect ([9]). This new formalism is the auxiliary field (AF) formalism in which one introduces the auxiliary field $T(x,y)$ as a composite field directly into the path-integral of the theory and couples sources to it. In this case, a loop expansion in $T(x,y)$ can be developed. I will go through the derivation of this effective potential formalism.

We can alter the functional integral without changing the physics by inserting:

$$\text{Const.} = \int [dT]\; e^{\frac{1}{2}\mathrm{Tr}\,(T-\Psi\overline{\Psi})\,G\,(T-\Psi\overline{\Psi})} \tag{2.39}$$

This introduces an auxiliary composite field $T(x,y)$. $G(x,y)$ is an arbitrary function to be chosen later.

Consider then the generating functional with the auxiliary field introduced:

$$e^{iW_{AF}[K]} = \int [d\Psi d\bar{\Psi} dT]\, e^{i[\bar{\Psi} i S_0^{-1}\Psi + I_{int} - \frac{i}{2}\mathrm{Tr}(T - \bar{\Psi}\Psi)\, G\, (T - \bar{\Psi}\Psi) + \mathrm{Tr}\, KT]} \tag{2.40}$$

where $K(x,y)$ is the source coupled to auxiliary field $T(x,y)$. In the case of QED where the interaction term is given by (2.29), one can choose the arbitrary function $G(x,y)$ to be $D(x-y)$ to cancel the interaction term in the path integral. Then the exponential in (2.40) becomes bilinear in the fermions. After integrating out the fermions, Eq.(2.40) becomes:

$$e^{iW_{AF}[K]} = \int [dT]\, e^{i[-i\,\mathrm{Tr}\,\mathrm{Ln}(S_0^{-1} - DT) - \frac{i}{2}\mathrm{Tr}(TDT) + \mathrm{Tr}(KT)]} \tag{2.41}$$

We can Legendre transform this just as before:

$$\frac{\delta W_{AF}[K]}{\delta K} \equiv T_C \tag{2.42a}$$

$$\Gamma_{AF}[T_C] = W[K] - \mathrm{Tr}(KT_C) \tag{2.42b}$$

$$\frac{\delta \Gamma_{AF}[T_C]}{\delta T_C} = -K \tag{2.42c}$$

where T_C is the vacuum expectation value of the auxiliary field $T(x,y)$. If we do a tree approximation in the field $T(x,y)$, we can get the effective action Γ analogous to (2.37). Expanding (2.41) around $T_C(x,y)$ gives:

$$e^{i W_{AF}[K]}$$

$$= \int [dT]\, e^{i[-i \operatorname{Tr}\operatorname{Ln}(S_0^{-1} - DT_C) - \frac{i}{2}\operatorname{Tr}(T_C D T_C) + \operatorname{Tr}(KT_C)]}$$

$$\cdot\, e^{i[-i(T-T_C)(-D)(S_0^{-1} - DT_C)^{-1} + i(T-T_C)DT_C + (T-T_C)K + \cdots]}$$

$$\approx e^{i[-i \operatorname{Tr}\operatorname{Ln}(S_0^{-1} - DT_C) - \frac{i}{2}\operatorname{Tr}(T_C D T_C) + \operatorname{Tr}(KT_C)]}$$

$$\cdot \int [dT]\, e^{i\left\{(T-T_C)[iD(S_0^{-1} - DT_C)^{-1} - iDT_C + K]\right\}}$$

$$= e^{i[-i \operatorname{Tr}\operatorname{Ln}(S_0^{-1} - DT_C) - \frac{i}{2}\operatorname{Tr}(T_C D T_C) + \operatorname{Tr}(KT_C)]} \qquad (2.43)$$

where the stationary phase condition is applied

$$iD(S_0^{-1} - DT_C)^{-1} - iDT_C + K = 0 \qquad (2.44a)$$

or $$-K = iD(S_0^{-1} - DT_C)^{-1} - iDT_C \qquad (2.44b)$$

Therefore under the tree approximation, we have

$$\Gamma_{AF}[T_C] = W_{AF}[K] - \operatorname{Tr}(KT_C)$$

$$= -i \operatorname{Tr}\operatorname{Ln}(S_0^{-1} - DT_C) - \frac{i}{2}\operatorname{Tr}(T_C D T_C) \qquad (2.45a)$$

$$\frac{\delta\Gamma_{AF}[T_C]}{\delta T_C} = -K = iD(S_0^{-1} - DT_C)^{-1} - iDT_C \qquad (2.45b)$$

$$\frac{\delta^2\Gamma_{AF}[T_C]}{\delta T_C\,\delta T_C} = i\,D\,(S_0^{-1} - DT_C)^{-2} D - i\,D \tag{2.45c}$$

$$W_{AF}[K] = -i\,\mathrm{Tr\,Ln}\,(S_0^{-1} - DT_C) - \frac{i}{2}\mathrm{Tr}\,(T_C\,D\,T_C) + \mathrm{Tr}\,(KT_C) \tag{2.45d}$$

where $T_C = T_C(K)$ is given by tha stationary phase condition (2.44).

The relation of T_C to the fermion propagator at this level will be clarified. Equation (2.41) tells us that the second derivative given by Eq.(2.45c) is the inverse of the T free propagator. Hence when we set the momentum equal to zero in that expression, which corresponds to taking a translation invariant part of $T(x,y)$, this second derivative gives us minus $[\text{mass}]^2$ of the field $T(x,y)$.

The essential difference between the CJT and AF formalisms is in the coupling of the source to $\Psi(x)\overline{\Psi}(y)$ or to $T(x,y)$ respectively. We can get an expression for $W_{CJT}[J]$ analogous to $W_{AF}[K]$ by doing the same thing as we did in the AF formalism, namely, inserting Eq.(2.39) into the path integral Eq(2.28) and integrating out the fermion field without changing the physics:

$$e^{iW_{CJT}[T]} = \int [dT]\; e^{i[-i\,\mathrm{Tr\,Ln}(S_0^{-1} - DT + i\,J) - \frac{i}{2}\mathrm{Tr}\,(TDT)]} \tag{2.46}$$

If we use the invariance of the volume element under translations we can change the integration variable T to $T + i\,D^{-1}J$ and hence find

$$W_{CJT}[J] = W_{AF}[J] + \frac{i}{2}\,\mathrm{Tr}\,(J\,D^{-1}\,J) \tag{2.47}$$

The whole effect of introducing the auxiliary field and coupling a source to it is to add a term with quadratic J dependence to W_{CJT}. This term is responsible for changing the boundedness properties of V.

Eq.(2.37) and Eq.(2.45) describe the same Lagrangian but with different sources. That is, they describe the different composite fields to which the sources are linearly attached. The relationship between the two different composite fields, namely, $\Psi(x)\overline{\Psi}(y)$ and T(x,y), can be clarified as follows. Let us define S_C:

$$S_C \equiv T_C + i D^{-1} K \tag{2.48}$$

Then, the stationary condition in AF formalism, Eq.(2.45b) becomes:

$$i S_C^{-1} = i S_0^{-1} - K - i D S_C \tag{2.49}$$

This is identified with the CJT stationary condition (2.37b) if we equate S (the full fermion propagator) to S_C and J (the CJT source) to K in this equation. This means that the tree approximation in T fields in the AF formalism is equivalent to the lowest non-trivial (two-loop) order approximation in the CJT formalism, up to the first derivative of the effective actions Γ.

Taking sources J and $K \equiv -i D^{-1} J$ for CJT and AF cases respectively, we can summarize the results here:

$$W_{CJT}[J] = -i \,\mathrm{Tr\,Ln}\, S^{-1} - \mathrm{Tr}\, S\,(iS_0^{-1} - J) + \frac{i}{2}\mathrm{Tr}\,(SDS) \tag{2.50a}$$

$$W_{AF}[K] = -i \,\mathrm{Tr\,Ln}\, S^{-1} - i\,\mathrm{Tr}\, SS_0^{-1} + \frac{i}{2}\mathrm{Tr}\,(S+K)\,D\,(S+K) \tag{2.50b}$$

The fermion self-energy Σ is defined as

$$\Sigma \equiv -i\,(S^{-1} - S_0^{-1}) \tag{2.51}$$

Then the Legendre transform variables are the fermion propagator S and the fermion self-energy Σ for CJT and AF formalisms respectively:

$$\frac{\delta W_{CJT}}{\delta J} = S \tag{2.52a}$$

$$\frac{\delta W_{AF}}{\delta K} = i\,D\,(S + K) = -i\,S^{-1} + i\,S_0^{-1} = \Sigma \tag{2.52b}$$

Finally the corresponding effective actions can be obtained by doing Legendre transforms:

$$\Gamma_{CJT}[\,S\,] = -i\,\mathrm{Tr}\,(\,\mathrm{Ln}\,S^{-1} + S_0^{-1}\,S\,) + \frac{i}{2}\,\mathrm{Tr}\,(S\,D\,S) \tag{2.53a}$$

$$\Gamma_{AF}[\,\Sigma\,] = -i\,\mathrm{Tr}\,\mathrm{Ln}\,(\,i\,S_0^{-1} - \Sigma) + \frac{i}{2}\,\mathrm{Tr}\,(\Sigma\,D^{-1}\,\Sigma) \tag{2.53b}$$

Having obtained the effective actions, it is an easy matter to derive the effective potentials from the effective actions. Since we are interested in translation invariant solutions, we let $S(x,y)$ and the source $J(x,y)$ be functions only of the relative coordinate $(x-y)$. For this case, the effective potential may be defined in the standard way:

$$\Gamma[\,S\,(x,y\,]\,\Big|_{\text{translation invariant}} = -\Omega\,V\,[\,S\,(\,x-y\,)\,] \tag{2.54}$$

where $\Omega = \int (\,dx\,)$ is the volume of space-time.

For example, the series for effective potential $V_{CJT}(S)$ can be obtained from the effective action given by Eq.(2.53a) by Fourier transforming the propagators:

$$S(x-y) = \int \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, S(p)$$

$$S_0(x-y) = \int \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, S_0(p) \qquad (2.55)$$

$$\delta^3(x-y) = \int \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}$$

Substituting (2.55) into (2.53a) gives:

$$-V_{CJT}\int d^3x = -i\int d^3x\, d^3y\, \mathrm{Tr}\left\{\mathrm{Ln}S^{-1}(x-y)\,\delta^{(3)}(x-y) + S_0^{-1}(x-y)\,S(y-x)\right\}$$

$$+\frac{i}{2}\int d^3x\, d^3y\, \mathrm{Tr}\left\{S(x-y)\,D(x-y)\,S(y-x)\right\}$$

$$= -i\int d^3x\, d^3y \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\, e^{ip(x-y)}\, e^{iq(y-x)}\, \mathrm{Tr}\left\{\mathrm{Ln}\,S^{-1}(p) + S_0^{-1}(p)\,S_0(q)\right\}$$

$$+\frac{i}{2}\int d^3x\, d^3y \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\, e^{ip(x-y)}\, e^{iq(x-y)}\, e^{ik(y-x)}\, \mathrm{Tr}\,[S(p)D(q)S(k)]$$

$$= \left\{-i\int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr}\,[\mathrm{Ln}\,S^{-1}(p) + S_0^{-1}(p)\,S(p)] + \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\, \mathrm{Tr}\,[S(p)\,D(p-k)S(k)]\right\}$$

$$\cdot\int d^3x \qquad (2.56)$$

and finally we have

$$V_{CJT}[S] = i\int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr}\,[\mathrm{Ln}\,S^{-1}(p) + S_0^{-1}(p)\,S(p)]$$

$$-\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\, \mathrm{Tr}\,[S(p)\,D(p-k)\,S(k)] \qquad (2.57a)$$

Similarly,

$$V_{AF}[\Sigma] = i\int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr\,Ln}\,(\gamma^\mu \cdot p_\mu - \Sigma)$$

$$- \frac{i}{2}\int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3}\, \mathrm{Tr}\,[\Sigma(p)\, D^{-1}(p-k)\, \Sigma(k)] \qquad (2.57b)$$

With these two equations, we end our sketch of the effective potential formalism.

Now we come to evaluating the effective potentials for our particular model---QED_3 with N fermion flavours. We recall here the Feynman rules of this theory:

(1). Gauge boson propagator corrected to the leading order of 1/N expansion:

$$i D_{\mu\nu}(p) = -i \frac{g_{\mu\nu} - p^\mu p^\nu / p^2}{p^2 [\, 1 + \Pi(p)\,]} \qquad (2.58a)$$

where $\Pi(p) \equiv \alpha / 8p$.

(2). Free fermion propagator:

$$i S_0(p) = \frac{i}{\not p} \qquad (2.58b)$$

(3). Full fermion propagator:

$$i S_f(p) = \frac{i}{\not p\,(1 + A(p)) - \Sigma(p)} \qquad (2.58c)$$

where $A(p)$ is the wave-function renormalization factor.

(4). Vertex corrected to the leading order of 1/N expansion:

$$-ie\Gamma^\mu = -ie\gamma^\mu \qquad (2.58d)$$

To make the Schwinger-Dyson equation, which is determined by the stationary condition of the effective potentials, tractable, we use some certain approximation in deriving the specific forms of the effective potentials for our model. That is, in (2.57), the full gauge boson propagator $D_{\mu\nu}$ is replaced by (2.58a)---the propagator corrected to the leading order of 1/N expansion, the full vertex Γ^{μ} appearing in the definition of D(p) (2.38) is approximated by (2.58d)---the leading behaviour in 1/N expansion and the wave-function renormalization factor A(p) in the full fermion propagator (2.58c) is neglected.

Making the above approximations in (2.57) and doing a straightforward computation, we obtain: (see Appendix C)

$$V_{CJT}[\Sigma] = \int p^2 dp \left[\frac{\Sigma^2(p)}{p^2+\Sigma^2(p)} - \frac{1}{2} Ln\left(1+\frac{\Sigma^2(p)}{p^2}\right)\right]$$

$$-\frac{1}{2}\int p^2 dp \int q^2 dq \, \frac{\Sigma(p)}{p^2+\Sigma^2(p)} \, M(p,q) \, \frac{\Sigma(q)}{q^2+\Sigma^2(q)} \qquad (2.59a)$$

$$V_{AF}[\Sigma] = -\frac{1}{2}\int p^2 dp \; Ln\left(1+\frac{\Sigma^2(p)}{p^2}\right)$$

$$+\frac{1}{2}\int p^2 dp \int q^2 dq \; \Sigma(p)\, M^{-1}(p,q)\, \Sigma(q) \qquad (2.59b)$$

where

$$M(p,q) = \frac{\alpha}{2\pi^2 N p q} \; Ln\left(\frac{p+q+\frac{\alpha}{8}}{|p-q|+\frac{\alpha}{8}} \right) \qquad (2.60)$$

and $M^{-1}(p,q)$ is defined by the following

$$\int dr \; p^2 M^{-1}(p,r)\, r^2 \, M(r,q) = \delta(p-q) \qquad (2.61)$$

here p, q, r are Euclidean momenta.

In the above derivation an assumption has been made to the fermion self-energy:

$$\Sigma_{\alpha\beta} = \Sigma \delta_{\alpha\beta} \quad , \quad \alpha \text{ and } \beta \text{ are indices of internal symmetry} \tag{2.62}$$

since we are only looking for a singlet solution of the S-D gap equation in the internal and Lorentz spaces, and a spherical symmetry in the momentum space for the mass function Σ is also assumed.

CHAPTER 3

SCHWINGER-DYSON EQUATION AND DYNAMICAL CHIRAL SYMMETRY BREAKING SOLUTIONS

Having obtained the explicit forms of the CJT and AF effective potentials, we can pursue our study of dynamical chiral symmetry breaking by analyzing the S-D equation which results from requiring the effective potentials to be stationary against variations of fermion self-energy $\Sigma(p)$. Whether this equation admits non-zero fermion self-energy solutions corresponding to dynamical chiral symmetry breaking is the central problem to be explored in this chapter.

Recall from last chapter that:

$$V_{CJT}[\Sigma] = \int p^2 dp \left[\frac{\Sigma^2(p)}{p^2 + \Sigma^2(p)} - \frac{1}{2} \mathrm{Ln}\left(1 + \frac{\Sigma^2(p)}{p^2}\right) \right]$$

$$- \frac{1}{2} \int p^2 dp \int q^2 dq \frac{\Sigma(p)}{p^2 + \Sigma^2(p)} M(p, q) \frac{\Sigma(q)}{q^2 + \Sigma^2(q)} \qquad (2.59a)$$

$$V_{AF}[\Sigma] = -\frac{1}{2} \int p^2 dp \ \mathrm{Ln}\left(1 + \frac{\Sigma^2(p)}{p^2}\right)$$

$$+ \frac{1}{2} \int p^2 dp \int q^2 dq \ \Sigma(p) M^{-1}(p, q) \Sigma(q) \qquad (2.59b)$$

and

$$M(p,q) = \frac{\alpha}{2\pi^2 N p q} \; Ln\left(\frac{p+q+\frac{\alpha}{8}}{|p-q|+\frac{\alpha}{8}} \right) \tag{2.60}$$

$$\int dr \; p^2 M^{-1}(p,r)\, r^2 \, M(r,q) = \delta(p-q) \tag{2.61}$$

Taking the first derivatives of the effective potentials given by (2.59) with respect to $\Sigma(p)$ gives:

$$\frac{\delta V_{CJT}[\Sigma]}{\delta\Sigma(p)}$$

$$= p^2 \left\{ \frac{[2\,\Sigma(p)\,(p^2+\Sigma^2(p)) - 2\,\Sigma^3(p)\,]}{[\,p^2+\Sigma^2(p)\,]^2} - \frac{\frac{\Sigma(p)}{p^2}}{1+\frac{\Sigma^2(p)}{p^2}} \right\}$$

$$- p^2 \int q^2 dq \; \frac{[p^2+\Sigma^2(p) - 2\,\Sigma^2(p)\,]}{[\,p^2+\Sigma^2(p)\,]^2} \; M(p,q) \; \frac{\Sigma(q)}{q^2+\Sigma^2(p)}$$

$$= \frac{p^2[\,p^2-\Sigma^2(p)\,]}{[\,p^2+\Sigma^2(p)\,]^2} \; \left\{ \Sigma(p) - \int q^2 dq \; M(p,q) \; \frac{\Sigma(q)}{q^2+\Sigma^2(q)} \right\} \tag{3.1a}$$

$$\frac{\delta V_{AF}[\Sigma]}{\delta\Sigma(p)} = -p^2 \frac{\frac{\Sigma(p)}{p^2}}{1+\frac{\Sigma^2(p)}{p^2}} + p^2 \int q^2 dq \; M^{-1}(p,q)\,\Sigma(q)$$

$$= - \frac{p^2\,\Sigma(p)}{p^2+\Sigma^2(p)} + p^2 \int q^2 dq \; M^{-1}(p,q)\,\Sigma(q)$$

$$= p^2 \; \left\{ \int q^2 dq \; M^{-1}(p,q)\,\Sigma(q) - \frac{\Sigma(p)}{p^2+\Sigma^2(p)} \right\} \tag{3.1b}$$

We can see from (3.1) that the stationary condition

$$\frac{\delta V}{\delta \Sigma} = 0$$

implies that

$$\Sigma(p) = \int q^2 dq \; M(p, q) \frac{\Sigma(q)}{q^2 + \Sigma^2(q)} \tag{3.2a}$$

$$\int q^2 dq \; M^{-1}(p, q) \, \Sigma(q) = \frac{\Sigma(p)}{p^2 + \Sigma^2(p)} \tag{3.2b}$$

Eq.(3.2b) is equivalent to Eq.(3.2a). In fact, applying (2.61) to Eq.(3.2b) and performing the integration $\int p^2 dp \, M(r, p)$ on both sides of Eq.(3.2b) give:

$$\int q^2 dq \int p^2 dp \; M(r, p) \; M^{-1}(p, q) \, \Sigma(q) = \int p^2 dp \; M(r, p) \frac{\Sigma(p)}{p^2 + \Sigma^2(p)} \tag{3.3}$$

Thus,

$$\int q^2 dq \, \frac{1}{r^2} \, \delta(r - q) \; \Sigma(q) = \int p^2 dp \; M(r, p) \frac{\Sigma(p)}{p^2 + \Sigma^2(p)} \tag{3.4}$$

Eq.(3.4) is actually (3.2a).

Eq.(3.2a) or Eq.(3.4) is the Schwinger-Dyson equation which we will utilize to study chiral symmetry breaking. Both V_{CJT} and V_{AF} yield the same S-D equation. This further clarifies the point that these two effective potentials are equivalent up to the stationary condition.

Eq.(3.2a) (S-D equation) is an integral equation that is presumably impossible to be solved analytically. The practical way to study this equation quantitatively is to solve it numerically. However, it would be helpful to do some simple analytical analysis and get qualitative ideas about how the solutions of the equation may behave in the asymptotic regime (large and small momentum p regime). To do this, it is convenient to break the momentum integration in Eq.(3.2a) into two regions and expand the logarithm appropriately for each region:

$$\Sigma(p) = \frac{\alpha}{2\pi^2 N p} \int_0^\infty dk \, \frac{k\,\Sigma(k)}{k^2 + \Sigma^2(k)} \, \mathrm{Ln} \left(\frac{p + k + \frac{\alpha}{8}}{|p-k| + \frac{\alpha}{8}} \right)$$

$$= \frac{\alpha}{2\pi^2 N p} \int_0^p dk \, \frac{k\,\Sigma(k)}{k^2 + \Sigma^2(k)} \, \mathrm{Ln} \left(\frac{p + k + \frac{\alpha}{8}}{p - k + \frac{\alpha}{8}} \right)$$

$$+ \frac{\alpha}{2\pi^2 N p} \int_p^\infty dk \, \frac{k\,\Sigma(k)}{k^2 + \Sigma^2(k)} \, \mathrm{Ln} \left(\frac{k + p + \frac{\alpha}{8}}{k - p + \frac{\alpha}{8}} \right) \qquad (3.5)$$

and

$$\mathrm{Ln}\,(p + k + \frac{\alpha}{8}) = \mathrm{Ln}\,(p + \frac{\alpha}{8}) + \frac{2k}{p + \frac{\alpha}{8}} + o\left(\frac{k}{p + \frac{\alpha}{8}}\right)^2 \qquad (3.6)$$

hence,

$$\mathrm{Ln} \left(\frac{p + k + \frac{\alpha}{8}}{p - k + \frac{\alpha}{8}} \right) = \frac{2k}{p + \frac{\alpha}{8}} + o\left(\frac{k}{p + \frac{\alpha}{8}}\right)^3 \qquad (3.7a)$$

$$\mathrm{Ln} \left(\frac{k + p + \frac{\alpha}{8}}{k - p + \frac{\alpha}{8}} \right) = \frac{2p}{k + \frac{\alpha}{8}} + o\left(\frac{p}{k + \frac{\alpha}{8}}\right)^3 \qquad (3.7b)$$

Eq.(3.5) then becomes:

$$\Sigma(p) = \frac{\alpha}{2\pi^2 N p} \int_0^p dk \, \frac{k\,\Sigma(k)}{k^2 + \Sigma^2(k)} \left\{ \frac{2k}{p + \frac{\alpha}{8}} + o\left(\frac{k}{p + \frac{\alpha}{8}}\right)^3 \right\}$$

$$+ \frac{\alpha}{2\pi^2 N p} \int_p^\infty dk \frac{k\Sigma(k)}{k^2+\Sigma^2(k)} \left\{ \frac{2p}{k+\frac{\alpha}{8}} + o\left(\frac{p}{k+\frac{\alpha}{8}}\right)^3 \right\} \qquad (3.8)$$

For both large p and small p (relative to $\alpha/8$ since α provides an intrinsic scale), asymptotic forms of $\Sigma(p)$ may be found by retaining only the first term in the expansion of the logarithm. In this approximation, the integral equation may be converted to a more manageable second order non-linear differential equation:

$$\frac{d\Sigma(p)}{dp} = \frac{\alpha}{\pi^2 N} \frac{d}{dp} \left[\frac{1}{p(p+\frac{\alpha}{8})} \right] \int_0^p k^2 dk \frac{\Sigma(k)}{k^2+\Sigma^2(k)}$$

$$= - \frac{\alpha}{\pi^2 N} \frac{2p+\frac{\alpha}{8}}{p^2(p+\frac{\alpha}{8})^2} \int_p^\infty k^2 dk \frac{\Sigma(k)}{k^2+\Sigma^2(k)} \qquad (3.9)$$

therefore,

$$\frac{d}{dp} \left[\frac{p^2(p+\frac{\alpha}{8})^2}{2p+\frac{\alpha}{8}} \frac{d\Sigma(p)}{dp} \right] = - \frac{\alpha}{\pi^2 N} \frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)} \qquad (3.10)$$

In the limit $p \ll \alpha/8$, this equation can be simplified as

$$\frac{d}{dp} \left[p^2 \frac{d\Sigma(p)}{dp} \right] = - \frac{8}{\pi^2 N} \frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)} \qquad (3.11)$$

If we further assume that $p \gg \Sigma(p)$ (this comes from the assumption that there exists a hierarchy between generated mass $\Sigma(p)$ and the intrinsic scale α), Eq.(3.11) may be linearized to the follwing form:

$$\frac{d}{dp} \left[p^2 \frac{d\Sigma(p)}{dp} \right] = - \frac{8}{\pi^2 N} \frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)} \qquad (3.12)$$

One thing that has to be made clear here is that the use of this linear differential equation is self-consistent only if the hierarchy $\Sigma(p) << \alpha/8$ emerges from the full non-linear integral equation---the S-D equation. This will be checked shortly using a numerical analysis of the S-D equation.

It is then not difficult to see that the linear differential equation (3.12) has solutions of the form

$$\Sigma(p) = C\,p^{a} \qquad \text{where} \quad a = \frac{1}{2}\left(-1 \pm \sqrt{1 - \frac{32}{\pi^2 N}}\right) \tag{3.13}$$

For large N, the two solutions in the asymptotic regime are

$$\Sigma_1(p) \sim \frac{1}{p^{8/\pi^2 N}} . \qquad \text{and} \qquad \Sigma_2(p) \sim \frac{1}{p^{1-8/\pi^2 N}} \tag{3.14}$$

$\Sigma_1(p)$ barely falls asymptotically while $\Sigma_2(p)$ is roughly of order $1/p$. For $N<32/\pi^2$ (=3.2), the solutions fall like $1/\sqrt{p}$ times a function that oscillates in $Ln(p)$. Whether the oscillatory behaviour is seen in the solutions of the full integral equation depends on the range (hierarchy) available between $\Sigma(p)$ and α.

We now turn to the other regime $p>>\alpha/8$. In this limit, Eq.(3.10) becomes:

$$\frac{d}{dp}\left[p^3 \frac{d\Sigma(p)}{dp}\right] = -\frac{2\alpha}{\pi^2 N}\,\Sigma(p) \tag{3.15}$$

Eq.(3.15) admits two possible asymptotic solutions:

$$\Sigma_A(p) = \frac{A}{p^2}\left\{1 + a\frac{\alpha}{p} + \ldots\ldots\right\}, \qquad \text{where} \quad a = -\frac{2}{3\pi^2 N}$$

$$\Sigma_B(p) = B\left\{1 + b\frac{\alpha}{p} + \ldots\ldots\right\}, \qquad \text{where} \quad b = \frac{2}{\pi^2 N} \tag{3.16}$$

It can be seen, however, that the solution $\Sigma_B(p)$, corrsponding to a bare mass, is not compatible with the homogeneous equation(3.2). A bare mass has simply been banished from the theory by not including an inhomogeneous term in this equation. Therefore, in the large momentum ($p>>\alpha/8$) regime, one would expect that $\Sigma(p)$ exhibits approximately $1/p^2$ behaviour.

In the above analytical discussion, we have made the assumption that the dynamically generated mass is much smaller than the intrinsic scale $\alpha\equiv Ne^2$. The consistency of this assumption must be checked by solving the S-D equation completely. Yet, the explicit solutions we have discussed so far are only asymptotic and qualitative, in other words, their behaviour over all momenta regime and their magnitudes are not determined until the non-linearities are taken into account. A numerical analysis is, therefore, necessary to be performed for a complete study of the non-linear Schwinger-Dyson integral equation (3.2). Written down explicitly, Eq(3.2) is

$$\Sigma(p) = \frac{\alpha}{2\pi^2 N p} \int_0^\infty dq \, \mathrm{Ln}\left(\frac{p+q+\frac{\alpha}{8}}{|p-q|+\frac{\alpha}{8}} \right) \frac{q\,\Sigma(q)}{q^2+\Sigma^2(q)} \tag{3.2}$$

Recalling that both $\Sigma(p)$ and α here have dimensions of mass, we define

$$x \equiv \frac{p}{\alpha}, \qquad y \equiv \frac{q}{\alpha}$$

$$B(x) \equiv \frac{\Sigma(p)}{\alpha}, \qquad B(y) \equiv \frac{\Sigma(q)}{\alpha} \tag{3.17}$$

to be dimensionless variables and functions for the convenience of doing numerical analysis.

Thus, Eq.(3.2) becomes:

$$B(x) = \frac{1}{2\pi^2 N x} \int_0^\infty dy \, Ln \left(\frac{x + y + \frac{1}{8}}{|x - y| + \frac{1}{8}} \right) \frac{y B(y)}{y^2 + B^2(y)} \tag{3.18}$$

The fermion flavour number N is now the only parameter in this integral equation. The right hand side of this equation contains an explicit factor of 1/N. A solution $\Sigma(p)$ will then clearly have to exhibit some N dependence.

We further rescale the function $\overline{B}(x)$

$$B(x) = C \, \overline{B}(x) \tag{3.19}$$

where C is an adjustable constant.

Substituting (3.19) into (3.18), we have an integral equation for $\overline{B}(x)$:

$$\overline{B}(x) = \frac{1}{2\pi^2 N x} \int_0^\infty dy \, Ln \left(\frac{x + y + \frac{1}{8}}{|x - y| + \frac{1}{8}} \right) \frac{y \overline{B}(y)}{y^2 + C^2 \overline{B}^2(y)} \tag{3.20}$$

With the existence of this scaling parameter C, we may achieve reasonable sensitivity to the shape of the solutions.

To make Eq.(3.20) more manageable, we use an ultraviolet cutoff $\Lambda >> \alpha/8$ to the integral. A reasonable cutoff will be the one to which the solutions are quite insensitive.

The numerical method we have used for solving this integral equation is the "quadrature method" in which an appropriate quadrature rule may be used to approximate the integration in the integral equation. Here repeated Simpson's quadrature rule has been used and the numerical procedures are as follows:

$$\overline{B}(x) \approx \frac{1}{2\pi^2 N} \sum_{n=0}^{M-1} \frac{1}{2} h \left\{ f(y_n, x, \overline{B}(y_n)) + f(y_{n+1}, x, \overline{B}(y_{n+1})) \right\} \tag{3.21}$$

where the integration is approximated by Simpson's summation. M is the number of intervals to which the integration range is divided and h is the step size. $\{ y_i, i=0, \ldots\ldots, M \}$ are the mesh points.

$$f(y_i, x, \overline{B}(y_i)) = \frac{y}{x\,[\, y_i^2 + C^2 \overline{B}^2(y_i) \,]} \mathrm{Ln}\left(\frac{y_i + x + \frac{1}{8}}{|y_i - x| + \frac{1}{8}} \right) \overline{B}(y_i) \tag{3.22}$$

Eq.(3.21) should hold for $x = y_m$, $m = 0, 1, 2, \ldots\ldots, M$.

Let $Z_i = B(y_i)$, $i = 0, 1, 2, \ldots\ldots, M$, we have a set of non-linear algebraic equations approximating the original integral equation:

$$Z_m = \frac{1}{2\pi^2 N} \sum_{n=0}^{M-1} \left(\frac{1}{2} h\right) \left\{ f(y_n, y_m, Z_n) + f(y_{n+1}, y_m, Z_{n+1}) \right\} \tag{3.23}$$

where m runs over $0, 1, 2, \ldots\ldots, M$.

In principle, we may get solutions $\{Z_m\}$ or $\{B(y_m)\}$---the numerical solutions of the S-D equation---by solving this set of algebraic equations. Eq.(3.23) is, however, non-linear and therefore not easily solved. Practically, we have employed Newton's method to linearize Eq.(3.23), that is, Newton's iterative procedure has been used.

In Newton's method a set of non-linear algebraic equations

$$F_i(\{Z_j\}) = 0 \qquad i, j = 1, \ldots\ldots, n \tag{3.24}$$

can be reduced to a linear one by using the following iterative procedure:

$$F_i(\{Z_j^{(m)}\}) + \sum_{k=1}^{n} (Z_k^{(m+1)} - Z_k^{(m)}) \frac{\partial}{\partial Z_k} F_i(\{Z_j\}) \Big|_{Z_j = Z_j^{(m)}} = 0 \tag{3.25}$$

$m = 0, 1, 2, \ldots\ldots, K$ indicates the order of iterating steps. Eq.(3.25) can be written in a more transparent form:

$$\sum_{k=1}^{n} (F)_{ik} Z_k^{(m+1)} = -F_i + \sum_{k=1}^{n} (F)_{ik} Z_k^{(m)} \tag{3.26}$$

where

$$(F)_{ik} = \frac{\partial}{\partial Z_k} F_i(\{Z_j\}) \Big|_{Z_j = Z_j^{(m)}} \tag{3.27}$$

and

$$F_i = F_i(\{Z_j^{(m)}\}) \tag{3.28}$$

If we start from one appropriate initial set of numerical values $\{Z_j^{(0)}\}$, the linear equations for the iterates $\{Z_j^{(1)}\}$ can be easily solved. Once $\{Z_j^{(1)}\}$ is obtained, we can solve (3.27) again for $\{Z_j^{(2)}\}$ and so on until the iterates converge to some $\{Z_j^{(*)}\}$. $\{Z_j^{(*)}\}$ will then be the solution of the non-linear equations (3.24). Newton's method simplifies the problem since solving linear algebraic equations is not difficult.

From (3.23), we have the non-linear equations

$$F_m(\{Z_i\}) = \sum_{i=0}^{M} \delta_{mi} Z_i - \frac{1}{2\pi^2 N} \sum_{i=0}^{M-1} (\frac{1}{2}h) \{f(y_i, y_m, Z_i) + f(y_{i+1}, y_m, Z_{i+1})\} = 0 \qquad (3.29)$$

To linearize (3.29) following (3.27), we need to compute the matrix elements $(F)_{mk}$ defined by (3.27).

$$\frac{\partial}{\partial Z_k} f(y_i, y_m, Z_i) \equiv \frac{\partial}{\partial Z_k} [\frac{y_i Z_i}{(y_i^2 + C^2 Z_i^2)} \frac{1}{y_m} Ln (\frac{y_i + y_m + \frac{1}{8}}{|y_i - y_m| + \frac{1}{8}})]$$

$$= \frac{y_i (y_i^2 - C^2 Z_i^2)}{(y_i^2 + C^2 Z_i^2)^2} \frac{1}{y_m} Ln (\frac{y_i + y_m + \frac{1}{8}}{|y_i - y_m| + \frac{1}{8}}) \delta_{ik} \qquad (3.30)$$

hence

$$(F)_{m0} = \delta_{m0} - \frac{1}{2\pi^2 N} (\frac{1}{2}h) \frac{1}{y_m} Ln (\frac{y_0 + y_m + \frac{1}{8}}{|y_0 - y_m| + \frac{1}{8}}) \frac{y_0 (y_0^2 - C^2 Z_0^2)}{(y_0^2 + C^2 Z_0^2)^2} \qquad (3.31a)$$

$$(F)_{mk} = \delta_{mk} - \frac{1}{2\pi^2 N} (h) \frac{1}{y_m} Ln (\frac{y_k + y_m + \frac{1}{8}}{|y_k - y_m| + \frac{1}{8}}) \frac{y_k (y_k^2 - C^2 Z_k^2)}{(y_k^2 + C^2 Z_k^2)^2} \qquad (3.31b)$$

$$k = 1, 2, \ldots\ldots, M - 1$$

$$(F)_{mM} = \delta_{mM} - \frac{1}{2\pi^2 N} (\frac{1}{2}h) \frac{1}{y_m} Ln (\frac{y_m + y_M + \frac{1}{8}}{|y_m - y_M| + \frac{1}{8}}) \frac{y_M (y_M^2 - C^2 y_M^2)}{(y_M^2 + C^2 Z_M^2)^2} \qquad (3.31c)$$

Finally, we come to a set of linear equations which takes the form:

$$\sum_{k=0}^{M} (F)_{mk} Z_k^{(n+1)} = G_m , \qquad m = 0, 1, 2, \ldots\ldots, M \qquad (3.32)$$

where $(F)_{mk}$ are given by (3.31) and

$$G_m = \sum_{k=0}^{M} (F)_{mk} Z_k^{(n)} - F_m(\{Z_l^{(n)}\}) \tag{3.33}$$

For solving Eq.(3.32), SFU MTS computer system was used and a Fortran subroutine was called. Practically, the number of mesh points was taken to be 700. The 700 points were not equally spaced because of the consideration of the possible rapidly changing behaviour of the solutions in the infrared region as suggested by the previous analysis of the S-D equation.

Computations were perfomed for different value of the fermion flavour number N. In fact, for $N = 0.5, 1.0, 1.2, 1.4, \ldots\ldots\ldots, 2.6, 2.8$, the S-D integral equation was numerically solved following the procedures stated above and the non-zero solutions $\Sigma(p)$ are found. The numerical results $\Sigma(p)/\alpha$ vs p/α for $N = 0.5, 1.0, 1.2, 2.4$ are plotted in Fig.5, Fig.6, Fig.7 and Fig.8.

The same computations were also done for larger N values. However, the numerical round-off error became problematic because the magnitudes of the solutions are very small when N increases. Therefore no numerically stable solutions were found for larger N values.

For checking the reliability of the solutions found, several different sets of initial values $\{ Z_j^{(0)} \}$ were used. Different initial values, however, only resulted in the difference of the computing time taken and the final results from different initial values all converge to same solution.

In order to check the numerical method we also adopted the simple, self-consistent iterative procedure, namely, the "repeated substitution", to solve the non-linear algebraic equations (3.23). Although this iterative method was more time-consuming than the Newton's iterative method, it also provided the convergent results---the numerical solutions which are the same as those obtained by using Newton's method.

The numerical solutions, as shown in Fig.5, 6, 7, and Fig.8, have the qualitative behavior we discussed before. The hierarchy between the fermion generated mass $\Sigma(0)$ and the intrinsic scale α does emerge from the non-linear S-D equation since $\Sigma(0)/\alpha$ for each value N is much smaller than 1. And as the value of N increases, the hierarchy becomes larger. The appearance of such hierarchy is very interesting but not surprising since a hierarchy between the fermion generated mass and the intrinsic scale appears in some models of four dimensional theories where a small dimensionless parameter, like $1/N$ in our model, exists. Recalling the asymptotic solutions of the S-D equation within the region $\Sigma(p) \ll p \ll \alpha$ for $N<3.2$, one may notice that the numerical solutions do not exhibit the oscillatory behaviour which is suggested by the analytical asymptotic solutions. This is, however, understandable. Although there is a hierarchy between $\Sigma(0)$ and α, this gap may not be so large that the oscillatory solutions found will obviously play a role in the region $\Sigma(p) \ll p \ll \alpha/8$. When it comes to the ultraviolet (large momentum) asymptotic region, one can see that the numerical solutions go to zero following

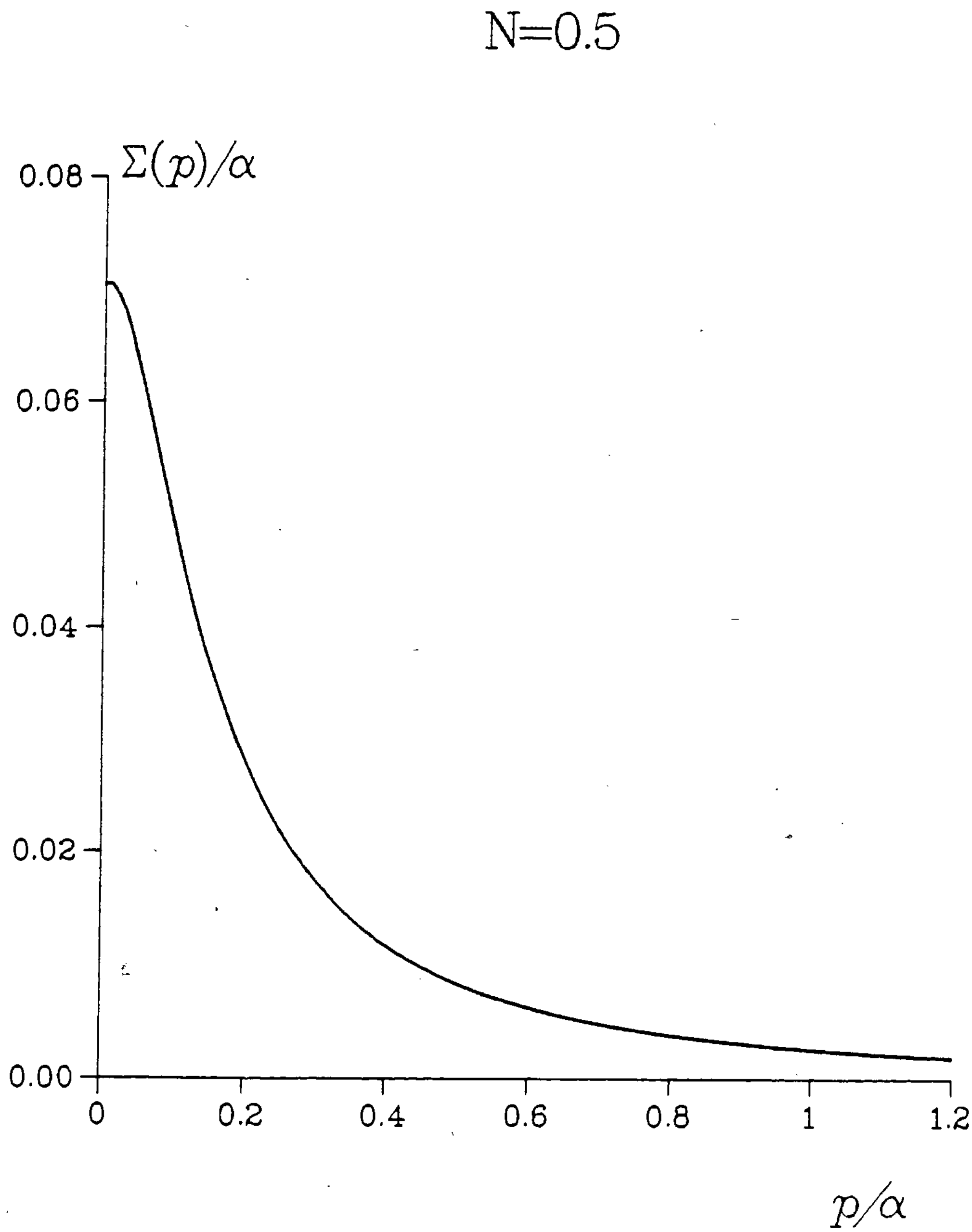


Figure 5: Solution of the S-D equation for fermion flavor number N=0.5.

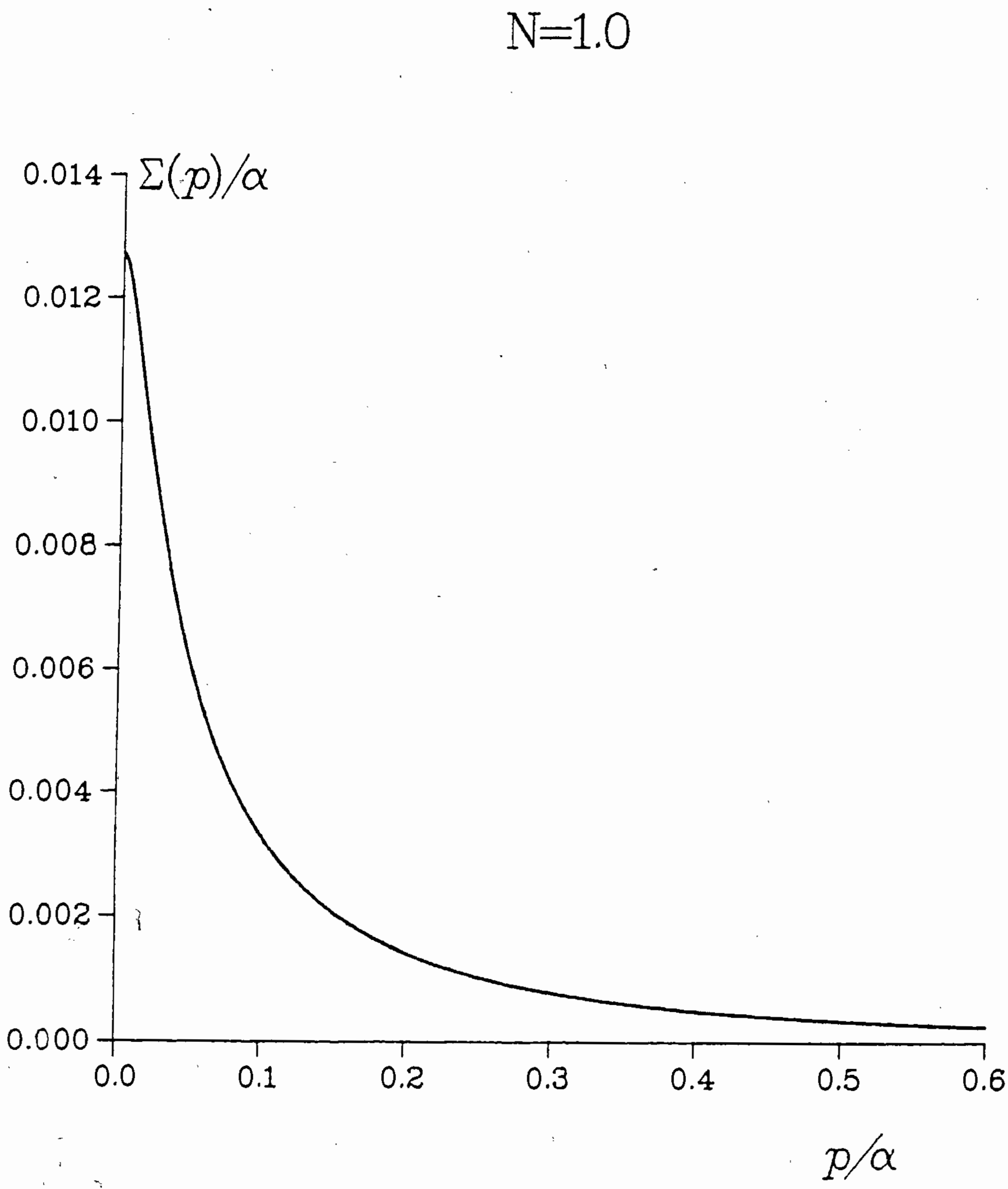


Figure 6: Solution of the S-D equation for fermion flavor number N=1.0.

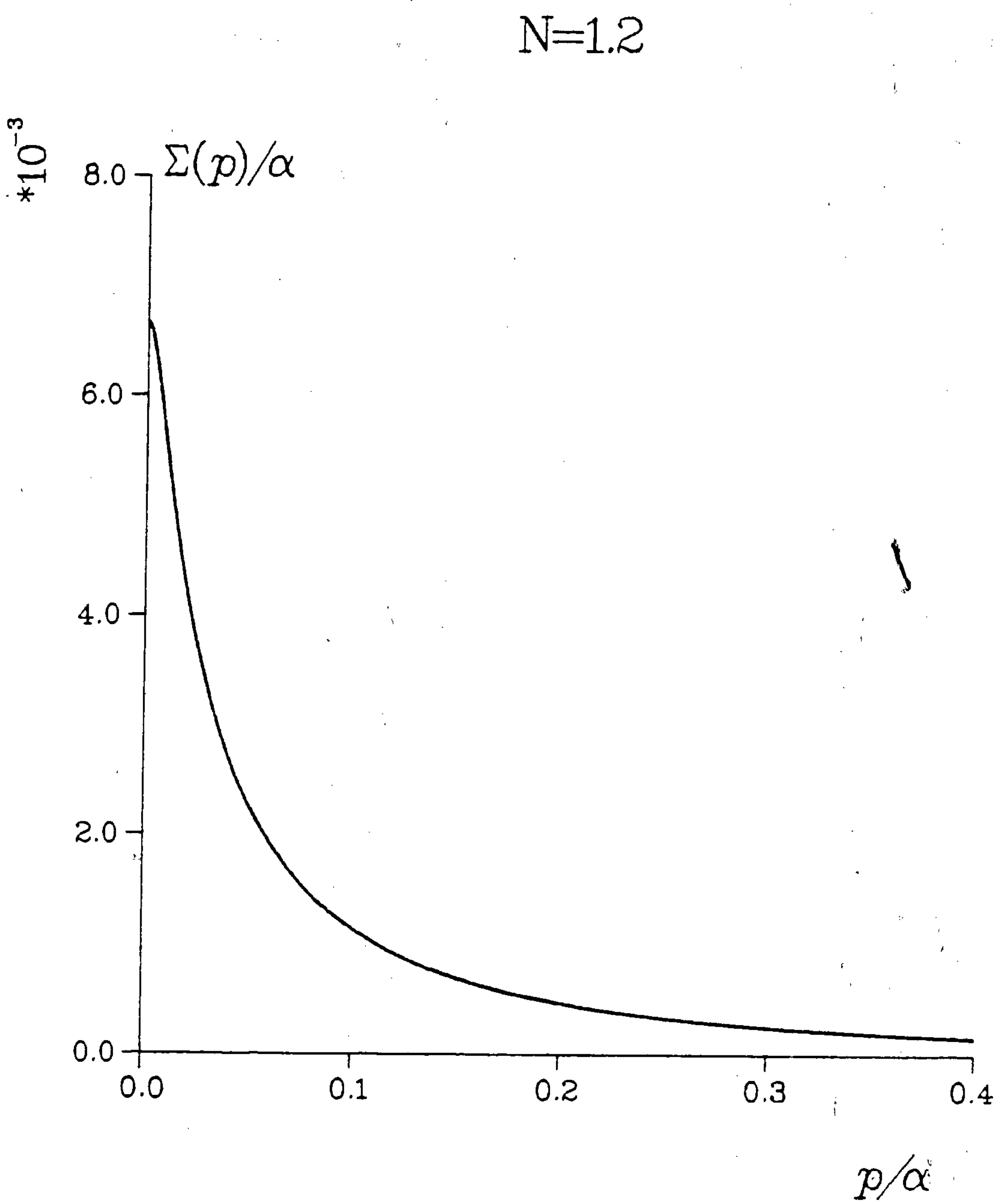


Figure 7: Solution of the S-D equation for fermion flavor number N=1.2.

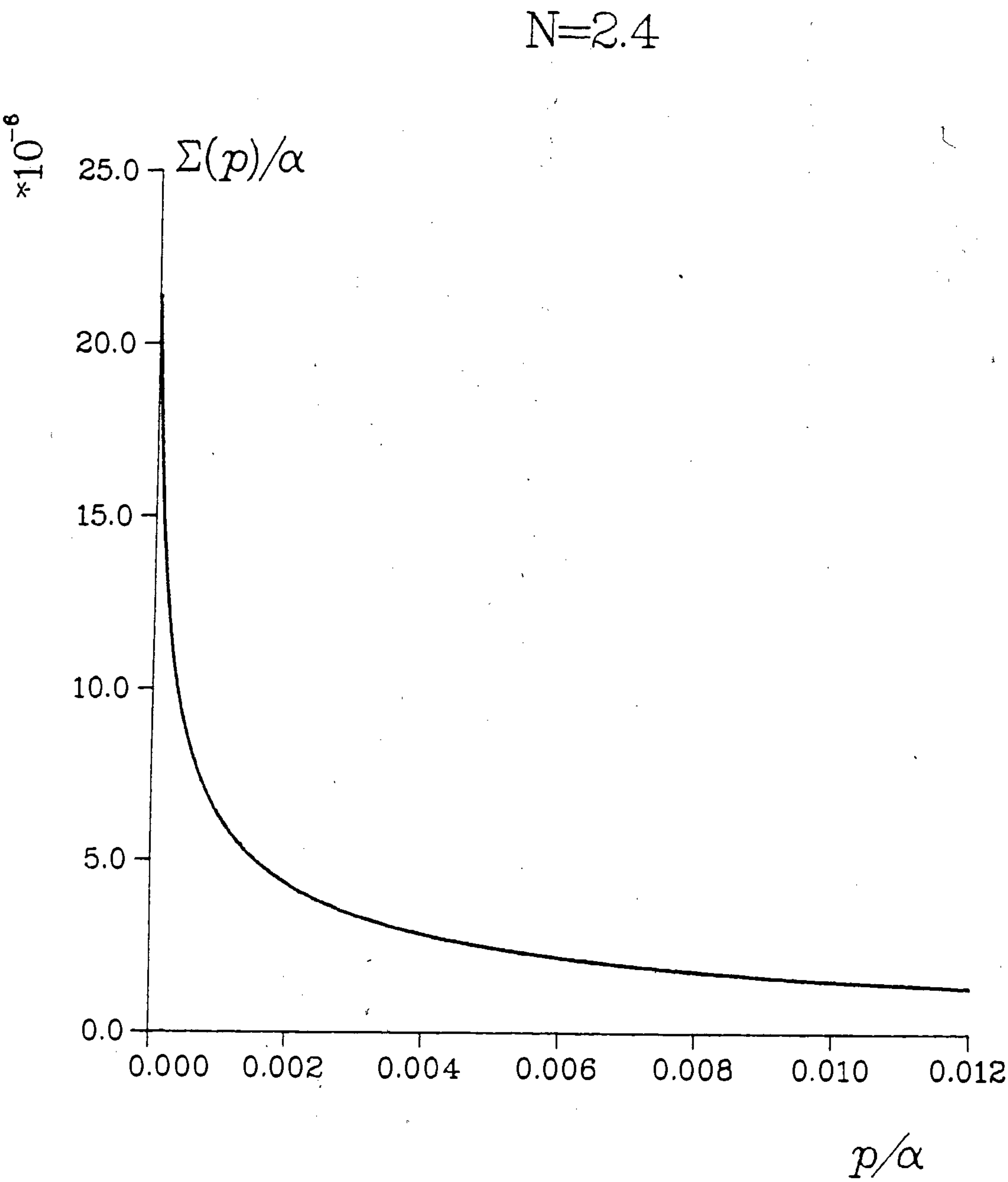


Figure 8: Solution of the S-D equation for fermion flavor number N=2.4.

approximately the $1/p^2$ behaviour which is given by the analytical analysis.

What we can conclude from the above analysis is as follows. Up to the stage of approximations we have made, we found non-zero numerical solutions of the Schwinger-Dyson equation for the values of N from 0.5 to 2.8. This indicates that the chiral symmetry of this model may possibly be broken at least for these values of fermion flavour number. Although we have the practical difficulty in finding solutions for larger values of N as we explained above, it seems to us that there is no reason why one should not expect non-trivial solutions for large N values. Dynamical chiral symmetry breaking may take place for all values of the fermion flavour number N.

Whether or not these non-trivial solutions which indicate the possibility of dynamical symmetry breaking actually correspond to the stable vacuum configuration of the theory is the question to be answered in the next chapter.

CHAPTER 4

STABILITY ANALYSIS

§4.1 Stability of The Chiral Symmetry Breaking Solutions:

Given the solutions obtained in the last chapter, we are left with the question of whether or not these chiral symmetry breaking solutions are stable. That is, under small arbitrary variations about the stationary points, do the solutions diverge from or return to the stationary points, or in other words, do the stationary points correspond to the local minima of the effective potentials? To see this feature, we expand the effective potential $V[\Sigma]$ about the stationary point $\Sigma(p)$:

$$V[\Sigma+\delta\Sigma] = V[\Sigma] + \int dp\, \delta\Sigma(p) \frac{\delta V[\Sigma]}{\delta\Sigma(p)} + \frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q) \tag{4.1}$$

At the stationary point, the first derivative satisfies:

$$\frac{\delta V[\Sigma]}{\delta\Sigma(p)} = 0$$

Eq.(4.1) then becomes:

$$V[\Sigma+\delta\Sigma] = V[\Sigma] + \frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q) \tag{4.2}$$

The answer to the stability question lies then in the second quadratic term of Eq.(4.2). This term actually is the expectation value of the functional second derivative operator $\delta^2 V[\Sigma] / \delta\Sigma(p)\delta\Sigma(q)$ ---the so-called stability operator or curvature operator at non-vanishing solution $\Sigma(p)$. If the expectation value is always positive for every physically allowable variation $\delta\Sigma(p)$, the corresponding stationary point (chiral symmetry breaking solution) is stable; if the expectation value can be either positive or negative, the corresponding chiral symmetry breaking solution is then an unstable saddle point. The term "physically allowable $\delta\Sigma(p)$" means the set of functions $\delta\Sigma(p)$ which keeps the effective potentials continuous and satisfies certain boundary conditions, namely, the conditions that $\delta\Sigma(p)$ should preserve the finiteness of $\Sigma(p)$ as $p \to 0$ and the $1/p^2$ behavior of $\Sigma(p)$ as $p \to \infty$.

For the CJT and AF effective potentials given by (2.59):

$$V_{CJT}[\Sigma] = \int p^2 dp \left[\frac{\Sigma^2(p)}{p^2 + \Sigma^2(p)} - \frac{1}{2} \mathrm{Ln}\left(1 + \frac{\Sigma^2(p)}{p^2}\right)\right]$$

$$- \frac{1}{2} \int p^2 dp \int q^2 dq \frac{\Sigma(p)}{p^2 + \Sigma^2(p)} M(p, q) \frac{\Sigma(q)}{q^2 + \Sigma^2(q)} \qquad (2.59a)$$

$$V_{AF}[\Sigma] = -\frac{1}{2} \int p^2 dp\, \mathrm{Ln}\left(1 + \frac{\Sigma^2(p)}{p^2}\right) + \frac{1}{2} \int p^2 dp \int q^2 dq\, \Sigma(p) M^{-1}(p,q) \Sigma(q) \qquad (2.59b)$$

taking the functional second derivatives of the effective potentials with respect to $\Sigma(p)$, we have, the two stability operators respectively:

$$\frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} = D(p)\left[\,\delta(p-q) - M(p,q)D(q)\,\right] \qquad (4.3)$$

$$\frac{\delta^2 V_{AF}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} = p^2 M^{-1}(p,q) q^2 - D(p)\delta(p-q) \qquad (4.4)$$

where $D(p)$ is defined as

$$D(p) = \frac{p^2(p^2 - \Sigma^2(p))}{[p^2 + \Sigma^2(p)]^2} \tag{4.5}$$

§4.2 Stability Analysis Of The Non-Symmetry-Breaking Solution $\Sigma(p)=0$

Since $\Sigma(p)=0$ is always a "trivial" solution of the S-D equation which corresponds to the vacuum configuration with no chiral symmetry breaking, it is quite interesting and non-trivial to study the local stability of this vacuum configuration. In the model of four-dimensional QED, the local stability of the vacuum of unbroken symmetry was studied ([17]) and a critical value for the dimensionless parameter---coupling constant g---was determined such that when the coupling constant is larger than this critical value, the vacuum with unbroken symmetry is unstable and when the coupling constant is smaller than the critical value, this vacuum is stable. In our model there also is a dimensionless parameter---1/N. Our aim here, therefore, is to see whether or not a critical value for this parameter can be found in the same sense as in the four-dimensional model.

For the solution $\Sigma(p)=0$---the origin of the functional space ---the two stability operators given by (4.3) and (4.4) take the simpler forms as follows

$$\frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} = \delta(p-q) - M(p,q) \tag{4.6}$$

$$\frac{\delta^2 V_{AF}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} = p^2 M^{-1}(p,q)\, q^2 - \delta(p-q) \tag{4.7}$$

where the definition of the inverse $M^{-1}(p,q)$ is given by

$$\int dr\, p^2 M^{-1}(p,r)\, r^2 M(r,q) = \delta(p-q) \tag{2.61}$$

The major purpose of doing stability analysis here is then to justify whether or not the expectation values of the above stability operators are always positive for any physically allowable departures $\delta\Sigma(p)$ from $\Sigma(p)=0$ when the parameter N varies.

Since at the origin, the two effective potentials should provide the same information about the local stability and the CJT stability operator (4.6) looks easier to deal with, we now concentrate on this operator denoted by H:

$$H \equiv \frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} = \delta(p-q) - M(p,q) \tag{4.8}$$

§4.2.1 Functional Analysis

It is well-known that the vector space $L^2(0,\infty)$ which contains all the square-integrable functions is a Hilbert space. However, physical conditions on variations $\delta\Sigma(p)$ as stated above require us to start from a subspace of $L^2(0,\infty)$ on which the operator H can be physically well-defined. This subspace is called the domain of definition of H.

Let us consider a dense subspace D included in domain $D(p^{-1/2})$ (the domain of functions which, multiplied by $p^{-1/2}$, are square-integrable) in which the operators H and $p^{-1/2}$ are well-defined and symmetric, that is, for any $f \in D \subset D(p^{-1/2})$,

$Hf \in L^2(0,\infty)$ or more specifically $Mf \in L^2(0,\infty)$

and

$(1/p)f \in L^2(0,\infty)$.

In reference [16], A. Amer, A. Le Yaouanc, L. Oliver, O. Péne and J-C Raynal proved that the kernel

$$K_1(p,q) = \frac{1}{\sqrt{p}} \mathrm{Ln}\left(\frac{p+q}{|p-q|}\right) \frac{1}{\sqrt{q}}$$

defines a bounded symmetric operator in $L^2(0,\infty)$ with a finite norm:

$$\| K_1 \| = B(1) = \int_0^\infty \frac{dp}{p} \mathrm{Ln}\left(\frac{p+1}{|p-1|}\right) \tag{4.9}$$

Borrowing this result and observing that the kernel

$$K_2(p,q) = \frac{1}{\sqrt{p}} \mathrm{Ln}\left(\frac{p+q+\frac{1}{8}}{|p-q|+\frac{1}{8}}\right) \frac{1}{\sqrt{q}}$$

relevant to our problem is always less than the kernel $K_1(p,q)$, one sees that $K_2(p,q)$ also defines a bounded operator in $L^2(0,\infty)$ with finite norm

$$\| K_2 \| = C \leq \| K_1 \| = B(1) \tag{4.10}$$

For any $f \in D$, the expectation value of H is

$$(f, Hf) = (f, f) - (f, Mf)$$

$$\equiv \int dp\, f^2(p) - \frac{1}{2\pi^2 N} \int dp \int dq\, f(p) \left[\frac{1}{p} \mathrm{Ln} \left(\frac{p+q+\frac{1}{8}}{|p-q|+\frac{1}{8}} \right) \frac{1}{q} \right] f(q)$$

$$= (f, f) \left[1 - \frac{(f, Mf)}{(f, f)} \right] \qquad (4.11)$$

Defining

$$f'(p) \equiv \frac{1}{\sqrt{p}} f(p)$$

where $f'(p) \in L^2(0, \infty)$

one has

$$\frac{(f, Mf)}{(f, f)} = \frac{1}{2\pi^2 N} \frac{(f', K_2 f')}{(f', f')} \frac{(f', f')}{(f, f)} = \frac{1}{2\pi^2 N} \frac{(f', K_2 f')}{(f', f')} \frac{(f, \frac{1}{p} f)}{(f, f)}$$

The expectation value (4.11) then becomes

$$(f, Hf) = (f, f) \left[1 - \frac{1}{2\pi^2 N} \frac{(f', K_2 f')}{(f', f')} \frac{(f, \frac{1}{p} f)}{(f, f)} \right] \qquad (4.12)$$

Since it is shown that $K_2(p,q)$ is a bounded operator on $L^2(0,\infty)$ with finite norm C, it is then true that $(f', K_2 f')/(f', f')$ is always finite. However if one analyzes the factor $(f, 1/p\, f)/(f, f)$, one finds that even though $(f, 1/p\, f)$ is finite, this factor is not bounded from above. That is, given any positive value A, one can find some $f \in D$ such that

$$\frac{(f, \frac{1}{p} f)}{(f, f)} \geq A \qquad (4.13)$$

An example of this kind of f is

$$f_\alpha(p) = \sqrt{p}\, \exp(-\alpha p) \qquad \alpha > 0 \qquad (4.14)$$

which yields

$$\frac{(f_\alpha, \frac{1}{p} f_\alpha)}{(f_\alpha, f_\alpha)} = 2\alpha \tag{4.15}$$

Therefore, one can always choose α to make (4.15) arbitrarily large.

Following this fact, one observes that, given any finite value N, it is possible that there exists $f_0 \in D$ so that

$$\frac{1}{2\pi^2 N} \frac{(f_0', K_2 f_0')}{(f_0', f_0')} \frac{(f_0, \frac{1}{p} f_0)}{(f_0, f_0)} > 1$$

Hence

$$(f_0, Hf_0) < 0 \tag{4.16}$$

(4.16) then indicates that the expectation value of the operator H may not be positive definite for any given finite value of the fermion flavour number N.

We then suggest that the chiral symmetry-preserving solution $\Sigma(p) = 0$ may correspond to the locally unstable point of the effective potentials for any given fermion number N, which means there may not exist a critical value of the fermion flavour number in the sense we stated at the beginning of this section.

§4.2.2 Numerical Analysis

The numerical analysis of the stability operator defined in (4.8) provides support to the suggestion drawn from the functional analysis in the previous section.

In doing the numerical analysis, we study the following integral eigenvalue equation of the stability operator

$$H\Phi_n \equiv \int dq\, [\, \delta(p-q) - M(p,q)\,]\, \Phi_n(q) = \lambda_n \Phi_n(p) \tag{4.17}$$

that is, we solve this equation numerically and determine the possible eigenvalues λ_n. The equivalent form of Eq.(4.17) is

$$M\Phi_n = \int dq\, M(p,q)\Phi_n(q) = \rho_n \Phi_n(p) \tag{4.18a}$$

where $$\rho_n = 1 - \lambda_n \tag{4.18b}$$

For solving an integral equation like (4.18a), the simplest and the most recommended numerical method is based on quadrature rules. The quadrature rule we use here is that of Gauss-Laguerre. Noticing the discontinuity of the first derivative of the kernel, we adopt a modified quadrature method to remedy the possible defect due to this discontinuity. Applying the Gauss-Laguerre quadrature rule and the modified method to Eq.(4.18) and going through the procedures similar to those performed in solving the S-D equation, we have an algebraic eigenvalue problem which approximates the integral eigenvalue equation (4.18)

$$\sum_{j=1}^{n} A_{ij} Z_j^{(n)} = \rho_n Z_i^{(n)} \tag{4.19}$$

where $\{\rho_n\}$ are the approximate eigenvalues and

$$A_{ij} = \begin{cases} \int_0^\infty dy\, M(y_i, y) - \sum_{k\neq i}^{n} M(y_i, y_k)\, w_k & i = j \\ [w_r]^{1/2} M(y_i, y_j)\, [w_j]^{1/2} & i \neq j \end{cases} \tag{4.20a}$$

and

$$Z_i^{(n)} \equiv \Phi_n(y_i) \tag{4.20b}$$

In (4.20) $\{ w_i, i = 1, 2, ..., n \}$ are Gauss-Laguerre weight factors and $\{ y_i, i = 1, 2, ..., n \}$ are Gauss-Laguerre abscissae which are chosen not to be zero. This is understandable if one notices the singularity of the kernel $M(p,q)$ as $p \to 0$, $q \to 0$ and remembers the fact that we are only working with the domain D of functions defined previously.

The algebraic eigenvalue equation (4.20) is solved and the approximate eigenvalues are determined for the integral eigenvalue equation (4.17) by using a 64-point Gauss-Laguerre quadrature rule and calling a subroutine in the NAG Fortran Library. This numerical procedure is performed for different Gauss- Laguerre abscissae $\{ y_i \}$ and different values of fermion flavor number $N = 1.0, 1.2, 1.4, ..., 2.8, 3.0, ...$. Some of the numerical results are listed in Table 1 (see page 79). The numerical calculations show that, for each value N given above, by carefully choosing the abscissae $\{y_i\}$ which may correspond to the possible eigenfunctions with important infrared behavior, one can get some negative approximate eigenvalues for the integral eigenvalue equation. This fact seems to indicate that the eigenvalue spectrum of the stability operator H has a negative part for any finite value N. The numerical analysis therefore matches with the result (4.16) obtained in the previous functional analysis.

What we may be able to conclude from above analysis of the local stability of the solution $\Sigma(p) = 0$ is that the vacuum configuration with unbroken chiral symmetry is unstable and the chiral symmetry breaking solutions may be energetically preferred by the vacuum of this theory. This is to be verified by the work to be explained in the next section.

§4.3 Stability Analysis Of The Chiral Symmetry Breaking Solutions

With the non-trivial chiral symmetry breaking solutions $\Sigma(P)\neq 0$ obtained by solving the S-D equation numerically, the local stability of these non-trivial solutions can be analyzed by studying the expectation values of the stability operators (4.3) and (4.4) at the stationary points. Since this expectation value can be expressed in terms of the eigenvalues of the stability operators, the problem is reduced to finding the eigenvalues of the stability operators.

In fact, the eigenvalue problem is:

$$\int dq \frac{\delta^2 V[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \Phi_n(q) = \lambda_n \Phi_n(p) \tag{4.21}$$

Expanding the arbitrary variation $\delta\Sigma$ of the fermion generated mass in terms of the eigenfunctions $\{\Phi_n\}$, one has

$$\delta\Sigma(p) = \sum_n C_n \Phi_n(p) \tag{4.22}$$

Therefore, the expectation value of the stability operator is given as

$$\frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q)$$

$$= \frac{1}{2}\int dp\, dq\, [\sum_m C_m \Phi_m(p)] \frac{\delta^2 V[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} [\sum_n C_n \Phi_n(q)]$$

$$= \frac{1}{2}\sum_{m,n} C_m C_n \lambda_n \int dp\, \Phi_m(p)\Phi_n(p) = \frac{1}{2}\sum_n C_n^2 \lambda_n \tag{4.23}$$

where the orthogonality relation of the eigenfunctions $\{\Phi_n\}$ has been used

$$\int dp\, \Phi_m(p)\Phi_n(p) = \delta_{mn} \tag{4.24}$$

Consequently, as we can see from (4.23), if all the eigenvalues are positive, then the stationary point $\Sigma(p)\neq 0$ corresponds to a local, stable minimum of the effective potential; if at least one eigenvalue is negative, the point is then unstable.

For the two stability operators given by (4.3) and (4.4), the corresponding eigenvalue problems are

$$\int dq\, D(p)[\delta(p-q) - M(p,q)D(q)]\, f_n(q) = \lambda_n f_n(p) \tag{4.25}$$

and

$$\int dq\, [p^2 M^{-1}(p,q)q^2 - D(p)\delta(p-q)]\, g_m(q) = \rho_m g_m(p) \tag{4.26}$$

In principle, these two eigenvalue problems can be solved for each non-trivial symmetry breaking solution. However, we find significant difficulty in attempting to solve these two equations directly whereas the following eigenvalue problem is much easier to deal with.

$$\int dq\, [\delta(p-q) - M(p,q)D(q)]\, \Phi_n(q) = \Lambda_n \Phi_n(p) \tag{4.27}$$

It is possible to extract the stability information from this latter equation as we will show in the following steps.

The orthogonality relation of the eigenfunctions $\{\Phi_n\}$ is given by

$$\int dp\, dq\, \Phi_m(p)D(p)M(p,q)D(q)\Phi_n(q) = \delta_{mn} \tag{4.28}$$

Actually, from (4.27), one has

$$\int dq\, [\delta(p-q) - M(p,q)D(q)]\, \Phi_m(q) = \Lambda_m \Phi_m(p) \tag{4.29a}$$

$$\int dp\, [\delta(p-q) - M(q,p)D(p)]\, \Phi_n(p) = \Lambda_n \Phi_n(q) \tag{4.29b}$$

Multiplying

$$\int dq'\, \Phi_n(q')D(q')M(q',p)D(p)$$

to (4.29a) and integrating over p gives:

$$\int dp\, dq\, dq'\, \Phi_n(q')D(q')M(q',p)D(p)\, [\delta(p-q) - M(p,q)D(q)]\, \Phi_m(q)$$

$$= \Lambda_m \int dp\, dq'\, \Phi_n(q')D(q')M(q',p)D(p)\Phi_m(p) \tag{4.30a}$$

Multiplying

$$\int dq'\, \Phi_m(q')D(q')M(q',q)D(q)$$

to (4.29b) and integrating over q gives:

$$\int dp\, dq\, dq'\, \Phi_m(q')D(q')M(q',q)D(q)\, [\, \delta(p-q) - M(q,p)D(p)\,]\, \Phi_n(p)$$

$$= \Lambda_n \int dq\, dq'\, \Phi_m(q')D(q')M(q',q)D(q)\Phi_n(q) \tag{4.30b}$$

Substracting (4.30b) from (4.30a), we have

$$(\Lambda_m - \Lambda_n) \int dp\, dq\, \Phi_n(q)D(q)M(q,p)D(p)\Phi_m(p) = 0 \tag{4.31}$$

which implies (4.28).

Now we evaluate the expressions of the expectation values for the AF and CJT stability operators in terms of the eigenvalues

$\{\Lambda_n\}$ satisfying Eq.(4.27).

A straightforward calculation using (4.27),(4.28) and (2.61) gives:

$$\int dp\, \Phi_m(p) D(p) \Phi_n(p) = \frac{\delta_{mn}}{1-\Lambda_n} \tag{4.32}$$

and

$$\int dq\, p^2 M^{-1}(p,q) q^2\, \Phi_n(q) = D(p)\Phi_n(p) \frac{1}{1-\Lambda_n} \tag{4.33}$$

therefore,

$$\frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{AF}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q)$$

$$= \frac{1}{2}\int dp\, dq \sum_{m,n} C_m \Phi_m(p) \left[p^2 M^{-1}(p,q) q^2 - D(p)\delta(p-q) \right] C_n \Phi_n(q)$$

$$= \frac{1}{2}\sum_{m,n} C_m C_n \left[\frac{\delta_{mn}}{(1-\Lambda_n)^2} - \frac{\delta_{mn}}{1-\Lambda_n} \right] = \frac{1}{2}\sum_n C_n^2 \frac{\Lambda_n}{(1-\Lambda_n)^2}$$

and

$$\frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q)$$

$$= \frac{1}{2}\int dp\, dq \sum_{m,n} C_m \Phi_m(p) \left[\delta(p-q) - M(p,q)D(q) \right] C_n \Phi_n(q)$$

$$= \frac{1}{2}\sum_{m,n} C_m C_n \left[\frac{\delta_{mn}}{1-\Lambda_n} - \delta_{mn} \right] = \frac{1}{2}\sum_n C_n^2 \frac{\Lambda_n}{1-\Lambda_n}$$

Finally, we have

$$\frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{AF}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q) = \frac{1}{2}\sum_n C_n^2 \frac{\Lambda_n}{(1-\Lambda_n)^2} \tag{4.34}$$

$$\frac{1}{2}\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q) = \frac{1}{2}\sum_n C_n^2 \frac{\Lambda_n}{(1-\Lambda_n)} \tag{4.35}$$

It was established ([15,9]) that in the model of four-dimensional QCD with a running coupling constant, the solutions of the Schwinger-Dyson equation correspond to the locally stable minima in the AF formalism while the same solutions are the saddle points in the CJT formalism. With (4.34) and (4.35) having been obtained, we will show, by solving the eigenvalue equation (4.27), that the similar results hold in our model---QED_3.

Before doing any numerical computations, we establish an interesting argument to see why the chiral symmetry breaking solutions correspond to the saddle points of the CJT effective potential.

Recall from (4.2) and (4.3) that the expectation value of the CJT stability operator has the form

$$\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q)$$

$$= \int dp\, \delta\Sigma(p) D(p) \delta\Sigma(p) - \int dp\, dq\, \delta\Sigma(p) D(p) M(p,q) D(q) \delta\Sigma(q) \tag{4.36}$$

We shall show that this expectation value of the CJT stability operator can be either negative or positive by choosing appropriate variations $\delta\Sigma(p)$.

The solutions $\Sigma(p)$ of the Schwinger-Dyson equation obtained in the third chapter have the following asymptotic behaviors:

$$\Sigma(p) \to \text{finite constant} \qquad \text{as } p \to 0$$
$$\Sigma(p) \to \frac{1}{p^2} \qquad \text{as } p \to \infty \tag{4.37}$$

therefore, the asymptotic behavior of the function $D(p)$ defined as (4.5) is obtained as

$$D(p) \to -p^2 / \Sigma^2(0) \qquad \text{as } p \to 0$$
$$D(p) \to 1 \qquad \text{as } p \to \infty \tag{4.38}$$

Since the sign of $D(p)$ is different in the two asymptotic regions, there is at least one vanishing point for the function $D(p)$ in momentum space. Suppose there is only one vanishing point $p = p_0$. It is possible for us to take a positive variation $\delta\Sigma(p)$ which contributes mainly in the infrared momentum region $p < p_0$. Therefore,

$$\int dp\, \delta\Sigma(p)D(p)\delta\Sigma(p) - \int dp\, dq\, \delta\Sigma(p)D(p)M(p,q)D(q)\delta\Sigma(q)$$
$$\approx \int_0^{p_0} dp\, \delta\Sigma(p)D(p)\delta\Sigma(p) - \int_0^{p_0} dp \int_0^{p_0} dq\, \delta\Sigma(p)D(p)M(p,q)D(q)\,\delta\Sigma(q) < 0 \tag{4.39}$$

for this particular variation. That is, along this particular direction in the functional space, the symmetry breaking solution corresponds to the locally unstable point of the CJT effective potential.

We can also show that there are variations that give positive expectation value to the CJT stability operator. Since the operator $D(p)M(p,q)D(q)$ is bounded from above, by taking

$$\delta\Sigma(p) = \delta_\varepsilon(p - \bar{p}) \tag{4.40}$$

where δ_ε is a smeared delta function of width ε. We can make the first term in (4.36) dominate over the second term by choosing width ε appropriately. Then, if $\bar{p} > p_0$

$$\int dp\, dq\, \delta\Sigma(p) \frac{\delta^2 V_{CJT}[\Sigma]}{\delta\Sigma(p)\delta\Sigma(q)} \delta\Sigma(q) > 0 \tag{4.41}$$

Having proven that the expectation value of the CJT stability operator can be either positive or negative along different directions in the functional space, we conclude that chiral symmetry breaking solutions correspond to the saddle points of the CJT effective potential. This argument will be supported by the numerical results to be presented later.

Numerical analysis is performed to solve the eigenvalue equation (4.27) and determine those eigenvalues from which we can get the stability information.

Substituting into (4.5) the numerical solutions of the S-D equation, one gets the numerical data for the characteristic function D(p). D(p) for $N = 0.5, 1.0, 1.2$ are plotted in Fig.9, Fig.10, and Fig.11. As we can see from the plots, D(p) does have a negative value region although this negative value region is very small because of the small magnitude of the solution $\Sigma(p)$. Knowing the behavior of the function D(p), we now solve the characteristic eigenvalue problem (4.27).

Naturally the quadrature method is used here since only the

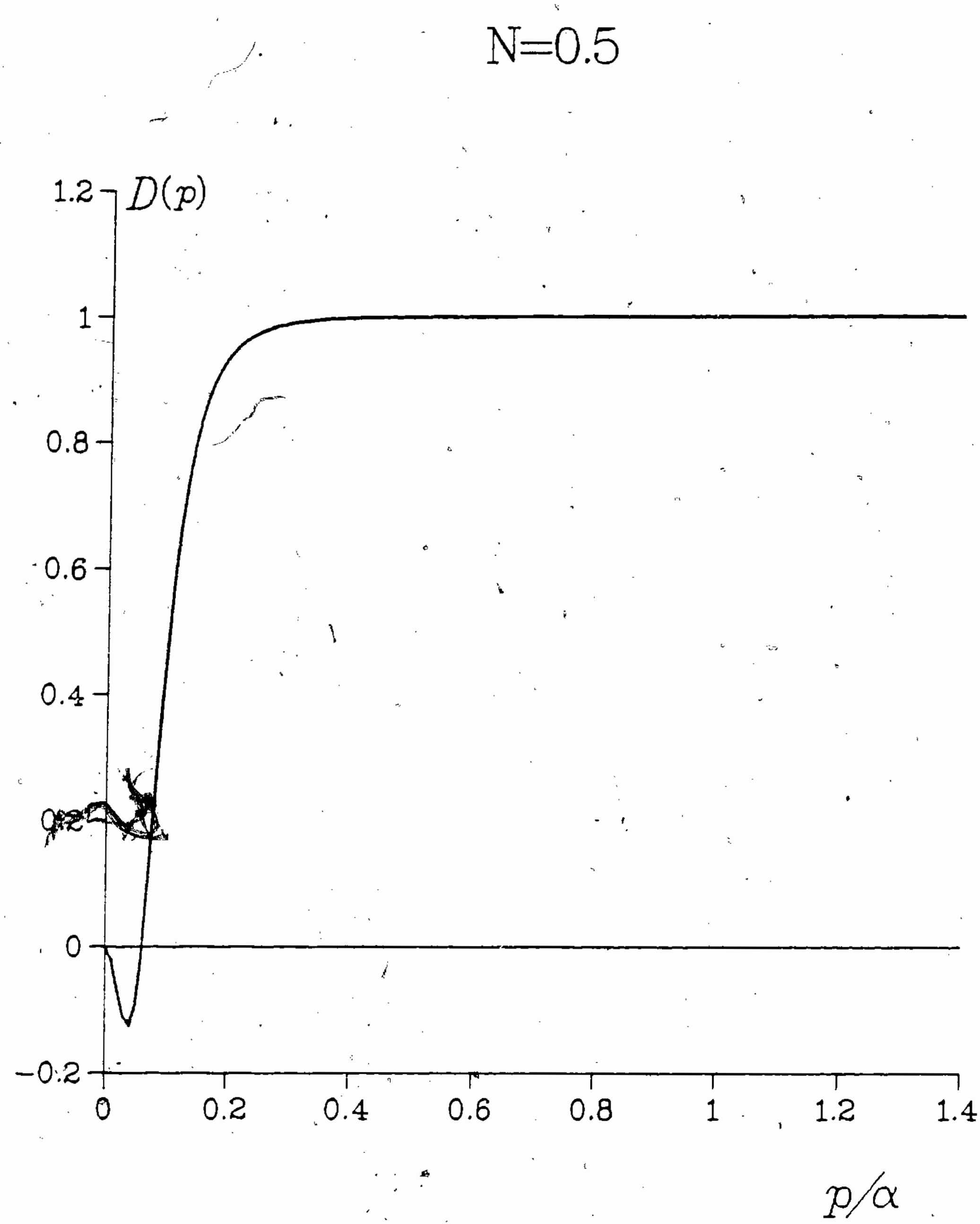


Figure 9: Characteristic function D(p) for N=0.5.

N=1.0

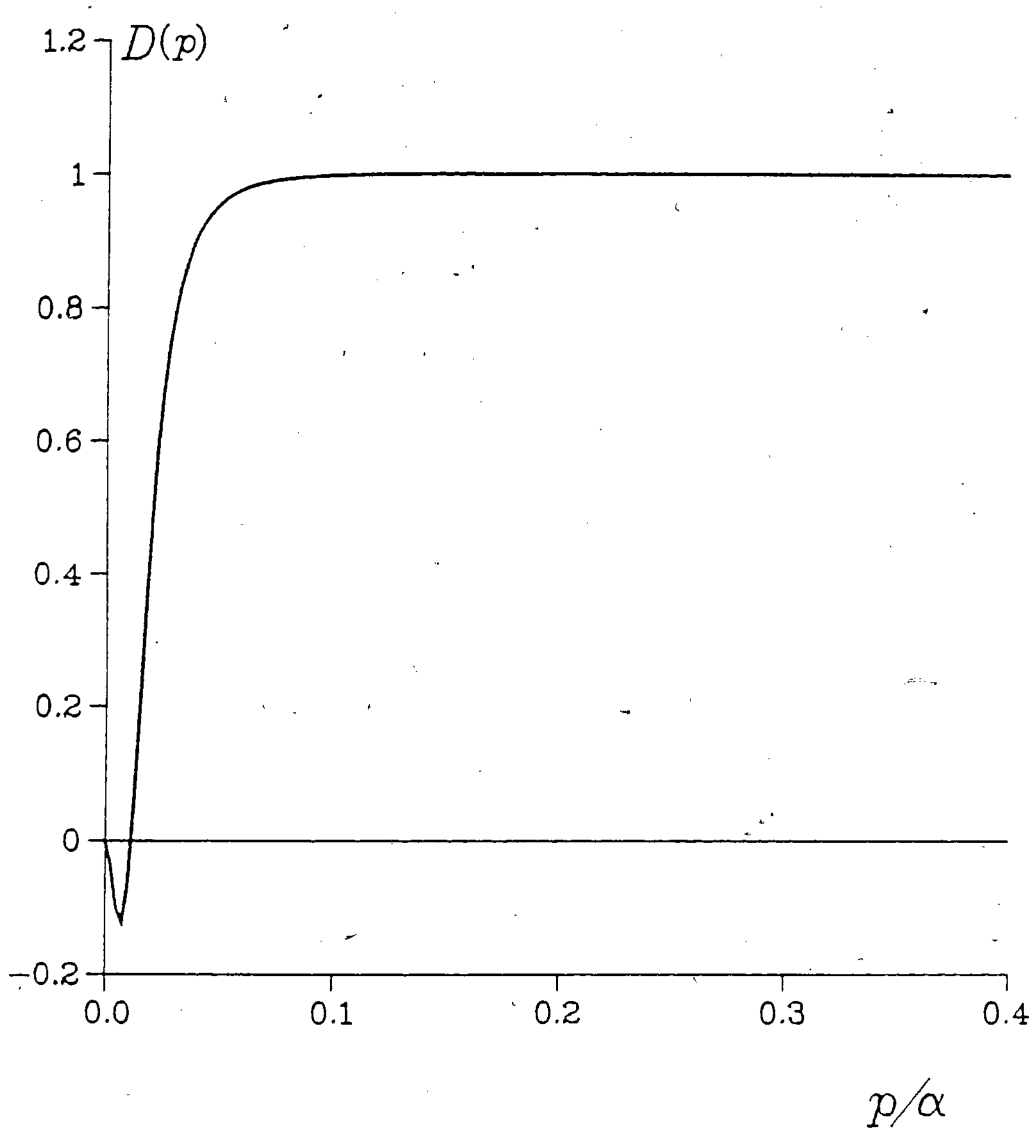


Figure 10: Characteristic function D(p) for N=1.0.

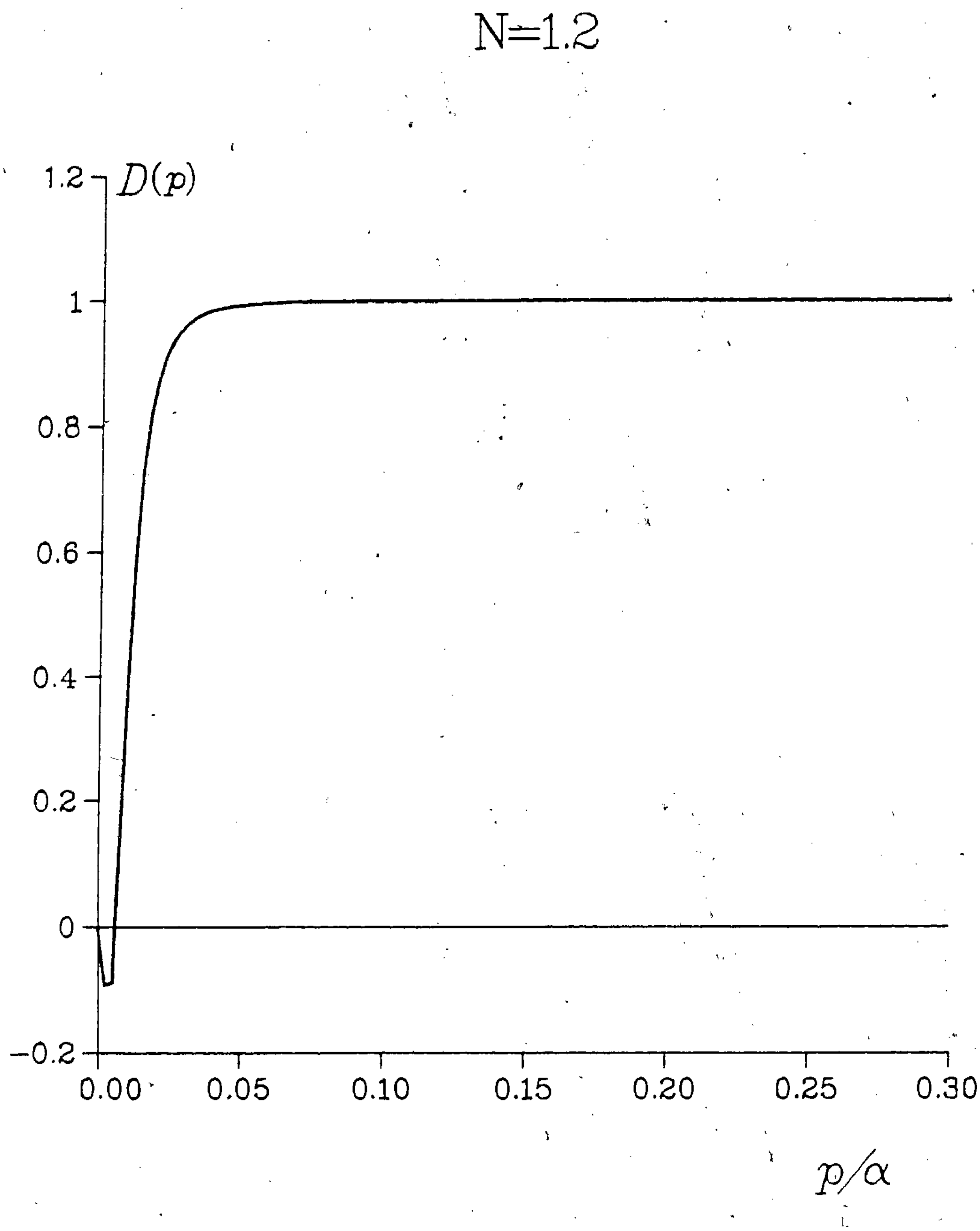


Figure 11: Characteristic function D(p) for N=1.2.

numerical data for $D(p)$ was obtained. Noticing that the negative value region of $D(p)$ is very small, we use a much smaller integration grid for this region than the grid size used for positive value region in order to achieve a sensitivity for the infrared bahaviours of $D(p)$ and eigenfunctions. Considering the boundary condition that has to be satisfied in the ultraviolet region by the variations, we use an ultraviolet cutoff for the integral appearing in Eq.(4.27). The eigenvalue equation (4.27) is solved using the numerical techniques stated above for all those non-trivial solutions of the S-D equation. The approximate eigenvalues for $N = 0.5, 1.0, 1.2, 2.0$ are listed in Table 2.

We note from Table 2 that all the eigenvalues Λ_n are positive and some of them are even greater than 1. Referring to the expressions given by (4.34) and (4.35), we see that all the eigenvalues of the AF stability operator are positive. Therefore the expectation value of this operator is positive definite whereas the expectation value of the CJT stability operator is indefinite since some of its eigenvalues are positive and some of them are negative.

We now conclude that the chiral symmetry broken vacuum configurations are stable according to the improved AF effective potential and preferred energetically by the vacuum of this theory. However, in the CJT effective potential formalism, as we have discussed before, the same vacuum configurations correspond to the saddle points of this effective potential. This is only given for comparison since the CJT effective potential does not provide the correct stability information.

Table 1. Some of the eigenvalues of the stability operator (4.8) at $\Sigma(p)=0$. Negative eigenvalues are responsible for the local instability.

Λ_n (N = 1.0)	Λ_n (N = 1 4)	Λ_n (N = 2.0)	Λ_n (N = 2.4)
-1.61	-0.87	-0.30	-0.09
-0.58	-0.13	0.21	0.34
0.44	0.35	0.54	0.62
0.63	0.60	0.72	0.77
0.74	0.73	0.87	0.84
0.86	0.81	0.91	0.89
.	.	.	.
.	.	.	.

Table 2. Some of the eigenvalues of (4.27). All eigenvalues are positive and some are even greater than 1.

Λ_n (N = 0.5)	Λ_n (N = 1.0)	Λ_n (N = 1.2)	Λ_n (N = 2.0)
0.29	0.24	0.18	0.06
0.75	0.74	0.71	0.59
0.87	0.86	0.85	0.78
.	.	.	.
.	.	.	.
1.010	1.006	1.008	0.96
1.022	1.014	1.018	1.006
1.110	1.076	1.077	1.035

CHAPTER 5

SUMMARY

In this thesis we study the dynamical chiral symmetry breaking in quantum electrodynamics with N fermion flavors in 2+1 dimensions.

This work was motivated by observing that the underlying theories of the strong interaction must undergo dynamical chiral symmetry breaking and the physical world resulting from the QCD Lagrangian is then non-perturbative. Since the QED_3 model is simple and genuine so that we may pursue a systematic treatment to the non-perturbative feature of dynamical chiral symmetry breaking, we hope that this will shed some light on the more complicated cases such as the gauge theories of the strong interaction.

For systematic study of the chiral symmetry breaking, the effective potential formalism proposed by Cornwall, Jackiw and Tomboulis, which is capable of describing the non-perturbative features of field theories, was used. However, since this effective potential has certain defects so that it can not be used to test the stability of the vacuum configurations with broken symmetry, we adopted an improved effective potential. Up to the stationary condition, namely, the condition that the first functional derivatives of the effective potentials with respect to the fermion self-energy are zero as required by the physical processes, these two effective potentials are equivalent and give same physics: Schwinger-Dyson (S-D) equation.

In fact, there is another choice of the effective potential. This effective potential is originally due to Casalbuoni, De Curits, Domonici and Gatto (CDDG) ([18]) and contains the same physics as the original CJT formulation. It is shown in the work done by J. Otu and K.S. Viswanathan ([19]) that this modified effective potential is also bounded from below and is stable against fluctuations to the stationary conditon, namely, Schwinger-Dyson equation for the QCD-like gauge theories.

In chapter 3, the possibility of dynamical chiral symmetry breaking in this model treated within the framework of 1/N expansion was explored by analyzing the S-D equation analytically and numerically. The non-trivial chiral symmetry breaking solutions with expected hierarchy between the generated fermion mass $\Sigma(p)$ and the intrinsic energy scale α were found numerically for fermion flavor number $N = 0.5, 1.0, 1.2, \ldots, 2.6, 2.8$. Although we had practical difficulty in finding non-trivial solutions for larger fermion flavour number N because of the very small magnitude of $\Sigma(0)/\alpha$, we suggested that non-trivial solutions exist for any finite value of N. The dynamical chiral symmetry breaking may then occur in this model. Further work can be done to find solutions for large N by using a more sophisticated numerical technique to deal with the difficulty.

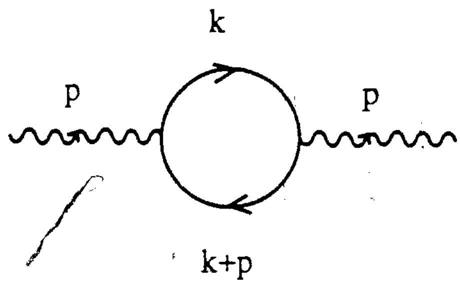
In chapter 4, we studied the local stability of the chirally symmetric vacuum corresponding to the solution $\Sigma(p)=0$ and that of the vacuum configurations with broken symmetry corresponding to the non-vanishing $\Sigma(p)\neq 0$. It is suggested by the functional analysis and the numerical analysis that the vacuum configuration with perfect chiral symmetry is unstable for any finite number of fermion flavours N and the vacuum of this theory may then prefer other locally stable configurations which may correspond to the symmetry breaking solutions. It is also shown, by studying the stability operator resulting from the modified effective potential which is believed to provide correct stability information, that the non-trivial solutions obtained in chapter 3 are the locally stable points for the effective potential.

In conclusion, we have found that the chiral symmetry of our model treated within the framework of $1/N$ expansion may be broken for any number of fermion flavours N and the configurations with broken chiral symmetry are locally stable and preferred energetically by the vacuum of this model.

APPENDIX A

The leading order correction to photon propagator in 1/N expansion.

Photon vacuum polarization is given by the following Feynman graph:



$$= i\Pi_{\mu\nu}$$

$$= (-1)(-ie)^2 N \int (dk)\, \mathrm{Tr}\left[\frac{i}{\gamma\cdot k}\, \gamma_\mu\, \frac{i}{\gamma\cdot(k+p)}\, \gamma_\nu \right]$$

$$= -\alpha \int (dk)\, \frac{\mathrm{Tr}\left[(\gamma\cdot k)\,\gamma_\mu\,(\gamma\cdot k + \gamma\cdot p)\,\gamma_\nu\right]}{k^2 (k+p)^2} \qquad \text{(A.1)}$$

Using the well-known formula

$$\mathrm{Tr}(\gamma_\lambda \gamma_\mu \gamma_\delta \gamma_\nu) = 4\eta_{\lambda\mu}\eta_{\delta\nu} + 4\eta_{\mu\delta}\eta_{\lambda\nu} - 4\eta_{\lambda\delta}\eta_{\mu\nu} \qquad \text{(A.2)}$$

and noticing

$$\frac{1}{k^2 (k+p)^2} = \int_0^1 dx\, \frac{1}{\left[k^2 + 2x\,p\cdot k + x p^2\right]^2} \qquad \text{(A.3)}$$

we can write (A.1)

(A.1)

$$= -\alpha \int (dk) \int_0^1 dx\, \frac{4\left[k_\mu (k+p)_\nu + k_\nu (k+p)_\mu - \eta_{\mu\nu}\, k\cdot(k+p)\right]}{\left[k^2 + 2x\,p\cdot k + x p^2\right]^2}$$

$$= -\alpha \int (dk) \int_0^1 dx \, \frac{4\left[k_\mu (k+p)_\nu + k_\nu (k+p)_\mu - \eta_{\mu\nu} k \cdot (k+p)\right]}{\left[(k+xp)^2 - x(x-1)p^2\right]^2} \qquad (A.4)$$

Note that the volume element is invariant under the translation $k \to k+xp$, (A.4) therefore becomes:

(A.4)

$$= -4\alpha \int (dk) \int_0^1 dx \, \frac{\{2k_\mu k_\nu - 2x(1-x)p_\mu p_\nu - \eta_{\mu\nu}[k^2 - x(1-x)p^2]\}}{[k^2 - x(x-1)p^2]^2}$$

$$= -4\alpha \int (dk) \int_0^1 dx \, \frac{\{\frac{2}{D} k^2 \eta_{\mu\nu} - 2x(1-x)p_\mu p_\nu - \eta_{\mu\nu}[k^2 - x(1-x)p^2]\}}{[k^2 - x(x-1)p^2]^2}$$

$$= -4\alpha \int (dk) \int_0^1 dx \, \frac{\{\frac{2-D}{D} k^2 \eta_{\mu\nu} - 2x(1-x)p_\mu p_\nu - \eta_{\mu\nu}[k^2 - x(1-x)p^2]\}}{[k^2 - H]^2}$$

(A.5)

where $H=-x(1-x)p^2$ (A.6)

and D is the dimension of the space-time, D=3 here.

Further computation of (A.6) gives

$$(A.6) = -4\alpha \int (dk) \int_0^1 dx \{ \frac{2-D}{D} \frac{\eta_{\mu\nu}}{(k^2-H)} + \frac{x(1-x)}{(k^2-H)^2} [\eta_{\mu\nu} \frac{2(D-1)p^2}{D} - 2p_\mu p_\nu]\}$$

$$= \frac{-4\alpha i}{(2\pi)^D} \int d\Omega^D \int_0^1 dx \int_0^\infty k^{D-1} dk \left\{ \frac{D-2}{D} \frac{\eta_{\mu\nu}}{k^2+H} \right.$$

$$\left. + \frac{x(1-x)}{(k^2+H)^2} \left[\eta_{\mu\nu} \frac{2(D-1)p^2}{D} - 2p_\mu p_\nu \right] \right\}$$

$$= -\frac{4\alpha i}{(2\pi)^D} \int d\Omega^D \int_0^1 dx \left\{ \left(\frac{D-2}{D} \eta_{\mu\nu} \right) \left[\int_0^\infty dk \frac{k^{D-1}}{[k^2+H]^2} \right] \right.$$

$$\left. + x(1-x) \left[\eta_{\mu\nu} \frac{2(D-1)p^2}{D} - 2p_\mu p_\nu \right] \left(\int_0^\infty dk \frac{k^{D-1}}{[k^2+H]^2} \right) \right\}$$

(A.7)

Recalling the formula:

$$\int_0^\infty dt \frac{t^{m-1}}{(1+bt^a)^{m+n}} = a^{-1} b^{-\frac{m}{a}} B\left(\frac{m}{a}, m+n-\frac{m}{a}\right) \quad (a>0, b>0) \qquad \text{(A.8)}$$

and

$$B(p,q) = B(q,p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad \text{(A.9)}$$

we get

$$\int_0^\infty dk \frac{k^{D-1}}{k^2+H} = \frac{1}{2} H^{\frac{D-2}{2}} \frac{\Gamma(\frac{D}{2})\Gamma(1-\frac{D}{2})}{\Gamma(1)}$$

$$= -\frac{\pi}{2} H^{\frac{1}{2}} \qquad (D=3) \qquad \text{(A.10)}$$

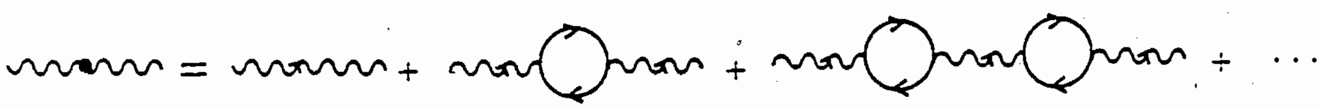
$$\int_0^\infty dk \, \frac{k^{D-1}}{(k^2+H)^2} = \frac{1}{2} H^{\frac{D}{2}-2} \, \frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{\Gamma(2)} \qquad (A.11)$$

Substituting (A.10) and (A.11) into (A.7) and setting D=3:

$$(A.7) = -\frac{i\alpha}{\pi}\left\{(-p^2)^{-1/2}[\eta_{\mu\nu}p^2 - p_\mu p_\nu]\left(\int_0^\infty dx\,[x(1-x)]^{1/2}\right)\right\}$$

$$= i\,\frac{-\alpha}{8(-p^2)^{\frac{1}{2}}}\,[\eta_{\mu\nu}p^2 - p_\mu p_\nu] \qquad (A.12)$$

The leading order correction to the photon propagator in 1/N expansion is given by the sum of the following Feynman graphs.



Symbolically, this can be written in a compact form:

$$iD'_{\mu\nu}(p^2) = iD_{\mu\nu}(p^2) + [iD_\mu^{\lambda}(p^2)][i\Pi_{\lambda\sigma}(p^2)][iD'^{\sigma}_{\nu}(p^2)] \qquad (A.13)$$

here $iD_{\mu\nu}'(p^2)$ represents the corrected photon propagator and it takes the form

$$iD'_{\mu\nu}(p^2) = A(p^2)\eta_{\mu\nu} + B(p^2)p_\mu p_\nu \qquad (A.14)$$

Replacing $i\Pi_{\lambda\sigma}(p^2)$ with (A.12) and using (A.14), we obtain

$$iD'_{\mu\nu}(p^2) = -\frac{i}{p^2\left[1+\frac{\alpha}{8\sqrt{-p^2}}\right]}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \qquad (2.23)$$

APPENDIX B

The Cancellation of Two-Particle-Reducible Diagrams

In this appendix we will show how the two-particle reducible diagrams are cancelled in the CJT approach at the non-trivial order for the real scalar model for the sake of simplicity:

$$L = \frac{1}{2} \Phi\, iG_0^{-1}\, \Phi + I_{int} \tag{B.1}$$

with the interaction given by

$$I_{int}[\Phi(x)\,\Phi(y)] = \frac{1}{2}\, \Phi^2(x)\, \Delta(x-y)\, \Phi^2(x) \tag{B.2}$$

The connected Green functional for the composite operator $\Phi(x)\Phi(y)$ is given by

$$e^{iW[J]} = \int [d\Phi]\, e^{i\,[\frac{1}{2}\Phi\, iG_0^{-1}\, \Phi + I_{int} + \frac{1}{2}\Phi J \Phi\,]} \tag{B.3}$$

Defining the Legendre transform in the usual way gives:

$$\frac{\delta W[J]}{\delta J} = \frac{1}{2} G \tag{B.4a}$$

$$\Gamma[G] = W - \frac{1}{2} \mathrm{Tr}\, JG \tag{B.4b}$$

$$\frac{\delta \Gamma[G]}{\delta G} = -\frac{1}{2} J \tag{B.4c}$$

The Legendre variable G conjugate to J is the full scalar propagator $\langle\Phi\Phi\rangle$ as can be seen from Eq.(B.3). The corresponding expressions to Eq.(2.32) and (2.35) for the scalar case are given by

$$\Gamma = \frac{i}{2} \mathrm{Tr}\, \mathrm{Ln} G^{-1} + \frac{i}{2} \mathrm{Tr} G_0^{-1} G + \Gamma_2 \tag{B.5}$$

$$\Gamma_2 - \mathrm{Tr}\, G\frac{\delta\Gamma_2}{\delta G} = -i\,\mathrm{Ln}\frac{\int[d\Phi]\, e^{i[\frac{1}{2}\Phi(iG^{-1} - 2\frac{\delta\Gamma_2}{\delta G})\Phi + I_{int}]}}{\int[d\Phi]\, e^{i\frac{1}{2}\Phi iG^{-1}\Phi}} \tag{B.6}$$

The right hand side of Eq.(B.6) is further simplified as

$$-i\,\mathrm{Ln}\frac{e^{iI_{int}[-2\delta/\delta\overline{G}^{-1}(x,y)]}\int[d\Phi]\, e^{i\frac{1}{2}\Phi\, i\overline{G}^{-1}\Phi}}{\int[d\Phi]\, e^{i\frac{1}{2}\Phi\, iG^{-1}\Phi}}$$

$$= -i\,\mathrm{Ln}\frac{e^{iI_{int}[-2\delta/\delta\overline{G}^{-1}(x,y)]}\exp(-\frac{1}{2}\mathrm{Tr\,Ln}\overline{G}^{-1})}{\exp(-\frac{1}{2}\mathrm{Tr\,Ln}G^{-1})} \tag{B.7}$$

where

$$\overline{G} = (G^{-1} + 2i\frac{\delta\Gamma_2}{\delta G})^{-1} = (1 + 2i\,G\frac{\delta\Gamma_2}{\delta G})^{-1}G \tag{B.8}$$

We will check consistency of Eq.(B.6) given the lowest order of Γ_2 (see Fig.12). To do this, we need to expand $\exp(iI_{int})$ up to second order in Δ and keep terms of second order in Δ in Eq.(B.7). That is, we will see that 2PI diagrams in Fig.13.a remain and the two-particle reducible diagrams in Fig.13.b are cancelled. We further assume that tadpole diagrams like Fig.13.c vanish and hence such a term is not included in Γ_2 from the outset even though it is a 2PI diagram. The lowest order Γ_2 is given by

$$\Gamma_2^{(1)} = \Delta_{xy}G_{xy}G_{yx}\ , \tag{B.9}$$

where notations are self-consistent. We give the results in the order of Δ for the expansion $\exp(i I_{int})$ after performing the functional differentiation and expanding the inside of the logarithmic function in Eq.(B.7) as

$$-i\,\mathrm{Ln}\left[(1 + i\,1st + i\,2nd + \ldots\ldots)\, e^{i0th}\right] = 0th + 1st + 2nd - \frac{i}{2}(1st)^2 + \ldots \qquad \text{(B.10)}$$

Each term is given by

$$0th = -\frac{i}{2}\,\mathrm{Tr}\,\mathrm{Ln}(\overline{G}G^{-1}) = -\mathrm{Tr}\,G\frac{\delta\Gamma^{(1)}_2}{\delta G} + 4\,i\,\Delta_{xy}\Delta_{zw}G_{xy}G_{yz}G_{zw}G_{wx} \qquad \text{(B.11a)}$$

$$1st = \Gamma^{(1)}_2 - 8\,i\,\Delta_{xy}\Delta_{zw}G_{xy}G_{yz}G_{zw}G_{wx} \qquad \text{(B.11b)}$$

$$2nd = \frac{i}{2}(\Gamma^{(1)}_2)^2 + i\,\Delta_{xy}\Delta_{zw}(4\,G_{xy}G_{yz}G_{zw}G_{wx} + 2\,G_{xz}G_{zy}G_{yw}G_{wx} + G_{xz}G_{zx}G_{yw}G_{wy}) \qquad \text{(B.11c)}$$

$$-\frac{i}{2}(1st)^2 = -\frac{i}{2}(\Gamma^{(1)}_2)^2 \qquad \text{(B.11d)}$$

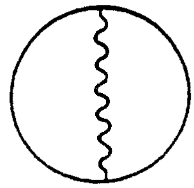
Hence the total sum of all these terms is

$$\Gamma^{(1)}_2 + \Gamma^{(2)}_2 - \mathrm{Tr}\,G\frac{\delta\Gamma^{(1)}_2}{\delta G} \qquad \text{(B.12)}$$

where

$$\Gamma^{(2)}_2 = i\,\Delta_{xy}\Delta_{zw}(2\,G_{xz}G_{zy}G_{yw}G_{wx} + G_{xz}G_{zx}G_{yw}G_{wy}) \qquad \text{(B.13)}$$

This completes the consistency checking of (B.6) at the lowest order. Generalization to fermions is straightforward.



$$= \frac{i}{2} \, \mathrm{Tr}\, SDS$$

Figure 12: Lowest order of Γ_2.

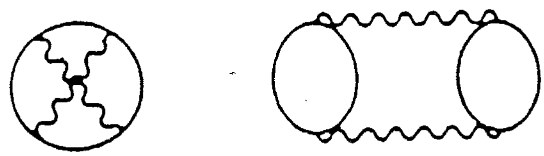


Figure 13.a: Examples of two-particle irreducible diagrams.

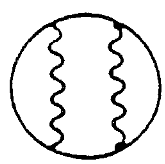


Figure 13.b: Example of two-particle reducible diagrams.

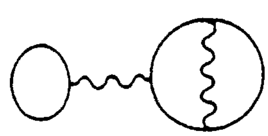


Figure 13.c: Example of tad-pole diagrams.

APPENDIX C

1. Evaluation of the CJT effective potential:

(2.57a) gives the general form of the CJT effective potential:

$$V_{CJT}[\Sigma] = i\ \mathrm{Tr}\left[\mathrm{Ln}\, S^{-1} + S_0^{-1} S\right] - \frac{i}{2}\ \mathrm{Tr}\,(SDS) \tag{2.57a}$$

Then, after a constant renormalization,

$$V_{CJT}[\Sigma] = -i\ \mathrm{Tr}\left[\mathrm{Ln}\, S_0^{-1} S - S_0^{-1} S + 1\right] - \frac{i}{2}\ \mathrm{Tr}\,(SDS) \tag{C1.1}$$

$$\begin{aligned}
V_0 &\equiv -i\,\mathrm{Tr}\left[\mathrm{Ln}\, S_0^{-1} S - S_0^{-1} S + 1\right] \\
&= -i \int (dP)\ \mathrm{Tr}\left[\mathrm{Ln}\, S_0^{-1} S - S_0^{-1} S + 1\right] \\
&= -i \int (dP)\ \mathrm{Tr}\left[\mathrm{Ln}\left(\frac{\not{p}}{\not{p} - \Sigma}\right) - \frac{\not{p}}{\not{p} - \Sigma} + 1\right] \\
&= -i \int (dP)\ \mathrm{Tr}\left[\mathrm{Ln}\left(\frac{p^2 + \not{p}\,\Sigma(p)}{p^2 - \Sigma^2(p)}\right) - \frac{\Sigma^2(p)\ (\not{p} + \Sigma(p))}{p^2 - \Sigma^2(p)}\right] \\
&= -i \int (dP)\ \mathrm{Tr}\left[\mathrm{Ln}\left(\frac{p^2 + \not{p}\,\Sigma}{p^2}\right) + \mathrm{Ln}\left(\frac{p^2}{p^2 - \Sigma^2(p)}\right) - \frac{\Sigma(p)\ (\not{p} + \Sigma(p))}{p^2 - \Sigma^2(p)}\right]
\end{aligned} \tag{C1.2}$$

here and later on, P represents the momentum vector and p represents the magnitude of P.

$$\mathrm{Ln}\left(\frac{p^2+\not{p}\Sigma}{p^2}\right) = \mathrm{Ln}\left(1+\frac{\not{p}\Sigma}{p^2}\right) = -\sum_n \left(\frac{1}{n}\right)\left(-\frac{\not{p}\Sigma}{p^2}\right)^n$$

$$= -\sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{\Sigma^{2n}}{p^{2n}}\right) + \sum_{n=1}^{\infty} \frac{1}{2n+1}\, \not{p}\, \frac{\Sigma\,(p\,\Sigma)^{2n}}{(p^2)^{2n+1}}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\Sigma^2}{p^2}\right)^n + \sum_{n=1}^{\infty} \frac{1}{2n+1}\, \not{p}\, \frac{\Sigma\,(p\,\Sigma)^{2n}}{(p^2)^{2n+1}}$$

$$= -\frac{1}{2}\,\mathrm{Ln}\left(1-\frac{\Sigma^2}{p^2}\right) + \sum_{n=1}^{\infty} \frac{1}{2n+1}\, \not{p}\, \frac{\Sigma\,(p\,\Sigma)^{2n}}{(p^2)^{2n+1}} \qquad \text{(C1.3)}$$

Substituting (C1.3) into (C1.2) and noticing that

$$\mathrm{Tr}\,\not{p} = 0$$

one has

$$V_0 = 4\,i \int (dP)\left[\frac{1}{2}\,\mathrm{Ln}\left(1-\frac{\Sigma^2}{p^2}\right) + \frac{\Sigma^2(p)}{p^2-\Sigma^2(p)}\right]$$

$$= -4\,i \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}\,\mathrm{Ln}\left(1-\frac{\Sigma^2}{p^2}\right) + \frac{\Sigma^2(p)}{p^2-\Sigma^2(p)}\right]$$

Doing a Wick rotation, the above equation can be written in terms of the integration over the Euclidean momentum space:

$$V_0 = \frac{2}{\pi^2}\int p^2\,dp\left[\frac{\Sigma^2(p)}{p^2+\Sigma^2(p)} - \frac{1}{2}\,\mathrm{Ln}\left(1+\frac{\Sigma^2(p)}{p^2}\right)\right] \qquad \text{(C1.4)}$$

in (C1.4) p now is Euclidean momentum.

$$V_1[\Sigma] = -\frac{i}{2}\,\mathrm{Tr}(SDS)$$

$$= -\frac{i}{2}\int (dP)\,(dK)\,(-i\,e)^2\,\mathrm{Tr}\left[\gamma^\mu\,(i\,S(P)\,\gamma^\nu\,(i\,S(K))\right]\left[i\,D_{\mu\nu}(P-K)\right]$$

$$= -\frac{i}{2} \int (dP)\,(dK)\; (-i e)^2 \; \mathrm{Tr}\left[\gamma^{\mu} \frac{i}{\not{P} - \Sigma(p)} \gamma^{\nu} \frac{i}{\not{K} - \Sigma(k)} \right]$$

$$\cdot \left\{ -i \left[\eta_{\mu\nu} - \frac{(P-K)_{\mu} (P-K)_{\nu}}{(P-K)^2} \right] \frac{1}{(P-K)^2 \left[1 + \frac{\alpha}{8\sqrt{-(P-K)^2}} \right]} \right\}$$

$$= -\frac{\alpha}{2N} \int (dP)\,(dK)\; \frac{1}{p^2 - \Sigma^2(p)} \; \frac{1}{k^2 - \Sigma^2(k)} \; \mathrm{Tr}\left[\gamma^{\mu} (\not{P} + \Sigma(p))\, \gamma^{\nu} (\not{K} + \Sigma(k)) \right]$$

$$\cdot \left\{ \left[\eta_{\mu\nu} - \frac{(P-K)_{\mu} (P-K)_{\nu}}{(P-K)^2} \right] \frac{1}{(P-K)^2 \left[1 + \frac{\alpha}{8\sqrt{-(P-K)^2}} \right]} \right\}$$

(C1.5)

$$\mathrm{Tr}\left[\gamma^{\mu} (\not{P} + \Sigma(p))\, \gamma^{\nu} (\not{K} + \Sigma(k)) \right]$$

$$= 4 P^{\mu} K^{\nu} + 4 P^{\nu} K^{\mu} + 4 \eta^{\mu\nu} \Sigma(p)\, \Sigma(k) - 4 \eta^{\mu\nu} P \cdot K \qquad \text{(C1.6)}$$

A simple calculation also shows that

$$\mathrm{Tr}\left[\gamma^{\mu} (\not{P} + \Sigma(p))\, \gamma^{\nu} (\not{K} + \Sigma(k)) \right] \left[\eta_{\mu\nu} - \frac{(P-K)_{\mu} (P-K)_{\nu}}{(P-K)^2} \right]$$

$$= 8\, \Sigma(p)\, \Sigma(k) - \frac{8 \left[P \cdot (P-K) \right] \left[K \cdot (P-K) \right]}{(P-K)^2} \qquad \text{(C1.7)}$$

With the second term in (C1.7), $V_1[\Sigma]$ is a constant at $\Sigma = 0$. For simplicity, we require

$$V_1[\Sigma] = 0 \qquad \text{at } \Sigma = 0$$

Therefore, we substract that constant from V_1 and get

$$V_1[\Sigma] = -\frac{\alpha}{2N} \times 8 \int (dP)\,(dK)\; \frac{\Sigma(p)}{p^2 - \Sigma^2(p)}\; \frac{\Sigma(k)}{k^2 - \Sigma^2(k)}$$

$$\cdot \left\{ \frac{1}{(P-K)^2 \left[1 + \frac{\alpha}{8\sqrt{-(P-K)^2}}\right]} \right\}$$

$$= -\frac{4\alpha}{N} \int i\, \frac{p^2\, dp\, d\Omega_p}{(2\pi)^3} \int i\, \frac{k^2\, dk\, d\Omega_k}{(2\pi)^3}\; \frac{\Sigma(p)}{p^2 + \Sigma^2(p)}\; \frac{\Sigma(k)}{k^2 + \Sigma^2(k)}$$

$$\cdot \left\{ \frac{1}{(P-K)^2 \left[1 + \frac{\alpha}{8\sqrt{(P-K)^2}}\right]} \right\}$$

$$= -\frac{4\alpha}{(2\pi)^6 N}\, 4\pi \int p^2 dp\, k^2 dk\, \frac{\Sigma(p)}{p^2+\Sigma^2(p)}\; \frac{\Sigma(k)}{k^2+\Sigma^2(k)}$$

$$\cdot \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi\, \sin\varphi\, \frac{1}{(p^2+k^2-2p\cdot k\cos\varphi)^{1/2}\left[\frac{\alpha}{8} + (p^2+k^2-2p\cdot k\cos\varphi)^{1/2}\right]}$$

$$= -\frac{32\alpha\,\pi^2}{(2\pi)^6 N} \int p^2 dp\, k^2 dk\, \frac{\Sigma(p)}{p^2+\Sigma^2(p)}\; \frac{\Sigma(k)}{k^2+\Sigma^2(k)}\; \frac{1}{pk}\, \mathrm{Ln}\left(\frac{p+k+\frac{\alpha}{8}}{|p-k| + \frac{\alpha}{8}} \right)$$

$$= -\frac{2}{\pi^2} \times \frac{1}{2} \int p^2 dp\, k^2 dk\, \frac{\Sigma(p)}{p^2+\Sigma^2(p)}\; \frac{\Sigma(k)}{k^2+\Sigma^2(k)}\, M(p,q) \qquad \text{(C1.8)}$$

where

$$M(p,k) = \frac{\alpha}{2\pi^2 Npk} \, Ln \, (\frac{p+k+\frac{\alpha}{8}}{|p-k|+\frac{\alpha}{8}}) \qquad (C1.9)$$

From (C1.4) and (C1.8), we finally have

$$V_{CJT}[\Sigma] = V_0 + V_1$$

$$= \frac{2}{\pi^2} \{ \int p^2 dp \, [\frac{\Sigma^2(p)}{p^2+\Sigma^2(p)} - \frac{1}{2} Ln \, (1+\frac{\Sigma^2(p)}{p^2})]$$

$$- \frac{1}{2} \int p^2 dp \, k^2 dk \, \frac{\Sigma(p)}{p^2+\Sigma^2(p)} \, M(p,k) \, \frac{\Sigma(k)}{k^2+\Sigma^2(k)} \} \qquad (C1.10)$$

Except for the constant $2/\pi^2$, (C1.10) is exactly (2.59a).

2. Evaluation of the AF effective potential:

Following from (2.57b), one has the AF effective potential:

$$V_{AF}[\Sigma] = -i \, Tr \, Ln \, (S_0^{-1} S) - \frac{i}{2} Tr \, [\, \Sigma(p) \, D^{-1}(P-K) \, \Sigma(k) \,]$$

$$= -i \int (dP) \, Tr \, Ln \, [S_0^{-1}(P) \, S(P) \,] - \frac{i}{2} \int (dP) \int (dK) \, Tr \, [\, \Sigma(p) \, D^{-1}(P-K) \, \Sigma(k)$$

$$(C2.1)$$

Simply,

$$V_0[\Sigma] = -i \int (dP) \, Tr \, Ln \, [S_0^{-1}(P) \, S(P) \,]$$

$$= -\frac{1}{\pi^2} \int p^2 dp \, Ln \, (1+\frac{\Sigma^2(p)}{p^2}) \qquad (C2.2)$$

We now compute

$$V_1[\Sigma] = -\frac{i}{2}\int(dP)\int(dK)\,\mathrm{Tr}\,[\,\Sigma(p)\,D^{-1}(P-K)\,\Sigma(k)\,] \tag{C2.3}$$

where $D^{-1}(P-K)$ is the inverse of $D(P-K)$, that is,

$$\int(dK)\,D^{-1}_{\alpha\beta',\gamma\delta}(P-K)\,D_{\beta\beta,\delta\delta}(K-Q) = \delta^{(3)}(P-Q)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} \tag{C2.4}$$

and the definition of $D(P-K)$ is given by (2.38), explicitly,

$$D_{\alpha\delta,\gamma\beta}(K-Q) = i\,e^2\,(\gamma^\mu)_{\alpha\beta}\,(\gamma^\nu)_{\gamma\delta}\,[\,\eta_{\mu\nu} - \frac{(K-Q)_\mu(K-Q)_\nu}{(K-Q)^2}\,]\,\frac{1}{(K-Q)^2\,[1+\frac{\alpha}{8\sqrt{-(K-Q)^2}}\,]} \tag{C2.5}$$

Since we are only looking for a singlet fermion self-energy solution

$$\Sigma_{\alpha\beta}(p) = \Sigma(p)\,\delta_{\alpha\beta}$$

the photon propagator given in (C2.4) can be replaced with

$$D_{\alpha\delta,\gamma\beta}(K-Q) = i\,e^2\,\delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\frac{1}{(K-Q)^2\,[1+\frac{\alpha}{8\sqrt{-(K-Q)^2}}\,]} \tag{C2.6}$$

therefore,

$$D^{-1}_{\alpha\delta,\gamma\beta}(K-Q) = \bar{D}^{-1}(K-Q)\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} \tag{C2.7}$$

one then has, following (C2.4),

$$\int(dK)\,\bar{D}^{-1}(P-K)\,D(K-Q) = \delta^{(3)}(P-Q) \tag{C2.8}$$

Writing out the above equation in terms of the Euclidean momenta and performing the integration over the angular part yields:

$$\int \frac{i\, k^2 dk}{(2\pi)^3} \bar{D}^{-1}(P-K) \int d\Omega_k \frac{(2i\, e^2)}{-(k^2+q^2-2kq\cos\theta_k)\left[1+\frac{\alpha}{8\sqrt{k^2+q^2-2kq\cos\theta_k}}\right]}$$

$$= \int k^2 dk\, \bar{D}^{-1}(P-K) \frac{\alpha}{2\pi^2 N} \frac{1}{pq} \operatorname{Ln}\left(\frac{p+q+\frac{\alpha}{8}}{|p-q|+\frac{\alpha}{8}}\right)$$

$$= \int k^2 dk\, \bar{D}^{-1}(P-K)\, M(k,q) = \delta^{(3)}(P-Q) \qquad \text{(C2.9)}$$

Integrating over the angular part of both sides of (C2.9):

$$\int \frac{i\, d\Omega_p}{(2\pi)^3} \int k^2 dk\, \bar{D}^{-1}(P-K)\, M(k,q) = \int \frac{i\, d\Omega_p}{(2\pi)^3} \delta^{(3)}(P-Q) \qquad \text{(C2.10)}$$

Define

$$M^{-1}(p,k) = \int \frac{i\, d\Omega_p}{(2\pi)^3} \bar{D}^{-1}(P-K) \qquad \text{(C2.11)}$$

The right hand side of (C2.10) is given as

$$\int \frac{i\, d\Omega_p}{(2\pi)^3}\, \delta^{(3)}(P-Q) = \frac{1}{p^2}\, \delta(p-q) \qquad \text{(C2.12)}$$

here p and q on the right hand side of (C2.12) are all Euclidean momenta.

(C2.12) comes from the fact that

$$\int (dP)\, \delta^{(3)}(P-Q) = 1 \qquad \text{(C2.13)}$$

Hence, we have from (C2.10) that

$$\int k^2 dk\, M^{-1}(p,k)\, M(k,q) = \frac{1}{p^2}\, \delta(p-q) \qquad \text{(C2.14)}$$

$$V_1[\Sigma] = -\frac{i}{2}\int\frac{i\,p^2dp\,d\Omega_p}{(2\pi)^3}\int\frac{i\,k^2dk\,d\Omega_k}{(2\pi)^3}\;4\,\Sigma(p)\,\Sigma(k)\,\overline{D}^{-1}(P-K)$$

$$= \frac{8\pi}{(2\pi)^3}\int p^2dp\int k^2dk\,\Sigma(p)\,\Sigma(k)\int\frac{i\,d\Omega_p}{(2\pi)^3}\,\overline{D}^{-1}(P-K)$$

$$= \frac{1}{\pi^2}\int p^2dp\int k^2dk\,\Sigma(p)\,M^{-1}(p,k)\,\Sigma(k) \qquad (C2.15)$$

where definition (C2.11) is used.

Finally, we obtain

$$V_{AF}[\Sigma] = V_0 + V_1$$

$$= \frac{2}{\pi^2}\Big[-\frac{1}{2}\int_0^{\infty}p^2dp\,\mathrm{Ln}\Big(1+\frac{\Sigma^2(p)}{p^2}\Big) + \frac{1}{2}\int_0^{\infty}p^2dp\int_0^{\infty}k^2dk\,\Sigma(p)\,M^{-1}(p,k)\,\Sigma(k)\Big]$$

(C2.16)

This gives the potential (2.59b).

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