RESOLVABLE PATH DESIGNS OF COMPLETE GRAPHS

by

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B.Sc., Fudan University, 1983

Ext. Dip., Simon Fraser University, 1985

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics and Statistics

SIMON FRASER UNIVERSITY

August 1987

APPROVAL

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Degree: MASTER OF SCIENCE

Title of thesis: Resolvable Path Designs of complete graphs

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RESOLVABLE PATH DESIGNS OF COMPLETE GRAPHS

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ABSTRACT

A resolvable path design is a decomposition of λ copies of K_n (λK_n) into edge-disjoint subgraphs such that each subgraph consists of n/k vertex-disjoint paths of length k-1, each with k vertices. We also call a resolvable path design a P_k-factorization of λK_n and each subgraph a P_k-factor.

J. D. Horton found necessary conditions, and conjectured that they were also sufficient for the existence of resolvable path designs of λK_n. He proved that for λ=1 the conditions are asymptotically sufficient (that is, for each value of k the design exists if n is sufficiently large) and that they are sufficient for any λ when k=3. In this thesis we prove that the conjecture is sufficient when k is even and λ=1, and for all values of k when λ=2.

In the second part of the thesis we investigate the following two questions:

(1) For given integers s and t, under what conditions can λK_n be decomposed into s 1-factors and t P_3-factors?

(2) For given integers s and t, under what conditions can λK_n be decomposed into s 1-factors and t P_4-factors?

Necessary and sufficient conditions are found for both questions.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to Dr. K. Heinrich for suggesting this topic and guidance. This work would not have been possible without her tremendous help and encouragement.

I would also like to thank Gillian Nonay and Shelly Wismath for their advice. Finally my special thanks to Mrs. Sylvia Holmes for her help during my M.Sc. studies.

The financial support provided by Simon Fraser University is also gratefully acknowledged.

DEDICATION

To my parents and sisters

TABLE OF CONTENTS

LIST OF FIGURES

Figure | Page

PART A

INTRODUCTION

The complete graph on n vertices, in which each vertex is joined precisely once by an edge to each other vertex, is denoted by K_n. We denote by λK_n the graph with n vertices in which each vertex is joined by precisely λ edges to each other vertex.

A G-*factorization* of λK_n is a partition of the edges of λK_n into disjoint spanning subgraphs each of which is the vertex disjoint union of $n/|V(G)|$ copies of G. These spanning subgraphs are called G-*factors* of λK_n. In the special case when $G=P_k$, where P_k is a path with k vertices, we call such a partition a *resolvable path design* or a P_k-*factorization*. If C_k denotes the cycle with length k, then we similarly define a C_k-*factorization*. When G is regular of degree r, we may also call the G-factorization (G-factor) an r-*factorization* (r-*factor*). If the graph H has a G-factorization, then we say that H is G-*factorable*. Factorizations are often called decompositions.

The Oberwolfach problem was formulated by G. Ringel and first mentioned in [2]. It asks: Given integers $r_1,r_2,\ldots,r_t$ all at least 3 and so that $r_1+\ldots+r_t=n$ where n is odd, is it possible to find a 2-factorization of K_n so that each 2-factor contains a cycle of each length $r_1,\ldots,r_t$? This problem has been studied for a long time and there are many papers on it (see for example [1], [4], and [7]). One of the more interesting cases of the Oberwolfach problem is when $r_i=r_j$, where $1\le i,j\le t$. This has been solved in [1], where the authors also show that, provided n is even and not equal to 4m, m is odd, and $K_n\backslash F$ (the complete graph with a 1-factor removed) has a C_m-factorization whenever m divides n. The case precluded remains unsettled.

J. D. Horton [6] considered the related question for path decompositions of λK_n. These are resolvable path designs or path factorizations. The question is, given an integer $k \geq 2$, is it possible to find a P_k-factorization of λK_n? The difference between this problem and the Oberwolfach problem is that Horton considered paths of the same length instead of cycles and with the complete multigraph. He made the following conjecture.

Conjecture. A resolvable path design of λK_n with path length k exists (or λK_n is P_k-factorable), if and only if k divides n and $n(k-1)/k$ divides $\lambda n(n-1)/2$.

It is not difficult to see that these conditions are necessary. The first is obtained by counting the number of vertices, and the second by counting the number of edges. Some special cases of this conjecture have been known for a long time. For example, when $k=2$, it asks for a 1-factorization and when $k=n$, for a Hamilton path decomposition. Horton also gave answers for some special cases of this conjecture. The following two theorems are the two main theorems in his paper.

Theorem. Let k be any integer greater than 1. Then there exists a constant $c(k)$ such that if $n>c(k)$, then K_n is P_k-factorable if and only if $n \equiv k^2$ (modulo $\mathrm{lcm}(2k-2,k)$), where $\mathrm{lcm}(a,b)$ denotes the lowest common multiple of a and b.

Theorem. The graph λK_n is P_3-factorable if and only if,

(A) when $\lambda \equiv 1$ or $3 \pmod 4$, then $n \equiv 9 \pmod{12}$,

(B) when $\lambda \equiv 2 \pmod 4$, then $n \equiv 3 \pmod 6$, and

(C) when $\lambda \equiv 0 \pmod 4$, then $n \equiv 0 \pmod 3$.

The proof of the first theorem is based on the fact that for all $k>4$, the existence of resolvable block designs is known asymptotically due to a result of D. Ray-Chaudhuri and R. Wilson [8]. When $k=2$ a resolvable block design on n vertices is simply a 1-factorization of K_n, which exists if and only if n is even. When $k=3$, we have Kirkman Triple Systems which exist if and only if $n \equiv 3 \pmod 6$ [9] and when $k=4$, the resolvable designs exist if and only if $n \equiv 4 \pmod{12}$ [3]. The proof of the second theorem uses Bose's method of pure and mixed differences on the appropriate group. P. Hell and A. Rosa also used this method in [5] to construct several examples of resolvable path designs.

In this thesis, we will provide more evidences for Horton's conjecture by proving the following two results which are the two main theorems in the first part of the thesis.

Theorem. The graph K_n is P_{2k}-factorable if and only if $n \equiv 2k \pmod{2k(2k-1)}$.

Theorem. The graph $2K_n$ is P_k-factorable if and only if ,

(a) when $k=2m$, $n \equiv 2m \pmod{2m(2m-1)}$ and

(b) when $k=2m+1$, $n \equiv (2m+1) \pmod{2m(2m+1)}$.

In the Oberwolfach problem, one case of special interest is that when all cycles have length 3. As we mentioned before these designs are called Kirkman Triple Systems, or C_3-factorizations. R. Rees [10] considered a generalization of this factorization and obtained the following theorem.

Theorem. Let $n \equiv 0 \pmod 6$ and $n \geq 18$. The complete graph K_n can be decomposed into t C_3-factors and s 1-factors if and only if $2t+s=n-1$.

Motivated by Rees' work, we investigated the following two questions.

(1) For given integers s and t, under what conditions can λK_n be decomposed into s 1-factors and t P_3-factors?

(2) For given integers s and t, under what conditions can λK_n be decomposed into s 1-factors and t P_4-factors?

We can think of such path decompositions as resolvable path designs with mixed path lengths. For convience, we introduce the following definition.

An $(s,t)_k$-*factorization* of λK_n is a partition of the edges of λK_n into s P_2-factors (also called 1-factors) and t P_k-factors. In particular, an $(s,0)_k$-factorization is a 1-factorization and a $(0,t)_k$-factorization is a P_k-factorization. We also call λK_n $(s,t)_k$-*factorable* if there exists such factorizations.

The two theorems which answer the above two questions and are discussed in the second part of the thesis can be stated as follows.

Theorem. The graph λK_n has an $(s,t)_3$-factorization if and only if, either

(A) $s=0$, and

(1) $\lambda \equiv 1, 3 \pmod 4$, $n \equiv 9 \pmod{12}$ and $t=3\lambda(n-1)/4$, or

(2) $\lambda \equiv 2 \pmod 4$, $n \equiv 3 \pmod 6$ and $t=3\lambda(n-1)/4$, or

(3) $\lambda \equiv 0 \pmod 4$, $n \equiv 0 \pmod 3$ and $t=3\lambda(n-1)/4$,

(B) $t=0$, $n \equiv 0 \pmod 2$ and $s=\lambda(n-1)$, or

(C) $st \neq 0$, $3s+4t=3\lambda(n-1)$ and $n \equiv 0 \pmod 6$.

Theorem. The graph λK_n is $(s,t)_4$-factorable if and only if, either

(A) $s=0$, and

(1) $\lambda \equiv 0 \pmod 3$, $n \equiv 0 \pmod 4$ and $t=2\lambda(n-1)/3$, or

(2) $\lambda \equiv 1, 2 \pmod 3$, $n \equiv 4 \pmod{12}$ and $t=2\lambda(n-1)/3$,

(B) $t=0$, $n \equiv 0 \pmod 2$ and $s=\lambda(n-1)$, or

(C) $st \neq 0$, $2s+3t=2\lambda(n-1)$ and $n \equiv 0 \pmod 4$.

PART B

RESOLVABLE PATH DESIGNS

As mentioned in the introduction, Horton conjectured necessary and sufficient conditions for the existence of resolvable path designs. We will prove two special cases of his conjecture here. The first case is when $\lambda=1$ and the path length is odd (the path has an even number of vertices) and the second is when $\lambda=2$ and there is no restriction on the path length.

For the convenience of the proof, we introduce the following definition.

In the complete bipartite graph $K_{m,m}$, where $V(K_{m,m})=A\cup B$, $A=\{a_1,\ldots,a_m\}$ and $B=\{b_1,\ldots,b_m\}$, the 1-factor of *distance t* consists of the edges $\{(a_i,b_{i+t}): 1\leq i\leq m\}$, where subscript addition is modulo m and on residues $1,\ldots,m$.

Before we present the two main theorems in this chapter, some lemmas are first proved.

Lemma 1.1. The graph $K_{2m,2m}$ can be decomposed into m P_{2m}-factors and one 1-factor.

Proof. Let $V(K_{2m,2m})=U\cup V$, where $U=\{u_1,\ldots,u_{2m}\}$ and $V=\{v_1,\ldots v_{2m}\}$. To find the P_{2m}-factor we will use the fact that K_{2m} can be decomposed into m Hamilton paths, say $H_1,\ldots,H_m$. Assuming that the vertex set of K_{2m} is $\{1,2\ldots 2m\}$ we construct a P_{2m}-factor of $K_{2m,2m}$ from each H_i as follows. If (k,j) is an edge in H_i, then the P_{2m}-factor contains the edges (u_k,v_j) and (v_k,u_j). It is not difficult to see that the P_{2m}-factors of $K_{2m,2m}$ obtained from H_i and H_j, $i\neq j$, are edge-disjoint.

Repeating this procedure for all H_i, $i=1,\ldots,m$, we obtain m edge-disjoint P_{2m}-factors in $K_{2m,2m}$. On deleting from $K_{2m,2m}$ the edges of these P_{2m}-factors, what remains is a 1-factor in $K_{2m,2m}$ with distance 0. So we are done.

■

Lemma 1.2. The graph $K_{m(2m-1),m(2m-1)}$ is P_{2m}-factorable.

Proof. Let $V(K_{m(2m-1),m(2m-1)})=U\cup V$ where $U=\{u_1,\ldots,u_{m(2m-1)}\}$ and $V=\{v_1,\ldots,v_{m(2m-1)}\}$. Let f_i be the 1-factor of $K_{m(2m-1),m(2m-1)}$ with distance i. We claim that the set of edges obtained from the union of 2m-1 1-factors with consecutive distances $a,a+1,\ldots,a+2m-2$ can be decomposed into m P_{2m}-factors. It is not difficult to see that the union of two consecutive 1-factors, $f_i \cup f_{i+1}$, forms a Hamilton cycle. We can delete 2m-1 independent edges on the Hamilton cycle and obtain 2m-1 2m-paths, a P_{2m}-factor.

Now to verify the claim, we divide each of U and V into 2m-1 blocks so that the vertex labels in each block are consecutive; that is, the blocks are $\{v_{km+i}: 1\le i\le m\}$ and $\{u_{km+i}: 1\le i\le m\}$ where $0\le k\le 2m-1$. Now we divide the proof into two parts.

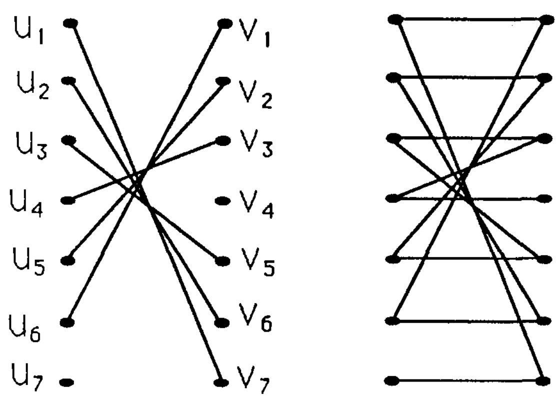
(1) $m=2p+1$.

We consider the 1-factors f_i of $K_{m(2m-1),m(2m-1)}$, where $-(m-1)\le i\le (m-1)$. Pairing f_i and f_{i+1}, where $i=1,3,\ldots,m-2$, and f_{i-1} and f_i, $i=-1,-3,\ldots,-(m-2)$, we obtain m-1 Hamilton cycles. Let $e^1_{i,k}=(u_{km+p-(i-1)/2}, v_{km+p+2+(i-1)/2})$ and

$e^2_{i,k} = (u_{km+p+1-(i+1)/2}, v_{km+p+(i+1)/2})$. From $f_i \cup f_{i+1}$, $i=1,3,\ldots,m-2$, delete the edges of $E^1_i = \{e^1_{i,k}: 0 \le k \le 2m-2\}$, which all have distance $i+1$. Moreover, $(f_i \cup f_{i+1}) \setminus E^1_i$ is a P_{2m}-factor of $K_{m(2m-1),m(2m-1)}$.

From $f_i \cup f_{i-1}$, $i=-1,-3,\ldots,-(m-2)$, delete the edges of $E^2_i = \{e^2_{i,k}: 0 \le k \le 2m-2\}$, which all have distance i. Moreover, $(f_i \cup f_{i-1}) \setminus E^2_i$ is a P_{2m}-factor of $K_{m(2m-1),m(2m-1)}$.

Now let us study the edges in $E^1_i \cup E^2_i$. First we partition them into sets $E(k)$ where $E(k) = \{e^1_{i,k}: i=1,3,\ldots,m-2\} \cup \{e^2_{i,k}: i=-1,-3,\ldots,-(m-2)\}$, $0 \le k \le 2m-2$. For example, when $m=7$, $E(0)$ is as shown in Figure 1.1(a).



(a) (b)

Fig. 1.1

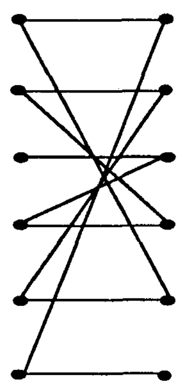


Fig. 1.2

It is clear that for each value of k, E(k) is isomorphic to this graph and disjoint from it. Now add the edges of f_0. It is easy to check that $f_0 \cup \{E(k): 0 \le k \le 2m-2\}$ is a P_{2m}-factor (see Figure 1.1(b)).

(2) $m=2p$.

Let $e^1_{i,k}=(u_{km+p-1-(i-1)/2}, v_{km+p+1+(i-1)/2})$ and $e^2_{i,k}=(u_{km+p-(i-1)/2}, v_{km+p+1+(i-1)/2})$. Consider the 1-factors f_i where $i=-m,-(m-1),..,-1,0,1,..,m-3,m-2$. Pair these 1-factors as $f_i \cup f_{i+1}$, $i=1,3,..,m-3$, and $f_i \cup f_{i-1}$, $i=-1,-3,..,-(m-1)$. From each of the first pairings, we respectively delete the edges in set $E^1_i=\{e^1_{i,k}: 0\le k\le 2m-2\}$, $i=1,3,..,m-3$, and from the second pairings delete (respectively) $E^2_i=\{e^2_{i,k}: 0\le k\le 2m-2\}$, $i=-1,-3,..,-(m-1)$. It is not difficult to check that each of $(f_i \cup f_{i+1})\backslash E^1_i$, $i=1,3,..m-3$, and $(f_{i-1} \cup f_i)\backslash E^2_i$, $i=-1,-3,..,-(m-1)$ is a P_{2m}-factor. Let $E(k)=\{e^1_{i,k}: i=1,3,...,m-3\}\cup\{e^2_{i,k}: i=-1,-3,...,-(m-1)\}$ where $0\le k\le 2m-2$. As before, it is easy to see that $f_0 \cup \{E(k): 0\le k\le 2m-2\}$ is also a P_{2m}-factor in $K_{m(2m-1),m(2m-1)}$. The case $m=6$ is shown in Figure 1.2.

We have proved that the set of edges obtained from the union of 2m-1 1-factors with consecutive distances can be decomposed into m P_{2m}-factors. It is not difficult to see that the edge set of $K_{m(2m-1),m(2m-1)}$ is the union of m edge sets each of which is formed from 2m-1 1-factors with consecutive distances and they are all isomorphic. Therefore, the desired result follows immediately and this completes the proof.

■

The following lemma is an immediate result of Lemma 1.2.

Lemma 1.3. The graph $K_{n,n}$ is P_{2m}-factorable if and only if

$n \equiv 0(\mathrm{mod}(m(2m-1)))$.

Proof: Suppose that $K_{n,n}$ is P_{2m}-factorable then $2n$ is divisible by $2m$ and n^2 is divisible by $n(2m-1)/m$. These two conditions imply $n \equiv 0(\mathrm{mod}(m(2m-1)))$. So we have proven the necessity.

Now we are going to prove that the conditions are sufficient.

We divide the two parts of $K_{n,n}$ into blocks of size $m(2m-1)$. Contracting each block into one vertex, we obtain a $K_{p,p}$ where $n=m(2m-1)p$. We know that $K_{p,p}$ has a 1-factorization and each 1-factor corresponds to p copies of $K_{m(2m-1),m(2m-1)}$ in $K_{n,n}$. By Lemma 1.2, it follows that $K_{n,n}$ is P_{2m}-factorable.

■

Lemma 1.4. The graph K_{4m^2} is the union of a K_{2m}-factor and $2m^2$ P_{2m}-factors.

Proof. We divide $V(K_{4m^2})$ into $2m$ blocks each of size $2m$. Contracting each block into one vertex we obtain a K_{2m}.

We know that K_{2m} has a 1-factorization, say $f_1,\ldots,f_{2m-1}$. Each f_i corresponds to n copies of $K_{2m,2m}$. By Lemma 1.1, $K_{2m,2m}$ can be decomposed into m P_{2m}-factors and one 1-factor. We can assume that the 1-factor left in each copy of $K_{2m,2m}$ has distance 0 and all these edges forms a K_{2m}-factor which is P_{2m}-factorable. Deleting all P_{2m}-factors obtained in this way (total $(2m-1)m$ +m P_{2m}-factors), we are left with a K_{2m}-factor of

K_{4m^2} which is the subgraphs corresponding to the blocks. This completes the proof.

■

Now we present the first main result of this chapter.

Theorem 1.5. The graph K_n is P_{2m}-factorable if and only if $n \equiv 2m(\mathrm{mod}(2m(2m-1)))$.

Proof. Suppose that K_n is P_{2m}-factorable, then n is divisible by 2m and n(n-1) is divisible by n(2m-1)/m. These two conditions imply $n \equiv 2m(\mathrm{mod}(2m(2m-1)))$.

Suppose that $n=2m+2m(2m-1)p$. We will show that K_n is P_{2m}-factorable. To do so we divide the proof into two parts depending on the parity of p.

If $p=2s+1$, then $n=2m(2m-1)(2s+1)+2m$.

We divide $V(K_n)$ into 2s+2 blocks in which 2s+1 of them have size 2m(2m-1) and one has size 2m. Contracting each block into one vertex, we obtain a K_{2s+2}. Taking a 1-factorization of K_{2s+2} yields 2s+1 1-factors, say $f_1,\ldots,f_{2s+1}$ and each f_i corresponds to s disjoint copies of $K_{2m(2m-1),2m(2m-1)}$ and one copy of $K_{2m(2m-1),2m}$ in K_n. For each f_i, we take a P_{2m}-factorization of the subgraph corresponding to the s copies of $K_{2m(2m-1),2m(2m-1)}$. By Lemma 1.3, this is possible and yields $2m^2$ P_{2m}-factors in that subgraph. In the graph $K_{2m(2m-1),2m}$, if we include the edges in both $K_{2m(2m-1)}$ and K_{2m}, we have a K_{4m^2} which , by lemma 1.4, can be factored into $2m^2$

P_{2m}-factors and a K_{2m}-factor one component of which corresponds to the block of size 2m. We delete the $2m^2$ P_{2m}-factors obtained from the $K_{2m(2m-1),2m(2m-1)}$ subgraphs and the K_{4m^2}. Doing this for each f_i, we see that we are left with a K_{2m}-factor in K_n. But K_{2m} is P_{2m}-factorable and hence K_n is P_{2m}-factorable.

If p=2s, the construction is somewhat more complicated. Here $n=2m(2m-1)2s+2m=2m(2s(2m-1)+1)$. We divide $V(K_n)$ into $2s(2m-1)+1$ blocks each of which has size 2m. Contracting each block into one vertex, we obtain a $K_{2s(2m-1)+1}$ which has a near 1-factorization, say $f_1,\ldots,f_{(2m-1)2s+1}$ and each f_i corresponds to a K_{2m} and $s(2m-1)$ copies of $K_{2m,2m}$ in K_n. By Lemma 1.1, $K_{2m,2m}$ can be decomposed into m P_{2m}-factors and one 1-factor. We also know that K_{2m} can be decomposed into m P_{2m}-factors. Therefore, in the subgraph corresponding to f_i, we delete the m P_{2m}-factors so that we are left with an 1-factor in each $K_{2m,2m}$ and 2m isolated vertices. Repeating this procedure for all the near 1-factors, we obtain a graph in which there is only one 1-factor between each pair of blocks in K_n. It is important to note that in obtaining this graph, we were free to choose the 1-factors between pairs of blocks independently.

We label the blocks from 1 to $2s(2m-1)+1$ and for the block i, we label its vertices from $v(i,1)$ to $v(i,2m)$. Now we are going to prove that the graph as described above is P_{2m}-factorable. (There will be $s(2m-1)+s$ P_{2m}-factors.)

Consider the contracted graph $K_{2s(2m-1)+1}$, we know that it can be decomposed into s(2m-1) Hamilton cycles. Fix one of these Hamilton cycles, say (1,2,...,2s(2m-1)+1,1), where i is the label of the corresponding block. For each edge (i,i+1), $1 \le i \le 2s(2m-1)$, and (2s(2m-1)+1,1) in the cycle, we choose the corresponding 1-factor between the two blocks to be the 1-factor with distance 1 (think of the vertices on the cycle being ordered by their positions on the cycle). If we now delete all of these edges between vertex sets {v(1,i),...,v(2s(2m-1)+1,i)} and {v(1,i+1),...,v(2s(2m-1)+1,i+1)} for a fixed i, where $1 \le i \le 2m$, then what remains will be a P_{2m}-factor. In Figure 1.3 the case m=2, s=1 and i=1 is shown. Repeating this procedure for each of the s(2m-1) Hamilton cycles, we obtain s(2m-1) P_{2m}-factors.

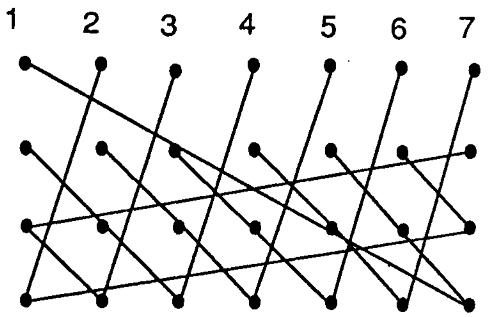


Fig. 1.3

Notice that the edges deleted from each Hamilton cycle are independent. Now we are going to prove that if we suitably choose the sets of independent edges for each Hamilton cycle, the union of them will form s P_{2m}-factors.

We divide the s(2m-1) Hamilton cycles into s groups so that each group has 2m-1 Hamilton cycles. We claim that in each group, the union of independent edge sets, if choosen suitably, forms a P_{2m}-factor. In each group, we label the Hamilton cycles from 1 to 2m-1 and then we choose, from the cycle labelled i, the independent edge set as the edges between $\{v(1,i),v(2,i),...,v(2s(2m-1)+1),i)\}$ and $\{v(1,i+1),v(2,i+1),...,v(2s(2m-1)+1,i+1)\}$. It is not difficult to check that union of the 2m-1 independent edge sets from the cycles in each group will form a P_{2m}-factor. This can be seen from Figure 1.4 in the case when m=2 and s=1. Since there are s groups, we obtain s P_{2m}-factors.

This completes the proof.

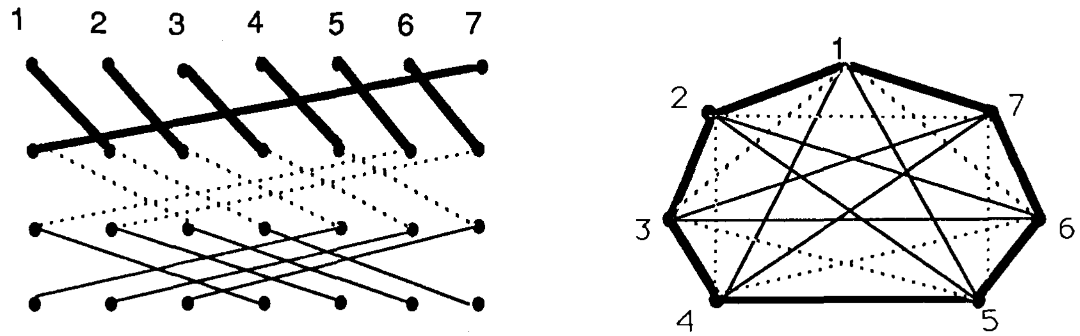


Fig. 1.4.

■

The following lemmas are used for proving Theorem 1.12, the second main result of the chapter. The idea used here is similar to that used before, but the construction is somewhat more complicated.

Lemma 1.6. The graph K_{2n+1} can be decomposed into n edge-disjoint Hamilton paths and an one near 1-factor.

Proof. Let $V(K_{2n+1})=\{0,1,2,...,2n\}$. We arrange the vertices 1 to 2n in a cycle with 0 as the centre and the vertices labelled is in increasing order in a clockwise dirction. Let H be the Hamilton cycle (0,1,2,2n,3,2n-1,...,n,n+2,n+1,0). It is not difficult to see that K_{2n+1} can be obtained by, fixing the vertices of K_{2n+1} and rotating the edges of H n-1 times through an angle π/n about the centre vertex 0. If we delete the edge $([(n+1)/2]+1, 2n+2-[(n+1)/2])$ from H we obtain a Hamilton path (where [x] denotes the largest integer which does not exceed x) Deleting the corresponding edge (under the rotation) from each of the other cycle yields n Hamilton paths and a near 1-factor. Figure 1.5 shows the case when n=5.

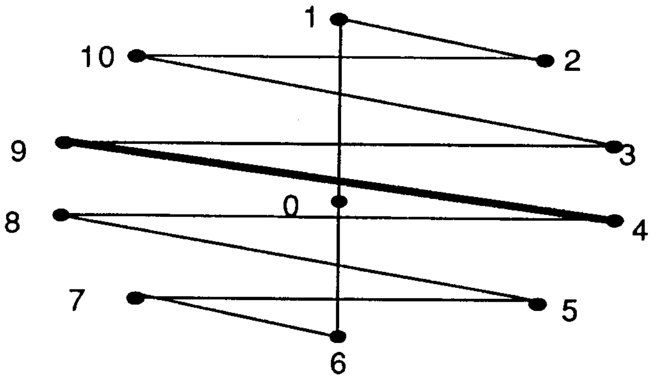


Fig. 1.5

■

Lemma 1.7. The graph $2K_{2m+1}$ is P_{2m+1}-factorable.

Proof. We know that K_{2m+1} can be decomposed into m edge-disjoint Hamilton paths and one near 1-factor (see Lemma 1.6). This near 1-factor can be chosen arbitarily. So we take such a decomposition of each of the two copies of K_{2m+1} so that the union of the two near 1-factors yields a Hamilton path. Hence, $2K_{2m+1}$ can be decomposed into 2m+1 Hamilton paths (P_{2m+1}-factors).

∎

Lemma 1.8. The graph $2K_{2m+1,2m+1}$ can be decomposed into (2m+1) P_{2m+1}-factors and two 1-factors.

Proof. By Lemma 1.7, $2K_{2m+1}$ is P_{2m+1}-factorable. Therefore, we can use the same method as in the proof of Lemma 1.1 to get the desired result.

∎

Lemma 1.9. The graph $2K_{(2m+1)^2}$ is the union of a $2K_{2m+1}$-factor and $(2m+1)^2$ P_{2m+1}-factors.

Proof. We arrange the vertex set of $2K_{(2m+1)^2}$ in a (2m+1)x(2m+1) array. The vertices of each row and column form a copy of $2K_{2m+1}$. We take the 2m+1 copies of $2K_{2m+1}$ defined by the columns of the array as the $2K_{2m+1}$-factor. Now we need to prove that the graph obtained on deleting this $2K_{2m+1}$-factor is P_{2m+1}-factorable. Let 2G be the graph obtained from deleting both the $2K_{2m+1}$-factor and the 2m+1 P_{2m+1}-factors obtained from edges define by the 2m+1 rows (see Lemma 1.7). If we consider each column as a block and contract it into a vertex, we obtain a copy of K_{2m+1} which we know has m edge disjoint Hamilton cycles. It is easy to see that if we can prove that the subgraph in G corresponding to one of the Hamilton cycles is P_{2m+1}-factorable, then we are done. We label the blocks on the cycle from 1 to 2m+1 and let the vertices in block i be v(i,1),...,v(i,2m+1). Now we construct the P_{2m+1}-factor as follows. Consider the bipartite graph formed by the edges between blocks 1 and 2. We construct the path (v(1,m+1), v(2,m+2), v(1,m), v(2,m+3),... v(1,2), v(2,2m+1), v(1,1)). (Figure 1.6(a) shows the case for m=2.) We see that each edge of this (2m+1)-path has a different distance and moreover all distances 1,2,...,2m occur on these edges. We take a copy of this subgraph in each bipartite subgraph corresponding to an edge of the Hamilton cycle under consideration. It is easy to check that the resulting graph is a P_{2m+1}-factor (see Figure 1.6(b)). For each i, $1 \le i \le 2m$, we replace the edges (v(a,b),v(c,d)) of the P_{2m+1}-factor by the edges (v(a,b+i),v(c,d+i)), where addition is modulo 2m+1 on the residues 1,2,...,2m+1, so obtaining another 2m P_{2m+1}-factors. Applying this procedure to each Hamilton cycle, we get m(2m+1) P_{2m+1}-factors which together constitute a P_{2m+1}-factorization of G. Duplicate this to obtain the factorizatin of 2G. This completes the proof.

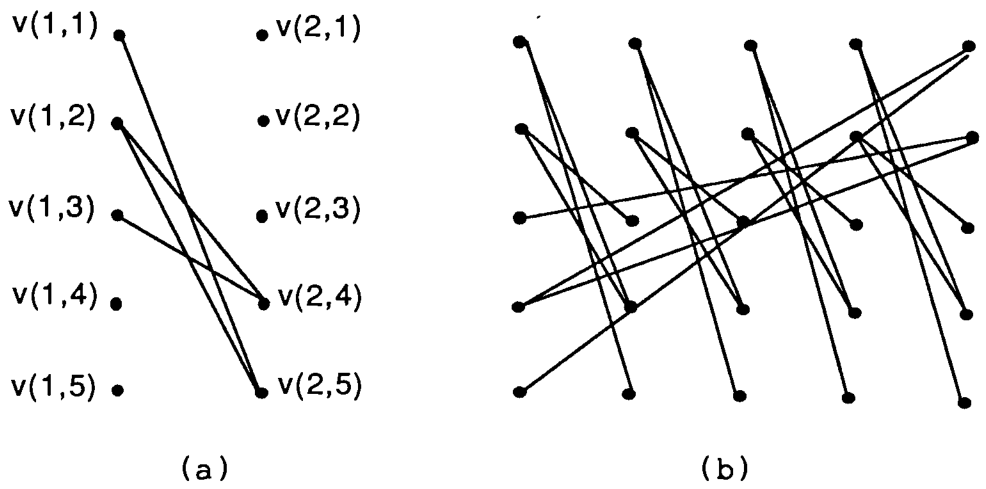


Fig. 1.6

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Lemma 1.10. Let f_i be the 1-factor with distance i in $K_{2m(2m+1),2m(2m+1)}$. Then the subgraph induced by the edges of $\{f_i \mid i \in S\}$ where $S=\{0,1,(2m+1),(2m+1)+1,\ldots,i(2m+1),i(2m+1)+1,\ldots,4m^2-1, 4m^2\}$, is P_{2m+1}-factorable.

Proof. Let $V(K_{(2m+1)2m,(2m+1)2m})=U \cup V$, where $U=\{u_i \mid 1 \le i \le (2m+1)2m\}$ and $V=\{v_i \mid 1 \le i \le (2m+1)2m\}$. Pairing the 1-factors $f_{i(2m+1)}$ and $f_{i(2m+1)+1}$, $0 \le i \le 2m-1$, we obtain 2m Hamilton cycles. For any Hamilton cycle we can delete 4m independent edges so that from the remaining edges in the cycle we obtain a P_{2m+1}-factor. (Each of the two 1-factors has 2m edges removed.) If we apply this procedure to each Hamilton cycle, we are left with 2m(4m) edges which is the same as the number of edges in a P_{2m+1}-factor of $K_{2m(2m+1),2m(2m+1)}$.

The question is whether we can suitably choose such sets of independent edges so that their union is a P_{2m+1}-factor. We will show that the sets can be so chosen. Consider the ordered sets $S_e=\{0,(2m+1)+1,\ldots,2i(2m+1),(2i+1)(2m+1)+1,\ldots, 2(m-1)(2m+1), 4m^2\}$ and $S_o=\{1, (2m+1),\ldots,2i(2m+1)+1, (2i+1)(2m+1),\ldots, 2(m-1)(2m-1)+1, 4m^2-1\}$. It is easy to see that $S=S_o \cup S_e$ and $|S_o|=|S_e|=2m$.

We now show that by suitably removing 2m edges from each of the 1-factors with distances in S_e, we can obtain 2m vertex disjoint (2m+1)-paths. First we choose edge (u_1,v_1) from f_0. Fixing v_1 as an end vertex of the path , we extend this path at u_1 to a (2m+1)-path by using one edge from each of the 1-factors with distances in S_e so that the ith edge in the path is from the 1-factor with the ith element of S_e as its distance. (It will be $(i-1)(2m+1)$ if i is odd and $(i-1)(2m+1)+1$ if i is even.) We call this path P. Construct 2m other (2m+1)-paths from P in the following way. If (u_k,v_t) is an edge of P, let $(u_{k+(2m+1)i},v_{t+(2m+1)i})$ be an edge of the path P_i, $1\le i\le 2m$, where subscript addition is modulo $2m(2m+1)$ on the residues $1,2,\ldots,2m(2m+1)$.

Now we need to show that these paths are indeed vertex disjoint. Suppose that each of U and V are divided into 2m blocks each and each block has 2m+1 vertices. Let the vertices in block i of U be $u_{i+(i-1)(2m+1)},\ldots,u_{2m+1+(i-1)(2m+1)}$ and in block i of V be $v_{i+(i-1)(2m+1)},\ldots,v_{2m+1+(i-1)(2m+1)}$, $1\le i\le 2m$, and denote them by the 1st, 2nd,..., and (2m+1)th positions. As defined before, the ith element in S_e is $(i-1)(2m+1)\equiv 0(\text{mod}(2m+1))$, if i is odd and $(i-1)(2m+1)+1\equiv 1(\text{mod}(2m+1))$ if i is even. This implies that P is incident with vertices in different positions in each of the blocks and in each bipartitions. (Figure 1.7 shows the paths in the case m=2.) Therefore, paths $P,P_1,\ldots,P_{2m}$ are all vertex disjoint. Furthermore, we notice that these paths cover only vertices with position numbers 1 to m in the blocks of U and 1 to m+1 in the blocks of V.

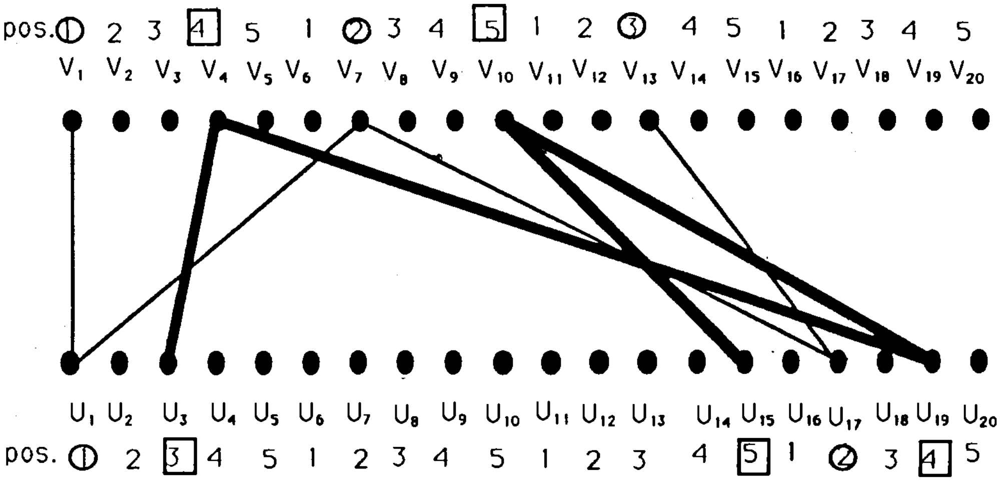

Fig. 1.7

Now we construct another 2m (2m+1)-paths, this time using edges with distances from S_o. We choose the first edge as (u_{m+1},v_{m+2}) and fix u_{m+1} as an end vertex. Using the same method as before we extend it to a (2m+1)-path which we call P'. As before, we obtain 2m (2m+1)-paths. It can be seen that these paths are vertex disjoint. (In Figure 1.7 the bold path shows the edges of P' in the case m=2.)

We notice that these paths cover all vertices with position numbers m+1 to 2m+1 in the blocks of U and m+2 to 2m+1 in the blocks of V. Thus it immediately follows that the 4m (2m+1)-paths form a P_{2m+1}-factor. Now we only need to verify that after we delete the edges in this way, the remaining edges in each Hamilton cycle form a P_{2m+1}-factor.

By the above construction we find that the ith edge in P and the ith edge in P' belong to the same Hamilton cycle as defined what we call in the beginning of the proof. Now we define a modular graph for paths P and P' and we call it G, where $V(G)=S \cup T$ and $S=\{s_1,...,s_{2m+1}\}$ and $T=\{t_1,...,t_{2m+1}\}$. If (u_k,v_j) is an edge in P or P', then (s_a,t_b) is in G where a and b are the values of k and j respectively, modulo 2m+1. (Here the residues are 1,2,...,2m+1.) In this way, we obtain the graph G which is shown in Figure 1.8. It can be seen that corresponding to P and P', we have vertex disjoint paths Q and Q' in G and they all have the same length 2m. In addition, the edges of Q and Q' are all in the Hamilton cycle of G formed by 1-factors with distances 0 and 1. The ith edges of P and P' are also the ith edges of Q and Q'. It is not difficult to find that the length of the path between the ith edges of P and P' on the Hamilton cycle $H_i=f_{(i-1)(2m+1)} \cup f_{(i-1)(2m+1)+1}$ is the same as the path length between the ith edges of Q and Q' on the Hamilton cycle formed by 1-factors with distances 0 and 1. (This can be seen clearly if we construct a modular graph for the corresponding path.) Therefore, from the structure of G we see that the length of the path between the ith edges of P and P' in H_i is 2m. Since i is general, this completes the proof.

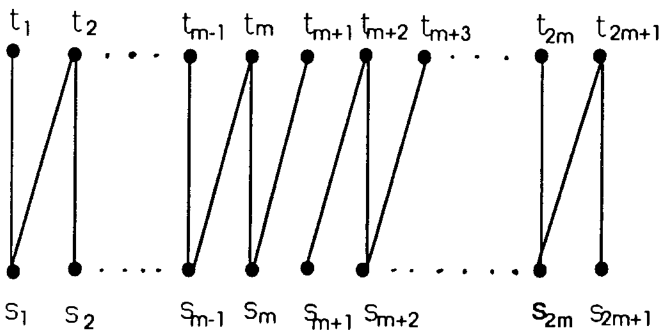

Fig. 1.8

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Lemma 1.11. The graph $2K_{2m(2m+1),2m(2m+1)}$ is P_{2m+1}-factorable.

Proof. We first consider one copy of $K_{2m(2m+1),2m(2m+1)}$. By Lemma 1.10 we know that the subgraph G<E> induced by the edges of $E=\{f_{i(2m+1)},f_{i(2m+1)+1} \mid 0\le i\le 2m-1\}$ can be decomposed into 2m+1 P_{2m+1}-factors. Let $E_j=\{f_{i(2m+1)+2j},f_{i(2m+1)+2j+1}: 0\le i\le 2m-1\}$, $1\le j\le m-1$. It is not difficult to see that $G<E_j>=G<E>$ and that $K_{2m(2m+1),2m(2m+1)}$ is the union of the subgraphs $G<E_j>$, $1\le j\le m-1$, G<E> and $\{f_{(2m+1)i+2m} \mid 0\le i\le 2m-1\}$. Thus $K_{2m(2m+1),2m(2m+1)}$ can be decomposed into m(2m+1) P_{2m+1}-factors and the 2m 1-factors given by $\{f_{i(2m+1)-1} \mid 1\le i\le 2m\}$. In the second copy, we apply the same procedure except that the 2m 1-factors are $\{f_{i(2m+1)} \mid 1\le i\le 2m\}$. This can be done by relabelling the graph. It is not difficult to see that the subgraph $G<E'>$ induced by edge set $E'=\{f_{i(2m+1)-1}, f_{i(2m+1)} | 1\le i\le 2m\}$ is isomorphic $G<E>$. Therefore, $2K_{2m(2m+1),2m(2m+1)}$ is P_{2m+1}-factorable. This completes the proof.

■

By using the preceeding lemmas and Theorem 1.5, we can resolve the case $\lambda=2$ in Horton's conjecture. This is Theorem 1.12 and the techniques used in its proof are similar to those used in the proof of Theorem 1.6.

Theorem 1.12. The graph $2K_n$ is P_k-factorable if and only if
(a) when $k=2m$, $n\equiv 2m(\mathrm{mod}(2m(2m-1)))$ and
(b) when $k=2m+1$, $n\equiv(2m+1)(\mathrm{mod}(2m(2m+1)))$.

Proof. Suppose $2K_n$ is P_k-factorable. When $k=2m$, n must be divisible by 2m and $n(n-1)$ must be divisible by $(2m-1)n/(2m)$. These two conditions imply $n\equiv 2m(\mathrm{mod}(2m(2m-1)))$. When $k=2m+1$, n must be divisible by 2m+1 and $n(n-1)$ must be divisible by $2mn/(2m+1)$. These two conditions imply $n\equiv(2m+1)(\mathrm{mod}(2m(2m+1)))$. So we have proven the necessity. Now we are going to show that the conditions are also sufficient.

When $k=2m$, the result immediately follows from Theorem 1.5.

Assume $k=2m+1$ and $n=2m+1+2m(2m+1)p$. The proof is divided into two parts depending on the parity of p.

If $p=2s+1$, we divide $V(2K_n)$ into 2s+2 blocks in which 2s+1 of them have size $2m(2m+1)$ and one has size 2m+1. Contracting each block into one vertex, we obtain a copy of K_{2s+2}. Taking a 1-factorization of K_{2s+2} yields 2s+1 1-factors, say $f_1,\ldots,f_{2s+1}$ and each f_i corresponds to s disjoint copies of $2K_{2m(2m+1),2m(2m+1)}$ and one copy of $2K_{2m(2m+1),2m+1}$ in $2K_n$. For each f_i, we take a P_{2m+1}-factorization of the subgraph corresponding to the s copies of $2K_{2m(2m+1),2m(2m+1)}$. By Lemma 1.11, this is possible and yields $(2m+1)^2$ P_{2m+1}-factors in that subgraph. In the graph $2K_{2m(2m+1),2m+1}$, if we include the edges in both $2K_{2m(2m+1)}$ and $2K_{2m+1}$, we have a $2K_{(2m+1)^2}$ which is the union of a $2K_{2m+1}$-factor and $(2m+1)^2$ P_{2m+1}-factors. (This is Lemma 1.9.) We delete the $(2m+1)^2$ P_{2m+1}-factors obtained from all $2K_{2m(2m+1),2m(2m+1)}$ and the $2K_{(2m+1)^2}$. We specify that this remaining $2K_{2m+1}$-factor includes the K_{2m+1} which is one of the blocks. Having done this for each f_i, we see that we are left with a $2K_{2m+1}$-factor in $2K_n$. But $2K_{2m+1}$ is P_{2m+1}-factorable and hence $2K_n$ is P_{2m+1}-factorable if $n=2m(2m+1)(2s+1)+2m+1$.

If p=2s, the construction is somewhat more complicated. Here $n=2m(2m+1)2s+2m+1=(2m+1)(4ms+1)$. We divide $V(2K_n)$ into 4ms+1 blocks each of which has size 2m+1. Contracting each block into one vertex, we obtain a copy of K_{4ms+1} which has a near 1-factorization, say $f_1,\ldots,f_{4ms+1}$, and each f_i corresponds to a $2K_{2m+1}$ and 2ms copies of $2K_{2m+1,2m+1}$ in $2K_n$. By Lemma 1.8, $2K_{2m+1,2m+1}$ can be decomposed into 2m+1 P_{2m+1}-factors and two 1-factors (in fact 2m+1 copies of $2K_2$). We also know that $2K_{2m+1}$ can be decomposed into 2m+1 P_{2m+1}-factors. Therefore, in the subgraph corresponding to f_i, we delete the 2m+1 P_{2m+1}-factors so that we are left with two 1-factors in each $2K_{2m+1,2m+1}$ and 2m+1 isolated vertices. Repeating this procedure for all the near 1-factors, we obtain a graph in which there are only two 1-factors (as described above) between each pair of blocks in $2K_n$. It is important to note that in obtaining this graph, we were free to choose the 1-factors between pairs of blocks independently. Now we are going to prove this graph is P_{2m+1}-factorable. We know this graph is a multigraph with multiplicity two. In the following proof we only consider a single copy of it.

We label the blocks from 1 to 4ms+1 and for each block i, we label its vertices from v(i,1) to v(i,2m+1).

Consider the contracted graph K_{4ms+1}, we know that it can be decomposed into 2ms Hamilton cycles. Fix one of these Hamilton cycles, say (1,2,...,4ms+1,1), where i is the label of the block. For each edge (i,i+1), $1 \le i \le 4ms$, and (4ms+1,1) in the cycle, we choose the corresponding 1-factor between the two blocks to be the 1-factor with distance 1. If we now delete all of these edges between vertex sets {v(1,j),...,v(4ms+1,j)} and {v(1,j+1),...,v(4ms+1,j+1)} for a fixed j where $1 \le j \le 2m+1$, then what remains will be a P_{2m+1}-factor. Repeating this procedure for each of the 2ms cycles, we obtain 2ms P_{2m+1}-factors.

Notice that the edges deleted from each Hamilton cycle are independent. Now we are going to prove that if we suitably choose the sets of independent edges from each Hamilton cycle, the union of them will form s P_{2m+1}-factors.

We divide the 2ms Hamilton cycles into s groups so that each group has 2m Hamilton cycles. We claim that in each group, the union of independent edge sets, if choosen suitably, forms a P_{2m+1}-factor. In each group, we label the Hamilton cycles from 1 to 2m and then we choose, from the cycle labelled i, the independent edge set as the edges between $\{v(1,i), v(2,i),...,v(4ms+1,i)\}$ and $\{v(1,i+1), v(2,i+1),..., v(4ms+1,i+1)\}$. It is not difficult to check that the union of the 2m independent edge sets from the cycles in each group will form a P_{2m+1}-factor. Since there are s group, we obtain s P_{2m+1}-factors. In total, we have $(2m+1)(4ms+1)+2(2ms+s)=(2m+1)(4ms+2s+1)$ P_{2m+1}-factors which is a P_{2m+1}-factorization of $2K_n$, where $n=(2m+1)(4ms+1)$.

This completes the proof.

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PART C

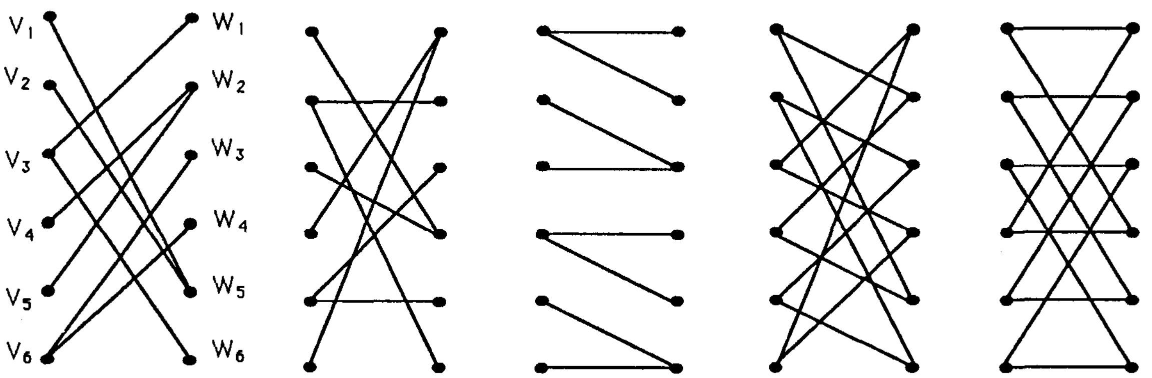
RESOLVABLE PATH DESIGNS WITH MIXED PATH LENGTHS

In this chapter, we prove necessary and sufficient conditions for the existence of $(s,t)_3$-factorizations and $(s,t)_4$-factorizations of λK_n.

The main theorems are Theorem 2.8 and 2.18 and both are proved using recursive constructions. We begin with some lemmas.

Lemma 2.1. Let $V(K_{6,6})=V\cup W$ and $V=\{v_1,..,v_6\}$, $W=\{w_1,..,w_6\}$. The subgraphs $K_{6,6}\backslash\{f_i,f_{i+3}\}$, where $i\in\{0,1,2\}$ and $f_i=\{v_jw_{j+i} : 1\le j\le 6\}$ with all addition modulo 6 and on residues 1,...,6, are both P_3-factorable and C_4-factorable.

Proof. This can be done by direct construction as shown in Figure 2.1 when i=2. It is not difficult to see that $f_i\cup f_{i+3}$ is isomorphic to $f_j\cup f_{j+3}$, $i,j\in\{0,1,2\}$ and each is three 4-cycle. (Figure 2.1(a) is a P_3-factorization and Figure 2.1(b) is a C_4-factorization.)



(a) (b)

Fig. 2.1

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Lemma 2.2. The graph K_6 is $(s,t)_3$-factorable if $3s+4t=15$.

Proof. The non-negative integer solutions of $3s+4t=15$ are $(s,t)=(1,3)$ and $(5,0)$. The latter case is just a 1-factorization which is trivial. When $(s,t)=(1,3)$, we give the following direct construction as shown in Figure 2.2.

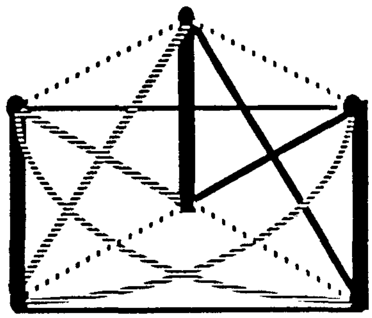

Fig. 2.2

■

Lemma 2.3. The graph K_{12} is $(s,t)_3$-factorable if $3s+4t=33$.

Proof. All the possible non-negative integer solutions of $3s+4t=33$ are $(s,t)=(11,0)$, $(7,3)$ and $(3,6)$. The first case is trivial. When $(s,t)=(7,3)$, we take a 1-factorization of $K_{6,6}$ and two copies of a $(1,3)_3$-factorization of K_6 as given in Lemma 2.2. When $(s,t)=(3,6)$, we take a $(2,3)_3$-factorization of $K_{6,6}$ and two copies of a $(1,3)_3$-factorization of K_6. By Lemmas 2.1 and 2.2, this is possible.

Hence we have the desired decompositions. This completes the proof.

■

Lemma 2.4. Let G be a graph. If G can be decomposed into two C_3-factors, then it can also be decomposed into three P_3-factors.

Proof. Let V_1 and V_2 be two C_3-factors. We consider each 3-cycle in V_1 or V_2 as a vertex of a graph in which two vertices are connected by an edge if and only if the corresponding two 3-cycles have one common vertex. So this is a 3-regular bipartite graph with V_1 and V_2 as the parts of the bipartition. This graph has a 1-factorization with 1-factors f_1, f_2 and f_3. It is not difficult to see that each vertex in G is in the intersection of precisely two 3-cycles; one from each C_3-factor. Hence each vertex in G corresponds to an edge in the above bipartite graph.

Now we label the vertices of G as follows. If $x \in V(G)$ corresponds to the edge belonging to f_i, then we label it i. Each 3-cycle will have its vertices labelled 1,2 and 3.

We decompose each 3-cycle into one 2-path (an edge) and one 3-path. In V_1, we let the 2-path in each 3-cycle be (1,2). On removing these edges from V_1, the subgraph left over forms a P_3-factor in G. In V_2, we let the 2-path be (2,3) and again edges of V_2 left over form another P_3-factor. After we delete the two P_3-factors, all vertices labelled 2, still have degree 2 and the rest labelled 1 and 3 have degree 1. It is easy to see that this graph is a P_3-factor. Therefore, G can be decomposed into three P_3-factors.

∎

(The ideas used in above proof are based on those of Horton in [6]. But the result proved here is slightly different from his.)

Lemma 2.5 [10]. Let $n \equiv 0 \pmod 6$ and $n \geq 18$. K_n can be decomposed into t C_3-factors and s 1-factors if and only if $2t+s=n-1$.

∎

The following theorem is a special case of Theorem 2.8. It will simplify the proof of the Theorem 2.8 if we give it separately.

Theorem 2.6. If $st \neq 0$, then K_n is $(s,t)_3$-factorable if and only if $3s+4t=3(n-1)$ and $n \equiv 0 \pmod 6$.

Proof. Suppose K_n is $(s,t)_3$-factorable. Since $st \neq 0$, n must be divisible by both 2 and 3. This implies that $n \equiv 0 \pmod 6$. By counting the number of edges, it is easy to see that the given equation must be satisfied. So we have proved the necessary conditions.

Now we prove that the conditions are also sufficient.

By Lemmas 2.2 and 2.3, the result is true for $n=6,12$. Now we just consider the case for $n \geq 18$. Let (s,t) be a solution of $3s+4t=3(n-1)$. It is not difficult to see that $t \equiv 0 \pmod 3$ and so we can assume $t=3p$. Then $4(3p)+3s=3(n-1)$ which is $2(2p)+s=n-1$. By Lemma 2.5, there exist a decomposition of K_n into 2p

C_3-factors and s 1-factors. By Lemma 2.4, these 2p C_3-factors can be decomposed into 3p P_3-factors. Hence we are done.

■

Lemma 2.7. The graph $K_{2n,2n,2n}$ is P_3-factorable.

Proof. We name the three parts of $V(K_{2n,2n,2n})$, U, V and W. Let $U=\{u_i \mid 1\le i\le 2n\}$, $V=\{v_i \mid 1\le i\le 2n\}$ and $W=\{w_i \mid 1\le i\le 2n\}$. Now we construct the P_3-factorization as follows.

Define edge sets $S_i=\{u_jv_{j+i}: 1\le j\le 2n\}$, $Q_i=\{v_jw_{j+i}: 1\le j\le 2n\}$ and $R_i=\{w_ju_{j+i}: 1\le j\le 2n\}$. It is not difficult to see that $S_i\cup Q_{i+1}$, $Q_i\cup R_{i+1}$ and $R_i\cup S_{i+1}$ are three P_3-factors of $K_{2n,2n,2n}$. (The case n=2 and i=0 is shown in Figure 2.3) Letting i=0,2,4,...,2n-2 we obtain a P_3-factorization.

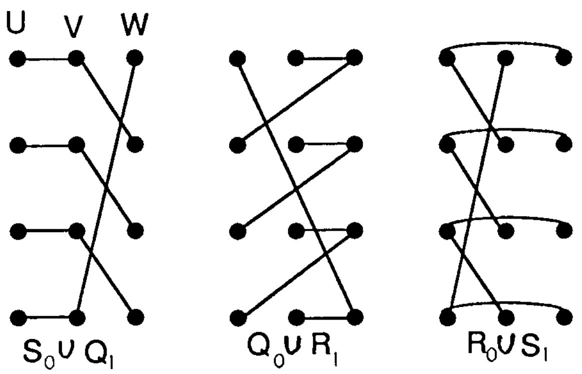

Fig. 2.3

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Actually, for the proof of Theorem 2.8 we only need the result that $K_{4,4,4}$ is P_3-factorable. Since the construction of the proof is easily extended to the general case that is what we have done.

Now we presented the first main theorem of this chapter.

Theorem 2.8. The graph λK_n is $(s,t)_3$-factorable if and only if one of the following holds.

(A) $s=0$, and

(1) $\lambda\equiv 1,\ 3 \pmod 4$, $n\equiv 9 \pmod{12}$ and $t=3\lambda(n-1)/4$,

(2) $\lambda\equiv 2 \pmod 4$, $n\equiv 3 \pmod 6$ and $t=3\lambda(n-1)/4$,

(3) $\lambda\equiv 0 \pmod 4$, $n\equiv 0 \pmod 3$ and $t=3\lambda(n-1)/4$,

(B) $t=0$, $n\equiv 0 \pmod 2$ and $s=\lambda(n-1)$,

(C) $st\neq 0$, $3s+4t=3\lambda(n-1)$ and $n\equiv 0 \pmod 6$.

Proof. For the necessary conditions, (A) is shown in [6] and (B) is quite trivial. By counting the number of edges, we obtain (C).

Now we prove that the conditions are sufficient.

(A). This was done by Horton [6].

(B). This is simply asking for a 1-factorization and is trivial.

(C). First we give the proof for $\lambda=2,3$ and 4. Then we shall extend them for general λ. Let M be the maximum value such that $3N+4M=n-1$, where N, M are non-negative integers. If (s,t) is a solution of $3x+4y=3\lambda(n-1)$ and $t\leq M\lambda$, then the decomposition can be obtained easily by Theorem 2.6. We write $t=t_1+\ldots+t_\lambda$, $t_i\leq M$, and $s=s_1+\ldots+s_\lambda$ so that (s_i,t_i) is a non-negative integer solution of $3x+4y=n-1$. Otherwise, we divide the proof into following two cases.

Case1. $n\equiv 6 \pmod{12}$.

Let $n=12p+6$. It is not difficult to find that when $\lambda=1$, $s\equiv1(\text{mod}4)$, $t\equiv0(\text{mod}3)$ and $M=9p+3$. K_n is $(1,9p+3)_3$-factorable.

For $\lambda=2,3$, t must be no larger than $M\lambda$. The reason is that if $t>M\lambda$, then $t=M\lambda+3k$ where k is non-negative integer. From $3s+4t=3\lambda(n-1)$ we get, on substituting, $12k+3s=3\lambda$ which is impossible when $k>0$. Therefore, we only need to consider $\lambda=4$. In this case, $(s,t)=(0,M\lambda+3)$ is the only solution for which we cannot combine $\lambda=1$ solutions. But this has already been dealt with in (A).

Case2. $n\equiv0(\text{mod}12)$.

Let $n=12p$. In this case, when $\lambda=1$, $s\equiv3(\text{mod}4)$, $t\equiv(\text{mod}3)$ and $M=9p-3$. We know that K_n is $(3,9p-3)_3$-factorable by Theorem 2.6. It is necessary for us to know the structure of the three 1-factors as for $\lambda>1$ we want to get P_3-factors by combining these 1-factors. For this purpose, we give a specific construction of a $(3,9p-3)_3$-factorization. We must consider separately the cases p even and p odd.

Suppose that p is even, so $p=2m$. We divide the vertex set of K_{24m} into 2m blocks of size twelve. By contracting each block into one vertex, we obtain a K_{2m} which has a 1-factorization, say $f_1,\ldots, f_{2m-1}$. Each f_i corresponds to p disjoint copies of $K_{12,12}$. From Lemma 2.3 we have a P_3-factorization of $K_{12,12}$ and a $(3,6)_3$-factorization of K_{12}. Combining these we have the desired factorization of K_{24m}. Now we know the structure of the three 1-factors: in the subgraphs corresponding to each block, two 1-factors are edges of $K_{6,6}$ and the other 1-factor has three edges in each of the K_6 which make up the K_{12}. Note that the third 1-factor can be as any 1-factor in the two K_6 subgraphs.

We now give the proof for $\lambda=2,3$ and 4 when $n=24m$. In each case continue to think of K_{24m} as 2m blocks of size 12.

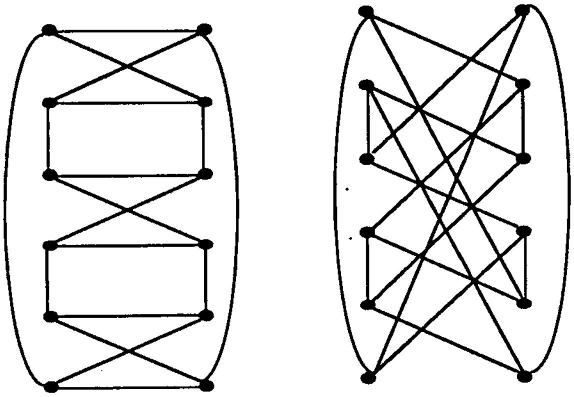
(a) $\lambda=2$.

Here, if $t>2M$, then $t=2M+3$. So if we can show that $2K_n$ is $(2,2M+3)_3$-factorable, then we are done. Take a $(3,M)_3$-factorization of each copy of K_n so that in copy 1, we choose two 1-factors of $K_{6,6}$ in each block with distances 0 and 3 and in copy 2, we choose two 1-factors of $K_{6,6}$ of each block with distances 1 and 4 and we know that the third 1-factor in each copy is not important. By Lemma 2.1, we find that the graph obtained by combining these four specified 1-factors can be decomposed into three P_3-factors. Therefore, $2K_n$ is $(2,2M+3)_3$-factorable.

(b) $\lambda=3$.

Here if $t>3M$, then $(s,t)=(5,3M+3)$ or $(1,3M+6)$. In both cases, we first take a $(3,M)_3$-factorization of each of the three copies of K_n. If $(s,t)=(5,3M+3)$, then we use the same method as in (a) on two of the copies to get the desired decomposition. If $(s,t)=(1,3M+6)$, then we choose the three 1-factors in copy 1 as in Figure 2.4(a) and choose the three 1-factors in copy 2 as in Figure 2.4(b), choose the three 1-factors in copy 3 as in Figure

2.4(c).



(a) (b) (c)

Fig. 2.4

Combining these 1-factors we obtain a graph which is the union of $K_{6,6}\backslash\{f_2, f_5\}$ which is P_3-factorable (see Figure 2.1(a)) and the graph shown in Figure 2.5 which is $(1,3)_3$-factorable. Therefore, we obtain a $(1,3M+6)_3$-factorization of $3K_{24m}$.

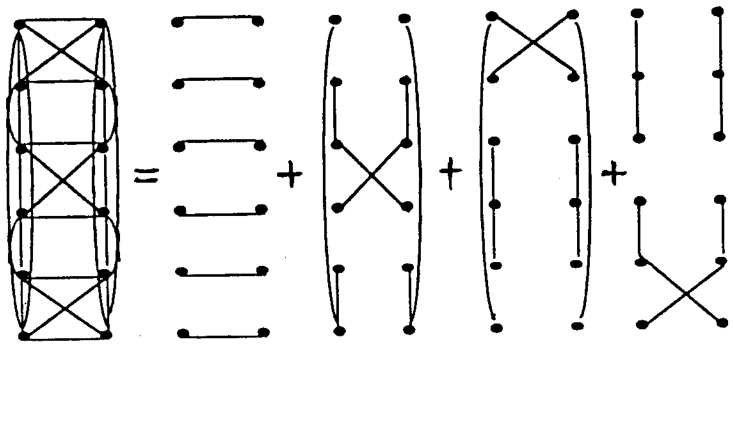


Fig. 2.5

(c) $\lambda=4$.

Here (s,t) must be one of (8,3M+3),(4,3M+6) or (0,3M+9) as all other cases are covered by Theorem 2.6. The first two cases can be done by the same methods as in parts (a) and (b), and the third case is covered by (A).

So now suppose that p is odd, or equivalently $n\equiv12 \pmod{24}$. Let $n=24m+12=4(2m+1)3$. As before, we first give a construction for the extreme case which is a $(3,M)_3$-factorization in K_n.

We divide the vertex set of K_{24m+12} into 3(2m+1) blocks each of size four. By contracting each block into one vertex, we obtain a $K_{3(2m+1)}$. We know there is a 2-factorization of $K_{3(2m+1)}$ in which each 2-factor is a union of 3-cycles (or equivalently, a Kirkman triple system on 6m+3 elements). The subgraph corresponding to each 3-cycle is a $K_{4,4,4}$ in K_n. By Lemma 2.7, $K_{4,4,4}$ is P_3-factorable. Deleting all P_3-factors obtained in this way, we find that the remaining edges consitute 3(2p+1) copies of K_4 which is $(3,0)_3$-factorable. Now we know the structure of the three 1-factors. Before we prove the result for $\lambda=2,3$ and 4, we divide 3(2m+1) K_4 into 2m+1 groups so that each group consists of twelve vertices. We only need to be concerned with the subgraph corresponding to each group.

(a) $\lambda=2$.

As in the case p even, all we need to find is a $(2,2M+3)_3$-factorization. By Lemma 2.1, we know that $K_{6,6}$ has a four regular bipartite subgraph which is not only C_4-factorable, but P_3-factorable as well. Now in each K_{12} we have two copies of three K_4 subgraphs (made up of the 1-factors in the $(3,M)_3$-factorization). If we delete one 1-factor from each of them, we get two copies of three 4-cycles. It is not difficult to see that we can suitably choose the vertices of the three K_4 in each copy so that the resulting two C_4-factors will form the $K_{6,6}$ subgraphs as described above. There are only two 1-factors left. Therefore we are done.

(b) $\lambda=3$.

As before, we only need to find a $(1,3M+6)_3$-factorization to be done. We wish to choose the C_4-factor in each of two copies as in Figure 2.1(b) so that the remaining 1-factors in the three K_4 form the 2-regular graph shown in Figure 2.6.

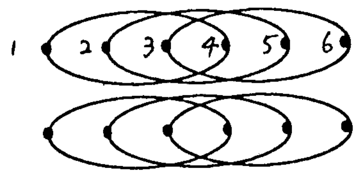


Fig. 2.6

Now we choose the third copy of the three K_4's so that together with the graph in Figure 2.6, it forms the graph of Figure 2.5 which is $(1,3)_3$-factorable. Therefore, we have a $(1,3M+6)_3$-factorization.

(c) $\lambda=4$.

In this case, as before, (s,t) must be one of $(8,3M+3)$, $(4,3M+6)$ and $(0,3M+9)$ as all other cases are covered by Theorem 2.6. Therefore, the desired decompositions immediately follows from the previous proofs.

Up to now, we have proven only that the result is true for $\lambda=1,2,3$ and 4. Now we give the proof for the general λ.

Let $\lambda=4q+i$ where $0\le i\le 3$. When $i=0$, it is not difficult to prove this case by induction on q. Therefore, we assume $1\le i\le 3$. Now let (s,t) be a solution of $3s+4t=3\lambda(n-1)$. If $t\le 3q(n-1)$, we only need to take an $(s-i(n-1),t)_3$-factorization of $4qK_n$ and a 1-factorization of the i other copies of K_n. Otherwise, take a P_3-factorization of $4qK_n$ and a $(s,t-3q(n-1))_3$-factorization of iK_n to yield the desired result. Therefore, we have completed the proof.

■

Now we prove the necessary and sufficient conditions of for the existence of $(s,t)_4$-factorizations of λK_n. As before, we first prove some lemmas which will be used to prove the main theorem. Note that in the proof of the following lemmas, we ignore the 1-factorization case which is trivial there.

Lemma 2.9. The graph $K_{6,6}$ is $(s,t)_4$-factorable if $2s+3t=12$.

Proof. We first find that all possible non-negative integer solutions of $2s+3t=12$. They are $(s,t)=(0,4)$ and $(3,2)$ (provide

$t \neq 0$). The direct contruction proof is shown in Figure 2.7.

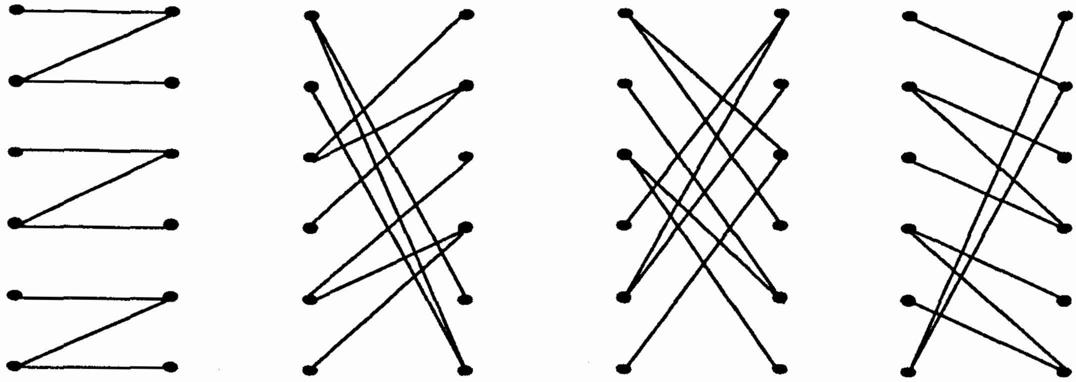


Fig.2.7

Note: the union of the first (third) and second (forth) graph in the Figure 2.7 is a 3-regular subgraphs and consequently each of them can be decomposed into three 1-factors.

■

Lemma 2.10. Let $n \equiv 0 \pmod 6$. The graph $K_{n,n}$ is $(s,t)_4$-factorable if $2s+3t=2n$.

Proof. Let $n=6m$. We divide each part of the vertex set into m blocks of size six. By contracting each blocks into one vertex, we obtain a $K_{m,m}$ which has a 1-factorization, say $f_1,...,f_m$ and each of them corresponds to m disjoint copies of $K_{6,6}$ in $K_{6m,6m}$.

Let (s,t) be a solution of equation $2s+3t=2n=12m$. It is clear that t must be even so we let $t=2p$. Take a P_4-factorization of each of the subgraphs corresponding to $f_1,...,f_{[p/2]}$ yields $4[p/2]$ P_4-factors in $K_{6m,6m}$. If p is even, we complete the $(s,t)_4$-factorization by taking a 1-factorization of the remaining graph. Otherwise, we use Lemma 2.9 to take a $(3,2)_4$-factorization of the subgraph corresponding to $f_{[p/2]+1}$ and then 1-factorize the remaining graph. This yields s 1-factors and 2p P_4-factors.

■

We next construct all $(s,t)_4$-factorizations of K_{12} and K_8.

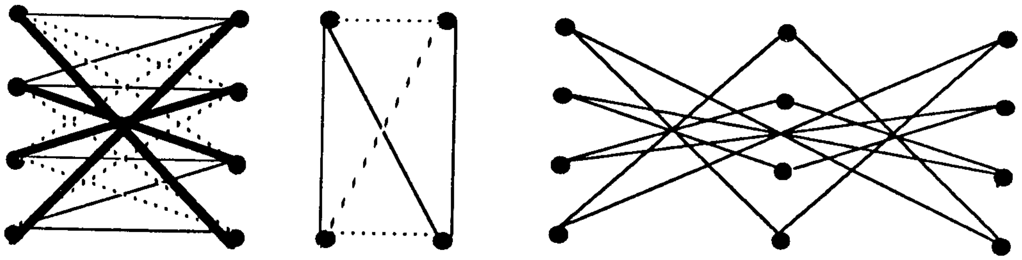
Lemma 2.11. The graph K_{12} is $(s,t)_4$-factorable if and only if $2s+3t=22$.

Proof. The necessity follows by counting the edges.

It is easy to see that all non-negative integer solutions (provided $t\neq 0$) of $2s+3t=22$ are $(s,t)=(2,6)$, $(5,4)$ and $(8,2)$.

(1) $(s,t)=(2,6)$.

We divide the vertex set of K_{12} into three blocks of size four. By contracting each block into one vertex, we obtain K_3 which has a near 1-factorization, say f_1, f_2 and f_3. The subgraph of K_{12} corresponding to each near 1-factor in K_3 ($K_{4,4} \cup K_4$) can be decomposed into two P_4-factors and four independent edges as shown in Figure 2.8(a). These four independent edges can be chosen in such a way that after deleting all the P_4-factors obtained from each f_i, $1\le i\le 3$, (see Figure 2.8(b)) the remaining graph is a 2-factor made up of two 6-cycles which is 1-factorable. So we are done.



(a) (b)

Fig. 2.8

(2) $(s,t)=(5,4)$.

By using Lemma 2.9, we see that $K_{6,6}$ can be decomposed into four P_4-factors. Since K_6 is 1-factorable, we are done.

(3) $(s,t)=(8,2)$.

From Lemma 2.9, $K_{6,6}$ can be decomposed into two P_4-factors and three 1-factors. Since K_6 is 1-factorable, the desired decomposition can be easily obtained.

■

Lemma 2.12. The graph K_8 is $(s,t)_4$-factorable if and only if $2s+3t=14$.

Proof. The necessity follows by counting the edges.

All non-negative integer solutions (provided that $t\neq0$) of $2s+3t=14$ are $(s,t)=(4,2)$ and $(1,4)$.

(1) $(s,t)=(4,2)$.

We know each K_4 can be decomposed into two 4-paths and $K_{4,4}$ is 1-factorable. Thus we are done.

(2) $(s,t)=(1,4)$.

We know $K_{4,4}$ can be decomposed into two P_4-factors and one 1-factor. (See Figure 2.8(a).) So we can get our desired decomposition.

■

The main theorem will be proved by a series of three lemmas each of which deals with one of the residues classes of n modulo 12 where n is divisible by 4.

Lemma 2.13. Let $n\equiv 0 \pmod{12}$. The graph K_n is $(s,t)_4$-factorable if and only if $2s+3t=2(n-1)$.

Proof. The necessity follows by counting the edges.

We divide the proof into two cases. Let (s,t) be a solution of $2s+3t=2(n-1)$. It is not difficult to see that $t\equiv 0 \pmod 2$ and $s\equiv 2 \pmod 3$.

Case1. $n=12(2p)$

We divide $V(K_n)$ into 2p blocks each containing twelve vertices. By contracting each block into one vertex we obtain a K_{2p} which has a 1-factorization, say $f_1, f_2, \ldots, f_{2p-1}$. Each f_i corresponds to p disjoint copies of $K_{12,12}$ in K_n.

In this case, t can be any even number no more than $(2n-6)/3=16p-2$.

If (s,t) satisfies $2s+3t=2(n-1)$ and $t\le 16p-8$, then we take a P_4-factorization of each of the subgraphs corresponding to f_1, $f_2,\ldots,f_{[t/8]}$. This yields $8[t/8]$ P_4-factors in K_n. Next we take a $(12-3(t-8[t/8])/2,t-8[t/8])_4$-factorization of the subgraph corresponding to $f_{[t/8]+1}$ (see Lemma 2.10) and 1-factorize both the subgraphs corresponding to $f_{[t/8]+2},\ldots,f_{2p-1}$ and the 2p disjoint copies of K_{12}.

If $16p-8<t\le 16p-2$, then we take a P_4-factorization of each of subgraphs corresponding to $f_1,\ldots,f_{2p-1}$ and decompose the remainder of the graph (the 2p disjoint copies of K_{12}) into $t-(16p-8)$ P_4-factors and s 1-factors.

Case2. $n=12(2p+1)=6(4p+2)$

As in case1, we find that t can be any even number no more than $(2n-6)/3=16p+6$.

We divide $V(K_n)$ into $(4p+2)$ blocks so that each of the blocks contains six vertices. By contracting each block into one vertex, we obtain a K_{4p+2} which has a 1-factorization, say f_1, $f_2,\ldots,f_{4p+1}$. Each f_i corresponds to $2p+1$ disjoint copies of $K_{6,6}$ in K_n.

If $t\le 16p+4$, then we take a P_4-factorization of each of the subgraphs corresponding to f_1, $f_2,\ldots,f_{[t/4]}$. This yields $4[t/4]$ P_4-factors in K_n. Now take a

$(6-3(t-4[t/4])/2, t-4[t/4])_4$-factorization of the subgraph corresponding to $f_{[t/4]+1}$ (see Lemma 2.10) and 1-factorize each of the subgraphs corresponding to $f_{[t/4]+2}, \ldots, f_{4p+1}$ and each K_6.

If $t>16p+4$, then $t=16p+6$. In this situation, our construction is as follows.

Take a P_4-factorization of each of the subgraphs corresponding to $f_1, \ldots, f_{4p}$ yielding 16p P_4-factors in K_n. By combining the (4p+2) disjoint copies of K_6 and the subgraph corresponding to f_{4p+1}, we obtain (2p+1) disjoint copies of K_{12}. By applying Lemma 2.11, K_{12} is $(2,6)_4$-factorable. Therefore, we obtain a $(2,16p+6)_4$-factorization.

■

The following lemma is proved by Hanani, Ray-chandhure and R. Wilson. We are going to use it to prove Lemma 2.15.

Lemma 2.14 [3]. The graph K_n is K_4-factorable if and only if $n \equiv 4 \pmod{12}$.

■

Lemma 2.15. Let $n \equiv 4 \pmod{12}$. K_n is $(s,t)_4$-factorable if and only if $2s+3t=2(n-1)$.

Proof. The necessity follows by counting the edges.

Using Lemma 2.14, we can decompose K_n into (4p+1) K_4-factors if $n=12p+4$.

Let $(s,t)=(s,2m)$ be a solution of $2s+3t=2(n-1)=2(12p+3)$. From this we see that $s\equiv 0 \pmod 3$ and $t\leq 2(4p+1)$.

If $m=4p+1$, we simply take a P_4-factorization of each K_4-factor. (Each K_4 can be decomposed into two 4-paths). Otherwise, we choose m ($m<4p+1$) K_4-factors in K_n and decompose them into 2m P_4-factors; the remaining K_4-factors are 1-factorized.

∎

Lemma 2.16. Let $n\equiv 8 \pmod{12}$. The graph K_n is $(s,t)_4$-factorable if and only if $2s+3t=2(n-1)$.

Proof. The necessity follows by counting the edges.

Let $n=12p+8=6(2p+1)+2$ and $(s,t)=(s,2m)$ be a solution of $2s+3t=2(n-1)=2(12p+7)$. We find that $s\equiv 1 \pmod 3$ and $t\leq 8p+4$.

We divide $V(K_n)$ into $2p+2$ blocks of which $2p+1$ blocks have size six and one block has size two. Let the vertices in the block of size two be x and y. By contracting each block into one vertex we obtain K_{2p+2} which has a 1-factorization, say $f_1,\ldots,f_{2p+1}$. Each f_i corresponds to p disjoint copies of $K_{6,6}$ and one copy of $K_{6,2}$. Note that in each f_i the block of size six in the $K_{6,2}$ is distinct.

We decompose each of the subgraph of K_n corresponding to the 1-factors $f_1,\ldots,f_{[m/2]}$ into four P_4-factors. This decomposition needs to be specified as follows. Each $K_{6,6}$ is $(0,4)_4$-factorable. For the $K_{6,2}$, we include the edges in K_2 and

K_6 to get a K_8. Since K_8 is $(1,4)_4$-factorable by Lemma 2.12, there is a subgraph $G=K_8-f$, where f is a 1-factor containing the edge (x,y), so that G has a P_4-factorization (there are four P_4-factors). We thus obtain 4[m/2] P_4-factors and we remember that we still have a 1-factor in each of [m/2] copies of K_6 and edge (x,y).

If m=0(mod2), then we only need to prove that the remaining subgraph of K_n is 1-factorable. We do so as follows.

For each i, $[m/2]+1 \le i \le 2p+1$, we decompose the subgraph corresponding to f_i into six 1-factors. Again the decomposition needs to be specified. Ecah $K_{6,6}$ has six 1-factors. With the $K_{6,2}$ we include the edges in K_2 and K_6 and obtain a K_8 which has seven 1-factors. But we choose only six of them and the remaining one is the one containing the edge (x,y). Doing this for each i we obtain 6(2p-[m/2]+1) 1-factors. On deleting all of these 1-factors, the resulting graph consists of the edge (x,y) and one 1-factor in each K_6. This gives us another one 1-factor. Therefore, we obtain 6(2p-[m/2]+1)+1 1-factors and 4[m/2] P_4-factors in K_n when m is even.

If m=1(mod2), we decompose the subgraph corresponding to $f_{[m/2]+1}$ into two P_4-factors and three 1-factors such that edge (x,y) belongs to one of the 1-factors. This is possible, because K_8 is $(4,2)_4$-factorable. Then by using the same method as before, we can prove that the remaining graph is 1-factorable.

This completes the proof.

■

Because we need the result for $\lambda=1$ to prove the case that λ is general, we take this special case as theorem 2.17.

Theorem 2.17. The graph K_n is $(s,t)_4$-factorable if and only if, either

(A) $s=0$, $n\equiv4 \pmod{12}$ and $t=2(n-1)/3$,

(B) $t=0$, $n\equiv0 \pmod 2$ and $s=(n-1)$, or

(C) $st\neq0$, $2s+3t=2(n-1)$ and $n\equiv0 \pmod 4$.

Proof. Both the necessary and sufficiency of conditions of this theorem follow by Lemmas 2.13, 2.15 and 2.16.

■

In Theorem 2.18 we generalize Theorem 2.17 to arbitrary values for λ.

Theorem 2.18. The graph λK_n is $(s,t)_4$-factorable if and only if, either

(A) $s=0$,

and (1) $\lambda\equiv0 \pmod 3$, $n\equiv0 \pmod 4$ and $t=2\lambda(n-1)/3$,

or (2) $\lambda\equiv1, 2 \pmod 3$, $n\equiv4 \pmod{12}$ and $t=2\lambda(n-1)/3$,

(B) $t=0$, $n\equiv0 \pmod 2$ and $s=\lambda(n-1)$, or

(C) $st\neq0$, $2s+3t=2\lambda(n-1)$ and $n\equiv0 \pmod 4$.

Proof. By counting the number of edges, it is not difficult to prove the necessity of this theorem. We only need now to show the sufficiency of these conditions.

(A) s=0.

We know that K_n is P_4-factorable if $n \equiv 4 \pmod{12}$. So this will be true for all value of λ. Now we are going to prove that when $n \equiv 0 \pmod{12}$ or $n \equiv 8 \pmod{12}$, $3K_n$ is P_4-factorable. Let K_n^1, K_n^2 and K_n^3 be the three copies of K_n.

First, let $n=12p$. By Lemma 2.13, K_n is $(2,8p-2)_4$-factorable. Moreover, in that construction (using Lemma 2.11) the union of the two 1-factors is a 2-factor in which each cycle has length six. Take such a $(2,8p-2)_4$-factorization of K_n^1, This yields 2p 6-cycles. We can think of them as p pairs of 6-cycles, say $(x_1^i,\dots,x_6^i)$ and $(y_1^i,\dots,y_6^i)$ where $i=1,\dots,p$, and $V(K_n)=\{x_j^i,y_j^i: 1\le i\le p,\ 1\le j\le 6\}$. Now we take a $(2,8p-2)_4$-factorization on K_n^2 and such that one of the 1-factors is formed by the edges (x_1^i,y_6^i), (x_2^i,y_5^i), (x_3^i,y_4^i), (x_6^i,y_1^i), (x_5^i,y_2^i) and (x_4^i,y_3^i) where $i=1,\dots,p$. By adding this 1-factor to the p pairs of 6-cycles from K_n^1, we obtain a three factor with p identical components as shown in Figure 2.9. Observe that this 3-regular subgraph is P_4-factorable.

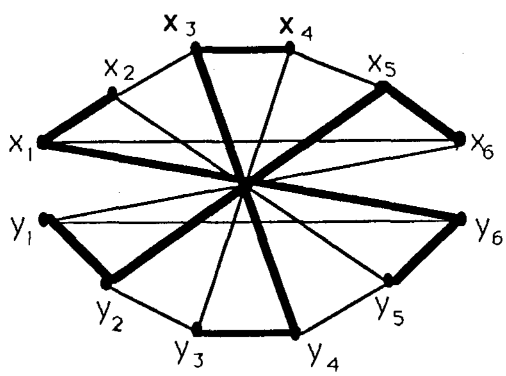


Fig. 2.9

Now we take a $(2,8p-2)_4$-factorization on K_n^3 so that the one 1-factor left in K_n^2 and the p pairs of 6-cycle obtained in K_n^3 will again form a 3-factor as above. Therefore, $3K_n$ is P_4-factorable if $n \equiv 0 \pmod{12}$.

Suppose now that $n=12p+8$. By Lemma 2.16, K_n is $(1,8p+4)_4$-factorable. Now take a $(1,8p+4)_4$-factorization of each of the three copies of K_n so that the three 1-factors will form a K_4-factor of K_n. This subgraph is P_4-factorable.

Therefore, the graph K_n is P_4-factorable if $n \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 3$ and the number of P_4-factors is $2\lambda(n-1)/3$.

(B) $t=0$. This is just a 1-factorization and so it is trivial.

(C) $st \neq 0$.

When $\lambda=1$, this is Theorem 2.17. Suppose that $\lambda=2$. Let (s,t) be a solution of $2s+3t=4(n-1)$ and M be the maximum number of P_4-factors in K_n. If $t \leq 2M$, we can obtain M P_4-factors from K_n^1 and $t-M$ P_4-factors from K_n^2, and 1-factorize the remaining graph. By Theorem 2.17, this is possible. Otherwise, the only possibility occurs when $n \equiv 0 \pmod{12}$. The reason is that if $n \equiv 4,8 \pmod{12}$, then 2M is the maximum number of P_4-factors possible. Now when $n \equiv 0 \pmod{12}$, the maximum number of P_4-factors possible is $2M+2$. So we apply the same method as in (a) by first taking a $(2,8p-2)_4$-factorization of each copy of $K_{12p} (n=12p)$ and then combining the four 1-factors to obtain the two more P_4-factors.

Now we consider the case of general λ. For any given λ, we consider its value modulo 3. Since we know that λK_n is P_4-factorable if $n \equiv 0 \pmod 4$ and $\lambda \equiv 0 \pmod 3$, and that when $\lambda = 1, 2$, λK_n is $(s,t)_4$-factorable if and only if $2s+3t=2\lambda(n-1)$ and $n \equiv 0 \pmod 4$, Then the desired result follows immediately.

This completes the proof.

■

PART D

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