TYPICAL PROPERTIES OF CONTINUOUS FUNCTIONS

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Linda Szpitun

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Name: Linda Szpitun

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Examining Committee:

Chairman: Dr. G. Bojadziev

Dr. B.S. Thomson
Senior Supervisor

Dr. J.J. Sember

Dr. T.C. Brown

Dr. G.A.C. Graham
External Examiner

Date Approved: July 19, 1988

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Author:

(signature)

Linda Szpitun

(name)

July 25, 1988

(date)

ABSTRACT

In this thesis we survey the typical properties of continuous functions defined on [0,1]. A property is typical if the set of functions which have this property is the complement of a set of first category in C[0,1]. We begin by focusing on typical differentiation properties of continuous functions. We see that nondifferentiability is typical, not only in the ordinary sense but with regard to several generalized derivatives. We then discuss typical intersection sets of continuous functions with functions in several families. We look at these sets in terms of perfect sets and isolated points and in terms of porosity. We review the Banach-Mazur game and see how it has been applied in proofs of typical properties. Finally we indicate some related areas of research and some open questions regarding typical properties in C[0,1].

DEDICATION

To Ted for his support and perseverance.

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CHAPTER I

INTRODUCTION

The first example of a continuous nowhere differentiable function seems to be due to Weierstrass, in about 1872. Other examples of "pathological" functions followed by Dini, Darboux and others. In the 1870's Thomae and Schwarz defined examples of functions of two real variables, continuous relative to each variable separately but not continuous. Examples of "pathological" functions led to the study of the properties of arbitrary functions. René Baire was one of those who investigated this area. Baire was strongly influenced by Cantor's set theory and made use of some new set theoretic notions. In 1899 Baire's Doctoral Thesis, "Sur les fonctions de variables réelles", appeared in Annali di Matematica Pura et Applicata. In order to characterize limits of convergent sequences of continuous functions (and their limits and so on) Baire introduced the concept of category. A subset of **R** is said to be of first category in **R** when it is the union of countably many nowhere dense sets in **R**. A set which is not of first category is said to be of second category. Baire proved the following results.

(i) **R** is not of first category in itself.

(ii) The set of points of discontinuity of a function in the first class of Baire (noncontinuous functions which are limits of continuous functions) is of first category.

(iii) For every function f which belongs to a Baire class there is a set E of first category such that f restricted to the complement of E is continuous.

The introduction of hyperspaces by Fréchet, Hausdorff and others, especially the introduction of $\mathbf{R}^I$, had great influence on the proofs of existence theorems. Result (i) above was extended to complete metric spaces in what is now called the Baire Category Theorem. In 1931 Mazurkiewicz and Banach separately used this theorem to prove the existence of continuous nowhere differentiable functions.

The Baire Category Theorem has often been used to prove that some certain subset of a complete metric space is not empty. This is usually accomplished by defining a countable number of subsets of the space so that the desired subset is the complement of the union of these subsets, and then showing that each of these subsets is nowhere dense. Then, since a complete metric space is of second category the desired subset is not empty. Such a set, the complement of a first category set, is said to be residual.

The advantage of this method of proof is that it produces a whole class of examples, not just one. It also often simplifies the problem by allowing concentration on the essential properties of the desired example. In principle a specific example can always be constructed by successive approximations, although this may be difficult.

In about 1928 Mazur invented a mathematical game which can also be used to provide proofs of the existence of a residual set, all of whose elements have certain properties. The game is between two players, A and B. A is given an arbitrary subset $\mathcal{A}$ of a closed interval $I_0 \subset \mathbf{R}$. The complementary set, $I_0 \backslash \mathcal{A}$ is given to B. Now A chooses any closed interval $I_1 \subset I_0$; then B chooses a closed interval $I_2 \subset I_1$. They continue this way, alternately choosing closed intervals. If the set $\cap_n^\infty I_n$ intersects the set $\mathcal{A}$ then A wins; otherwise B wins. Banach showed that there is a strategy by which B can always win if and only if $\mathcal{A}$ is of first category. Recently the Banach-Mazur game has been used by several analysts, including L. Zajíček, to develop proofs that certain differentiation properties are typical in C[0,1]. A property is said to be typical in the complete metric space, C[0,1], if the set of functions in C[0,1] which possess this property is residual.

The main purpose of this thesis is to survey the properties which are typical of continuous functions in [0,1]. Chapter 2 is concerned with differentiation properties which are typical in C[0,1]. In Chapter 3 we will look at the intersection of typical continuous functions with various families of continuous functions. We will also see that these results clarify some of the results developed in Chapter 2. Chapter 4 surveys the porosity characteristics of some of these intersection sets and finally Chapter 5 provides some recent results and alternate proofs for known results by use of the Banach-Mazur game.

Definitions and notation

We now begin with a few necessary definitions and two elementary results relating to category. By C[0,1] we mean the complete metric space of all continuous real valued functions defined on the interval [0,1] and supplied with the sup norm, $\|f\| = \sup\{|f(x)|: x\in[0,1]\}$. For convenience we will also use the notation C for C[0,1] when no confusion can occur. We have already defined category, residual sets and typical properties. The phrase "a typical continuous function has property P" can be interpreted to mean that there is a residual set of functions in C[0,1] which have property P. This language is perhaps a bit dangerous, (a function may be typical in one sense while not in others), but is in wide use in the literature.

The closure of a set A will be denoted by $\bar{A}$. The interior will be denoted A°. A set A in a topological space X is dense in X if $\bar{A}$ is all of X. A set A is nowhere dense in X if $\bar{A}$ contains no nonempty open set. We will use $\lambda(A)$ to denote the Lebesgue measure of A. A set A has the property of Baire if it can be represented in the form $A = G\Delta P = (G\cup P)\setminus(G\cap P)$, where G is open and P is of first categgory.

THEOREM 1.1. The Baire Category Theorem. Let X be a complete metric space and A a first category subset of X. Then the complement of A in X is dense in X. Thus X is of second category in itself.

THEOREM 1.2. Any subset of a set of first category is of first category. The union of any countable family of first category sets is of first category.

CHAPTER II

DIFFERENTIATION PROPERTIES

The existence of continuous nowhere differentiable functions was first proved by the category method in 1931 by Mazurkiewicz [33] and Banach [1]. Banach's proof clearly demonstrates the use of the category method.

THEOREM 2.1.1. The set of functions in C which have no finite right hand derivative at any point is residual in C.

PROOF: (cf. [29] pp. 420-421) For each natural number n let N_n denote the set of functions, f, in C for which there exists $x \in [0,1]$ such that:

$$\left| \frac{f(x + h) - f(x)}{h} \right| \leq n \qquad \forall\, h \in (0,1-x)$$

Let $N = \cup_{n=1}^{\infty} N_n$. We will show that for each n, N_n is nowhere dense. To show that N_n is closed, consider any sequence $\{f_k\}$ in N_n that converges to $f \in C$. Then there is a sequence $\{x_k\}$ in [0, 1] such that $|f_k(x_k + h) - f_k(x_k)| \leq nh$, $\forall h \in (0, 1-x_k)$. Passing to a subsequence if necessary we may assume that $x_k \to x$ for some $x \in [0,1]$. Then it is easy to see that $|f(x + h) - f(x)| \leq nh$, $\forall h \in (0,1-x)$. Thus $f \in N_n$ and so we see that N_n is closed.

Now, to show that N_n is nowhere dense in C it suffices to show that N_n contains no open sphere in C. Let $\epsilon>0$ be given and $f \in C$. Then we can choose a piecewise linear function h such that $\|f - h\| < \epsilon$ and each segment of h has a slope with absolute value greater than n. Then $h \notin N_n$ so N_n contains no sphere in C. Thus N_n is nowhere dense.

The set $C \setminus \bigcup_{n=1}^{\infty} N_n$ is residual and any function in this set has no finite right hand derivative at any point.□

In 1925 Besicovich had constructed a continuous function with no finite or infinite one sided derivative at any point. The question then arose whether a category proof is available to prove the existence of such functions. In our terminology the question is whether Besicovich functions were also typical. This was answered in the negative by Saks in 1932 [40]. What Saks showed was that a typical continuous function has an infinite derivative on an uncountable set. Saks' proof was not elementary. He showed that the set of functions with a one-sided derivative (finite or infinite) at some point is of second category in every sphere of C. He went on to show (using results of Banach, Tarski and Kuratowski) that this set is analytic and so residual in C. We will see in Chapter 5 that the second part of Saks' proof is not needed and that the first part is an example of the use of the Banach-Mazur game to prove a category result. A simpler but still not elementary proof is attributed to Preiss by Bruckner [4]. In 1986 Carter [11] supplied an even simpler proof.

Dini Derivatives

There are many results regarding the relationships among the Dini derivatives of various classes of real valued functions. The most famous of these is now known as the Denjoy-Young-Saks theorem which gives relationships among the Dini derivatives of an arbitrary function.

THEOREM 2.2.1. Let f be defined on an interval, I. Then all $x \in I$, excepting at most a set of measure zero, are in one of the following four sets:

$$\{x : f'(x) \text{ exists and is finite}\}$$

$$\{x : D^+f(x) = D_-f(x) \text{ are finite, } D_+f(x) = -\infty,\ D^-f(x) = +\infty\}$$

$$\{x : D_+f(x) = D^-f(x) \text{ are finite, } D^+f(x) = +\infty,\ D_-f(x) = -\infty\}$$

$$\{x : D^+f(x) = D^-f(x) = +\infty,\ D_+f(x) = D_-f(x) = -\infty\}.$$

This result was proved for continuous functions by Denjoy in 1915, for measurable functions by Young in 1916 and for arbitrary functions by Saks in 1924. In 1933 Jarnik [25] proved results regarding the relationships among the Dini derivatives of a residual set of continuous functions. We shall need another definition before stating Jarnik's results.

DEFINITION 2.2.2. Let $f \in C$ and $x \in (0,1)$. We say that x is a _knot point of f_ if $\underline{D}f(x) = -\infty$ and $\overline{D}f(x) = +\infty$.

THEOREM 2.2.3. The set of functions f in C with the following properties is residual in C.

(i) For all $x \in (0,1)$ $[D_-f(x), D^-F(x)] \cup [D_+f(x), D^+f(x)] = [-\infty, \infty]$.

(ii) The set of points which are not knot points of f has measure zero.

In 1970 Garg [17] strengthened Jarnik's results while incorporating those of Banach and Mazurkiewicz and Saks. We need some notation. Let $f \in C$ and let r be a real number. Then define:

$$E(f) = \{x: D^+f(x)=D^-f(x)=+\infty,\ D_+f(x)=D_-f(x)=-\infty\}$$

$$E_1(f)\ [E_2(f)] = \{x: f'_+(x)=+\infty[-\infty],\ D^-f(x)=+\infty,\ D_-f(x)=-\infty\}$$

$$E_3(f)\ [E_4(f)] = \{x: f'_-(x)=+\infty[-\infty],\ D^+f(x)=+\infty,\ D_+f(x)=-\infty\}$$

$$E_{1r}(f) = \{x: D^+f(x)=r \geq D_-f(x),\ D_+f(X)=-\infty,\ D^-f(x)=+\infty\}$$

$$E_{2r}(f) = \{x: D_-f(x)=r \leq D^+f(x),\ D_+f(x)=-\infty,\ D^-f(x)=+\infty\}$$

$$E_{3r}(f) = \{x: D_+f(x)=r \leq D^-f(x),\ D^+f(x)=+\infty,\ D_-f(x)=-\infty\}$$

and

$$E_{4r}(f) = \{x: D^-f(x)=r \geq D_+f(x),\ D^+f(x)=+\infty,\ D_-f(x)=-\infty\}.$$

THEOREM 2.2.4. There exists a residual set of functions f in C such that each $x \in (0,1)$ is in one of the sets $E(f)$, $E_i(f)$ $(i=1,2,3,4)$ and $E_{ir}(f)$ $(i=1,2,3,4,\ r \in \mathbf{R})$, and

(i) $E(f)$ is residual in $(0,1)$ and has measure one,

(ii) Each of the sets $E_i(f)$ $(i=1,2,3,4)$ and $E_{ir}(f)$

$(i=1,2,3,4,\ r \in \mathbf{R})$ is a first category set of measure zero and has the power of the continuum in each subinterval of $(0,1)$, and

(iii) for each $r \in \mathbf{R}$ the sets $E_{1r}(f) \cap E_{2r}(f)$ and $E_{3r}(f) \cap E_{4r}(f)$ are both dense in $(0,1)$.

Path derivatives

The question of the differentiability of typical continuous functions can be extended to several generalized derivatives. Generalized derivatives have often been defined because complete differentiability is not required and some weaker property is sufficient for the development of the theory under consideration. Most of the generalized derivatives are obtained by a restriction of the limit of a difference quotient to specific classes of sets. The concept of path derivative, introduced by Bruckner, O'Malley and Thomson [10] can be used to define several generalized derivatives.

DEFINITION 2.3.1. Let $x \in \mathbf{R}$. A path leading to x is a set $E_x \subset \mathbf{R}$ such that $x \in E_x$ and x is an accumulation point of E_x. Then a collection $E = \{E_x : x \in \mathbf{R},$ E_x a path leading to $x\}$ is called a system of paths.

DEFINITION 2.3.2. Let $f : \mathbf{R} \to \mathbf{R}$ and let $E = \{ E_x : x \in \mathbf{R}\}$ be a system of paths. If

$$\lim_{y\to x,\ y\in E_x} \frac{f(y) - f(x)}{y - x} = g(x)$$

is finite then f is E-differentiable at x and we write $f'_E = g(x)$. If f is E-differentiable at every point x then f is E-differentiable.

DEFINITION 2.3.3. Let A be a measurable subset of **R** and $x \in$ **R**. Let

$$\overline{d}(A,x) = \overline{\lim}_{h,k\to 0+,h+k\neq 0} \frac{\lambda(A\cap[x-h,x+k])}{h+k}.$$

Then $\overline{d}(A,x)$ is called the upper density of A at x. The lower density of A at x, $\underline{d}(A,x)$ is similarly defined. If $\overline{d}(A,x) = \underline{d}(A,x)$ we call this number the density of A at x and denote it by $d(A,x)$. When $d(A,x) = 1$ we say that x is a point of density of A.

DEFINITION 2.3.4. If each $E_x \in E$ has density 1 at x the E-derivative is called the approximate derivative.

DEFINITION 2.3.5. If each $E_x \in E$ has upper density 1 at x then the E-derivative is called the essential derivative.

DEFINITION 2.3.6. If each $E_x \in E$ has lower right and lower left density both greater that 1/2 at x the E-derivative is called the preponderant derivative.

Of course if each $E_x \in E$ is a neighbourhood of x then we have the ordinary derivative. The one sided derivatives are given by restricting E_x to one side of x and if we replace limit in definition 2.3 by lim sup or lim inf we obtain the corresponding extreme derivatives.

A complete analogue to the Denjoy-Young-Saks theorem holds for the approximate derivative (see [5] p.149). In 1934 Jarnik proved several properties of typical continuous functions with regard to these generalized derivatives.

THEOREM 2.3.7. The set of functions f in C with the following properties is residual in C.

(i)[27] The set of points which are not essential knot points of f has measure zero.

(ii)[24] For each $x \in (0,1)$ at least one of ∞, $-\infty$ is a right essential derived number, and at least one of ∞, $-\infty$ is a left essential derived number.

(iii)[24] For each $x \in (0,1)$ both $-\infty$ and ∞ are derived numbers of f at x with symmetric upper density greater than or equal to 1/2.

(iv)[26] For each $x \in (0,1)$ there is one side where at least two of $-\infty$, 0, ∞ are derived numbers of f at x with density on that side greater than or equal to 1/4.

We can see that these results imply that the typical continuous function has no preponderant derivative, no finite approximate derivative, no finite unilateral preponderant derivative, and does not have both unilateral approximate derivatives for any x in (0,1).

Other generalized derivatives

In 1972 Kostyrko [28] showed that a typical continuous function is nowhere symmetrically differentiable.

THEOREM 2.4.1. The set of functions f in C with the following property is residual in C:
For each $x \in (0,1)$

$$\limsup_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = +\infty, \text{ and}$$

$$\liminf_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = -\infty.$$

In 1974 Evans [14] proved the analogous result for the approximate symmetric derivative. The selective derivative was introduced by O'Malley in 1977.

DEFINITION 2.4.2. A function of two variables s(x,y) for which s(x,y) = s(y,x) and s(x,y) is between x and y is called a *selection*. We define the *selective derivative* ${}_sf'(x)$ of the function f(x) by

$${}_sf'(x) = \lim_{y\to x} \frac{f(s(x,y)) - f(x)}{s(x,y) - x}$$

In 1983 Lazarow [31] showed that all measurable functions have a selective derivative on a set of the cardinality of the continuum. She then went on to show:

THEOREM 2.4.3. The set of functions f in C such that, for every selection s, $_sf'$ exists (possibly infinite) only for a first category set of measure zero, is residual in C.

Universal properties

We have seen that the typical continuous function is very nondifferentiable with regard to several generalized derivatives. But Marcinkiewicz in 1935 [32] proved that the typical continuous function is differentiable almost everywhere in a certain sense. Moreover we can choose the derivative in advance.

THEOREM 2.5.1. Let $\{h_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} h_n = 0$. The set of functions f in C(0,1) with the following property is residual in C(0,1):
For each measurable function $\phi(x)$ on (0,1) there corresponds a subsequence $\{h_{n_k}\}$ such that

$$\lim_{k\to\infty} \frac{f(x + h_{n_k}) - f(x)}{h_{n_k}} = \phi(x)$$

almost everywhere.

We note that if $\phi(x) = r \in R$ the Marcinkiewicz' theorem shows that r is a derived number of every typical continuous function almost everywhere and, in fact, is a derived number relative to a fixed sequence. Bruckner [3] has reported that Scholz has obtained a stronger result than theorems 2.8 and 2.9.

THEOREM 2.5.2. Let ϕ be an arbitrary measurable function. Then the set of functions f in C with the following property is residual in C:
There exists a set E_f having upper density 1 at the origin such that

$$\lim_{h \to 0, h \in E_f} \frac{f(x + h) - f(x)}{h} = \phi(x)$$

almost everywhere.

We note that this is a strong essential derivative in the sense that the same set, E_f, is used for each $x \in [0,1]$. From these last results we get some idea how strongly nondifferentiable a typical continuous function is.

CHAPTER III

INTERSECTIONS

The results mentioned in Chapter 2 show that typical continuous functions exhibit a great deal of pathological behaviour. But they also show much regularity. In this chapter we will see some of this regularity in the way that typical continuous functions intersect lines and other classes of functions.

In section 2 we shall see some of the properties of the intersection sets of straight lines with non-monotonic functions and then with typical continuous functions. In section 3 this will be extended to polynomials. Section 4 deals with the intersection of a typical continuous function with what will be called 2-parameter families. A few definitions and notations are needed. They are from [7, 8].

DEFINITION 3.1.1. For $f,g \in \mathbf{R}^{[0,1]}$ the set $\{x:f(x) = g(x)\}$ is called the <u>intersection set of f and g</u>. For each $c,\lambda \in \mathbf{R}$ the set $\{x:f(x) = \lambda x + c\}$ is called a <u>level of f in the direction λ</u>. If $\lambda = 0$ this set is called a <u>horizontal level of f</u>.

DEFINITION 3.1.2. A function $f \in C$ is <u>nondecreasing</u> at $x \in [0,1]$ if there exists $\delta>0$ such that $f(t) \leq f(x)$ for $t \in [0,1]\cap(x-\delta,x)$ and $f(t) \geq f(x)$ when $t \in [0,1]\cap(x,x+\delta)$. A function f is <u>nonincreasing</u> at x if -f is nondecreasing at x.

A function f is <u>monotone</u> at x if it is either nonincreasing or nondecreasing at x. A function f is of <u>monotonic type at x</u> if there exists $\lambda \in \mathbf{R}$ such that $f_{-\lambda} = f(x) - \lambda x$ is monotone at x. If f is not of monotonic type at any point in [0,1] then it is of <u>nonmonotonic type</u>. If f is not of monotonic type in any interval then it is <u>nowhere of monotonic type</u>.

DEFINITION 3.1.3. Let $f \in C$, I a subinterval of [0,1] and L a line given by $y = \lambda x + c$. The line L is said to <u>support the graph of f in I (or support f in I) from above [below]</u> if $f(x) \leq \lambda x + c$ $[f(x) \geq \lambda x + c]$ for all $x \in I$ and there exists a point x_0 in I such that $f(x_0) = \lambda x_0 + c$. Also if the point x_0 is not unique then L <u>supports f at more than one point in I</u>.

<u>Intersections with lines</u>

K. Garg proved in 1963 [16] two results regarding the horizontal levels of nonmonotonic functions. First he showed that if $f \in C$ is nowhere monotone then the horizontal levels of f are non-dense for every $y \in \mathbf{R}$ and are nonempty perfect sets for a set of values of y residual in the range of f. Secondly he showed that if f is nowhere of monotonic type then for any $\lambda \in \mathbf{R}$

there exists a residual set, $H_\lambda \subset \mathbf{R}$ such that for all $c \in H_\lambda$ the line $y = \lambda x + c$ intersects the graph of f in a nowhere dense perfect set (possibly empty). Also in 1963 he showed [16] that every nondifferentiable function is nowhere of monotonic type. Since the nondifferentiable functions form a residual set in C he was able to show the following results.

THEOREM 3.2.1. There is a residual set of functions in C such that for every $\lambda \in \mathbf{R}$ the level sets in the direction λ are perfect sets of linear measure zero except for at most a countable set of values of c (depending on λ and f).

THEOREM 3.2.2. There is a residual set of functions in C such that for every $\lambda \in \mathbf{R}$ the level sets in the direction λ have the power of the continuum for all values of c between the two extreme values.

THEOREM 3.2.3. There is a dense set A of functions in C such that every horizontal level is perfect.

These results raised the question of whether the set of functions in Theorem 3.2.3 form a residual set. Bruckner and Garg [7] answered this question in the negative by giving a description of the structure of the level sets in all directions for typical continuous functions. We need one more definition.

DEFINITION 3.2.4. We denote by $\alpha_{f,\lambda}$ and $\beta_{f,\lambda}$:

$$\alpha_{f,\lambda} = \inf\{f(x) - \lambda x: 0 \leq x \leq 1\}$$

$$\beta_{f,\lambda} = \sup\{f(x) - \lambda x: 0 \leq x \leq 1\}.$$

Let $f \in C$ and $\lambda \in \mathbf{R}$. If there exists a countable dense set, $E_{f,\lambda}$, in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$ such that the level $\{x: f(x) = \lambda x + c\}$ is

(i) a perfect set when $c \notin E_{f,\lambda} \cup \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$,

(ii) a single point when $c = \alpha_{f,\lambda}$ or $c = \beta_{f,\lambda}$, and

(iii) the union of a nonempty perfect set, P, with an isolated point, $x \notin P$, when $c \in E_{f,\lambda}$ (P depends on f, λ, and c)

then the levels of f in the direction λ are said to be <u>normal</u>.

THEOREM 3.2.5. The functions of nonmonotonic type form a dense G_δ set in C.

PROOF: ([7], pp. 309-311) For each natural number n let A_n denote the set of functions $f \in C$ such that there exists $\lambda \in [-n,n]$ and $x \in [0,1]$ so that $f_{-\lambda}$ is nondecreasing on $(x-1/n, x+1/n) \cap [0,1]$. Then $A = \bigcup_{n=1}^{\infty} A_n$ is the set of functions $f \in C$ for which there exists $\lambda \in \mathbf{R}$ such that f_λ is nondecreasing for some $x \in [0,1]$. We will show that each A_n is closed and nowhere dense in C.

Let n be given and let $\{f_k\}$ be a sequence of functions in A_n that converge uniformly to a function $f \in C$. For each k there exists $\lambda_k \in [-n,n]$ and $x_k \in [0,1]$ such that $f_k - \lambda_k$ is nondecreasing on $(x_k-1/n, x_k+1/n) \cap [0,1]$. Then there is a sequence $\{k_i\}$ of natural numbers such that $\{\lambda_{k_i}\}$ converges to some $\lambda \in [-n,n]$ and $\{x_{k_i}\}$ converges to $x \in [0,1]$. Then it is easy to see that $f_{-\lambda}$ is nondecreasing on $(x-1/n, x+1/n) \cap [0,1]$. Thus $f \in A_n$ and so A_n is closed.

Now it is necessary to show that A_n is nowhere dense. Let U be a nonempty open subset in C. Then there is a polynomial g and $\epsilon>0$ such that the open sphere centered at g with radius ϵ, $B(g,\epsilon)$, is contained in U. Then a "saw-tooth" function $f \in C$ can be constructed with $f \in B(g,\epsilon)$ and $f \notin A_n$. This is accomplished by defining a piecewise linear function h with intervals of monotonicity small enough and slope great enough that if $f = g + h$ then if $\lambda \in [-n,n]$ $f_{-\lambda}$ is not monotonic on any interval of length $2/n$. Thus A_n is nowhere dense.

We see that A is an F_σ set of first category. Let B be the set of functions f in C such that there exists $\lambda \in \mathbf{R}$ and $x \in [0,1]$ with $f_{-\lambda}$ nonincreasing at x. Then $B = \{-f : f \in A\}$ so B is also a first category F_σ. Hence $C \setminus (A \cup B)$ is a dense G_δ set in C.□

Theorem 3.2.5 together with a result by Garg [18] that a function of nonmonotonic type has a knot point everywhere yields a variant of the Mazurkiewicz-Banach theorem: the set of functions in C which have no finite or infinite derivative at any point is residual in C.

LEMMA 3.2.6. There is a residual set of functions f in C such that no horizontal level of f contains more than one point of extrema of f.

PROOF: ([7], p.312) Let I and J be any two disjoint closed subintervals of [0,1] with rational endpoints. Let $A_{I,J}$ be the set of all functions f in C such that neither the supremum nor the infimum of f on I equals either the supremum or the infimum of f on J. Let $A = \cap A_{I,J}$, where the intersection is taken over all pairs of disjoint closed rational subintervals of [0,1].

Fix I and J. Let $E_1 = \{f \in C : \sup\{f(x) : x \in I\} \neq \sup\{f(x) : x \in J\}\}$ and let E_2, E_3 and E_4 be the analogous sets obtained by interchanging sup with inf on I or J or both. Let $f \in E_1$, $\alpha = \sup\{f(x) : x \in I\}$ and $\beta = \sup\{f(x) : x \in J\}$. Then $\alpha \neq \beta$. Let $\epsilon = |\alpha - \beta| > 0$. Clearly $B(f, \epsilon/2) \subset E_1$ so that E_1 is open in C. Also it is clear that E_1 is dense in C and so it is residual. It can be shown similarly that E_2, E_3 and E_4 are residual so that $A_{I,J} = \cup_{i=1}^{4} E_i$ is residual and hence A is residual.□

THEOREM 3.2.7. There exists a residual set of functions in C whose horizontal levels are normal.

PROOF: ([7], pp. 312-313) Let A be the intersection of the two residual sets given by Theorem 3.2.5 and Lemma 3.2.6. Then A is residual and we will show that all functions in A have the required properties.

Let $f \in A$ and $\alpha = \inf\{f(x):0\leq x\leq 1\}$ and $\beta = \sup\{f(x):0\leq x\leq 1\}$. By (3.2.5) f is of nonmonotonic type, so for every $c \in \mathbf{R}$, x is an isolated point of the level $f^{-1}(c)$ if and only if x is a point of proper extremum of f. By (3.2.6) each horizontal level of f contains at most one point of extremum of f so that every extremum of f is a proper extremum.

Let D be the set of extrema of f. Then since f is continuous and nowhere monotone and every point of D is proper, D must be countable and f(D) a countable dense subset of $[\alpha,\beta]$. Also we have $\alpha,\beta \in f(D)$. Let $E = f(D)\setminus\{\alpha,\beta\}$.

Let $c \in \mathbf{R}$. When $c \notin E\cup\{\alpha,\beta\}$, $f^{-1}(c)$ has no isolated points so $f^{-1}(c)$ is perfect. Clearly $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are singletons. When $c \in E$, $f^{-1}(c)$ contains exactly one point of extremum of f, say x_0, and x_0 is isolated. Since f is continuous it satisfies the Darboux property so that $f^{-1}(c) \setminus \{x_0\}$ is not empty and must be perfect. Thus the horizontal levels of f are normal.□

A corollary to this theorem is that given a function $g \in C$ there is a residual set A in C such that for every $f \in A$, the horizontal levels of $f - g$ are normal. By letting $g(x) = \lambda x$ we see that for every sequence $\{\lambda_n\}$ in **R** there is a residual set of functions in C whose levels are normal in each of the directions λ_n. But note that the set of exceptional levels does not vary with the function. This is not the same as Theorem 3.2.13 to follow for there the set of directions depends on the function.

LEMMA 3.2.8. For every function $f \in C$ there are at most countably many lines that support the graph of f in 2 or more disjoint open subintervals of [0,1].

PROOF: ([7], pp. 314-315) Let L denote the set of lines that support the graph of f in at least 2 disjoint open subintervals of [0,1]. Let I and J be two disjoint open subintervals of [0,1]. Then it is easy to see that there is at most one line which supports f from above in both I and J. Similarly there is at most one line which supports the graph of f from below in both I and J, one line that supports f from below in I and above in J and one line that supports f from above in I and from below in J. Let $L_{I,J}$ denote the set of lines that support the graph of f in both I and J. Then $L_{I,J}$ has at most 4 elements. Clearly $L = \cup_{I,J} L_{I,J}$ where the union is taken over all pairs of disjoint open rational subintervals in [0,1], and it follows that L is countable.□

Note that this lemma concerns all continuous functions, not just a residual set of them. From it we obtain the following theorem regarding the level sets of all functions in C.

THEOREM 3.2.9. For every function $f \in C$ there exists a countable set, Λ, in $\mathbf{R}$ such that for every $\lambda \in \mathbf{R}\setminus\Lambda$:
(i) the levels $\{x:f(x)=\lambda x+\alpha_{f,\lambda}\}$ and $\{x:f(x)=\lambda x+\beta_{f,\lambda}\}$ consist of single points, and
(ii) there is a dense set of points, c, in $(\alpha_{f,\lambda},\beta_{f,\lambda})$ such that $\{x:f(x)=\lambda x+c\}$ contains at least one isolated point.

PROOF: ([7], p. 315) Let $f \in C$ and let Λ be the set of slopes of lines that support the graph of f in at least two disjoint open subintervals of [0,1]. Then Λ is countable by Lemma 3.2.8. Let $\lambda \in \mathbf{R}\setminus\Lambda$.

Let $c \in \mathbf{R}$ and $x_0 = \{x:f(x)=\lambda x+c\} = \{x:f_{-\lambda}(x)=c\}$. Then $y = \lambda x+c$ supports the graph of f in some neighbourhood of x_0 if and only if $f_{-\lambda}$ has an extremum at x_0. Since $\lambda \in \mathbf{R}\setminus\Lambda$ there is no horizontal level of $f_{-\lambda}$ with more than one point of extremum of $f_{-\lambda}$ so $f_{-\lambda}$ has only proper extrema. The function $f_{-\lambda}$ attains $\alpha_{f,\lambda}$ and $\beta_{f,\lambda}$ in [0,1] so $\{x:f(x)=\lambda x+\alpha_{f,\lambda}\}$ and $\{x:f(x)=\lambda x+\beta_{f,\lambda}\}$ are single points.

Let (a,b) be any open subinterval of $(\alpha_{f,\lambda},\beta_{f,\lambda})$. Let I be a connected component of $\{x:a<f_{-\lambda}(x)<b\}$. Then I is an open subinterval of [0,1] and $f_{-\lambda}$ is not constant on any

subinterval of I. Now, if $f_{-\lambda}$ is monotone on I then for each $c \in f_{-\lambda}(I)$ we have $c \in (a,b)$ and $\{x:f_{-\lambda}(x)=c\}$ is isolated. If $f_{-\lambda}$ is not monotone in I it has a proper extremum at some point x_0 in I. Let $c = f_{-\lambda}(x_0)$. Then $a<c<b$ and x_0 is isolated in $\{x:f(x)=\lambda x+c\}$.□

This lemma disproves a claim of Gillis [19] to have constructed a continuous real-valued periodic function f on **R**, all of the levels of which are perfect and, for f restricted to [0,1], are all infinite. Garg [16] had used the Gillis function to show that there is a dense set of functions in C which have in each direction all but a finite number of levels perfect. This too is now seen to be false.

We now need three more technical lemmas regarding lines of support, in order to prove the main result on level sets in all directions of typical continuous functions.

LEMMA 3.2.10. There exists a residual set of functions f in C such that, for every rational open interval $I \subset [0,1]$ the slopes of the lines that support the graph of f in I from above at more than one point form a dense set in **R**. The same holds true for lines of support from below.

PROOF: ([7], pp. 316-318) We shall prove the result for one fixed open rational interval in [0,1]. Since the set of rational intervals in [0,1] is countable this will prove the theorem.

For each natural number n let A_n denote the set of functions f in C for which there is a line with slope in $(-1/n, 1/n)$ which supports f in I from above at more than one point. We shall show that A_n is residual in C.

Let n be fixed and let U be a nonempty open set in C with $f \in U$. Then there is a nonempty open sphere, $B(f,\epsilon) \subset U$, centered at f and with radius ϵ. Let $\alpha = \sup\{f(x):x\in I\}$. Then there is a nondegenerate subinterval $J = (a,b)$ of I such that $f(x) > \alpha - \epsilon/4$ for $x \in J$. Let $\delta > 0$ be such that the length of J is greater than 4δ. We can find 5 points, x_i $(i=0,1,2,3,4)$, such that $a<x_0<\ldots<x_4<b$ and $x_i - x_{i-1} = \delta$, $(i=1,2,3,4)$. Define $g:[0,1]\to\mathbf{R}$ by

$g(x) = f(x)$ for $x \in [0,x_0]\cup[x_4,1]$

$g(x_1) = g(x_3) = \alpha + \epsilon/2$

$g(x_2) = \alpha$

and g is linear in each interval $[x_{i-1},x_i]$ and continuous on $[0,1]$.

Then it can be shown that there is a real number $\eta<\epsilon/8$ such that the open sphere $B(g,\eta)$ is contained in A_n. (The details are tedious.) Thus A_n contains a dense open subset of C and so is residual in C.

Let $\{r_n\}$ be an enumeration of the set of rational numbers. For each natural number n let B_n denote the set of all functions f in C such that $f_{-r_n} \in A_n$. Then B_n is residual in C and so the set $B = \cap B_n$ is also.

Now, given (p,q), an open interval in **R**, we will show that if $f \in B$ then there is a line with slope in (p,q) which supports f in more than one point of I. Let $\delta = q - p$. Then choose 5 points, $x_0,\ldots,x_4$, so that $p=x_0<\ldots<x_4=q$ and $x_{i-1} - x_i = \delta/4$ $(i=0,1,2,3)$. Now we can find a natural number n such that $n>4/\delta$ and $r_n \in (x_1,x_3)$. If $f \in B_n$, $f_{-r_n} \in A_n$ and so there is a line given by $y = sx + b$, $s \in (-1/n,1/n)$ which supports f_{-r_n} in I from above at more than one point. Let M be the line given by $y = (s+r_n)x + b$. Then M supports the graph of f in I from above at more than one point. Also $s+r_n \in (p,q)$.

The proof for support from below is similar.□

LEMMA 3.2.11. There exists a residual set of functions f in C for which there is no line that supports the graph of f in more than two mutually disjoint open intervals.

PROOF: ([7], p. 318) Let A denote the set of functions f in C for which there exists a line which supports f in three or more disjoint open intervals. For each triple (I,J,K) of disjoint open intervals in [0,1] let $A_{I,J,K}$ denote the set of functions f in C for which there is a line which supports f in I, J and K. Then $A = \cup_{I,J,K} A_{I,J,K}$ where the union is taken over all triples of disjoint rational open intervals in [0,1]. We will show that each $A_{I,J,K}$ is nowhere dense and so A is of first category.

Now fix I, J and K and let E denote the set of functions f in $A_{I,J,K}$ for which a line supports f from above in all three subintervals. Let U be an open subset of C and suppose there exists $f \in U \cap E$. Then there is a line $y = \lambda x + c$ which supports f in I, J and K from above and there is an open sphere, $B(f,\epsilon)$, centered at f and with radius ϵ contained in U. We can assume that J is between I and K. Let $g \in B(f,3\epsilon/4)$ such that $g(x) = f(x)$ for $x \in I \cup K$ and $g = f - \epsilon/2$ in J. Now let $h \in B(g,\epsilon/4)$ and assume there is a line, L, which supports h from above in I and K. Then in the interval J, L is strictly above the line $y = \lambda x + c - \epsilon/4$ and h is below this line. Thus L does not support the graph of h in J so $h \notin E$. Then $b(g,\epsilon/4) \subset U \backslash E$ and so E is nowhere dense in C.

The other seven subsets of $A_{I,J,K}$ obtained by replacing above by below in one or more of I, J or K can be shown similarly to be nowhere dense. Then $A_{I,J,K}$ is nowhere dense and so A is of first category.□

LEMMA 3.2.12. There exists a residual set of functions f in C for which there are no two distinct parallel lines such that each of these lines supports the graph of f in 2 disjoint open subintervals of [0,1].

The proof of this lemma follows the pattern of the previous one and so is omitted. We now come to the main result of this section.

THEOREM 3.2.13. There exists a residual set N in C such that if $f \in N$ then there is a countable dense set Λ_f in **R** such that:

(i) the levels of f are normal in each direction $\lambda \in \mathbf{R}\setminus\Lambda_f$, and

(ii) if $\lambda \in \Lambda_f$ the levels are normal except for a unique element, $c_{f,\lambda}$ of $E_{f,\lambda} \cup \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ for which $\{x : f(x) = \lambda x + c_{f,\lambda}\}$ contains two isolated points.

PROOF: ([7], pp. 319-320) Let N be the intersection of the residual sets determined by 3.2.5, 3.2.10, 3.2.11, and 3.2.12. Then N is residual in C.

Let $f \in N$ and let Λ_f denote the set of slopes of lines that support f in at least two disjoint open subintervals of [0,1]. Then by 3.2.8 and 3.2.10 Λ_f is a countable dense subset of **R**. Let $\lambda \in \mathbf{R}$. Then for each $c \in \mathbf{R}$ $\{x : f(x) = \lambda x + c\} = \{x : f_{-\lambda}(x) = c\}$. By 3.2.5 $f_{-\lambda}$ is not monotone at any point of [0,1]. As a result a point x_0 in $\{x : f_{-\lambda}(x) = c\}$ is isolated if and only if $f_{-\lambda}$ has a proper extremum at x_0. Also such an x_0 is a proper extremum of $f_{-\lambda}$ if and only if the line $y = \lambda x + c$ supports the graph of f in some neighbourhood of x_0.

Let $\lambda \in \mathbf{R}\backslash\Lambda_f$. Then there is no line with slope in Λ_f that supports the graph of f in two disjoint open subintervals of [0,1]. Hence all horizontal levels of $f_{-\lambda}$ contain at most one point of extremum of $f_{-\lambda}$. Thus $f_{-\lambda}$ is a continuous nowhere monotone function with only proper extrema and we can follow the same arguments as in the proof of 3.2.7 to see that all the horizontal levels of $f_{-\lambda}$ are normal so that the levels of f are normal in the direction λ. Part (i) of the theorem is proved.

Now let $\lambda \in \Lambda_f$. By 3.2.11 no line supports the graph of f in more than two mutually disjoint open subintervals of [0,1]. Then no horizontal level of $f_{-\lambda}$ contains more than two points of extremum of $f_{-\lambda}$ so that $f_{-\lambda}$ has only proper extrema. Since $\lambda \in \Lambda_f$ there is a $c_0 \in \mathbf{R}$ such that $y = \lambda x + c_0$ supports the graph of f in at least two disjoint open subintervals of [0,1]. Hence $\{x:f_{-\lambda}(x)=c_0\}$ contains exactly two points of extremum of $f_{-\lambda}$. By 3.2.12, for $c \neq c_0$ $\{x:f_{-\lambda}(x)=c\}$ contains at most one point of extremum of $f_{-\lambda}$. Now, $f_{-\lambda}$ is a continuous nowhere monotone function with only proper extrema so that we can again use the argument of 3.2.7 to complete the proof.□

If we remove the requirement that Λ_f be dense we can see that the first part of the proof of the above theorem uses only the property that f is of nonmonotonic type. Thus we have the following theorem.

THEOREM 3.2.14. If a function $f \in C$ is of nonmonotonic type then its levels are normal in all but a countable set of directions.

A number of questions arise from these results. One type of problem arises by replacing the family of lines with polynomials of a given degree, or with 2-parameter families of functions. We will see in sections 3 and 4 the results of consideration of these problems. Another type of problem arises by replacing C[0,1] with some other complete metric space such as the space of Darboux functions of the first class of Baire, or approximately continuous functions. Many results have been obtained for these spaces (see [13], [15], [34], [35], [37]) but they are beyond the scope of this survey.

Polynomials

In 1981 Ceder and Pearson [12] investigated the intersection sets of typical continuous functions with polynomials. The following corollary is an immediate consequence of Theorem 3.2.5.

COROLLARY 3.3.1. There is a residual set of functions f in C such that if p is a polynomial and x_0 is isolated in $\{x:f(x)=p(x)\}$ then p supports f in a subinterval of [0,1].

The next lemma is a generalization of lemma 3.2.11 and the proof is similar.

LEMMA 3.3.2. For any n there exists a residual set of functions f in C such that no polynomial of degree less than or equal to n supports f in more than n+1 mutually disjoint open subintervals of [0,1].

The next theorem states that a typical continuous function intersects "most" polynomials in a perfect set. This is a generalization of the result of Bruckner and Garg given in Theorem 3.2.13.

THEOREM 3.3.3. There exists a residual set of functions f in C such that the following properties hold:
(i) for $n \geq 1$ and $j \leq n$ and a fixed n-tuple, $(a_0, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$ in $\mathbf{R}^n$ and a polynomial p given by $p(x) = \sum_{i=0}^{n} a_i x^i$, the intersection set $\{x: f(x)=p(x)\}$ is a perfect set except for countably many values of a_j. For each of these exceptional values of a_j the intersection set is a perfect set union a set containing at most n+1 points.
(ii) for each n the set $(a_0, \ldots, a_n)$ in $\mathbf{R}^{n+1}$ such that the intersection set $\{x: f(x)=p(x)\}$ where $p(x) = \sum_{i=0}^{n} a_i x^i$ fails to be a perfect set is a first category null set in $\mathbf{R}^{n+1}$.

PROOF: We follow [12], pp.258-259 with some modifications.

Let $M = N \cap \bigcap_{n=0}^{\infty} N_n$ where N is the residual set given by Corollary 3.3.1 and the N_n are the residual sets given by Lemma 3.3.2. Then M is residual.

By the lemma if $f \in M$ and p is a polynomial of degree n then $\{x:f(x)=p(x)\}$ is a perfect set union a set of no more than n+1 elements.

Now fix $j \leq n$ and an n-tuple $(a_0,\ldots,a_{j-1},a_{j+1},\ldots,a_n)$. Let $p_\alpha(x) = \sum_{i=0}^{n} a_i x^i$ where $a_j = \alpha \in \mathbf{R}$. Let $A = \{x_\alpha : x_\alpha$ is isolated in $\{x:f(x)=p_\alpha(x)\}\}$ Then for each $x_\alpha \in A$ there is a positive integer n_α such that either

$f(x_\alpha) = p_\alpha(x_\alpha)$ and $f(x)<p_\alpha(x)$ for $0<|x-x_\alpha|<1/n_\alpha$
or $f(x_\alpha) = p_\alpha(x_\alpha)$ and $f(x)>p_\alpha(x)$ for $0<|x-x_\alpha|<1/n_\alpha$.

Now suppose that A is uncountable. Then, without loss of generality there is a positive integer m such that the set $A_m = \{x_\alpha \in A : n_\alpha = m,\ f(x)<p_\alpha(x) \text{ for } 0<|x-x_\alpha|<1/m\}$ is uncountable. Since A_m is uncountable there exists $x_\beta \in A_m$ which is a condensation point for A_m with $x_\beta \neq 0$. Choose $x_\alpha \in A_m$ so that $|x_\alpha - x_\beta|<1/m$.

Now the polynomials p_α and p_β can intersect only at x=0 so for x>0 either $p_\alpha<p_\beta$ or $p_\alpha>p_\beta$. But $f(x)\leq p_\beta(x)$ for $|x-x_\beta|<1/m$ so if $\{x:p_\alpha(x)=f(x)\} \neq \phi$ then $p_\alpha(x)<p_\beta(x)$ for x>0. We have $0<|x_\alpha-x_\beta|<1/m$ and $f(x_\beta) = p_\beta(x_\beta) > p_\alpha(x_\beta)$ so $x_\alpha \notin A_m$, a contradiction. Hence A is countable.

To prove part (ii) choose n. Then, for a given $a=(a_1,...,a_n) \in \mathbf{R}^n$ there exist countably many values of a_0 for which the polynomial $p_a(x)=\sum_{i=0}^{n} a_i x^i$ fails to intersect f in a perfect set. Let A denote the set of all $(a_0, ..., a_n) \in \mathbf{R}^{n+1}$ such that $p_a(x)$ fails to intersect f in a perfect set. Then, considering $(a_0,...,a_n)$ as the point $(a_0(a_1,...,a_n))$ in $\mathbf{R}\times\mathbf{R}^n$, each horizontal section of A in $\mathbf{R}\times\mathbf{R}^n$ is countable. Accordingly A will be of first category and measure zero if A has the property of Baire and is measurable [36, th.15.4]. This will be so if A is an analytic set [29 pp.94-95, 482]. We will show now that A is analytic.

Now $(a_0,...,a_n) \in A$ if and only if there is some $x\in[0,1]$ and $h>0$ such that $p_a(x)=f(x)$ and if $0<|x-z|<h$ then $p_a(z)\neq f(z)$. It is clear that:

$$(a_0,...,a_n)\in A \Leftrightarrow \exists x\in[0,1]\times\mathbf{R}\ \exists h\in\mathbf{R}\ [(h>0 \wedge x_2=p_a(x_1)=f(x_1)) \wedge\forall z\in[0,1]\times\mathbf{R}((z=x) \vee (|z_1-x_1|>h) \vee (z_2>p_a(x_1)) \vee (z_2<p_a(x_1)) \vee (z_2\neq f(z_1)))]$$

Thus A is the projection onto $\mathbf{R}^{n+1}$ of the set of $(h,x,(a_0,...,a_n))$ given by:

$$\bigvee_x \bigvee_h \{(h>0) \wedge (x_2=p_a(x_1)=f(x_1)) \wedge \bigwedge_z((z=x) \vee (|z_1-x_1|>h) \vee (z_2>p_a(x_1)) \vee (z_2<p_a(x_1)) \vee (z_2\neq f(z_1)))\}$$
$$= \bigvee_x \bigvee_h (\alpha(h,x,a) \wedge \bigwedge_z (\beta\vee\gamma\vee\delta\vee\xi\vee\eta))$$

It is easy to see that α is a G_δ, β is F_0 and γ, δ, ξ, and η are G_0. Then $\wedge_z\ \beta\vee\gamma\vee\delta\vee\xi\vee\eta$ is a G_δ [see 29 20.V.7b]. Thus we have $\vee_x\ \vee_h\ \theta(x,h,a)$ where $\theta(x,h,a)$ is a G_δ set. Then [see 29 38.viii.4] this is an analytic set since a projection corresponds to a continuous function. Thus A is analytic and part (ii) is proved. □

We see, then, that typical continuous functions intersect "most" polynomials in perfect sets. Ceder and Pearson state that an analogous result to theorem 3.3.3 for piecewise linear functions also holds. A perfect analogue does not exist. To see this, consider a piecewise linear function of 2 pieces. By Theorem 3.2.9 for any continuous function f there are countably many piecewise linear functions such that the intersection of the first piece with f has an isolated point. Now for any of these, no matter how we may vary the slope of the second piece we will not have an intersection set which is perfect. We provide, instead, a partial analogue.

THEOREM 3.3.4. There is a residual set of functions f in C such that, for each n, if h is a piecewise linear function with n intervals of linearity, the set $\{x:f(x)=h(x)\}$ is a perfect set (possibly empty) union a set of at most 3n-1 isolated points.

PROOF: Let N be the residual set of functions given by Theorem 3.2.13. Let $f \in N$ and $E_{f,\lambda}$, $\alpha_{f,\lambda}$ and $\beta_{f,\lambda}$ be as in Theorem 3.2.13. Fix n and let H_n be the set of all piecewise linear functions in [0,1] with n intervals of linearity.

For each $h \in H_n$ and for i=1,...,n let h_i be the i^{th} linear segment of h, extended linearly to all of [0,1]. Then $h_i = \lambda x + c$ for some $\lambda, c \in \mathbf{R}$. If $\{x:h_i(x)=f(x)\}$ is perfect then $\{x:h_i(x)=f(x)\} \cap I_i$, where I_i is the i^{th} interval of linearity of h, is perfect, except possibly for isolated points at the endpoints of the interval. Hence if x_0 is isolated in $\{x:h(x)=f(x)\}$ it is isolated in one of the sets $\{x:h_i(x)=f(x)\}$ or it is the endpoint of one of the intervals of linearity of h.

By theorem 3.2.13 if $c \notin E_{f,\lambda} \cup \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ $\{x:h_i(x)=f(x)\}$ is perfect; otherwise $\{x:h_i(x)=f(x)\}$ is a perfect set (possibly empty) union a set of at most 2 isolated points. Thus $\{x:h(x)=f(x)\}$ is a perfect set (possibly empty} union a set of at most 3n-1 isolated points. □

This is consistent with the following conjecture put forth by Ceder and Pearson: for any closed nowhere dense subset N of C there is a residual set A of functions in C such that for each f in A there exists a residual subset N_f of N such that for all g in N, the intersection set of f with g is perfect if and only if g is in N_f.

Two parameter families

Zygmunt Wojtowicz investigated the intersection sets of typical continuous functions with functions in 2-parameter families in 1985 [43].

DEFINITION 3.4.1. A family of functions $H \subset C$ is called a 2-parameter family if for all $x_1, x_2 \in [0,1]$ with $x_1 \neq x_2$ and for all $y_1, y_2 \in \mathbf{R}$ there exists a unique function h in H such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

We denote by $h_{\lambda,c}$ the unique function $h \in H$ such that $h(0) = c$ and $h(1) - h(0) = \lambda$. We call λ the increase of the function $h_{\lambda,c}$. Let H_λ denote the set of functions $h \in H$ such that the increase in h is λ. Clearly $H_{\lambda_1} \cap H_{\lambda_2} = \phi$ for $\lambda_1 \neq \lambda_2$ and $\cup_{\lambda \in \mathbf{R}} H_\lambda = H$.

Wojtowicz proved several properties of 2-parameter families. These will be given without proof.

LEMMA 3.4.2. If $(x_0,y_0) \in [0,1]\times\mathbf{R}$ and $h_1, h_2 \in H$, $h_1 \neq h_2$ and $h_1(x_0) = h_2(x_0) = y_0$ then either $h_1(x) < h_2(x)$ for $0 \le x < x_0$ and $h_1(x) > h_2(x)$ for $x_0 < x \le 1$, or $h_1(x) > h_2(x)$ for $0 \le x < x_0$ and $h_1(X) < h_2(x)$ for $x_0 < x \le 1$.

LEMMA 3.4.3. For every triple $(x_0,y_0,\lambda) \in [0,1]\times\mathbf{R}\times\mathbf{R}$ there exists a unique function $h \in H_\lambda$ such that $h(x_0) = y_0$.

COROLLARY 3.4.4. If h_1, $h_2 \in H_\lambda$ and $h_1 \neq h_2$ then $h_1(x) \neq h_2(x)$ for every $x \in [0,1]$. In particular $h_1(0) > h_2(0)$ if and only if $h_1(x) > h_2(x)$ for all $x \in [0,1]$.

LEMMA 3.4.5. $\text{Lim}_{n\to\infty} |h_{\lambda_n,c_n} - h_{\lambda,c}| = 0$ if and only if $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} \lambda_n = \lambda$.

LEMMA 3.4.6. For each natural number n let (x_n',y_n'), (x_n'',y_n''), (x',y') and (x'',y'') be in $[0,1]\times\mathbf{R}$ with $x_n' \neq x_n''$ and $x' \neq x''$. Let h_{λ_n,c_n}, $h_{\lambda,c} \in H$ be such that $h_{\lambda_n,c_n}(x_n') = y_n'$, $h_{\lambda_n,c_n}(x_n'') = y_n''$, $h_{\lambda,c}(x') = y'$ and $h_{\lambda,c}(x'') = y''$. Then if $\lim_{n\to\infty} (x_n',y_n') = (x',y')$ and $\lim_{n\to\infty} (x_n'',y_n'') = (x'',y'')$ we have $\lim_{n\to\infty} \|h_{\lambda_n,c_n}-h_{\lambda,c}\| = 0$.

The methods used by Wojtowicz to develop the properties of the intersection sets of typical continuous functions with 2-parameter families parallel those of Bruckner and Garg for lines. The next lemma is proved by an argument parallel to that of Lemma 3.2.8.

LEMMA 3.4.7. For every function f in C there is at most a countable set of functions in H whose graphs support the graph of f in two or more disjoint open subintervals of $[0,1]$.

Now if $f \in C$ and $\lambda \in \mathbf{R}$ let

$$\alpha_{f,\lambda} = \inf\{c\in\mathbf{R}: \{x:f(x)=h_{\lambda,c}(x)\}\neq\phi\}$$

and $\beta_{f,\lambda} = \sup\{c\in\mathbf{R} :\{x:f(x)=h_{\lambda,c}(x)\}\neq\phi\}$.

LEMMA 3.4.8. For every function f in C and every number λ in $\mathbf{R}$, the graphs of the functions $h_{\lambda,\alpha_{f,\lambda}}$ and $h_{\lambda,\beta_{f,\lambda}}$ support the graph of f in $[0,1]$ from below and above respectively, at least at one point.

PROOF: ([43], p. 75) Let $f \in C$ and $\lambda \in \mathbf{R}$. Since $h_{\lambda,\alpha_{f,\lambda}}(x) \leq f(x)$ for every $x \in [0,1]$ it is sufficient to show that $\{x:f(x)=h_{\lambda,\alpha_{f,\lambda}}(x)\} \neq \phi$.

Assume $\{x:f(x)=h_{\lambda,\alpha_{f,\lambda}}(x)\}=\phi$. Then for all $x \in [0,1]$, $h_{\lambda,\alpha_{f,\lambda}}(x) - f(x) < 0$. Let $d = \min\{f(x)-h_{\lambda,\alpha_{f,\lambda}}(x):0\leq x\leq 1\}$. By Lemma 3.4.5 there is a function h_{λ,c_1} in the open sphere $B(h_{\lambda,\alpha_{f,\lambda}},d)$ such that $h_{\lambda,c_1} \neq h_{\lambda,\alpha_{f,\lambda}}$ and $c_1 > \alpha_{f,\lambda}$. Clearly then $h_{\lambda,c_1}(x) < f(x)$ for every $x \in [0,1]$ and $\{x:f(x)=h_{\lambda,c}(x)\} = \phi$ for all $c \in (\alpha_{f,\lambda},c_1)$. This contradicts the definition of $\alpha_{f,\lambda}$ and so we have $\{x:f(x)=h_{\lambda,\alpha_{f,\lambda}}(x)\} \neq \phi$.

Similarly $\{x:f(x)=h_{\lambda,\beta_{f,\lambda}}(x)\} \neq \phi$ and since $h_{\lambda,\beta_{f,\lambda}}(x) \geq f(x)$ for all $x \in [0,1]$ the lemma is proved.□

From the above lemma we obtain the following result regarding the existence of isolated points in the intersection sets of any continuous function with functions in a 2-parameter family. This is a close parallel to Theorem 3.2.9.

THEOREM 3.4.9. For every function f in C there is at most a countable set $\Lambda_f \subset \mathbf{R}$ such that if $\lambda \in \mathbf{R}\setminus\Lambda_f$ then:
(i) the sets $\{x:f(x)=h_{\lambda,\alpha_{f,\lambda}}(x)\}$ and $\{x:f(x)=h_{\lambda,\beta_{f,\lambda}}(x)\}$ consist of single points, and
(ii) the set $E_{f,\lambda}$ of numbers $c \in \mathbf{R}$ such that $\{x:f(x)=h_{\lambda,c}(x)\}$ is not perfect, is dense in $(\alpha_{f,\lambda},\beta_{f,\lambda})$.

PROOF: ([43], pp. 75-76) Let $f \in C$ and let Λ_f be the set of increases of functions in H whose graphs support the graph of f in at least two disjoint open intervals of [0,1]. Then Λ_f is countable by Lemma 3.4.7. Let $\lambda \in \mathbf{R}\setminus\Lambda_f$. Then the graphs of $h_{\lambda,\alpha_{f,\lambda}}$ and $h_{\lambda,\beta_{f,\lambda}}$ each support f at a unique point and (i) is proved.

Let (a,b) be any open subinterval of $(\alpha_{f,\lambda},\beta_{f,\lambda})$. Let I be a connected component of $\{x:h_{\lambda,a}(x)<f(x)<h_{\lambda,b}(x)\}$. Then I is an open subinterval of [0,1] and for every $c \in (a,b)$ $f - h_{\lambda,c} \neq 0$ in every subinterval of I. Now if for all $c \in (a,b)$ the set $\{x:f(x)=h_{\lambda,c}(x)\}\cap I$ consists of a single point, x_0, then x_0 is an isolated point of $\{x:f(x)=h_{\lambda,c}(x)\}$.

If this is not the case then there is a number $c \in (a,b)$ such that $\{x:f(x)=h_{\lambda,c}(x)\}\cap I$ contains at least two different points, say x_1 and x_2 with $x_1 < x_2$. Let

$$c_1 = \inf\{c\in\mathbf{R}: \{x\in[x_1,x_2]:f(x)=h_{\lambda,c}(x)\}\neq\phi\}$$

and $c_2 = \sup\{c\in\mathbf{R}: \{x\in[x_1,x_2]:f(x)=h_{\lambda,c}(x)\}\neq\phi\}$.

Then, as in the proof of Lemma 3.4.8, there are numbers $x_3,x_4 \in [x_1,x_2]$ such that $f(x_3) = h_{\lambda,c_1}(x_3)$ and $f(x_4) = h_{\lambda,c_2}(x_4)$ so c_1, $c_2 \in (a,b)$. Now $\lambda \notin \Lambda_f$ so there exists $x' \in (x_1,x_2)$ such that $f(x') = h_{\lambda,c_1}(x')$ and $f(x) > h_{\lambda,c_1}(x)$ or $f(x') = h_{\lambda,c_2}(x')$ and $f(x) < h_{\lambda,c_2}(x)$, for all $x \in (x_1,x_2)\backslash\{x'\}$. Then the graph of h_{λ,c_1} or h_{λ,c_2} supports f in (x_1,x_2) and x' is an isolated point of the intersection set. Thus $c_1 \in E_{f,\lambda}$ or $c_2 \in E_{f,\lambda}$. This proves that $E_{f,\lambda}$ is dense in $(\alpha_{f,\lambda},\beta_{f,\lambda})$ and so part (ii) is proved.□

The next lemma is analogous to Lemma 3.2.10. Although the proof is not difficult the details are tedious and we will only outline it here.

LEMMA 3.4.10. There exists a residual set of functions f in C such that for every rational open interval $I \subset [0,1]$ the increases of functions in H which support the graph of f from above at more than one point, form a dense set in **R**. The same holds true for functions in H which support f from below.

PROOF: ([43], pp. 476-478) It suffices to prove the result for one fixed open rational subinterval I in [0,1]. Let $\{\lambda_n\}$ be an enumeration of the rational numbers. For each pair of natural numbers, (n,m) let A_{nm} denote the set of functions f in C for which there is a function in H with increase $\lambda \in (\lambda_n - 1/m, \lambda_n + 1/m)$ which supports the graph of f in I from above at more that one point. We shall show that $A_{n,m}$ is residual in C.

Let n, m be fixed and let U be a nonempty open set in C with $f \in U$. Then there exists $\epsilon > 0$ such that the open sphere $B(f,\epsilon) \subset U$. Let $\alpha_n = \sup\{c \in \mathbf{R}: \{x \in I: f(x) = h_{\lambda_n,c}(x)\} \neq \phi\}$. Then by Lemma 3.4.5 there exists a function $h_{\lambda_n,c_1} \in H$ such that $h_{\lambda_n,c_1} \in B(h_{\lambda_n,\alpha_n}, \epsilon/4)$ and $h_{\lambda_n,c_1}(x) < h_{\lambda_n,\alpha_n}(x)$ for all $x \in [0,1]$. Let J be a subinterval of I such that $f > h_{\lambda_n,c_1}$ on J. We can find five points, x_i (i=0,...,4), in J such that $x_0 < \ldots < x_4$. Then by Lemma 3.4.5 there is a function $h_{\lambda_n,c}$ in H such that $h_{\lambda_n,c} \in B(h_{\lambda_n,\alpha_n}, \epsilon/2)$ and $h_{\lambda_n,c}(x) > h_{\lambda_n,\alpha_n}(x)$ for all $x \in [0,1]$. Define $g:[0,1] \to \mathbf{R}$ by:

$g(x) = f(x)$ for $x \in [0,x_0] \cup [x_4,1]$

$g(x_1) = h_{\lambda_n,c}(x_1)$, $g(x_3) = h_{\lambda_n,c}(x_3)$,

$g(x_2) = h_{\lambda_n,\alpha_n}(x_2)$

and g = h for some $h \in H$ in each of the intervals $[x_i, x_{i+1}]$, (i=0,1,2,3). Then $g \in B(f,\epsilon)$. It can be shown that there is a real number $\eta < \epsilon/8$ such that $B(g,\eta) \subset A_n$. Thus $A_{n,m}$ contains a dense open subset of C and so it is residual in C. Hence $A = \bigcap_{n=1}^{\infty} \bigcap_{n=1}^{\infty} A_{n,m}$ is residual in C.

Now given $(p,q) \subset \mathbf{R}$ and $f \in A$ there exists a rational number $\lambda_n \in (p,q)$ and a natural number m such that $(\lambda_n - 1/m, \lambda_n + 1/m) \subset (p,q)$. $f \in A_{n,m}$ so there exists a function $h_{\lambda,c} \in H$ which supports the graph of f from above at more than one point and $\lambda \in (\lambda_n - 1/m, \lambda_n + 1/m)$. Hence the set of increases of functions in H which support f from above at more than one point is dense in $\mathbf{R}$.

The proof of support from below is similiar.□

We need three more technical lemmas as well as another definition before arriving at our main result for this section. The first two lemmas are proved by arguments parallel to those for lemmas 3.2.11 and 3.2.12.

LEMMA 3.4.11. There exists a residual set of functions f in C for which there is no function in H which supports f in more than two points.

LEMMA 3.4.12. There exists a residual set of functions f in C for which there does not exist $\lambda \in \mathbf{R}$ with two distinct functions in H_λ each of which support f in 2 distinct points.

DEFINITION 3.4.13. A two parameter family H of continuous functions is <u>almost</u> <u>uniformly</u> <u>Lipschitz</u> if for all c, $\lambda \in \mathbf{R}$

there exists $L_{\lambda,c} > 0$ such that for all $x_1, x_2 \in [0,1]$ $|h_{\lambda,c}(x_1)-h_{\lambda,c}(x_2)| \le L_{\lambda,c}|x_1-x_2|$ and for every rational number n, $M_n = \sup\{L_{\lambda,c}:\lambda, c \in [-n,n]\} < +\infty$.

The next theorem is analogous to Theorem 3.2.5 which says that functions of nonmonotonic type form a dense G_δ set in C. The proof is also similar.

THEOREM 3.4.14. Let H be a two parameter family of continuous functions which is almost uniformly Lipschitz. Then there exists a residual set of functions f in C such that for every h in H the function f-h is not monotone at any point $x \in [0,1]$.

PROOF: ([43], pp. 478-480) For each natural number n let A_n denote the set of functions f in C for which there exist numbers λ, $c \in [-n,n]$ and $x \in [0,1]$ such that $f-h_{\lambda,c}$ is nondecreasing on $(x-1/n,x+1/n)\cap[0,1]$. Then $A = \bigcup_{n=1}^{\infty} A_n$ is the set of functions f in C for which there exists a function h in H such that f-h is nondecreasing at some point of [0,1]. We will show that each A_n is closed and nowhere dense.

Let n be given and let $\{f_k\}$ be a sequence of functions in A_n that converges uniformly to a function $f \in C$. Then for each k there exist λ_k, $c_k \in [-n,n]$ and $x_k \in [0,1]$ such that for $h_k = h_{\lambda_k, c_k}$, the function $f_k - h_k$ is nondecreasing on $(x_k - 1/n, x_k + 1/n) \cap [0,1]$. Moreover there is a sequence $\{k_i\}$ of natural numbers such that $\{\lambda_{k_i}\}$ converges to some $\lambda \in [-n,n]$, $\{c_{k_i}\}$ converges to some $c \in [-n,n]$ and $\{x_{k_i}\}$ converges to some $x \in [0,1]$. By Lemma 3.4.5 the sequence $\{h_{k_i}\}$ converges uniformly to the function $h_{\lambda,c}$. It is easy to see that $f - h_{\lambda,c}$ is nondecreasing on $(x-1/n, x+1/n) \cap [0,1]$. Thus $f \in A_n$ and A_n is closed.

Now we will show that A_n is nowhere dense. Let U be a nonempty open subset of C. Then there is a polynomial g and $\epsilon > 0$ such that $B(g,\epsilon) \subset U$. Then a "saw toothed" function $f \in C$ can be constructed with $f \in B(g,\epsilon)$ and $f \notin A_n$. To do this let $0 < \alpha < \epsilon$ and $\beta = \sup\{|g'(x)| : 0 \le x \le 1\}$ and let $m > \max\{2M_n, 2n, 3(\beta + M_n)/\alpha\}$ where M_n is as in Definition 3.4.12. Define a function s by:

$s(x) = \alpha$ for $x = 2i/m$ $(i = 0, 1, \ldots, (m-1)/2)$

$s(x) = 0$ for $x = (2i+1)/m$ $(i = 0, 1, \ldots, (m-1)/2)$

s is linear for $x \in [i/m, (i+1)/m]$ $(i = 0, 1, \ldots, m-1)$.

Then s is continuous and $f = g + s \in B(g,\epsilon)$. For $x \in [0,1]$ there exists an integer i such that $0 \le i \le (m-1)/2$ and $2i/m \le x \le (2i+2)/m$. Let λ, $c \in [-n,n]$. Then it is easy to show that $f - h_{\lambda,c}$ is not nondecreasing for any $x \in [0,1]$ and so $f \notin A_n$. Thus A_n is nowhere dense and so A is of first category in C.

The set of functions f in C for which there is a function $h \in H$ such that f-h is nonincreasing at some point of [0,1] can be shown in the same way to be of first category. Thus the theorem is proved.□

We now define the notion of normal in the context of 2-parameter families.

DEFINITION 3.4.15. Let $f \in C$, $\lambda \in \mathbf{R}$ and H a 2-parameter family of functions $h_{\lambda,c}$ in C. If there exists a countable dense set $E_{f,\lambda}$ in $(\alpha_{f,\lambda},\beta_{f,\lambda})$ such that the set $\{x:f(x)=h_{\lambda,c}(x)\}$ is:

(i) a perfect set when $c \notin E_{f,\lambda}\cup\{\alpha_{f,\lambda},\beta_{f,\lambda}\}$

(ii) a single point when $c=\alpha_{f,\lambda}$ or $c=\beta_{f,\lambda}$, and

(iii) the union of a nonempty perfect set and an isolated point when $c \in E_{f,\lambda}$.

Then the intersection sets of f with functions in H_λ are said to be <u>normal</u>.

THEOREM 3.4.16. Let H be a 2-parameter family of continuous functions which is almost uniformly Lipschitz. Then there exists a residual set of functions f in C for which there is a countable dense set $\Lambda_f \subset \mathbf{R}$ such that:

(i) the intersection sets of f with functions in H_λ is normal for each $\lambda \in \mathbf{R}\setminus\Lambda_f$ and

(ii) for any $\lambda \in \Lambda_f$ the intersection sets of f with functions in H_λ are normal but for a unique number $c_{f,\lambda} \in E_{f,\lambda} \cup \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ for which the intersection set contains two isolated points in place of one.

PROOF: ([43], pp. 480-481) Let A be the intersection of the residual sets determined by Lemmas 3.4.10, 3.4.11, 3.4.12 and Theorem 3.4.14. Then A is residual in C. Let $f \in A$ and let Λ_f denote the set of increases of functions h in H which support the graph of f in at least two disjoint open subintervals of [0,1]. By Lemmas 3.4.7 and 3.4.10 Λ_f is a countable dense subset of **R**. For $\lambda \in \mathbf{R}$ and $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$ we have $\{x:f(x)=h_{\lambda,c}(x)\} = \{x:f(x)-h_{\lambda,c}(x)=0\}$. By theorem 3.4.14 $f-h_{\lambda,c}$ is not monotone at any point of [0,1]. Thus x is isolated in $\{x:f(x)=h_{\lambda,c}(x)\}$ if and only if $f-h_{\lambda,c}$ has a proper extremum equal to zero at x. This is so if and only if $h_{\lambda,c}$ supports f at x. Since $f-h_{\lambda,c}$ is continuous $\{x:f(x)=h_{\lambda,c}(x)\}$ is a nonempty closed set for all $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$. Let $E_{f,\lambda}$ be the set of numbers $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda})$ such that $\{x:f(x)=h_{\lambda,c}(x)\}$ is not perfect. By Lemma 3.4.9 $E_{f,\lambda}$ is dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$.

Let $c_0 \in E_{f,\lambda}$ and let x_0 be a point at which a proper maximum of $f-h_{\lambda,c_0}$ is equal to zero. Then there is a rational open subinterval $I \subset [0,1]$ such that $x_0 \in I$ and $f(x_0)=h_{\lambda,c_0}(x_0)$ and $f(x)<h_{\lambda,c_0}(x)$ for $x \in I\setminus\{x_0\}$. Then (x_0,c_0) is unique in I. Similarly if a proper minimum of $f-h_{\lambda,c_0}$ is equal to zero at x_0 then there is an open

rational subinterval $J \subset [0,1]$ such that for J the pair (x_0,c_0) is unique. Hence the set of pairs, (x,c) for which x is an isolated point of $\{x:f(x)=h_{\lambda,c}(x)\}$ is countable so that $E_{f,\lambda}$ is countable.

Let $\lambda \in \mathbf{R}\backslash\Lambda_f$. Let $c \in [\alpha_{f,\lambda},\beta_{f,\lambda}]$. Then $h_{\lambda,c}$ supports the graph of f at most at one point. Hence $\{x:f(x)=h_{\lambda,c}(x)\}$ contains at most one isolated point. If $c \in \{\alpha_{f,\lambda},\beta_{f,\lambda}\}$ then, by Lemma 3.4.8, $\{x:f(x)=h_{\lambda,c}(x)\}$ consists of a single point. If $c \in (\alpha_{f,\lambda},\beta_{f,\lambda})\backslash E_{f,\lambda}$ then $\{x:f(x)=h_{\lambda,c}(x)\}$ is a perfect set. Let $c \in E_{f,\lambda}$ and let x_0 be an isolated point of $\{x:f(x)=h_{\lambda,c}(X)\}$. Since $\lambda \in \Lambda_f$, x_0 is the unique isolated point of this set. Clearly $\{x:f(x)=h_{\lambda,c}(X)\}\backslash\{x_0\}\neq\phi$. This proves part (i).

Let $\lambda \in \Lambda_f$. By Lemma 3.4.11 if $c \in [\alpha_{f,\lambda},\beta_{f,\lambda}]$ the set $\{x:f(x)=h_{\lambda,c}(x)\}$ contains at most two isoated points. By Lemma 3.4.12 there is a unique number $c_{f,\lambda} \in [\alpha_{f,\lambda},\beta_{f,\lambda}]$ such that $\{x:f(x)=h_{\lambda,c_{f,\lambda}}(x)\}$ contains two isolated points. For $c\neq c_{f,\lambda}$, $\{x:f(x)=h_{\lambda,c_{f,\lambda}}(x)\}$ contains at most one isolated point. The rest follows as in the proof of part (i).□

Wojtowicz extends this result to families of continuous functions which are homeomorphic images of 2-parameter families.

THEOREM 3.4.17. If H is a 2-parameter family for which

Theorem 3.4.16 holds and Ψ is a homeomorphism, then Theorem 3.4.16 holds for the family $\Psi(H)$.

Conclusion

We have seen that although typical continuous functions exhibit "pathological" properties such as nondifferentiability and nonmonotonicity there is also a great deal of regularity in their behaviour. For each typical continuous function this regularity is exhibited in the pattern of the intersection sets of the function with lines, polynomials, and functions in some 2-parameter families. There is also a regularity within the family of typical continuous functions in that the pattern of intersection sets is the same.

Many of these results derive from the fact that a typical continuous function is of nonmonotonic type. This means that the function is not monotonic in a very strong way, much stronger than simple nowhere monotone. A function of nonmonotonic type is nowhere monotone but the converse does not necessarily hold. If we notice that a function of nonmonotonic type cannot cross a straight line in a "simple' way then we see how nonmonotonicity is related to the structure of the intersection sets of typical continuous functions with straight lines. Also since it is clear that functions of nonmonotonic type have $\underline{f}' = +\infty$ and $\overline{f}' = +\infty$ we see that they are nowhere differentiable.

CHAPTER IV

POROSITY

In Chapter 3 we looked at some of the geometric properties of typical continuous functions. One of the results of that chapter was that any typical continuous function intersects every line in a nowhere dense set. Nowhere denseness can be seen as a measure of the "smallness" of a set. Other ways to measure smallness include Lebesgue measure and porosity. In this chapter we will look mostly at the porosity of intersection sets of typical continuous functions with certain families of functions. In section 2 we shall see results regarding the size of intersection sets of typical continuous functions with families of functions controlled by a modulus of continuity and with horizontal lines. Section 3 deals with the intersection sets of typical continuous functions with functions in σ-compact subsets of C. In section 4 we look at results obtained by Haussermann relating the porosity of intersection sets to the modulus of continuity controlling the class of functions with which the intersections are taken. Section 5 concerns [g]-porosity and [g]-knot points.

The concept of porosity was first intoduced by Denjoy in 1941, in the form of an index. Dolzenko introduced the term porosity in 1967 and it is his notation that we will use. Porosity measures the relative size of the "gaps" in a set so that if the porosity is high then the set is relatively "thin".

DEFINITION 4.1.1. Let E be a subset of **R** and a, b $\in$ **R** with a<b. Then by $\lambda(E,a,b)$ we mean the length of the largest open interval of (a,b) that contains no point of E. For convenience we will also write $\lambda(E,a,b)$ for b<a as well as a<b.

DEFINITION 4.1.2. Let E be a subset of **R** and $x_0 \in$ **R**. Then the right hand porosity of E at x_0 is defined as:

$$p_+(E,x_0) = \overline{\lim}_{h\to 0+} \frac{\lambda(E,x_0,x_0+h)}{h}.$$

Similarly the left hand porosity of E at x_0 is defined as:

$$p_-(E,x_0) = \overline{\lim}_{h\to 0-} \frac{\lambda(E,x_0,x_0+h)}{|h|},$$

and the bilateral porosity of E at x_0 is defined as:

$$p(E,x_0) = \overline{\lim}_{h\to 0} \frac{\lambda(E,x_0,x_0+h)}{|h|}.$$

DEFINITION 4.1.3. Let E be a subset of **R** and $x_0 \in$ **R**. Then we will say E is porous at x_0 if $p(E,x_0)>0$; porous on the right (left) if $p_+(E,x_0)>0$ ($p_-(E,x_0)>0$); strongly porous at x_0 if $p(E,x_0)=1$; right (left, bilaterally) strongly porous at x_0 if $p_+(E,x_0)=1$ ($p_-(E,x_0)=1$, $p_+(E,x_0)=p_-(E,x_0)=1$ respectively); nonporous at x_0 if $p(E,x_0)=0$. We say that E is porous (porous on the right, strongly porous, etc.) if for every $x \in E$, E is porous (porous on the right, strongly porous, etc.) at x.

We see then that a porous set is both nowhere dense and of measure zero. This is because porosity measures the size of the gaps in a set and if a set has gaps near each of its points then it is nowhere dense and if the gaps are large enough then no point can be a point of density so the set has measure zero. We need two more definitions.

DEFINITION 4.1.4. Let σ be a real valued function defined for all nonnegative real numbers such that σ is increasing and $\lim_{x\to 0+} \sigma(x) = \sigma(0) = 0$. Then σ is a <u>modulus of continuity</u>.

DEFINITION 4.1.5. Let σ be a modulus of continuity. Then by <u>$C(\sigma)$</u> we mean the set of functions f in C such that for every $x, y \in [0,1]$, $|f(x)-f(y)| \le \sigma(|x-y|)$. We call $C(\sigma)$ the <u>equicontinuous family determined by σ</u>.

<u>Intersections with $C(\sigma)$ and with horizontal lines</u>

In 1963 C. Goffman [20] showed that for any modulus of continuity σ, the typical continuous function intersects every function in the equicontinuous class $C(\sigma)$ in a set of measure zero. To see this we first need two technical lemmas.

LEMMA 4.2.1. Let σ be a modulus of continuity. For each ϵ, $1>\epsilon>0$, and positive integer n there exists δ and η where $1/n>\delta>0$ and $\eta>0$, and a function $f \in C$, with $\|f\|\le\epsilon$, such that for each $x \in [0,1-1/n]$ and every function g with $\|f-g\|<\eta$ we have either:

$$|g(x)-g(y)| > \sigma(|x-y|),\ \forall y \in [x-\delta/2,x-\delta/4] \text{ or}$$
$$|g(x)-g(y)| > \sigma(|x-y|),\ \forall y \in [x+\delta/4,x+\delta/2].$$

PROOF: ([20] pp. 741-742) Choose $\delta < 1/n$ so that $\sigma(\delta) < \epsilon/8$. Then there is a positive integer k such that $k\delta < 1 \le (k+1)\delta$ and $[0,k\delta] \supset [0,1-1/n]$. Define a continuous function f as follows: $f(0) = 0$, for $m\le k$, $f(m\delta) = \epsilon$ if m is odd, $f(m\delta) = 0$ if m is even and f is linear in between such that each segment has slope $\pm\epsilon/\delta$.

For every $x \in [0,k\delta]$, $(y,f(y))$ is on the same line segment of the graph of f as $(x,f(x))$, either for all $y \in [x-\delta/2,x-\delta/4]$ or for all $y \in [x+\delta/4,x+\delta/2]$. Then for all such y,

$$\begin{aligned}|f(x)-f(y)| = \epsilon/\delta|x-y| &> 8\sigma(|x-y|)|x-y|/\delta \\ &\ge 2\sigma(|x-y|)\end{aligned}$$

Let $\eta = 1/2\ \sigma(\delta/4)$ and suppose $\|f-g\| < \eta$ for all x and y as above, then:

$$|g(x)-g(y)| > |f(x)-f(y)| -2\eta > 2\sigma(|x-y|)-\sigma(\delta/4)$$
$$\geq 2\sigma(|x-y|) - \sigma(|x-y|) = \sigma(|x-y|). \square$$

LEMMA 4.2.2. Let σ be a modulus of continuity. For every positive integer n, the set E_n of functions f in C such that there exists δ (depending on f) with $0<\delta<1/n$, such that for all $x \in [0,1-1/n]$, $|f(x)-f(y)| > \sigma(|x-y|)$ either for every $y \in [x-\delta/2,x-\delta/4]$ or for every $y \in [x+\delta/4,x+\delta/2]$, contains a dense open set in C.

PROOF: ([20], pp. 742-743) Let $g \in C$ and α be such that $0<\alpha<1$. Then there exists a polynomial, p, such that $p \in B(g,\alpha/2)$. Let $M = \max\{|p'(x)|:x\in[0,1]\}$. Let $\alpha/2$ take the place of ϵ in 4.2.1 and $\omega(x) = \sigma(x) + Mx$ take the place of σ in 4.2.1. Then there exists δ and η with $0<\delta<1/n$ and $\eta>0$ and a continuous function h with $\|h\|\leq\alpha/2$, such that if $\|k-h\|<\eta$ then, for all $x \in [0,1-1/n]$, $|k(x)-k(y)| > \omega(|x-y|)$ either for every $y \in [x-\delta/2,x-\delta/4]$ or for every $y \in [x+\delta/4,x+\delta/2]$. For all such x and y we have:

$$|(p+k)(x)-(p+k)(y)| \geq |k(x)-k(y)| - |p(x)-p(y)|$$
$$> \omega(|x-y|) - M|x-y|$$
$$= \sigma(|x-y|).$$

Now $\|g-(p+h)\| \leq \|g-p\| + \|h\| < \alpha$. Thus $B(g,\alpha)$ contains an open subset of E_n and so E_n contains a dense open subset of C.$\square$

The main result of this section follows easily.

THEOREM 4.2.3. For each modulus of continuity σ the set of functions f in C such that for every $g \in C(\sigma)$, the Lebesgue measure of $\{x:f(x)=g(x)\}$ equals zero, is residual in C.

PROOF: ([20], p.743) For each natural number n let E_n be the set defined in Lemma 4.2.2. Let $E = \cap_{n=1}^{\infty} E_n$. Then E is residual in C.

Let $f \in E$. Then for each n there is a δ_n such that $0<\delta_n<1/n$ and for all $x \in [0,1/n]$, $|f(x)-f(y)| > \sigma(|x-y|)$ either for every $y \in [x-\delta_n/2, x-\delta_n/4]$ or for every $y \in [x+\delta_n/4, x+\delta_n/2]$. Let $g \in C(\sigma)$ and let $A = \{x:f(x)=g(x)\}$. A is measurable and so we assume $m(A)>0$. Let $x \in (0,1)\cap A$ such that the density of A at x, $d(A,x)$, is 1. Then if $B = \{y:|f(x)-f(y)|>\sigma(|x-y|)\}$, $d(B,x) = 0$. Now choose n_0 so that $x < 1-1/n_0$. Then for $n \geq n_0$, B contains either the set $[x-\delta_n/2, x-\delta_n/4]$ or the set $[x+\delta_n/4, x+\delta_n/2]$ and so B has relative measure of at least 1/4 on $[x-\delta_n/2, x+\delta_n/2]$. But $\lim_{n\to\infty}\delta_n = 0$ so this contradicts the assertion that $d(B,x)=0$. Hence $\lambda(A)=0$.□

In 1981 B. Thomson [41] showed that the level sets of a typical continuous function are strongly porous on both sides.

THEOREM 4.2.4. There is a residual set of functions in C all of whose horizontal levels are strongly porous on both sides.

PROOF: ([41], p.189) For $x \in [0,1]$ let $L_x(f) = f^{-1}(f(x))$. For each pair of rational numbers p and δ such that $\delta>0$ and $p \in (0,1)$ let $A_{p,\delta}$ denote the set of all functions f in C Such that there is a point x (depending on p, δ, and f) so that $\lambda(L_x(f),x,x+h) \le ph$ for $0<h<\delta$. We will show that each $A_{p,\delta}$ is nowhere dense in C.

It is easy to see that no $A_{p,\delta}$ contains a neighbourhood in C since every neighbourhood contains many functions whose graphs are nonhorizontal line segments and so are not in $A_{p,\delta}$. Then if $A_{p,\delta}$ is closed we see that it is nowhere dense.

Let $\{f_n\}$ be a sequence of functions in $A_{p,\delta}$ which converges uniformly to a function $g \in C$. Let $\{x_n\}$ be the sequence of points associated with the corresponding f_n so that $\lambda(L_x(f_n),x_n,x_n+h) \le ph$ for $0<h<\delta$. Then, passing to a subsequence if necessary, $\{x_n\}$ converges to a point z in [0,1]. Suppose that $\lambda(L_z(g),z,z+h) > ph$ for some h such that $0<h<\delta$. Then there is an interval J in (z,z+h) for which $|J|>ph$ and $J \cap L_z(g) = \phi$. Then there is a closed subinterval $[a,b] \subset J$ with $b-a > ph$ and $|g(x)-g(z)| \ge \epsilon > 0$ for all $x \in [a,b]$. Choose n sufficiently large so that

$[a,b] \subset (x_n, x_n+h)$ and $|f_n(t)-g(t)|<\epsilon/3$ for all $t \in [0,1]$ and $|g(z)-f_n(x_n)|<\epsilon/3$. Then for $x \in [a,b]$ we would have $|f_n(x)-f_n(x_n)|\geq\epsilon/3$. But this would imply that $L_{x_n}(f_n)\cap[a,b] = \phi$ which gives $\lambda(L_{x_n}(f_n),x_n,x_n+h) \geq b-a > ph$. This contradicts the choice of f_n and x_n and so we have shown that $\lambda(L_z(g),z,z+h) \leq ph$. Hence $A_{p,\delta}$ is closed and so nowhere dense.

Let $A = \cup_{p,\delta}A_{p,\delta}$. Then A is of first category. The case for strong porosity on the left is similarly shown and so the theorem follows.□

Intersections with σ-Compact Sets of Functions

In 1985 Bruckner and Haussermann [9] generalized the results of Goffman and Thomson to show that for any σ-compact class of functions F, a typical continuous function will intersect every function from F in a bilaterally strongly porous set. This leads to some new results and new proofs of known results regarding the differentiability of typical continuous functions.

First we require two technical lemmas which are variants of Lemmas 4.2.1 and 4.2.2 given by Goffman. The proofs are similar.

LEMMA 4.3.1. Let σ be a modulus of continuity. For each ϵ,

$1/2>\epsilon>0$ and each positive integer $n\geq 2$, there exists $\delta_n \in (0,1/n)$ and $\eta>0$ such that:

(i) there exists $f \in C$ with $\|f\|\leq\epsilon$, such that for each $x \in [0,1-1/n]$ and every function $g \in C$ with $\|f-g\|<\eta$, $|g(x)-g(y)| > \sigma(|x-y|)$ either for all $y \in [x+\delta_n/2n^2, x+\delta_n/2n]$ or for all $y \in [x+\delta_n/2n^2+\delta_n/n, x+\delta_n]$; and

(ii) there exists $h \in C$, with $\|h\|\leq\epsilon$, such that for each $x \in [1/n,1]$ and every function $g \in C$ with $\|h-g\|<\eta$, $|g(x)-g(y)| > \sigma(|x-y|)$either for all $y \in [x-\delta_n/2n, x-\delta_n/2n^2]$ or for all $y \in [x-\delta_n, x-\delta_n/2n^2-\delta_n/n]$.

LEMMA 4.3.2. Let σ and n be as in 4.3.2 and let

$E_n^1 = \{f\in C:\exists\delta_n\in(0,1/n)$ so that $\forall x\in(0,1-1/n]$,
$|f(x)-f(y)|>\sigma|x-y|$ either for all $y\in[x+\delta_n/2n^2, x+\delta_n/2n]$
or for all $y\in[x+\delta_n/2n^2+\delta_n/n, x+\delta_n]\}$

$E_n^2 = \{f\in C:\exists\delta_n\in(0,1/n)$ so that $\forall x\in[1/n,1]$,
$|f(x)-f(y)|>\sigma(|x-y|)$ either for all
$y\in[x-\delta_n/2n, x-\delta_n/2n^2]$
or for all $y\in[x-\delta_n, x-\delta_n/2n^2-\delta_n/n]\}$.

Then E_n^1 and E_n^2 contain dense open subsets of C.

The next lemma extends Theorem 4.2.3.

LEMMA 4.3.3. For each modulus of continuity σ the set of functions f in C, such that for every $g \in C(\sigma)$, $\{x:f(x)=g(x)\}$ is bilaterally strongly porous, is residual in C.

PROOF: ([9], pp. 9-10) Let E_n^1 and E_n^2 be as in 4.3.2 and let $E^1 = \cap_{n=2}^{\infty} E_n^1$ and $E^2 = \cap_{n=2}^{\infty} E_n^2$. Then E^1 and E^2 are residual in C and so is $E = E^1 \cap E^2$.

Let $f \in E^1$, $g \in C(\sigma)$ and $H = \{t: f(t)=g(t)\} \cap [0,1)$. We will show that H has right porosity equal to one at each point of H. Let $x \in H$, N be a positive integer such that $x \in [0,1-1/N]$, and $B = \{y: |f(x)-f(y)| > \sigma(|x-y|)\}$. Then $B \subset [0,1] \setminus H$. For $n \geq N$ we have either:

(i) $[x+\delta_n/2n^2, x+\delta_n/2n] \subset B$ for infinitely many n. Now $\delta_n/2n \to 0$ so

$$p_+(H,x) = \overline{\lim}_{h\to 0+} \frac{\lambda(H,x,x+h)}{h} \geq \overline{\lim}_{n\to\infty} \frac{\lambda([0,1]\setminus B, x, x+\delta_n/2n)}{\delta_n/2n}$$

$$\geq \overline{\lim}_{n\to\infty} \frac{\delta_n/2n(1-1/n)}{\delta_n/2n} = 1; \text{ or}$$

(ii) $[x+\delta_n/2n^2+\delta_n/n, x+\delta_n] \subset B$ for infinitely many n. Then $p_+(H,x) \geq \lim_{n\to\infty}(1-1/n-1/2n^2) = 1$.

Thus H has right porosity 1 at each of its points. Similarly for $f \in E^2$ and $g \in C(\sigma)$ the set $\{t: f(t)=g(t)\} \cap (0,1]$ has left porosity 1 at each of its points. Hence if $f \in E$ and $g \in C(\sigma)$ then $\{t: f(t)=g(t)\}$ is bilaterally strongly porous.□

Lemma 4.3.3 together with Ascoli's theorem implies the following theorem.

THEOREM 4.3.4. Let K be a σ-compact subset of C. The set of functions $f \in C$ such that if $g \in K$ then $\{x:f(x)=g(x)\}$ is bilaterally strongly porous is residual in C.

There are several interesting consequences of Theorem 4.3.4.

THEOREM 4.3.5. Let L_α be the class of Lipschitz functions of order α on [0,1]. The set of functions f in C such that if $g \in L_\alpha$ then $\{x:f(x)=g(x)\}$ is bilaterally strongly porous, is residual in C.

An immediate consequence of this theorem is that the graph of a typical continuous function will intersect the graph of a function which has a bounded derivative in a bilaterally strongly porous set. Bruckner and Haussermann went on to show that this is also true for functions with finite, rather than bounded, derivatives.

THEOREM 4.3.6. The set of functions f in C such that if g is differentable on [0,1] then $\{x:f(x)=g(x)\}$ is bilaterally strongly porous, is residual in C.

This result can now be applied to obtain other results regarding generalized derivatives. Consider a sequence $\{x_n\}$ converging to $x \in [0,1]$ such that $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x}$ exists and is finite. Then we can construct a differentiable function g such that $f(x_n)=g(x_n)$ for each n. Then by 4.3.6 $\{x_n\}$ must be bilaterally strongly porous at x.

DEFINITION 4.3.7. Let $E = \{E_x : x \in R\}$ be a system of paths. If each E_x is residual in some neighbourhood of x then the E-derivative is called the <u>qualitative derivative</u>.

THEOREM 4.3.8. Let $E = \{E_x : x \in [0,1]\}$ be a system of paths such that each E_x is not bilaterally strongly porous at x. Then there is a residual set of functions in C which are nowhere E-differentiable. Hence a typical continuous function is nowhere unilaterally preponderantly differentiable, nowhere qualitatively differentiable and nowhere unilaterally approximately differentiable.

By looking at the porosity of the paths concerned we see how these results clarify results in chapter 2. In chapter 3 we saw that a typical continuous function intersects "most" straight lines in nowhere dense perfect sets. Now we have also seen that these sets must also be bilaterally strongly porous.

<u>Generalized Porosity</u>

We have seen several results regarding the "smallness" of the intersection sets of typical continuous functions with continuous functions in a fixed class. The "smallness" has been gauged by density, Lebesgue measure or porosity. The fixed classes have been lines, equicontinuous families or σ-compact subsets of C. In his doctoral thesis in 1984 Haussermann [21] asked if the idea of smallness could be strengthened and what the relationship is between the fixed class and how we measure smallness. To strengthen the idea of smallness he formulated a generalized porosity. To define the relationship between the class and the measure of smallness he used the modulus of continuity. We will begin with several definitions.

DEFINITION 4.4.1. Let ϕ be a strictly increasing function in $C[0,1]$ such that $\phi(0)=0$. Then ϕ is called a porosity function.

A family $\Phi = \{\phi_\alpha:\alpha\in J\}$ of porosity functions indexed by an open subinterval J of $(0,\infty)$ is called a porosity family.

DEFINITION 4.4.2. Let E be a subset of $\mathbf{R}$, $x_0 \in \mathbf{R}$ and ϕ a porosity function. Then E is ϕ-porous at x_0 if there is a sequence of intervals $\{I_n\}$ with each I_n in the complement of $E\cup\{x_0\}$, $|I_n| < 1$ and the distance from I_n to x_0, $d(I_n,x_0)$ decreases to zero and for all n, $d(I_n,x_0) < \phi(|I_n|)$. Right, left and bilateral ϕ-porosity are similarly defined. E is ϕ-porous if E is ϕ-porous at all of its points. If $\Phi = \{\phi_\alpha:\alpha\in J\}$ is a porosity family we say that E is Φ-porous at x_0 if there is a $\phi\in\Phi$ so that E is ϕ-porous at x_0. Right, left and bilateral Φ-porosity are similarly defined. E is strongly Φ-porous at x_0 if for every $\phi\in\Phi$ E is ϕ-porous at

x_0. Right, left and bilateral strong Φ-porosity are similarly defined. We say E is Φ-porous if it is Φ-porous at each of its points.

If we take $\phi_\alpha(x) = \alpha x$ for $\alpha \in (o,\infty)$ then bilateral strong porosity is equivalent to ordinary porosity. We see that the faster $\phi(x)$ tends to zero with x the stronger is the resulting porosity.

DEFINITION 4.4.4. Let $\Phi = \{\phi_\alpha : \alpha \in J\}$ be a porosity family. Then if for all $\alpha, \beta \in J$ with $\alpha<\beta$ there is a positive number δ such that $\phi_\alpha(x) < \phi_\beta(x)$ for all $x \in (0,\delta)$, we say that Φ is ordered. The family Φ is refined if, given $\{t_n\}$, a strictly decreasing sequence converging to zero in such a manner that $\{t_n\}$ is not right ϕ_α-porous at zero for some $\alpha \in J$, then there exists $\beta \in J$ such that $\{t_{2n}\}$ is not right ϕ_β-porous at zero.

By using ordered porosity families we are assured that we need deal only with a countable number of porosity functions. This is so because for an ordered family Φ there is always a sequence of porosity functions $\{\phi_{\alpha_n}\}$ such that if a set is ϕ_{α_n}-porous for each n then it is strongly Φ-porous. Using refined porosity families ensures that the family contains functions of the same order of growth near zero and functions which are arbitrarily small near zero.

In the remainder of this section we will require that for any modulus of continuity σ, $D_+\sigma(0)>0$.

DEFINITION 4.4.5. Let σ be a modulus of continuity, and $g \in C$. If there is a positive number, M, such that for all x and y in [0,1], $|g(x)-g(y)| \leq M\sigma(|x-y|)$ then we say g is Lipschitz-σ and write $g \in L(\sigma)$. When σ is the identity we obtain the Lipschitz functions.

In section 3 we saw that if the porosity function is linear then no restriction is necessary on the modulus of continuity to ensure that a typical continuous function will intersect every function in the class $C(\sigma)$ in a bilaterally strongly porous set. Haussermann proves the reciprocal of this.

THEOREM 4.4.6. Let Φ be an ordered porosity family. Then there exists a residual set of functions in C which intersect every Lipschitz function in a bilaterally strongly Φ-porous set.

We will now look at several conditions on σ and ϕ which will guarantee the "smallness" of the intersection sets.

THEOREM 4.4.7. Let σ be a modulus of continuity and ϕ a porosity function. For values of x near zero define $h(x) = \phi^{-1}(2\phi^{-1}(x)+3x) + 2\phi^{-1}(x) + 3x$. If $D_+(h\sigma\circ h)(0) = 0$

then the set of functions in C which intersect every Lipschitz-σ function in a bilaterally ϕ-porous set is residual in C.

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The condition specified in this theorem is quite complicated. A much simpler condition can be given if we give up bilateral porosity. This is done in the next theorem.

THEOREM 4.4.8. Let σ be a concave modulus of continuity and ϕ be a porosity function. If $D_+(\sigma\phi^{-1})(0) = 0$ then the set of functions in C which intersect every Lipschitz-σ function in a ϕ-porous set is residual in C.

The condition that the modulus of continuity be concave is not much of a restriction. For, if ω is any modulus of continuity, let σ be the concave upper boundary of the convex hull of ω. Then σ is concave and $L(\sigma)\supset L(\omega)$. Then if the conclusion of 4.4.8 is true of $L(\sigma)$ it will also be true of $L(\omega)$. Haussermann goes on to show that if $D_+(\omega\phi^{-1})(0)=0$ then $D_+(\sigma\phi^{-1})(0)=0$ and so the conclusion of 4.4.8 holds for σ and hence for ω as well.

The next theorem is the main result of Haussermanns thesis. In it we are given conditions so that for a fixed class of functions and a measure of "smallness" the typical continuous function will intersect every function in the fixed class in a "small" set and we are given conditions so that no function in C will intersect every function in the fixed class in a "small" set.

THEOREM 4.4.9. Let σ be a concave modulus of continuity and let Φ be an ordered porosity family. If $D_+(\sigma\phi^{-1})(0) = 0$ for each $\phi \in \Phi$ then the set of functions in C which intersect every function in $L(\sigma)$ in a strongly Φ-porous set is residual in C.

Let Φ also be refined and $D^+\phi(0) < \infty$ for each $\phi \in \Phi$. Then if there is a $\phi \in \Phi$ such that $D_+(\sigma\phi^{-1})(0) > 0$, then there is no function in C which will intersect every function in $L(\sigma)$ in a strongly Φ-porous set.

This theorem leads to several results about the differentiability of typical continuous functions. The first is stated in terms of path derivatives.

THEOREM 4.4.10. Let Φ be a refined and ordered porosity family such that $D^+\phi(0) < \infty$ for all $\phi \in \Phi$. Then for every system of paths, E, such that for each $x \in [0,1]$ there is a $\phi \in \Phi$ so that E_x is (unilaterally) ϕ-nonporous at x, there exists a residual set of functions f in C such that f is nowhere E-differentiable.

We can now apply Theorem 4.4.10 to two generalized derivatives. The preponderant derivative we have already defined. It was introduced by Denjoy and is a generalization of the approximate derivative. In 1968 Sindalovski introduced the congruent derivative.

DEFINITION 4.4.11. Let Q be a fixed set of real numbers with 0 as a limit point. Let E be a system of paths such that for each $x \in [0,1]$, $E_x = (x+Q) \cap [0,1]$. Then the E-derivative is called the <u>congruent derivative</u>.

THEOREM 4.4.12. The set of functions in C which are nowhere preponderantly differentiable is a residual subset of C.

THEOREM 4.4.13. Let Q be a fixed set of real numbers with 0 as a limit point. The set of functions in C which are nowhere congruently differentiable (with respect to Q) is a residual subset of C.

Jarnik had shown [27] that the typical continuous function has every extended real number as an essential derived number almost everywhere. Haussermann extended this result.

THEOREM 4.4.14. The set of functions in C such that $-\infty$ or ∞ is a right essential derived number and $-\infty$ or ∞ is a left essential derived number at every point of [0,1] is residual in C.

Another way in which the "smallness" of a set can be measured is by Hausdorff dimension. We need two more definitions.

DEFINITION 4.4.15. Let E be a subset of **R** and let ϕ be a porosity function. The <u>Hausdorff ϕ-measure of E</u> is

$$\mu^{\phi}(E) = \sup_{0<\delta<1} \inf_{\{I_n\}} \Sigma_n \, \phi(|I_n|)$$

where I_n is an open interval such that $|I_n|<\delta$ and $E \subset \cup I_n$

If ϕ is the identity function the Hausdorff ϕ-measure is the same as the Lebesgue (outer) measure. If E is ϕ-porous then the Hausdorff ϕ^{-1}-measure of E is zero.

DEFINITION 4.4.16. Let E be a subset of **R** and let $r \in (0,\infty)$. Define $\phi_r(x) = x^r$. If $d = \inf\{r \in (0,\infty)$: the Hausdorff ϕ_r-measure of E is $0\}$. Then the <u>Hausdorff dimension of E</u> is d.

THEOREM 4.4.17. Let $0 < \beta \leq 1$. The set of functions in C which intersect every Lipschitz-β function in a set of Hausdorff dimension less than or equal to $1 - \beta$ is residual in C.

Thus we have been able, for a fixed $\beta \in (0,1]$, to give an upper bound on the Hausdorff dimension of the intersection set of a typical continuous function with any Lipschitz-β function.

In 1985 P. Humke and M. Laczkovich [23] continued the study of the porosity of intersection sets. Haussermann has shown that for a porosity function ϕ, the typical continuous function intersects every Lipschitz function in a bilaterally strongly ϕ-porous set. Humke and Laczkovich show that the class of Lipschitz functions can be replaced by the class of monotone functions. Using an argument of Bruckner they also show that this is not true for the class of absolutely continuous functions. Their proofs make use of the idea of a proper pair of sequences.

DEFINITION 4.4.18. Let $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be a pair of sequences of real numbers. If $\{\beta_n\} \to 0$ and $0<\alpha_n<\beta_n$ for all $n = 1,2,\ldots$ then (α,β) is called a <u>proper pair</u> of sequences. Let (α,β) be a proper pair of sequences and $x = \{x_n\}$ be a sequence of real numbers that converge to x_0. If $\alpha_n \le x_n - x_0 \le \beta_n$ for $n = 1,2,\ldots$ the x is an <u>(α,β) sequence</u>.

Now if (α,β) is a proper pair define:

$I_n = \{f \in C : \exists\ (\alpha,\beta)$ sequence x with f increasing on $\{x_i\}_{i=n}^{\infty}\}$ and $D_n = \{f \in C : \exists\ (\alpha,\beta)$ sequence x with f decreasing on $\{x_i\}_{i=n}^{\infty}\}$. We need two lemmas regarding proper pairs of sequences and I_n and D_n.

LEMMA 4.4.19. If (α,β) is a proper pair then both I_n and D_n are closed for all n.

PROOF: ([23], p.245) Let N be fixed and suppose $\{f_k\}$ is a sequence in I_N which converges uniformly to f. Then for each $k = 1,2,\ldots$, $f_k \in I_N$ so that there is a sequence $\{x_n^k\}_{n=1}^{\infty}$ converging to x_0^k such that $\alpha_n \le x_n^k - x_0^k \le \beta_n$ for each n and f_k is increasing on $\{x_n^k\}_{n=N}^{\infty}$. Thus there is a subsequence $\{k_i\}_{i=1}^{\infty}$ such that for each n=0,1,... the sequence $\{x_n^{k_i}\}$ converges (say to x_n'). Since $\alpha_n \le x_n^{k_i} - x_0^{k_i} \le \beta_n$ we have $\alpha_n \le x_n' - x_0' \le \beta_n$. Thus the sequence $x' = \{x_n'\}$ is an (α,β) sequence. Since $\{f_k\} \to f$ uniformly and each f_k is increasing on $\{x_n^k\}_{n=N}^{\infty}$, f is increasing on $\{x_n'\}_{n=N}^{\infty}$. Hence $f \in I_n$ and so I_n is closed. Since $D_n = \{f : -f \in I_n\}$, D_n is also closed.□

LEMMA 4.4.20. If (α,β) is a proper pair, the both I_n and D_n are nowhere dense for each n=1,2,....

PROOF: ([23], p. 245) It suffices to prove the result for I_n, and by Lemma 4.4.19 we need only show that I_n contains no nonempty open sphere. Let N be fixed, $f \in C$ and $\epsilon > 0$. We will show that there is a $g \in C \setminus I_N$ within ϵ of f.

Partition [0,1] into congruent closed intervals $\{J_k : k=1,2,\ldots,K\}$ so that each has length $d < \epsilon/2$ and the oscillation of f on each interval is less than $\epsilon/2$. Choose

$n_1, n_2 \geq N$ such that $0<\alpha_{n_2}<\beta_{n_2}<\alpha_{n_1}<\beta_{n_1}<d$. Let $\delta=\min\{\alpha_{n_2}, (d-\beta_{n_1})/2\}$. We define g on each interval as follows:

for a fixed k let $J_k=[a,b]$, then:

(i) if $f(a)>f(b)$ let $g(a)=f(a)$, $g(b)=f(b)$ and g is linear on J_k,

(ii) if $f(a)\leq f(b)$ let $g(a)=f(a)$, $g(b)=f(b)$ and g is linear on $[a,a+\delta]$ and $[a+\delta,b]$ and has slope of 1 on $[a+\delta,b]$.

Then $|f(x)-g(x)| < \epsilon$ for all $x \in [0,1]$.

Let $\{x_n\}$ be an (α,β) sequence which converges to $x_0 \in (0,1)$. Then there is a unique k such that $x_0 \in J_k$ but x_0 is not the right endpoint of J_k. Now let $J_k = [a,b]$. If $f(a)>f(b)$ then g is decreasing on J_k and so is eventually decreasing on $\{x_n\}$. Suppose, on the other hand, that $f(a)<f(b)$. If $x_0 \in [a+\delta,b)$ the g is decreasing on $[a+\delta,b]$ and eventually on $\{x_n\}$. If $x_0 \in [a,a+\delta)$ then $a+\delta \leq a+\alpha_{n_2} \leq x_0+\alpha_{n_2} \leq x_{n_2} \leq x_0+\beta_{n_2} < x_0+\alpha_{n_1} \leq x_{n_1} \leq x_0+\beta_{n_1} < a+\delta+\beta_{n_1} < b$. However g is decreasing on $[a+\delta,b]$ and $x_{n_1}<x_{n_2}$ so $g(x_{n_2})>g(x_{n_1})$. Thus g is not increasing on $\{x_n\}_{n=N}^{\infty}$ and so $g \notin I_n$.□

Using these two lemmas we are now able to prove easily the following theorem.

THEOREM 4.4.21. Let ϕ be a porosity function. Then the set of functions in C which intersect every monotone function in a bilaterally strongly ϕ-porous set is residual in C.

PROOF: ([23], p. 246) Define a proper pair of sequences (α,β) as follows. Let $\beta_0=1$ and if $\beta_n \in (0,1]$ has been defined, define β_{n+1} by $0<\beta_{n+1}<\min\{\beta_n, 1/(n+1)\}$ such that $\phi(\beta_n-\beta_{n+1})>n\beta_{n+1}$. Let $\alpha_n=\beta_{n+1}$ for $n=0,1,2,\ldots$.

Then for each n let I_n and D_n be as given above and let $A = (\cup_n I_n)\cup(\cup_n D_n)$. Then $C\backslash A$ is residual in C. By the lemmas if $f \in C\backslash A$ and f is monotone on a set $M \subset [0,1]$ then for each $x \in M$, $M\cap[x+\alpha_{n_i}, x+\beta_{n_i}] = \phi$ for some subsequence $\{n_i\}$ of natural numbers. For each natural number i let $J_i = [x+\alpha_{n_i}, x+\beta_{n_i}]$. Then $J_i \subset (x,x+1/i)\backslash M$ and

$$d(x,J_i) = \alpha_{n_i} = \beta_{n_i+1} < \frac{\phi(\beta_{n_i}-\beta_{n_i+1})}{n_i} = \frac{\phi(|J_i|)}{n_i}$$

for $i=1,2,\ldots$. Then $\lim_{i\to\infty} \frac{d(x,J_i)}{\phi(|J_i|)} = 0$. Strong ϕ-porosity on the left is shown similarly. This proves that M is bilaterally strongly ϕ-porous.□

We will show now that the class of monotone functions cannot be replaced by the class of absolutely continuous functions. We need one more lemma before proving this result. The lemma is essentially due to Haussermann ([22] Theorem 2.16) although Humke and Laczkovich have rewritten it in terms of (α,β) sequences.

LEMMA 4.4.22. Let (α,β) be a proper pair of sequences and let σ be a positive increasing function on $(0,1]$ such that $\lim_{n\to\infty}\frac{\sigma(\alpha_n)\beta_n}{\alpha_n} = \infty$. If $f \in C$, then for almost all $x \in [0,1]$ there is an (α,β) sequence $\{y_n\} \to x$ such that $|f(y_n)-f(x)| \le \sigma(y_n-x)$ for n sufficiently large.

THEOREM 4.4.23. Given $\delta>0$ and $f \in C$, there is an absolutely continuous function g such that $\{x:f(x)=g(x)\}$ is not bilaterally strongly $x^{1+\delta}$-porous.

PROOF: ([23], pp. 247-248) For each natural number n define $\beta_n = (n!)^{-1-\delta}$ and $\alpha_n = \beta_{n+1}$, and let $\sigma(x) = n^{-1-\delta/2}$ if $x \in (\alpha_n,\beta_n]$. Then

$$\frac{\sigma(\alpha_n)\beta_n}{\alpha_n} = (n+1)^{-1-\delta/2}(n+1)^{1+\delta} \to \infty.$$

Thus by Lemma 4.4.22 for each $f \in C$ there is an $x_0 \in [0,1)$ and an (α,β) sequence $\{x_n\}\to x_0$ such that $x_n<1$ and $|f(x_n)-f(x_0)| \le \sigma(x_n-x_0) \le \sigma(\beta_n) = n^{-1-\delta/2}$ for n sufficiently large, say $n\ge n_0$.

Define $g \in C$ to be linear on the intervals $[0,x_0]$, $[x_{n_0},1]$ and $[x_{n+1},x_n]$ $(n\ge n_0)$ and to agree with f for x_n, $(n\ge n_0)$. Then $\Sigma_{n=1}^{\infty} n^{-1-\delta/2} < \infty$ and we see that g is absolutely continuous on $[0,1]$. Let $H = \{x:f(x)=g(x)\}$. Then for each interval $J\subset(x_0,1)\backslash H$ there is an n such that $J\subset(x_{n+1},x_n)\subset(x_0+\alpha_{n+1},x_0+\beta_n)$. Then

$$\frac{d(x_0,J)}{|J|^{1+\delta}} \ge \frac{[(n+2)!]^{-1-\delta}}{(n!)^{-(1+\delta)^2}} = \left(\frac{(n!)^{\delta}}{(n+1)(n+2)}\right)^{1+\delta} \to \infty. \quad \square$$

[g]-porosity and [g]-knot points

In 1934 Jarnik proved that for a typical continuous function f the set of points which are not knot points of f is of first category and measure zero. He improved this result to show that for a typical continuous function f almost all points of [0,1] are essental knot points of f (see Chapter 2). Recently, L. Zajíček stated that Petruska has proved that for a typical continuous function the set of points which are not knot points is σ-bilaterally strongly porous. Zajíček [44]has improved this result with the ideas of [g]-porosity and [g]-knot points. Let G denote the family of positive increasing functions g on $(0,\infty)$ for which $g(x)>x$ for all x.

DEFINITION 4.5.1. Let $g \in G$ and $E \subset \mathbf{R}$. Then E is [g]-porous from the right (from the left) at a point x $\in \mathbf{R}$ if there is a sequence of positive numbers $\{h_n\}$ which decreases to 0 and such that $g(\lambda(E,x,x+h_n)) > h_n$ ($g(\lambda(E,x-h_n,x)) > h_n$) for all n. E is [g]-porous if it is [g]-porous at each of its points.

DEFINITION 4.5.2. For $a \in \mathbf{R}$ and $h>0$ a system of the form $D = \{[a+nh,a+(n+1)h];n \text{ an integer}\}$ is called an equidistant division of $\mathbf{R}$ with norm h.

Let $g \in G$ and $E \subset \mathbf{R}$. If for any $\epsilon>0$ there exists an equidistant division D of $\mathbf{R}$ with norm less than ϵ such that

$g(\lambda(E,I)) > |I|$ (where $\lambda(E,I)$ is the length of the largest open interval in E disjoint from I) for any $I \in D$, then we say that E is [g]-totally porous.

Thus we see that the concept of [g]-totally porous is stronger than the concept of ordinary porosity. Zajíček strengthens the concept of knot points as well and then goes on to show that for a residual set in C the set of non-[g]-knot points is [g]-totally porous.

DEFINITION 4.5.3. Let $g \in G$ and f a real valued function on **R**. Let y be an extended real number. Then y is a right (left) [g]-derived number of f at a point $x \in \mathbf{R}$ if there is a set $E \subset \mathbf{R}$ such that

$$\lim_{z \to x, z \in E} \frac{f(z)-f(x)}{z-x} = y$$

and $\mathbf{R} \setminus E$ is [g]-totally porous from the right (left) at x.

We shall say that $x \in \mathbf{R}$ is a [g]-knot point of f if every extended real number is a bilateral [g]-derived number of f at x.

In order to prove the main result of this section we shall need to define two constructions of functions and prove two lemmas. In the constructions let $g \in G$ be fixed and continuous.

CONSTRUCTION 1: Let h be a Lipschitz-K function on [0,1], $c \in \mathbf{R}$, n a natural number and v a real number such that $0<v<1/n$. Then we let $f = f(h,c,n,v)$ denote the function on [0,1] uniquely determined as follows:

(i) $f(k/n) = h(k/n)$ $(k=0,1,\ldots,n)$

(ii) $f(x) = h(x)+c(x-k/n)$ for $x \in [k/n,(k+1)/n-v]$ $(k=o,1,\ldots,n-1)$

(iii) f is linear on the intervals $[(k+1)/n-v,(k+1)/n]$ $(k=0,1,\ldots,n-1)$

LEMMA 4.5.4. Let h,K,c,n,v,f be as in construction 1. Then $\|f-h\| \le \dfrac{2K+|c|}{n}$.

PROOF: ([44], p. 8) If $x \in [k/n,(k+1)/n-v]$ for some $k \in \{0,1,\ldots,n-1\}$ then

$$|f(x)-h(x)| \le |f(x)-f(k/n)| + |h(x)-h(k/n)| \le \frac{K+|c|)}{n}.$$

If $x \in [(k+1)/n-v,(k+1)/n]$ for some $k \in \{0,1,\ldots,n-1\}$ then

$$|f(x)-h(x)| \le \max\{|h(x)-f((k+1)/n)|,|h(x)-f((k+1)/n-v)|\}.$$

Now $|h(x)-f((k+1)/n)| \le K/n$ and $|h(x)-f((k+1)/n-v)| \le |h(x)-h((k+1)/n-v)| + |h((k+1)/n-v)-f((k+1)/n-v)| \le K/n + (K+|c|)/n$. □

CONSTRUCTION 2: Let p be a polynomial and $a<b$ be real numbers and $\delta>0$ be a real number. Let $c = (a+b)/2$ and $d = (b-a)/2$. Then define real numbers K,n,v and ϵ and an open sphere $B \subset \mathbf{C}$ as follows: Let $K=\sup\{|p'(x)|:x \in [0,1]\}$. Choose n such that $(2K+|c|)/n < \delta/2$ and $1/n < \delta$. Then find

$v > 0$, $0 < q < v$, $0 < \epsilon$ such that:

(i) $v < 1/(2n)$, $g(1/n-2v) > 1/n$,

(ii) $g(v-q) > v$, and

(iii) $\epsilon < \delta/2$, $2\epsilon/q < d$.

Now let $B = B(a,b,p,\delta)$ be the open ball with centre $f = f(p,c,n,v)$ (given in construction 1) and radius ϵ.

LEMMA 4.5.5. Let the notation be as in construction 2. Then:

(i) for any $h \in B(a,b,p,\delta)$ we have $\|h-p\| < \delta$, and

(ii) for any $h \in B(a,b,p,\delta)$ and $x \in \cup_{k=0}^{n-1}[k/n,(k+1)/n-2v]$ we have $g(\lambda(\{y:\frac{h(y)-h(x)}{y-x} \notin [a,b]\},[x,x+v])) > v$.

PROOF: ([44] p. 9) (i) follows from Lemma 4.5.4 and construction 2. To prove (ii) suppose that $x \in [k/n,(k+1)/n-2v]$ for some $k \in [o,1,\ldots,n-1\}$. By (ii) of construction 2 it is sufficient to show that $\{y:\frac{h(y)-h(x)}{y-x} \notin [a,b]\} \cap [x+q,x+v] = \phi$. Choose $y \in [x+q,x+v]$ and consider $f=f(p,c,n,v)$. Then by the construction of f, the definition of $B(a,b,p,\delta)$ and by (iii) of construction 2 we have

$$\left|\frac{h(y)-h(x)}{y-x}-c\right| \leq \left|\frac{f(y)-f(x)}{y-x}-c\right| + \left|\frac{h(y)-f(y)}{y-x}\right| + \left|\frac{f(x)-h(x)}{y-x}\right|$$

$$\leq 0+\epsilon/q+\epsilon/q < d.$$

Hence $g(\lambda(\{y:\frac{h(y)-h(x)}{y-x} \notin [a,b]\},[x,x+v])) > v$.

With these constructions and lemmas we are now in a position to prove the main result of this section.

THEOREM 4.5.6. Let $g \in \mathcal{G}$. Then the set of functions f in C such that the set of points in [0,1] which are not [g]-knot points of f is σ-[g]-totally porous, is residual in C.

PROOF: ([44], p. 9-10) We can assume that g is continuous for if it is not then there is a continuous function $g' \in \mathcal{G}$ such that $g' \leq g$. Let $P = \{p_n\}_{n=1}^{\infty}$ be a dense set of polynomials in C. For a<b let V(a,b) denote the set $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} B(a,b,p_k,1/k)$ where $B(a,b,p_k,1/k)$ is given by construction 2. Now let $V = \cap\{V(a,b):a<b,\ a,b \text{ are rational}\}$. Then by Lemma 4.5.5 (i) each V(a,b) is residual in C and so V is also.

Let $f \in V$. For each natural number m and each pair of rational numbers a and b with a<b let A(a,b,m) denote the set of all $x \in [0,1)$ for which

$$g(\lambda(\{y:\frac{f(y)-f(x)}{y-x} \notin [a,b],(x,x+h)\})) \leq h$$

whenever $0<h\leq\min\{1/m,1-x\}$. Let A denote the set of all $x \in [0,1)$ for which there is an extended real number y which is not a right [g]-derived number of f at x. Then $A \subset \cup\{A(a,b,m):m \text{ is rational, } a<b \text{ are natural numbers}\}$ and it is sufficient to show that each A(a,b,m) is [g]-totally porous (the proof for left [g]-totally porous is handled

similarly).

Fix a,b,m and choose $\epsilon>0$. Now $f \in V(a,b)$ so we can choose a positive integer j such that $1/j < \min\{\epsilon,1/m\}$ and $f \in B(a,b,p_j,1/j)$. By construction 2, $v < 1/n < 1/j < 1/m$. Thus from Lemma 4.5.5 (ii) we see that $A(a,b,m) \cap \bigcup_{k=0}^{n-1}[k/n,(k+1)/n-2v] = \phi$. Consequently $g(\lambda(A(a,b,m),[k/n,(k+1)/n])) > 1/n$ for $k \in \{0,1,\ldots,n-1\}$. Since the norm of the division $\{[k/n,(k+1)/n]\}_{k=0}^{n-1}$ is $1/n < 1/j < \epsilon$, A(a,b,m) is [g]-totally porous.□

Haussermann and Humke and Laczkovich have shown that the typical continuous function f has the property that the intersection set $\{x:f(x)=h(x)\}$ of f with a Lipschitz function or a monotone function, h, is bilaterally [g]-porous. Zajíček proved a similar result for the class of functions, h, such that the set of knot points of h is σ-[g]-porous.

THEOREM 4.5.7. Let $g \in G$. Then the set of functions f in C such that $\{x:f(x)=h(x)\}$ is σ-[g]-porous whenever the set of knot points of $h:[0,1]\to R$ is σ-[g]-porous.

PROOF: ([44] pp. 11-12) Let A be the residual set given by Theorem 4.5.6 and let $f \in A$. Let N(f) denote the set of points in [0,1] which are not [g]-knot points of f. Then N(f) is σ-[g]-totally porous and so it is easily shown that N(f) is σ-[g]-porous. Let h be a function on [0,1] such that

$K(h)$, the set of knot points of h is σ-[g]-porous. Let $M = \{x:f(x)=h(x)\}\setminus(N(f)\cup K(h))$. Then it suffices to show that M is [g]-porous. Let $x \in M$. Then $x \notin K(h)$ so there exists real numbers c and d with $c<d$ such that the set $A = \{x\}\cup\{y:\dfrac{h(y)-h(x)}{y-x} \notin (c,d)\}$ is a one sided neighbourhood of x. Also $x \notin N(f)$ so that the set $B = \{y:\dfrac{f(y)-f(x)}{y-x} \notin (c,d)\}$ is bilaterally [g]-porous at x. Then $M \subset (M\setminus A)\cup B$ so that M is (unilaterally) [g]-porous at x.□

Conclusion

In this chapter we have seen that the typical intersection sets of continuous functions with functions in various classes can be said to be "small" in several ways. Sections 2 and 3 dealt with equicontinuous classes, horizontal levels and σ-compact sets. Section 4 generalized these results by the use of a generalized definition of porosity. The characteristics of the intersection sets led to several results regarding the differentiability of typical continuous functions. We saw that there is a residual set of functions which are nowhere qualitatively, preponderantly or congruently differentiable. In section 5 we saw that the set of non [g]-knot points of a typical continuous function is σ-[g]-totally porous.

CHAPTER V

THE BANACH-MAZUR GAME

The Banach-Mazur game was described in Chapter 1. There, the game was described as played on a closed interval of **R**. The game, of course, can also be played in the space C[0,1]. In this case the first player, *A*, is given an arbitrary subset, A, of C. *A* then chooses a closed sphere, S_1, in C. The second player, *B*, then chooses a closed sphere, $S_2 \subset S_1$; Then *A* chooses a closed sphere $S_3 \subset S_2$; and so on, *A* and *B* alternately choosing closed spheres. If $(\cap_{i=1}^{\infty} S_i) \cap A \neq \phi$ then *A* wins; otherwise *B* wins.

Clearly if A is of first category there is always a way for *B* to win. Banach proved that if the second player has a certain strategy to win then A must be of first category in C. A strategy for *B* is a sequence $\{f_n\}$ of functions whose values are closed spheres in C, such that $f_n(S_1, S_2, \ldots, S_{2n-1}) = S_{2n} \subset S_{2n-1}$. The function f_n must be defined for all (2n-1)-tuples of closed spheres with the property that $S_1 \supset S_2 \supset \ldots \supset S_{2n-1}$. Then $\{f_n\}$ is a winning strategy for B if and only if $(\cap S_i) \cap A = \phi$ for all sequences $\{S_i\}$ of closed spheres with $S_i \supset S_{i+1}$ and $f(S_1, S_2, \ldots, S_{2n-1}) = S_{2n}$.

For the next theorem we follow Banach's proof (as given in [36]) but generalize to an arbitrary metric space. This result seems to be well known but we have been unable to find a proof in the literature.

THEOREM 5.1.1. Let X be a arbitrary metric space. There exists a strategy by which the second player in the Banach-Mazur game is sure to win if and only if A is of first category in X.

PROOF: If $A=\cup_{n=1}^{\infty}A_n$ where A_n is nowhere dense, B need only choose $S_{2n} \subset S_{2n-1}\backslash A_n$ for each n. Thus if A is of first category, B has a winning strategy.

Conversely suppose that $\{f_n\}$ is a winning strategy for *B*. Given f_1, a transfinite sequence of closed spheres $\{J_\alpha\}$ can be defined such that the closed spheres $K_\alpha = f_1(J_\alpha)$ are disjoint and the union of their interiors is dense in C. This can be done letting S be a transfinite sequence consisting of all the open spheres in X. Let J_1 be the first term of S. Then for each β let J_β be the first term contained in $C\backslash\cup_{\alpha<\beta}K_\alpha$. Now for each transfinite ordinal, α, let $\{J_{\alpha,\beta}\}_\beta$ be a transfinite sequence of closed spheres contained in the interior of K_α such that the spheres $K_{\alpha,\beta} = f_2(J_\alpha,K_\alpha,J_{\alpha,\beta})$ are disjoint and the union of their interiors is dense in K_α. Then the union, $\cup_{\alpha,\beta}K^\circ_{\alpha,\beta}$ is dense in X.

Continuing in this way, define two families of closed spheres, $J_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ and $K_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ where n is a positive integer and α_i is a transfinite ordinal such that $K_{\alpha_1,\alpha_2,\ldots,\alpha_n} = f_n(J_{\alpha_1}, K_{\alpha_1}, J_{\alpha_1,\alpha_2}, K_{\alpha_1,\alpha_2}, \ldots, J_{\alpha_1,\alpha_2,\ldots,\alpha_n})$ and $J_{\alpha_1,\alpha_2,\ldots,\alpha_{n+1}} \subset K^\circ_{\alpha_1,\alpha_2,\ldots,\alpha_n}$. For each n, the spheres $K_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ are disjoint and the union of their interiors is dense in X.

Now consider an arbitrary sequence of ordinals, α_n, and define

(*) $S_{2n-1} = J_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, $S_{2n} = K_{\alpha_1,\alpha_2,\ldots,\alpha_n}$, $n=1,2,\ldots.$

Then the sequence $\{S_n\}$ is a possible playing of the game consistent with the given strategy for B. Hence $(\cap S_n) \cap A = \phi$.

For each n define $E_n = \cup_{\alpha_1,\alpha_2,\ldots,\alpha_n} K^\circ_{\alpha_1,\alpha_2,\ldots,\alpha_n}$. Let $E = \cap_n E_n$. Then for each $x \in E$ there is a unique sequence $\{\alpha_n\}$ such that $x \in K_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ for every n. We now use this sequence to define (*). Then $x \in \cap S_n$. This shows that $E \cap A = \phi$ so that $A \subset X \setminus E = \cup_n (X \setminus E_n)$. Each of the sets $X \setminus E_n$ is nowhere dense and so A is of first category.□

It is easy to see that there is a strategy by which the first player is certain to win if and only if $S \setminus A$ is of first category for some sphere $S \subset C$.

In this chapter we shall see several applications of the Banach-Mazur game to prove the existence of residual sets of continuous functions with certain differentiation properties.

Non-Besicovich Functions

It was stated in Chapter 2 that Sak's proof that the set of non-Besicovich functions is residual in C is actually an example of the use of the Banach-Mazur game (although Saks did not use it). We now give the first part of Saks' proof [40] and indicate how the Banach-Mazur game is applied.

LEMMA 5.2.1. Let $f(x)$ be continuous in an interval (a,b) and let $|f(x)-mx+n| < (b-a)/8\epsilon$ $(m>0, \epsilon>0)$ for all $a \le x \le b$. Then there exists in the interval $(a,(a+b)/2)$ a nondenumerable set of points c with the property that $f(x)-f(c) \ge (m-\epsilon)(x-c)$ for every $c \le x \le b$.

The proof of this lemma is not difficult and so it is omitted.

THEOREM 5.2.2. The set of functions f in C such that the right hand derivative of f exists and equal $+\infty$ in a set of the power of the continuum, is of second category in every open sphere in C.

PROOF: ([40], pp. 215-217) Let K be an arbitrary open sphere in C and let $\{A_n\}$ be a sequence of nowhere dense sets in C.

We shall define a system of subintervals, $\{I_{n_1,n_2,\ldots,n_j} = (a_{n_1,n_2,\ldots,n_j}, b_{n_1,n_2,\ldots,n_j})\}$, in $(0,1)$, a sequence of functions, $\{f_j\}$ in C and a sequence of open spheres K_j in C satisfying the following conditions:

(i) $K_0=K$, $K_j \subset K_{j-1}$, $\overline{K_j} \cap \overline{A_j} = \phi$, $(j \geq 1)$

(ii) f_j is the centre of K_j, $(j=0,1,\ldots)$

(iii) $I_{n_1,n_2,\ldots,n_j} \subset I_{n_1,n_2,\ldots,n_{j-1}}$; $I_{n_1,n_2,\ldots,n_{j-1}0} \cap I_{n_1,n_2,\ldots,n_{j-1}1} = \phi$, $|I_{n_1,n_2,\ldots,n_j}| \leq 1/j$, $(j \geq 1,\ n_j=0,1)$

(iv) f_j is linear with slope j in each interval $I_{n_1,n_2,\ldots,n_j}$ of the j^{th} order, and

(v) if $x \in I_{n_1,n_2,\ldots,n_j}$ and $y \in (b_{n_1,n_2,\ldots,n_i}, b_{n_1,n_2,\ldots,n_{i+1}})$, $(1<i<j)$, then

$$\frac{f_j(y)-f_j(x)}{y-x} > i-2.$$

Suppose that the functions f_j, the spheres K_j and the subintervals $I_{n_1,n_2,\ldots,n_j}$ have been determined for $j=1,2,\ldots,r$ and satisfy conditions (i) to (v). The set A_r is nowhere dense so there exists in each neighbourhood of f_r a continuous function g in $K_r \backslash \overline{A_r}$. By condition (iv) (for $j=r$) f_r is linear and has slope r in each interval $I_{n_1,n_2,\ldots,n_r}$ of the r^{th} order. We can choose g sufficiently close to f_r so that by Lemma 5.2.1 there exists in each interval $I_{n_1,n_2,\ldots,n_r}$ two points, say $b_{n_1,n_2,\ldots,n_r1}$ and $b_{n_1,n_2,\ldots,n_r0}$, such that

$$\frac{g(x)-g(b_{n_1,n_2,\ldots,n_{r+1}})}{x-b_{n_1,n_2,\ldots,n_{r+1}}} > r-1$$

for $b_{n_1,n_2,\ldots,n_{r+1}} < x < b_{n_1,n_2,\ldots,n_r}$, $(n_{r+1}=0,1)$.

We can now modify g to obtain a function h, linear and with slope j+1 in a pair of distinct intervals,

$d_{n_1,n_2,\ldots,n_r}$, $d'_{n_1,n_2,\ldots,n_r}$, in $I_{n_1,n_2,\ldots,n_r}$ whose right endpoints are $b_{n_1,n_2,\ldots,n_r0}$ and $b_{n_1,n_2,\ldots,n_r1}$ respectively. We can choose these subintervals sufficiently small, less than $1/(j+1)$, and h sufficiently close to g so that $h \in K_r\backslash\overline{A_r}$ and

$$\frac{h(x)-h(b_{n_1,n_2,\ldots,n_{r+1}})}{x-b_{n_1,n_2,\ldots,n_{r+1}}} > r-1$$

for $b_{n_1,n_2,\ldots,n_{r+1}} < x < b_{n_1,n_2,\ldots,n_r}$, $(n_{r+1}=0,1)$.

Then the function f_{r+1} is defined as h and the intervals $I_{n_1,n_2,\ldots,n_r0}$ and $I_{n_1,n_2,\ldots,n_r1}$ as the intervals $d_{n_1,n_2,\ldots,n_r}$ and $d'_{n_1,n_2,\ldots,n_r}$ respectively. Now choose an open sphere K_{r+1} with centre f_{r+1} and radius less than $1/(r+1)$ such that $K_{r+1}\subset K_r$ and $\overline{K_{r+1}}\cap\overline{A_{r+1}} = \phi$. Conditions (i) to (v) are satisfied for $j=r+1$.

Now by conditions (i) and (ii) the sequence $\{f_j\}$ converges uniformly to a continuous function, $f \in K\backslash\cup_n A_n$. For each sequence $\{n_j\}$, $(n_j=0,1)$ set $x_{n_1,n_2,\ldots,n_k,\ldots} = I_{n_1}\cap I_{n_1,n_2,}\cap\ldots\cap I_{n_1,n_2,\ldots,n_j}\cap\ldots$. Then we have

$$\frac{f_j(y)-f_j(x_{n_1,n_2,\ldots,n_k,\ldots})}{y-x_{n_1,n_2,\ldots,n_k,\ldots}} > i-2$$

for every sequence $\{n_k\}$, $(n_k=0,1)$, $j>i>1$, and $b_{n_1,n_2,\ldots,n_i}>y>x_{n_1,n_2,\ldots,n_k,\ldots}$. It follows that

$$\frac{f(y)-f(x_{n_1,n_2,\ldots,n_k,\ldots})}{y-x_{n_1,n_2,\ldots,n_k,\ldots}} \geq i-2$$

for $i=1,2,\ldots$ and $b_{n_1,n_2,\ldots,n_i}>y>x_{n_1,n_2,\ldots,n_k,\ldots}$. Thus $f'_+(x_{n_1,n_2,\ldots,n_k,\ldots}) = +\infty$ for each $x_{n_1,n_2,\ldots,n_k,\ldots}$. This set of points is clearly perfect and so has the power of the continuum.□

Saks goes on to show that the set of functions f in C with right derivative equal to $+\infty$ in a set of the power of the continuum, is analytic in C and hence has the property of Baire. It is thus a residual set. It is easy to see that this second part of the proof is not needed, for the first part can be seen as a winning strategy for the second player in the Banach-Mazur game.

The Weak Preponderant Derivative

In Chapter 2 several differentiation properties of typical continuous functions were presented. Recently the Banach-Mazur game has been used to prove several new differentiation properties. L. Zajíček has proved [46] a new result regarding the preponderant derivative by use of the game.

DEFINITION 5.3.1. Let $E = \{E_x : x\in[0,1]\}$ be a system of paths. If for each $E_x \in E$ there is a $\delta>0$ such that $\frac{\lambda(E_x \cap I)}{|I|} > 1/2$ for all open intervals, I, with $x\in I$ and $|I| < \delta$ then the E derivative is called a <u>weak preponderant derivative</u>.

Zajíček shows that a typical continuous function f has a point $x \in (0,1)$ where the weak preponderant derivative of f is $+\infty$. The proof is lengthy and we will only outline it here. First we need a lemma which has a straightforward proof which is omitted.

LEMMA 5.3.2. Let $f \in C$, $x \in (0,1)$, $\{a_n\}$ be an increasing sequence converging to x and $\{b_n\}$ be a decreasing sequence converging to x, such that:

$$y \in [a_k, a_{k+1}] \Rightarrow \lambda\{z \in (y,x): \frac{f(z)-f(x)}{z-x} > k\} > 1/2(x-y), \text{ and}$$

$$y \in [b_{k+1}, b_k] \Rightarrow \lambda\{z \in (x,y): \frac{f(z)-f(x)}{z-x} > k\} > 1/2(y-x)$$

for all natural numbers, n. Then the weak preponderant derivative of f is $+\infty$.

THEOREM 5.3.3. There exists a residual set A of functions in C such that if $f \in A$ then there exists $x \in (0,1)$ so that the weak preponderant derivative of f at x is $+\infty$.

PROOF: Consider the Banach-Mazur game in C. The first player chooses an open sphere $B(g_1, \delta_1)$, centered at g_1 with radius δ_1, then the second player chooses $B(f_1, \epsilon_1) \subset B(g_1, \delta_1)$, and so on. We can suppose that all of the functions, f_n and g_n $(n=1,2,\ldots)$, are piecewise linear. If the second player has a strategy so that $\bigcap_{n=1}^{\infty} B(f_n, \epsilon_n)$ consists of a single function f for which there is a point x in (0,1) where the weak preponderant derivative of f is $+\infty$ then the theorem is proved.

The strategy is as follows. In his n^{th} move the second player will construct $f_n \in C$, $\epsilon_n > 0$, $0 < a_n < b_n < 1$ so that, letting $x_n = (a_n + b_n)/2$ and $z_n = (b_n - a_n)/100$, the following conditions hold:

(i) $[a_n, b_n] \subset (x_{n-1} - 4z_{n-1}, x_{n-1} + 4z_{n-1})$ for $n > 1$,

(ii) f_n is linear on $[x_n - 5z_n, x_n + 5z_n]$ and constant on $[a_n, x_n - 5z_n]$ and $[x_n + 5z_n, b_n]$, and

$$\frac{f_n(b_n) - f_n(a_n)}{b_n - a_n} > 10(n+1),$$

(iii) for $x \in [x_n - 4z_n, x_n + 4z_n]$, $k \in \{1, 2, \ldots, n-1\}$, and $f \in B(f_n, \epsilon_n)$ we have

$$y \in [a_k, a_{k+1}] \Rightarrow \lambda\{w \in (y,x) : \frac{f(x) - f(w)}{x - w} > k\} > 1/2(x-y) \text{ and}$$

$$y \in [b_{k+1}, b_k] \Rightarrow \lambda\{w \in (x,y) : \frac{f(x) - f(w)}{x - w} > k\} > 1/2(y-x),$$

(iv) and $\overline{B(f_n, \epsilon_n)} \subset B(g_n, \delta_n)$ and $\epsilon_n < 1/2z_n$.

If the second player uses this strategy then (i) and (iv) imply that $\cap B(f_n, \epsilon_n)$ consists of one function f, and $\cap [a_n, b_n]$ consists of one point x. Lemma 5.2.2 and (iii) imply that the weak preponderant derivative is $+\infty$ at x.□

In Chapter 2 we saw that the following properties are typical of continuous functions:

(i) for each $x \in [0,1]$, $\max\{|D^+f(x)|, |D_+f(x)|\} = \infty$ and

$\max\{|D^- f(x)|, |D_- f(x)|\}=\infty$,

(ii) for each $x\in(0,1)$, $[D_- f(x), D^- f(x)] \cup [D_+ f(x), D^+ f(x)] = [-\infty,\infty]$, and

(iii) for any $r\in\mathbf{R}$, $D^+ f(x)=\infty$, $D_- f(x)=-\infty$, and $D_+ f(x)=D^- f(x)=r$ for a dense set of $x\in[0,1]$.

L. Zajíček [45] has reported that, about 6 years ago Preiss used the Banach-Mazur game to extend these results.

THEOREM 5.3.4. Let D^+, D_+, D^-, and D_- be extended real numbers such that $\max(|D^+|, |D_+|\} = \max\{|D^-|, |D_-|\} = \infty$ and $[D_-, D^-] \cup [D_+ D^+] = [-\infty,\infty]$. Then there is a residual set N of functions in C such that if $f\in N$ then there is a c-dense set A in (0,1) such that $D^+ f(x)=D^+$, $D_+ f(x)=D_+$, $D_- f(x)=D_-$ and $D^- f(x)=D_-$ for all $x\in A$.

The proof of this theorem consists of devising a strategy for the second player in a Banach-Mazur game played in C, where the set given to the first player consists of all those functions which do not satisfy the conditions of the theorem.

The *N*-Game

In Saks' proof (Theorem 5.1.3) a system of subintervals of [0,1] was defined in such a way that the union of the intersections of decreasing sequences in the system formed a perfect set. Saks also defined a sequence of functions which converge to a function having the required property on this set of points. Zajíček [45] has defined a game which generalizes this method and he has used it to prove several results regarding differentiation properties of typical continuous functions.

DEFINITION 5.4.1. Let F be a nonempty set of the form $[a_1,b_1]\cup\ldots\cup[a_n,b_n]$ where $0\le a_1<b_1<a_2<b_2<\ldots<b_n\le 1$. Then we call F a <u>figure</u> and we define the <u>norm of F</u>, n(F) as $\max\{a_1,b_1-a_1,a_2-b_1,\ldots,b_n-a_n,1-b_n\}$.

DEFINITION 5.4.2. Let N be a σ-ideal of subsets of [0,1]. We define an <u>N-game</u>, between two players, the F-player and the E-player, as follows. In the first move let the E-player choose $\epsilon_1>0$. In the second move let the F-player choose a figure F_1 such that $n(F_1))\le\epsilon_1$. In the $(2n-1)^{th}$ move let the E-player choose $\epsilon_n>0$ and in the $(2n)^{th}$ move let the F-player choose a figure F_n such that $n(F_n)\le\epsilon_n$. If $\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}F_n\in N$ then the F-player wins. Otherwise the E-player wins.

Zajíček notes the following results of the N-game.

(i)If N is the system of all σ-bilaterally strongly porous sets then the F-player has a winning strategy.

(ii)If N is the system of all σ-[g]-totally porous sets then the F-player has a winning strategy.

(iii)If μ is a σ-finite Borel measure on [0,1] and N is the system of all μ-null sets then the F-player has a winning strategy.

(iv)A set $S \subset \mathbf{R}$ is superporous if $S \cup P$ is porous for all porous sets, P. Now if N is the system of all σ-superporous sets then the E-player has a winning strategy.

Using the N game with the Banach-Mazur game Zajíček has developed several new results.

THEOREM 5.4.3. Let the F-player have a winning strategy for the N game. Then there is a residual set of functions f in C such that the set of points $x \in (0,1)$ which are not essential knot points of f belongs to N.

Essential knot points can be replaced in this theorem by [g]-knot points. This, together with result (ii) above is Theorem 4.5.6. An improvement of Preiss' theorem (Theorem 5.3.4) is also obtained by use of the N game.

THEOREM 5.4.4. Let the E-player have a winning strategy for the N game and let D^+, D_+, D^-, and D_- be extended real numbers such that $\max(|D^+|, |D_+|\} = \max\{|D^-|, |D_-|\} = \infty$ and $[D_-,D^-] \cup [D_+D^+] =[-\infty,\infty]$. Then there is a residual set M of functions in C such that if $f \in M$ then there exists a set A in (0,1), $A \notin N$ such that $D^+f(x)=D^+$, $D_+f(x)=D_+$, $D_-f(x)=D_-$ and $D^-f(x)=D_-$ for all $x \in A$.

It is also possible to show, using the N game that the following properties are typical of functions f in C.

(i)for all x in (0,1) there exists a bilateral essential derived number of f at x,

(ii)there exists a c-dense set P such that $+\infty$ is a weak preponderant derivative of f at x, for each x in P,

(iii)there exists $x \in [0,1]$ such that f has no finite derived number with positive upper density.

Conclusion

In this chapter we have seen how the Banach-Mazur game has been used to prove that certain differentiation properties are typical of continuous functions. The introduction of the N-game by Zajíček leads to a general method for determining some properties of sets where a typical continuous function does not have an essential knot point.

CHAPTER VI

CONCLUSION

Nondifferentiability is a typical property of continuous functions. This property has often been said to be pathological and many of the other typical properties of continuous functions might also be described in this way. Despite this we have seen that typical continuous functions display a great deal of regularity. Not only are they nondifferentiable almost everywhere in several generalized senses but their intersection sets with various families of functions are similar.

In Chapter 2 we saw several results regarding the differentiability of typical continuous functions. They have knot points and essential knot points almost everywhere. They are nowhere differentiable, nowhere approximately, symmetrically or preponderantly differentiable. On the other hand in certain senses they are differentiable almost everywhere and in such a way that we can choose the derivatives ahead of time.

In Chapter 3 we saw how the typical continuous function intersects lines, polynomials and 2 parameter families of functions. The characteristics of these intersection sets are similar in all these cases. Seeing how these functions intersect with lines helps to clarify some of the results in Chapter 2.

Chapter 4 continued the study of intersection sets but in this case in the context of porosity and generalized porosity.

We saw that the intersection sets with various families of functions are "small" in the context of the various types of porosity and in Hausdorff dimension. This leads to several more differentiation results, and different proofs of some results from Chapter 2.

In Chapter 5 we looked at the Banach-Mazur game and saw how it could be used to develop differentiation results. We saw that the introduction of the N-game produces some general results regarding knot points.

Related results and open questions

Several areas of investigation arise from the known typical properties of continuous functions. These can be divided into two types: those regarding typical properties in C[0,1] and those regarding typical properties in other spaces.

We begin with the second type. A great deal of work has recently been done regarding typical properties in the spaces of bounded Darboux Baire-1 (bDB_1) functions, bounded approximately continuous (bA) functions, bounded derivatives (bΔ), bounded Baire-1 (bB_1) functions and bounded functions in the Zahorski classes (bM_i i=1,2,3,4,5). Some results are also known for the spaces of bounded upper and lower semi-continuous Darboux functions (bDusc, bDlsc). Summaries of many of these results have been given by I. Mustafa [34] and G. Petruska [37]. In 1986 Mustafa [35] used a general approach to prove typical properties in the spaces bDusc, bDlsc, dΔ, and bM_i. For these spaces he has been able to prove results analogous to many of those we have seen in Chapters 2 and 3 for continuous functions. We have found no results regarding the porosity of intersection sets or sets of non-knot points (similar to results in Chapters 4 and 5) in these spaces.

Bruckner [5] has suggested the investigation of typical properties of intersection sets of functions of several variables with lines. This might lead to a greater understanding of the differentiablity properties of such functions.

In Chapter 3 we cited a suggestion of Ceder and Pearson for the investigation of typical properties of the intersection sets of continuous function with functions in closed nowhere dense subsets of C[0,1].

Haussermann has proposed a question regarding Theorem 4.4.9. He asks whether, for a concave modulus of continuity σ and an ordered porosity family Φ, there is a residual set of functions in C[0,1] which will intersect every function in $L(\sigma)$ in a bilaterally strongly Φ-porous set. This actually is two questions; firstly, can we add bilateral under the hypotheses in part (i) of the theorem and secondly can we remove the requirements that Φ be refined and that $D_+(\sigma\phi^{-1})(0) < \infty$ from part (ii).

Bruckner stated in 1978 [5] that he had never encountered a function meeting the conditions in Theorem 3.2.13. We have been unable to find such a function, let alone one which meets the conditions of the other theorems cited as well. This, despite the "typicalness" of such functions.

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