

**SOME CONTRIBUTIONS TO THE THEORY OF GROWTH WITH EXHAUSTIBLE  
RESOURCES**

by

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Some Contributions to the Theory of Growth with Exhaustible Resources

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## ABSTRACT

The question of the survival of an economy endowed with a finite stock of resources has drawn much attention from economists in recent years. It is now believed that the exhaustibility of some stocks of natural resources may not be a threat to the growth of an economy. Substitution, technical progress and returns to scale may adequately make up for the dwindling stock of resources. The objective of this work is to examine two other means by which resource scarcity may be offset: they are by exploration and by trade.

One part of this thesis is concerned with optimal growth for an economy where a scarce stock of a natural resource input can be augmented by costly exploration. Four different assumptions about the exploration technology are examined. First, the yields from exploration are assumed to depend on cumulative discoveries and also on some other input representing exploratory effort. The other input may be an amount of output (first assumption) or labour (second assumption) or capital (third assumption) used in the exploration process. The fourth assumption investigated is similar to the first except that the yields from exploration here depend on the existing stock of the resource. In the first case, it is shown that per capita consumption will increase indefinitely even if there is no exogenous technical progress. By contrast, in the absence of exploration facilities, a sufficiently large rate of exogenous technical progress is needed to maintain a steadily growing per capita consumption. The minimum level of exogenous technical progress necessary for a rising per capita consumption is higher for the case in which labour is used in the exploration process than for the case without exploration. Per capita consumption increases indefinitely when capital is utilized in the exploration process. Similar conclusions are obtained under the fourth assumption.

The four cases mentioned earlier assume that the yields from exploration are known with certainty. A case where the yields are uncertain is also examined. It is discovered that

uncertainty with respect to the yields from exploration leads to rates of change of consumption, resource use and exploratory effort which are higher than the rates obtained under certain yields.

The thesis also re-examines optimum growth paths with exhaustible resources in the context of an open economy. This is achieved by making some minor modifications in two famous models used for this purpose. We re-examine the Dasgupta, Eastwood and Heal model which treats foreign assets as one source of income for the economy. We assume that there are no foreign assets. Instead it is assumed that the imported commodity can be used to augment home consumption and investment. Moussavian incorporated a non-traded sector in the DEH model. He also assumed an exogenous rate of return on foreign assets. We rework the Moussavian model by making the rate of return endogenous. The possibility of a total extinction of home production of the traded good is significantly reduced under the altered assumptions.

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**DEDICATION**

**TO  
ALMIGHTY ALLAH**



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## CHAPTER 1

### INTRODUCTION

The study of the economics of exhaustible resources goes back to Lewis Gray (1914) when he formulated the theory of mine. Harold Hotelling made the first rigorous analysis of exhaustible resource extraction in his 1931 seminal article. He discovered the famous "Hotelling rule" which is still regarded as the most fundamental principle of resource economics. Simply put, the rule says that the net price  $\pi$  (marginal profit) of a resource in a competitive (monopolistic) market is expected to grow at a rate equal to the market rate of interest. Economists did not pay much attention to resource economics, however, until very recently. For almost four decades since the publication of the article by Hotelling, there prevailed a vacuum in the literature. Some extensions on the early works of Gray and Hotelling were finally made by Herfindahl (1967) and Scott (1967). The energy crisis of the seventies paved the way of a resurgence of interest in resource economics. Since then the literature has grown tremendously. While the new generation of resource economists found new and a multitude of ways of refining the pioneering work of Hotelling, the element of exhaustibility also came to be considered by other branches of economic science. One such branch is optimal macro-economic growth models. Two factors stimulated the phenomenal upsurge in work on growth models with exhaustible resources. First, the research in the area was more or less instigated by the widely debated "Club of Rome" studies. Forrester's (1971) "World Dynamics" and Meadow's (1972) "The Limits to Growth" portrayed a very pessimistic scenario of the future of the world. They hypothesized that the economy of the world is going to be doomed sooner or later because of a shortage of natural resources. Second, the energy crisis of the seventies made people aware of their vulnerability to any kind of resource shock. Consequently, the question of growth with a finite stock of a resource became a topic of intensive discussion in the academic arena.

For example, a symposium on growth with exhaustible resources was held under the auspices of the Review of Economic Studies in 1974.

Studies made during the seventies and the early eighties suggest, however, that the exhaustibility of natural resources may not pose a threat to the growth of an economy. The impediment scarce resources impose to achieving a constant or steadily growing level of consumption can be overcome by any combination of three factors. These three factors are substitution, technical progress and returns to scale. Assuming a Cobb-Douglas production function, for example, Stiglitz (1974) showed that an economy with a finite stock of resources and no technical progress can maintain a constant level of per capita consumption if the share of capital is larger than the share of resources in total production costs. Solow (1974) and Stiglitz (1974) also showed that with a sufficiently large rate of technical progress, a steadily growing per capita consumption is possible.

Extensions on the pioneering works of Solow and Stiglitz have been made in a number of directions. One line of direction is to allow for endogenous technical progress. It is true that technical progress does not originate in a vacuum. Some productive resources have to be put aside for use in the R&D sector of the economy in order to enjoy the fruits of technical progress. These productive resources may take the form of output or some factor of production, viz., capital or labour. Characteristics of the optimal paths of consumption and resource extraction have also been studied in models where there is uncertainty about the size of the stock of the resource, or on market conditions. Other models have been constructed which investigate the optimal growth paths of an open economy with exhaustible resources.

In this thesis, we will examine some extensions of the works of Solow and Stiglitz. First, growth patterns are examined where an alternative view of the resource generation process is taken. Most of the studies mentioned earlier assume a fixed amount of initial

resources. It is more realistic to assume that the stock of the resource can be augmented by exploration and development activities.<sup>1</sup> This view comes from the realization that the stock of a resource is not homogenous but consists of a variety of deposits of different qualities. As time passes and economic conditions change, more and more of the resource stock becomes economically accessible. It is really this pattern of rising exploration and development costs as the more accessible deposits of the resource are depleted that limits the use of the resource by economy rather than the finiteness of the stock. Our modelling of exploration, therefore, assumes that as cumulative discoveries increase, the returns to exploration activities become less and less. Another aspect of the exploration process is taken into consideration. Exploration is not, of course, costless. It entails the expense of some output or some factor of production like capital. Thus, more expenses made for exploration should lead to more discoveries subject, however, to diminishing returns. Such an exploration technology can be represented by a myriad of functional forms. We use four different functional forms to study growth in the presence of exploration. In the first three models, we assume that exploration returns depend positively on exploratory effort and negatively on the size of the cumulative discoveries. In the first model, the input of exploration is some portion of the economy's output. In the second model, we assume that a portion of labour is used in the exploration process. In the third model, a portion of capital is expended for exploration. The fourth model assumes that the yields from exploration depend positively on both the output expended for exploration and the existing stock of the resource. We find that in most cases under the presence of exploration, per capita consumption can increase indefinitely even if there is no exogenous technical progress. By contrast, a sufficiently large rate of technical progress is required to maintain a steadily growing per capita consumption in the absence of exploration.

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<sup>1</sup> Hereafter, this will be referred to simply as exploration activities but the process should include the costs of developing the newly discovered stocks up to some common standard.

We also consider the question of growth with an augmentable resource stock when the returns to exploration activities are uncertain. The returns to exploration activities are assumed to follow a Wiener distribution with mean zero and a variance proportional to time lapse between the present and the future. We discover that uncertainty with respect to the yields from exploration leads to higher expected rates of change of consumption, resource extraction and exploratory effort.

Second, we study the nature of growth experienced by an open economy possessing a finite supply of resources. Particularly, we rework the models employed by Dasgupta, Eastwood and Heal (1978)<sup>2</sup> and Moussavian (1985). According to the DEH model, the gross total wealth of an economy accrues from three distinct sources, viz., output produced in the home country, interest earnings from foreign assets and the proceeds from sale of an exhaustible resource in the world market. A portion of gross total wealth is used for financing the consumption needs of the people. We assume away foreign assets as a component of the total wealth of the economy. Rather, we postulate that home country's consumption is supplemented by an imported commodity. Moussavian added one non-tradeable sector to the DEH model. He, however, assumed an exogenous rate of return on foreign assets. We allow the rate of return to be endogenous depending negatively on the quantity of foreign assets. The implications of these changes in the key assumptions are noted in each case. It is demonstrated that the possibility of the gradual extinction of the home production of the consumption good is significantly reduced under the altered assumptions in both the models.

In this thesis, we make use of the optimal control theory in the derivation of the optimal growth paths of the models developed in chapters 3 and 5. We use the dynamic programming technique of optimisation in chapter 4. The optimal control theory is a set of techniques used in dynamic optimisation problems. It is an improvement over the classical

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<sup>2</sup> Henceforth abbreviated by DEH

theory of the calculus of variations applied in non-linear dynamic optimisation problems. The maximum principle of the optimal control theory can be used for both the non-linear and linear optimisation problems and for the cases of inequality constraints which cannot be handled by the classical techniques.<sup>3</sup>Dynamic programming is an alternative technique of dynamic optimisation. Known as Bellman's Principle of Optimality, it says, "an optimal policy has the property that, whatever the initial state and control are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."<sup>4</sup>

The organisation of this thesis is as follows. As usual, a systematic analysis of the theoretical literature on growth models with exhaustible resources is presented in chapter 2. Chapter 3 examines the growth behaviour of an economy when it can increase the stock of resources by costly exploration. As discussed above, four different functional forms are considered. In another section of chapter 3, we compare our results with those results obtained from some closely related models. In chapter 4, we incorporate the uncertainty about the yields from exploration in a model similar to that employed in section 5 of chapter 3. We present the discussion on the modified DEH and Moussavian models in chapter 5. The concluding chapter of the thesis summarizes the conclusions and discusses the limitations of this research. Finally, future areas of research are suggested.

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<sup>3</sup>See Miller (1979) for a detailed discussion on the calculus of variations and the maximum principle.

<sup>4</sup>See Bellman (1957).



## CHAPTER 2

### GROWTH WITH EXHAUSTIBLE RESOURCES: A SURVEY

#### 2.1 INTRODUCTION

In this chapter, we make a survey of the works published in the field of growth with exhaustible resources. The literature has grown tremendously during the last 15 years. Accordingly, a large number of articles are available on this broad subject. It is very difficult to define a criterion on the basis of which the published articles can be grouped among different categories. This is true in the cases of both within and among the different branches of resource economics. Therefore, we had to narrow the topic down for reasons of economy of space. The selected papers are those which use the criterion of maximising a stream of discounted utility over a time horizon subject to constraints relating to capital formation and resource depletion. The capital formation constraint yields investment as the residual of total output over the consumption needs. The resource depletion constraint, on the other hand, stipulates that the cumulative use of the resource in the production process and elsewhere<sup>1</sup> cannot exceed the size of the initial stock of the resource plus any additional reserves discovered. Usually one question is pertinent in such a paper: what are the characteristics of the optimal time paths of consumption, output, capital stock and resource utilization rate in an economy with a finite stock of the resource?

For expositional convenience, we have grouped the published articles in five categories: (1) The Utilitarian Approach, (2) The Equity Approach, (3) Endogenous Technical Progress and Exploration, (4) Open Economies and (5) With Uncertainty. We mainly follow the chronological order in surveying the articles within each category.

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<sup>1</sup> In an open economy model, the resource can also be exported abroad.

## 2.2 THE UTILITARIAN APPROACH

Before we review the articles in this category, a brief discussion of the difference between the utilitarian and equity approaches should be made. The utilitarian approach assumes an additive social welfare function in the sense that a loss of utility to an individual can be compensated by a gain of an equal amount to another individual. It ignores the question of the distribution of wealth or utility in society. In contrast, the equity or Rawlsian (1971) approach stipulates that an improvement in social welfare occurs when there is an improvement in the position of the poorest individual or individuals in the society.

As an example, we will review the Solow (1956,1970) neoclassical growth model<sup>2</sup> which uses the utilitarian approach. The articles that are discussed later in this section and the models later in this thesis are extensions of this basic model. This review, therefore, introduces the questions that all the later works examine. It also illustrates the methodology that these works usually employ. The single sector neoclassical growth model allows for two inputs of production, capital and labour, to be combined in varying proportions. The production possibilities of a society can be represented by a smooth two input neoclassical production function. Let Y stand for the sole homogenous commodity produced. This output is produced with the help of two inputs, capital (K) and labour (L), according to a production function

$$(2-2-1) \quad Y = F(t, K, L)$$

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<sup>2</sup> For a textbook exposition of the neoclassical growth model, see Neher (1969). As a matter of fact, the neo-classical growth model was developed to remove some of the shortcomings implicit in the Harrod-Domar growth model. The latter assumes that labour is always combined with capital in a fixed ratio. The outcome is that the economy needs to maintain a delicate balance between the actual and desired rates of growth of investment. In the absence of balancing of the two rates, the economy experiences either a shortage of capacity when the actual rate is larger than the desired rate or a surplus capacity when the actual growth of investment lags behind the required rate.

The time element,  $t$ , in (2-2-1) allows for the presence of an exogenous technical progress. The following assumptions with regard to the production function are made. This production function is continuous and twice differentiable. The marginal productivity of each factor is positive and diminishing. Thus,

$$(2-2-2) \quad f_K > 0 \quad f_L > 0 \quad f_{KK} < 0 \quad f_{LL} < 0 \quad \text{where } f_K = \partial f(K, L) / \partial K, \text{ etc.}$$

It is also assumed that both marginal products start at infinity and diminish to zero. That is,

$$(2-2-3) \quad \lim_{i \rightarrow 0} f_i = \infty \quad \lim_{i \rightarrow \infty} f_i = 0 \quad i = K, L$$

The assumptions given in (2-2-3) are called the Inada conditions. These conditions are necessary to ensure that the maximisation process does not necessitate a zero level of use of the inputs of production. Moreover, the production function (2-2-1) displays constant returns to scale. In other words, a proportionate increase in all the inputs will cause the output to increase at the same proportionate rate. The assumption of constant returns to scale with respect to an aggregate production function as specified in (2-2-1) is plausible if there are no scarce factors in the economy.

It is assumed that all of society's stocks of labour and capital are fully employed. Therefore, growth in the economy can only take place through the growth of these two stocks. Labour growth is usually assumed to be exogenous and is assumed to grow exponentially at a given rate, say  $n$ . That is,

$$(2-2-4) \quad \hat{L} = n$$

where circumflex ( $\hat{\quad}$ ) over a variable signifies the growth rate of that variable. On the other hand, capital stock growth occurs by setting aside some of society's output to be used to

augment this stock. Thus,

$$(2-2-5) \quad Y = C + I$$

where  $C$  = Aggregate Consumption,  $I$  = Investment, and hence

$$\dot{K} = I - \tau K \quad \text{where } K = dK/dt \text{ etc. \& } \tau = \text{the depreciation rate.}$$

Now let,  $k = K/L$        $c = C/L$        $y = Y/L$

The time rate of change of per capita capital stock of the economy can then be shown to be given by

$$(2-2-6) \quad \dot{k} = f(k) - nk - c^3$$

where  $f(k) = F(k,1)$ . Equation (2-2-6) is called the fundamental differential equation of the neoclassical growth model. Assume as well that (2-2-1) takes the explicit form of a Cobb-Douglas production function embodying exogenous technical progress. Then

$$(2-2-7) \quad Y = e^{\lambda t} K^{\alpha} L^{\gamma}$$

Here  $\alpha + \gamma = 1$  and  $\lambda$  stands for the rate of exogenous technical progress. In per capita terms, this production function becomes

$$(2-2-8) \quad y = e^{\lambda t} k^{\alpha}$$

Now suppose the central planner of this hypothetical economy believes that social welfare is best described by a utility function which gives utility at any point of time as a function of the standard of living as measured by consumption per worker:

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<sup>3</sup> In equation (2-2-6), the depreciation rate is assumed to be zero. That, however, does not affect the basic results of the model.

$$(2-2-9) \quad U = U(c(t))$$

The utility function should satisfy the following assumptions:

(1) It is twice differentiable with positive but diminishing marginal utility. That is,

$$(2-2-10) \quad dU(c)/dc > 0 \quad d^2U(c)/dc^2 < 0 \quad \text{for all } c, \quad 0 < c < \infty$$

(2) It satisfies the limit conditions:<sup>4</sup>

$$(2-2-11) \quad \lim_{c \rightarrow 0} U'(c) = \infty \quad \lim_{c \rightarrow \infty} U'(c) = 0$$

We use the optimal control theory to derive the optimal growth paths of this economy. The problem the social planner faces is to choose a per capita consumption path so as to maximise the sum of discounted utility of per capita consumption over all time. Hence he wishes to

$$(2-2-12) \quad \text{Max } \Omega = \int_0^{\infty} e^{-\delta t} U(c) dt$$

subject to

$$(2-2-13) \quad \dot{k} = y - nk - c$$

$$(2-2-14) \quad k(0) = k_0$$

$$(2-2-15) \quad 0 \leq c \leq f(k) = e^{\lambda t} k^{\alpha}$$

---

<sup>4</sup> Unless stated otherwise, we will stick to the same notation throughout the rest of the thesis. The same assumptions will be made for the utility functions in all the models developed in chapters 3, 4 and 5. The assumptions for the production function will also remain the same except that the resource will be considered as an additional input of production. All the properties pertaining to the marginal productivity of capital or labour also apply to the marginal productivity of the resource.

(2-2-16)  $c(t)$  piecewise continuous

Condition (2-2-14) stipulates that the objective functional has to be maximised subject to a given initial stock of capital. Equation (2-2-15) says that per capita consumption cannot be negative or greater than per capita output. The last condition, on the other hand, postulates that the consumption function need not be continuous throughout the tenure of the program. It may change by jumps if necessary.

The current value Hamiltonian for the planner's problem is given by

$$H = U(c) + \mu_1 [y - c - nk]$$

The necessary conditions for optimisation are as follows:

$$(2-2-17) \quad U'(c) = \mu_1$$

$$(2-2-18) \quad \dot{\mu}_1 = \delta + n - (\alpha y/k)$$

Taking the time derivative of (2-2-17) and substituting for  $\dot{\mu}_1$ , we find the time rate of change of per capita consumption to be

$$(2-2-19) \quad \dot{c} = (c/\sigma) [(\alpha y/k) - \delta - n]$$

where  $\sigma = -U''(c)c/U'(c) > 0$ . where  $U'(c) = dU(c)/dc$  etc.<sup>5</sup>

The optimal growth paths of the economy then satisfy differential equations (2-2-13) and (2-2-19). We can analyse the behaviour of this differential equation system in two ways. We can examine the phase diagram of this system directly. This method has one disadvantage. If we assume a production function which also entails an exogenous technical progress, the differential equation system becomes non-autonomous. The system then cannot possess a

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<sup>5</sup> For tractability, it will be assumed that  $\sigma$  is constant.

stationary point. Alternatively, we can introduce the ratios of output to capital and consumption to capital and make the analysis in terms of these ratios. This method alleviates the problem of the presence of a time element in the production function. Analytically, the steady state of the transformed system represents balanced growth paths where all the aggregate as well as per capita variables grow at the same rate. The balanced growth paths are important elements of the solution to the planner's problem.

Thus, define  $X = y/k = Y/K$  and  $V = c/k = C/K$ . The time rate of change of per capita consumption can be expressed in terms of  $X$  and  $V$ :

$$(2-2-20) \quad \hat{c} = (1/\sigma)(\alpha X - \delta - n).$$

The differential equations (2-2-13) and (2-2-19) can then easily be transformed into the following differential equations for  $X$  and  $V$ :

$$(2-2-21) \quad \hat{X} = -\gamma X + \gamma V + \lambda + \gamma n$$

$$(2-2-22) \quad \hat{V} = \{(\alpha - \sigma)/\sigma\}X + V - (1/\sigma)(\delta + n - \sigma n).$$

There are four equilibrium points for this system. These can occur when [I]  $X > 0, V > 0$   
 [II]  $X = 0, V > 0$  [III]  $X > 0, V = 0$  and [IV]  $X = 0, V = 0$ .

The importance of these equilibrium points is that the optimal trajectory will usually converge asymptotically to one of these points. However, not all of these equilibrium points need to be considered as relevant to the optimisation problem. The second equilibrium point can be rejected because it contradicts the common sense view that the ratio of output to capital must be greater than the ratio of consumption to capital. It is also an unstable equilibrium. The optimal trajectory also cannot converge to the last case  $X^* = 0, V^* = 0$  as  $\dot{X} > 0$  when  $X$  and  $V$  are very small. We can also rule out the third possibility by using the transversality

condition for capital which is<sup>6</sup>

$$(2-2-23) \quad \text{Limit } e^{-\delta t} \mu_1(t) k(t) = 0 \\ t \rightarrow \infty$$

When enough time has passed, the above product must thus be a decreasing function of time. Since all the variables here are positive, this condition requires then that

$$(2-2-24) \quad \lim_{t \rightarrow \infty} -\delta + \hat{\mu}_1 + \hat{k} \leq 0$$

provided this second limit exists. In the problem here, using (2-2-13) and (2-2-18), this means that

$$(2-2-25) \quad (1-\alpha)X^* - V^* = \nabla \leq 0$$

Trajectories which converge to  $X^* > 0$  and  $V^* = 0$  such as case [III] do not satisfy this property. Only the first point qualifies as a feasible solution here. The stationary state values of  $X$  and  $V$  are easily found for this case by setting  $\hat{X} = 0$  and  $\hat{V} = 0$ . They are

$$(2-2-26) \quad X^* = [\sigma\lambda + \gamma(\delta+n)]/\alpha\gamma$$

$$(2-2-27) \quad V^* = [\gamma\delta + \gamma n(1-\alpha) + \lambda(\sigma-\alpha)]/\alpha\gamma$$

$X^*$  is always positive and  $V^*$  is positive if  $\gamma\delta + \gamma n(1-\alpha) + \sigma\lambda > \alpha\lambda$ . For a logarithmic utility function, where  $\sigma=1$ ,  $V^*$  is always positive. The rate of growth of per capita consumption tends asymptotically to

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<sup>6</sup> This transversality condition is one requisite for the satisfaction of the sufficient condition for the maximisation problem. See Arrow and Kurz (1970).



$$(2-2-28) \quad \hat{c} \approx \sigma\lambda/\gamma$$

Moreover, all the per capita variables such as consumption, capital and output grow at the same positive rate at the equilibrium point. It is obvious from (2-2-28) that without an exogenous technical progress ( $\lambda=0$ ), there is a "golden rule" optimal per capita consumption level which the social planner aims for. The golden rule values of X and V are given by

$$(2-2-29) \quad X^* = k^{*\alpha-1} = (\delta+n)/\alpha$$

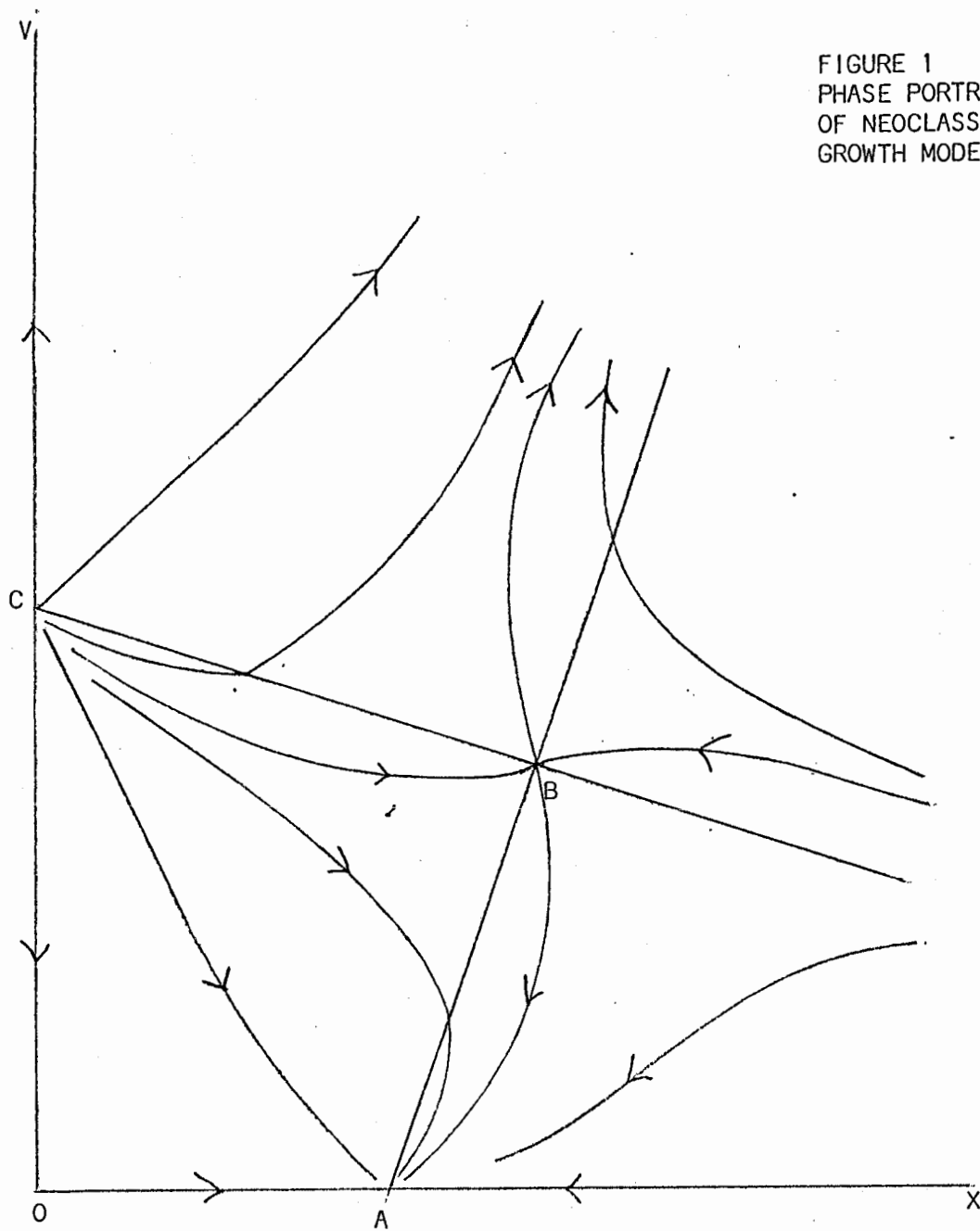
$$(2-2-30) \quad c^* = V^* k^* = \left\{ (\delta+n)/\alpha \right\}^{-1/(1-\alpha)} \left[ \left\{ \delta+n(1-\alpha) \right\} / \alpha \right]$$

With technical change ( $\lambda>0$ ), however, per capita consumption will grow for ever under optimal management. The equilibrium point is a saddle point which will be approached in the long run whatever the initial condition is.

Perhaps, the nature of the optimal growth path can be better understood with the help of a phase diagram. Figure 1 on next page presents the phase diagram of the differential equation system (2-2-21) and (2-2-22) in V-X surface. In this diagram, we observe three possible candidates for the long run equilibrium point as shown by points A, B and C. B refers to the saddle point found for case 1. C is, however, totally unstable. Moreover, it can be rejected because V can never be greater than X. A represents a stable equilibrium point. It can, however, be verified that the optimal growth path does not tend to the point A. We can rule out point A on the basis of the transversality condition associated with capital. It is argued that the optimal trajectory must converge to  $(X^*, V^*)$  whatever the initial value of X is.<sup>7</sup> The reason is that each segment of a trajectory is optimal for some combination of T (finite time), X(0) and X(T). As T tends to  $\infty$  with X(T) fixed, a segment of any trajectory will tend to one of the separatrices converging to the middle equilibrium point. For

<sup>7</sup> The initial value of  $V=C_0/K_0$  is optimally determined by choosing the optimum initial level of C,  $C_0$ .

FIGURE 1  
PHASE PORTRAIT  
OF NEOCLASSICAL  
GROWTH MODEL



example, suppose both  $X(0)$  and  $X(T)$  are large. When  $T$  increases, the lower segment of the relevant trajectory gets extended and it comes close to the equilibrium point,  $B$ . In the limit when  $T$  tends to infinity, it converges to the equilibrium point.

In the neoclassical growth model, all the inputs of production are assumed to be augmentable. In other words, the stock of all inputs of production are assumed to be replenishable over time. There are some factors of production which are, however, non-replenishable and their stocks may get exhausted in finite time through continued uses in the production process. The best example of this kind of input is natural oil which is required in the production of many different commodities. The growth models surveyed in the rest of this chapter assume that one of the factors of production is exhaustible or non-renewable. In that sense, these growth models are more realistic than the neoclassical growth models. With this brief review of the neoclassical growth model in the background, we now turn to the discussion of the papers which have adopted the utilitarian approach to study optimal growth patterns of an economy subject, among other things, to a resource constraint.

The basic issues addressed in these articles challenge the very pessimistic view about the world economy presented by the "Club of Rome" studies.<sup>8</sup> In 1972, D. L. Meadows and his associates<sup>9</sup> argued that the world economy will soon face a total extinction that will be brought about by the problem of resource scarcity. A growing population and an advancing technology only worsen the problem. This will happen because most of the resources available in this world are non-renewable and their stocks are going to be exhausted sooner or later in the future. A growing population enhances the destruction process of an economy endowed with a fixed resource base in two ways. First, a growing population applied to a constant

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<sup>8</sup> The "Club of Rome" is a body of academicians, politicians and businessmen who are concerned about the progress of mankind in the earth. The club was formed in 1968.

<sup>9</sup>See Meadows et. al. (1972) The Limits to Growth

stock of the resource causes per head output to fall. Second, a population increase leads to a greater level of industrial output which in turn necessitates a higher rate of use of the non-replenishable resource. As a consequence, the stock of the resource is depleted more quickly. Technological development increases per capita productivity and in the process fuels the engine of the destruction of the economy. This occurs because more and more non-renewable resources are substituted for agricultural inputs as the state of technology advances. Thus, even technological progress cannot provide a breathing space in the extinction process caused by the resource scarcity. The aforesaid conclusions derived by Meadows and his colleagues are, however, based on some naive assumptions about the production function. For example, they assumed a fixed proportion production function so that there is no scope for substitution among the factors. The growth paths they used in their simulation experiments were ad hoc in the sense that those were not derived from any dynamic optimisation problem where some criterion function is maximised. In short, their analysis is not based on a sound footing. The papers that we will discuss next, filled in the important gaps which the "Club of Rome" studies did not pay much attention to. The later authors addressed more rigorously the question of the survival of an economy possessing only a finite stock of a resource. In particular, all the papers examine the circumstances under which a constant or steadily increasing per capita consumption is possible in an economy with an exhaustible resource.

Whether an economy with a finite supply of a resource will face total annihilation or not depends a lot on the resource being an essential factor in society's production process. An input is essential when society cannot produce its output without the use of this input. The assumption of essentiality can be introduced, however, in two different ways. The essential factor can be a substitute for the other factors or it can hold a complementary relationship with the level of output and the other factors of production. Take, for example, the model used by Anderson (1972). He wrote the first article on growth with a

non-replenishable resource. He set up his model as follows:

$$(2-2-31) \quad Y(t) = \text{Min}[F(K,L), e^{\theta t} \dot{S}(t)]$$

where  $F(\cdot)$  is a neoclassical production function which satisfies the Inada conditions.  $S(t)$  stands for the stock of the resource at time  $t$ . Equation (2-2-31) is a Leontief production function having the neoclassical production function and the resource flow as its two inputs. In other words, labour and capital can be used in varying proportions here. The resource flow is, however, combined in fixed proportions with  $F(K,L)$ . Assuming

$$(2-2-32) \quad F(K,L) < e^{\theta t} \dot{S}(t)$$

we get,

$$(2-2-33) \quad Y = F(K,L)$$

$$(2-2-34) \quad \dot{S} = -e^{-\theta t} F(K,L)$$

Equation (2-2-34) implies a constant ratio of the resource utilization rate to the output level at any given point of time. Technological progress, however, leads to more efficient use of the resource and hence a diminishing ratio of the resource utilization rate to the output level over time. Output is allocated between consumption and capital formation. The central planner's problem is to find the optimum path of saving that maximises the discounted sum of per capita consumption over a finite time horizon. Anderson demonstrated two effects of the resource constraint. As compared to the unconstrained case, the resource constrained economy faces a lower savings rate and a lower capital labour ratio along the turn-pike path it follows. Second, the resource constrained economy spends more time on the turn-pike path.

A related model was discussed by Withagon (1983). He analysed the time paths of consumption and capital in a model where the resource input and capital are complementary

to each other. In particular, he assumed a constant elasticity of demand for the resource with respect to the capital input. The production function incorporates exogenous technical progress. The objective is to maximise the social welfare from consumption over an infinite time horizon. Using this model, he showed that the capital stock diminishes to zero in the long run. The economy can, however, sustain a growing rate of consumption if there is a sufficiently large rate of technical progress.

#### THE 1974 SYMPOSIUM ARTICLES (1974)

The papers mentioned so far are not totally satisfactory. Either the roles of capital and labour as factors of production are ignored or the resource is not treated as a direct input of the production function. The 1974 symposium articles addressed this problem. In 1974, the Review of Economic Studies published a set of articles given at a symposium on growth models. The papers by Stiglitz (1974), Solow (1974) and Dasgupta and Heal (1974) made a breakthrough in the field of growth with exhaustible resources. Discussion of Solow's paper is postponed until a later section where we address the question of equity. Here we focus on the results found in Stiglitz (1974), Dasgupta and Heal (1974), Beckman (1974) and Weinstein and Zeckhauser (1974).

Stiglitz assumed a linear homogenous Cobb-Douglas production function which embodies exogenous technical progress. The three factors of production are capital, labour and the resource flow. At any point of time, total output is divided between consumption and capital formation. The labour force grows at a constant exogenous rate. The stock of the resource is depleted for use in the production function. Thus,

$$(2-2-35) \quad \dot{K} = Y - C$$

$$(2-2-36) \quad Y = e^{\lambda t} K^{\alpha} R^{\beta} L^{\gamma}, \quad \alpha + \beta + \gamma = 1$$

$$(2-2-37) \quad \dot{\hat{L}} = n$$

$$(2-2-38) \quad \dot{S} = -R$$

Stiglitz's (1974) model is indeed a modification of the single sector neoclassical model. The modification is made in two places. The production function now has three inputs: capital, labour and the resource flow instead of just capital and labour as in (2-2-7). Most importantly, the stock of the resource is finite and depletes according to equation (2-2-38). Stiglitz analysed his model in two ways. First, efficient growth paths were derived. Efficient growth paths satisfy the intertemporal efficiency condition that the growth rate of the marginal productivity of the resource must equal the marginal productivity of capital. The efficiency condition mentioned above dictates that the return to the non-replenishable resources must grow at the rate of marginal productivity of capital. Otherwise, the owners of the resources will find it profitable to dispose off the stock of the resource immediately and invest the proceeds in capital investment. It is also assumed that consumption is growing exponentially at a constant rate. The characteristics of the efficient growth paths are as follows:

1. With suitable initial conditions, the output-capital ratio, the savings rate and the resource flow to stock ratio will converge to stationary values in the long run. Along the efficient path, the rate of resource use will fall continuously to zero.
2. Per capita consumption will be increasing, constant and decreasing according as to whether the ratio of the rate of technical progress to the labour growth rate ( $\lambda/n$ ) is greater, equal or smaller than the share of the resource in total output ( $\beta$ ). With constant output, constant population and no technical progress, a constant level of consumption can be maintained if the share of capital in total output is larger than that of the resource ( $\alpha > \beta$ ).

Second, optimal growth paths were derived using the same discounted utility maximising

criterion as in Solow (1956). The planner's problem is:

$$(2-2-39) \quad \text{Max } \Omega = \int_0^{\infty} e^{-\iota t} U(c) dt^{10}$$

subject to equations (2-2-35) through (2-2-38) and

$$(2-2-40) \quad U(c) = c^{1-\sigma}/(1-\sigma) \quad c = C/L$$

Define the additional normalized variables:

$$\tau = R/L \quad s = S/L \quad Z = R/S = \tau/s.$$

Using the optimal control theory, as in the discussion of the neoclassical model, it can be shown that the optimal growth path satisfies

$$(2-2-41) \quad \hat{c} = (1/\sigma) [\alpha X - \delta]$$

Further, these optimal growth paths of the economy can be shown to satisfy the following differential equations:

$$(2-2-42) \quad \hat{X} = \{1/(1-\beta)\} [-(1-\alpha)(1-\beta)X + \gamma V + \gamma n + \lambda]$$

$$(2-2-43) \quad \hat{V} = [\{(\alpha/\sigma) - 1\}X + V + n - \delta/\sigma]$$

$$(2-2-44) \quad \hat{Z} = \{1/(1-\beta)\} [-\alpha V + (1-\beta)Z + \gamma n + \lambda]$$

There are eight possible equilibrium points in this model. Six of them, however, do not possess the properties of a feasible limit of the optimal trajectory and hence can be ignored.

The following two cases qualify as possible long run equilibrium points:

[I]  $X > 0$ ,  $V > 0$  and  $Z > 0$  [II]  $X > 0$ ,  $V > 0$  and  $Z = 0$ .

<sup>10</sup> In most cases  $\iota = \delta$ . Stiglitz actually used an  $\iota = \delta - n$  and his notation is followed here. In general, the two forms of discount rates are mathematically equivalent as long as  $\delta > n$  which is important in Stiglitz (1974).



It should be noted here that Stiglitz (1974) analysed only the first case. He ignored the second case. Later, we will show that the equilibrium point considered by Stiglitz is non-stable in Z. The stationary state values of X, V and Z for the first case can be obtained by setting  $\hat{X}=0$ ,  $\hat{V}=0$ , and  $\hat{Z}=0$ . These are

$$(2-2-45) \quad X^* = [\delta\gamma + \lambda\sigma] / \alpha(\sigma\beta + \gamma)$$

$$(2-2-46) \quad V^* = [(\gamma + \alpha\beta)(\delta - \sigma n) - (\alpha - \sigma)(\lambda + \gamma n)] / \alpha(\sigma\beta + \gamma)$$

$$(2-2-47) \quad Z^* = [(1 - \alpha)(\delta - \sigma n) - (\lambda + \gamma n)(1 - \sigma)] / (\sigma\beta + \gamma)$$

The stationary state values of all the variables will be positive if we assume, like Stiglitz (1974), a logarithmic utility function ( $\sigma=1$ ) and a positive discount rate ( $\delta > n$ ). In that case, the determinant value of the Jacobian of the differential equation system (2-2-42) through (2-2-44) at  $(X^*, V^*)$  is  $[-\alpha(1 - \alpha) / (1 - \beta)] X^* V^* Z^* < 0$  and the associated trace is  $2(\delta - n) > 0$ . The Jacobian has only one negative characteristic root. The equilibrium here is a saddle point. It can easily be deduced that the economy cannot converge to the above mentioned equilibrium point. Because of the recursive nature of the differential equation system, we can solve for equilibrium values of X and V independently of the differential equation for Z. We can substitute  $V^*$  for V in (2-2-44) to obtain an approximation to the differential equation for Z in terms of Z only which is valid near  $V^*$ . This equation takes the form

$$(2-2-48) \quad \dot{Z} = \{Z / (1 - \beta)\} [-\alpha V^* + (1 - \beta)Z + \gamma n + \lambda]$$

Since  $-\alpha V^* + \gamma n + \lambda = -(1 - \beta)Z^* < 0$ , its stable equilibrium value is at  $Z=0$ , not at  $Z=Z^*$ . The equilibrium of case [II] with  $Z=0$  is still a saddle point in X-V space. The optimal growth paths of the economy can converge to this equilibrium point provided the initial conditions are suitable. Stiglitz (1974) was undoubtedly wrong in asserting that the economy

will converge to the first equilibrium point.

Finally, the following results with respect to per capita consumption hold for any iso-elastic utility function. The optimisation model here yields the conclusion that optimal per capita consumption will be increasing or decreasing asymptotically according to the following condition:

$$(2-2-49) \quad \hat{c} \geq 0 \quad \text{as} \quad (\lambda/\delta) \geq \beta$$

The conclusion is in sharp contrast with the conclusion found in the single sector neoclassical growth model. Compare equation (2-2-28) with equation (2-2-49). In the neoclassical growth model, the presence of any exogenous technical progress is sufficient for a growing per capita consumption. In the Stiglitz (1974) model, a steadily growing per capita consumption can be obtained only when the rate of exogenous technical progress is sufficiently large. In the absence of an exogenous technical progress, the economy can maintain a constant per capita consumption for ever in the neoclassical model but declines exponentially to zero in the Stiglitz (1974) model.

Dasgupta and Heal (1974) analysed the same problem as in Stiglitz's model except that the more general C.E.S. production function is used instead of a Cobb-Douglas production function and they do not allow for technological progress. The central planner's problem is again to maximise the sum of discounted utility of consumption in the presence of the capital formation and resource constraints. The population is also assumed to be constant and hence is suppressed in the model. The C.E.S. production function can be written as:

$$(2-2-50) \quad F(K, R) = [\theta K^{(\eta-1)/\eta} + (1-\theta) R^{(\eta-1)/\eta}]^{\eta/(\eta-1)}$$

They noted the following specific results:

1.  $\eta \leq 1$  : The exhaustible resource is an essential factor of production. In the long run,

both consumption and capital must decay to zero. The case when  $\eta = 1$ , which is a Cobb-Douglas production function, thus gives the same results as in Stiglitz (1974).

2.  $\eta > 1$  : The exhaustible resource is not essential to the production process in this case. The optimal time paths of consumption and capital may be increasing or decreasing depending on the relative magnitudes of different parameters of the model.

In both the cases mentioned above, the rate of use of the exhaustible resource diminishes continuously to zero.

Weinstein and Zeckhauser (1974) used a discrete time model to analyze the optimal time paths of consumption and resource extraction. The assumptions they made are somewhat different from the assumptions usually made in the standard model. Their objective is to maximise the non-discounted sum of the utility of total consumption made over time. The consumption good is produced by using the resource and the consumption good itself. Capital formation is absent in the model though the economy has to face the resource constraint. Population is assumed to be constant in their model. They derived the conclusion that when the resource is an essential factor of production, both optimal output and consumption (total) fall continuously to zero. The resource use rate declines continuously to zero. Unlike Stiglitz, they did not assume an exogenous rate of technical progress. As we noted earlier, in Stiglitz (1974), per capita consumption eventually falls over time in the absence of an exogenous technical progress. In that sense, the results in the two places agree with each other. A falling per capita consumption, however, does not necessarily imply a falling total consumption. Perhaps, the absence of capital formation in Weinstein and Zeckhauser (1974) makes the results of their model look more pessimistic.

A number of later authors have discussed variants of the models presented at the symposium. Cigno (1978) reworked Stiglitz's efficient growth model and showed that the saddle point property of the long run equilibrium of Stiglitz's model is caused by the assumption of a constant population growth rate. He, instead, assumed an endogenous

population growth rate depending negatively on the degree of industrialization, i.e., per capita capital stock, and positively on per capita consumption. The model has a stable long run equilibrium if the population growth rate depends more on per capita capital than on per capita consumption. Moreover, in these circumstances, it does not require a critical level of technical progress for maintaining a constant per capita consumption.

In another article, Beckmann (1976) investigated the growth behaviour of an economy plagued by the twin problems of exhaustion and environmental pollution. The environment gets polluted by the resource extraction process. He used a logarithmic utility function which has both consumption goods and environmental products as its arguments. A generalized technology is represented by a Cobb-Douglas production function which uses different kinds of capital and exhaustible resources in the production process. An exogenous technical progress amplifies the level of output as time passes on. The production process involves a one-period lag. He also derived the result that a critical level of technical progress is necessary for stopping the predicament of the society brought about by a depletable resource.<sup>11</sup>

Ingham and Simmons (1975) selected a slightly different ethical criterion. The objective functional contains the undiscounted utility function multiplied by an exponentially growing population. Per capita consumption is the argument of the utility function. Using a C.E.S production function, they discovered that with an elasticity of substitution less than one, per capita consumption will increase throughout the optimal program while the capital stock will increase initially and then fall or this stock will fall continuously to zero. For the Cobb-Douglas case, the results are the same except that capital will be single peaked in this case. They got the striking result that resource depletion may rise initially instead of falling continuously to zero. Ingham and Simmon's (1975) results can be compared with the results

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<sup>11</sup> Another discrete model was analysed by Mitra (1978). He characterised the efficient paths of growth with an exhaustible resource in a neo-classical model. Since we are more interested in optimal growth paths than in efficient growth paths, we skip the discussion of the results obtained in Mitra (1978).

of Dasgupta and Heal (1974). Both the papers work with a C.E.S production function. In Dasgupta and Heal, aggregate consumption falls for the Cobb-Douglas case. Per capita consumption increases under the same case in Ingham and Simmons. The reason may be that in Ingham and Simmons (1975), utility is not discounted as is done in Dasgupta and Heal (1974).

In most of the growth models with a non-replenishable resource, population is either assumed to be constant or is growing at an exogenously given rate. A growing population aggravates the problem caused by the exhaustion of the resource. It seems that there is a critical level of population growth rate beyond which the economy cannot maintain a constant per capita consumption. Mitra (1983) came to this conclusion using both the utility maximising criterion and the Rawlsian maximin criterion of equity.

### MULTI-SECTOR MODELS

So far our discussion has been concerned with growth models of an economy in which only one type of good is produced. Recently, several economists have extended the analysis to allow for additional production sectors in the economy. The assumptions relevant to a one-sector model are accordingly adjusted to suit a two-sector model. Two sectors of the economy, viz., consumption and investment goods sectors, are normally assumed. Each sector is supposed to possess a constant returns to scale production function with three inputs: capital, labour and the resource. The total income of the economy is then given by the sum of the total nominal value of the consumption good and the investment good.<sup>12</sup> A constant fraction of total income is saved. The stock of the resource is depleted for use as inputs in the two sectors. Population grows at a constant rate. Hu (1978) examined the efficient growth paths of a two-sector economy endowed with a finite stock of the resource. He observed that the

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<sup>12</sup> The investment good is regarded as the numeraire good.

ratios; total income to total capital<sup>13</sup> and resource flow to resource stock, will converge to a steady state in the long run. Moreover, he showed that both the consumption good and the investment good per unit of labour will decline over time. The above result is in agreement with the characteristics of efficient growth paths derived in Stiglitz (1974).<sup>14</sup> Using a model similar to Hu (1978), Chiarella (1980) demonstrated that the optimal growth paths of (i) the output to capital ratio of the investment good sector, (ii) the ratio of the capital in the investment good sector to total capital and (iii) the resource flow to stock ratio, will converge to a saddle point equilibrium in the long run. The consumption good sector can experience a positive growth rate if one parametric restriction is satisfied. The restriction is, however, complicated and does not admit of any economic interpretation.

### 2.3 THE EQUITY APPROACH

So far our discussion has concentrated on the implications of exhaustibility of a natural resource for the growth of an economy under the utilitarian approach. As mentioned earlier, the utilitarian approach ignores the question of justice with regard to the distribution of wealth and utility among the members of the society. The equity approach, on the other hand, stipulates that a social policy change is justified only because it improves the position of the poorest individual or individuals in the society. This Rawlsian (1971) criterion of equity is applicable only in the case of intra-generational comparisons. It is difficult to envisage a similar principle of justice for intergenerational comparisons because of the asymmetry involved in the direction of intergenerational transfers. The current generation can leave bequests for the future generations. The future generations cannot, however, transfer

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<sup>13</sup>Total capital is the sum of the capital used in the two sectors

<sup>14</sup>Recall that in Stiglitz (1974), along the efficient paths per capita consumption is falling if  $(\lambda/n) < \gamma$ . Here  $\lambda$ (exogenous technical progress)=0

wealth to the current generation in posteriority when the principle of justice among the generations demands that kind of transfer. Solow contended that a constant per capita consumption for all generations would ensure justice among them.

Solow (1974) was the first economist to introduce the question of equity in a growth model with a non-replenishable resource. He advocated the use of a maximin criterion to determine the maximum level of per capita consumption that people of all generations can enjoy while the cumulative size of the resources used up in the production process does not exceed the size of the initial stock of the resource. Assuming a constant level of population and no exogenous technical progress, he proved that this maximin problem possesses a solution if the share of capital in total output is larger than the share of the resource ( $\alpha > \beta$ ). Hartwick (1977) discovered that with no capital depreciation, per capita output and consumption will remain constant over time if society's investment is just equal to the proceeds from the sale of exhaustible resources. In a subsequent paper, Hartwick (1978) generalized his model to allow for the use of different kinds of the exhaustible resources. Intergenerational equity in the sense of a constant per capita consumption for all the generations can be maintained if all proceeds from the sale of the resources net of extraction costs are invested. A more general model was developed by Dixit, Hammond and Hoel (1980). Hartwick's rule gets changed to "keep the total value of net investment under competitive pricing equal to zero." <sup>15</sup>Strict adherence to the above rule yields constant utility over time.

The papers discussed in the preceding paragraph assume constant population and the absence of technical progress. Okuguchi (1980) relaxed these assumptions and found that intergenerational equity can be achieved whenever the rate of technical progress is not equal to or not greater than (i) the product of the share of capital in total output and the population growth rate or (ii) the product of the share of the resource and the population

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<sup>15</sup>See Dixit, Hammond and Hoel (1980).

growth rate. When the rate of technical progress is equal to the first product, the equilibrium savings rate turns out to be equal to the share of the resource input in total output. In other words, investment then accords to the Hartwick savings rule. The equilibrium obtained here, however, will be unstable in the sense that the actual savings rate will never converge to that particular savings rate. The case of many exhaustible resources is similar.

Chiarella and Okuguchi (1982) were able to synthesize the optimal, maximin and Hartwick growth paths. The optimal paths are those obtained by choosing a consumption level that maximises the discounted utility of per capita consumption over an infinite time horizon. The maximin paths are derived by choosing a savings rate such as to hold per capita consumption constant over time. Hartwick's paths result from investing the proceeds of the exhaustible resource extraction in capital formation. The intergenerational equity path is shown to be the limiting case of the optimal consumption path as the elasticity of marginal utility of per capita consumption, i.e.,  $\sigma$ , tends to infinity. Similarly, the Hartwick path can be obtained as a limiting case of the intergenerational equity path as the rate of technical progress tends to the product of the share of capital in total output and the growth rate of labour.

Dasgupta and Mitra (1983) derived the conditions necessary for the existence of intergenerational equity paths using a general production function with three inputs: capital, labour and the resource. One such condition is that both the capital and the resource inputs must be essential to the production of output. In other words, the economy cannot produce the output without the resource or the capital input. They also demonstrated that in a discrete time model, the competitive conditions, the intergenerational equity conditions and Hartwick's conditions are incompatible with each other when the production function is strictly concave.



## 2.4 ENDOGENOUS TECHNICAL PROGRESS AND EXPLORATION

From the preceding discussion, it seems clear that the impediment to a constant level of consumption brought about by the use of an exhaustible resource in the production can be overcome by the two factors: substitutability and technical progress. The articles reviewed so far assume an exogenous technical progress. The assumption of exogenous technical progress is not, however, realistic because technical progress does not originate out of a vacuum. Some expense must be made in order to achieve technical progress. Some economists have incorporated this phenomenon in growth models with exhaustible resources.

Suzuki (1976) set up a model where technical progress depends on output devoted to research and development efforts. The objective is to minimise the sum of the resource use over an infinite time horizon subject to non-negative investment and to a given rate of growth of total consumption:

$$(2-4-1) \quad \text{Min} \quad \Omega = \int_0^{\infty} R dt$$

subject to

$$(2-4-2) \quad \dot{K} = Y - C_0 e^{\zeta t} - D \quad \zeta = \text{growth rate of aggregate consumption}$$

$$(2-4-3) \quad Y = A^{\lambda} K^{\alpha} R^{\beta} L^{\gamma}$$

$$(2-4-4) \quad \dot{A} = D \quad A = \text{level of technological knowledge}$$

$$(2-4-5) \quad K \geq 0$$

$$(2-4-6) \quad \hat{L} = n$$

$$(2-4-7) \quad \int_0^{\infty} R dt \leq S_0$$

Using the above model, he derived the following necessary and sufficient condition that

guarantees a long run equilibrium for the ratios: output to capital, resource flow to resource stock and consumption to capital.

$$(2-4-8) \quad (\lambda - \beta)n + (\lambda - \beta - \gamma)\xi > 0 \quad \text{provided } n \geq 0.$$

The optimal growth paths tend to a saddle point equilibrium given suitable initial conditions. He also concluded that per capita consumption can be growing if and only if there exists a  $\xi > 0$  such that the restriction (2-4-8) is satisfied. Since the restriction can also be satisfied with  $\xi < 0$ , the above mentioned condition does not provide a clear idea about the optimal growth path of per capita consumption. We can, however, be sure about the possibility of a growing per capita consumption when  $\lambda > \beta$  provided  $n > 0$ . Compare this result with Stiglitz's (1974) result which says that per capita consumption will be growing along the optimal path if the ratio of the rate of exogenous technical progress to the discount rate is greater than the share of resource in total output [ $\lambda/\delta > \beta$ ]. Since  $(\lambda/\delta) > \lambda$  for reasonable discount rates, it implies that as soon as we assume an endogenous technical progress, the condition necessary for obtaining a steady growth in per capita consumption becomes more restrictive.

Output devoted to the research and development sector enhances the stock of knowledge leading to an invention. The invention of a new technology might render the exhaustible resource non-essential to the production of the consumption good. The timing of the new technology, however, may be uncertain. This was the view taken by Dasgupta, Heal and Majumdar (1977) and Kamien and Schwartz (1978). Dasgupta, Heal and Majumdar allowed for a non-linear relationship between the time rate of change of the stock of knowledge and the output expended for research. Thus, they had

$$(2-4-9) \quad \dot{K} = F(K, R) - C - D$$

$$(2-4-10) \quad \dot{S} = -R$$

$$(2-4-11) \quad \dot{A} = \psi(D)$$

The probability that the arrival date of new technology,  $T$ , will occur between the time points "a" and "b" was represented by

$$(2-4-12) \quad \int_a^b g(A(t)) dt$$

where  $g$  is the probability density function.

From  $T$  onwards, the non-replenishable resource ceases to be an essential factor of production. Let  $\Omega$  stand for the maximum level of discounted utility of total consumption that accrues during the time period after  $T$ :

$$(2-4-13) \quad \Omega(K_T, S_T) = \text{Max} \int_T^\infty U(C) e^{-\delta(t-T)} dt$$

The problem is then to maximise the expected value of discounted utility over two segments of the planning horizon:

$$(2-4-14) \quad \text{Max} \quad E \left[ \int_0^T U(C) e^{-\delta t} dt + \Omega e^{-\delta T} \right]$$

subject to equations (2-4-9) through (2-4-11).

Optimal time paths for different key economic variables up to time  $T$  were derived by assuming specific forms for the different functions in the model. The results they obtained for the time interval up to the arrival date of new technology are as follows:

1. Output and consumption diminish continuously over time.
2. Expenditure on R&D falls continuously. The rate at which it falls is higher with a higher discount rate and lower with a more risk averse society.
3. The capital stock increases initially and then falls as time passes on.

Using a model very close to Dasgupta, Heal and Majumdar (1977), Kamien and Schwartz (1978) showed that the expenditure on research and development initially increases to a peak level and then falls. Aggregate consumption falls continuously or grows initially and then declines depending on the satisfaction of a restriction expressed in terms of the production function and the discounting factor,  $\delta$ .

So far the assumption has been that a portion of total output is expended for R&D purposes. Some factor of production might instead be used to finance R&D expenditures. Davison (1978) thought that the level of knowledge depends on the proportion of capital devoted to R&D effort. In other words, total capital is allocated among two uses. A fraction of total capital is used in the material production process, the remainder being spent in the R&D process. Again the arrival date of the new technology is uncertain. He demonstrated that both the consumption and R&D expenditure fall continuously over time. Another example is Robson (1980) who considered a deterministic model with endogenous technical progress similar to Suzuki (1976). Unlike Suzuki, he maximizes the discounted sum of utility over time. The rate of growth of the technology parameter, 'A', in his model, depends proportionally on the fraction of labour devoted to the research sector.<sup>16</sup> That is, he assumed,

$$(2-4-15) \quad \hat{A} = aL_i/L$$

where the remaining labour  $L_q = L - L_i$  is used in the production of the economy's output. Let  $\theta$  stand for the fraction of labour used in the research sector of the economy. He proved that the variables  $\theta$ ,  $X$  and  $V$  converge to a steady state if two restrictions on parameters are satisfied:

$$(2-4-16) \quad a(1-\sigma) < \delta(1-\alpha)$$

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<sup>16</sup> Here "A" signifies Hick's neutral progress embodied in a Cobb-Douglas production technology.

$$(2-4-17) \quad \gamma(1-\alpha)\delta \leq a[1-\alpha-\beta(1-\sigma)]$$

These conditions cannot be compared with those of Suzuki (1976) directly because of the differing treatment of technical progress, but at least they show again that with technical progress, steadily growing per capita consumption is possible.

Chiarella (1980) also worked with a model very close to Suzuki (1976). The only difference is that the time rate of change of technological level is assumed to be an exponential function of, instead of being equal to, output devoted to R&D expenditures. Labour is not a factor of production in Chiarella (1980). Moreover, unlike Suzuki, Chiarella assumed nonincreasing returns to scale with respect to capital and the non-renewable resource. Along the optimal path, the consumption to capital ratio, the output to capital ratio, the resource flow to stock ratio, the research investment to capital ratio and the technological level growth rate fall or rise monotonically to stationary values if some restrictions are satisfied. The restrictions are very complicated and hence the results here can also not be compared directly with the results of Suzuki (1976).

Takayama (1980) considered a model almost identical to Robson (1980). The only difference between the models lies in the set-up of the objective functional. In Takayama, the central planner's objective is to maximise the stream of discounted utility of per capita consumption rather than the discounted utility of total consumption, as in Robson (1980). Moreover, Robson analysed his model for the general class of iso-elastic utility functions. Takayama has  $\sigma=1$ . Thus, in Takayama (1980) the objective is

$$(2-4-18) \quad \text{Max} \quad \Omega = \int_0^{\infty} U(C/L) e^{-\delta t} dt$$

Assuming a logarithmic utility function he proved that on the optimal growth path the ratios; output to capital, consumption to capital, labour used in the production process to total labour and resource use to resource stock, can have positive asymptotic limits if "a" is

positive. Moreover, the model yields a saddle point equilibrium for the first three ratios. Growing per capita consumption is obtained if

$$(2-4-19) \quad a > \beta n + (\gamma - \alpha) \delta$$

Some general observations can be made on the basis of our previous discussion of the papers in sections 2 and 4 of this chapter. All the articles reviewed there except for Suzuki (1976) accept the utilitarian approach as the measuring rod of social welfare. The only difference is that technical progress is assumed to be exogenous in the former and endogenous in the latter. The main thrust of the papers in section 2 is that an economy endowed with an exhaustible resource can survive in the long run if and only if it possesses at the same time a level of technical progress that grows exogenously at a sufficiently high rate. The conditions guaranteeing a constant or steadily growing per capita consumption become more stringent when technical progress is assumed to be endogenous depending on some productive resources. Thus, the prospect of the survival of an economy endowed with a limited supply of an input of production looks less optimistic. Nevertheless, we may get over this pessimistic view if we introduce the more realistic assumption that the stock of the resource can be augmented through exploration process. Our next discussion focuses on the issue of exploration.

### EXPLORATION

Pindyck (1978) analysed a model where exploration is present. The economy is capable of augmenting its stock of the resource by costly exploration. In his model, the stock of the resource is subject to both decumulation and accumulation occurring at the same time. The stock decumulates as it is depleted for the resource flow used as an input in the production function. New discoveries of resource reserves made possible by costly exploration activities add to the existing stock of the resource. It is assumed that more exploratory efforts lead to

more discoveries of new reserves. The additions to the stock of the resource, however, get smaller and smaller as the cumulative size of new discoveries increases. It should be mentioned in this context that we use similar assumptions for our first model developed in third chapter of this thesis. Pindyck's model, however, did not deal with optimal growth in the utility of consumption in the presence of exploration. In his model, the objective was instead to maximise the revenue from the sale of the resources net of extraction costs in a market economy. Pindyck was mainly concerned with the time profile of the price of the resource when the economy can increase the stock of the resource by costly exploration. He showed that the price pattern will be "U"-shaped if the initial reserve endowment is small. Moreover, the exploratory effort will decrease during the end phase of the program.

The fact that the stock of the resource can be increased by costly exploration was incorporated in a growth model with exhaustible resources by Takayama (1980). He assumed that the growth rate of the stock of the resource is linearly related to the proportion of labour used in the exploration process. The model is

$$(2-4-20) \quad Y = K^\alpha R^\beta L_m^\gamma$$

$$(2-4-21) \quad \dot{K} = Y - C$$

$$(2-4-22) \quad L_m + L_e = L$$

$$(2-4-23) \quad \hat{L} = n$$

$$(2-4-24) \quad \dot{S} = -R + Sa(L_e/L)$$

The objective is to maximise the discounted utility of per capita consumption subject to above equations. A steady state solution can be obtained where output, capital and consumption grow at the same rate and the resource growth and utilization rates are equal. A necessary and sufficient condition for growing per capita consumption is

$$(2-4-25) \quad a > (n+\delta)+\gamma(\delta/\beta)$$

Ignoring the optimal allocation of output between investment and consumption and presupposing instead a constant growth rate of consumption, the above condition is changed to

$$(2-4-26) \quad a > n$$

It should be mentioned here that Takayama (1980) was the first author to consider the possibility of augmenting the resource stock by exploration activities in a standard growth model with an exhaustible resource. His model, however, misses some important points about the nature of the exploration technology. The second model in chapter three takes Takayama's model as the starting point and makes improvements on its representation of the exploration technology.

## 2.5 OPEN ECONOMIES

All the growth models discussed earlier represented a closed economy. International trade, however, plays an important role in determining the growth pattern of an economy. An economy may be an exporter or an importer of an exhaustible resource. An exporting country, for example, may use the proceeds from the sale of an exhaustible resource to enhance capital formation in the home country and thus foster economic growth. As soon as we allow for an open economy, there may be no ultimate scarcity of the resource, possibly because the resource can be imported at the going price for ever from abroad. More importantly, the resource based economy can specialise in the production of the resource



intensive commodities at home. It can export the resource intensive commodities to foreign countries and import the non-resource intensive commodities from abroad. Thus, it can reap the benefits of international trade and offset the effects of the resource constraint faced at home. As the resource becomes scarce, prices of the resource based commodities should rise allowing the resource exporting country to maintain its export revenues. On the whole, it can be said that international trade is another means of circumventing the problem of resource scarcity. It is, therefore, imperative that any growth model with non-replenishable resources should also include the foreign sector of the economy. Some authors have incorporated the foreign sector in growth models with exhaustible resources. We now review the papers in this category.

Vousden (1974) investigated the nature of the optimal extraction paths of an exhaustible resource in an open economy. The economy produces a manufacturing good using the resource as the only input of production. Besides being an input of production, the resource can be used for two other purposes. It can be exported or imported at a given price expressed in terms of the manufacturing good. It also serves as one of the consumption goods of the economy. There is an upper limit on the amount of the resource that can be extracted at any point of time. Consumption of both the manufacturing and the resource goods can be increased (decreased) by imports (exports). A commodity cannot be imported and exported simultaneously. The country's international payments are always balanced. The objective is to maximise the discounted utility of consumption over a finite time horizon. Utility depends on the total consumption of both the resource and the manufacturing good. The analysis falls into two cases. First, the world price of the resource is less than the maximum but greater than the minimum domestic price.<sup>17</sup> There is no difference between the

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<sup>17</sup> Suppose  $J=\psi(R)$  represents the product transformation curve of the economy where the resource good is used to produce the manufacturing commodity. Let  $P$  denote the world price of the resource expressed in terms of the manufacturing good. Let  $E$  stand for the maximum amount of the resource that can be extracted at any point of time. The first case is then denoted by  $-\psi'(0) < P < -\psi'(E)$

static and optimal levels of extraction of the resource if the planning period is short and the initial endowment of the resource is large.<sup>18</sup> In the opposite situation of a longer planning period and a smaller initial endowment, the optimal level of extraction will be less than the static level and will decline continuously over time until exhaustion. Moreover, the optimal policy sequences are shown to depend on the relative price of the resource and the properties of the utility function. In the second case where the world price of the resource is less than the minimum domestic price, the optimal action is to specialize in the production of the manufacturing good.

Kemp and Suzuki (1975) also used the criterion of maximising the discounted utility of consumption to find the nature of specialisation in an open economy with a depletable resource. The assumptions of the model are as follows. There are two commodities, the consumption good and the raw material. Labour (l) and raw material (v) are used to produce the consumption good. The raw material is produced with the help of labour and the resource. Home consumption can be supplemented by consumption of foreign goods made possible by selling a portion of the raw material at the price P, in terms of the consumption good. Labour is assumed to be constant and is normalized to 1. The model gives rise to total specialisation in the production of either commodity and after some time, only the consumption good is produced. The resource is imported from abroad when the economy specialises in the production of the consumption good. A portion of total stock of the resource is left unextracted at the end of the program. They extended the model to allow for artificial and natural replenishments. The time rate of change of the stock of the resource looks as follows:

$$(2-5-1) \quad \dot{S} = -l_2 g(S) + \psi(S) + \theta(1 - l_1 - l_2)$$

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<sup>18</sup>The static level of extraction is determined from a static optimisation problem where the objective is to maximise the total earnings from the resource at a point in time. Total earnings is the sum of total revenue obtained from the sale of the resource and the amount of the manufacturing good produced with the help of the resource.

$$(2-5-2) \quad c = f(l_1, v) + p[l_2 g(S) - v]$$

$\psi(S)$  = natural replenishment       $\theta(\cdot)$  = artificial replenishment

$l_1$  and  $l_2$  stand for labour used in the consumption good and raw material producing industries respectively.

Consider two cases in this extended model. First,  $g(S) > \psi(S)$  for all  $S > 0$ . In this case, the optimal path converges to a stationary point with incomplete specialisation. The raw material in combination with either the consumption good or the resource good (but not both) is produced. Second, consider the case when all labour is devoted to the production of the raw material. The stock of the resource still grows because of the natural replenishment. It may be optimal to specialise in the production of the raw material now. But in all cases, there will not be complete specialisation in the production of the consumption good at the stationary point.

Later authors have considered economies where capital is also a factor of production. Suppose such a country possesses a stock of the resource and a constant labour force. It produces a raw material using the inputs of labour, capital and the resource. The raw material is sold abroad at a world price. The country purchases consumption goods and capital from the world market. The country can borrow and lend at a given interest rate. Capital is needed for the extraction of the resource. The objective is, as usual, to maximise the stream of discounted utility of consumption over an infinite time horizon. The problem can be solved in two different ways. First, it can be tackled in a two-steps procedure. Initially, the country maximises the present value of the mine by choosing an optimal path of borrowing capital at the given interest rate. In the second stage, the country can choose the optimal paths of consumption and amortization so that the discounted utility of consumption is maximised. The second solution for the country is to sell the mine initially and use the proceeds from it to finance consumption over time. Kemp (1976) demonstrated

that the optimal paths of consumption and extraction would be identical under the two methods. If the initial extent of the resource is unknown, it will be optimal for a risk-averse country to sell the mine at the outset. Kemp, however, did not have much to say about the nature of the optimal growth path.

Aarrestad (1978) examined the optimal path of extraction of the exhaustible resource and the optimal savings rate when an economy sells the resource abroad but does not use it in the home production function. The model in per capita terms is as follows:

$$(2-5-3) \quad \text{Max} \quad \Omega = \int_0^{\infty} U(c) e^{-\delta t} dt$$

subject to

$$(2-5-4) \quad c = (1-b)[f(k)+\pi]+c^0$$

$$(2-5-5) \quad \dot{k} = b[f(k)+\pi]-(r+n)k$$

$$(2-5-6) \quad \pi = pr-\Psi(r)$$

$$(2-5-7) \quad -\dot{s} = r+ns$$

$$(2-5-8) \quad 0 \leq b \leq 1$$

$$(2-5-9) \quad 0 \leq r \leq r^0$$

$$(2-5-10) \quad \delta, n, \tau, c^0 \text{ and } r^0 \text{ are exogenously given.}$$

Here  $b$  = the savings rate,  $\pi$  = profit,  $P$  = price of the resource exported,  $\psi$  = cost of extraction, and  $\tau$  = the depreciation rate.

The objective functional here is maximised with respect to  $r$ ,  $c$  and  $b$ . It should be noted in this context that Aarrestad's model is different from Stiglitz's (1974) model. In Aarrestad's

model, the resource is not used as an input of production of the consumption good. Thus, Aarrestad's results are very similar to the those in the single sector neoclassical model. On the basis of the above model, he obtained the following main conclusions:

1. Along the optimal path, per capita consumption will be growing, falling or remain constant when per capita capital stock is less than, more than or equal to the modified golden rule capital intensity,  $k^*$  where

$$f'(k^*) = \delta + r + n$$

2. Extraction may remain constant initially and then fall or may fall always when the price of the resource is constant and per capita capital stock is less than or equal to modified golden rule level. Extraction may be increasing over time and it may be optimal to leave a part of the resource unextracted when the price of the resource increases exponentially over time.
3. The savings rate will fall if there is a new discovery of the resource.

At the same time as Aarrestad (1978), Dasgupta, Eastwood and Heal (1978) developed a simple model where the resource has the two fold uses of being an input in the home production process and at the same time being an exportable good. Their model is

$$(2-5-11) \quad \text{Max} \quad \Omega = \int_0^{\infty} U(c) e^{-\delta t} dt$$

subject to

$$(2-5-12) \quad \dot{S} = -[R(t) + E(t)]$$

$$(2-5-13) \quad \dot{W} = F(K, R, t) + r(W - K) + P(E)E - C$$

where  $W$  is total wealth,  $F$  is a production function which also embodies exogenous technical progress and  $K$  is home capital.  $W - K$  is thus foreign assets.  $r$  is the rate of return on these foreign assets.  $E$  is the exported resource.  $P(E)$  is the inverse foreign demand function

for the home resource.

It should be noted here that in the DEH (1978) model, capital plays the role of a control variable. In other words, capital is a flow variable here. In this model,  $\dot{W}$  equation replaces the normal  $\dot{K}$  equation. As such, the objective functional here is maximised with respect to  $C$ ,  $R$ ,  $K$  and  $E$ . Thus, equation (2-5-13) here says that the time rate of change of an economy's total wealth is the sum of total output produced at home, interest earnings from foreign assets, total revenue from the sale of the resource abroad, all net of consumption.

Explicit solutions for key economic variables were obtained by assuming the specific forms for utility, production and foreign demand functions which are shown below:

$$(2-5-14) \quad -U''(C)C/U'(C) = \sigma$$

$$(2-5-15) \quad P(E) = \xi E^{\xi-1}$$

$$(2-5-16) \quad F(K, R, t) = e^{\lambda t} K^{\alpha} R^{\beta}$$

It is found that the two uses of the resource fall over time. The time rate of change of capital stock of the economy satisfies

$$(2-5-17) \quad \dot{K} \geq 0 \text{ iff } \lambda \geq \beta r$$

Consumption will be increasing, constant or decreasing as long as the rate of interest on foreign assets is larger than, equal to or smaller than the discount rate. As we know, in Stiglitz (1974), per capita consumption falls in the absence of exogenous technical progress. Here we find that aggregate consumption falls only when the discount rate is high. In other words, aggregate consumption can grow even when there is no technical progress. Thus, the DEH model finds another factor that may offset the effects of increasing scarcity of a

resource.

Working with a slightly different model, Kemp and Okuguchi (1979) obtained similar results. They assumed that all the capital necessary in home production is borrowed from abroad and that a part of total output is spent on discharging foreign debts. In another section of the same paper, they assumed that there is no home production of the consumption good. The extraction of the resource, however, needs capital and the resource itself. The net extracted resource is used for multiple purposes: consumption, purchase of foreign assets and discharge of foreign debts. The time rate of change of total assets is given by

$$(2-5-18) \quad \dot{W} = R(K, e)P(R) + rW - C - iK$$

where  $C$  is the consumption of the resource and  $e$  is the gross total amount of the resource extracted at any point of time. Here  $i$  is the interest payment on borrowed capital. The difference between  $R$  and  $e$  is the amount of the resource lost in the extraction process. The conclusions of the model are as follows. The depletion rate of the resource always falls. The capital stock will be growing if  $R_{ek} \leq 0$ <sup>19</sup> and the price of the resource is constant. With a Cobb-Douglas production function and a constant elasticity of foreign demand, the capital stock is always falling. When the initial stock of the resource is not known with certainty, the depletion of the resource may not fall monotonically.

Moussavian (1985) added one non-traded sector to the DEH (1978) model. The demand and supply of the non-traded good sector are assumed to be in equilibrium at all times. In the DEH model, home production of the consumption good falls over time if the rate of exogenous technical progress is not sufficiently large. Moussavian, however, shows that the presence of a non-traded sector prevents the economy from collapsing since the value and quantity of the non-traded good will increase over time unless the discount rate is very

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<sup>19</sup>  $R_{ek} = (\partial/\partial e)(\partial R/\partial k)$

large. The presence of a non-traded sector dampens the effects of an increase in the world interest rate on the domestic economy. The non-traded sector works as a destabilizer by magnifying the domestic repercussions of a change in the world interest rate when labour is a factor of production in both the sectors. On the other hand, if the price of the resource is growing, then there is a pro-industrialization effect, contrary to the conclusions of the Dutch-Disease theory.

Before we review the remaining articles in this section, it is worth-while mentioning a few other points about the DEH model. The DEH model is more interesting than any other model discussed so far in this section. There are at least two reasons for this. First, the DEH model considers explicitly the two uses of an exhaustible resource in an open economy viz., as an input of production and as an exportable commodity. More importantly, it does so using a very simple model. By contrast, in Kemp and Suzuki (1975), the resource is used only indirectly in producing the consumption good. Moreover, investment in capital formation is absent in the Kemp and Suzuki model. Second, the DEH model recognizes the earnings from foreign assets as an additional source of income. The Moussavian model (1985) is an improvement over the DEH model. In this thesis, we rework the DEH model and the Moussavian model. The objective is to examine the robustness of the main conclusions obtained from these models to some minor changes in the basic assumptions of these models.

Aarrestad (1979) examined the optimal paths of borrowing and extraction of the resource using a model slightly different from his previous one. He found that along the optimal path, per capita consumption will be growing (falling) as long as the real rate of return on foreign assets is higher (lower) than the discount rate plus the rate of growth of population. When a borrowing restriction<sup>20</sup> applies, per capita consumption is increasing (decreasing) according as to whether the real rate of return from keeping the resource in the ground is higher (lower) than the discount rate plus the population growth rate.

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<sup>20</sup>Aarrestad assumed that there is a limit on the amount an economy can borrow.



So far we have discussed the characteristics of growth paths in open economies with an exhaustible resource. Not all the countries of the world are equally rich in resources. It is also interesting to examine the interaction between a resource poor and a resource rich country. Consider two countries. The first country is rich in the resource but lacks the technological know-how necessary for the production of the consumption good. It only sells the resource to the resource poor country which, on the other hand, possesses all the technical know-how for producing the consumption good. Both the countries seek to maximise the sum of discounted utility of consumption over an infinite time horizon. The resource rich country's consumption is financed by the sale proceeds of the resource. The resource poor country's consumption is equal to the output it produces minus a portion of that output it expends to buy the resource. The other two factors of production are assumed constant and are, therefore, suppressed. Kemp and Long (1979) utilized the above mentioned model to arrive at the following conclusions:

1. When both the countries are price takers, consumption of the second country and the price of the resource in terms of consumption good move in the opposite directions over time. The consumption of the first country may be increasing or decreasing.
2. When either of the countries is aggressive, the optimal consumption profile cannot be specified precisely.

Chiarella (1979) extended the model of Kemp and Long (1979) to allow the variables capital and labour to be two other factors of production. He demonstrated for the second country that the ratios of consumption to its aggregate capital stock, aggregate output to aggregate capital stock and resource use to resource stock converge to a stationary state with suitable initial conditions.

A very similar model was considered by Kemp and Long (1982). Suppose a country is totally dependent on her trading partner for the supply of a resource that is fed immediately into the production of the consumption good. The price of the resource is growing

exponentially at a rate equal to the interest rate. The importing country has a Cobb-Douglas production function possessing a resource augmenting technical progress. International trade is always balanced. Kemp and Long showed that the importing country can maintain a positive and constant level of consumption if the rate of technical progress is at least as large as the growth rate of the resource price. The same conclusion holds even if there are returns to scale in the production function.

We can compare the results of the interaction models as discussed above with the results of the single-country open economy models. In case of the interaction models, the optimal per capita consumption may not be increasing for both the countries at the same time. In the single-country open economy models, normally it is found that the economy may experience a growing consumption even when there is no technical progress.

Before we leave this section, it is better to make an overall review of the papers discussed here. By contrast with the articles discussed under the utilitarian approach, the articles in this section address a more diversified range of issues. Some papers are concerned with the degree of specialisation in the production of the resource intensive commodities. Vousden (1974), and Kemp and Suzuki (1975) fall in this category. Aarrestad considers two models where the resource is not used in the production process. Only DEH (1978) and Moussavian (1985) examine the issues of survival of an economy directly. DEH show that home production of the consumption good may come to an end very soon when the rate of exogenous technical progress is not sufficiently large. Moussavian, on the other hand, proved that the presence of a non-traded sector in the economy may prolong the process of the extinction of home production of the consumption good. Few articles here address the question of maintaining a steadily growing per capita consumption in the long run. A common thread of argument, however, follows through all the papers discussed in this section. It is that an open economy has more means of circumventing the problem of resource scarcity faced at home than a closed economy has.

## 2.6 WITH UNCERTAINTY

Most of the articles reviewed so far assume that the world we face now and in the future is certain in all respects. However, the real world is full of uncertainties. For example, the initial stock of the resource may not be known exactly or the demand for the resource in future may be uncertain, etc. In recent years, researchers of growth theory have started to incorporate different elements of uncertainty in their models.

We now turn to the discussion of the main articles published in this branch of the literature on growth with exhaustible resources. It should, however, be mentioned here that all of these articles consider only the utility generated by consuming the resource. In other words, there is no capital stock or technology to produce a consumer good as in the optimal growth models discussed earlier. All the papers discussed in this section are concerned with the generic problem of eating a cake under uncertainty. Under certainty, the solution of the cake eating problem is easily perceived: consumption decays continuously to zero.

Dasgupta and Heal (1974) set up a model where the arrival date of a new technology or a perfect substitute is uncertain. The discovery of a perfect substitute renders the existing stock of the resource and capital useless economically. The effect of uncertainty about the timing of the new technology is shown to be equivalent to discounting future consumption at a higher rate. The amount added to the discount rate is equal to the conditional probability that the substitute will arrive at a particular point of time given it has not occurred already. In this model, aggregate consumption will increase initially and then begin to fall afterwards. Then there is a jump in the consumption level at the advent of the new technology at time  $T$ . From  $T$  onwards, it will increase asymptotically to a stationary level.

Hoel (1979) took an alternative view of the uncertainty. He postulated that the arrival date of the new technology is known exactly. The cost of perfect substitute, on the other

hand, is not known until the time it becomes available. The social optimisation problem is to maximise the expected value of the stream of discounted utility net of the costs of the substitute. Utility, of course, depends on the consumption of the resource initially and then the substitute afterwards. He demonstrated that the optimal path of extraction will be identical in the cases with and without uncertainty about cost if the total stock is exhausted at time T, the same time at which the substitute becomes available. It is also shown that risk aversion reduces the optimal level of extraction but increases the level of competitive extraction before T if some resource is left unextracted at that time.

Now suppose that the initial stock of the resource is uncertain. Society attaches a subjective probability to each conceivable size of the stock. It wishes to formulate an optimal policy with the presumption that consumption will be nil when exhaustion occurs. Kemp (1977) showed that under these circumstances, the consumption of an exhaustible resource may not fall monotonically. With no discounting, the extraction of the resource might be increasing always if the size of the initial stock of the resource is unknown.

Loury (1978) analysed a model with the same assumption about initial stock of the resource. The total initial reserve is a random variable with a given probability distribution at the initial date. The problem is to maximise the expected value of the discounted utility of consumption. A stochastic analogue of the golden rule of consumption is obtained. Consumption will be growing if the expected marginal rate of return to deferred consumption is larger than the discount rate and vice versa. The expected marginal rate of return is the marginal rate of return adjusted for the probability of exhaustion. When the probability of exhaustion is very high, people appreciate that the return from deferred consumption will be very high in future. This will induce them to follow an increasing consumption path.

Gilbert (1979) also assumed that the initial stock of the resource is unknown. People consume the resource as it is extracted from the ore. No information about the stock is

available except that the initial endowment cannot be less than what has already been extracted. Gilbert then demonstrated that uncertainty about the stock of the resource leads to a more conservative extraction policy in comparison with the case where there is no such uncertainty.

Arrow and Chang (1980) used a slightly different model in order to explain why the mineral prices fail to rise at the market rate of interest. They assumed that natural resources are randomly dispersed among different spatial locations according to a Poisson distribution. The objective is to maximise the sum of discounted utility over an infinite time horizon. The utility function is separable in the utility of consumption and the costs of exploration.

Deshmukh and Pliska (1980) considered a model where uncertainty prevails with respect to both the timing and magnitude of new discoveries of the resource. Additions to the stock of the resource can be brought about by costly exploration activities. Both the levels of consumption and exploration effort depend on the level of proven reserves. The degree of uncertainty about the returns to exploration activities, however, is controllable. More output expended for exploration leads to a more prompt discovery of a larger amount of reserves. The yields from exploration are assumed not to depend on the remaining stock of the resource. Production is absent in the model. Consumption is equal to the extracted resource. The central planner's problem is to select consumption and exploration rates so as to maximise the discounted utility of consumption minus the costs of exploration. The social benefit function is separable in the utility of consumption and exploration costs. They then apply a dynamic programming technique of optimisation to the expected return function which is

$$(2-6-1) \quad V_{\pi}(G) = E_{\pi} \left[ \int_0^{\infty} e^{-\delta t} (U(c(G)) - h(e(G))) dt \mid G_0 = G \right] \quad G \geq 0$$

where  $G$  is the level of proven reserves and  $h(\dots)$  is the cost of exploration function.

The resource on hand is likened to a store the size of which is controlled by the consumption and exploration policies. They showed that along the optimal path, consumption is positive and non-decreasing while exploration is non-increasing with respect to the level of proven reserves. Later, Deshmukh and Pliska (1983) extended their previous model to incorporate environmental uncertainty. Environment uncertainty affects both the supply and demand for the resource. For example, due to environmental shifts, fewer and smaller discoveries including exhaustion might take place in future. On other hand, the costs of extraction might fall abruptly due to advances in exploration technology. On the demand side, the development of new substitutes for the resource might affect the future demand for it. Accordingly, the return function is modified to include a reward function which depends on the environment, the level of proven reserves and the level of consumption. The conclusions are identical to those in their previous model.

A common feature of the papers discussed in this section is that they ignore production of the consumption good. The resource is consumed as it is extracted from the ore. This thesis develops an uncertainty model which incorporates the production sector of the economy. Output is produced with the help of two inputs, capital and the resource. The objective is to maximise the expected discounted utility of consumption over time. The model examines the effects of uncertainty with regard to the yields from exploration on consumption, resource extraction and exploratory effort.

## 2.7 CONCLUSIONS

In this chapter, we have reviewed the works published in the field of optimal growth with an exhaustible resource. It seems clear that exhaustibility of natural resources is not always a threat to the continued growth of an economy. In most cases, we met the conclusion that a constant level of total or per capita consumption may be maintained under certain restrictions. Substitution and technical progress can adequately make up for the dwindling stock of the resource. In open economies, earnings from exports or holdings of foreign assets may alleviate the problem of increasing resource scarcity.

This thesis tries to fill up some important gaps in the literature. First, the question of growth with an exhaustible resource has not been considered seriously when the stock of the resource can be augmented through exploratory efforts. As we mentioned earlier, Takayama (1980) was the first author to set up a utilitarian optimal growth model where exploration is present. There were other writers who also considered the possibility of increasing the stock of the resource by exploration activities. For example, Pindyck (1978), Deshmukh and Pliska (1978,1983), Arrow and Chang (1980), etc. have developed models which incorporate exploration efforts leading to new discoveries of resource reserves. None of these models, however, can be considered complete. Either production is ignored or some important issues regarding the exploration technology are missing in each of these models. The problem with Takayama's model is that the yields from exploration are assumed to be linearly and positively related to the current stock of the resource. However, common sense dictates that the returns to exploration activities should be related to the cumulative discoveries of the resource and not to current remaining stock. Moreover, these returns should fall as the level of cumulative discoveries increases.

In the next chapter, we incorporate this type of relationship between the returns to exploration activities and the cumulative discoveries in some simple optimal growth models.

We discover that in most cases, when the economy can augment its stock of the resource by costly exploration, optimal per capita consumption will increase indefinitely even in the absence of exogenous technical progress. In particular, this happens when some output or a part of capital is expended for exploration purposes. By contrast, per capita consumption can increase only with a sufficiently large rate of technical progress when exploration is absent. In a third model, however, when a portion of labour is used in exploration, we also find that a sufficiently large rate of technical progress is needed to obtain a non-decreasing per capita consumption. It is strange that the minimum rate required in this model is actually higher than the minimum rate when there is no exploration. Another model was set up where it was assumed that the additions to the stock of the resource depend positively on the existing stock of the resource. Though such an assumption is contrary to our assertion about the exploration technology, the model was devised as a means of correcting for one unavoidable technical problem that beset the earlier models. The model yields balanced growth paths for consumption, capital stock and total output in some circumstances. Once again we find that per capita consumption can increase indefinitely in the absence of technical progress.

The papers that consider different kinds of uncertainty have the defect of ignoring the production of the consumption good. In chapter 4, we devise a model to incorporate uncertainty with regard to the returns to exploration activities. The model, however, does not ignore the production of the consumption good. We find that uncertainty with regard to the returns to exploration activities leads to higher expected rates of change of consumption, resource extraction and exploratory effort.

Finally in chapter 5 of this thesis, we present a modified version of the DEH model. We assume away foreign assets in the DEH model. Instead, we assume that the economy imports a foreign commodity to supplement its consumption of the home produced commodity. In the Moussavian model, the rate of return is assumed to be exogenous. We assume that the rate of return on foreign assets depends negatively on the quantity of the foreign assets.



Our purpose in chapter 5 is to examine the implications of the changed assumptions in these models. It is demonstrated that the possibility of gradual extinction of home production of the consumption good is significantly reduced under the altered assumptions of both the models. Moreover, unlike DEH, we are able to show the existence of a saddle point equilibrium in the modified DEH model. We prove that, under some circumstances, the optimal growth path tends to Stiglitz's (1974) equilibrium in the long run.

Although the theoretical literature abounds in proofs showing that an economy endowed with an exhaustible resource can survive in the long run, the literature has some limitations. First, the issue has not been dealt with comprehensively when different kinds of uncertainties are encountered. The reason is obviously technical. Second, the conditions necessary for a constant or growing level of consumption may not be satisfied in the real world. Sometimes, the restrictions are too vague to admit of any economic interpretation. Third, in the process of reviewing the literature, we came across with two broad types of articles. Most of the authors only derive the conditions for the optimal growth paths without analysing their asymptotic behaviour. Few papers endeavor to prove the existence of a saddle point equilibrium in terms of some economic variables or their ratios. The implication of a saddle point equilibrium is that with suitable initial conditions, the optimal paths of the economy will tend to the equilibrium. However, the initial conditions may not be suitable to start with.

In addition to the above mentioned limitations, the current literature has two other shortcomings. First, the criterion of convergence for a differential equation system has not been used very carefully. In the literature, it is normally argued that the optimal growth path will converge to the equilibrium point whenever the equilibrium turns out to be a saddle point. The only requirement for convergence with a saddle point equilibrium is that the initial conditions be suitable. Using Stiglitz's (1974) model in this chapter and some models developed in chapter 3 of this thesis, we show that this may not be true for higher

dimensional models. In a 3 by 3 differential equations system, it is really difficult to conclude that the optimal growth path will converge to the equilibrium point even if the equilibrium is a saddle point. Second, most of the works published in this branch of growth literature make little use of the transversality conditions that must be satisfied in an optimisation problem. In this thesis, we make use of the transversality conditions associated with different state variables while characterising the optimal growth path of the economy.

Finally, the theoretical conclusions obtained in the literature have not been verified empirically. The reason is that, most often, the theoretical model envisaged deviates very much from the real world it wants to explain. As illustrated earlier, the literature depends heavily on the mathematical tools of the optimal control theory and the calculus of variations. The abstract nature and technical difficulties associated with these mathematical tools limit their applications to the real world phenomena. More importantly, we do not know as yet exactly whether a market economy generates the same growth paths as those produced by an optimal growth model. But despite all its limitations as noted above, it can be concluded that the literature on growth with exhaustible resources has successfully refuted the "Doomsday" premises advocated by the "Club of Rome" studies.

## CHAPTER 3

### GROWTH IN THE PRESENCE OF COSTLY EXPLORATION OF RESOURCES

#### 3.1 INTRODUCTION

In this chapter, we study the optimal growth paths of an economy capable of increasing its stock of the resource through costly exploration. There is one reason why a theoretical analysis of the optimal growth paths in the presence of exploration needs to be pursued. First, though the assumption of a fixed amount of initial reserves reflects an acute problem of scarcity and hence makes an interesting economic problem, it does come short of the real world phenomenon. It cannot explain where the initial reserves came from in the first place. Even at the latter stages of exploration process, the stock of the resource may not be necessarily constant. It is more realistic to assume that exploration leading to the discovery of new resource reserves is possible.

It seems that we face an apparent dilemma here. If the stock of the resource can be increased by means of exploration activities, then the resource may not be scarce any more. However, this is not the real fact. Some expenses in the form of output or factors of production must be made for exploration purposes. The costs of exploration may more than outweigh the benefits accruing from it. It is possible that the economy may not benefit much from exploration if the costs of exploration are high. It is, therefore, interesting to investigate the characteristics of the optimal growth paths of such an economy under different assumptions about the costs of exploration. These assumptions are embedded in the functional forms used to represent the different types of exploration technologies.

We mentioned in the introductory chapter that a possible characterization of the exploration technology must consider two facts separately or in combination. First, more exploration efforts should lead to more new discoveries. Second, the returns to exploration

activities should be negatively related to the size of cumulative discoveries. Most of the models discussed below ignore these two facts. In the previous literature, only Pindyck (1978) related new discoveries to the size of cumulative discoveries. He was not, however, interested in characterising the optimal growth path for the economy as is done here.

This chapter can also be regarded as an extension of the works made by Pindyck (1978), Takayama (1980) and Deshmukh and Pliska (1980,1983). They considered the possibility of augmenting the stock of the resource by exploration activities. Pindyck was mainly concerned with the nature of time paths of the price of the resource in such an environment. Takayama derived a growth model where exploration is present. He assumed a proportional relationship between the growth rate of the resource and the proportion of labour used for exploration purposes. Deshmukh and Pliska analysed a growth model where the returns to exploration activities are uncertain. Production is absent in their model. People consume the resource as it is extracted from the ore. None of the above models, however, is complete. Each model either ignores the production of the consumption good or misses some important issues regarding the exploration technology.

The plan of this chapter is as follows. The optimal growth paths in the presence of exploratory activities are examined in the next four sections each applying a different functional form for the exploration technology. The first three models assume that the yields from exploration decline as the cumulative discoveries increase. The difference in three models is in the input required by the exploration technology. In the second section, this input is a portion of the economy's output. In the third and fourth section, it is a portion of the economy's stock of labour and stock of capital respectively. The final model of this chapter modifies the first model to make the exploration yields depend on the existing stock of the resource rather than the cumulative discoveries. The sixth section of this chapter then makes a comparative analysis of the results of this chapter and those found in Takayama (1980), Arrow and Chang (1980) and Deshmukh and Pliska (1980,1983). The final section gives the

conclusions obtained in this chapter.

It is evident that most of the assumptions pertaining to the models that we will study next will remain identical in all the sections. The equation representing the time rate of change of the stock of the resource will, however, differ among the sections. The common assumptions are discussed next.

The central planner's objective is assumed to be the maximisation of the sum of discounted utility of per capita consumption accumulated over an infinite time horizon. A homogenous output is produced with the help of three inputs: capital, the resource and labour. The production technology is characterised by a constant returns to scale Cobb-Douglas production function. The production function embodies exogenous technical progress. Labour is assumed to grow exponentially at a given rate. At any point of time, total output is allocated among three uses: consumption, investment and exploration. The assumption is a little bit different in sections 3 and 4. The stock of the resource is depleted and used in the material production process. The stock of the resource also grows due to exploration activities. In all cases, the utility function is assumed to be iso-elastic with positive but diminishing marginal utility. On the whole, these models are extensions of Stiglitz (1974). The extensions are made in the equation representing the time rate of change of the stock of the resource.

### 3.2 OUTPUT REQUIRED FOR EXPLORATION

Here we assume that the yields from exploration depend on the level of output used for exploration and cumulative discoveries. Cumulative discoveries is the sum of the resources used up so far in the production process and the "proven stock" of the resource currently

lying in the ground. It is not true that nature abounds with an infinite amount of the resource available at constant cost. The problem of scarcity would not have been the way of human life if it did so. In reality, additions to the stock of a resource brought about by exploration diminish with increases in the size of cumulative discoveries. In other words, the costs of exploration increase with the increase in the size of cumulative discoveries. There is one reason for rising exploration costs. The earth's surface abounds in different types of natural resources. That means, resources of different grades are available to an economy. Natural resources are extracted in order of their quality with the high grade resources being extracted before the low grade resources. Normally, the crude resource found in the ore cannot be used in the production process instantaneously. Some expenses have to be made in the process of working up the resources to a standard quality. In all our models, we assume that the resource in the production process,  $R$ , is of the same quality regardless of the kind of resource reserve it is extracted from. Consequently, the costs of extraction and refinement of the resource increase as resources of lower and lower grades are extracted. That means, the costs of exploration and development also increase as more reserves of the resource are developed. It is also obvious that more output used for exploration brings in more new discoveries of resource reserves. We assume as well diminishing returns to output used for exploration.

The specification of the exploration technology mentioned in the preceding paragraph has two shortcomings. First, the sufficient conditions for dynamic maximization of an objective functional requires that the Hamiltonian of an optimization problem should be concave in all the state variables.<sup>1</sup> Capital, the stock of the resource and the cumulative level of resource used in the production process play the role of state variables in the current model. We work with a multiplicatively separable exploration technology for ease of manipulations. We

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<sup>1</sup>There is another sufficient condition. It is that the transversality conditions associated with different state variables must be satisfied. We will discuss the transversality conditions later. See Arrow and Kurz (1970).

also assume a negative relationship between the returns to exploration activities and the size of cumulative discoveries. The exploration technology should also be such as to make the Hamiltonian concave in all the state variables. Our model, however, becomes complicated analytically if the Hamiltonian is to be concave in the last two state variables notwithstanding all other assumptions about the exploration technology. This is a technical problem and no tractable alternative could be found which satisfies the above sufficiency condition. Perhaps, this is the reason why Pindyck (1978) suppressed the question of sufficient conditions in his paper. He estimated an exploration function, for oil in the Permian region of Texas over the period 1965-1974, which also violates the sufficient condition.<sup>2</sup> The problem is that the dynamic maximisation problem may or may not possess an optimal solution when the sufficient condition is not satisfied. There are two ways of alleviating this problem. First, we can obtain numerical solutions and thus check for the existence of a local optimum. Alternatively, we can investigate whether some kind of feasible, if not optimal, solution exists within the framework of such an exploration technology. We show in the appendix that a feasible solution exists when a constant level of per capita output is used for exploration purposes. The requirement that the Hamiltonian be concave in all the state variables is indeed a very strong condition to be satisfied. We will assume here that our problems have solutions provided the Hamiltonian is a concave function of the control variables only.

The second shortcoming of the exploration technology as envisaged here is that it allows the size of cumulative discoveries to be infinite. Our exploration technology only assumes a negative relationship between the returns to exploration activities and the size of cumulative discoveries. It does not, however, restrict the size of the stock of the resource eventually available to the economy to be finite. The more plausible assumption is that the ultimate size of cumulative discoveries is finite. However, it is difficult to envisage an

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<sup>2</sup>Pindyck (1978) used the following exploration function:

$$f(G, D) = AD^\alpha e^{-\beta G}$$

where D is exploratory effort and G is cumulative discoveries.

exploration technology which sets a ceiling on the size of the stock of the resource eventually available to an economy without disturbing the other assumptions about the exploration technology. Again, the matter becomes a technical problem and no better alternative could be devised to tackle it. Moreover, all the previous writers who incorporated the phenomenon of exploration in their models also allowed the ultimate stock of the resource to be infinite. For example, Pindyck (1978) thinks that the potential reserves of resources are unlimited. Similarly, Deshmukh and Pliska (1980) assume that any amount of resources can be discovered eventually. Takayama (1980) also designs a model where the stock of the resource can grow indefinitely. Pindyck (1978) coined the word "non-renewable" for this kind of resource rather than calling it "exhaustible". In short, our model reflects an expanding economy though the rate of expansion decreases with the increase in the size of cumulative discoveries. Next, we present the first model and the necessary conditions for maximisation that follow from it.

### 3.2.1 The Model and Necessary Conditions

The central planner's objective is to

$$(3-2-1) \quad \text{Max} \quad \Omega = \int_0^{\infty} e^{-\delta t} U(C/L) dt$$

subject to

$$(3-2-2) \quad \dot{K} = Y - C - D$$

$$(3-2-3) \quad \dot{B} = R$$

$$(3-2-4) \quad \dot{S} = -R + a(S+B)^{\epsilon} D^{\eta} \quad \epsilon < 0, \quad 0 < \eta < 1$$

$$(3-2-5) \quad Y = e^{\lambda t} K^{\alpha} R^{\beta} L^{\gamma} \quad \alpha + \beta + \gamma = 1$$



$$(3-2-6) \quad \hat{L} = n$$

where

D = output used for exploration,

B = cumulative use of the resource in the production process and the other variables are as defined in the previous chapter.

The problem can be rephrased in per capita terms as follows:

$$(3-2-1) \quad \text{Max} \quad \Omega = \int_0^{\infty} e^{-\delta t} U(c) dt$$

subject to

$$(3-2-7) \quad \dot{k} = y - c - nk - d$$

$$(3-2-8) \quad \dot{b} = r - bn$$

$$(3-2-9) \quad \dot{s} = -r - sn + a(s+b)^\epsilon d^\eta L^{\epsilon+\eta-1}$$

$$(3-2-10) \quad y = e^{\lambda t} k^\alpha r^\beta$$

where<sup>3</sup>

$$(3-2-11) \quad d = (D/L) \text{ and } b = (B/L).$$

The maximisation is with respect to c, r and d. The current value Hamiltonian is then

$$H = U(c) + \mu_1 [y - c - nk - d] + \mu_2 [-r - sn + a(s+b)^\epsilon d^\eta L^{\epsilon+\eta-1}] + \mu_3 [r - bn]$$

where  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are the shadow prices of investment, resource accumulation (or decumulation) and resource use in the material production function respectively. Assuming an

<sup>3</sup>As in the previous chapter, lower case letters are used to denote per capita variables.

interior solution, we get the following necessary conditions:

$$(3-2-12) \quad U'(c) = \mu_1$$

$$(3-2-13) \quad \mu_1 \beta(y/r) + \mu_3 = \mu_2$$

$$(3-2-14) \quad \mu_1 = \mu_2 a \eta (S+B) \epsilon D^{\eta-1}$$

The differential equations for the co-state variables are as follows:

$$(3-2-15) \quad \hat{\mu}_1 = \delta + n - \alpha (y/k)$$

$$(3-2-16) \quad \hat{\mu}_2 = \delta + n - a \epsilon (S+B) \epsilon^{-1} D^\eta > 0$$

$$(3-2-17) \quad \hat{\mu}_3 = \mu_3 \delta - \mu_2 a \epsilon (S+B) \epsilon^{-1} D^\eta + \mu_3 n.$$

We find a relationship between  $\mu_2$  and  $\mu_3$  using equations (3-2-13) and (3-2-14) as follows:

$$(3-2-18) \quad \mu_2 = \mu_3 / [1 - a \eta (S+B) \epsilon D^{\eta-1} \beta(y/r)].$$

Inserting the above equation in (3-2-17), we obtain,

$$(3-2-19) \quad \hat{\mu}_3 = \delta + n - [a \epsilon (S+B) \epsilon^{-1} D^\eta] / [1 - a \eta (S+B) \epsilon D^{\eta-1} \beta(y/r)].$$

Now taking the time derivative of (3-2-12) and substituting for  $\hat{\mu}_1$ , we get the time rate of change of per capita consumption:

$$(3-2-20) \quad \hat{c} = -(1/\sigma) [\delta + n - \alpha (y/k)]$$

where  $\sigma = -U''(c)c/U'(c) > 0$ .

Multiplying (3-2-16) by  $\mu_2$  and subtracting (3-2-17) from the product, we note that,

$$(3-2-21) \quad \{\mu_2 - \mu_3\} / \{\mu_2 - \mu_3\} = \delta + n$$

We differentiate (3-2-10) and (3-2-13) with respect to time and make use of (3-2-21) in the resulting expression to obtain,

$$(3-2-22) \quad \alpha \hat{k} - (1-\beta) \hat{r} + \lambda = \alpha y/k.$$

Similar time differentiation of (3-2-14) and substitution for  $\hat{\mu}_1$ ,  $\hat{\mu}_2$  etc. yield,

$$(3-2-23) \quad \hat{D} = \alpha(y/k)/(1-\eta) > 0.$$

Thus, the growth rates of the different per capita variables, viz., consumption, resource utilization, etc. can conveniently be expressed in terms of X, V and M where  $X = y/k$ ,  $V = c/k$  as in Stiglitz (1974) and  $M = d/k$ . The expressions obtained are

$$(3-2-24) \quad \hat{c} = (1/\sigma)[\alpha X - \delta - n]$$

$$(3-2-25) \quad \hat{y} = [\lambda + \alpha X(1-\beta) - \alpha V - \alpha M - \alpha n]/(1-\beta)$$

$$(3-2-26) \quad \hat{r} = [\lambda - \alpha V - \alpha M - \alpha n]/(1-\beta)$$

$$(3-2-27) \quad \hat{d} = [\alpha X - n(1-\eta)]/(1-\eta).$$

These equations can then be used to show that the optimal growth path of the economy is described by the following differential equations:

$$(3-2-28) \quad \hat{X} = \{1/(1-\beta)\}[\lambda - (\gamma + \alpha\beta)X + \gamma V + \gamma M + \gamma n]$$

$$(3-2-29) \quad \hat{V} = \{(\alpha - \sigma)/\sigma\}X + V + M - (1/\sigma)(\delta + n - \sigma n),$$

$$(3-2-30) \quad \hat{M} = \{(\alpha + \eta - 1)/(1-\eta)\}X + V + M.$$

### 3.2.2 The Stationary State Solution

We now look at the stationary states of the system of differential equations (3-2-28) through (3-2-30). The reason is that the optimal growth path of the economy usually converges asymptotically to one of these stationary states. In this model there are 8 such states as enumerated below:

[I]  $X > 0$ ,  $V > 0$  and  $M > 0$ . [II]  $X = 0$ ,  $V > 0$  and  $M > 0$ . [III]  $X > 0$ ,  $V = 0$  and  $M > 0$ . [IV]  $X > 0$ ,  $V > 0$  and  $M = 0$ . [V]  $X = 0$ ,  $V = 0$  and  $M > 0$ . [VI]  $X = 0$ ,  $V > 0$  and  $M = 0$ . [VII]  $X > 0$ ,  $V = 0$  and  $M = 0$ . [VIII]  $X = 0$ ,  $V = 0$  and  $M = 0$ .

The optimal growth path of the economy here is such that it cannot converge to a stationary state with positive values for all the variables, i.e., case [I]. This is because when the growth rates are equated to zero, the coefficient matrix of the system of linear equations becomes singular. Cases [II], [V], [VI] and [VIII] can be ruled out since these points are totally unstable or partially unstable in a way which does not allow the optimal trajectory to converge to these points.<sup>4</sup> Moreover, these points can be rejected on the ground that  $V$  or  $M$  cannot be greater than  $X$  if we do not allow for negative investment. Case [III] leads to inconsistent equations and hence can be ignored. Case [VII] can be rejected by using the transversality condition for capital which is

$$(3-2-31) \quad \lim_{t \rightarrow \infty} e^{-\delta t} \mu_1(t) k(t) = 0.$$

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<sup>4</sup>The unstable nature of these stationary states can be determined by replacing the three variables,  $X$ ,  $V$  and  $M$ , in the differential equations by their equilibrium values. For example, the point  $(0,0,0)$  is ruled out since near this point  $\hat{X}$  is approximately  $(\gamma n + \lambda) / (1 - \beta)$ .  $0$  is an unstable equilibrium of this differential equation. Thus, although  $(0,0,0)$  is a saddle point for the full system of equations, the only trajectories which converge to this steady state must have  $X=0$  and these trajectories are clearly not optimal.

This transversality condition can be rewritten in terms of X, V and M as follows:<sup>5</sup>

$$(3-2-32) \quad (1-\alpha)X^* - V^* - M^* = \nabla \leq 0.$$

Case [VII] does not meet this requirement and hence can be ruled out as a possible limit of the optimal growth path. Apparently case [IV] can be accepted as a possible long run equilibrium point. The two cases, [VII] and [IV], in fact correspond to points A & B respectively in figure 1 used in the discussion of one sector neoclassical model in the previous chapter. Case [IV] also corresponds to the equilibrium obtained in Stiglitz (1974). This case would be the long run equilibrium if there was no exploration at all. However, maximisation of the Hamiltonian yields the result that some exploration is optimal here. The values of  $X^*$  and  $V^*$  for case [IV] can be obtained by replacing  $\delta$  by  $\delta+n$  in the stationary values of Stiglitz (1974) on page 22. Alternatively, these values can be obtained by setting  $\hat{X}=0$  and  $\hat{V}=0$  in (3-2-28) and (3-2-29) respectively:

$$(3-2-33) \quad X^* = [\sigma\lambda + \gamma(\delta+n)] / \alpha(\sigma\beta + \gamma).$$

$$(3-2-34) \quad V^* = [\{\gamma(1-\alpha) + \alpha\beta(1-\sigma)\}n + \delta(\alpha\beta + \gamma) - \lambda(\alpha - \sigma)] / \alpha(\sigma\beta + \gamma)$$

$X^*$  is always positive.  $V^*$  is positive provided  $\alpha \leq \sigma \leq 1$  or if  $\lambda$  is not too large.

### 3.2.3 Stability Analysis

We now investigate further the stability nature of the equilibrium point determined above. The Jacobian of the differential equations system at the equilibrium has one negative and two positive characteristic roots. This is because the determinant of the Jacobian has a negative determinant and a positive trace at the equilibrium point. This kind of equilibrium point is known as a saddle point equilibrium. It is conventionally argued that the optimal

<sup>5</sup>See page 13 for a discussion of this condition

growth path of the economy should converge to a saddle point equilibrium provided the initial values of the stock variables are suitable. The above contention, however, is true only for a 2 by 2 differential equations system. For a 3 by 3 differential equations system, that may not be necessarily true.<sup>6</sup> A closer look at the differential equation (3-2-30) reveals the fact that the optimal growth path of the economy cannot converge to the equilibrium point mentioned earlier. It is obvious from (3-2-30) that M will keep on increasing for ever whenever

$$(3-2-35) \quad \{\alpha X / (1-\eta)\} + V + M > X$$

which is very likely to happen. A value of  $\alpha > (1-\eta)$  is sufficient for the condition to be satisfied. Moreover, we can rewrite (3-2-30) as follows:

$$(3-2-36) \quad [ \{ \alpha / (1-\eta) \} - \alpha ] X + (\alpha - 1) X + V + M$$

The transversality condition for capital (3-2-32) requires that the sum of the last three terms in (3-2-36) be non-negative. The first term, on the other hand, is also positive. Together they make the whole expression positive. In other words, (3-2-30) must be positive in the long run if the transversality condition is to be satisfied. That means, M increases indefinitely and so do X and V.<sup>7</sup>

From the preceding discussion, it is clear that the optimal growth path of the economy in this model will not converge to any steady state. The system of the differential equations system is such that X, V and M will ultimately keep on increasing for ever. In other words, aggregate as well as per capita consumption will eventually be increasing continuously in such a world even if there is no exogenous technical progress. Compare this

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<sup>6</sup>One example of this can be the Stiglitz (1974) model. We explained why the point ( $X > 0$ ,  $V > 0$  and  $Z > 0$ ), though a saddle point equilibrium, cannot be approached asymptotically by the optimal trajectory.

<sup>7</sup>The variable M appears with a positive coefficient in other two differential equations.

result with the result of Stiglitz (1974) given in (2-2-49). Per capita consumption falls for ever when the rate of exogenous technical progress is not sufficiently large. The reason for this marked difference in results of the two models is that in the presence of exploration facilities, the problem of resource scarcity is not so acute as it is in the absence of exploration. Exploration is absent in Stiglitz (1974) model. The fact is that exploration is not a free gift from heaven. It is expensive. The gains from exploration in the current model, however, outweigh the costs of exploration for ever. It should be mentioned here that the equilibrium point discussed earlier is actually identical to the equilibrium point found in Stiglitz (1974). As such, the condition guaranteeing a growing per capita consumption is identical in both places. We have explained why this equilibrium point cannot be obtained in the long run. It can be claimed that the current model is a more general model than Stiglitz's model which can be shown to be a special case of the current model. The current model explains why Stiglitz's equilibrium point fails to be a limiting case of the optimal growth path in a more general perspective where exploration is present.

### 3.2.4 Sufficient Condition

There are two requirements for the satisfaction of the sufficient conditions in a dynamic maximisation problem.<sup>8</sup> First, the Hamiltonian must be concave in all the state variables at the maximizing values of the control variables given the multipliers. Second, the transversality conditions for different state variables must be satisfied. An examination of the Hamiltonian here reveals that it is not concave in  $S$  and  $B$  because of the assumption  $\epsilon < 0$ . We had to assume a negative  $\epsilon$  in order to have a negative relationship between the returns to exploration activities and the size of cumulative discoveries. In other words, the concavity condition and the format of this model are not compatible with each other. In the

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<sup>8</sup>See Arrow and Kurz (1970)

last section of this chapter, we set up a model which satisfies the concavity condition but forgoes the assumption of a negative relationship between the yields from exploration and cumulative discoveries.

Now let us consider the transversality conditions. As we showed earlier, the equilibrium point ( $X>0$ ,  $V>0$  and  $M=0$ ) cannot be the limit of the optimal trajectory because of the transversality condition for capital even though it satisfies the condition. There is a transversality condition associated with the stock of the resource:

$$(3-2-37) \quad \lim_{t \rightarrow \infty} e^{-(\delta+n)t} \mu_2(t) S(t) = 0.$$

(3-2-16) gives  $d\{\mu_2 e^{-(\delta+n)t}\}/dt > 0$ . Moreover,  $\mu_2 > 0$  by (3-2-14). Therefore, the application of this transversality condition gives

$$(3-2-38) \quad \lim_{t \rightarrow \infty} S(t) = 0$$

According to equation (3-2-38), the stock of the resource must ultimately be decreasing over time so that asymptotically it will be exhausted. This conclusion reflects the exhaustibility of the natural resource. There is a third transversality condition:

$$(3-2-39) \quad \lim_{t \rightarrow \infty} e^{-\delta t} \mu_3 b(t) = 0$$

This condition does not yield the same conclusion as before since  $\mu_3 < 0$  is probable and indeed does not seem to provide any additional information about the optimal growth path. Henceforth, we will not discuss these two transversality conditions as the same results appear in each model.



### 3.2.5 Shadow Prices

In this subsection, we examine the behaviour of the present value shadow prices of per capita investment ( $\mu_1 e^{-\delta t}$ ), per capita resource accumulation or decumulation ( $\mu_2 e^{-\delta t}$ ) and per capita resource use in the material production ( $\mu_3 e^{-\delta t}$ ). The shadow price of investment which signifies the change in the present value of the utility functional resulting from an additional investment, is positive according to equation (3-2-12). Equation (3-2-15), on the other hand, shows that the growth rate of  $\mu_1 e^{-\delta t}$  falls over time as  $X$  increases indefinitely. This indicates that the shadow price of investment may increase for a while but eventually starts to fall. Accordingly, investment in capital formation may be growing initially but then it must decline. The shadow price of resource accumulation or decumulation is positive according to equation (3-2-14) and increases at a positive but declining rate as cumulative discoveries increase over time (3-2-16). With a fixed stock of resources, the shadow price of resource decumulation grows at a rate equal to the sum of rate of growth of population and the rate of discount. From equation (3-2-16), we see that it grows at a higher rate when the economy can augment the stock of the resource through costly exploration. In the case of a fixed stock of resources, the opportunity cost of one unit of the resource extracted is equal to the foregone gain which comes through the appreciation in the value of the resource. In the presence of exploration facilities, the opportunity cost is equal to the appreciation in the value of the resource plus the foregone gains which are obtained through the indirect use of one unit of the resource in exploration process of the economy. When one unit of the resource is left in the ground, it can be used to produce output in future. A part of that output can in turn be used in the exploration process to augment the stock of the resource. This is why the shadow price of one unit of the resource left in the ground should increase at a rate higher than the sum of discount rate and the population growth rate. But as cumulative discoveries increase, the yields from exploration decrease over time. Consequently, the growth rate of the shadow price of resource

accumulation decreases with the increase in the size of cumulative discoveries. The shadow price of resource use in the material production process also increases over time. The rate of growth of the shadow price becomes smaller and smaller as the size of cumulative discoveries increases.

### 3.2.6 Final Remarks

In this section, we have examined the behaviour of the optimal growth path of a resource based economy. The economy can increase the stock of the resource by costly exploration activities. We assumed a negative relationship between the yields from exploration activities and the size of cumulative discoveries. We found that output used for exploration increases along the optimal path. Also the size of cumulative discoveries is increasing as time passes. However, the returns to exploration activities become smaller and smaller as the size of cumulative discoveries increases. It may be that the use of the resource in the production process increases for a while. Eventually, however, it starts to fall. In the later stages of the program, capital is likely to be substituted for the resource input in the production process. Consequently, per capita output and consumption will increase indefinitely even if there is no exogenous technical progress. This model has one technical difficulty with regard to the satisfaction of the sufficient condition for maximisation. In the last section, we develop an alternative model that does not have this problem.

In this subsection, we also use the notation employed by Pindyck (1978) to study the price behaviour of the resource in an economy possessing the opportunity of increasing the stock of the resource by exploratory effort. We assume away, like Pindyck, labour as a factor of production. The optimal growth problem (which Pindyck did not fully discuss) is

$$(3-2-40) \quad \text{Max} \quad \Omega = \int_0^{\infty} e^{-\delta t} U(C) dt$$

subject to

$$(3-2-41) \quad \dot{K} = Y - C - D$$

$$(3-2-42) \quad \dot{S} = -R + f(G, D) \quad f_G < 0, \quad f_D > 0 \quad \text{where} \quad f_G = \partial f / \partial G \quad \text{etc.}$$

$$(3-2-43) \quad \dot{G} = f(G, D) \quad G = \text{cumulative reserve additions}$$

$$(3-2-44) \quad Y = K^\alpha R^\beta$$

The model gives rise to the following equations of the optimal paths:

$$(3-2-45) \quad \hat{C} = -(1/\sigma) [\delta - \alpha(Y/K)]$$

$$(3-2-46) \quad \alpha \hat{K} - (1 - \beta) \hat{R} = \alpha(Y/K)$$

$$(3-2-47) \quad \hat{f}_D + \alpha(Y/K) = f_G$$

Notationally, the difference between Pindyck's formulation and our formulation is that he uses  $G=S+B$  as the third stock variable in his model. Indeed our model can be viewed as an example of the above with the additional assumption of  $f(G, D) = H(G)D^\eta$ . We can compute (3-2-47) for this case using (3-2-43). It becomes

$$(3-2-48) \quad \hat{D} = \{ \alpha Y / (1 - \eta) k \}$$

which is same as (3-2-23) of our model. Thus, our model by specifying a separable function form for  $f(\cdot)$ , gets much more revealing insights into the optimal growth path. Moreover, it can be shown that with the above general formulation, the optimal growth paths for  $X$ ,  $V$  and  $M$  do not depend on the specification of  $H$ . Thus, the same results are obtained whether  $dH/dG < 0$  as is economically plausible or when  $dH/dG > 0$ . Next, however, we look at alternative specifications of the input needed for exploration activity.

### 3.3 LABOUR REQUIRED FOR EXPLORATION

So far we have assumed that the returns to exploration activities depend positively on the level of output used for exploration. An alternative formulation is to assume instead that exploration requires the use of one of the factors of production. This section and the following section are devoted to the study of the optimal growth path when some factor of production is used in the exploration process. In this section, we assume that a portion of labour is used for exploration purposes. A similar model was analysed by Takayama (1980). He assumed, however, that the returns to exploration activities depend positively both on the existing stock of the resource (not on cumulative discoveries) and on the proportion of total labour (rather than the amount) used in the exploration process. Takayama's model has two shortcomings. First, it contains a "learning by doing process" by assuming a positive relationship between the level and the increments to the stock of the resource. The economy gets more from each dose of exploratory effort as the proven stock of the resource increases in size. The economy will never face the problem of resource scarcity in this kind of world. Second, his model implies that the stock of the resource will not increase in size if the rate of growth of exploratory effort is equal to the rate of growth of population. Both of these assumptions appear to be unrealistic. In this section, we are, therefore, trying to correct for these drawbacks in Takayama's model. Thus, total labour is used for two purposes: material production and exploration. We assume,

$$(3-3-1) \quad Y = e^{\lambda t} K^{\alpha} R^{\beta} L_m^{\gamma}$$

$$(3-3-2) \quad \dot{S} = -R + a(S+B)^{\epsilon} L_e^{\eta} \quad \epsilon < 0$$

where  $L_m$  = labour used in material production, and  $L_e$  = labour used in exploration.

Full employment is still assumed so that

$$(3-3-3) \quad L_m + L_e = L$$

The optimal growth path here might be of a different nature from that obtained in the earlier section. The reason is that the expenses involved in exploration activities will be quite different in the two cases.

### 3.3.1 The Model and Necessary conditions

The model in per capita variables is now as follows:

$$(3-3-4) \quad \text{Max} \quad \Omega = \int_0^{\infty} e^{-\delta t} U(c) dt$$

subject to

$$(3-3-5) \quad y = e^{\lambda t} k^{\alpha} r^{\beta} \theta^{\gamma} \quad \text{where } \theta = L_m/L$$

$$(3-3-6) \quad \dot{k} = y - c - nk$$

$$(3-3-7) \quad \dot{s} = -r - sn + a(s+b)^{\epsilon} (1-\theta)^{\eta} L^{\epsilon+\eta-1}$$

$$(3-3-8) \quad \dot{b} = r - bn$$

The current value Hamiltonian is then

$$H = U(c) + \mu_1 [y - c - nk] \\ + \mu_2 [-r - sn + a(s+b)^{\epsilon} (1-\theta)^{\eta} L^{\epsilon+\eta-1}] + \mu_3 [r - bn]$$

For this model, the control variables are  $c$ ,  $r$  and  $\theta$ . The following are the necessary conditions for maximisation assuming an interior solution:

$$(3-3-9) \quad U'(c) = \mu_1$$

$$(3-3-10) \quad \mu_1 \beta (y/r) + \mu_3 = \mu_2$$

$$(3-3-11) \quad \mu_1 \gamma Y / \theta = a \eta \mu_2 (s+b)^\epsilon (1-\theta)^{\eta-1} L^{\epsilon+\eta-1}$$

$$(3-3-12) \quad \hat{\mu}_1 = \delta + n - \alpha (y/k)$$

$$(3-3-13) \quad \hat{\mu}_2 = \delta + n - a \epsilon (s+b)^{\epsilon-1} (1-\theta) \eta L^{\epsilon+\eta-1}$$

$$(3-3-14) \quad \hat{\mu}_3 = \mu_3 (\delta + n) - \mu_2 a \epsilon (s+b)^{\epsilon-1} (1-\theta) \eta L^{\epsilon+\eta-1}$$

Taking logarithmic differentiation of (3-3-10) and making use of the fact that equations (3-5-13) and (3-5-14) imply  $\{\mu_2 - \mu_3\} / \{\mu_2 - \mu_3\} = \delta + n$ , we obtain,

$$(3-3-15) \quad \hat{y} - \hat{r} - \alpha (y/k) = 0$$

Similarly, logarithmic differentiation of (3-3-11) yields,

$$(3-3-16) \quad \hat{y} - \{(1-\eta\theta)/(1-\theta)\} \hat{\theta} - \alpha (y/k) + (1-\eta)n = 0$$

The asymptotic growth rates of per capita resource extraction, consumption and output can now be expressed in terms of X, V and  $\theta$  as shown below:

$$(3-3-17) \quad \hat{r} = [\lambda - \alpha(V+n) + \gamma\hat{\theta}] / (1-\beta)$$

$$(3-3-18) \quad \hat{c} = (1/\sigma)(\alpha X - \delta - n)$$

$$(3-3-19) \quad \hat{y} = \hat{r} + \alpha X$$

The preceding equations can then be used to derive the differential equations for X, V and  $\theta$ :

$$(3-3-20) \quad \hat{X} = \{1/(1-\beta)\} [ -(\gamma + \alpha\beta)X + \gamma V + \lambda + \gamma n + \gamma\hat{\theta} ]$$

$$(3-3-21) \quad \hat{V} = \{(\alpha - \sigma)/\sigma\}X + V - (1/\sigma)[\delta + n - \sigma n]$$

$$(3-3-22) \quad \hat{\theta} = \{[(1-\theta)\{\lambda - \alpha V - \alpha n + (1-\beta)(1-\eta)n\}]/[\alpha(1-\theta) + (1-\eta)(1-\beta)\theta]\}$$

### 3.3.2 The Stationary State Solution

There are twelve possible equilibrium points with non-negative values for all the variables. However, it can be shown that  $\theta$  must be 1 or 0 at an equilibrium.<sup>9</sup> Moreover, only the stationary states with  $X \neq 0$  and  $V \neq 0$  will be discussed here because the other equilibrium points do not satisfy some of the conditions for optimality. The following two cases qualify to be feasible solutions:

$$[1] X > 0 \quad V > 0 \quad \theta = 1$$

$$[2] X > 0 \quad V > 0 \quad \theta = 0$$

We now turn to the discussion of these stationary states.

$$\text{Case 1 : } X > 0, V > 0 \text{ and } \theta = 1$$

This case of all labour being used in material production gives rise to a equilibrium which is identical to the equilibrium found in Stiglitz (1974) and also in case [IV] of previous section. Later it will be shown that the optimal growth path does not converge to this point.

$$\text{Case 2 : } X > 0, V > 0 \text{ and } \theta = 0$$

Here it can be shown that as  $\theta$  tends to 0, the growth rate of  $\theta$  tends to

$$(3-3-23) \quad \hat{\theta} \approx [\lambda - \alpha V - \alpha n + (1-\beta)(1-\eta)n]/\alpha$$

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<sup>9</sup>See the appendix

We substitute (3-3-23) in (3-3-20) to obtain the approximate growth rate of X when  $\theta$  tends to 0:

$$(3-3-24) \quad \hat{X} \approx -(1-\alpha)X + \{\lambda + (1-\eta)n\} / \alpha$$

The differential equation for V does not change here. The stationary state values of X and V in this case are found to be

$$(3-3-25) \quad X^* = [\lambda + \gamma(1-\eta)n] / \alpha(1-\alpha)$$

$$(3-3-26) \quad V^* = [-(\alpha-\sigma)\{\lambda + \gamma n(1-\eta)\} + \alpha(1-\alpha)\{\delta + n(1-\sigma)\}] / \alpha(1-\alpha)\sigma$$

$X^*$  is always positive.  $V^*$  is positive as long as  $\sigma \leq 1$ . The asymptotic growth rate of per capita consumption, output and capital in this case is

$$(3-3-27) \quad \hat{c} (= \hat{k} = \hat{y}) \approx [\lambda + \gamma(1-\eta)n - (\delta+n)(1-\alpha)] / (1-\alpha)\sigma$$

Per capita consumption will, therefore, be growing in the long run provided

$$(3-3-28) \quad \lambda > \gamma(\delta+n\eta) + \beta(\delta+n)$$

In this case, per capita resource use is always falling with

$$(3-3-29) \quad \hat{r} = -(\delta+n)$$

In Stiglitz's (1974) case, only  $\lambda > \beta(\delta+n)$  is required for a growing per capita consumption. It is strange that the minimum rate here is higher than the minimum rate needed when exploration is absent. The reason for this result may be that here an increasing amount of labour must be utilized in the exploration process. While more and more labour is being put in the exploration process, other inputs will have to be substituted for labour in the production process if a growing per capita consumption is to be maintained. The substitution



by other factors, however, may not adequately make up for a decreasing proportional use of the labour input. The consequence may be a lower growth rate for aggregate output. Thus, the rate of growth of exogenous technical progress in this case must be higher in order to maintain a growing per capita consumption.

Another reason for a falling per capita consumption in the absence of exogenous technical progress here may be as follows. Since people cannot control the growth rate of labour, a portion of which is used in the exploration process, the increase in future consumption and investment due to exploration may be limited. Nevertheless, more consumption is preferred to less so that some exploration shall be carried out at all times. However, people would like to enjoy the benefits of exploration more in present than in future due to time preference associated with consumption. In the process, the accumulated size of the utility functional can be larger than if they had not consumed more at present. This explains why future consumption will be falling in the absence of technical progress. The same cannot be true when a portion of total output or the capital input is used in the exploration process. The growth rate of the amount of output or capital spent for exploration can be optimally controlled at all times. Consequently, per capita consumption and capital stock may increase continuously in these circumstances.

### 3.3.3 Stability Analysis

Here we assume a logarithmic utility function, i.e.,  $\sigma=1$  for the reason of tractability. The Jacobian of the differential equation system has two negative and one positive eigenvalues in the second case. In the first case, it has two positive and one negative characteristic roots. It implies that both the equilibrium points analysed above are saddle points. It can, however, be shown that the optimal growth path of the economy can approach asymptotically to the second equilibrium point only. This is because  $\partial \theta / \partial \theta$  evaluated at equilibrium is found to be positive for the first case and negative for the

second case. The  $\partial \theta / \partial \theta$  equation for the model is given by

$$(3-3-30) \quad \partial \theta / \partial \theta = (A/D^2) [D(1-2\theta) - \theta(1-\theta) \{ (1-\eta)(1-\beta) - \alpha \}]$$

where  $A = \lambda - \alpha V - \alpha n + (1-\beta)(1-\eta)n$  and

$$D = \alpha(1-\theta) + (1-\eta)(1-\beta)\theta .$$

Using equilibrium values obtained in the two cases, we find that  $\partial \theta / \partial \theta$  tends to  $\{(\delta + \eta n) / (1-\eta)\} > 0$  when  $\theta$  tends to 1 and to  $-(\delta + \eta n) < 0$  when it tends to 0. Moreover, it can be noted that as  $\hat{\theta}$  tends to zero, equation (3-3-20) ensures that  $X$  tends to  $X^*$  of case [II] so that indefinitely increasing per capita consumption is not possible here. Thus, only convergence to this second steady state is possible.

### 3.3.4 Sufficient Condition

The sufficient condition concerning the concavity of the Hamiltonian here cannot be satisfied for the reason explained in section 2 of this chapter. The transversality condition for capital is satisfied in both cases. This transversality condition takes the following form here:

$$(3-3-31) \quad (1-\alpha)X^* - V^* \leq 0$$

In particular, for the second case we have  $\nabla = -\{\delta + n(1-\sigma)\} / \sigma$ , which is negative provided  $\sigma \leq 1$ .

### 3.3.5 Shadow Prices

In this model, the asymptotic behaviour of the shadow price of investment cannot be determined exactly. With a sufficiently high rate of exogenous technical progress, it is likely to be falling at the equilibrium found for the second case. The shadow prices of resource accumulation or decumulation and resource use in the production process follow time paths

similar to those found for the first model.

### **3.3.6 Final Remarks**

In this section, we assumed that an amount of the labour input is used in the exploration process. We discovered that the optimal growth path of the economy converges to a stationary state where the proportion of labour spent in the exploration activities tends to 1. Along the optimal path, the use of labour in the production process will decrease proportionally over time. As time passes, proportionally more and more of the labour input is used in the exploration process. This leads to more and more discoveries of new resource reserves. The returns to exploration activities, however, decrease with the increase in the size of cumulative discoveries. Once the economy has built up a sufficiently large stock of the resource and has approached a stationary state, the use of the resource in the production process will fall and that of capital will rise. Consequently, output and consumption will grow at the same rate over time. The minimum rate of exogenous technical progress necessary for a rising per capita consumption in this case turns out to be higher than the rate for the case when there is no exploration.

## **3.4 CAPITAL REQUIRED FOR EXPLORATION**

Though capital, like labour, is an input of production, there is one important difference between the two inputs. Labour is assumed to grow exponentially at a given rate. No such assumption is made for capital. The rate of growth of capital is, on the other hand, controllable. A separate section where capital is used in the exploration process is, therefore, of much interest. In this section, we assume that a part of aggregate capital is used in the exploration process. The returns to exploration activities are assumed to be positively related

to the amount of capital used for exploration and negatively to cumulative discoveries. Thus, we assume,

$$(3-4-1) \quad Y = e^{\lambda t} K_m^\alpha R^\beta L^\gamma$$

$$(3-4-2) \quad \dot{S} = -R + a(S+B)^\epsilon K_e^\eta$$

where  $K_m$  = capital used in material production, and  $K_e$  = capital used in the exploration process. Since all capital is used,  $K_e + K_m = K$ . Next, we present the model in per capita variables and derive the necessary conditions of optimization.

### 3.4.1 The model and Necessary Conditions

$$(3-4-3) \quad \text{Max } \Omega = \int_0^\infty e^{-\delta t} U(c) dt$$

subject to

$$(3-4-4) \quad y = e^{\lambda t} \theta^\alpha k^\alpha r^\beta \quad \text{where } \theta = K_m/K$$

$$(3-4-5) \quad \dot{k} = y - c - nk$$

$$(3-4-6) \quad \dot{s} = -r - sn + a(s+b)^\epsilon (1-\theta)^\eta L^{\epsilon+\eta-1} k^\eta$$

$$(3-4-7) \quad \dot{b} = r - bn$$

The current value Hamiltonian is given by

$$H = U(c) + \mu_1 [y - c - nk] \\ + \mu_2 [-r - sn + a(s+b)^\epsilon (1-\theta)^\eta L^{\epsilon+\eta-1} k^\eta] + \mu_3 [r - bn]$$

The following necessary conditions are obtained assuming an interior solution exists maximizing H with respect to c, r and  $\theta$ :

$$(3-4-8) \quad U'(c) = \mu_1$$

$$(3-4-9) \quad \mu_1 \beta (y/r) + \mu_3 = \mu_2$$

$$(3-4-10) \quad \mu_1 \alpha y / \theta = a \eta \mu_2 (s+b)^\epsilon (1-\theta)^{\eta-1} k^\eta L^{\epsilon+\eta-1}$$

$$(3-4-11) \quad \hat{\mu}_1 = \delta + n - (\alpha y / \theta k)$$

$$(3-4-12) \quad \hat{\mu}_2 = \delta + n - a \epsilon (s+b)^{\epsilon-1} (1-\theta)^\eta k^\eta L^{\epsilon+\eta-1}$$

$$(3-4-13) \quad \hat{\mu}_3 = \mu_3 (\delta + n) - \mu_2 a \epsilon (s+b)^{\epsilon-1} (1-\theta)^\eta k^\eta L^{\epsilon+\eta-1}$$

Taking logarithmic differentiation of (3-4-9) and noting again that the difference of the equations (3-4-12) and (3-4-13) yields  $\{\hat{\mu}_2 - \hat{\mu}_3\} / \{\mu_2 - \mu_3\} = \delta + n$ , we obtain,

$$(3-4-14) \quad \hat{y} - \hat{r} - (\alpha y / \theta k) = 0$$

Similarly, logarithmic differentiation of (3-4-10) and use of (3-4-6), (3-4-7), (3-4-11) and (3-4-12) yield,

$$(3-4-15) \quad \hat{y} - \left\{ \frac{(1-\eta)\theta}{(1-\theta)} \right\} \hat{\theta} - \hat{\eta} k + (1-\eta)n - \alpha (y/\theta k) = 0$$

In this section, we make a slight change in the definition of X for technical reasons. Let X now stand for the ratio of output to the portion of capital used in the production process, i.e.,  $X = Y/K_m$ . The growth rates of per capita consumption, resource use and output then satisfy

$$(3-4-16) \quad \hat{c} = \{\alpha X - (\delta + n)\} / \sigma$$

$$(3-4-17) \quad \hat{r} = [1/(1-\beta)] [\lambda - \alpha(V+n) + \alpha\hat{\theta} + \alpha X(\theta-1)]$$

$$(3-4-18) \quad \hat{y} = \hat{r} + \alpha X$$

These equations can be used to show that the optimal growth paths of the economy satisfy the following differential equations for X, V and  $\theta$ :

$$(3-4-19) \quad \hat{X} = \{1/(1-\beta)\}[\lambda - (\gamma\theta + \alpha\beta)X + \gamma V + \gamma n - \gamma\hat{\theta}]$$

$$(3-4-20) \quad \hat{V} = \{(\alpha - \sigma\theta)/\sigma\}X + V - (1/\sigma)[\delta + n - \sigma n]$$

$$(3-4-21) \quad \hat{\theta} = (1-\theta)[\lambda + \{\alpha\theta - (1-\beta)\eta\theta - \alpha\}X - \{\alpha - (1-\beta)\eta\}V + \gamma n] / [\gamma(1-\theta) + (1-\eta)(1-\beta)\theta]$$

### 3.4.2 The Stationary State Solution

As usual, the stationary state is obtained by setting  $\dot{X}=0$ ,  $\dot{V}=0$  and  $\dot{\theta}=0$ . There are twelve possible equilibrium points in this model. Only two cases, however, meet the requirements of a feasible long run equilibrium. These solutions are [I]  $X > 0$ ,  $V > 0$  and  $\theta = 1$  [II]  $X > 0$ ,  $V > 0$  and  $\theta = 0$ . It can be easily verified that the stationary values of the variables X and V for case [I] are identical to the values of X and V obtained in Stiglitz (1974) and case I of last section. The point is, however, shown not to be the asymptotic limit of the optimal trajectory. The second case here is interesting and needs further scrutiny.

$$\text{Case [II] } X > 0, V > 0 \text{ and } \theta = 0$$

Here we note that as  $\theta$  tends to zero, the growth rates for X, V and  $\theta$  tend to the following approximate growth rates:

$$(3-4-22) \quad \hat{X} \approx \alpha X + (1-\eta)V$$

$$(3-4-23) \quad \hat{V} \approx (\alpha X/\sigma) + V - \{\delta + n(1-\sigma)\}/\sigma$$

$$(3-4-24) \quad \hat{\theta} \approx [\lambda - \alpha X - \{\alpha - (1 - \beta)\eta\}V + \gamma n] / \gamma$$

A close look at (3-4-22) reveals the fact that here  $X$  must be increasing when  $\theta$  approaches 0. The same is true for  $V$  since the coefficient of  $X$  in (3-4-23) is positive. It is clear that the optimal growth path here cannot converge to a stationary state. This is because  $X$  and  $V$  increase indefinitely while  $\theta$  falls for ever.

### 3.4.3 Stability Analysis

Here again, we make the analysis for a logarithmic utility function. For the first case, the Jacobian of the differential equation system has only one negative characteristic root. Though the equilibrium point here is a saddle point, it can be ruled out because  $\partial \theta / \partial \theta$  is found to be positive in this case. In other words, the equilibrium point is not stable in  $\theta$ . We are then left with the second case where  $X$  and  $V$  increase indefinitely. It can be verified that  $\partial \theta / \partial \theta$  will eventually turn out to be negative for the second case.<sup>10</sup>

### 3.4.4 Sufficient Condition

The transversality condition for capital for the second case is given by  $\nabla = (\theta - \alpha)X^* - V^*$ . As  $\theta$  approaches 0,  $X$  tends to be increasing indefinitely. This ensures that  $\nabla$  will eventually be negative.

### 3.4.5 Shadow Prices

All the shadow prices here behave in the same way as they do in the first model of this chapter.

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<sup>10</sup> This is shown in appendix.

### 3.4.6 Final remarks

When capital is used for exploration, the equilibrium point is obtained when all capital is used in the exploration process. As more and more capital is used proportionally in the exploration process, the returns to exploration activities decrease with the increase in the size of cumulative discoveries. Aggregate resource use in the production process initially increases for a while and then it will fall. At later stages of the program, more and more labour will be substituted for resource in the production process. Once again, like the first model, we find that per capita consumption will increase indefinitely even when exogenous technical progress is absent.

## 3.5 EXPLORATION DEPENDS ON THE STOCK OF THE RESOURCE

The model presented in the second section of this chapter has one shortcoming. The Hamiltonian of the dynamic optimisation model cannot be a concave function of all the state variables. In this section, we try to circumvent this problem by replacing cumulative discoveries by the stock of the resource in the time rate of change of the stock of the resource equation. The implied Hamiltonian then may be a concave function of the stock of the resource. This specification of the exploration technology, however, is not free from flaws. Though the relationship between cumulative discoveries and the yields from exploration is negative, the same is not true between the proven stock of the resource and the returns to exploration activities. This is because the stock of the resource may accumulate or run down in comparison with its initial size during the process of exploration. The exploration results will decline over time if the stock actually runs down. Consequently, the relationship between the stock of the resource and the yields from exploration will be positive. In this section, we assume that the yields from exploration depend on both the stock of the resource and



exploration output. In particular, we assume,

$$(3-5-1) \quad \dot{S} = -R + aS^\epsilon D^\eta.$$

We find that the above equation when used in a model of intertemporal utility maximisation yields an optimal growth path when  $\epsilon$  lies between 0 and 1 and  $\eta=1$ . The positive values of  $\epsilon$  indicate that the economy gains more from exploration activities as the size of the proven stock of the resource increases. The model has the possibility that the stock of the resource will be increasing during the exploration process. In that sense, the model implies a "learning by doing" process. This model embodies an exploration technology which is less expensive than the one in section 2. A value of  $\epsilon$  less than 1 means that the growth rate of the stock of the resource will fall with the increase in the size of the stock of the resource. As the stock increases indefinitely, the growth rate tends to zero. In other words, the economy will never face resource scarcity, though the stock of the resource cannot become infinite in finite time. Life should be much easier in this kind of world.

Though the model developed here resembles Takayama's (1980) model, it has superiority over Takayama's model in two respects. First, Takayama assumes that the returns to exploration activities will not increase when exploratory efforts increase at the rate of growth of population. The assumption is unrealistic. Second, in Takayama, the returns to exploration activities hold a positive linear relationship to the stock of the resource. In other words, the growth rate of the stock of the resource remains constant as the stock increases in size. Our model, however, shows that the growth rate falls with the increase in size of the stock of the resource. Next, we present the formal analysis of the model.

### 3.5.1 The Model and Necessary Conditions

The model in per capita terms is

$$(3-5-2) \quad \text{Max} \quad \Omega = \int_0^{\infty} e^{-\delta t} U(c) dt$$

subject to

$$(3-5-3) \quad \dot{k} = y - c - nk - d$$

$$(3-5-4) \quad \dot{s} = -r - ns + as^{\epsilon} d^{\eta} L^{\epsilon + \eta - 1}$$

$$(3-5-5) \quad y = e^{\lambda t} k^{\alpha} r^{\beta}$$

The current-value Hamiltonian is, therefore,

$$H = U(c) + \mu_1 [y - c - nk - d] + \mu_2 [-r - ns + as^{\epsilon} d^{\eta} L^{\epsilon + \eta - 1}]$$

$\mu_1$  and  $\mu_2$  are multipliers representing the shadow prices of investment and accumulation (or decumulation) of the resource. We obtain the following necessary conditions for an interior maximum with respect to  $c$ ,  $r$  and  $d$ :

$$(3-5-6) \quad U'(c) = \mu_1$$

$$(3-5-7) \quad \mu_1 \beta (y/r) = \mu_2$$

$$(3-5-8) \quad \mu_1 = \mu_2 a \eta S^{\epsilon} D^{\eta - 1}$$

$$(3-5-9) \quad \hat{\mu}_1 = \delta + n - \alpha (y/k)$$

$$(3-5-10) \quad \hat{\mu}_2 = \delta + n - a \epsilon S^{\epsilon - 1} D^{\eta}$$

Use of (3-5-7) and (3-5-8) yields

$$(3-5-11) \quad r/y = \eta \beta a S^\epsilon D^{\eta-1}.$$

Time differentiation of (3-5-6) and use of (3-5-9) give the time rate of change of per capita consumption:

$$(3-5-12) \quad \hat{c} = -(1/\sigma) [\delta + n - \alpha(y/k)].$$

Similar time differentiation of (3-5-7) and substitution for  $\hat{\mu}_1$ ,  $\hat{\mu}_2$ ,  $\hat{y}$ , etc. give,

$$(3-5-13) \quad \lambda + \alpha \hat{k} + (\beta - 1) \hat{r} - \alpha(y/k) + \epsilon a S^{\epsilon-1} D^\eta = 0.$$

Making the same type of manipulations for (3-5-8), we get,

$$(3-5-14) \quad (\eta - 1) \hat{D} + \alpha(y/k) = \epsilon(r/s)$$

It is very difficult to analyse the equations (3-5-3) through (3-5-14). To simplify things, henceforth we assume  $\eta = 1$ . This assumption makes the Hamiltonian linear in  $d$ . That means, from now on our discussion will focus on the characteristics of the singular path. We solve (3-5-14) for  $r/y$  and then eliminate  $r/y$  using (3-5-11) and (3-5-14) to obtain

$$(3-5-15) \quad S^\epsilon = [\alpha/a\beta\epsilon](s/k).$$

Clearly this can hold if  $\epsilon > 0$ . We then rearrange (3-5-1) to find an expression for  $D/K = d/k$ . Next, we logarithmically differentiate (3-5-15). Making use of the result thus obtained and of (3-5-15) in the equation for  $D/K$ , we obtain the following expression for  $d/k = D/K$  in terms of  $\hat{k}$  and  $y/k$ :

$$(3-5-16) \quad d/k = D/K = [\beta(n + \hat{k})\epsilon/\alpha(1 - \epsilon)] + \beta(y/k).$$

Note that,

$$(3-5-17) \quad \hat{k} = x - v - (d/k) - n$$

$$(3-5-18) \quad \hat{X} = \lambda + (\alpha - 1)\bar{K} + \beta\hat{r}$$

$$(3-5-19) \quad \hat{V} = \hat{c} - \bar{K}.$$

We next solve (3-5-11) and (3-5-14) for  $S$ , and then take logarithmic differentiation of the resulting equation with respect to time to obtain an expression for  $\hat{r}$  in terms of  $\bar{K}$ :

$$(3-5-20) \quad \hat{r} = [\bar{K}\{\epsilon + \alpha(1-\epsilon)\} + \epsilon n + \lambda(1-\epsilon)] / (1-\beta)(1-\epsilon)$$

Then we find an expression for  $d/k$  in terms of  $X$  and  $V$  by making use of (3-5-17) in (3-5-16) and solving the resulting equation for  $d/k$ :

$$(3-5-21) \quad D/K = [\beta\{\alpha(1-\epsilon) + \epsilon\}X - \beta\epsilon V] / [\beta\epsilon + \alpha(1-\epsilon)].$$

Substituting for  $\bar{K}$ ,  $\hat{r}$  and  $d/k$  in (3-5-18) and (3-5-19) we finally obtain a differential equation system which the normalized variables,  $X$  and  $V$ , must satisfy:

$$(3-5-22) \quad \hat{X} = ([\{\beta\epsilon - \gamma(1-\epsilon)\}\alpha(1-\beta)X - \{\beta\epsilon - \gamma(1-\epsilon)\}\alpha V] / [(1-\beta)\{\beta\epsilon + \alpha(1-\epsilon)\}]) + (n\gamma + \lambda) / (1-\beta)$$

$$(3-5-23) \quad \hat{V} = ([\alpha\{\beta\epsilon + \alpha(1-\epsilon)\}X - \alpha\sigma(1-\epsilon)(1-\beta)X + \sigma\alpha(1-\epsilon)V] / [\sigma\{\beta\epsilon + \alpha(1-\epsilon)\}]) + (1/\sigma)\{n(\sigma-1) - \delta\}$$

Henceforth, we assume a logarithmic utility function, i.e.  $\sigma = 1$ . Then (3-5-23) reduces to

$$(3-5-24) \quad \hat{V} = \{\alpha[\beta\epsilon - \gamma(1-\epsilon)]X / [\beta\epsilon + \alpha(1-\epsilon)]\} + \{(1-\epsilon)\alpha V / [\beta\epsilon + \alpha(1-\epsilon)]\} - \delta.$$

### 3.5.2 The Stationary State Solution

There are four stationary states in this model: [I]  $X > 0, V > 0$  [II]  $X = 0, V > 0$   
[III]  $X > 0, V = 0$  [IV]  $X = 0, V = 0$ .

The second point can be ruled out if we assert that  $X$  must be greater than  $V$ . The third case violates the transversality condition for capital.<sup>11</sup> The last point can never be attained as the limiting case of the optimal path for stability reasons. Only the first case can apparently be accepted as an equilibrium point. We now turn to the discussion of this stationary state. The stationary values of  $X$  and  $V$  are obtained by setting  $\dot{X} = 0$  and  $\dot{V} = 0$  in (3-5-22) and (3-5-24) respectively. These values are

$$(3-5-25) \quad X^* = (\delta/\alpha) + [(\gamma n + \lambda)(1 - \epsilon)] / \alpha [\gamma(1 - \epsilon) - \beta \epsilon]$$

$$(3-5-26) \quad V^* = [\delta(1 - \beta) + \gamma n + \lambda] / \alpha.$$

Now we digress on the values of  $\epsilon$  which will warrant an economically meaningful solution. The latter must satisfy the following conditions:

$$[I] \quad X^* > 0, V^* > 0$$

$$[II] \quad (X^* - V^*) > 0 \text{ or } (X^*/V^*) > 1$$

$$[III] \quad D^*/K^* = d^*/k^* \geq 0$$

$$[IV] \quad Z^* = R^*/S^* \geq 0.$$

Condition [I] says that the stationary values of ratio of total output ( $Y$ ) to capital ( $K$ ) and ratio of consumption ( $C$ ) to capital ( $K$ ) must be positive. Since capital must be positive in the stationary state, the ratio of output to consumption must be greater than 1 [II].

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<sup>11</sup>See section 3.5.4.

Similarly, the ratio of exploration output (D) to capital (K) must be positive. [III]

Condition [IV] indicates that the resource-utilization rate must be positive.

We find that  $V^*$  is always positive.  $X^*$  is positive only if

$$(3-5-27) \quad \epsilon < \gamma/(\beta+\gamma)$$

Or

$$(3-5-28) \quad \epsilon > \{\gamma(\delta+n)+\lambda\}/\{\beta\delta+\gamma(\delta+n)+\lambda\}$$

It can be shown that  $X^* - V^* > 0$  if (3-5-27) is satisfied. Substituting for  $X^*$  and  $V^*$  in (3-5-21), we can solve for the asymptotic value of  $d/k$ :

$$(3-5-29) \quad (d/k)^* = (\beta/\alpha)[\delta + \{\gamma n + \lambda\}]/\{\gamma(1-\epsilon) - \beta\epsilon\}$$

which can be positive if (3-5-27) is satisfied. Now we consider condition [IV]. The stationary value of resource utilization rate is given by (3-5-14).

$$(3-5-30) \quad Z^* = R^*/S^* = (\alpha/\epsilon)X^*.$$

Since  $Z^*$  must be positive, it implies that

$$(3-5-31) \quad \epsilon > 0 \text{ for any } X^* > 0$$

The asymptotic growth rate of per capita capital, hence per capita consumption and output is equal to

$$(3-5-32) \quad \hat{k} = \hat{c} = \hat{y} = [\beta\epsilon n + \lambda(1-\epsilon)]/[\gamma(1-\epsilon) - \beta\epsilon].$$

The above growth rate is positive if

$$(3-5-33) \quad \epsilon < \gamma/(\beta+\gamma) < 1$$

Now conditions [I] through [III] are satisfied, among other things,<sup>12</sup> if restriction (3-5-27) or (3-5-33) is satisfied. Thus, we come to the conclusion: an economically meaningful solution exists if (3-5-33) is satisfied.

The preceding analysis was based on the assumption that  $\eta$  is one. In other words, the returns to exploration activities with respect to output used for exploration are constant. A look at equations (3-5-14) and (3-5-30) reveals that this value of  $\eta$  forces  $\epsilon$  to be positive. Suppose, we allow  $\eta$  to be anything less than one. According to equation (3-5-14),  $\epsilon$  can be negative in that circumstance. This analysis suggests that a steady state may exist when the returns to exploration activities are negatively related to the stock of the resource. The model need not necessarily involve a "learning by doing" process. However, the model becomes very complicated when  $\eta < 1$  or  $\epsilon < 0$ .

### 3.5.3 Stability Analysis

We now examine the stability property of the steady state solution. The determinant value of the Jacobian matrix of differential equations (3-5-22) and (3-5-24) evaluated at equilibrium is

$$(3-5-34) \quad |J| = [\alpha^2 \{ \beta \epsilon - \gamma(1-\epsilon) \} / (1-\beta) \{ \beta \epsilon + (1-\epsilon)\alpha \}] X^* V^*$$

The trace is equal to  $\delta > 0$ . A few points about the nature of the equilibrium can be made. Since the trace is always positive, the equilibrium can never be stable. It will be a saddle point equilibrium if  $|J| < 0$ . This holds if (3-5-27) is valid. Thus, we conclude: the model generates balanced growth paths for per capita capital, consumption, output and exploration-output if  $0 < \epsilon < \{ \gamma / (\beta + \gamma) \}$ .

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<sup>12</sup>Each of these conditions is satisfied if  $\epsilon$  is less than some positive number less than one or  $\epsilon$  is larger than some positive number, not always less than one.

### 3.5.4 Sufficient Condition

We have two state variables in this model, K and S. It is easily seen that the Hamiltonian is concave in these two variables since

$$|H_{kk}| = \mu_1 Y_{kk} < 0,$$

$$\begin{vmatrix} H_{kk} & H_{ks} \\ H_{sk} & H_{ss} \end{vmatrix} = \mu_1 \mu_2 a \epsilon (\epsilon - 1) Y_{kk} S^{\epsilon-2} DL > 0$$

Moreover, the transversality condition for capital is satisfied here with  $\nabla = (1-\alpha)X^* - V^* = -\delta$ . However, the transversality condition for the stock of the resource may not be satisfied here. This is because the stock of the resource will never be finite in this model.

### 3.5.5 Shadow Prices

Here again, the asymptotic behaviour of shadow price of capital investment is indeterminate. The shadow price of the resource accumulation or decumulation for this model will be growing at a rate less than the rate at which it grows when the stock of the resource is fixed. The reason is that this model involves a "learning by doing" process in the exploration technology. As the stock of the resource increases in size, the economy gets more from each additional unit of exploratory effort. That means, the resource becomes less and less scarce with the increase in the size of the stock of the resource. Therefore, the utility value or shadow price of one unit of the resource unextracted will be growing at a lower rate in this model.



### 3.5.6 Final Remarks

In this section, we discovered that balanced growth paths for all the variables can be obtained if  $0 < \epsilon < \gamma/(\beta+\gamma)$ . For  $\epsilon$  in this range, per capita consumption, output and capital grow at the same positive rate even in the absence of an exogenous technical progress. There are, however, conceptual problems with positive values of  $\epsilon$ . A positive value of  $\epsilon$  implies that the returns to exploration activities increase with the increase in the size of the resource stock. As a matter of fact, this model portrays an economy where the problem of resource scarcity is not very severe. One reason for the existence of a steady state for positive values of  $\epsilon$  may be the assumed constant returns to exploratory efforts ( $\eta=1$ ). A steady state for negative values of  $\epsilon$  only may be obtained with diminishing returns to exploration ( $\eta<1$ ). It is unfortunate that the model gets very complicated and non-tractable with  $\eta<1$ . This model presents the scenario of a "golden age" economy where per capita consumption can increase indefinitely. Though the model presented in this section contains a "learning by doing process" and as such is less interesting, it carries some academic interests. It provides an example of a model which is elegant mathematically but less interesting conceptually.

We can compare the results from this model with the results of the first model developed in this chapter. Here we find that per capita consumption grows at a steady rate only when  $\epsilon < \gamma/(\beta+\gamma)$ . By contrast, per capita consumption increases indefinitely in the first model. It is difficult to explain the difference in the results as the models are structurally different. Both the models; however, share one common conclusion: per capita consumption can increase indefinitely in the absence of exogenous technical progress.

### 3.6 COMPARISON OF RESULTS

In this section, we compare our results with those obtained from some related models. Though the literature on growth with exhaustible resources has experienced a vast expansion during the last decade, few attempts have been made to incorporate the phenomenon of exploration in those growth models.

Pindyck (1978) analysed the optimal paths of exploratory activity and resource extraction using a profit maximizing model. Particularly, he showed that the price profile of the resource will be 'U'-shaped. Resource extraction rate will rise initially and then will fall as the level of exploration falls. The model used in section 2 of this chapter is very much similar to that in Pindyck (1978). The only difference is that Pindyck does not use a separable functional form for reserve additions as is done here. Therefore, his model cannot be used to make as complete an analysis of the optimal extraction path as the current model is able to do. His model leads to a declining level of exploration activities during the end phase of the program whereas in our case, it is always increasing. A comparison of results, however, cannot be made because his analysis was concerned with competitive growth paths rather than the optimal growth path.

Deshmukh and Pliska (1980) assumed that the uncertainty involved in the timing and magnitudes of new discoveries of the resource can be controlled through exploratory activities. Both the levels of exploration and consumption depend on the level of proven reserves. The objective is to maximize the expected sum of discounted utility over an infinite time horizon. They showed that the optimal consumption is non-decreasing and exploration is non-increasing in the size of known deposits of the resource. Deshmukh and Pliska (1983) extended their previous model to incorporate environmental uncertainty that affects both the demand and supply aspects of the resource. The results are very much similar in both the cases. The Deshmukh and Pliska (1980,1983) models suffer from three deficiencies. First, they do not

consider the fact that the additions to the stock of the resource made possible by exploratory efforts diminish as the size of cumulative discoveries increases. Second, production of the consumption good is absent in their models. It is assumed that the extracted resource can either be consumed or used to discover new sources of the resource. Thus, the issues of substitution between the factors of production are not addressed in their models. As a matter of fact, they were mainly concerned with the effect of uncertainty with regard to timing and size of the discoveries on the optimum consumption profile of the economy. Third, they assume a social benefit function separable in consumption utility and exploration costs. The models developed in this chapter make improvements on the Deshmukh and Pliska models in these three respects.

Takayama (1980) used a growth model where he assumed a proportional relationship between the growth rate of the resource and the proportion of labour used in the exploration process<sup>13</sup>. He demonstrated the existence of a saddle point equilibrium for the ratios; output to capital, consumption to capital and resource flow to resource stock. Moreover, per capita consumption can be growing even in the absence of an exogenous technical progress. Takayama's model also has a number of drawbacks. First, like Deshmukh and Pliska (1980,1983), he ignores the effect of the size of cumulative discoveries on the size of the returns to exploration activities. Instead he, like our model in section 5, has these returns depend only on the size of the stock of the resource. Second, the equilibrium point derived in Takayama (1980) is not the correct equilibrium point for the same reason as explained in connection with Stiglitz (1974) model in chapter 2. According to Takayama (1980), the proportion of labour used in the exploration process tends to a non-negative value at the so-called equilibrium point. He gets this value because the ratio of the resource use to the resource stock turns out to be a non-negative value at his equilibrium point. The

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<sup>13</sup> He did not use the word "exploration process". Instead he termed it as the "portion of labour used in the research sector". But "research" leads to the augmentation of the resource stock

equilibrium point shown in Takayama (1980) cannot be an stable equilibrium point for the the resource use to the resource stock ratio. The only stable equilibrium point for this ratio is where the ratio is zero in value. In other words, his resource utilization rate tends to zero in the long run. That means, the proportion of labour used in the exploration process should be zero at his equilibrium point. In section 3 of this chapter, we showed that the economy converges to the equilibrium when the proportion of labour utilized in exploration is 1. The difference in the results is clearly due the fact that in our model there are decreasing returns to exploration activities while in his model there are constant returns to the output used for exploration.

Arrow and Chang (1980) also used a model where costly exploration leads to new discoveries. For them, however, the returns to exploration activities are uncertain. Using their model, they explained why mineral prices may fail to rise at the market rate of interest. Like Pindyck, they did not analyze the optimal growth paths of the economy. On the basis of preceeding discussion, we can conclude that this thesis makes the most rigorous study yet of the optimal growth path in the presence of the possibility of exploring to augment stocks of resources.

### 3.7 CONCLUSIONS

This chapter investigated the nature of the optimal growth paths experienced by an economy capable of augmenting its stock of the resource by exploration activities. Four models each with a different type of exploration technology were examined. First, the returns to exploration activities are assumed to be related positively to the level of output used for exploration and negatively to the size of cumulative discoveries. It was demonstrated that per capita consumption will increase indefinitely even if there is no exogenous technical progress.

By contrast, in a model where exploration is absent, per capita consumption cannot increase in the absence of exogenous technical progress. The result obtained here makes much economic sense. When an economy can increase its stock of the resource by costly exploration, it should not face any resource scarcity problem. The fact is that the benefits from exploration may more than outweigh the costs of exploration. Second, the returns to exploration activities depend positively on the portion of labour used for exploration and negatively on the cumulative discoveries. A long run equilibrium solution can be obtained when all labour is used in the exploration process. The minimum rate of exogenous technical progress necessary for a rising per capita consumption is higher than the rate for the case with no exploration. This shows, in contrast to the first model, that the costs of exploration can be so high in comparison to the benefits as to make it more difficult to achieve a rising per capita consumption. Third, the returns are related negatively to the size of cumulative discoveries and positively to the portion of capital used for exploration. An equilibrium is found to exist only when all capital is used in the exploration process. Per capita consumption increases indefinitely in this case. Finally, we assumed that the yields from exploration depend positively on both the stock of the resource and the level of output used for exploration. An equilibrium point with positive values for all the variables is derived under the additional assumption of constant returns to exploration output. Balanced growth paths for per capita consumption, output and capital were obtained for some values of the parameters. Per capita consumption can be growing even in the absence of exogenous technical progress. However, the model in this case implies a "learning by doing" process in exploration. Such a process in exploration activities is very unlikely.

## CHAPTER 4

### GROWTH WITH UNCERTAIN YIELDS FROM EXPLORATION

#### 4.0.1 Introduction

The last decade has witnessed a rising interest in growth models with exhaustible resources. The early works of Stiglitz (1974) and Solow (1974) have been extended in many directions. Recently the element of uncertainty has been incorporated into these growth models. Optimal consumption paths have been analysed in growth models with exhaustible resources and uncertainty about (i) the new supply of resources by exploration, (ii) the size of the stock of the resource and (iii) the timing of the development of a perfect substitute. Of particular interest is the class of growth models that combine exploration with uncertainty. Arrow and Chang (1982) consider a model where the size of the stock of the resource in a spatial location follows a Poisson distribution. In their model, costly exploration is a means of increasing the level of proven reserves and reducing the uncertainty about the stock of the resource. In Deshmukh and Pliska (1980), exploration effort determines the uncertainty in the timing of discoveries as well as their magnitudes. In a subsequent paper, Deshmukh and Pliska consider two kinds of uncertainty. One source of uncertainty involves the resource discovery process and another is about the economic environment that affects the productivity of exploratory effort and the demand conditions for the resource in future.

This chapter examines the growth behaviour of an economy with costly exploration under a different kind of uncertainty. It differs from Deshmukh and Pliska in two respects. First, the returns to exploration activities depend on the level of output used for exploration, the existing stock of the resource and an uncertain element ( $\theta$ ). This uncertain element follows a Wiener stochastic process with zero mean and a variance proportional to the magnitude of time lapse between the present and the future. Deshmukh and Pliska used an

additive Markov process. Second, in both Arrow and Chang (1982) and Deshmukh and Pliska (1980), the role of capital is absent. In fact, they do not have a production function. People consume the resource as it is extracted from the ore. The present model, on the other hand, allows for a production function and capital investment.

The type of uncertainty envisaged here is different from the usual one where some parameters or variables follow a probability distribution. The inclusion of an uncertain element in exploration technology implies that the returns to exploration activities shift randomly according to a stochastic process. The shift is, of course, continuous. Though the returns to exploration activities are known today with certainty, the future returns may be high or low. But on the average, the upward shifts will cancel the downward shifts, leaving a net zero level of expected shifts. The variance of the shifts, however, increases with the time horizon as people are more unaware of distant future than near future.

The model used in this chapter is an extension of one of the models developed in Pindyck (1980). The model developed here, however, is structurally different from Pindyck's model. In our model, the central planner's objective is to maximise the expected utility of consumption over time. The economy faces the constraints relating to capital formation and resource depletion. Pindyck, however, maximises the expected profit obtained from the sale of resources net of costs of extraction and exploration. Production of the consumption good is absent in his model. There is another difference between the two models. Instead of cumulative discoveries, we use existing stock of the resource as an argument of exploration technology. We do this for ease of manipulations.

The basic questions raised in this chapter emanate from the stochastic nature of the problem. It is interesting to examine the behaviour of expected consumption, exploration and resource utilization in the presence of uncertainty. Will these expected values be different from the deterministic case? Does uncertainty lead to increased or decreased rates of

consumption, exploration activities and resource utilization?

It is shown in this chapter that the expected rate of change of consumption is stimulated by uncertainty in the yields from exploration. Moreover, with constant returns to scale in production technology and diminishing returns in exploration technology, the expected rates of change of exploratory effort and resource utilization will also increase due to uncertainty.

The second section of the chapter sets up the model. The necessary conditions for maximisation are derived and analysed in the third section. In the fourth section, the effect of uncertainty is analysed on the basis of the presence of uncertainty coefficient in different expected value formulations. In the next section, the results of this chapter are compared with those in a deterministic version of the same model. The concluding section summarizes the main findings of this chapter. As tools, we make use of Stochastic Dynamic programming and Ito's Lemma.

#### 4.0.2 The Model

We assume an economy with a Cobb-Douglas production technology. The production function is characterised by constant returns to scale. The inputs of production are capital (K), the resource (R) and labour (L). We assume that the labour force is constant and normalized to 1. Thus, we have,

$$(4-1) \quad Y = Y(K, R)$$

The production function is assumed to be continuous and thrice differentiable. Total output is



allocated among three uses, namely, consumption, investment and exploration activities. The rate of capital formation is given by

$$(4-2) \quad dK = (Y-C-D)dt$$

The stock of the resource (S) decumulates because some resource is used up in the production process. It also accumulates due to exploratory efforts. The additions to the stock of the resource depend on the amount of output redirected to exploration activities (D), the stock of the resource and the uncertainty element. Normally, the additions to the stock should depend positively on D and negatively on cumulative discoveries. Cumulative discoveries is equal to the cumulative amount of the resource used up in the production process plus the amount lying in the ground (S) at any point of time. Here we use S as a proxy for cumulative discoveries. The time rate of change of the stock of the resource is, therefore, given by

$$(4-3) \quad dS = [-R+f(D,S,\theta)]dt, \quad f_D > 0, \quad \text{where } f_D = \partial f / \partial D \text{ etc.}$$

$\theta$  signifies the uncertainty element in the exploration technology. It is a stochastic process of the form

$$(4-4) \quad d\theta = \sigma(\theta)dz \quad \Rightarrow \quad \theta(t_1) - \theta(t_2) \sim N(0, \sigma^2 dt) \quad t_1 - t_2 = dt$$

$z$  is a Weirner process defined on some probability space.  $dz$  is distributed normally with mean zero and variance  $dt$ . Hence  $d\theta$  is also normally distributed with mean 0 and variance  $\sigma^2 dt$ . We do not make any assumption about how  $\theta$  affects  $f(\cdot)$ . The presence of  $\theta$  in equation (4-3) makes the rate of increase in the stock of the resource stochastic. The uncertainty of stochastic process  $\theta$  increases as time goes on. This is because one is more uncertain about more distant values of  $\theta$  than near future values of  $\theta$ . The fluctuations in the yields from exploration occur continuously. Equation (4-4) also precludes any kind of

jumps.

Given the types of circumstances regarding capital formation and resource accumulation, the economy has an objective of maximizing the stream of discounted utility with respect to C, R and D over a finite time horizon. That is,

$$(4-5) \quad \text{Max} \int_0^T U(C) e^{-\delta t} dt = \int_0^T U(C, t) dt$$

subject to conditions (4-1) through (4-4). The utility function, U(C), is assumed to be continuous and thrice differentiable.

#### 4.0.3 Necessary Conditions

The optimal value function in the framework of Stochastic Dynamic Programming technique can be defined as:

$$(4-6) \quad J = J(\theta, K, S, t) = \text{Max} \int_t^T U(C, t) d\tau$$

Bellman's equation of optimality is

$$(4-7) \quad \text{Max} \{U(C, t) + J_t + (Y - C - D)J_K + [f(D, S, \theta) - R]J_S + (1/2)\sigma^2(\theta)J_{\theta\theta}\} = 0$$

We have applied Ito's Lemma in the above equation of Stochastic Dynamic Programming.<sup>1</sup> Maximizing with respect to D, R and C, subject to (4-1) through (4-4), we get,

$$(4-8) \quad J_S = Y_R J_K$$

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<sup>1</sup>See Fischer (1975), Chow (1979) and Kushner (1967) for a detailed discussion of these techniques.

$$(4-9) \quad J_K = f_D J_S$$

$$(4-10) \quad \partial U(C, t) / \partial C = J_K$$

Use of equations (4-8) and (4-9) yields,

$$(4-11) \quad f_D Y_R = 1$$

Equation (4-8) implies the equality between the shadow price of the resource ( $J_S$ ) and the gain in utility achieved from the use of the resource in the production function ( $Y_R J_K$ ). Equation (4-9) stipulates that the opportunity cost of exploration ( $J_K$ ) should be equal to the gain in utility from using one unit of output for exploration ( $f_D J_S$ ). Equation (4-10) shows the equality between the gain ( $\partial U(C, t) / \partial C$ ) and the opportunity cost ( $J_K$ ) of consuming one unit of output. Equation (4-11) can be interpreted in the following way. Suppose we apply one unit of output to exploration and use the additional reserves discovered to raise output. Along the optimal path, the increase in output due to the newly discovered reserves will be equal to the original unit of output.

The preceding discussion focused on the derivation of the necessary conditions for maximisation and their economic interpretations. Next, we will examine the effect of uncertainty about the returns to exploration activities on the expected rate of change of consumption. Now differentiating (4-7) with respect to  $K$ , we get,

$$(4-12) \quad \{ J_{tK} + Y_K J_K + (Y - C - D) J_{KK} + [ f(D, S, \theta) - R ] J_{SK} + (1/2) \sigma^2(\theta) J_{\theta\theta K} \} = 0$$

Along the optimal path,  $J_K$  depends on  $K$ ,  $S$  and  $\theta$ . Applying Ito's Lemma to find  $dJ_K$  along the optimal path and taking its expectation, we obtain,

$$(4-13) \quad (1/dt)E_t(dJ_K) = J_{Kt} + (Y-C-D)J_{KK} + [f(D, S, \theta) - R]J_{KS} + (1/2)J_{K\theta\theta}\sigma^2$$

The expression for  $(1/dt)E_t(dJ_K)$  can be used to rewrite (4-12) as follows:

$$(4-14) \quad Y_K J_K + (1/dt)E_t d(J_K) = 0$$

Similarly, we differentiate (4-7) with respect to  $S$  and then use Ito's Lemma to expand  $dJ_S$ . That yields,

$$(4-15) \quad J_S f_S + (1/dt)E_t d(J_S) = 0$$

We cannot take time derivatives of equation (4-8), (4-9) and (4-10), because both sides of those equations involve stochastic processes. We apply Ito's differential operator  $(1/dt)E_t$  on both sides of (4-8), (4-9) (4-10) and (4-11), and thus obtain,

$$(4-16) \quad (1/dt)E_t d(J_S) = (1/dt)E_t d(Y_R J_K)$$

$$(4-17) \quad (1/dt)E_t d(J_K) = (1/dt)E_t d(f_D J_S)$$

$$(4-18) \quad (1/dt)E_t d[\partial U(C, t)/\partial C] = (1/dt)E_t d(J_K)$$

$$(4-19) \quad (1/dt)E_t d(f_D Y_R) = 0$$

Now use of equations (4-10), (4-14) and (4-18) yields,

$$(4-20) \quad (1/dt)E_t d[\partial U(C, t)/\partial C] = -Y_K [\partial U(C, t)/\partial C]$$

The expected rate of change of marginal utility can be obtained by expanding equation (4-20) as follows:

$$(4-21) \quad (1/dt)E_t dU_C = U_C [\delta - Y_K]$$

Along the optimal path, C depends on K, S,  $\theta$ , and t. That allows us to find an expression for  $(1/dt)E_t dU_C$  using Ito's Lemma. The process yields,

$$(4-22) \quad (1/dt)E_t dC = [U_C/U_{CC}] (\delta - Y_K) - [U_{CCC}/2U_{CC}] C^2 \sigma^2$$

Equation (4-22) gives the expected rate of change of consumption. As we see from the equation, the expected rate of change of consumption increases due to uncertainty. Next, we derive expressions for expected rates of change of resource utilization and exploration activities. Using equations (4-15) and (4-16), we get,

$$(4-23) \quad Y_R (1/dt)E_t d(J_K) + J_K (1/dt)E_t dY_R = -J_S f_S$$

Substituting for  $(1/dt)E_t d(J_K)$  and  $J_S$  from (4-14) and (4-8) respectively in the above expression and dividing both sides by  $J_K$ , we have,

$$(4-24) \quad (1/dt)E_t d(Y_R) = Y_R (Y_K - f_S)$$

Marginal productivity of resource  $Y_R$  has two arguments, capital (K) and resource (R). Along the optimal path, R is a function of K, S,  $\theta$  and t of which  $\theta$  is a stochastic process. Hence R is also stochastic. This suggests the use of Ito's Lemma for finding  $dY_R$ :

$$(4-25) \quad dY_R = Y_{RK} dK + Y_{RR} dR + (1/2) \{ Y_{RRR} (dR)^2 + 2Y_{RRK} dR dK + Y_{RKK} (dK)^2 \}$$

Ito's Lemma can again be used to calculate  $(dR)^2$  in the above expression. Finally, the expansion of (4-23) gives the expected rate of change of resource utilization:

$$(4-26) \quad (1/dt)E_t dR = \{[-Y_R(f_S - Y_K) - Y_{RK}(Y - C - D)]/Y_{RR}\} \\ - \{\sigma^2 Y_{RRR} R^2 / 2Y_{RR}\}$$

Expanding (4-19), we get,

$$(4-27) \quad (1/dt)E_t d(f_D) = (-f_D/Y_R)(1/dt)E_t dY_R$$

We can use Ito's Lemma to derive  $d(f_D)$ . Finally, expansion of (4-27) yields,

$$(4-28) \quad (1/dt)E_t dD = (-1/f_{DD})\{f_D(Y_K - f_S) + f_{DS}[f(D, S, \theta) - R]\} \\ - (\sigma^2/2f_{DD})[f_{DDD}D^2 + f_{D\theta\theta} + 2f_{DD\theta}D\theta]$$

#### 4.0.4 Effects Of Uncertainty

First, we observe from (4-20) that the expected rate of change of marginal utility of consumption is not affected by uncertainty. On the other hand, the expected rate of change of consumption gets stimulated by uncertainty (4-22). This, of course, depends on  $U_{CC}$  and  $U_{CCC}$  being negative and positive respectively. Those two conditions are satisfied for all iso-elastic utility functions with diminishing marginal utility. The rate of resource utilization will increase due to uncertainty. This result hinges on the assumption of constant returns to scale in the production function. With constant returns to scale, output and the inputs of production increase at the same proportionate rate. Because of the assumption, we have  $Y_{RRR} > 0$ , which in turn implies that the rise in marginal productivity of resource due to a one unit decrease in the use of the resource will be more than the fall in the marginal productivity for a one unit increase in the use of the same. Stochastic changes in the stock of the resource leads to changes in the use of the resource. With zero mean fluctuations in the growth rate of the stock of the resource, the average marginal productivity of the

resource will be higher. Since these relations obtain at all points of time, the social planner finds always an incentive to increase the rate of the resource use (R) faster than in the certainty case if at least the same level of Y-C-D is used. Similarly, uncertainty will increase the expected rate of change of exploratory effort if we assume

- (i)  $f_{DD} < 0$       (ii)  $f_{DDD} > 0$       (iii)  $f_{D\theta} > 0$  and  
 (iv)  $f_{D\theta\theta} > 0$       (v)  $f_{DD\theta} > 0$       (vi)  $D_{\theta} > 0$

The first assumption implies diminishing returns to output used for exploration. The second assumption means that the returns to exploration activities decrease at decreasing rates as exploration continues. The third assumption stipulates that the uncertainty element  $\theta$  raises the marginal productivity of exploratory effort.<sup>2</sup> The fourth assumption says that the increase in marginal productivity due to one unit increase in  $\theta$  will be more than the decrease in marginal productivity of exploratory effort due to equal decrease in  $\theta$ . Zero mean of fluctuations in  $\theta$  will, on the average, make the marginal productivity of exploratory effort higher. This will lead to an increase in the expected rate of change of exploratory effort. The remaining two conditions can also be assumed to be satisfied. Since the expected rates of change of both the resource use and exploration activities increase under uncertainty, the expected rate of change of output is also likely to increase under such a situation. This explains why the expected rate of change of consumption increases when there is uncertainty with regard to the yields from exploration.

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<sup>2</sup> We could also assume  $f_{D\theta} < 0$ . The basic results, however, do not change.

#### 4.0.5 Comparison With Deterministic Model

We can compare the results of this chapter with those of the fourth model of the last chapter where we had

$$\dot{S} = -R + f(D, S, \theta) = -R + aS^\epsilon D$$

with

$$f_S = a\epsilon S^{\epsilon-1}D, \quad f_{DD} = 0.$$

It was shown that with  $0 < \epsilon < 1$ , balanced growth paths can be obtained for capital, consumption and total output. Moreover, we found that the resource use rate (R) and the stock of the resource (S) grow at the same positive rate over time. In the current model also, the resource use rate increases over time. Allowing  $\sigma^2$  to be zero in equation (4-26), we find that the (expected) rate of change of resource utilization will be positive if

$$f_S > Y_K, \quad \text{or} \quad Y_{RK}(Y - C - D) > Y_R(Y_K - f_S) \quad \text{and} \quad Y_K > f'_S.$$

The two models also produce similar results with regard to the consumption profile of the economy. But the levels of exploratory effort in two models cannot be compared since a value of  $f_{DD} = 0$  (as assumed in the earlier chapter) will make the expression  $(1/dt)E_t dD$  undefined in current chapter.

#### 4.0.6 Conclusion

We analysed the effects of uncertainty on expected rates of change of consumption, resource utilization and exploratory effort. The uncertainty as modelled in this chapter has a distinctive interpretation. Here we assume that the present returns to exploration activities are



always known but the future returns shift upward or downward randomly and continuously. This kind of uncertainty is different from the case where even the present situation is not known with certainty.

The results obtained here are precise. The levels of consumption, exploratory effort and resource utilization rate are expected to change at a higher rate due to uncertainty. The last model of chapter 3 can be considered as a special case of the present model. With  $\sigma^2 = 0$  in this chapter, we get almost identical results for consumption and resource utilization rates in both the models. The levels of exploratory effort could not be compared for technical reasons.

## CHAPTER 5

### GROWTH WITH EXHAUSTIBLE RESOURCES IN AN OPEN ECONOMY

#### 5.1 INTRODUCTION

The previous chapters explored the nature of growth with an exhaustible resource in the context of a closed economy. The assumption of a closed economy, however, is not realistic. International trade accounts for a significant portion of the total GNP of every modern economy. An open economy with an exhaustible resource can be more optimistic about its prospect for survival. For example, a closed economy endowed with a finite stock of the resource and possessing no other factors of production and no technical progress cannot survive in the long run. But an open economy under the same situation may export the resource and use the proceeds obtained from these exports to finance capital formation at home. The analysis of the growth of an open economy facing the constraint of a limited supply of a resource has not received much attention from economists until very recently. A few papers dealing with models of the interactions among trade, investment and resource use have been published in recent years. Quite a number of them evolve around a very simple but pragmatic model developed by Dasgupta, Eastwood and Heal (1978).

The objective of this chapter is to examine the implications of some changed assumptions in the models developed by DEH and Moussavian (1985). They analysed the optimal growth paths of an open economy which owns a finite stock of a resource. The resource has two uses: it can be used in the production of a material good at home or it can be exported abroad. The country also invests in foreign assets. The total income of the

economy then is the sum of the interest income from foreign assets, the proceeds from resource exports and the value of the home country production. A portion of total income is consumed and the residual is invested. The objective is to maximise the sum of the discounted utility of consumption. Dasgupta, Eastwood and Heal (1978) derived explicit optimal time paths for consumption, capital, the exported resource and the resource used in the production process. They came to two main conclusions. First, they showed that domestic production in such an economy may be falling over time. Second, they proved that the resource depletion rate is independent of the discount rate and the elasticity of marginal utility. Under autarky, on the other hand, these two parameters play a dominant role in determining the optimal rate of depletion. Moussavian (1985) added one non-traded sector to the DEH model. He proved that the presence of a non-traded sector makes the possibility of a total diminution of production activities at home less likely.

In this chapter, we test the robustness of these results against some minor modifications of these models. In the first case, we drop foreign assets from the original DEH model. Instead, we incorporate the idea that home consumption and investment can be boosted by importing the consumption good. The DEH model ignored this possibility. In the second case, we introduce an endogenous rate of return on foreign assets in the Moussavian model. He assumed an exogenous rate of return.<sup>1</sup>

The plan of this chapter is as follows. The next section deals with the modified DEH model. We then discuss the results of the modified Moussavian model in the third section. In each case, we make a comparative analysis of the results obtained from the original model and from its modified form.

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<sup>1</sup> We discussed the conclusions obtained in these papers in section 5 of chapter 2.

## 5.2 THE MODIFIED DEH MODEL

### 5.2.1 Assumptions

Here we briefly describe the assumptions of the modified DEH model. The time rate of change of the stock of the resource is given by

$$(5-2-1) \quad \dot{S} = -(R+E)$$

where

R is the amount of the resource used in the production process and E is the amount of the resource exported. In each period, the country sells E units of the resource at a price P. It uses the proceeds to import a commodity at price  $\pi$ . Prices P and  $\pi$  are both denoted in foreign currency. The amount of the imported commodity is, therefore,  $PE/\pi$ . Both the home produced goods and the imported goods can be used for the dual purposes of consumption and investment. National income Y is then given from the production side as follows:

$$(5-2-2) \quad Y = (PE/\pi) + Y_d$$

where  $Y_d$  is the value of the home produced goods. We assume that the domestic production technology can be represented by

$$(5-2-3) \quad Y_d = e^{\lambda t} K^\alpha R^\beta L^\gamma \quad \alpha + \beta + \gamma = 1$$

Labour is assumed to grow at an exogenously given rate, n. The demand behaviour of the resource importing country is given by a demand function:

$$(5-2-4) \quad E = E(P) \quad E' = dE/dP < 0$$

The price of the exported resource is set so that demand and supply of the resource are equal at all points of time. Finally, total national income is allocated between consumption and investment. Thus,

$$(5-2-5) \quad \dot{K} = Y - C$$

Substituting for Y in the above equation, we get,

$$(5-2-6) \quad \dot{K} = (PE/\pi) + Y_d - C$$

Given the assumptions (5-2-1) through (5-2-6), the objective is to maximise the sum of discounted utility of per capita consumption over time with respect to c, r and E/L.

### 5.2.2 The Model and Necessary Conditions

The model can be cast in per capita variables as follows:

$$(5-2-7) \quad \text{Max } \Omega = \int_0^{\infty} e^{-\delta t} U(c) dt$$

subject to

$$(5-2-8) \quad \dot{k} = (PE/\pi L) + f(k, r, t) - nk - c$$

$$(5-2-9) \quad \dot{s} = -\{(E/L) + r\} - sn$$

$$(5-2-10) \quad f(k, r, t) = e^{\lambda t} k^{\alpha} r^{\beta}$$

The home country is assumed to be a price taker with respect to exports of this resource. It is, therefore, assumed to know in advance the time path of P. Since we have assumed competitive equilibrium in the world market for the resource, this time path is determined by

the additional market clearing condition:

$$(5-2-11) \quad P = P(E)$$

It should be noted here that  $P=P(E)$  is not a constraint on the optimisation.

The current value Hamiltonian is given by

$$H = U(c) + \mu_1 [ (PE/\pi L) + f(k, r, t) - nk - c ] + \mu_2 [ -\{ (E/L) + r \} - sn ]$$

The following necessary conditions are obtained for an interior maximum:

$$(5-2-12) \quad U'(c) = \mu_1$$

$$(5-2-13) \quad \mu_1 P/\pi = \mu_2^2$$

$$(5-2-14) \quad \mu_1 f_r = \mu_2 \quad \text{where } f_r = \partial f(k, r)/\partial r \text{ etc.}$$

$$(5-2-15) \quad \hat{\mu}_1 = \delta + n - f_k$$

$$(5-2-16) \quad \hat{\mu}_2 = \delta + n$$

Now use of equations (5-2-13) and (5-2-14) yields,

$$(5-2-17) \quad P/\pi = f_r$$

Equation (5-2-17) says that along the optimal path, the marginal rate of return to the resource from both the domestic and external uses should be the same. Logarithmic differentiation of (5-2-14) and application of (5-2-15) and (5-2-16) yield,

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<sup>2</sup>The Hamiltonian is linear in  $E/L$ . Hence, we are assuming that the singular path should be followed here.

$$(5-2-18) \quad f_k = \hat{f}_r$$

Equation (5-2-18) is the famous Solow-Stiglitz efficiency condition which says that the growth rate of the return to the resource in its domestic use should equal the marginal productivity of capital. In other words, the rates of return on the two assets, the exhaustible resource and the capital input, should be equal to each other. Logarithmic differentiation of (5-2-13) and use of (5-2-15) and (5-2-16) give,

$$(5-2-19) \quad \hat{P} = f_k + \hat{\pi}$$

According to equation (5-2-19), the price of exports will be rising and the amount of exports will be falling as long as the price of the imported commodity is rising or remains constant.

Taking the time derivative of (5-2-12) and substituting for  $\hat{\mu}_1$ , we get,

$$(5-2-20) \quad \hat{c} = -(1/\sigma)(\delta+n-f_k)$$

Finally, differentiation of (5-2-4) yields,

$$(5-2-21) \quad \hat{E} = \epsilon \hat{P} \quad \epsilon = PE'(P)/E(P) < 0$$

For a better understanding of the optimal growth paths obtained here, we derive the differential equations for  $X$ ,  $V$  and  $W$  where  $X=Y_d/K$ ,  $W=PE/\pi K$  and  $V$  is as in previous chapters. We then examine whether the optimal paths converge to a steady state. The optimal growth paths of the economy here satisfy the following differential equations:<sup>3</sup>

$$(5-2-22) \quad \hat{X} = [-(\gamma+\alpha\beta)X + \gamma V - \gamma W + \lambda + \gamma n] / (1-\beta)$$

<sup>3</sup>We use equations (5-2-8), (5-2-10) and (5-2-18) to derive (5-2-22). Equations (5-2-20) and (5-2-8) are used to obtain (5-2-23). Finally, the use of equations (5-2-8), (5-2-19) and (5-2-21) gives (5-2-24).

$$(5-2-23) \quad \hat{V} = \{(\alpha - \sigma) / \sigma\} X + V - W - \{(\delta + n - \sigma n) / \sigma\}$$

$$(5-2-24) \quad \hat{W} = \{(1 + \epsilon) \alpha - 1\} X + V - W + \epsilon \pi$$

Finally, the rates of growth of resources used in the home production process and exported abroad can be expressed in terms of X, V and W as follows:

$$(5-2-25) \quad \hat{R} = \{\lambda - \alpha(V - W) + \gamma n\} / (1 - \beta)$$

$$(5-2-26) \quad \hat{E} = \epsilon \{\alpha X + \hat{\pi}\}$$

### 5.2.3 The Stationary State

The equations of the optimal paths derived here are such that they do not give rise to a stationary state with positive values for all the variables. The point ( $X > 0$ ,  $V > 0$  and  $W = 0$ ) is the only feasible equilibrium here. It can be verified that this equilibrium is stable in W provided

$$(5-2-27) \quad \epsilon < -[\lambda(\sigma - 1) + \delta(\beta + \gamma) + \beta n(1 - \sigma)] / [\sigma\lambda + \gamma(\delta + n) + \pi(\sigma\beta + \gamma)]$$

We conjecture that the optimal growth path does converge to this equilibrium when  $\epsilon$  satisfies this condition. Thus, trade allows society to augment its consumption but not by substantial amounts asymptotically. We also conjecture that in the cases where  $\epsilon$  does not satisfy this condition, W tends to  $\infty$ . In effect, when  $\epsilon > -1$  in magnitude, total revenues from export sales actually increase as extraction diminishes. If W tends to  $\infty$ , then from (5-2-24) it follows that  $V > W$ . It is likely that when  $\epsilon$  is sufficiently small in magnitude, ever increasing per capita consumption can be maintained through ever increasing export revenues.



Returning to the condition where  $\epsilon$  satisfies (5-2-27), it can also be shown that the transversality condition for capital is satisfied at the conjectured equilibrium. Moreover, the equilibrium is identical to Stiglitz's (1974) equilibrium. In other words, the long run behaviour of the system is to tend to Stiglitz's equilibrium with  $W$  decreasing to zero over time. More insights into the nature of the optimal paths can be obtained by using the transversality condition for capital which, for this model, can be conveniently put as

$$(5-2-28) \quad \nabla = (1-\alpha)X^* - V^* + W^* \leq 0$$

asymptotically. Now the differential equation (5-2-24) can be rewritten as

$$(5-2-29) \quad \hat{W} = (\alpha-1)X+V-W+\alpha\epsilon X+\epsilon\hat{\pi}$$

The sum of the first three terms in the above equation is positive when (5-2-28) is satisfied. The fourth term of the expression, however, is negative. The last term is negative as long as price of the imported good is not falling. That means,  $W$  may be continuously decreasing to zero as long as the price of the imported good is not falling very sharply. Thus, the optimal growth paths here may converge to the equilibrium of Stiglitz (1974). Along the optimal trajectory,  $W$  will be falling over time. Now consider the transversality condition for  $S$  which is

$$(5-2-30) \quad \lim_{t \rightarrow \infty} e^{-(\delta+n)t} \mu_2(t) S(t) = 0.$$

Since  $\mu_2 e^{-(\delta+n)t}$  is constant according to (5-2-16), this implies that

$$(5-2-31) \quad \text{Limit } S(t) = 0$$

$$t \rightarrow \infty$$

Equation (5-2-31) implies that the stock of the resource will be exhausted

asymptotically.

#### 5.2.4 Comparison of Results

Here we compare our results with those obtained by DEH (1978). The following points can be noted for this purpose:

1. In the DEH model, the time rate of change of consumption depends on the sign and magnitude of the difference between the rate of return on foreign assets and the discount rate. According to the modified model, the price elasticity of the demand for the exports and the growth rate of the price of the imported commodity play the critical role in determining the optimal time path of consumption. This result has more appeal to the economic sense. At least, the economy has more means of increasing per capita consumption in the current model than it has in the DEH model.
2. The DEH model predicts that export of the resource will fall continuously over time. This is not necessarily the case in the present version of the model. Equation (5-2-21) shows that exports of the resource may increase if the price of the imported commodity is falling quickly enough.
3. The DEH model allows for exogenous technical progress. The domestic use of the resource will be increasing or decreasing or remain constant depending on the relative rates of technical progress, the rate of return on foreign assets and the share of capital in total output. In the absence of technical progress, it will be decreasing at all times. In the modified version of the model, the rate of resource extraction may not be falling along the optimal path in the absence of exogenous technical progress. It will, however, be decreasing to zero at the conjectured equilibrium.
4. According to the DEH model, the time rate of change of capital is given by

$$(5-2-32) \quad \dot{K} \geq 0 \quad \text{iff} \quad \lambda \geq \beta b$$

where

$\lambda$  = the rate of technical progress,  $b$  = the rate of return from foreign assets and  $\beta$  = the share of resource in total output.

In the absence of technical progress, capital will always be falling. Consequently, home production of the consumption good will be falling over time. Here we find that the optimal growth path of the economy will converge to an equilibrium with positive values for  $X$  and  $V$  under certain circumstances. In the current model, per capita capital will be falling asymptotically unless  $\lambda$  is sufficiently high. Aggregate capital, however, may be rising under the same circumstances. In other words, the chance of gradual extinction of home production of the consumption good is significantly reduced here. If the absolute value of  $\epsilon$  is high, exporting the resource will be phased out over time.

5. They proved that the rate of optimal depletion of the resource could be determined independently of the discount rate and the elasticity of marginal utility if the rate of return on foreign assets is exogenously given. Here also we find that the optimal depletion rate of the resource does not depend on the discount rate and the elasticity of marginal utility.

### 5.3 THE MODIFIED MOUSSAVIAN MODEL

In 1985, Moussavian modified the DEH model by adding one non-traded good producing sector to it. The demand for the non-traded good is always in equilibrium with its supply. In the DEH model, the production of the traded commodity may be falling continuously over time. The reason is that the domestic use of the capital input will be falling over time when the rate of exogenous technical progress is not sufficiently large. The assumed equilibrium in the non-tradeable sector in the Moussavian model, on the other hand,

reduces the possibility of a total extinction of all production activities in the home country. Moussavian showed that the production of the non-traded commodity will not cease even when that of the traded commodity does. Our objective in this section is to test the afore-said result in the light of a slightly different assumption about the rate of return on foreign assets. Moussavian assumed an exogenous rate of return. We relax that assumption by allowing it to be a negatively sloped function of total foreign assets held domestically. This modified assumption can be justified as follows. Suppose, there is a portfolio of foreign investments with different rates of return. Domestic people will purchase assets with the highest rate of return first. As they buy more foreign assets, they switch to the asset with the next lower rate of return & so on. Thus, the rate of return on foreign assets decreases as the amount of foreign assets held domestically increases. The remaining assumptions of the Moussavian model have been kept intact here.

### 5.3.1 Assumptions

In the Moussavian (1985), the country produces two commodities, a traded good and a non-traded good using Cobb-Douglas production technologies. Both the production functions embody exogenous technical progress. The inputs of production are capital and the resource in both the cases. Like Moussavian (1985), we assume two labour forces of fixed amounts with skills specific to both industries. Thus,

$$(5-3-1) \quad Q_1 = F(K_1, R_1, t) = e^{\lambda_1 t} K_1^{\alpha_1} R_1^{\beta_1} \quad \alpha_1 + \beta_1 < 1$$

$$(5-3-2) \quad Q_2 = G(K_2, R_2, t) = e^{\lambda_2 t} K_2^{\alpha_2} R_2^{\beta_2} \quad \alpha_2 + \beta_2 < 1$$

The first commodity is the traded good and the second commodity is the non-traded good. The stock of the resource is depleted for three uses: as inputs of production in two sectors

and as exports.

$$(5-3-3) \quad S = -(R_1 + R_2 + E)$$

Foreign demand for the home resource is a non-linear function of price:

$$(5-3-4) \quad E(P) = Z e^{-\rho t} P(t)^\epsilon \quad \epsilon < -1$$

where

$P$  is the price of the resource,  $\epsilon$  is the price elasticity of foreign demand for the resource, and  $\rho$  represents a secular downward shift in the demand for the resource. It is to be noted here that the  $\rho$  of current model plays the role of  $\pi$  of the previous model. Equation (5-3-4) also stipulates that the foreign demand for the home resource should be elastic with respect to its price. In other words, a change in the price of the resource would lead to a more than proportionate change in the quantity of the resource demanded.

Society can use its output in three ways; consume it, invest it in foreign assets or use it to purchase capital goods for domestic use. Total domestic capital is allocated between the two sectors. Thus,

$$(5-3-5) \quad K = K_1 + K_2$$

Like Moussavian, we assume that foreign assets can be instantaneously transformed into domestic capital or vice versa so that  $K_1$  and  $K_2$  are control variables here. The economy allocates its stock of wealth between investment in foreign assets and domestic capital for each industry. Thus, domestic capital is now a flow rather than a stock variable. The country also invests in foreign assets. We posit here that the rate of return  $b$  on foreign assets is a negative function of the amount of foreign assets:

$$(5-3-6) \quad b = f(A-K) = f(w) \quad b' = df/dw < 0$$

where A denotes total assets, K is total home capital and hence  $A-K = w$  is foreign assets. The time rate of change of the total wealth of the economy is then given by

$$(5-3-7) \quad \dot{A} = B(w) + P(E)E + F(K-K_2, R_1, t) - C_1$$

where

$$C_1 = \text{consumption of the traded good} \quad B(w) = f(w)w$$

The country has a utility function where utility depends on consumption of both the goods:

$$(5-3-8) \quad U(C) = [1/(1-\theta)]C^{(1-\theta)} \quad C = C_1^\gamma C_2^{1-\gamma} \quad 0 < \gamma < 1 \quad \theta > 0$$

When  $\theta=1$ , the utility function reduces to a log linear consumption function.

Since the second commodity cannot be traded, by definition,

$$(5-3-9) \quad Q_2 = C_2$$

It can be noted that  $P_2 = (\partial U / \partial C_2) / (\partial U / \partial C_1)$  can be thought of as the domestic price of good 2 in terms of good 1.

### 5.3.2 The Model and Necessary Conditions

Given the assumptions (5-3-1) through (5-3-8), the objective is to maximise the sum of discounted utility of consumption over an infinite time horizon. Thus, the problem is

$$(5-3-10) \quad \text{Max} \quad \Omega = \int_0^\infty e^{-\delta t} U(C) dt$$

The control variables are  $C_1$ ,  $R_1$ ,  $R_2$ ,  $E$ ,  $K_2$ , and  $K$ . The state variables are  $A$  and  $S$ . The current value Hamiltonian is defined as

$$H = U(C) + \mu_1 [-(R_1 + R_2 + E)] \\ + \mu_2 [B + P(E)E + F(K - K_2, R_1, t) - C_1]$$

The following necessary conditions are obtained assuming an interior solution:

$$(5-3-11) \quad U_1 = \mu_2 \quad \text{where } U_1 = \partial U / \partial C_1 \text{ etc.}$$

$$(5-3-12) \quad \mu_1 = \mu_2 F_R \quad \text{where } F_R = \partial F / \partial R_1 \text{ etc.}$$

$$(5-3-13) \quad \mu_1 = U_2 G_R$$

$$(5-3-14) \quad \mu_1 = \mu_2 MR_E \quad \text{where } MR_E = \partial \{EP(E)\} / \partial E$$

$$(5-3-15) \quad U_2 G_K = \mu_2 F_K$$

$$(5-3-16) \quad F_K = B' \quad B' = b + b'w$$

$$(5-3-17) \quad \hat{\mu}_1 = \delta$$

$$(5-3-18) \quad \hat{\mu}_2 = \delta - B'$$

Taking time derivative of (5-3-12) and substituting for  $\hat{\mu}_1$  and  $\hat{\mu}_2$ , we get,

$$(5-3-19) \quad \hat{R}_1 = \hat{Q}_1 - B'$$

Similar manipulations of (5-3-16) yield,

$$(5-3-20) \quad \hat{K}_1 = \hat{Q}_1 - \hat{B}'$$

Taking logarithmic differentiation of (5-3-1) and substituting for  $\hat{K}_1$  and  $\hat{R}_1$  in the resulting equation, we can solve for the growth rate of production of the first commodity:

$$(5-3-21) \quad \hat{Q}_1 = [\lambda_1 - \beta_1 B' - \alpha_1 \hat{B}'] / (1 - \alpha_1 - \beta_1)$$

We can substitute for  $\hat{Q}_1$  in (5-3-19) and (5-3-20) to solve for  $\hat{R}_1$  and  $\hat{K}_1$  respectively:

$$(5-3-22) \quad \hat{R}_1 = [\lambda_1 - (1-\alpha_1)B' - \alpha_1\hat{B}'] / (1-\alpha_1-\beta_1)$$

$$(5-3-23) \quad \hat{K}_1 = [\lambda_1 - \beta_1 B' - (1-\beta_1)\hat{B}'] / (1-\alpha_1-\beta_1)$$

We differentiate (5-3-11) with respect to time to obtain

$$(5-3-24) \quad \{\gamma(1-\theta) - 1\}\hat{C}_1 + (1-\theta)(1-\gamma)\hat{Q}_2 = \delta - B'$$

Similarly, logarithmic differentiation of (5-3-13) yields,

$$(5-3-25) \quad \gamma(1-\theta)\hat{C}_1 + (1-\theta)(1-\gamma)\hat{Q}_2 - \hat{R}_2 = \delta$$

Again we differentiate (5-3-15) with respect to time and obtain,

$$(5-3-26) \quad \gamma(1-\theta)\hat{C}_1 + (1-\gamma)(1-\theta)\hat{Q}_2 - \hat{K}_2 = \delta - B' - \hat{B}'$$

Subtracting (5-3-25) from (5-3-24), we get,

$$(5-3-27) \quad \hat{C}_1 - \hat{R}_2 = B'$$

Similar subtraction of (5-3-26) from (5-3-24) yields,

$$(5-3-28) \quad \hat{C}_1 - \hat{K}_2 = \hat{B}'$$

We multiply (5-3-27) by  $\beta_2$  and (5-3-28) by  $\alpha_2$ . Then we add the resulting equations to obtain,

$$(5-3-29) \quad \hat{Q}_2 - (\alpha_2 + \beta_2)\hat{C}_1 = \lambda_2 - \beta_2 - \alpha_2\hat{B}'$$

Now we solve equations (5-3-24) and (5-3-29) simultaneously for  $\hat{C}_1$  and  $\hat{Q}_2$ :



$$(5-3-30) \quad \hat{C}_1 = [ (1-\gamma)(1-\theta) \{ \lambda_2 - \beta_2 B' - \alpha_2 \hat{B}' \} - (\delta - B') ] / D$$

$$(5-3-31) \quad \hat{C}_2 = \hat{Q}_2 = [ \{ 1 - \gamma(1-\theta) \} \{ \lambda_2 - \beta_2 B' - \alpha_2 \hat{B}' \} - (\alpha_2 + \beta_2)(\delta - B') ] / D$$

where

$$D = [ \{ (1-\gamma) + \theta\gamma \} - (1-\gamma)(1-\theta)(\alpha_2 + \beta_2) ]$$

$$B' = b + b'w \quad \hat{B}' = \dot{w}(2b' + b''w) / (b + b'w)$$

Finally, we find the growth rate of the export of the resource using equations (5-3-4), (5-3-14), (5-3-17) and (5-3-18):

$$(5-3-32) \quad \hat{E} = \epsilon(b + wb') - \rho.$$

### 5.3.3 Comparison of Results.

As is evident from the equations derived here, the assumption of the endogeneity of the rate of return on foreign assets makes the analysis complicated. Specifically, the level and the rate of change of the quantity of foreign assets are found to be among the factors determining the growth rates of consumption and production of the two commodities. The effect of endogeneity of the rate of return will depend on the signs and magnitudes of  $\dot{w}$  and  $b''$ . The growth rate of the traded good in this case will be higher if we assume (i)  $\dot{w}=0$  or (ii)  $\dot{w}>0$ ,  $b'' \leq 0$  and  $b + b'w > 0$  or (iii)  $\dot{w} < 0$ ,  $b'' \leq 0$  and  $b + b'w < 0$ . The effect of an endogenous rate of return on the growth rates of consumption and production of the second commodity and consumption of the first commodity cannot be determined unambiguously. Due to endogenous rate of return on foreign assets, the price of the resource rises and demand falls less quickly than they do when the rate is exogenously

given.

It is easily seen that the results of Moussavian (1985) follow automatically if we set  $b' = 0$  in all the equations derived here. In that sense, the modified model here can be considered as a more general model. Below we discuss the implications of an endogenous rate of return on foreign assets:

1. As in Moussavian (1985), the growth rate of the production of the first commodity is independent of the production technology of the second commodity.
2. The most important implication of the endogeneity assumption is that the production of the traded good will not fall continuously even if exogenous technical progress is not sufficiently large ( $\lambda_1 < \beta_1 b$ ) or it is absent. This production can still be increasing if any of the following conditions is satisfied:
  - (i) A constant quantity of foreign assets is held.
  - (ii) An increasing quantity of foreign assets is held with  $b'' \leq 0$  and  $b + b'w > 0$ .
  - (iii) A decreasing quantity of foreign assets is held with  $b'' \leq 0$  and  $b + b'w < 0$ .

It should be noted in this regard that both DEH and Moussavian showed that the production of the traded good will phase out over time when exogenous technical progress is not sufficiently large. Moussavian made an improvement on the result of DEH by pointing out that while the production of the traded good will continue to fall, the production of the non-traded good may rise. Here we find that the production of the traded good can increase even if exogenous technical progress is absent.

3. The Moussavian findings about the growth rate of the non-traded good are still true, though some qualifications need to be added here. It is now more likely that the production and consumption of the second commodity will be growing at a faster rate in this situation.

## CHAPTER 6

### CONCLUSIONS

In this chapter, we summarize the main conclusions derived in earlier chapters. The objective of this thesis was to fill some important gaps in the theory of growth with exhaustible resources. It dealt with issues of growth and survival of an economy in two different contexts: exploration and open economy.

It investigated growth behaviour of an economy which can increase its stock of the resource by costly exploration. Quite a number of models using different assumptions about exploration technology were set up for this purpose. First, we assumed that some output is used in the exploration process. Moreover, the returns to exploration activities decrease with the increase in the size of cumulative discoveries. It was found that per capita consumption will increase indefinitely even when there is no exogenous technical progress. The same results hold when a portion of capital is used for exploration activities. We obtained quite a different result when we assumed that a portion of labour is used in the exploration process. We found that the minimum rate of exogenous technical progress necessary for a steadily growing per capita consumption in this case becomes higher than the rate required for the same purpose when there is no exploration. One reason for this result may be that the implied costs of exploration in this case are very high. All these models suffer from one drawback: the Hamiltonian cannot be a concave function of all the state variables in these models. Second, as a means of circumventing the problem of sufficient condition in the first three models, we replaced cumulative discoveries by the stock of the resource in the fourth model. We proved that a steady state exists if one restriction is satisfied. It is that the growth rate of the stock of the resource due to exploration depends negatively on the stock of the resource. If the above condition is satisfied, per capita output, consumption, and capital will be growing at the same constant rate. Moreover, at the steady state, the growth

rates of accumulation and decumulation of the resource will be just equal to each other. Though these results were derived assuming a logarithmic utility function, they are equally good for any utility function of iso-elastic type. We introduced the assumption of uncertain yields from exploration in chapter 4. The returns to exploration activities shift downward and upward continuously. We found that this kind of uncertainty with respect to the yields from exploration leads to increased expected rates of change of consumption, exploratory effort and resource utilization.

The thesis also examined growth paths of an open economy by modifying the models developed by DEH and Moussavian. In the original DEH model, consumption is increasing or decreasing according as the rate of return is less than or greater than the discount rate. The growth rates of prices of the exported resource and the imported commodity play the major role in determining the optimal time path of per capita consumption in the modified DEH model. Moreover, the probability of a total collapse of home production is less in the altered model. This is true even if we do not assume a non-traded sector. Assuming an endogenous rate of return in the Moussavian model, we showed that the production of the traded good need not be necessarily decreasing under a prespecified situation as shown by both DEH and Moussavian.

The preceding paragraphs summarized the basic contributions made in this thesis. Nevertheless, the thesis has some limitations as well. In chapter 3, we discussed in detail the two limitations embedded in the first three models developed there. First, the sufficient condition for dynamic maximisation is not satisfied. Second, the exploration technology is such that it allows the stock of the resource to be infinite. In most cases, the models were analysed for logarithmic utility functions. Incorporation of other types of utility functions makes the analysis too complicated. Needless to say, we used some simple models to make theoretical analysis of the optimal growth paths in the presence of exploration. More rigorous assumptions about uncertainty with regard to the yields from exploration can also be

incorporated. The models can also be generalized by using C.E.S production function.

A final point about the future avenues of research in the field is in order. Future research can be made to find ways of tackling the problems of the sufficient condition and lack of finiteness of the resource stock embedded in the models developed in this thesis. There exists a big vacuum of empirical studies in this area. The reason is very simple. The theoretical models developed and studied are too specific to draw any general inference about growth behaviour of a market economy possessing a finite stock of the resources. Use of more realistic assumptions makes a model more suitable for estimation.<sup>1</sup> Our objective in this thesis was to design a better model in this perspective. We hope more pragmatic research in this area will eventually lead to empirically testable conclusions.

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<sup>1</sup>For example, Pindyck (1978) used his model to estimate one specific exploration function using U.S. data.

Feasible Solution

In this part of the appendix, we show that a feasible solution to our model presented in section 3.2 exists. Suppose the objective of the central planner of the economy is to minimise the use of resources used up in the production process over time subject to three state equations. The first state equation gives total capital investment as the portion of total output saved after the expenses for consumption and exploration have been met. We also assume that the levels of per capita output used for consumption and exploration are fixed at all times. The second state equation relates to the time rate of change of the stock of the resource. The stock of the resource is depleted by the resource flow used as an input in the production process. The stock of the resource also grows over time because a constant level of per capita output is used in the exploration process. However, the returns to exploration activities decrease with the increase in the size of cumulative discoveries. Cumulative discoveries is the sum of the "proven stock" of the resource lying in the ground and the cumulative use of the resource in the production process. The third state equation gives the time rate of change of the cumulative use of the resource in the production process. The economy's production technology is given by a constant returns to scale Cobb-Douglas production function which also embodies exogenous technical progress. The three inputs of production are capital, the resource and labour. Labour is assumed to grow exponentially at a given rate,  $n$ . We also normalize the initial level of labour to 1 for analytical convenience.

Given these assumptions, the problem is then to minimise

$$(A-1-1) \quad \text{Min} \quad \Omega = \int_0^{\infty} R dt$$

subject to

$$(A-1-2) \quad \dot{K} = Y - (C+D) = Y - C_1 L \quad \text{where } C_1 = (C+D)/L = \text{a constant.}$$

$$(A-1-3) \quad \dot{B} = R$$

$$(A-1-4) \quad \dot{S} = -R + a(S+B) \epsilon D^\eta \quad \epsilon < 0, \quad 0 < \eta < 1$$

$$(A-1-5) \quad \hat{L} = n$$

$$(A-1-6) \quad Y = e^{\lambda t} K^\alpha R^\beta L^\gamma = e^{(\lambda + \gamma n)t} K^\alpha R^\beta$$

The present value Hamiltonian is given by

$$H = R + \mu_1 [Y - C_1 L] + \mu_2 [-R + a(S+B) \epsilon D^\eta] + \mu_3 [R]$$

The control variable is R and the state variables are K, S and B. Assuming an interior minimum, we get the following necessary conditions for minimisation:

$$(A-1-7) \quad \mu_1 \beta (Y/R) + \mu_3 = \mu_2 - 1$$

$$(A-1-8) \quad \hat{\mu}_1 = -\alpha (Y/K)$$

$$(A-1-9) \quad \dot{\mu}_2 = -\mu_2 a \epsilon (S+B) \epsilon^{-1} D^{\eta-1}$$

$$(A-1-10) \quad \dot{\mu}_3 = -\mu_3 a \epsilon (S+B) \epsilon^{-1} D^{\eta-1}$$

Subtracting (A-1-10) from (A-1-9), we get,

$$(A-1-11) \quad \dot{\mu}_2 = \dot{\mu}_3$$

Taking time differentiation of (A-1-7) and making use of (A-1-8) and (A-1-11), we obtain,

$$(A-1-12) \quad \hat{R} = \{1/(1-\beta)\} \{\lambda + \gamma n - \alpha(C_1 L/K)\}$$

We define the following normalized variables:

$$X = Y/K \quad V = L/K$$

We differentiate (A-1-6) with respect to time and make use of (A-1-2) and (A-1-12) to derive the differential equations for the variables X and V. The feasible growth paths of the economy satisfy the following differential equations for X and V:

$$(A-1-13) \quad \hat{X} = \{1/(1-\beta)\} [\lambda + \gamma n - (\gamma + \alpha\beta)X + \gamma C_1 V]$$

$$(A-1-14) \quad \hat{V} = -X + C_1 V + n$$

We find the stationary values of X and V by setting  $\hat{X}=0$  and  $\hat{V}=0$  in the above two equations. The stationary values are

$$(A-1-15) \quad X^* = \lambda / \alpha\beta$$

$$(A-1-16) \quad V^* = (\lambda - \alpha\beta n) / \alpha\beta C_1$$

We find that  $X^*$  is always positive.  $V^*$  is positive provided  $\lambda > \alpha\beta n$ . In other words, a sufficiently large rate of technical progress is needed for a positive value of  $V^*$ . The Jacobian of the differential equation system (A-1-13) and (A-1-14) has a negative determinant and a positive trace at the equilibrium. The determinant and trace values are

$$(A-1-17) \quad |J| = -\alpha\beta C_1 X^* V^* / (1-\beta)$$

$$(A-1-18) \quad \text{Trace} = (\lambda + \beta n) / \beta$$

In other words, the stationary point obtained here is a saddle point equilibrium. At the



equilibrium, both total output and capital will grow at the rate of growth of population. Thus, an economy which expends constant levels of per capita output for consumption and exploration purposes may converge to a long run equilibrium where both output and capital can grow at the same rate. However, such a steady state can be obtained only if the rate of exogenous technical progress is sufficiently large. The exploration technology used in this appendix is same as that used in the first model of chapter 3. The first model of chapter 3 yields the conclusion that per capita consumption will be increasing even when exogenous technical progress is zero. There is one reason for the difference in the results of the two models. In the feasible solution of this appendix, a constant level of per capita output is used for exploration. The optimal growth model of chapter 3, on the other hand, shows that the level of output used for exploration increases indefinitely. Nevertheless, we prove that a feasible solution with an exploration technology similar to that used in our first model may exist.

### Labour Required for Exploration

In this appendix, we make a detailed discussion of the stationary points found in the second model of chapter 3 where a part of labour is used in the exploration process. Here we show that only two stationary points out of a total of twelve can be accepted as possible long run equilibrium points. The optimal growth paths of the economy in this model satisfies the following differential equations:

$$(A-2-1) \quad \hat{X} = \{1/(1-\beta)\} [ -(\gamma+\alpha\beta)X + \gamma V + \lambda + \gamma n + \gamma \hat{\theta} ]$$

$$(A-2-2) \quad \hat{V} = \{(\alpha-\sigma)/\sigma\} X + V - (1/\sigma) [ \delta + n - \sigma n ]$$

$$(A-2-3) \quad \hat{\theta} = \frac{[(1-\theta)\{\lambda - \alpha V - \alpha n + (1-\beta)(1-\eta)n\}]}{[\alpha(1-\theta) + (1-\eta)(1-\beta)\theta]}$$

Let  $\Phi$  stand for the second parenthesized expression in the numerator of (A-2-3). As mentioned in section 3.3.2, there are twelve possible equilibrium points in this model. These can occur when

- [I]  $X > 0$ ,  $V > 0$  and  $\theta = 0$ . [II]  $X > 0$ ,  $V = 0$  and  $\theta = 0$ .  
 [III]  $X = 0$ ,  $V > 0$  and  $\theta = 0$ . [IV]  $X = 0$ ,  $V = 0$  and  $\theta = 0$ .  
 [V]  $X > 0$ ,  $V > 0$  and  $\theta = 1$ . [VI]  $X > 0$ ,  $V = 0$  and  $\theta = 1$ .  
 [VII]  $X = 0$ ,  $V > 0$  and  $\theta = 1$ . [VIII]  $X = 0$ ,  $V = 0$  and  $\theta = 1$ .  
 [IX]  $X > 0$ ,  $V > 0$  and  $\Phi = 0$ . [X]  $X > 0$ ,  $V = 0$  and  $\Phi = 0$ .  
 [XI]  $X = 0$ ,  $V > 0$  and  $\Phi = 0$ . [XII]  $X = 0$ ,  $V = 0$  and  $\Phi = 0$ .

Cases [II], [VI] and [X] can be ruled out on the basis of the transversality condition for capital which can be used here as follows:

$$(A-2-4) \quad \nabla = (1-\alpha)X^* - V^* \leq 0$$

It is obvious that  $\nabla$  turns out to be positive with  $V^* = 0$ . Hence the transversality condition (A-2-4) is not satisfied for these cases. Cases [III], [VII] and [XI] can be excluded for two reasons. First,  $V$  is greater than  $X$  at these points. However, this cannot be true if we do not allow for negative investment. Second, these points can be rejected for stability reasons. In case [III], the growth rate of  $X$  tends to

$$(A-2-5) \quad \hat{X} \approx [\lambda + \gamma n(1-\eta)]/\alpha > 0.$$

near the point where both  $X$  and  $\theta$  are zero. Hence  $X$  cannot approach to zero in the long run. Similarly, near the point  $(0, V, 1)$  of case [VII], the approximate growth rate of  $X$  is given by

$$(A-2-6) \quad \hat{X} \approx [\lambda + \gamma(V+n)] / (1-\beta) > 0$$

Similar results hold for case [XI]. Hence these three stationary points can be rejected as possible long run equilibrium points. Cases [IV], [VIII] and [XII] are also unstable equilibrium points. For case [IV], the growth rate of X tends to the rate given in (A-2-5). The approximate growth rate of X for case [VIII] is as follows:

$$(A-2-7) \quad \hat{X} \approx [\lambda + \gamma n] / (1-\beta) > 0$$

The growth rate of X tends to the same value as above for case [XII]. Hence, X cannot tend to zero in the long run at these points. Cases [IX] through [XI] yield inconsistent results and hence can be ignored. For example, we get two different values for V in case [IX]. The first value is obtained by setting  $\Phi=0$ .

$$(A-2-8) \quad V = [\lambda + \{(1-\beta)(1-\eta) - \alpha\}n] / \alpha.$$

We get the second value of V by setting  $\dot{X}=0$  and  $\dot{V}=0$  and solving the two linear equations thus obtained. The second value of V is:

$$(A-2-9) \quad V = [\{\gamma(1-\alpha) + \alpha\beta(1-\sigma)\}n + \delta(\alpha\beta + \gamma) - \lambda(\alpha - \sigma)] / \alpha(\sigma\beta + \gamma).$$

Obviously, the two values of V are not the same. Only cases [I] and [V] apparently qualify as asymptotic limit of the optimal growth path here. In other words,  $\theta$  can only take the value of 1 or 0.

### Capital Required for Exploration

Here also we first enumerate the twelve stationary points of the third model of chapter 3. In that model, we assume that a portion of capital is used in the exploration process. Next, we show how ten of twelve stationary points can be ruled out in this model. The system of differential equations in this case is given by

$$(A-3-1) \quad \hat{X} = \{1/(1-\beta)\}[\lambda - (\gamma\theta + \alpha\beta)X + \gamma V + \gamma n - \gamma\hat{\theta}]$$

$$(A-3-2) \quad \hat{V} = \{(\alpha - \sigma\theta)/\sigma\}X + V - (1/\sigma)[\delta + n - \sigma n]$$

$$(A-3-3) \quad \hat{\theta} = (1-\theta)[\lambda + \{\alpha\theta - (1-\beta)\eta\theta - \alpha\}X - \{\alpha - (1-\beta)\eta\}V + \gamma n] / \\ [\gamma(1-\theta) + (1-\eta)(1-\beta)\theta]$$

As before, we denote the second parenthesized expression in the numerator of (A-3-3) by  $\Phi$ .

The twelve potential equilibrium points for this model are as enumerated below:

[I]  $X > 0$ ,  $V > 0$  and  $\theta = 0$ . [II]  $X > 0$ ,  $V = 0$  and  $\theta = 0$ .

[III]  $X = 0$ ,  $V > 0$  and  $\theta = 0$ . [IV]  $X = 0$ ,  $V = 0$  and  $\theta = 0$ .

[V]  $X > 0$ ,  $V > 0$  and  $\theta = 1$ . [VI]  $X > 0$ ,  $V = 0$  and  $\theta = 1$ .

[VII]  $X = 0$ ,  $V > 0$  and  $\theta = 1$ . [VIII]  $X = 0$ ,  $V = 0$  and  $\theta = 1$ .

[IX]  $X > 0$ ,  $V > 0$  and  $\Phi = 0$ . [X]  $X > 0$ ,  $V = 0$  and  $\Phi = 0$ .

[XI]  $X = 0$ ,  $V > 0$  and  $\Phi = 0$ . [XII]  $X = 0$ ,  $V = 0$  and  $\Phi = 0$ .

We can rule out 10 cases out of 12 here by using several different arguments. Case [VI] can be ruled out because it violates the transversality condition for capital which here can be stated as follows:

$$(A-3-4) \quad \nabla = (\theta - \alpha)X^* - V^* \leq 0.$$

With  $\theta = 1$  and  $V = 0$ ,  $\nabla = (1 - \alpha)X^* > 0$  for any  $X^* > 0$ . Hence the transversality condition is

violated. The equilibrium value of  $\theta$  obtained for the case [X] is obtained by setting both  $\Phi$  and the right hand side of (A-3-1) equal to zero and solving the two linear equations thus obtained simultaneously. The value of  $\theta$  in this case is  $\alpha/(1-\eta) > 0$ , which is also less than 1 provided  $\alpha + \eta < 1$ . However, with this value of  $\theta$  and  $V=0$ ,  $\nabla$  becomes equal to  $\alpha\eta X^*/(1-\eta) > 0$ . Hence the transversality condition is violated for this case. Cases [III], [VII] and [XI] can be dropped because  $V$  is greater than  $X$  in these points. Moreover, these points are unstable. This is because  $X$  is growing near these points. In other words,  $X$  cannot tend to zero in the long run as required for convergence to these points. We find that the growth rate of  $X$  tends to  $(1-\eta)V > 0$  for case [III],  $\{\lambda + \gamma(V+n)\}/(1-\beta) > 0$  for cases [VII] and [XI]. Cases [II], [IV], [VIII] and [XII] are totally unstable and hence can be ruled out. For example,  $\hat{X}$  tends to  $\alpha X$  in case [II]. In other words,  $X$  is always increasing near this point. Since the variable  $X$  appears with a positive coefficient in the differential equation for  $V$ , it means  $V$  is also increasing near this point. Hence  $V$  cannot approach to zero in the long run as required to converge to this point.  $X$  is zero for cases [VIII] and [XII]. We find that  $\hat{X}$  tends to  $\{\lambda + \gamma n\}/(1-\beta) > 0$  near these points. Hence  $X$  cannot approach to these points in the long run. The value of  $\theta$  is zero for case [IV]. However,  $\partial\theta/\partial\theta$  tends to  $\{\lambda + \gamma n\}/\gamma > 0$  near this point. In other words, the value of  $\theta$  cannot decay to zero in the long run as is required to converge to this point. We can also ignore case [IX] because it gives rise to inconsistent results. To show this, we first assume  $\sigma=1$ . Then we set  $\Phi$  equal to zero and find an expression for  $\theta X$ . We find another expression for  $\theta X$  by setting  $V=0$ . We equate the two expressions for  $\theta X$  and then solve for  $X$  using the equation thus obtained:

$$(A-3-5) \quad X = [\delta\{\alpha - (1-\beta)\eta\} - (\lambda + \gamma)n] / \alpha\{\alpha - 1 + (\beta - 1)\eta\}$$

We find another value of  $X$  by setting  $\dot{X}=0$  and  $\dot{V}=0$  and solving the two linear equations thus obtained:

$$(A-3-6) \quad X = [\gamma\delta + \lambda + \gamma n] / \alpha(1-\beta)$$

It is obvious that the two values of  $X$  are not the same. We thus get inconsistent results in this case. Once again an equilibrium can be obtained when  $\theta$  is 1 or 0. The optimal growth paths can converge to the equilibrium point of case [I] or case [V].

It can easily be verified that the equilibrium point of case [V] (case [I] of text) is identical to the equilibrium point of Stiglitz (1974). However, it can be shown that the equilibrium point is not stable in  $\theta$  assuming a logarithmic utility function. The derivative of  $\dot{\theta}$  with respect to  $\theta$  in this model is

$$(A-3-7) \quad \partial \dot{\theta} / \partial \theta = [\Phi \{D(1-2\theta) - \theta(1-\theta)A\} + DC\theta(1-\theta)] / D^2$$

where

$$\Phi = \lambda + \{\alpha\theta - (1-\beta)\eta\theta - \alpha\}X - \{\alpha - (1-\beta)\eta\}V + \gamma n$$

$$D = \gamma(1-\theta) + (1-\beta)(1-\eta)\theta$$

$$A = (1-\beta)(1-\eta) - \gamma$$

$$C = \{\alpha - (1-\beta)\eta\}X$$

Using the equilibrium values of  $X$  and  $V$  for case [V] (text case [I]) and noting that  $\theta=1$  in this case, we find that  $\partial \dot{\theta} / \partial \theta$  here tends to

$$(A-3-8) \quad \partial \dot{\theta} / \partial \theta = [\eta(\lambda + \gamma n) + \delta \{(1-\eta)\beta + \gamma\}] / (1-\alpha)(1-\eta) > 0.$$

In other words,  $\theta$  cannot converge to the value of 1 if it is not equal to 1 to begin with. Only convergence to an equilibrium with  $\theta=0$  is possible in this model.

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