PROPERTIES OF K-TOURNAMENTS

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PROPERTIES OF K-TOURNAMENTS.

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Abstract

In this thesis, we investigate several properties of k-tournaments, where $k \geq 3$. These properties fall into three broad areas. The first contains properties related to the ranking of the participants in a k-tournament, including a representation theorem for posets. The second contains properties related to the representation of a finite group as the automorphism group of a k-tournament, with varying restrictions on the desired representation. The third area answers questions about regularity in k-tournaments.

Chapter 1 contains an introduction, and the definitions and notation.

In Chapter 2, we consider the ranking of the participants in a k-tournament. We introduce the notions of transitivity and quasitransitivity in a k-tournament, each of which extends the notion of transitivity in a tournament in a natural way, and we prove that every k-tournament on a sufficiently large number of vertices contains a quasitransitive sub-k-tournament on a given number of vertices, thus extending the analogous result for tournaments. We then consider ranking the participants in a general k-tournament. We define, for a general k-tournament, a binary relation on its vertex set, which corresponds to a partial ranking of the participants. We then show that any finite poset with cardinality at least $k+1$ can be represented by a k-tournament, in the sense that there is a k-tournament whose ranking relation is isomorphic to the given poset. The construction of this k-tournament suggests an interesting generalisation of the dimension of a poset.

In Chapter 3, we investigate the automorphism group of a k-tournament. We begin by characterising those finite groups G for which there exists a k-tournament whose automorphism group is isomorphic to G. This extends the theorem of Moon (1964) which characterises the finite groups admitting a representation as the automorphism

group of a tournament. We then consider the problem of finding the 'smallest' k-tournament whose automorphism group is isomorphic to G, where we determine how 'small' a k-tournament is by the number of orbits of its automorphism group acting on its vertex set. With this definition of size, our goal is to characterise those finite groups admitting a regular representation as the automorphism group of a k-tournament. We first construct, for each admissible group G of order at least k, a k-tournament whose automorphism group is isomorphic to G and has two orbits of vertices. We then show that every admissible cyclic group of order at least k, and every admissible group which has a minimal generating set with at least k elements, admits a regular representation as the automorphism group of a k-tournament.

Finally, in Chapter 4 we investigate regular and almost regular k-tournaments. We show that there is a regular k-tournament on n vertices if and only if $n \geq k$ and $\binom{n}{k} \equiv 0 \pmod{n}$, and that there is an almost regular k-tournament on n vertices for all n and k satisfying $n \geq k$. We then provide some explicit constructions of regular and almost regular k-tournaments.

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Contents

Chapter 1

Introduction

1.1 Definitions and notation

This section contains definitions and notation which will be used throughout the thesis. The definitions are classified into several subjects, roughly corresponding to the chapters of the thesis.

General

A *k-set* is a set with k elements, for a positive integer k. If X is a set, a *k-subset* of X is a subset of X which has k elements. We use $\binom{X}{k}$ to denote the set of k subsets of X.

Graphs

A *graph* G consists of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of distinct elements of $V(G)$, called *edges*. A graph G is *bipartite* if there is a partition of $V(G)$ into two sets X and Y such that every edge of G contains one vertex from each of X and Y.

A *path* in a graph is a sequence $(v_1, v_2, \ldots, v_l)$ of distinct vertices of G, with the property that $v_i v_{i+1} \in E(G)$ for $i = 1, \ldots, l-1$. A *cycle* in G is a path $(v_1, v_2, \ldots, v_l)$ with the additional properties that $l \geq 3$ and $v_l v_1 \in E(G)$.

A *matching* in a graph G is a set M of edges of G with the property that no two edges belonging to M have a vertex in common. If M is a matching in G and $u \in V(G)$, then u is said to be *M-saturated* if u belongs to an edge of M. If M is a matching in G, an *M-alternating path*, or simply an *alternating path*, in G is a path in which alternate edges belong to M.

A *directed graph*, or *digraph*, D consists of a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct elements of $V(D)$, called *arcs*. If D is a directed graph and $(u, v) \in A(D)$, then we say that u *dominates* v in D. The *outdegree* of a vertex v of a digraph D is the number of vertices dominated by v, and the *indegree* of v is the number of vertices which dominate v.

A *semicomplete digraph* D is one in which, given any pair $\{u, v\}$ of vertices of D, at least one of (u, v) and (v, u) belongs to $A(D)$. A *directed cycle* of length l in a directed graph D is a sequence $(v_0, v_1, \ldots, v_{l-1})$, $l \geq 2$, of vertices of D such that for each $i = 0, 1, \ldots, l-1$, $(v_i, v_{i+1}) \in A(D)$, where the subscripts are reduced modulo l.

A *tournament* T is a directed graph in which, given any unordered pair $\{u, v\}$ of distinct vertices of T, exactly one of (u, v) and (v, u) belongs to $A(T)$. A tournament T is *transitive* if whenever u dominates v and v dominates w in T, then u dominates w in T. Equivalently, a tournament is transitive if its vertices can be ordered $v_1, v_2, \ldots, v_n$ so that v_i dominates v_j in T if and only if $i < j$.

The *score* of a vertex v in a tournament T is the number of vertices dominated by v (equivalently, the outdegree of v); the *score sequence* of T is a list of the scores of its vertices, written in non-decreasing order. A tournament T is *regular* if for each vertex v of T, the indegree and the outdegree of v are equal. We say that a tournament is *almost regular* if the difference between the indegree and the outdegree of each vertex is at most one.

An *automorphism* of a tournament T is a permutation α of its vertex set such that $(u, v) \in A(T)$ if and only if $(\alpha(u), \alpha(v)) \in A(T)$. The *automorphism group* of T is the group of all automorphisms of T and is denoted $Aut(T)$.

k-tournaments

A *hypergraph* H consists of a set $V(H)$ of vertices and a set $E(H)$ of *hyperedges*, where each hyperedge is a subset of $V(H)$. A *k-uniform hypergraph* is one in which each hyperedge has cardinality k. A *complete k-uniform hypergraph* is a k-uniform hypergraph in which every k-subset of vertices determines a hyperedge.

Let $k \geq 3$. A *k-tournament* T consists of a set $V(T)$ of vertices and a set $A(T)$ of *arcs*. Each arc of T is a k-tuple of distinct vertices of T, and the set $A(T)$ has the property that for any k-subset S of $V(T)$, $A(T)$ contains exactly one of the $k!$ k-tuples whose entries belong to S.

We stress that whenever we use the term k-tournament, we are implicitly assuming that $k \geq 3$.

If T is a k-tournament and S is a k-subset of $V(T)$, then we say that the arc A of T *corresponds to* S if the entries of A are the elements of S. Thus for each k-subset S of $V(T)$, there is a unique element of $A(T)$ which corresponds to S.

The *order* of a k-tournament T is the cardinality of $V(T)$.

If $A = (v_1, v_2, \ldots, v_k)$ is an arc of a k-tournament T, we say that the vertex v_i *is the i^{th} co-ordinate of* A, or that v_i *is in co-ordinate i in* A, and we write $v_i = A(i)$.

The i^{th} *degree* of a vertex v in a k-tournament T is the number of arcs of T in which v is the i^{th} co-ordinate; and we use $\deg_i(v, T)$, or if there is no possibility of confusion, $\deg_i(v)$, to denote the i^{th} degree of v. The *degree vector* of a vertex v of T is the vector of length k whose i^{th} entry is $\deg_i(v, T)$. The *degree matrix* of a k-tournament T on n vertices is the $n \times k$ matrix whose (i, v)-entry is $\deg_i(v, T)$.

If T_1 and T_2 are k-tournaments, a bijection $f : V(T_1) \to V(T_2)$ is an *isomorphism* if $(v_1, v_2, \ldots, v_k) \in A(T_1)$ if and only if $(f(v_1), f(v_2), \ldots, f(v_k)) \in A(T_2)$. If $T_1 = T_2$, then f is called an *automorphism*. Again, the automorphism group of a k-tournament is the group of all automorphisms of T, and is denoted $Aut(T)$.

If T is a k-tournament and $A = (v_1, \ldots, v_k)$ is an arc of T, and f is a permutation of $V(T)$ (and, in particular, if $f \in Aut(T)$), we use $f(A)$ to denote the k-tuple $(f(v_1), \ldots, f(v_k))$.

A *sub-k-tournament* of a k-tournament T is a k-tournament T' such that $V(T') \subseteq V(T)$ and $A(T') \subseteq A(T)$. If T is a k-tournament and $U \subseteq V(T)$, the *sub-k-tournament of T induced by U* is the sub-k-tournament $T[U]$ of T where $V(T[U]) = U$ and $A(T[U]) = A(T) \cap U^k$.

Groups

For a finite group G, we use $|G|$ to denote the order of G. For an abstract group G, we use e to denote the identity of G.

We use Z_n to denote the cyclic group of order n.

If G is a group and $S \subseteq G$, we use gS to denote the set $\{gs : s \in S\}$. If G is a permutation group acting on a set X, and $S \subseteq X$, then we use $g(S)$ to denote the set $\{g(s) : s \in S\}$.

We use S_X to denote the symmetric group on a set X; and if $X = \{1, 2, \ldots, n\}$, we use S_n to denote the same group. We use ι to denote the identity element of S_X.

If G is a permutation group acting on a set X, then we can also view G as a group of permutations of the set $\binom{X}{k}$ of k-subsets of X, in which $g : S \rightarrow g(S)$, $S \in \binom{X}{k}$, for each $g \in G$.

If G is a permutation group acting on a set X, the *stabiliser* of an element x of X in G is the set $G_x = \{g \in G : g(x) = x\}$. If $g \in G_x$, we also say that x is a *fixed point* of g. We say that G is *semiregular*, or that *G acts semiregularly on X*, if for any $x \in X$, $G_x = \{e\}$. We say that $g \in G$ is *semiregular* if the group $\langle g \rangle$ is semiregular. It is easy to see that a permutation g is semiregular if and only if, when g is written as a product of disjoint cycles, all of its cycles have the same length.

An *orbit* of a permutation group G acting on a set X is a subset of X of the form $\{gx : g \in G\}$, for some $x \in X$. The orbits of G consititute a partition of X. It is well-known that if G is a permutation group acting on X and $x \in X$, and if $O(x)$ denotes the orbit of G which contains x, then $|G| = |G_x||O(x)|$; this result is known as the *Orbit-Stabilizer Theorem.*

A permutation group G acting on X is *transitive* if for any two elements x and y of X, there is some $g \in G$ such that $g(x) = y$. We say that G is *regular*, or that *G acts* *regularly* on X if G is both transitive and semiregular on X. Equivalently, G is regular on X if, for any elements x and y of X, there is a unique element $g \in G$ such that $g(x) = y$.

If G is a group, we use G_L to denote the left-regular representation of G; G_L is, therefore, the subgroup of S_G given by $G_L = \{g_L : g \in G\}$, where
$g_L : h \to gh, h \in G$, for each $g \in G$. Where there is no confusion, we use g to denote both the element g of G and the permutation g_L of G_L.

If G is a group, a *tournament regular representation*, or *TRR* of G is a tournament T whose automorphism group is isomorphic to G and acts regularly on $V(T)$. A *k-tournament regular representation*, or *k-TRR* of G is a k-tournament whose automorphism group is isomorphic to G and acts regularly on $V(T)$.

Partially ordered sets

A *partially ordered set*, or *poset*, $\mathcal{P} = (X, P)$ consists of a set X together with a reflexive, antisymmetric and transitive binary relation P defined on X. We also say that *P is a partial order on X*. If $x, y \in X$ and $(x, y) \in P$, we also say that $x \leq y$ in P (and if $x \neq y$ we say that $x < y$ in P). We say that x is *maximal* in P if there is no $y \in X$ for which $x < y$ in P. If either $(x, y) \in P$ or $(y, x) \in P$ then we say that x and y are *comparable* in P; otherwise x and y are *incomparable* in P, and we write $x \| y$ in P. We use $inc(\mathcal{P})$ to denote the set of pairs of points of X which are incomparable in P.

Let $\mathcal{P} = (X, P)$ be a poset. If $P = \emptyset$, then we say $\mathcal{P}$ is an *antichain*. If $inc(\mathcal{P}) = \emptyset$, then we say that $\mathcal{P}$ is a *chain*, or that P is a *linear order on X*. We say that a subset Y of X is a *chain* if the poset $(Y, P \cap (Y \times Y))$ is a chain.

An *extension* of $\mathcal{P} = (X, P)$ is a poset (X, P') such that $P \subseteq P'$. A *linear extension* of $\mathcal{P}$ is an extension $\mathcal{L} = (X, L)$ of $\mathcal{P}$ such that L is a linear order on X. A *subposet* of $\mathcal{P}$ is a poset (X', P') such that $X' \subseteq X$ and $P' \subseteq P$. If $Y \subseteq X$, we use $\mathcal{P}[Y]$ to denote the subposet of $\mathcal{P}$ induced by Y; that is, $\mathcal{P}[Y] = (Y, P \cap (Y \times Y))$.

If R is a binary relation on X, we use $tr(R)$ to denote the *transitive closure* of R, so that $tr(R)$ is the smallest (with respect to inclusion) transitive binary relation on X

which contains R. If P_1 and P_2 are partial orders on X, the *intersection* of P_1 and P_2 is the partial order P on X given by $x < y$ in P if and only if both $x < y$ in P_1 and $x < y$ in P_2.

Let $\mathcal{P} = (X, P)$ be a poset. An *alternating cycle* in $\mathcal{P}$ is a sequence $S = \{(x_1, y_1), \ldots, (x_l, y_l)\}$ of ordered pairs from $inc(\mathcal{P})$ such that $y_i \leq x_{i+1}$ for $i = 1, \ldots, l$ (where the subscripts are taken modulo l). An alternating cycle is *strict* if $y_i \leq x_j$ if and only if $j = i + 1$ (where again the subscripts are taken modulo l).

A *chain decomposition* of $\mathcal{P} = (X, P)$ is a partition of X into chains. The *width* of $\mathcal{P}$ is the smallest number of chains in a chain decomposition of $\mathcal{P}$. We say that a chain decomposition of $\mathcal{P}$ is *minimal* if the number of chains in the decomposition is equal to the width of $\mathcal{P}$. The *dimension* of $\mathcal{P}$ is the smallest number of linear orders on X whose intersection is P; it is, therefore, the smallest cardinality of a set $\{\mathcal{L}_1, \ldots, \mathcal{L}_m\}$ of linear orders on X with the property that $x < y$ in P if and only if $x < y$ in each of $L_1, \ldots, L_m$.

1.2 Introduction

The study of tournaments has generated a tremendous body of research in graph theory over the last half-century. Four major surveys of the subject have been written during the last thirty years: *The theory of round robin tournaments*([10]), by F. Harary and L. Moser, which appeared in 1966 and was the first such survey; the well-known book *Topics on Tournaments*, by J. W. Moon ([15]), published in 1968; the chapter on tournaments, by L. W. Beineke and K. B. Reid, in *Selected Topics in Graph Theory* ([6]), published in 1978; and a recent survey of results in the theory of paths, cycles and trees in tournaments, by J. Bang-Jensen and G. Gutin ([2]). Although much of the early research on tournaments was motivated by questions in areas outside pure mathematics (for example, the original characterisation of the score sequence of a tournament was motivated by research into hierarchies in animal societies ([14]), the subject has since developed into one of the fundamental areas of research in graph theory.

The results presented in this thesis are concerned with a natural generalisation of tournaments. In terms of graph theory, this is a generalisation of the notion of a tournament to hypergraphs; in this setting, we have a complete k-uniform hypergraph, each of whose hyperedges is replaced by a k-tuple whose entries are the elements of that hyperedge. In terms of round robin tournaments, we have some kind of game in which k players compete simultaneously; the 'round robin' tournament is now such that each set of k players competes in one game, and the k players are ranked (i.e., linearly ordered) by the outcome of that game. We use the term k-tournament to denote such a generalised tournament; although a tournament might reasonably be called a 2-tournament in this terminology, we reserve the use of the term k-tournament for occasions when $k \geq 3$. We present here the results of an investigation of how some of the well-known properties of tournaments generalise to the case of k-tournaments.

Our first area of investigation is the ranking of the participants in a k-tournament. As might be expected, ranking the participants in a k-tournament is significantly more complicated than ranking the participants in a tournament. To develop a ranking scheme, one has to contend with the fact that a ranking of the participants is a binary relation, while the relation defined by the outcomes of the games in the k-tournament is a k-ary relation. As a result it can happen, for example, that given two players in a k-tournament, neither beats the other in all the games in which both players participate. Even comparing the scores of two players in a k-tournament is a nontrivial task, since the 'score' of a player now consists of a vector of length k, whose i^{th} entry is the number of games in which the player placed i^{th}. Thus even the simplest ranking scheme for a tournament, in which the players are ranked as far as possible by their scores (i.e., the number of games won), and as many ties as possible are broken by considering the outcome of the game played by two players with equal scores, is no longer simple in a k-tournament. For this reason, we restrict our attention here to more general questions regarding the ranking of the participants.

A transitive tournament is one in which the vertices can be ranked so that, given any two vertices u and v, u is ranked ahead of v if and only if u dominates v in the tournament. Thus, a transitive tournament is the very simplest tournament from the point of view of ranking its participants. This is therefore a natural place to begin our task of generalising the ideas related to ranking in tournaments to k-tournaments. We discuss two ways in which the notion of transitivity in tournaments can be generalised to k-tournaments, and show that in each case, some properties of transitive tournaments are preserved, while others are not. The first results in the definition of a transitive k-tournament, in which, much as in a transitive tournament, the participants can be ranked so that u is ranked ahead of v if and only if u places ahead of v in every game in which both u and v participate. The second leads to the definition of a quasitransitive k-tournament. In this case the well-known Ramsey-type property of tournaments is preserved: there is a function $f(n, k)$ such that every k-tournament on at least $f(n, k)$ vertices contains a quasitransitive sub-k-tournament on n vertices. The corresponding result is not true of transitive k-tournaments.

If a k-tournament is not transitive, then it must contain two vertices u and v, neither of which places ahead of the other in every game in which the two participate (Proposition 2.2.10). With this in mind, given a k-tournament T, we define a binary relation on its vertex set by $u < v$ if and only if u places ahead of v in every game in which both u and v participate; our aim is to determine what form this relation can take, for an arbitrary k-tournament. To this end, we show in Theorem 2.3.2 that, given any finite partially ordered set of cardinality n and any integer k such that $n > k$, there is a k-tournament T for which the vertex set of T together with the relation $<$ defined above is isomorphic to the given poset. The construction of such a poset suggests an interesting generalisation of the dimension of a poset.

The second area in which we present results concerns the automorphism group of a k-tournament. An interesting feature of any structure is the symmetry which it might possess; thus we often examine the group of automorphisms of that structure. In the case of tournaments, it was shown by J. W. Moon in 1964 ([16]) that given a finite, abstract group G, there is a tournament whose automorphism group is isomorphic to G if and only if G has odd order. We show in Theorem 3.2.1 that there is a k-tournament whose automorphism group is isomorphic to G if and only if the order of G and k are relatively prime, thus extending Moon's result to all $k \geq 3$. As is the case with tournaments, the necessity of the condition is easily seen; the majority of the proof consists of showing that the required k-tournament exists.

Having characterised the finite, abstract groups which arise as the automorphism group of a k-tournament, it is natural to ask whether, given such a group G, we can find a regular representation of G as the automorphism group of a k-tournament. In other words, can we find a k-tournament whose automorphism group is isomorphic to G and is a regular permutation group? The number of vertices in such a k-tournament would necessarily be equal to the order of G. We therefore approach this question by asking, for a given finite, abstract group G, whose order is relatively prime to k: Over all k-tournaments T with automorphism group isomorphic to G, what is the minimum number of vertex orbits of the automorphism group of T? The analogous question for tournaments was answered by L. Babai and W. Imrich, who showed that every finite, abstract group of odd order, other than $Z_3 \times Z_3$, admits a regular representation as the automorphism group of a tournament ([1]). Their proof uses the Feit-Thompson theorem, which states that every finite group of odd order is solvable. Since it is not true for general k that the automorphism group of a k-tournament has odd order, their methods do not immediately lend themselves to a generalisation to k-tournaments. We therefore take a different approach in attempting to extend their result to k-tournaments.

We show first, in Section 3.3.1, that, given a finite group G whose order is relatively prime to k and strictly larger than k, there is a k-tournament on $2|G|$ vertices whose automorphism group is semiregular and is isomorphic to G. The automorphism group of this k-tournament, therefore, has two vertex orbits. We go on to show in Section 3.3.2 that if either G is cyclic, or $k \geq 4$ and G has a minimal generating set of cardinality at least k, then we can find a regular representation of G as the automorphism group of a k-tournament. Thus the real difficulty in completely solving this problem lies in finding a regular representation of G when either $k = 3$ and G is not cyclic, or $k > 3$ and every minimal generating set for G has cardinality lying strictly between 1 and k.

The third area which we discuss concerns the scores of the participants in a k-tournament. In a tournament, the score of a participant is defined to be the number of games won by that participant; in graph theoretic terms, the score of a vertex is its outdegree. The score sequence of a tournament is a list of the scores of its vertices, usually in nondecreasing order. As we have already pointed out, in a k-tournament the score of a vertex consists of a vector of length k, whose i^{th} entry is the number of arcs in which that vertex is the i^{th} co-odinate. The score sequence of a tournament is therefore generalised by the degree matrix of a k-tournament, which is an $n \times k$ matrix in which the entry in row v and column c is the number of arcs in which the vertex v is the c^{th} co-ordinate.

A tournament is said to be regular if the indegree of each vertex is equal to its outdegree. If we view the arcs of a tournament as ordered pairs, a tournament is then regular if and only if each vertex is the first co-ordinate in as many arcs as it is the second co-ordinate. Thus it is natural to define a regular k-tournament as one in which each vertex appears in each of the k co-ordinates in some fixed number d of arcs. It is easy to see that there exists a regular tournament on n vertices if and only if n is odd; and when n is even there exists a tournament in which the indegree and the outdegree of each vertex differ by at most one. With this in mind, we define an almost regular k-tournament on n vertices to be one in which the number of arcs in which any vertex appears in any of the k co-ordinates is one of two fixed integers d

and $d+1$. Of course, in both cases, the value of d is determined by n and k.

It is easy to see that if a k-tournament on n vertices is regular, then the number of arcs in the k-tournament is divisible by its number of vertices; that is, $\binom{n}{k}$ is divisible by n. E. Barbut and A. Bialostocki asked in [3] whether this condition is also sufficient for the existence of a regular k-tournament on n vertices. This question was partly motivated by the following problem of R. Graham ([9]). A *universal cycle* with parameters n and k is a sequence $a_1, a_2, \ldots, a_{\binom{n}{k}}$ of elements of $\{1, 2, \ldots, n\}$, of length $\binom{n}{k}$, with the property that the $\binom{n}{k}$ sets $\{a_i, a_{i+1}, \ldots, a_{i+k}\}$ (where the subscripts are reduced modulo n) are all distinct. The problem is to determine for which values of n and k a universal cycle exists. It is easy to see that if a universal cycle with parameters n and k does exists, then by interpreting the $\binom{n}{k}$ segments $(a_i, a_{i+1}, \ldots, a_{i+k})$ as the arcs of a k-tournament with vertex set $\{1, 2, \ldots, n\}$, the k-tournament we obtain will be regular. Since it is clear that not every regular k-tournament corresponds to a universal cycle, the existence of a regular k-tournament is weaker than the existence of a universal cycle. However, since the problem of determining for which n and k a universal cycle exists appears to be very difficult, Barbut and Bialostocki proposed determining those values of n and k for which a regular k-tournament on n vertices exists as a preliminary step.

We give an affirmative answer to Barbut and Bialostocki's question in Theorem 4.1.2; in fact we prove a stronger result, namely, that a regular k-tournament on n vertices exists for all n and k for which $n \geq k$ and $\binom{n}{k}$ is divisible by n, and that for all remaining values of n and k, with $n \geq k$, there exists an almost regular k-tournament on n vertices. Since proving this theorem, the author has discovered that the same result appear in the Journal of the Royal Statistical Society in 1948, with a similar proof ([11]); however, since the earlier paper is written in a different discipline and in entirely different terminology, we include here the author's proof.

The proof of the above theorem does not provide explicit constructions of regular and almost regular k-tournaments. This leaves open the problem of finding such explicit constructions. When the question of Barbut and Bialostocki appeared, the same authors had already found explicit constructions in some cases ([5]). In Section 4.2 we provide different constructions, also for selected cases; although the cases covered by our constructions and by those of Barbut and Bialostocki overlap, the constructions presented here are elementary while those of Barbut and Bialostocki are more intricate.

Chapter 2

Ranking the participants in a k-tournament

2.1 Introduction

In this chapter we consider some questions related to the ranking of the participants in a k-tournament. In Section 2.2 we discuss two ways in which the notion of transitivity in tournaments can be generalised to k-tournaments, and show that in each case some of the properties of transitive tournaments carry over to k-tournaments while others do not.

In Section 2.3 we define a binary relation on the vertex set of a k-tournament T, in which two vertices u and v are related if u precedes v in every arc of T which contains both u and v (where we say u precedes v in an arc A if u is in co-ordinate i and v in co-ordinate j of A, and $i < j$). We then construct, given a finite poset $\mathcal{P}$, a k-tournament for which the relation described above is isomorphic to $\mathcal{P}$. This construction leads to a generalisation of the dimension of a poset.

2.2 Transitivity

In this section we consider some questions related to transitivity, in the context of k-tournaments. The initial problem is to determine the most natural definition of a transitive k-tournament. The notion of transitivity occurs most often with respect to binary relations; the problem here is to find a reasonable extension to k-ary relations, where $k \geq 3$. Ideally we would like the notion of transitivity in a k-tournament with $k \geq 3$ to generalise that in a tournament; and we would hope that transitive k-tournaments might exhibit some of the same properties as transitive tournaments. Neither of the generalisations discussed below preserves all of the properties of transitive tournaments; however, it is arguable that the reason for this is simply that the case $k = 2$ is degenerate in some sense, since these two distinct notions coincide in that case.

2.2.1 A strict definition of transitivity

We begin by considering a notion of transitivity which is perhaps the most natural, but which at the same time is rather restrictive. We first note that a tournament T is transitive if there is a linear ordering of its vertices such that the vertex u dominates the vertex v in T if and only if u precedes v in this linear ordering. This approach suggests the following definition of a transitive k-tournament, which was also given by A. Bialostocki in [7].

Definition 2.2.1 Let $k \geq 3$, and let T be a k-tournament with vertex set $V(T)$. We say that T is *transitive* if there is a linear ordering $v_1 < \cdots < v_n$ of $V(T)$ such that $(v_{i_1}, \ldots, v_{i_k})$ is an arc of T if and only if $v_{i_1} < \cdots < v_{i_k}$.

Remark 2.2.2 Up to isomorphism, there is only one transitive k-tournament on n vertices for any fixed n and k with $n \geq k$.

We now investigate some of the properties of transitive k-tournaments. We begin by calculating the degree vectors of the vertices of a transitive k-tournament on n vertices. To this end, we let T be a transitive k-tournament with $V(T) = \{1, \ldots, n\}$

and with underlying linear order $1 < \cdots < n$. Then for $v \in V(T)$, the c^{th} degree of v is given by

$$\deg_c(v) = \binom{v-1}{c-1}\binom{n-v}{k-c}.$$

This gives us an easy proof of the following result.

Proposition 2.2.3 *Let T be a transitive k-tournament. If u and v are distinct vertices of T, then the degree vectors of u and v are distinct.*

Proof. Let T be a transitive k-tournament. Without loss of generality we can assume that $V(T) = \{1, \ldots, n\}$ with underlying linear order $1 < \cdots < n$. Let $u, v \in V(T)$ with $u \neq v$, and assume $u < v$. We want to show that there is some co-ordinate $c \in \{1, \ldots, k\}$ such that $\deg_c(u) \neq \deg_c(v)$.

To this end, let c be the smallest element of $\{1, \ldots, k\}$ such that $\deg_c(u) \neq 0$. If $\deg_c(v) = 0$ then c is the required co-ordinate. Thus we assume that $\deg_c(v) \neq 0$. Now if $c = 1$, then $\deg_c(u) = \binom{n-u}{k-1}$ and $\deg_c(v) = \binom{n-v}{k-1}$, and it is clear that $\deg_c(u) > \deg_c(v)$ (because we assumed that $u < v$). Otherwise $c > 1$. In this case, since $\deg_{c-1}(u) = 0$ we have either $u - 1 < c - 2$ or $n - u < k - c + 1$. But $\deg_c(u) > 0$ implies that $u - 1 \geq c - 1 > c - 2$ and $n - u \geq k - c$. It follows that $n - u = k - c$. Therefore $n - v < n - u = k - c$ and so $\binom{n-v}{k-c} = 0$; from this we get $\deg_c(v) = \binom{v-1}{c-1}\binom{n-v}{k-c} = 0$. Thus again we have $\deg_c(u) \neq \deg_c(v)$.

Therefore, in all cases, $\deg_c(u) \neq \deg_c(v)$, and the proposition follows. ■

Remark 2.2.4 Notice that if T is a transitive k-tournament with $V(T) = \{1, 2, \ldots, n\}$ and underlying linear order $1 < 2 < \cdots < n$, then the degree vector of the vertex i of T is the reverse of the degree vector of the vertex $n - i + 1$ of T. That is, for every $c = 1, 2, \ldots, k$ and every $i = 1, 2, \ldots, n$, $\deg_c(i) = \deg_{k-c+1}(n - i + 1)$.

Since it is clear that any automorphism of a k-tournament must map any vertex v to another vertex with the same degree vector, the next result follows immediately from Proposition 2.2.3.

Proposition 2.2.5 *Let T be a transitive k-tournament on n vertices. Then the only automorphism of T is the identity automorphism.*

As we shall see in Section 2.2.2, the transitive k-tournament is not alone in having the identity group as its automorphism group; in fact there are non-transitive k-tournaments with trivial automorphism group on any number $n > k$ of vertices. Its automorphism group does not, therefore, characterise the transitive k-tournament. Consequently we must look elsewhere for a characterisation.

It is well-known that a tournament on n vertices is transitive if and only if its score sequence is $n-1, n-2, \ldots, 2, 1, 0$. In other words, a tournament is transitive if and only if its vertices can be relabelled $v_1, v_2, \ldots, v_n$ so that the outdegree of v_i is $n-i$, $1 \leq i \leq n$. What would be the analogous result for k-tournaments? Since the analogue of the score sequence of a tournament is the degree matrix of a k-tournament, we ask the following question: If the vertices of a k-tournament, T on n vertices can be relabelled $v_1, \ldots, v_n$ so that $\deg_c(v_i) = \binom{i-1}{c-1}\binom{n-i}{k-c}$, $1 \leq c \leq k$, must T be transitive? In Proposition 2.2.7 below, we answer this question in the affirmative. We first define the matrix $\mathcal{D}_{n,k}$ to be the degree matrix of the transitive k-tournament on n vertices.

Definition 2.2.6 For $n \geq k \geq 3$, we let $\mathcal{D}_{n,k}$ be the $n \times k$ matrix whose (v, c)-entry is $\binom{v-1}{c-1}\binom{n-v}{k-c}$, $1 \leq v \leq n$, $1 \leq c \leq k$.

Proposition 2.2.7 *Let $n \geq k \geq 3$ and let T be a k-tournament on n vertices, with degree matrix $\mathcal{D}(T)$. Then T is transitive if and only if the rows of $\mathcal{D}(T)$ can be permuted to obtain the matrix $\mathcal{D}_{n,k}$.*

Proof. Let n, k and T be as in the statement of the proposition, and let $V(T) = \{v_1, \ldots, v_n\}$. By assumption the rows of $\mathcal{D}(T)$ can be permuted to obtain the matrix $\mathcal{D}_{n,k}$. Equivalently, there is a bijection $f : V(T) \to \{1, 2, \ldots, n\}$ such that $\deg_c(v, T) = \binom{f(v)-1}{c-1}\binom{n-f(v)}{k-c}$. Letting T' be the k-tournament defined by $V(T') = \{1, 2, \ldots, n\}$ and $A(T') = \{(f(v_1), f(v_2), \ldots, f(v_k)) : (v_1, v_2, \ldots, v_k) \in A(T)\}$, it is clear that T' is isomorphic to T and that $\deg_c(v, T') = \binom{v-1}{c-1}\binom{n-v}{k-c}$, where $1 \leq v \leq n$ and $1 \leq c \leq k$. We will show that T' is transitive, with underlying linear order $1 < 2 < \cdots < n$.

For each $v \in V(T')$ and each $c \in \{1, \ldots, k\}$, let $\mathcal{A}_{v,c}$ denote the set of arcs of T' which contain v and exactly $c-1$ elements of $\{1, \ldots, v-1\}$. (Note that $\mathcal{A}_{1,1}$ denotes the set of arcs of T' which contain 1, and that $\mathcal{A}_{1,c} = \emptyset$ if $c > 1$.)

We will prove, by induction on v and c, that if $A \in \mathcal{A}_{v,c}$, then $v = A(c)$.

First, if $v = c = 1$, then as observed above, $\mathcal{A}_{v,c}$ is the set of arcs of T' which contain the vertex 1. Since $\deg_1(1, T') = \binom{n-1}{k-1} = |\mathcal{A}_{1,1}|$ (and $\deg_c(1, T') = 0$ if $c > 1$), then 1 is the first co-ordinate in every arc of T' which contains it. Therefore, $1 = A(1)$ for every $A \in \mathcal{A}_{1,1}$.

Now let $v \in V(T')$ and $c \in \{1, \ldots, k\}$, where at least one of $v > 1$ and $c > 1$ holds, and assume that if either $v' < v$ and $c' \in \{1, \ldots, k\}$, or $v' = v$ and $c' < c$, then $v' = A(c')$ for every $A \in \mathcal{A}_{v',c'}$.

We want to show that $v = A(c)$ for every $A \in \mathcal{A}_{v,c}$. To do this, we first show that if $v \in A$ and $A \notin \mathcal{A}_{v,c}$, then $v \neq A(c)$. To this end, let S be *any* k-subset of $V(T)$ with $v \in S$, and let A be the corresponding arc of T. Let $L = \{s \in S : s < v\}$. Notice that $A \in \mathcal{A}_{v,|L|+1}$, and hence that $A \in \mathcal{A}_{v,c}$ if and only if $|L| = c - 1$. We will show, therefore, that if $|L| \neq c - 1$, then $v \neq A(c)$.

If $L < c-1$, then $|L|+1 < c$, and so $v \in \mathcal{A}_{v,c'}$ with $c' = |L|+1 < c$. By hypothesis, $v = A(c')$, and so $v \neq A(c)$. On the other hand, if $|L| > c - 1$, then there is some $v' < v$ such that $v' \in S$ and $A \in \mathcal{A}_{v',c}$; by hypothesis, $v' = A(c)$, and so $v \neq A(c)$. Thus $|L| \neq c - 1$ implies that $v \neq A(c)$, as desired.

Now $\deg_c(v, T') = \binom{v-1}{c-1}\binom{n-v}{k-c}$, and this is exactly the number of k-subsets S of $V(T')$ for which $v \in S$ and $|L| = c - 1$, where L is defined as above. It follows that if A is the arc of T' corresponding to such a set S, then $v = A(c)$. But these arcs are precisely the elements of $\mathcal{A}_{v,c}$. Therefore, if $A \in \mathcal{A}_{v,c}$, then $v = A(c)$, as desired.

It follows by induction that for all $v \in V(T')$ and all $c \in \{1, \ldots, k\}$, if $A \in \mathcal{A}_{v,c}$, then $v = A(c)$.

It now follows easily that T' is transitive, for if $S = \{x_1, \ldots, x_k\}$ is a k-subset of $V(T')$ with $x_1 < \cdots < x_k$, then the arc A of T' corresponding to S satisfies $A \in \mathcal{A}_{x_1,1} \cap \mathcal{A}_{x_2,2} \cap \cdots \cap \mathcal{A}_{x_k,k}$, so that $x_c = A(c)$, $1 \leq c \leq k$, and consequently $A = (x_1, x_2, \ldots, x_k)$.

Since $T' \cong T$, it now follows immediately that T is transitive. ■

Proposition 2.2.7 provides us with one characterisation of the transitive k-tournament on n vertices. However, as is the case with the transitive tournament, there are several characterisations of the transitive k-tournament; two of these will be presented later in this section.

One interpretation of a tournament T is as the outcome of a round robin tournament. In this setting, the vertices of the tournament represent the players, and vertex u dominates vertex v in T if and only if player u beats player v in the round robin tournament. If the tournament T is transitive, then there is a ranking of the players in the round robin tournament such that a player of higher rank will always beat a player of lower rank. We can interpret a k-tournament in a similar manner. In this case we have some kind of game in which k players compete simultaneously (for example, a running race); and every k-subset of players compete together exactly once. We then have an arc $(v_1, \ldots, v_k)$ in T if and only if when the players $v_1, \ldots, v_k$ compete together, player v_1 comes first, player v_2 comes second, and so on. In order to be able to rank all the players so that in any game, a player of higher rank always places ahead of a player of lower rank, we would need to know that given any two players, there is one who beats the other in every game in which the two both compete. It is easy to see that this is the case if T is a transitive k-tournament; and we show in Proposition 2.2.10 that this is the only case in which a k-tournament has this property.

Definition 2.2.8 Let T be a k-tournament. If A is an arc of T, we say that u *precedes* v *in* A if u is the i^{th} co-ordinate of A, v is the j^{th} co-ordinate of A, and $i < j$. We say that u *always precedes* v *in* T if u precedes v in every arc of T which contains both u and v.

Remark 2.2.9 Notice that the relation 'always precedes' is a binary relation on $V(T)$.

Proposition 2.2.10 *Let T be a k-tournament. Then T is transitive if and only if T has the property that given any two distinct vertices u and v of T, either u always precedes v in T or v always precedes u in T.*

Proof. It is clear that every transitive k-tournament has the stated property.

On the other hand, let T be a k-tournament which has this property. Then we can define a tournament T^* on the vertex set of T by letting $A(T^*) = \{(u, v) :$

u always precedes v in T}. Now if T^* is transitive, then there is a linear ordering of its vertices, say $v_1 < \cdots < v_n$, such that $(v_i, v_j) \in A(T^*)$ if and only if $i < j$. From the definition of $A(T^*)$, and the fact that $V(T^*) = V(T)$, it follows that such an ordering of $V(T^*)$ exists if and only if T is transitive. Therefore T^* is transitive if and only if T is transitive.

If T^* is not transitive, then it contains a directed cycle of length $l \geq 3$, which in turn implies that T^* contains a directed 3-cycle. Therefore, there are vertices u, v, w of T such that u always precedes v, v always precedes w and w always precedes u in T. But since $k \geq 3$, there is at least one arc A of T which contains all of u, v and w; and it is clearly impossible that u precedes v, v precedes w and w precedes u in A. Therefore T^* is transitive, and hence T is transitive. ■

A second characterisation of the transitive tournament is the following: A tournament is transitive if and only if it has no directed cycles. What can we say about cycles in the case of k-tournaments? Unfortunately, as is the case with hypergraphs, it is not clear how we ought to define a path or a cycle in a k-tournament. For the purposes of the current discussion, it turns out that one definition of a cycle in a k-tournament yields the result we seek. With this in mind, we begin by constructing, much as in the proof of Proposition 2.2.10, a digraph which encodes the relation 'always precedes' in a given k-tournament T.

Definition 2.2.11 Let T be a k-tournament with $k \geq 3$. We construct a digraph D as follows. We let $V(D) = V(T)$, and we let $(u, v) \in A(D)$ if and only if there is an arc of T in which u precedes v.

It is clear from the definition that given two vertices u and v of T, u always precedes v in T if and only if u dominates v but v does not dominate u in D. Also, for any pair of vertices $u, v \in V(T)$ there is at least one arc of T containing both u and v, and hence D is a semi-complete digraph (that is, D contains a tournament). Therefore T is a transitive k-tournament if and only if D is a transitive tournament. Now D is a transitive tournament if and only if D has no pairs of opposing arcs and no directed cycles. If we consider a pair of opposing arcs as a directed cycle of length two, then D is a transitive tournament if and only if D contains no directed cycles.

We now show that there is a notion of 'cycle' in a k-tournament with the property that each directed cycle of length $l \geq 2$ in D corresponds to a 'cycle' of length l in T.

Definition 2.2.12 Let T be a k-tournament, where $k \geq 3$. A *weak cycle of length l* in T is a sequence $(v_1, A_1, v_2, A_2, \ldots, v_l, A_l)$ of vertices and arcs of T, where the vertices $v_1, \ldots, v_l$ are all distinct, such that v_i precedes v_{i+1} in A_i for $i = 1, \ldots, l$ (where the subscripts are reduced modulo l).

Since it is clear that D contains a directed cycle of length $l \geq 2$ if and only if T contains a weak cycle of the same length l, the following result is immediate.

Proposition 2.2.13 *Let T be a k-tournament with $k \geq 3$. Then T is transitive if and only if T contains no weak cycles of length $l \geq 2$.*

A third well-known property of tournaments is the following: There is a function $f(n)$ such that every tournament on $f(n)$ vertices contains a transitive subtournament on n vertices. It is natural to ask whether such a result is true for k-tournaments. We show below that the answer is no: For any fixed $k \geq 3$ and $n \geq k$ there is a k-tournament T on n vertices which contains no transitive sub-k-tournament on more than k vertices. This is in direct contradiction to a pulished result of A. Bialostocki [7], who attempted to use Ramsey's theorem to show that there is a function $f(n, k)$ such that every k-tournament on at least $f(n, k)$ vertices contains a transitive sub-k-tournament on n vertices. His proof is a direct analogy of the corresponding result for tournaments. Given an arbitrary k-tournament T on N vertices, its vertices are linearly ordered $v_1 < \cdots < v_N$; and its arcs are coloured with the $k!$ elements of S_k, so that $(v_{i_1}, \ldots, v_{i_k})$ is coloured with $\pi \in S_k$ if and only if $v_{i_{\pi(1)}} < \cdots < v_{i_{\pi(k)}}$. It follows from Ramsey's theorem that if N is sufficiently large, there is a set U of vertices of T such that $|U| = n$ and the arcs of $T[U]$ are monochromatic. However, this does not imply that $T[U]$ is transitive; this is the mistake in the argument given. For example, if we let $V(T) = \{1, 2, 3, 4\}$, with underlying linear order $1 < 2 < 3 < 4$, and $k = 3$, then the arcs $(2, 1, 3), (2, 1, 4), (3, 1, 4)$ and $(3, 2, 4)$ are all coloured with the permutation $\pi = (12)$ of S_3, and yet the 3-tournament they define is not transitive.

To show that no such function $f(n,k)$ exists, we first make a simple observation which provides a straightforward construction of a k-tournament on an arbitrarily large number of vertices with no transitive sub-k-tournament on more than k vertices.

Lemma 2.2.14 *If T is a transitive k-tournament with $n > k \geq 3$ vertices, and if $2 \leq i \leq k-1$, then there is no vertex of T which occurs as the i^{th} co-ordinate of every arc which contains it.*

Proof. Let T be a transitive k-tournament, with $n > k \geq 3$ vertices, and suppose the vertices of T are ordered $v_1 < \cdots < v_n$. Let $v \in V(T)$, and suppose v occurs in co-ordinate i in some arc A of T, where $2 \leq i \leq k-1$. Let $A = (u_1, \ldots, u_k)$ with $u_i = v$. Note that $v \neq u_1, u_k$. Since $|V(T)| > k$, there is a vertex $w \in V(T) \setminus \{u_1, \ldots, u_k\}$. If $w < v$ then the arc corresponding to the set $\{w\} \cup \{u_1, \ldots, u_{k-1}\}$ has v in co-ordinate $i+1$; and if $w > v$ then the arc corresponding to the set $\{u_2, \ldots, u_k\} \cup \{w\}$ has v in co-ordinate $i-1$. Therefore v is not the i^{th} co-ordinate of every arc which contains it. ∎

This lemma gives us a simple way to construct arbitrarily large k-tournaments which contain no transitive sub-k-tournament larger than a single arc. The original proof of Proposition 2.2.15 was greatly simplified by R. Hochberg [13], who suggested the proof given below.

Proposition 2.2.15 *Let $k \geq 3$. For all $n > k$ there exists a k-tournament on n vertices which contains no transitive sub-k-tournament on more than k vertices.*

Proof. Let $n > k \geq 3$ be given. We construct a k-tournament T with no transitive sub-k-tournament on more than k vertices.

We let $V(T) = \{1, 2, \ldots, n\}$, and we order the vertices of T so that $1 < \cdots < n$. To define $A(T)$ we must assign a linear order to each k-subset of $V(T)$. Let $S \in \binom{V(T)}{k}$, where $S = \{v_1 < v_2 < \cdots < v_k\}$. Then we let the arc of T corresponding to S be $A = (v_2, v_1, v_3, \ldots, v_k)$. We do this for each k-subset S of $V(T)$.

We now show that T contains no transitive sub-k-tournament on more than k vertices.

Let $U \subseteq V(T)$, where $|U| > k$, and let u be the least element of U with respect to the natural order on $V(T)$. Consider $T[U]$, the sub-k-tournament of T induced by U. Let S be any k-subset of U which contains u. Then u is the least element of S, and so u appears in the second co-ordinate of the arc of T (and of $T[U]$) corresponding to S. Therefore whenever an arc of $T[U]$ contains u, u appears in the second co-ordinate. It now follows from Lemma 2.2.14 that $T[U]$ is not a transitive k-tournament. Consequently T contains no transitive sub-k-tournament on more than k vertices. ∎

2.2.2 Quasitransitive k-tournaments

We have shown in Section 2.2.1 that when $k \geq 3$ it is not true that every sufficiently large k-tournament contains a transitive sub-k-tournament larger than a single arc. This suggests the question 'Is there a set $\mathcal{T}$ of non-isomorphic k-tournaments on n vertices, and a function $f(n,k)$, such that every k-tournament on $f(n,k)$ vertices contains a sub-k-tournament isomorphic to an element of $\mathcal{T}$?' We show below that the answer to this question is yes, and we provide such a set of k-tournaments which is minimal with respect to this property.

Definition 2.2.16 Let $\pi \in S_k$, where $k \geq 3$, and let $n > k$. We define a k-tournament T_π^n on n vertices, as follows. We let $V(T_\pi^n) = \{1, \ldots, n\}$, and we think of $V(T_\pi^n)$ as being ordered by the natural order so that $1 < \cdots < n$.

To define $A(T_\pi^n)$, let $S = \{v_1, \ldots, v_k\} \subseteq V(T_\pi^n)$, where $v_1 < \cdots < v_k$. Then the arc of T_π^n corresponding to the set S is defined to be $A = (v_{\pi^{-1}(1)}, \ldots, v_{\pi^{-1}(k)})$.

If there is no ambiguity we omit n and write T_π.

We will show that the set $\mathcal{T}$ of k-tournaments of the form T_π^n, $\pi \in S_k$, has the property described above. For this reason we make the following definition.

Definition 2.2.17 Let $k \geq 3$. We say that a k-tournament T on n vertices is *quasitransitive* if $T \cong T_\pi^n$ for some $\pi \in S_k$.

Remark 2.2.18 Let T be a k-tournament on n vertices, where $k \geq 3$, and suppose $T \cong T_\pi^n$. Then for any subset $U \subseteq V(T)$, with $|U| = m$, $T[U] \cong T_\pi^m$ (recall that $T[U]$ denotes the sub-k-tournament of T induced by the vertices of U). Thus every sub-k-tournament of a quasitransitive k-tournament T_π is itself a quasitransitive k-tournament, with the same defining permutation π.

Remark 2.2.19 Let T be the transitive k-tournament with vertex set $V(T) = \{1, \ldots, n\}$ and underlying linear order $1 < \cdots < n$. Notice that we could equally define the k-tournament T_π^n as follows: Let $V(T_\pi^n) = V(T)$, and let $(v_1, \ldots, v_k) \in A(T_\pi^n)$ if and only if $(v_{\pi(1)}, \ldots, v_{\pi(k)}) \in A(T)$. Using this definition it is easy to calculate the i^{th} degree of a vertex v of T_π^n, since $\deg_{\pi(i)}(v, T_\pi^n) = \deg_i(v, T)$. From this, it follows in the same way as for the transitive k-tournament that the only automorphism of a quasitransitive k-tournament is the identity automorphism.

Before stating Theorem 2.2.21 we introduce notation for the Ramsey numbers needed in the proof.

Definition 2.2.20 Let n, l and k be integers. We let $R(n, l, k)$ denote the least integer N such that if f is any l-colouring of the k-subsets of an N-set X then there is a subset $Y \subseteq X$ with $|Y| = n$ such that f is constant on the k-subsets of Y.

The existence of the numbers $R(n, l, k)$ is guaranteed by Ramsey's Theorem ([18]).

We are now ready to show that the set $\mathcal{T}$ of quasitransitive k-tournaments has the property that there is a function $f(n, k)$ such that every k-tournament on $f(n, k)$ vertices contains a quasitransitive sub-k-tournament on n vertices.

Theorem 2.2.21 *There is a function $f(n, k)$ such that if $n > k \geq 3$, then every k-tournament on $f(n, k)$ vertices contains a quasitransitive k-tournament on n vertices.*

Proof. Let $n > k \geq 3$ be given, and let T be a k-tournament on N vertices, where $N = R(n, k!, k)$. We want to show that T contains a quasitransitive k-tournament on n vertices.

Without loss of generality we can assume that $V(T) = \{1, \ldots, N\}$. We first assign an artificial ordering to the elements of $V(T)$, so that $1 < 2 < \cdots < N$. We then define a colouring of the arcs of T with $k!$ colours: We let the colours be the elements of S_k, and given an arc $A = (v_1, \ldots, v_l)$ of T we assign the colour π to A if and only if $v_{\pi(1)} < \cdots < v_{\pi(k)}$.

Now by Ramsey's theorem and the choice of N, there is a subset U of $V(T)$ such that $|U| = n$ and all arcs induced by U have the same colour. Let this colour be π. Then for any $u_1, \ldots, u_k \in U$, we have $(u_1, \ldots, u_k) \in A(T[U])$ if and only if $u_{\pi(1)} < \cdots < u_{\pi(k)}$. It is therefore clear that the sub-k-tournament $T[U]$ of T induced by U is isomorphic to T^n_π. This is the required quasitransitive sub-k-tournament of T. ■

Notice that it follows from the proof of Theorem 2.2.21 that a k-tournament T on $f(n,k)$ vertices contains a quasitransitive sub-k-tournament on n vertices with respect to every initial ordering of the vertices of T.

Having shown that the set $\mathcal{T}$ of quasitransitive k-tournaments has the Ramsey-type property described above, we observe that since every sub-k-tournament of a quasitransitive k-tournament is itself quasitransitive, with the same defining permutation, the set $\mathcal{T}$ is clearly minimal with respect to this property.

We now investigate the set of quasitransitive k-tournaments in more detail. The first question we consider is the following: How many non-isomorphic quasitransitive k-tournaments on n vertices are there? This is answered in Proposition 2.2.23 below.

Definition 2.2.22 For $k \geq 3$ we define the permutation $\tau \in S_k$ by $\tau : i \to k+1-i$, $1 \leq i \leq k$.

Proposition 2.2.23 *Let $n > k \geq 3$, and $\pi, \sigma \in S_k$. Then $T^n_\pi \cong T^n_\sigma$ if and only if $\sigma = \pi$ or $\sigma = \pi\tau$.*

Proof. We first show that if $\sigma = \pi\tau$, then $T^n_\pi \cong T^n_\sigma$.

Recall that $V(T_\rho) = \{1, \ldots, n\}$ for each $\rho \in S_k$. We define a function $f : V(T_\pi) \to V(T_\sigma)$ by $f(i) = n+1-i$, $1 \leq i \leq n$. We claim that the function f is an isomorphism between T_π and T_σ. For we have

$$(v_1, \ldots, v_k) \in A(T_\pi)$$

if and only if

$$v_{\pi(1)} < \cdots < v_{\pi(k)},$$

which holds, by the definition of f, if and only if

$$f(v_{\pi(k)}) < \cdots < f(v_{\pi(1)}).$$

Letting $b_i = f(v_{\pi(\tau(i))})$, this last inequality is equivalent to

$$b_1 < \cdots < b_k,$$

which holds if and only if

$$(b_{(\pi\tau)^{-1}(1)}, \ldots, b_{(\pi\tau)^{-1}(k)}) \in A(T_{\pi\tau}).$$

Now

$$\begin{aligned} b_{(\pi\tau)^{-1}(i)} &= f(v_{\pi\tau(\pi\tau)^{-1}(i)}) \\ &= f(v_i), \end{aligned}$$

for each i, $1 \leq i \leq k$. Therefore we have $(v_1, \ldots, v_k) \in A(T_\pi)$ if and only if $(f(v_1), \ldots, f(v_k)) \in A(T_{\pi\tau}) = A(T_\sigma)$, and f is indeed an isomorphism between T_π and T_σ.

The remainder of the proof consists of showing that if $T_\pi^n \cong T_\sigma^n$, then $\sigma = \pi\tau$ or $\sigma = \pi$. Thus we assume that $T_\pi^n \cong T_\sigma^n$ for some $\pi, \sigma \in S_k$, and we let f be an isomorphism between them. Therefore we have $(v_1, \ldots, v_k) \in A(T_\pi)$ if and only if $(f(v_1), \ldots, f(v_k)) \in A(T_\sigma)$.

By definition, we have $V(T_\pi^n) = V(T_\sigma^n) = \{1, \ldots, n\}$, each with underlying linear order $1 < 2 < \cdots < n$.

Now

$$(v_1, \ldots, v_k) \in A(T_\pi)$$

if and only if

$$v_{\pi(1)} < \cdots < v_{\pi(k)},$$

and

$$(f(v_1), \ldots, f(v_k)) \in A(T_\sigma)$$

if and only if

$$f(v_{\sigma(1)}) < \cdots < f(v_{\sigma(k)}).$$

Thus we know that

$$v_{\pi(1)} < \cdots < v_{\pi(k)}$$

if and only if

$$f(v_{\sigma(1)}) < \cdots < f(v_{\sigma(k)}).$$

We will use this last equivalence to determine the function f.

Now in T_π, the vertex 1 appears only in co-ordinate $\pi(1)$, and is the only vertex which has this property. Therefore in T_σ, the vertex $f(1)$ appears only in co-ordinate $\pi(1)$ and is the only vertex of T_σ with this property. Now in any quasitransitive k-tournament T_ρ^n, there are only two vertices which appear in just one co-ordinate, and these are 1 and n. Therefore either $f(1) = 1$ or $f(1) = n$.

Similarly, in T_π if the arc A contains the vertex 2 then $2 = A(\pi(1))$ if 1 does not belong to A, and $2 = A(\pi(2))$ otherwise. Therefore in T_σ if the vertex $f(2)$ belongs to an arc A, then $f(2)$ is in co-ordinate $\pi(1)$ if $f(1)$ does not belong to A and in co-ordinate $\pi(2)$ otherwise. From above, $f(1) = 1$ or $f(1) = n$. If $f(1) = 1$, then the position of $f(2)$ in an arc depends only on the presence or absence of the vertex 1 in that arc, and hence it must be that $f(2) = 2$. Similarly, if $f(1) = n$, then the position of $f(2)$ in an arc depends only on the presence or absence of the vertex n in that arc and it follows that $f(2) = n - 1$.

Now assume, for some i, $1 \leq i < n$, that $f(1) = 1, \ldots, f(i) = i$. We want to show that $f(i+1) = i+1$. In T_π, if the vertex $i+1$ belongs to an arc A, then $i+1 = A(\pi(j))$ if and only if A contains exactly $j-1$ of the vertices $1, \ldots, i$, for any $j \in \{1, \ldots, i+1\}$ (and note that $i+1$ does not occur in any co-ordinate $\pi(c)$ for which $c > i+1$). Therefore in T_σ, if the vertex $f(i+1)$ belongs to an arc A, then $f(i+1) = A(\pi(j))$ if and only if A contains exactly $j-1$ of the vertices $\{f(1), \ldots, f(i)\} = \{1, \ldots, i\}$, for any $j \in \{1, \ldots, i+1\}$. Now $i+1$ is the only vertex of T_σ whose position in an arc depends only on the presence or absence of the vertices $1, 2, \ldots, i$ in that arc; and so $f(i+1) = i+1$.

Similarly, for any i with $2 \leq i < n$, if $f(1) = n, f(2) = n-1, \ldots, f(i) = n+1-i$, then $f(i+1) = n-i$.

Therefore, either $f(i) = i$, $1 \leq i \leq n$, or $f(i) = n+1-i$, $1 \leq i \leq n$. Having determined the isomorphism f, we are now able to relate the permutations π and σ.

If $f(i) = i$ for all $i = 1, \ldots, n$, then it is clear that $\pi = \sigma$. Otherwise we have $f(i) = n+1-i$ for $i = 1, \ldots, n$. Since f is an isomorphism between T_π^n and T_σ^n, we have

$$(v_1, \ldots, v_k) \in A(T_\pi^n) \text{ if and only if } (f(v_1), \ldots, f(v_k)) \in A(T_\sigma^n),$$

or equivalently,

$$v_{\pi(1)} < \cdots < v_{\pi(k)} \text{ if and only if } f(v_{\sigma(1)}) < \cdots < f(v_{\sigma(k)}).$$

Since $u < v$ if and only if $f(v) < f(u)$, this implies that

$$v_{\pi(1)} < \cdots < v_{\pi(k)} \text{ if and only if } v_{\sigma(k)} < \ldots < v_{\sigma(1)}.$$

But now replacing i by $\tau(k+1-i)$ gives us

$$v_{\pi(1)} < \cdots < v_{\pi(k)} \text{ if and only if } v_{\sigma(\tau(1))} < \cdots < v_{\sigma(\tau(k))}.$$

Since the above is true for any k-subset $\{v_1, \ldots, v_k\}$ of $\{1, \ldots, n\}$, it follows that $\pi = \sigma\tau$ and hence that $\sigma = \pi\tau$. ■

2.3 Ranking in non-transitive k-tournaments

This section is concerned with comparing the participants in a k-tournament, with a view to ranking the participants. In a transitive k-tournament, there is a natural way to rank the participants, namely, by using the underlying linear order of the vertices of the tournament. This ranking has the property that player u is ranked ahead of player v if and only if u places ahead of v in every game in which both players participate. This would appear to be a desirable property for any ranking to possess; however, as was seen in Proposition 2.2.10, if T has the property that either u always precedes v or v always precedes u for every pair $\{u, v\}$ of vertices of T, then T is transitive.

We therefore restrict our attention to k-tournaments which do not have this property, i.e., to non-transitive k-tournaments.

Despite this restriction, we might hope to find a partial ranking of the participants in a non-transitive k-tournament, in which player u is ranked ahead of player v if (but not only if) player u places ahead of player v in every game in which both players participate. Thus we would like to find a partial ranking of the participants in a k-tournament T, with the property that u is ranked ahead of v whenever u always precedes v in T. This leads naturally to the following question. Given a k-tournament T with $k \geq 3$, what can we say about the relation 'always precedes' on $V(T)$?

Notice that the relation 'always precedes' is antisymmetric. Therefore, this relation is a partial order if and only if it is a transitive relation. Now suppose u always precedes v and v always precedes w in some k-tournament T with $k \geq 3$. Then u precedes w in every arc which contains all three vertices u, v and w, but u need not precede w in any arcs which contain u and w but not v. It is therefore clear that the relation 'always precedes' need not be transitive. This suggests two further questions. If the relation 'always precedes' is a partial order on $V(T)$, can we say anything about that partial order? And if the relation 'always precedes' is not transitive, can we say anything more about it?

The answer to the first question is no, in the sense that there is no restriction on the partial orders which can arise from k-tournaments in the manner described above. We prove as much in Section 2.3.2. In answer to the second question, it is possible that there is again no restriction on the relations which can arise in this way; we hesitate, however, to conjecture as much.

Before stating our main theorem we need one more definition.

Definition 2.3.1 Let $\mathcal{P} = (X, P)$ be a finite poset, and let T be a k-tournament with $k \geq 3$. We say that T *represents* $\mathcal{P}$ if there is a bijection $f : V(T) \to X$ such that for any $u, v \in X$, $f(u) < f(v)$ in P if and only if u always precedes v in T.

Theorem 2.3.2 *Let $\mathcal{P} = (X, \leq)$ be a finite poset, and k an integer satisfying $3 \leq k \leq |X| - 1$. Then there is a k-tournament T such that T represents $\mathcal{P}$.*

We give the proof of Theorem 2.3.2 in Section 2.3.2. We first give some preliminary results which will be used in the proof of Theorem 2.3.2.

2.3.1 Partially ordered sets

This section consists of a brief discussion of some preliminary results concerning posets. We refer the reader to Section 1.1 for the relevant definitions and notation.

Recall that the dimension of a finite poset $\mathcal{P} = (X, P)$ is the smallest number of linear orders on X whose intersection is $\mathcal{P}$. Since the definition implies that each of these linear orders on X contains P, each corresponds to a linear extension of $\mathcal{P}$. Thus in finding the dimension of $\mathcal{P}$, we are interested in representing $\mathcal{P}$ by a collection of linear extensions of $\mathcal{P}$.

Suppose now that we have a k-tournament which represents a poset $\mathcal{P}$. Let $\mathcal{P} = (X, P)$, and assume for convenience that $X = V(T)$. Then given any k-subset S of X, the arc A of T corresponding to S has the property that for any $u, v \in S$, if $u < v$ in P, then u precedes v in A. We might, therefore, view A as a linear extension of the k-element subposet $\mathcal{P}[S]$ of $\mathcal{P}$. Viewing each arc of T in this manner allows us to view T itself as a collection of linear extensions of k-element subposets of $\mathcal{P}$; and this collection has the additional property that it contains exactly one linear extension of each such subposet. This suggests the following problem. Let us say that a collection $\mathcal{C}$ of linear extensions of k-element subposets of $\mathcal{P}$ *represents* $\mathcal{P}$ if $u < v$ in P if and only if $u < v$ in every element of $\mathcal{C}$ which contains both u and v. What is the minimum cardinality of such a collection $\mathcal{C}$? It is with this in mind that we make the following definition.

Definition 2.3.3 Let $\mathcal{P} = (X, P)$ be a finite poset, and k an integer such that $3 \leq k \leq |X| - 1$. The *k-dimension* of $\mathcal{P}$ is the minimum cardinality of a collection of linear extensions of k-element subposets of $\mathcal{P}$ which represents $\mathcal{P}$.

We note that 2-dimension would not be of any great interest since it is trivial to determine for a given finite poset, and that the $|X|$-dimension of $\mathcal{P} = (X, P)$ would simply be its dimension.

Theorem 2.3.2 shows that the k-dimension of a finite poset of cardinality n is bounded above by $\binom{n}{k}$, the number of k-element subposets of $\mathcal{P}$.

Before proceeding with the proof of Theorem 2.3.2, we give two lemmas which will be useful in what follows. The first is a result of W. T. Trotter and J. Moore [20], which tells us when we can add pairs (x, y) to a partial order P on X so that the resulting relation is again a partial order.

Lemma 2.3.4 *Let $\mathcal{P} = (X, P)$ be a poset, and $S \subseteq inc(\mathcal{P})$. Then $tr(P \cup S)$ is a partial order on X if and only if S contains no strict alternating cycle.*

The second lemma, which we also state without proof, is a result of T. Hiraguchi [12], which is used repeatedly in the constructions in Section 2.3.2.

Lemma 2.3.5 *Let $\mathcal{P} = (X, P)$ be a poset and C a chain in P. Then there are linear extensions $\mathcal{L}_1 = (X, L_1)$ and $\mathcal{L}_2 = (X, L_2)$ of $\mathcal{P}$ such that*

1. *If $x \in X$ and $c \in C$ and $x \| c$ in P, then $x \leq c$ in L_1.*
2. *If $x \in X$ and $c \in C$ and $x \| c$ in P, then $c \leq x$ in L_2.*

2.3.2 Proof of Theorem 2.3.2

In this section we give a proof of Theorem 2.3.2. The proof of the theorem is by induction on the number of points in the poset $\mathcal{P}$, for each fixed k. Lemma 2.3.11 below provides the induction base, and Lemma 2.3.12 provides the induction step. We begin with several preliminary lemmas which will be used in the proof. The first is a simple observation which is nevertheless quite useful.

Lemma 2.3.6 *Let $l \geq 2$ and let B be a set such that $|B| \geq l + 1$, and if $l = 2$ then $|B| \geq 4$. Then there are disjoint classes $\mathcal{C}_1$, $\mathcal{C}_2$ of l-subsets of B such that $\cup\{S : S \in \mathcal{C}_1\} = B = \cup\{S : S \in \mathcal{C}_2\}$.*

Proof. Let $m = |B|$ and write $m = ql + r$, with $0 \leq r < l$. Without loss of generality we assume $B = \{1, \ldots, m\}$. If $r \neq 0$, we let
$\mathcal{C}_1 = \{\{1, \ldots, l\}, \{l+1, \ldots, 2l\}, \ldots, \{(q-1)l+1, \ldots, ql\}, \{1, \ldots, l-r, ql+1, \ldots, m\}\}$
and

$$\mathcal{C}_2 = \{\{r+1,\ldots,r+l\},\{r+l+1,\ldots,r+2l\},\ldots,\{r+(q-1)l+1,\ldots,m\},$$
$$\{1,\ldots,r,r+2,\ldots,l+1\}\}.$$

Otherwise $r = 0$, and we let
$\mathcal{C}_1 = \{\{1,\ldots,l\},\{l+1,\ldots,2l\},\ldots,\{(q-1)l+1,\ldots,m\}\}$,
and

$$\mathcal{C}_2 = \{\{2,\ldots,l+1\},\{l+2,\ldots,2l+1\},\ldots,\{(q-2)l+2,\ldots,(q-1)l+1\},$$
$$\{1,(q-1)l+2,\ldots,m\}\}.$$

It is easy to check that in each case, the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ have the required property. ∎

The next lemma dispenses with the simple case of Theorem 2.3.2 in which the poset $\mathcal{P}$ is an antichain.

Lemma 2.3.7 *Let $\mathcal{P} = (X, P)$ be a finite antichain with $|X| \geq 4$. Then there is a k-tournament which represents $\mathcal{P}$ for every k satisfying $3 \leq k \leq |X| - 1$.*

Proof. Let $\mathcal{P} = (X, P)$ be as in the statement of the lemma, and let $X = \{v_1, \ldots, v_n\}$. If $k = 3$ and $|X| = 4$, then we let T be the following 3-tournament:

$$\begin{aligned} V(T) &= \{v_1, v_2, v_3, v_4\}, \\ \text{and } A(T) &= \{(v_1,v_2,v_3),(v_2,v_1,v_4),(v_3,v_4,v_1),(v_4,v_3,v_2)\}. \end{aligned}$$

It is easy to see that T represents $\mathcal{P}$.

Otherwise, using Lemma 2.3.6 we let $\mathcal{C}_1$ and $\mathcal{C}_2$ be disjoint classes of $(k-1)$-subsets of $\{v_1, \ldots, v_{n-1}\}$ such that $\cup\{S : S \in \mathcal{C}_i\} = \{v_1, \ldots, v_{n-1}\}$ for $i = 1, 2$. We then construct a k-tournament T as follows.

Let $V(T) = X$, and let

$$\begin{aligned} A(T) &= \{(v_{i_k},\ldots,v_{i_1}) : 1 \leq i_1 \leq \cdots \leq i_k \leq n-1\} \\ &\cup \{(v_n, v_{i_1},\ldots,v_{i_{k-1}}) : 1 \leq i_1 \leq \cdots \leq i_{k-1} \leq n-1 \text{ and } \{v_{i_1},\ldots,v_{i_{k-1}}\} \in \mathcal{C}_1\} \\ &\cup \{(v_{i_1},\ldots,v_{i_{k-1}}, v_n) : 1 \leq i_1 \leq \cdots \leq i_{k-1} \leq n-1 \text{ and } \{v_{i_1},\ldots,v_{i_{k-1}}\} \notin \mathcal{C}_1\}. \end{aligned}$$

We now need to show that T represents $\mathcal{P}$. To this end, let $v_i, v_j \in X$, with $i < j$. Since $\mathcal{P}$ is an antichain we need only show that there are arcs A_1 and A_2 of T such that v_i precedes v_j in A_1 and v_j precedes v_i in A_2.

First suppose $i, j \neq n$. Since $|X| > k$, then $n - 1 \geq k$ and so there is at least one k-subset of $\{v_1, \ldots, v_{n-1}\}$ which contains both v_i and v_j; so there is at least one arc of T of the form $(v_{i_k}, \ldots, v_j, \ldots, v_i, \ldots, v_{i_1})$ in which v_j precedes v_i. On the other hand, since $k \geq 3$ there is at least one k-subset of X which contains all of v_i, v_j, and v_n, and so there is at least one arc of T of the form $(v_n, v_{i_1}, \ldots, v_i, \ldots, v_j, \ldots, v_{i_{k-1}})$ or $(v_{i_1}, \ldots, v_i, \ldots, v_j, \ldots, v_{i_{k-1}}, v_n)$ in which v_i precedes v_j.

Now suppose $i = n$. Then there are $S_1 \in \mathcal{C}_1$ and $S_2 \in \mathcal{C}_2$ such that $v_j \in S_1 \cap S_2$. Then v_n precedes v_j in the arc of T corresponding to $S_1 \cup \{v_n\}$, and v_j precedes v_n in the arc of T corresponding to $S_2 \cup \{v_n\}$.

Therefore T represents $\mathcal{P}$, as desired. ■

We now show that if $\mathcal{P}$ contains an isolated point, by which we mean a point $x \in X$ which is incomparable to every other point of X, then we can use a $(k-1)$-tournament which represents $\mathcal{P}[X \setminus \{x\}]$ to construct a k-tournament which represents $\mathcal{P}$.

Lemma 2.3.8 *Let $\mathcal{P} = (X, P)$ be a finite poset and suppose there is $x \in X$ such that $x \| y$ for all $y \in X \setminus \{x\}$. Let $|X| \geq 5$ and $k = |X| - 1$. If there is a $(k-1)$-tournament representing $\mathcal{P}[X \setminus \{x\}]$, then there is a k-tournament representing $\mathcal{P}$.*

Proof. Let $\mathcal{P} = (X, P)$, x and k be as in the statement of the lemma. Let T' be a $(k-1)$-tournament representing $\mathcal{P}[X \setminus \{x\}]$. Without loss of generality we can assume that $V(T') = X \setminus \{x\}$.

By Lemma 2.3.6 there are classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of $(k-1)$-subsets of $X \setminus \{x\}$ such that $\cup\{S : S \in \mathcal{C}_1\} = X \setminus \{x\} = \cup\{S : S \in \mathcal{C}_2\}$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the sets of arcs of T' corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We now construct a k-tournament T which represents $\mathcal{P}$.

Let $V(T) = X$, and let

$$\begin{aligned} A(T) &= \{(x, u_1, \ldots, u_{k-1}) : (u_1, \ldots, u_{k-1}) \in \mathcal{A}_1\} \\ &\cup \{(v_1, \ldots, v_{k-1}, x) : (v_1, \ldots, v_{k-1}) \in \mathcal{A}_2\} \end{aligned}$$

$$\begin{aligned} &\cup \quad \{(w_1, \ldots, w_{k-1}, x) : (w_1, \ldots, w_{k-1}) \in A(T') \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)\} \\ &\cup \quad \{A\}, \end{aligned}$$

where A is any arc corresponding to $X \setminus \{x\}$ with the property that $u < v$ in P implies u precedes v in A.

This defines the k-tournament T; it remains to show that T represents $\mathcal{P}$.

First, if $y \in X \setminus \{x\}$, then x precedes y in at least one arc of the form $(x, u_1, \ldots, u_{k-1})$, where $(u_1, \ldots, u_{k-1}) \in \mathcal{A}_1$, since by the definition of $\mathcal{A}_1$ there is at least one arc of $\mathcal{A}_1$ containing y. Similarly there is at least one arc of $\mathcal{A}_2$ which contains y, and so y precedes x in at least one arc of the form $(v_1, \ldots, v_{k-1}, x)$, where $(v_1, \ldots, v_{k-1}) \in \mathcal{A}_2$. Thus for any $y \in X \setminus \{x\}$, x does not always precede y in T and y does not always precede x in T.

Now let $y, z \in X \setminus \{x\}$. It is clear from the definition of $A(T)$ that if $y < z$ in P, then y always precedes z in T. If $y \| z$ in P, then since T' represents $\mathcal{P}[X \setminus \{x\}]$ there are arcs A_1 and A_2 of T' such that y precedes z in A_1 and z precedes y in A_2. Without loss of generality we assume $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Let $A_1 = (u_1, \ldots, u_{k-1})$ and $A_2 = (v_1, \ldots, v_{k-1})$. Then y precedes z in the arc $(x, u_1, \ldots, u_{k-1})$ of T, and z precedes y in the arc $(v_1, \ldots, v_{k-1}, x)$ of T. Therefore y always precedes z in T if and only if $y < z$ in P.

It follows that T represents $\mathcal{P}$. ∎

We now investigate what we can require of a chain decomposition of a poset $\mathcal{P}$. The following lemma shows that if a minimal chain decomposition of $\mathcal{P}$ contains the smallest possible number of chains of cardinality one, then we have some information about the way in which the vertices belonging to chains of cardinality one are related (in P) to the remaining elements of X.

Lemma 2.3.9 *Let $\mathcal{P} = (X, P)$ be a finite poset of width n, and let $C_1, \ldots, C_n$ be a chain decomposition of $\mathcal{P}$ such that $|C_i| = 1$ if and only if $1 \le i \le t$, and t is least over all such decompositions. Let $C_i = \{v_i\}$, $1 \le i \le t$. Then $v_i \| c$ for every $c \in C_j$ with $|C_j| > 2$ and every $i \in \{1, \ldots, t\}$.*

Proof. Let $\mathcal{P}$ and $C_1, \ldots, C_n$ be as in the statement of the lemma.

Suppose $|C_j| > 2$, and $v_i < c$ for some $i \in \{1, \ldots, t\}$ and $c \in C_j$. Then letting $C_i' = \{v_i, c\}$ and $C_j' = C_j \setminus \{c\}$, it is easy to see that $(\{C_1, \ldots, C_n\} \setminus \{C_i, C_j\}) \cup \{C_i', C_j'\}$ is a set of n chains which partition X and with $t-1$ chains of cardinality 1, contradicting the minimality of t.

An analogous argument shows that $c \not< v_i$ for any $c \in C_j$ with $|C_j| > 2$ and $i \in \{1, \ldots, t\}$. ∎

Lemma 2.3.10 below is a technical result, concerning the construction of two particular extensions of a poset $\mathcal{P}$, which we will need for the proof of Lemma 2.3.11.

Lemma 2.3.10 *Let $\mathcal{P} = (X, P)$ be a finite poset of width n, and let $C_1, \ldots, C_n$ be a chain decomposition of $\mathcal{P}$, where $|C_i| = 1$ if and only if $1 \leq i \leq t$, and t is least over all such decompositions. Suppose further that $0 < t < n$. Let $C_i = \{v_i\}$, $1 \leq i \leq t$. Then there are posets $\mathcal{P}_1 = (X, P_1)$ and $\mathcal{P}_2 = (X, P_2)$ with the following properties.*

1. *Each of $\mathcal{P}_1$ and $\mathcal{P}_2$ is an extension of $\mathcal{P}$.*
2. *If $c \in C_n$ and $i \in \{1, \ldots, t\}$ and $v_i \| c$ in P, then $v_i < c$ in P_1.*
3. *If $i, j \in \{1, \ldots, t\}$ and $i < j$, then $v_i < v_j$ in P_1.*
4. *If $c \in C_n$ and $i \in \{1, \ldots, t\}$ and $v_i \| c$ in P, then $c < v_i$ in P_2.*
5. *If $i, j \in \{1, \ldots, t\}$ and $i < j$, then $v_i > v_j$ in P_2.*

Proof. Let $\mathcal{P}$ be as in the statement of the lemma. Let

$$\begin{aligned} S_1 &= \{(v_i, c) : 1 \leq i \leq t, c \in C_n \text{ and } v_i \| c \text{ in } P\} \\ &\cup \{(v_i, v_{i+1}) : 1 \leq i \leq t-1\}, \text{ and} \\ S_2 &= \{(c, v_i) : 1 \leq i \leq t, c \in C_n \text{ and } v_i \| c \text{ in } P\} \\ &\cup \{(v_{i+1}, v_i) : 1 \leq i \leq t-1\}. \end{aligned}$$

Notice that $S_1, S_2 \subseteq inc(\mathcal{P})$. Let $P_1 = tr(P \cup S_1)$ and $P_2 = tr(P \cup S_2)$. By Lemma 2.3.4, in order to show that each of $\mathcal{P}_1 = (X, P_1)$ and $\mathcal{P}_2 = (X, P_2)$ is a poset it suffices to show that neither S_1 nor S_2 contains a strict alternating cycle.

We distinguish two cases, depending on the cardinality of the chain C_n of $\mathcal{P}$.

Case 1: $|C_n| \geq 3$.

First let $Z = \{(x_1, y_1), \ldots, (x_l, y_l)\} \subseteq S_1$. By the definition of S_1, $x_h \in \{v_1, \ldots, v_t\}$ for each h, $1 \leq h \leq l$. Now if Z were a strict alternating cycle in S_1, we would have $y_1 \leq x_2$ in P, and so $y_1 \leq v_i$ in P for some $i \in \{1, \ldots, t\}$. By the minimality of t, $v_j \| v_i$ in P for any $j \neq i$ with $j \in \{1, \ldots, t\}$; so $y_1 \notin \{v_1, \ldots, v_t\}$. Similarly, since t is minimal and $|C_n| \geq 3$ we have by Lemma 2.3.9 that $v_i \| c$ for every $c \in C_n$. Therefore $y_1 \notin C_n$. But this implies that $(x_1, y_1) = (v_i, y_1) \notin S_1$, contrary to the definition of Z. It follows that Z is not a strict alternating cycle.

The same argument applied to the set S_2 shows that S_2 also contains no strict alternating cycle.

Case 2: $|C_n| = 2$.

Again we let $Z = \{(x_1, y_1), \ldots, (x_l, y_l)\} \subseteq S_1$, and observe that $x_h \in \{v_1, \ldots, v_t\}$ for each h, $1 \leq h \leq l$. Suppose Z is a strict alternating cycle. Then we have $y_h \leq x_{h+1}$, $1 \leq h \leq l$, in P (where the subscripts are reduced modulo l), and again by the minimality of t we know that $y_h \notin \{v_1, \ldots, v_t\}$. Therefore $y_1, \ldots, y_l \in C_n$. Let $C_n = \{c_1, c_2\}$, with $c_1 < c_2$ in P. Since for $h = 1, \ldots, l$, $y_h \in C_n$ and $y_h < x_{h+1} = v_i$ in P, then by the minimality of t we must have $y_h = c_1$, since otherwise $\{c_1, c_2, x_{h+1}\} = \{c_1, c_2, v_i\}$ is a chain in $\mathcal{P}$ (which would mean we could find a chain decomposition of $\mathcal{P}$ into $n-1$ chains). Thus we have $y_h = c_1$, $1 \leq h \leq l$, so that $y_1 = \cdots = y_l$.

Now suppose $l \geq 2$. Then $y_h \leq x_{h+1}$, $1 \leq h \leq l$, together with $y_1 = \cdots = y_l$, imply that $y_1 < x_h$ for each $h = 1, \ldots, l$. This contradicts the assumption that Z is a strict alternating cycle. Therefore $l = 1$; but in that case we get $y_1 \leq x_1$ in P which contradicts $(x_1, y_1) \in S_1$.

Therefore Z is not a strict alternating cycle, so we have shown that S_1 contains no strict alternating cycle, as desired. The same argument applied to S_2 shows that S_2 contains no strict alternating cycle.

Thus in either case, each of $\mathcal{P}_1$ and $\mathcal{P}_2$ is a poset; and it is clear that these posets have the required properties. This completes the proof of the lemma. ∎

We are now ready to prove Lemma 2.3.11, which provides the induction base for the proof of Theorem 2.3.2.

Lemma 2.3.11 *Let $\mathcal{P} = (X, P)$ be a finite poset with $|X| \geq 4$, and let $k = |X| - 1$. Then there is a k-tournament which represents $\mathcal{P}$.*

Proof. Let $\mathcal{P} = (X, P)$ be given, where $|X| \geq 4$, and let $k = |X| - 1$. Let n denote the width of $\mathcal{P}$.

Let $X = C_1 \cup \cdots \cup C_n$ be a chain decomposition of $\mathcal{P}$, where $|C_i| = 1$ if and only if $1 \leq i \leq t$, and t is least over all such decompositions.

Let $C_i = \{v_i\}$, $1 \leq i \leq t$. Without loss of generality we assume that $|C_{t+1}| \leq \cdots \leq |C_n|$.

The proof of the lemma is divided into several cases, depending on the cardinalities of the chains $C_1, \ldots, C_n$. We begin by observing that if $n = 1$, then $\mathcal{P}$ is itself a chain, and so the transitive k-tournament on vertex set X and with underlying linear order P represents $\mathcal{P}$. Thus we assume for the remainder of the proof that $n > 1$.

Case 1: $t = 0$, and $n > 1$.

In this case, $|C_j| \geq 2$ for all $j \in \{1, \ldots, n\}$. We construct a k-tournament T which represents $\mathcal{P}$. We let $V(T) = X$.

To define $A(T)$ we will define a linear extension $L(x)$ of the subposet $\mathcal{P}[X \setminus x]$, for each $x \in X$. We then define the arc A of T corresponding to $X \setminus x$ from the linear extension $L(x)$ in the obvious way: we let u precede v in A if and only if $u < v$ in $L(x)$. We therefore need to define the linear extensions $L(x)$ so that if $u \| v$ in P then there are $L(x_1)$ and $L(x_2)$ such that $u < v$ in $L(x_1)$ and $v < u$ in $L(x_2)$.

To do this, we will use Lemma 2.3.5. For each chain C_j in the decomposition of $\mathcal{P}$, let L_j be a linear extension of $\mathcal{P}$ in which $c \in C_j$ and $u \| c$ in P imply $u \leq c$ in L_i.

For each $x \in X$, let

$$L(x) = L_j[X \setminus x], \text{ where } x \in C_j.$$

Let the arc of T corresponding to $X \setminus x$ be $(x_1, \ldots, x_k)$, where $L(x) = \{x_1, \ldots, x_k\}$ with $x_1 < \cdots < x_k$.

This is enough to define T. We now need to show that T represents $\mathcal{P}$.

Let $u, v \in V(T)$. Since each L_j is a linear extension of $\mathcal{P}$, then each $L(x) = L_j[X \setminus x]$ is a linear extension of $\mathcal{P}[X \setminus x]$. It follows that if $u < v$ in P, then u always precedes v in T.

Now let $u \| v$ in P. Then u and v do not belong to the same chain in the decomposition of $\mathcal{P}$. Let $u \in C_j$ and $v \in C_l$. Since $|C_j|, |C_l| \geq 2$, there are $u' \in C_j$ and $v' \in C_l$ such that $u \neq u'$ and $v \neq v'$.

Consider L_j. We have $u \in C_j$ and $u \| v$ in P, so that $v < u$ in L_j. Since $u' \in C_j$, $L(u')$ is a subposet of L_j; and since $u, v \in X \setminus u'$, then $v < u$ in $L(u')$.

Similarly, since $v \in C_l$ and $u \| v$ in P, then $u < v$ in L_l; and since $v' \in C_l$ and $u, v \in X \setminus v'$, then $u < v$ in $L(v')$.

Therefore, letting A_1 be the arc of T corresponding to $L(u')$ and A_2 be the arc of T corresponding to $L(v')$, we have that v precedes u in A_1 and u precedes v in A_2.

Therefore T represents $\mathcal{P}$. This completes the proof of the lemma in Case 1.

Case 2: $t \geq 1$ and $n > t + 1$.

In this case X is partitioned into the chains $C_1, \ldots, C_t, C_{t+1}, \ldots, C_n$, where $C_i = \{v_i\}$ for $i \in \{1, \ldots, t\}$, and $|C_j| \geq 2$ for $j \in \{t+1, \ldots, n\}$. Note that since $n > t + 1$, there are at least two chains C_j of cardinality at least two.

In this case we will use the same approach as in Case 1. However the chains $C_1, \ldots, C_t$ must be dealt with differently; for although we can certainly find a linear extension L_i of $\mathcal{P}$ in which $u < v_i$ whenever $u \| v_i$ in P, we achieve nothing by letting $L(v_i) = L_i[X \setminus v_i]$ because there is no $v' \in C_i$ with $v' \neq v_i$. Thus the problem with the approach in Case 1 is that, for $u \in X$ and $v_i \in \cup_{i=1}^{t} C_i$ with $u \| v_i$ in P, there is no guarantee that there are arcs A_1 and A_2 of T such that u precedes v_i in A_1 and v_i precedes u in A_2.

We therefore modify the construction given in Case 1 in the following way. We select two chains C_{n-1} and C_n of cardinality at least two (this is the reason for the assumption that $n > t + 1$). We define the linear extension L_{n-1} of $\mathcal{P}$ corresponding to C_{n-1} in such a way that not only do $c \in C_{n-1}$, $u \notin \{v_1, \ldots, v_t\}$ and $u \| c$ in P imply $u < c$ in L_{n-1}, but also $u \in \cup_{j=t+1}^{n} C_j$ and $u \| v_i$ in P imply $u < v_i$ in L_{n-1} $(1 \leq i \leq t)$. Similarly, we define the linear extension L_n of $\mathcal{P}$ so that both $c \in C_n$, $u \notin \{v_1, \ldots, v_t\}$ and $u \| c$ in P imply $u < c$ in L_n, and also $u \in \cup_{j=t+1}^{n} C_j$ and $u \| v_i$ in P imply $v_i < u$ in L_n $(1 \leq i \leq t)$. This will be enough to make sure that for any $u \in \cup_{j=t+1}^{n} C_j$ such that $u \| v_i$ in P, there will be an arc of T in which v_i precedes u, and another in which u precedes v_i. This still does not take care of the incomparable pairs $\{v_{i_1}, v_{i_2}\}$,

$1 \leq i_1 < i_2 \leq t$; so we also require that $v_1 < \cdots < v_t$ in L_n and $v_t < \cdots < v_1$ in L_{n-1}.

We now give the precise construction. As before, for each chain C_i in the decomposition of $\mathcal{P}$ we define a linear extension L_i of $\mathcal{P}$; but the definition of this extension will now depend on the particular chain C_i in question.

If $1 \leq i \leq t$, we let L_i be any linear extension of $\mathcal{P}$.

If $t+1 \leq i \leq n-2$ (if any such i exists) we apply Lemma 2.3.5 to C_i and $\mathcal{P}$ to get a linear extension L_i of $\mathcal{P}$ in which $c \in C_i$ and $u \| c$ in P imply $u \leq c$ in L_i.

To define L_{n-1} and L_n, we let $\mathcal{P}_1$ and $\mathcal{P}_2$ be extensions of $\mathcal{P}$ defined as in Lemma 2.3.10. Recall that $\mathcal{P}_1$ has the properties

1. If $c \in C_n$ and $i \in \{1, \ldots, t\}$ and $v_i \| c$ in $\mathcal{P}$, then $v_i < c$ in $\mathcal{P}_1$, and

2. $v_1 < \cdots < v_t$ in $\mathcal{P}_1$,

and that $\mathcal{P}_2$ has the properties

3. If $c \in C_n$ and $i \in \{1, \ldots, t\}$ and $v_i \| c$ in $\mathcal{P}$, then $c < v_i$ in $\mathcal{P}_2$, and

4. $v_t < \cdots < v_1$ in $\mathcal{P}_2$.

Therefore $D_n = C_n \cup \{v_1, \ldots, v_t\}$ is a chain in each of $\mathcal{P}_1$ and $\mathcal{P}_2$; in $\mathcal{P}_1$ we have

$$v_1 < v_2 < \cdots < v_t < c_1 < \cdots < c_m$$

and in $\mathcal{P}_2$ we have

$$c_1 < \cdots < c_m < v_t < v_{t-1} < \cdots < v_1,$$

where $C_n = \{c_1, \ldots, c_m\}$ and $c_1 < \ldots < c_m$ in P.

We apply Lemma 2.3.5 to the chain C_{n-1} in $\mathcal{P}_2$ to obtain a linear extension L_{n-1} of $\mathcal{P}_2$ in which $c \in C_{n-1}$ and $u \| c$ in $\mathcal{P}_2$ imply $u \leq c$ in L_{n-1}. Note that since $\mathcal{P}$ is a subposet of $\mathcal{P}_2$, then L_{n-1} is also a linear extension of $\mathcal{P}$.

We apply Lemma 2.3.5 to the chain D_n in $\mathcal{P}_1$ to obtain a linear extension L_n of $\mathcal{P}_1$ (and hence of $\mathcal{P}$) in which $d \in D_n$ and $u \| d$ in $\mathcal{P}_1$ imply $u \leq d$ in L_n.

Having defined the linear extensions $L_1, \ldots, L_n$ of $\mathcal{P}$, we proceed as in Case 1 to define the linear extensions $L(x)$ of $\mathcal{P}[X \setminus x]$, for each $x \in X$:

$$L(x) = L_i[X \setminus x], \text{ where } x \in C_i.$$

Finally, we construct the k-tournament T by letting $V(T) = X$, and defining the arc of T corresponding to a k-subset $X \setminus \{x\}$ of $V(T)$ to be $A = (x_1, \ldots, x_k)$, where $X \setminus \{x\} = \{x_1, \ldots, x_k\}$ and $x_1 < \cdots < x_k$ in $L(x)$.

We now show that T represents $\mathcal{P}$.

Let $u, v \in X$. As in Case 1, each L_i is a linear extension of $\mathcal{P}$, and so if $u < v$ in P, then u always precedes v in T.

Now suppose $u \| v$ in P. Then u and v belong to different chains in the decomposition of $\mathcal{P}$.

First let $u \in C_{i_1}$ and $v \in C_{i_2}$, where $1 \leq i_1 < i_2 \leq t$. Then $u = v_{i_1}$ and $v = v_{i_2}$. Let $x_1 \in C_{n-1}$, and $x_2 \in C_n$. Since L_n is a linear extension of $\mathcal{P}_1$ and $v_{i_1} < v_{i_2}$ in P_1, it follows that $v_{i_1} < v_{i_2}$ in $L(x_1)$, and hence v_{i_1} precedes v_{i_2} in the corresponding arc of T. On the other hand, L_{n-1} is a linear extension of $\mathcal{P}_2$, and $v_{i_2} < v_{i_1}$ in P_2, so that $v_{i_2} < v_{i_1}$ in $L(x_2)$. Therefore v_{i_2} precedes v_{i_1} in the corresponding arc of T.

Thus if $u, v \in \{v_1, \ldots, v_t\}$ there is an arc of T in which u precedes v, and another in which v precedes u.

Now suppose $u \in C_i$, $1 \leq i \leq t$, and $v \in C_j$, $t + 1 \leq j \leq n$. Then $u = v_i$, and either $t + 1 \leq j \leq n - 2$ or $j \in \{n - 1, n\}$. In either case, $|C_j| \geq 2$; so we can find $x \in C_j \setminus \{v\}$. If $t + 1 \leq j \leq n - 2$, then since $v \in C_j$ and $u \| v$ in $\mathcal{P}$, we have $u < v$ in $L(x)$. Suppose $j = n - 1$. Since $u = v_i$ and $v_i \| v$ in P, and $v \notin C_n$, then also $u \| v$ in $\mathcal{P}_2$; it then follows from the definition of L_{n-1} that $u < v$ in L_{n-1}. Therefore $u < v$ in $L(x)$, and so u precedes v in the corresponding arc of T. Now let $j = n$. In this case $u = v_i < v$ in $\mathcal{P}_1$, so that $u < v$ in L_n and hence in $L(x)$. Therefore u precedes v in the arc of T corresponding to $L(x)$.

Thus if $u \in \{v_1, \ldots, v_t\}$ and $v \in \cup_{j=t+1}^{n} C_j$ there is an arc of T in which u precedes v. We now show that under the same conditions, there is an arc of T in which v precedes u.

If $v \in C_n$, then $v < v_i = u$ in P_2; since L_{n-1} is a linear extension of $\mathcal{P}_2$, then v precedes u in the arc corresponding to $L(x)$ for any $x \in C_{n-1}$. Otherwise $v \in \cup_{j=t+1}^{n-1} C_j$, and so $u = v_i \| v$ in P_1. Now $u = v_i \in D_n$, and L_n has the property that $d \in D_n$ and $v \| d$ in P_1 imply $v < d$ in L_n. Therefore $v < v_i = u$ in L_n. So v precedes u in the arc corresponding to $L(x)$, for any $x \in C_n$.

Therefore if $u \in \{v_1, \ldots, v_t\}$ and $v \in \cup_{j=t+1}^{n} C_j$ there is an arc of T in which u precedes v, and an arc of T in which v precedes u.

Finally, suppose $u, v \in \cup_{j=t+1}^{n} C_j$. Let $u \in C_{j_1}$ and $v \in C_{j_2}$, and let $x_1 \in C_{j_1} \setminus \{u\}$ and $x_2 \in C_{j_2} \setminus \{v\}$. Then $v < u$ in $L(x_1)$, and $u < v$ in $L(x_2)$; therefore u precedes v in the arc of T corresponding to $L(x_1)$ and v precedes u in the arc of T corrsponding to $L(x_2)$.

Thus whenever $u \| v$ in P, there are arcs A_1 and A_2 of T such that u precedes v in A_1 and v precedes u in A_2. It now follows that T represents $\mathcal{P}$.

This completes the proof of the lemma in Case 2.

Case 3: $t \geq 2$ **and** $n \leq t + 1$.

We observe that if $n = t$, then $\mathcal{P}$ is an antichain, and $|X| \geq 4$ implies $t \geq 4$. Therefore by Lemma 2.3.7 there is a k-tournament representing $\mathcal{P}$. We therefore assume that $n = t + 1$.

Notice that since $n = t + 1$, the chain decomposition of $\mathcal{P}$ consists of t chains of cardinality 1 and exactly one chain C_{t+1} of cardinality at least 2.

Again we define posets $\mathcal{P}_1$ and $\mathcal{P}_2$ from $\mathcal{P}$; in this case we require only that $v_1 < \cdots < v_t$ in $\mathcal{P}_1$ and $v_t < v_{t-1} < \cdots < v_1$ in $\mathcal{P}_2$. Therefore we let $S_1 = \{(v_i, v_{i+1}) : 1 \leq i \leq t-1\}$ and $S_2 = \{(v_{i+1}, v_i) : 1 \leq i \leq t-1\}$. It is easy to see that neither S_1 nor S_2 contains an alternating cycle; so by Lemma 2.3.4 each of $\mathcal{P}_1 = (X, tr(P \cup S_1))$ and $\mathcal{P}_2 = (X, tr(P \cup S_2))$ is a poset.

Let $x_1, x_2 \in C_{t+1}$.

We apply Lemma 2.3.5 to C_{t+1} in the poset $\mathcal{P}_1$ to obtain a linear extension L_1 of $\mathcal{P}_1$ (and so of $\mathcal{P}$) in which $c \in C_{t+1}$ and $v \| c$ in $\mathcal{P}_1$ imply $v < c$ in L_1. Therefore if $i \in \{1, \ldots, t\}$ and $c \in C_{t+1}$, and also $v_i \| c$ in P, then $v_i < c$ in L_1.

Next we apply Lemma 2.3.5 to C_{t+1} in $\mathcal{P}_2$ to obtain a linear extension L_2 of $\mathcal{P}_2$ (and hence of $\mathcal{P}$) in which $c \in C_{t+1}$ and $v \| c$ in $\mathcal{P}_2$ imply $v < c$ in L_2. Thus $v_i < c$ in L_2 whenever $i \in \{1, \ldots, t\}$, $c \in C_{t+1}$, and $v_i \| c$ in $\mathcal{P}$.

Finally we apply Lemma 2.3.5 to C_{t+1} in $\mathcal{P}$ to obtain a linear extension L of $\mathcal{P}$ in which $c \in C_{t+1}$ and $v \| c$ in P imply $c < v$ in L (notice that here we are using part 2 of Lemma 2.3.5).

We now define the linear extensions $L(x)$ of $\mathcal{P}[X \setminus x]$ for each $x \in X$.

$$\begin{aligned}
\text{We let } L(x_1) &= L_1[X \setminus x_1], \\
L(x_2) &= L_2[X \setminus x_2], \\
L(c) &= L_2[X \setminus c], \text{ for } c \in C_{t+1} \setminus \{x_1, x_2\}, \\
\text{and } L(v_i) &= L[X \setminus v_i], 1 \leq i \leq t.
\end{aligned}$$

Once again we let the arc A of T corresponding to the set $X \setminus \{x\}$ be $A = (x_1, \ldots, x_k)$, where $X \setminus \{x\} = \{x_1, \ldots, x_k\}$ and $x_1 < \cdots < x_k$ in $L(x)$.

This defines the k-tournament T. We now show that T represents $\mathcal{P}$.

First, since each of L_1, L_2 and L is a linear extension of $\mathcal{P}$, then it is clear that if $u < v$ in P, then u always precedes v in T.

Now let $u \| v$ in P. Then without loss of generality, either $u, v \in \{v_1, \ldots, v_t\}$ or $u \in \{v_1, \ldots, v_t\}$ and $v \in C_{t+1}$.

Suppose $u, v \in \{v_1, \ldots, v_t\}$. Let $u = v_i$ and $v = v_j$ with $i < j$. Then $u < v$ in L_1 and so in $L(x_1)$, while $v < u$ in L_2 and so in $L(x_2)$. Thus there is an arc of T in which u precedes v, and another in which v precedes u.

Suppose $u \in \{v_1, \ldots, v_t\}$ and $v \in C_{t+1}$. Then $u < v$ in both L_1 and L_2, so that $u < v$ in at least one of $L(x_1)$ and $L(x_2)$. Similarly $v < u$ in L, so that $v < u$ in $L(v_j)$, where $v_j \in \{v_1, \ldots, v_t\} \setminus \{u\}$ (here we are using the fact that $t \geq 2$). Thus again we have an arc of T in which u precedes v and another in which v precedes u.

It follows that T represents $\mathcal{P}$. This completes the proof of the proposition in Case 3.

Case 4: $t = 1$ and $n \leq t + 1$.

Notice that in this case $n > t$ since otherwise $\mathcal{P}$ consists of a single chain of cardinality one, contradicting $|X| \geq 4$. Therefore we assume that $n = t + 1 = 2$. So the chain decomposition of $\mathcal{P}$ consists of one chain $C_1 = \{v_1\}$ of cardinality one, and one other chain C_2 of cardinality at least three.

If $|C_2| \geq 4$, then $k - 1 \geq 3$ and the transitive $(k-1)$-tournament on vertex set C_2 represents C_2; so by Lemma 2.3.8 there is a k-tournament representing $\mathcal{P}$.

Otherwise $|C_2| = 3$. Let $C_2 = \{c_1, c_2, c_3\}$, where $c_1 < c_2 < c_3$. Then $X = \{v_1, c_1, c_2, c_3\}$ and the 3-tournament T represents $\mathcal{P}$, where T is given by:

$$\begin{aligned} V(T) &= X; \\ A(T) &= \{(c_1, c_2, c_3), (v_1, c_1, c_3), (c_1, v_1, c_2), (c_2, c_3, v_1)\}. \end{aligned}$$

This completes the proof of the lemma in Case 4. ■

Finally, we prove Lemma 2.3.12, which provides the induction step for the proof of Theorem 2.3.2.

Lemma 2.3.12 *Let $\mathcal{P} = (X, P)$ be a finite poset, where $|X| \geq 5$, and let k be an integer satisfying $3 \leq k \leq |X| - 2$. If every poset $\mathcal{P}' = (X', P')$ with $|X'| = |X| - 1$ can be represented by a k-tournament, then $\mathcal{P}$ can be represented by a k-tournament.*

Proof. Let $\mathcal{P}$ and k be as in the statement of the lemma, and assume that every finite poset $\mathcal{P}'$ on $|X| - 1$ vertices can be represented by a k-tournament.

We let $x \in X$ be maximal in $\mathcal{P}$, and let $\mathcal{P}'$ be the subposet of $\mathcal{P}$ induced by $X \setminus \{x\}$. Thus we have $\mathcal{P}' = (X', P')$, where $X' = X \setminus \{x\}$ and $u \leq v$ in P' if and only if $u, v \in X'$ and $u \leq v$ in P. By assumption there is a k-tournament T' which represents $\mathcal{P}'$, and we assume without loss of generality that $V(T') = X'$. Thus for any $u, v \in X'$, $u < v$ in $\mathcal{P}'$ if and only if u always precedes v in T'.

We will construct a k-tournament T, with $V(T) = X$, such that $A(T') \subseteq A(T)$, and such that T represents $\mathcal{P}$. Thus to define T we must define an arc corresponding to each k-subset of X which contains x. In order for T to represent $\mathcal{P}$, we must ensure that u always precedes x in T if and only if $u < x$ in P (recall that x is maximal in $\mathcal{P}$), and that if $u, v \in X'$ and $u < v$ in P, then u precedes v in any new arcs which contain both u and v. (Note that since T' represents $\mathcal{P}'$, if $u, v \in X'$ and $u \| v$ in P, then there are arcs A_1, A_2 of T' such that u precedes v in A_1 and v precedes u in A_2.)

Let $\mathcal{S} = \{S \subseteq X : |S| = k \text{ and } x \in S\}$. We will define a set $\mathcal{A}$ of arcs such that $\mathcal{A}$ contains one arc corresponding to each element of $\mathcal{S}$, and we will let $A(T) = A(T') \cup \mathcal{A}$.

Let $X' = U \cup I$, where

$$\begin{aligned} U &= \{u \in X' : u < x \text{ in } \mathcal{P}\} \text{ and} \\ I &= \{u \in X' : u \| x \text{ in } \mathcal{P}\}. \end{aligned}$$

Note that U and I are disjoint, and that if $u \in U$ and $v \in I$, then by the transitivity of P, $v \not< u$ in P.

We divide the remainder of the proof into two cases, depending on the cardinality of I. In each case, we construct the set $\mathcal{A}$ of arcs of T so that for every $A \in \mathcal{A}$,

1. If $u, v \in A$ and $u \in U$, $v \in I$, then u precedes v in A.

2. If $u, v \in A$ and $u < v$ in P, then u precedes v in A.

3. If $u \in A$ and $u \in U$ then u precedes x in A.

4. If $v \in I$ then there are $A_1, A_2 \in \mathcal{A}$ such that x precedes v in A_1 and v precedes x in A_2.

We first let L_U be a linear extension of $\mathcal{P}[U \cup \{x\}]$, and L_I be a linear extension of $\mathcal{P}[I]$. Notice that by the definition of U, $u < x$ in L_U for every $u \in U$.

Case 1: $|I| \geq k$.

If $|I| = k = 3$, then since $|X| \geq 5$, we have $|U| \geq 1$. We let $I = \{a_1, a_2, a_3\}$, and

$$\begin{aligned} \mathcal{A} &= \{(a_i, a_j, x) : 1 \leq i, j \leq 3, i \neq j, \text{ and } a_i < a_j \text{ in } L_I\} \\ &\cup \{(u, x, a_i) : 1 \leq i \leq 3 \text{ and } u \in U\} \\ &\cup \{(u, u', x) : u, u' \in U \text{ and } u < u' \text{ in } L_U\}. \end{aligned}$$

Otherwise either $|I| > k$ or $k \geq 4$, and by Lemma 2.3.6 we can find disjoint classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of $(k-1)$-subsets of I such that $\cup\{S' : S' \in \mathcal{C}_1\} = I = \cup\{S' : S' \in \mathcal{C}_2\}$. For each $S' \in \mathcal{C}_1$, let $A = (v_1, \ldots, v_{k-1}, x)$ where $S' = \{v_1, \ldots, v_{k-1}\}$ and $v_1 < \cdots < v_{k-1}$ in L_I, and let $A \in \mathcal{A}$. Thus A will be the arc of T corresponding to $S' \cup \{x\}$.

For each $S' \in \mathcal{C}_2$, let $A = (x, v_1, \ldots, v_{k-1})$ where $S' = \{v_1, \ldots, v_{k-1}\}$ and $v_1 < \cdots < v_{k-1}$ in L_I, and let $A \in \mathcal{A}$.

This defines the arc $A \in \mathcal{A}$ for every $S \in \mathcal{S}$ of the form $S = S' \cup \{x\}$ with $S' \in \mathcal{C}_1 \cup \mathcal{C}_2$. It also guarantees us an arc in which x precedes v and an arc in which v precedes x, for each $v \in I$.

Now let $S \in \mathcal{S}$ be such that $S \setminus \{x\} \notin \mathcal{C}_1 \cup \mathcal{C}_2$. If $S \cap U \neq \emptyset$, let $S \cap U = \{u_1, \ldots, u_i\}$, where $u_1 < \cdots < u_i$ in L_U; and if $S \cap I \neq \emptyset$, let $S \cap I = \{v_1, \ldots, v_j\}$, where $v_1 < \cdots < v_j$

in L_I. If $S \cap U \neq \emptyset$ and $S \cap I \neq \emptyset$, we let $A = (u_1, \ldots, u_i, x, v_1, \ldots, v_j)$. If $S \cap U = \emptyset$ but $S \cap I \neq \emptyset$, we let $A = (x, v_1, \ldots, v_j)$. Finally, if $S \cap U \neq \emptyset$ but $S \cap I = \emptyset$, we let $A = (u_1, \ldots, u_i, x)$. This defines an arc $A \in \mathcal{A}$ corresponding to each remaining element S of $\mathcal{S}$. We have therefore defined the set $\mathcal{A}$ of arcs of T. The k-tournament T is now given by:

$$\begin{aligned} V(T) &= X, \\ \text{and } A(T) &= A(T') \cup \mathcal{A}. \end{aligned}$$

We now show that T represents $\mathcal{P}$.

If $|I| = k = 3$, it is easy to see that T represents $\mathcal{P}$, so we assume that either $|I| > k$ or $k \geq 4$.

First, if $u < x$ in P then $u \in U$; therefore any k-subset S of X which contains both u and x is of the form $S = S' \cup \{x\}$, where $S' \notin \mathcal{C}_1 \cup \mathcal{C}_2$. Consequently u precedes x in the arc corresponding to S. Thus u always precedes x in T. On the other hand, if $v \| x$ in P then $v \in I$, so there are $S_1' \in \mathcal{C}_1$ and $S_2' \in \mathcal{C}_2$ such that $v \in S_1' \cap S_2'$. Therefore v precedes x in the arc corresponding to $S_1' \cup \{x\}$, and x precedes v in the arc corresponding to $S_2' \cup \{x\}$.

Now let $w_1, w_2 \in X' = X \setminus \{x\}$. If $w_1 \| w_2$ in $\mathcal{P}$, then also $w_1 \| w_2$ in $\mathcal{P}'$, and so there is an arc of T' in which w_1 precedes w_2 and another in which w_2 precedes w_1. Since $A(T') \subseteq A(T)$, the same is true of T. On the other hand, if $w_1 < w_2$ in P, then w_1 precedes w_2 in every arc of $\mathcal{A}$ which contains both w_1 and w_2; and since $w_1 < w_2$ also in $\mathcal{P}'$, then w_1 always precedes w_2 in T'. It follows that w_1 always precedes w_2 in T.

This completes the proof of the lemma in Case 1.

Case 2: $|I| \leq k - 1$.

In this case, there is at most one $(k-1)$-subset of I. We must, therefore, modify the construction given in Case 1. Once again we want to make sure that for each $v \in I$ we construct at least one arc in which v precedes x, and at least one more in which x precedes v. We achieve this in the following way. We distinguish a subset of U, whose cardinality is such that for each $v \in I$ we have at least one k-subset of X consisting of this distinguished subset of U, the element x, and a subset of I

containing v, and at least one more k-subset of X containing both x and v. We use these k-subsets of X to construct arcs in which v precedes x and x precedes v. The precise construction is given below.

First, let $m = max\{1, k-1-|I|\}$. We will distinguish a proper m-subset of U; we can do this provided $m < |U|$. We show in Claim 2.3.13 below that this is so.

Claim 2.3.13 *Let $m = max\{1, k-1-|I|\}$. Then $|U| > m$.*

Proof. First, we have $m = 1$ if and only if $k-1-|I| \leq 1$, which is the case if and only if $|I| \geq k-2$.

If $|I| = k-1$, then

$$\begin{aligned} |U| &= |X| - |I| - 1 \\ &= |X| - k \\ &\geq 2 \\ &> m. \end{aligned}$$

If $|I| = k-2$, then

$$\begin{aligned} |U| &= |X| - (k-2) - 1 \\ &= |X| - k + 1 \\ &\geq 3 \\ &> m. \end{aligned}$$

Finally, if $|I| \leq k-3$, then

$$\begin{aligned} |U| &= |X| - |I| - 1 \\ &\geq (k+2) - |I| - 1 \\ &= m + 2 \\ &> m. \end{aligned}$$

■

Thus we can find elements $u_1^*, \ldots, u_m^*$ of U, and we assume that $u_1^* < \cdots < u_m^*$ in L_U. We now proceed to define $\mathcal{A}$.

First let $S \in \mathcal{S}$ be of the form $S = \{u_1^*, \ldots, u_m^*\} \cup \{x\} \cup \{v_1, \ldots, v_{k-m-1}\}$, where $v_1, \ldots, v_{k-m-1} \in I$ and $v_1 < \cdots < v_{k-m-1}$ in L_I. (A simple calculation shows that $|I| \geq k - m - 1$.) We define the arc of $\mathcal{A}$ corresponding to S to be $A = (u_1^*, \ldots, u_m^*, x, v_1, \ldots, v_{m-k-1})$.

Now let $S \in \mathcal{S}$ be of the form $S = \{u_1, \ldots, u_m\} \cup \{x\} \cup \{v_1, \ldots, v_{k-m-1}\}$, where $v_1, \ldots, v_{m-k-1}$ are as in the preceding paragraph, and $u_1, \ldots, u_m \in U$, $\{u_1, \ldots, u_m\} \neq \{u_1^*, \ldots, u_m^*\}$ and $u_1 < \cdots < u_m$ in L_U. Such a set $\{u_1, \ldots, u_m\}$ exists since $|U| > m$. We define the arc of $\mathcal{A}$ corresponding to S to be $A = (u_1, \ldots, u_m, v_1, \ldots, v_{m-k-1}, x)$.

Finally let $S \in \mathcal{S}$ be such that $|S \cap U| \neq m$. If $S \cap U \neq \emptyset$, let $S \cap U = \{u_1, \ldots, u_i\}$; and if $S \cap I \neq \emptyset$, let $S \cap I = \{v_1, \ldots, v_j\}$. If $S \cap U \neq \emptyset$ and $S \cap I \neq \emptyset$, then we define the arc of $\mathcal{A}$ corresponding to S by $A = (u_1, \ldots, u_i, v_1, \ldots, v_j, x)$. If $S \cap U = \emptyset$ but $S \cap I \neq \emptyset$ then we define the arc of $\mathcal{A}$ corresponding to S by $A = (v_1, \ldots, v_j, x)$. Finally if $S \cap U \neq \emptyset$ but $S \cap I = \emptyset$, then we define A by $A = (u_1, \ldots, u_i, x)$.

This defines the set $\mathcal{A}$ of arcs of T. Once again we have defined T by

$$\begin{aligned} V(T) &= X, \\ \text{and } A(T) &= A(T') \cup \mathcal{A}. \end{aligned}$$

We now show that T represents $\mathcal{P}$.

First let $u \in U$. Then $u < x$ in P. It is clear from the definition of $\mathcal{A}$ that u always precedes x in T (note that every arc of T containing both u and x belongs to $\mathcal{A}$).

Now let $v \in I$. Then there are $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cap U = \{u_1^*, \ldots, u_m^*\}$ and $v \in S_1$, and $|S_2 \cap U| = m$, $S_2 \cap U \neq \{u_1^*, \ldots, u_m^*\}$ and $v \in S_2$. It is clear from the definitions of the elements of $\mathcal{A}$ that x precedes v in the arc corresponding to S_1, and v precedes x in the arc corresponding to S_2. It follows that for any $y \in X \setminus \{x\}$, $y < x$ in $\mathcal{P}$ if and only if y always precedes x in T.

Let $w_1, w_2 \in X \setminus \{x\}$, where $w_1 < w_2$ in $\mathcal{P}$. Then $w_1 \leq w_2$ in $\mathcal{P}'$ and so w_1 always precedes w_2 in T'. First suppose $w_1, w_2 \in U$ or $w_1, w_2 \in I$. Then w_1 precedes w_2 in every arc of $\mathcal{A}$ containing both w_1 and w_2, because w_1 precedes w_2 in L_U or L_I, whichever is appropriate. On the other hand, if $w_1 \in U$ and $w_2 \in I$ then it is clear from the definition of $\mathcal{A}$ that w_1 precedes w_2 in every arc of $\mathcal{A}$ containing both w_1 and w_2. Finally recall that by the transitivity of the relation P, we cannot have $w_1 \in I$ and $w_2 \in U$.

Now let $w_1, w_2 \in X \setminus \{x\}$, where $w_1 \| w_2$ in P. Then $w_1 \| w_2$ in $\mathcal{P}'$, so that there are arcs $A_1, A_2 \in A(T')$ such that w_1 precedes w_2 in A_1 and w_2 precedes w_1 in A_2. But $A(T') \subseteq A(T)$ so that $A_1, A_2 \in A(T)$.

It follows that T represents $\mathcal{P}$. This completes the proof of the lemma. ■

Theorem 2.3.2 now follows immediately from Lemmas 2.3.11 and 2.3.12. For convenience, we restate Theorem 2.3.2 below.

Theorem 2.3.2 *Let $\mathcal{P} = (X, \leq)$ be a finite poset, and k an integer satisfying $3 \leq k \leq |X| - 1$. Then there is a k-tournament T with $V(T) = X$ such that T represents $\mathcal{P}$.*

Chapter 3

The automorphism group of a k-tournament

The aim of this chapter is to determine those groups which admit a representation as the automorphism group of a k-tournament; we consider various restrictions on the representation required. To begin, we determine those finite, abstract groups G for which there exists a k-tournament whose automorphism group is isomorphic to G. This characterisation extends the well-known result of Moon ([16]), which states that there is a tournament whose automorphism group is isomorphic to a given finite, abstract group G if and only if G has odd order. We then consider the problem of finding the 'smallest' k-tournament whose automorphism group is isomorphic to G, where we measure how 'small' a k-tournament is by the number of orbits of its automorphism group acting on its vertex set. With this definition of size, our goal is to determine which groups admit a regular representation as the automorphism group of a k-tournament. For tournaments it was shown in [1] that every group of odd order, other than $Z_3 \times Z_3$, admits a regular representation as the automorphism group of a tournament. The construction given relies on the Feit-Thompson theorem, which states that every finite group of odd order is solvable. We cannot hope to use a similar construction for k-tournaments, however, because the groups which can be represented as the automorphism group of a k-tournament are not all solvable for general $k \geq 3$. Nevertheless we are able to show that if a group G admits a representation as the automorphism group of a k-tournament and has order larger than k, then there is a k-tournament whose automorphism group is semiregular, has two vertex orbits, and is isomorphic to G. This extends a result of L. Babai and W. Imrich [1]. Finally, in 3.3.2 we show that if either G is cyclic, or $k \geq 4$ and G has a minimal generating set with at least k elements, then G admits a regular representation as the automorphism group of a k-tournament.

3.1 Preliminaries

We give here some basic lemmas which will be used throughout the chapter. We refer the reader to Section 1.1 for the relevant definitions and notation.

We first state without proof a lemma of L. A. Nowitz and M. E. Watkins ([17]) which we will use repeatedly.

Lemma 3.1.1 *Let G be a group, and A a group of permutations of G which contains G_L (that is, $G_L \leq A \leq S_G$). Let H be a generating set for G. If $\alpha \in A_e$ implies that α fixes each element of H, then $A = G_L$.*

The following lemmas will also be useful in several constructions. Lemma 3.1.2 shows that, given a semiregular permutation group acting on a finite set U and an integer $k \geq 3$ which is relatively prime to the order of this permutation group, we can construct a semiregular permutation group acting on the set of k-subsets of a set V, where V consists of some number m of disjoint copies of the set U. We will use this lemma in constructing k-tournaments whose vertex sets consist of some number m of copies of a given finite group G, and whose automorphism groups have m vertex orbits.

Lemma 3.1.2 *Let A be a group of permutations of a finite set U, $|U| > 1$, let m be a positive integer, and let $k \geq 3$ be an integer such that $\gcd(|A|, k) = 1$. Let $V = \cup_{i=1}^{m} U^{(i)}$, where $U^{(i)} = \{u^{(i)} : u \in U\}$. Let $\overline{A}$ be the permutation group acting on V defined by $\overline{A} = \{\overline{\alpha} : \alpha \in A\}$ where $\overline{\alpha}$ is the permutation $\overline{\alpha} : u^{(i)} \mapsto (\alpha(u))^{(i)}$. If A acts semiregularly on U then $\overline{A}$ acts semiregularly on $\binom{V}{k}$.*

Proof. We must show that $\overline{\alpha}(S) = S$ implies $\alpha = \iota$, where ι is the identity element of A, for any k-subset S of V.

To this end, suppose that $\overline{\alpha}(S) = S$. Then for $i \in \{1, \ldots, m\}$, $\overline{\alpha}(S \cap U^{(i)}) = S \cap U^{(i)}$. Now $\overline{\alpha}$ acts on $U^{(i)}$ exactly as α acts on U; and α acts semiregularly on U. Thus the restriction of $\overline{\alpha}$ to $U^{(i)}$, when written as a product of disjoint cycles, consists of cycles of some fixed length t, and t is the order of α in A. Since $\overline{\alpha}$ fixes $S \cap U^{(i)}$ (setwise), then $S \cap U^{(i)}$ must be the union of the elements of some number of these t-cycles. Therefore $|S \cap U^{(i)}|$ is a multiple of t. Thus $k = |S| = \sum_{i=1}^{m} |S \cap U^{(i)}|$ is a multiple of t. But $|A|$ is also a multiple of t; thus $t = 1$, and so $\alpha = \iota$ as required. ■

Letting $m = 1$ in Lemma 3.1.2 gives us the following corollary.

Corollary 3.1.3 *Let A be a group of permutations of a finite set U, where $|U| \geq 1$, and let $k \geq 3$ be an integer such that gcd$(|A|, k) = 1$. If A acts semiregularly on U, then A acts semiregularly on $\binom{U}{k}$.*

Lemma 3.1.4 below establishes a useful property of a minimal generating set H of a finite group G. It gives us some information about the distribution of the k-subsets of $H \cup \{e\}$ among the orbits of G_L acting on $\binom{G}{k}$, where $k \geq 3$ and is relatively prime to the order of G.

Lemma 3.1.4 *Let G be a finite group, and $k \geq 3$ an integer, such that* $\gcd(|G|, k) =$ *1. Let H be a minimal generating set for G, where the minimality is with respect to inclusion, and assume that $|H| \geq k$. Let H^+ denote the set $H \cup \{e\}$. Then no two k-subsets of H^+ belong to the same orbit of G_L acting on $\binom{G}{k}$.*

Proof. Let $S = \{c_1, \ldots, c_k\}$ and $T = \{d_1, \ldots, d_k\}$ be distinct k-subsets of H^+. We want to show that $S = gT$, $g \in G$, implies $g = e$. We divide the proof into three cases.

Case 1: $e \notin S \cup T$. Let $S = \{c_1, \ldots, c_k\}$ and $T = \{d_1, \ldots, d_k\}$, and assume $S = gT$, where $g \in G$. Without loss of generality we can assume that $gc_i = d_i$ for $i = 1, \ldots, k$. Thus $g = d_1 c_1^{-1} = \cdots = d_k c_k^{-1}$.

First notice that if $c_i = d_i$ for some i then $g = e$, as desired. Thus we assume that $c_i \neq d_i$, $1 \leq i \leq k$.

Now let $i \neq j$. Since $d_i c_i^{-1} = d_j c_j^{-1}$, then $d_i = d_j c_j^{-1} c_i$. By the minimality of H, $d_i \in \{d_j, c_j, c_i\}$. But $d_i \neq d_j$, and we assumed $d_i \neq c_i$; it then follows that $d_i = c_j$.

Since $k \geq 3$, we now have that both $d_1 = c_2$ and $d_1 = c_3$. Since $c_2 \neq c_3$, this is the desired contradiction, and completes the proof for Case 1.

Case 2: $e \in S \cap T$. Let $S = \{e, c_2, \ldots, c_k\}$ and $T = \{e, d_2, \ldots, d_k\}$. Then $S = gT$ implies $S = \{e, c_2, \ldots, c_k\} = \{g, gd_2, \ldots, gd_k\}$. Assume $g \neq e$. Since $e \in gT \setminus \{g\}$, then $e = gd_i$ for some i, and so $g = d_i^{-1}$. We may assume without loss of generality that $g = d_2^{-1}$. Now $d_2^{-1} \neq e$ and $d_2^{-1} \in S$, so again without loss of generality we may assume that $d_2^{-1} = c_2$. By the minimality of H, $c_2 = d_2$, and so $g = d_2^{-1} = d_2 = c_2$.

Similarly, $gd_3 = d_2^{-1} d_3 \in S \setminus \{e, c_2\}$, and so without loss of generality $d_2^{-1} d_3 = c_3$. Equivalently, $d_2 d_3 = c_3$. By the minimality of H, $c_3 \in \{d_2, d_3\} = \{c_2, d_3\}$. Since $c_2 \neq c_3$, $c_3 = d_3$. But $c_3 = d_2 d_3$; consequently $d_2 = e$, a contradiction. This completes the proof for Case 2.

Case 3: $e \in S \setminus T$. In this case, let $S = \{e, c_2, \ldots, c_k\}$ and $T = \{d_1, d_2, \ldots, d_k\}$. Let $S = gT$. Then $\{e, c_2, \ldots, c_k\} = \{gd_1, gd_2, \ldots, gd_k\}$. In this case $g \neq e$ since $e \notin T$. Since $e \in gT$, without loss of generality $e = gd_1$, and so $g = d_1^{-1}$.

Now $c_2 \in \{gd_2, \ldots, gd_k\} = \{d_1^{-1} d_2, \ldots, d_1^{-1} d_k\}$, so we may assume that $c_2 = d_1^{-1} d_2$. By the minimality of H, $c_2 \in \{d_1, d_2\}$. If $c_2 = d_2$, then $d_1 = e$ and so $g = e$, a contradiction. If $c_2 = d_1$, then $c_2^2 = d_2$, which contradicts the minimality unless $c_2 = d_2$; but we have shown that this also leads to a contradiction. This completes the proof in Case 3 and so too the proof of the lemma. ■

3.2 A characterisation of those groups admitting a representation as the automorphism group of a k-tournament, $k \geq 3$

We show in this section that for a finite group G and an integer $k \geq 3$, there is a k-tournament whose automorphism group is isomorphic to G if and only if $|G|$ and k are relatively prime. We observe that this is a generalisation of the corresponding result for tournaments ([16]) which says that there is a tournament whose automorphism group is isomorphic to the finite group G if and only if $|G|$ is odd, or equivalently, is relatively prime to 2.

We state the main theorem here; Lemma 3.2.2 and Lemma 3.2.3 comprise its proof.

Theorem 3.2.1 *Let G be a finite group, and k an integer such that $k \geq 3$. There is a k-tournament whose automorphism group is isomorphic to G if and only if the order of G is relatively prime to k.*

We begin by showing the necessity of the condition.

Lemma 3.2.2 *If T is a k-tournament, then $|Aut(T)|$ and k are relatively prime.*

Proof. Let T be a k-tournament (recall that this implies $k \geq 3$) and suppose, towards a contradiction, that $|Aut(T)|$ and k share a common factor. Then $|Aut(T)|$ and k share a common prime factor, say p, and it follows that $Aut(T)$ contains an element α of order p. Consider the permutation of $V(T)$ induced by α, written in disjoint cycle notation. This permutation must consist of p-cycles, and possibly some fixed points. Construct a k-subset S of $V(T)$ as follows. If α contains at least k/p p-cycles, we let S be the union of the elements of exactly k/p p-cycles of α. If α contains $n < k/p$ p-cycles, we let S consist of the elements of these n p-cycles together with any $k - np$ fixed points of α.

Now the set S is fixed by α, but it is not fixed pointwise, since any vertex belonging to a p-cycle is not fixed. If we now let A be the unique arc of T corresponding to S, we see that $\alpha(A)$ is not an arc of T, because its elements are also the elements of S, but now in a different order. This contradicts $\alpha \in Aut(T)$. It follows that $|Aut(T)|$ and k must be relatively prime. ■

To show the sufficiency of the condition, we construct, given an integer $k \geq 3$ and a finite group G satisfying $\gcd(|G|, k) = 1$, a k-tournament whose automorphism group is isomorphic to G. In this section we are not concerned with the order of the k-tournament we construct, but rather with providing a general construction which is valid for all $k \geq 3$ and all finite groups satisfying $\gcd(|G|, k) = 1$. It will be shown in later sections that much smaller k-tournaments can be constructed in many cases. The construction below produces a k-tournament whose automorphism group has k orbits of vertices.

Lemma 3.2.3 *Let G be a finite group, and $k \geq 3$ an integer, such that* $\gcd(|G|, k) = 1$*. Then there is a k-tournament whose automorphism group is isomorphic to G.*

Proof. Let G and k be given, as in the statement of the lemma. Let H be a minimal generating set for G. We construct a k-tournament T with the required property.

We first define $V(T)$. Following Lemma 3.1.2, we let $G^{(i)} = \{g^{(i)} : g \in G\}$ for $i \in \{1, \ldots, k\}$, and we let $V(T) = \cup_{i=1}^{k} G^{(i)}$. We also let $\overline{G}_L = \{\overline{g} : g \in G\}$, where $\overline{g}$ is the permutation of $V(T)$ given by $\overline{g} : x^{(i)} \mapsto (gx)^{(i)}$. It follows from Lemma 3.1.2 that $\overline{G}_L$ acts semiregularly on $\binom{V(T)}{k}$, and consequently that if $S \in \binom{V(T)}{k}$ and $\overline{g}(S) = S$, then $g = e$.

In defining $A(T)$, we will find it useful to classify the k-subsets of $V(T)$ in the following way. What we are interested in is the manner in which a k-subset is distributed among the sets $G^{(1)}, \ldots, G^{(k)}$. Given a k-subset S of $V(T)$, we define the *partition* of S to be the multiset $\{|S \cap G^{(i)}| : 1 \leq i \leq m \text{ and } |S \cap G^{(i)}| > 0\}$. Notice that if S_1 and S_2 belong to the same orbit of $\overline{G}$ acting on $\binom{V(T)}{k}$, then S_1 and S_2 have the same partition (but the converse need not hold).

We partition $\binom{V(T)}{k}$ into several classes, each of which is fixed by $\overline{G}_L$; the order assigned to a k-subset of $V(T)$ will depend on the class to which it belongs.

Let $\mathcal{K}$ denote the family of k-subsets of $V(T)$ with partition $\{k\}$, if any such sets exist. These are exactly the k-sets all of whose vertices belong to the same set $G^{(i)}$. The class $\mathcal{K}$ will be empty if and only if $|G| < k$.

Let $\mathcal{L}$ denote the family of k-subsets of $V(T)$ with partition $\{\lambda_1, \ldots, \lambda_l\}$, where $1 < l < k$. Note that $l < k$ implies that $\lambda_j > 1$ for some j, so these are the k-subsets containing at least two vertices from some set $G^{(i)}$, but which are not contained within any $G^{(i)}$.

The k-subsets of $V(T)$ which do not belong to either $\mathcal{K}$ or $\mathcal{L}$ are those with partition $\{1, 1, \ldots, 1\}$, i.e., those whose elements all belong to distinct sets $G^{(i)}$. These k-subsets we will further classify as follows.

Let $\mathcal{S}$ be the family of k-subsets of $V(T)$ of the form $\{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}$. These are the k-subsets which consist of k copies of the same vertex v, one from each set $G^{(i)}$.

Let $\mathcal{D}$ be the family of k-subsets of $V(T)$ of the form $\{v_1^{(1)}, v_2^{(2)}, \ldots, v_k^{(k)}\}$, where $v_1, \ldots, v_k$ are all distinct. Thus $\mathcal{D}$ consists of those k-subsets whose vertices all come from different sets $G^{(i)}$ and all correspond to different elements of G.

Let $\mathcal{N}$ denote the family of k-subsets of $V(T)$ which have partition $\{1, \ldots, 1\}$ and which belong to neither $\mathcal{S}$ nor $\mathcal{D}$. These k-subsets have the form $\{v_1^{(1)}, v_2^{(2)}, \ldots, v_k^{(k)}\}$, where $1 < |\{v_1, \ldots, v_k\}| < k$.

Finally, we distinguish a subclass of $\mathcal{D} \cup \mathcal{N}$. Let $\mathcal{B}$ denote the family of k-subsets of $V(T)$ of the form $\{x^{(1)}, xh^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\}$, where $h \in H$ and $v_3, \ldots, v_k$ are all different.

We have now partitioned $\binom{V(T)}{k}$ into the classes $\mathcal{K}$, $\mathcal{L}$, $\mathcal{S}$, $\mathcal{D}$, and $\mathcal{N}$, with a distinguished subclass $\mathcal{B}$. Notice that each of these classes (including $\mathcal{B}$) is fixed (setwise) by $\overline{G}_L$.

We first define the arcs corresponding to elements of $\mathcal{L} \cup \mathcal{S}$. The ordering of the sets belonging to $\mathcal{L}$ is intended to force all automorphisms of T to map each set $G^{(i)}$ to itself. That the ordering we give here really has the desired effect will be shown below. The idea is to order each subset belonging to $\mathcal{L}$ so that any elements of $G^{(i)}$ precede any elements of $G^{(j)}$ whenever $i < j$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_t$ be the orbits of $\overline{G}_L$ acting on the elements of $\mathcal{L}$. For each orbit $\mathcal{O}_j$, choose a representative $S_j \in \mathcal{O}_j$, and let $G^{(i_1)}, \ldots, G^{(i_l)}$ be the sets $G^{(i)}$ for which $S_j \cap G^{(i)} \neq \emptyset$, where $i_1 < \cdots < i_l$. First, arbitrarily order S_j so that all elements of $G^{(i_1)}$ precede all elements of $G^{(i_2)}$, all elements of $G^{(i_2)}$ precede all elements of $G^{(i_3)}$, and so on, and call the resulting arc A_j. Now assign the order $\overline{g}(A_j)$ to each other element $\overline{g}(S_j)$ of $\mathcal{O}_j$ ($\overline{g} \in \overline{G}_L$). Thus $\mathcal{L}$ contributes to $A(T)$ the arcs $\{\overline{g}(A_j) : \overline{g} \in \overline{G}_L \text{ and } 1 \leq j \leq t\}$. Since $\overline{G}_L$ acts semiregularly on $\binom{V(T)}{k}$, this procedure assigns a unique order to each element of $\mathcal{L}$.

We now consider the elements of $\mathcal{S}$. The ordering of these k-subsets of $V(T)$ will force all automorphisms of T to be of the form $\overline{\pi}$ for some permutation π of G. If $S = \{v^{(1)}, \ldots, v^{(k)}\} \in \mathcal{S}$, then we let the corresponding arc of T be $(v^{(1)}, \ldots, v^{(k)})$.

The salient point is that the order assigned to S is given by the order of the sets $G^{(1)}, \ldots, G^{(k)}$.

The order assigned to the remaining elements of $\binom{V(T)}{k}$ will depend on the cardinality of H with respect to k. We distinguish two cases.

Case 1: $|H| > k$. Let $H^+ = \{h_0, h_1, \ldots, h_{|H|}\}$, where $h_0 = e$.

In this case, the class $\mathcal{K}$ is not empty. We want to define a k-tournament T^* with vertex set G, and put a copy of this k-tournament on each set $G^{(i)}$ of vertices. The only thing we require of T^* is that the subtournament of T^* induced by the set $H^+ = H \cup \{e\}$ be transitive, with underlying linear order $e = h_0 > h_1 > \cdots > h_{|H|}$.

Now we know by Lemma 3.1.4 that the k-subsets of H^+ all belong to different orbits of G_L acting on $\binom{G}{k}$. We therefore define $A(T^*)$ as follows. Let $\mathcal{O}$ be an orbit of G_L acting on $\binom{G}{k}$. If $\mathcal{O}$ contains a k-subset of H^+, then let S denote this k-subset; if $\mathcal{O}$ contains no such element, then arbitrarily select a k-set $S \in \mathcal{O}$. If $S \subseteq H^+$, say $S = \{h_{i_1}, \ldots, h_{i_k}\}$, where $i_1 < \cdots < i_k$, we let the corresponding arc of T^* be $A = (h_{i_1}, \ldots, h_{i_k})$. Otherwise $S = \{g_1, \ldots g_k\} \not\subseteq H^+$ and we order S arbitrarily to produce an arc A. Now let $\overline{g}(A)$ be the arc corresponding to $\overline{g}(S)$ for each element $\overline{g}(S)$ of $\mathcal{O}$.

This procedure will define an arc for each k-subset of G; we have thus defined the k-tournament T^*. Now let $S \in \mathcal{K}$, and write $S = \{v_1^{(i)}, v_2^{(i)}, \ldots, v_k^{(i)}\}$ where $(v_1, v_2, \ldots, v_k) \in A(T^*)$. We define the arc of T corresponding to S to be $(v_1^{(i)}, v_2^{(i)}, \ldots, v_k^{(i)})$.

It remains to define arcs corresponding to the k-sets belonging to $\mathcal{D} \cup \mathcal{N}$. In order to distinguish these sets from the elements of $\mathcal{S}$, the order assigned to them will be chosen expressly to conflict with the order of the sets $G^{(1)}, \ldots, G^{(k)}$. In addition we will use the arcs corresponding to the elements of $\mathcal{B}$ to distinguish the elements of H from the remaining elements of G.

We first order the sets belonging to the distinguished subclass $\mathcal{B}$. If $S \in \mathcal{B}$, then S has the form $S = \{x^{(1)}, xh^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\}$, where $h \in H$. We assign to S the arc $A = (v_k^{(k)}, \ldots, v_3^{(3)}, xh^{(2)}, x^{(1)})$.

Finally for any set $S \in (\mathcal{D} \cup \mathcal{N}) \setminus \mathcal{B}$ we order S so that so that the (unique) element of $S \cap G^{(2)}$ is in the first co-ordinate, the element of $S \cap G^{(1)}$ is in the second co-ordinate, and for each $i = 3, \ldots, k$ the element of $S \cap G^{(i)}$ is in the i^{th} co-ordinate. Thus a set $\{v_1^{(1)}, v_2^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\}$ would be assigned the order $(v_2^{(2)}, v_1^{(1)}, v_3^{(3)}, \ldots, v_k^{(k)})$.

This completes the definition of $A(T)$ in Case 1.

Case 2: $|H| \leq k-1$. Again we let $H = \{h_1, \ldots, h_{|H|}\}$ (notice that in this case we consider only H and not H^+).

In this case, the class $\mathcal{K}$ might or might not be empty, depending on whether $|G| \geq k$. If $|G| \geq k$, we will again define a k-tournament T^* with vertex set G, and put a copy of T^* on each of the sets $G^{(i)}$. However in this case we require of T^* only that $G_L \leq Aut(T^*)$. Thus we partition $\binom{G}{k}$ into orbits under the action of G_L, select a representative element S of each orbit, order S arbitrarily to produce an arc A, and assign the order $\overline{g}(A)$ to each remaining element $\overline{g}(S)$ of the orbit in question. This defines $A(T^*)$. Now given an element $S \in \mathcal{K}$, we write $S = \{v_1^{(i)}, \ldots, v_k^{(i)}\}$, where $(v_1, \ldots, v_k) \in A(T^*)$, and assign the order $(v_1^{(i)}, \ldots, v_k^{(i)})$ to S.

As in Case 1, the ordering assigned to the elements of $\mathcal{D} \cup \mathcal{N}$ is designed both to distinguish these k-sets from those belonging to $\mathcal{S}$, and to distinguish the elements of H from one another.

We again begin with the subclass $\mathcal{B}$. Let $S = \{x^{(1)}, xh^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\} \in \mathcal{B}$. The order assigned to S will depend on the element h of H. If $h = h_1$, the arc corresponding to S will be $(v_k^{(k)}, \ldots, v_3^{(3)}, xh^{(2)}, x^{(1)})$. If $h = h_i$, where $2 \leq i \leq k-2$, the arc corresponding to S will be $A = (v_k^{(k)}, \ldots, v_{i+1}^{(i+1)}, xh^{(2)}, v_i^{(i)}, \ldots, v_3^{(3)}, x^{(1)})$. If $h = h_{k-1}$, the arc corresponding to S will be $(xh^{(2)}, v_k^{(k)}, \ldots, v_3^{(4)}, x^{(1)})$. Thus if $h = h_i$ then the element $xh^{(2)}$ of S is in co-ordinate $k-i$ of A, $x^{(1)}$ is in co-ordinate k, and the elements $v_3^{(3)}, \ldots, v_k^{(k)}$ are ordered so that $v_i^{(i)}$ precedes $v_j^{(j)}$ if and only if $i > j$.

Finally, the elements of $(\mathcal{D} \cup \mathcal{N}) \setminus \mathcal{B}$ will be ordered in the same way as in Case 1.

This completes the definition of $A(T)$ in Case 2.

It is clear from the definitions that T is indeed a k-tournament, and also that $\overline{G}_L \leq Aut(T)$. It thus remains to show that $Aut(T) \leq \overline{G}_L$. This is done through a sequence of claims. The first two show that the ordering of the elements of $\mathcal{L}$ does indeed force all automorphisms of T to map each set $G^{(i)}$ to itself.

Claim 3.2.4 *Let $\alpha \in Aut(T)$. Suppose that for some $i \in \{1, \ldots, k\}$, $\alpha(G^{(i)}) \neq G^{(r)}$ for any r. Then there is $m < i$ such that $\alpha(G^{(m)}) \neq G^{(r)}$ for any r.*

Proof. Suppose $\alpha(G^{(i)}) \neq G^{(r)}$ for any r. Then there are $u^{(i)}, v^{(i)} \in G^{(i)}$ and $j < l$ such that $\alpha(u^{(i)}) \in G^{(j)}$ and $\alpha(v^{(i)}) \in G^{(l)}$. Since $|G^{(i)}| = |G^{(j)}|$, and $\alpha(G^{(i)}) \not\subseteq G^{(j)}$, there is $m \neq i$ and $w^{(m)} \in G^{(m)}$ such that $\alpha(w^{(m)}) \in G^{(j)}$. Now $\alpha(G^{(m)}) \neq G^{(j)}$ because $\alpha(u^{(i)}) \in G^{(j)}$; however, $\alpha(G^{(m)})$ does intersect $G^{(j)}$. Therefore, $\alpha(G^{(m)}) \neq G^{(r)}$ for any r, and it is sufficient to show that $m < i$.

Let S be any k-subset of $V(T)$ containing all of $u^{(i)}$, $v^{(i)}$ and $w^{(m)}$. Then $S \in \mathcal{L}$. Now $\alpha(S)$ contains all of $\alpha(u^{(i)})$, $\alpha(v^{(i)})$ and $\alpha(w^{(m)})$, which belong to $G^{(j)}$, $G^{(l)}$ and $G^{(j)}$, respectively. Therefore $\alpha(S) \in \mathcal{L}$ also. Since $j < l$, both $\alpha(u^{(i)})$ and $\alpha(w^{(m)})$ precede $\alpha(v^{(i)})$ in the arc of T corresponding to $\alpha(S)$. But since $\alpha \in Aut(T)$, this implies that both $u^{(i)}$ and $w^{(m)}$ precede $v^{(i)}$ in the arc of T corresponding to S. Since $S \in \mathcal{L}$, this can be the case only if $m < i$. ∎

Claim 3.2.5 *If $\alpha \in Aut(T)$, then $\alpha(G^{(i)}) = G^{(i)}$ for each $i \in \{1, \ldots, k\}$.*

Proof. We show first that $\alpha(G^{(1)}) = G^{(1)}$. By Claim 3.2.4, we know that $\alpha(G^{(1)}) = G^{(r)}$ for some r, $1 \leq r \leq k$. Consider the family of k-subsets of $V(T)$ which contain exactly two elements of $G^{(1)}$. There are $\binom{|G|}{2}\binom{(k-1)|G|}{k-2}$ such k-sets, and they all belong to $\mathcal{L}$; so in the corresponding arcs of T, the two elements of $G^{(1)}$ occupy the first two positions. Therefore for any such set S, the first two positions of $\alpha(S)$ are occupied by elements of $G^{(r)}$, and $\alpha(S)$ contains no other elements of $G^{(r)}$. So $\alpha(S)$ also belongs to $\mathcal{L}$. Since the first two positions of $\alpha(S)$ contain elements of $G^{(r)}$, $\alpha(S)$ contains no elements of $\bigcup_{i=1}^{r-1} G^{(i)}$. Now there are $\binom{|G|}{2}\binom{(k-r)|G|}{k-2}$ k-subsets of $V(T)$ which contain exactly two elements of $G^{(r)}$ and no elements of $\bigcup_{i=1}^{r-1} G^{(i)}$. But α induces a bijection on $\binom{V(T)}{k}$, and we have shown that each of the $\binom{|G|}{2}\binom{(k-1)|G|}{k-2}$ sets of the first type is mapped by α onto one of the $\binom{|G|}{2}\binom{(k-r)|G|}{k-2}$ sets of the second type. This is clearly impossible unless $r = 1$. Thus $\alpha(G^{(1)}) = G^{(1)}$ as desired.

We now proceed by induction on i to show that $\alpha(G^{(i)}) = G^{(i)}$ for each $i \in \{2, \ldots, k\}$. Suppose $\alpha(G^{(j)}) = G^{(j)}$ for each j with $1 \leq j \leq i - 1$. Then α maps the set $\bigcup_{j=i}^{k} G^{(j)}$ onto itself. Using this and Claim 3.2.4, we see that $\alpha(G^{(i)}) = (G^{(j)})$ for some $j \geq i$. But it now follows by the same counting argument as in the preceding paragraph that $\alpha(G^{(i)}) = G^{(i)}$. This proves the claim. ■

We now want to show that $Aut(T)$ is of the form $Aut(T) = \overline{A}$ for some group A of permutations of G. To do this we need to show that for any $x, y \in G$, and any $\alpha \in Aut(T)$, $\alpha(x^{(i)}) = y^{(i)}$ if and only if $\alpha(x^{(j)}) = y^{(j)}$ for any $i, j \in \{1, \ldots, k\}$. We show in Claim 3.2.6 below that this follows from the definitions of the arcs corresponding to the elements of $\mathcal{S}$ and to those of $\mathcal{D} \cup \mathcal{N}$.

Claim 3.2.6 *Let $\alpha \in Aut(T)$, $x, y \in G$ and $i, j \in \{1, \ldots, k\}$. Then $\alpha(x^{(i)}) = y^{(i)}$ if and only if $\alpha(x^{(j)}) = y^{(j)}$.*

Proof. Let α, x, y, i, and j be as in the statement of the claim. Since the k-set $S = \{x^{(1)}, x^{(2)}, \ldots, x^{(k)}\}$ belongs to $\mathcal{S}$, the corresponding arc of T is $A = (x^{(1)}, x^{(2)}, \ldots, x^{(k)})$. Now $\alpha(x^{(i)}) = y^{(i)}$ if and only if $\alpha(A)$ is of the form $(v_1^{(1)}, \ldots, v_{i-1}^{(i-1)}, y^{(i)}, v_{i+1}^{(i+1)}, \ldots, v_k^{(k)})$. Since the elements of $\mathcal{S}$ are the only sets with partition $\{1, 1, \ldots, 1\}$ which are ordered so that co-ordinate l contains an element of $G^{(l)}$, $l = 1, \ldots, k$, $\alpha(S)$ belongs to $\mathcal{S}$. Therefore $v_j = y$ for each $j = 1, \ldots, i-1, i+1, \ldots, k$, and $\alpha(x^{(j)}) = y^{(j)}$ as required. ■

Thus $Aut(T)$ is indeed of the form $Aut(T) = \overline{A}$ for some group A of permutations of G. Since $\overline{G}_L \leq Aut(T)$, then $G_L \leq A$. So we have $G_L \leq A \leq S_G$. Therefore if we can show that any automorphism of T which fixes $e^{(1)}$ (and hence $e^{(i)}$ for each $i = 2, \ldots, k$) also fixes each element of $H^{(1)}$, it will follow from Lemma 3.1.1 that $A = G_L$ and consequently that $Aut(T) = \overline{G}_L \cong G$. This constitutes the remainder of the proof of Lemma 3.2.3. Since by Claim 3.2.6 the group A is a faithful representation of $Aut(T)$, for the remainder of the proof we work with either A or $Aut(T)$, depending on which is more convenient. We again distinguish two cases, depending on the cardinality of H.

Case 1: $|H| \geq k$. Recall that in this case the elements of $H^{(i)}$ induce a transitive subtournament of T with underlying linear order $e^{(i)} = h_0^{(i)} > h_1^{(i)} > \cdots > h_{|H|}^{(i)}$, and that each element $\{x^{(1)}, xh^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\}$ of $\mathcal{B}$ corresponds to the arc $(v_k^{(k)}, \ldots, v_3^{(3)}, xh^{(2)}, x^{(1)})$ of T.

Claim 3.2.7 *If $\alpha \in Aut(T)$ and $\alpha(e) = e$, then $\alpha(H) = H$.*

Proof. The proof uses the definition of the arcs corresponding to the elements of $\mathcal{B}$.

Let $\alpha \in Aut(T)$ and $\alpha(e) = e$. Notice that the arcs corresponding to the elements of $\mathcal{B}$ are the only arcs of T containing vertices from both $G^{(1)}$ and $G^{(2)}$ in which an element of $G^{(1)}$ is in the k^{th} co-ordinate. Therefore these arcs are fixed, setwise, by α. Since $\alpha(e) = e$, the arcs corresponding to elements of $\mathcal{B}$ and which contain $e^{(1)}$ are also fixed setwise by α. Each of these arcs is of the form $(v_k^{(k)}, \ldots, v_3^{(3)}, h^{(2)}, e^{(1)})$, for some $h \in H$, and certainly each element of $H^{(2)}$ appears in co-ordinate $k-1$ in at least one of these arcs. This is enough to show that $\alpha(h^{(2)}) \in H^{(2)}$, and so that $\alpha(h) \in H$, for each $h \in H$. ■

To complete the proof in Case 1, we show that each automorphism of T which fixes e also fixes each element of H.

Claim 3.2.8 *Let $\alpha \in Aut(T)$ and let $\alpha(e) = e$. Then $\alpha(h) = h$ for each $h \in H$.*

Proof. By Claim 3.2.7, $\alpha(H) = H$. Therefore the restriction of α to $H^{(1)}$ is an automorphism of the subtournament of T induced by the elements of $H^{(1)}$. Since this subtournament is transitive, it has no non-trivial automorphisms, and so the restriction of α to $H^{(1)}$ is the identity. Thus $\alpha(h) = h$ for each $h \in H$. ■

An application of Lemma 3.1.1 completes the proof of Lemma 3.2.3 in Case 1.

Case 2: $|H| < k$. Recall that in this case the arc corresponding to an element $\{x^{(1)}, xh_i^{(2)}, v_3^{(3)}, \ldots, v_k^{(k)}\}$ of $\mathcal{B}$ has $x^{(1)}$ in co-ordinate k, and $xh_i^{(2)}$ in co-ordinate $k-i$, where $H = \{h_1, \ldots, h_{|H|}\}$. We show in Claim 3.2.9 below that this is enough to ensure that if $\alpha \in Aut(T)$ fixes e, then α fixes each element of H.

Claim 3.2.9 *Let $\alpha \in Aut(T)$. If $\alpha(e) = e$, then $\alpha(h) = h$ for each $h \in H$.*

Proof. As in Case 1, the arcs of T corresponding to the elements of $\mathcal{B}$ which contain $e^{(1)}$ are fixed, setwise, by α. However in this case, for any fixed $i \in \{1, \ldots, |H|\}$, the arcs corresponding to elements of $\mathcal{B}$ containing both $e^{(1)}$ and $h_i^{(2)}$ are the only arcs of T in which $e^{(1)}$ appears in co-ordinate k and an element of $G^{(2)}$ appears in co-ordinate

$(k-i)$. These arcs are, therefore, also fixed setwise by α. But since each of these arcs has h_i in co-ordinate $k-i$, it follows that $\alpha(h_i) = h_i$. Since the choice of i was arbitrary, $\alpha(h) = h$ for each $h \in H$. ∎

Again, an application of Lemma 3.1.1 completes the proof of Lemma 3.2.3 in Case 2. ∎

3.3 Minimizing the number of vertex orbits in a representation of a finite group G as the automorphism of a k-tournament

In this section we address the question of finding a 'small' k-tournament with automorphism group isomorphic to G, for a given finite group G satisfying $\gcd(|G|, k) = 1$. We begin by showing that the construction in Lemma 3.2.3 can be improved in this direction if the group G satisfies $|G| > k$. In Section 3.3.1 we construct a k-tournament with automorphism group isomorphic to G and inducing two orbits of vertices, under the above conditions; and in Section 3.3.2 we investigate conditions under which there is a regular representation of the group G as the automorphism group of a k-tournament.

3.3.1 Two orbits of vertices

This section contains a construction of a k-tournament on $2|G|$ vertices whose automorphism group is semiregular and is isomorphic to a given finite group G satisfying $\gcd(|G|, k) = 1$ and $|G| > k$. The automorphism group of the k-tournament we construct therefore has two vertex orbits.

Theorem 3.3.1 *Let G be a finite group and $k \geq 3$ an integer such that $|G| > k$ and $\gcd(|G|, k) = 1$. Then there is a k-tournament T such that $Aut(T)$ acts semiregularly on $V(T)$, is isomorphic to G, and has two orbits of vertices.*

Proof. First, let H be a minimal generating set for G (where the minimality is with respect to inclusion), and let $H = \{h_1, \ldots, h_{|H|}\}$.

We let $V(T) = G^{(1)} \cup G^{(2)}$, where $G^{(i)} = \{g^{(i)} : g \in G\}$. We let G_L act on $V(T)$ as in Lemma 3.1.2, so that $\overline{G}_L = \{\overline{g} : g \in G\}$, where $\overline{g}$ is the permutation of $V(T)$ given by $\overline{g}(x^{(i)}) = (gx)^{(i)}$. We will also want to consider the induced action of $\overline{G}$ on $\binom{V(T)}{k}$. We point out that it follows from Lemma 3.1.2 that if $\mathcal{O}$ is an orbit of $\overline{G}$ acting on $\binom{V(T)}{k}$, then for any $S \in \mathcal{O}$ we have $\mathcal{O} = \{\overline{g}(S) : \overline{g} \in \overline{G}\}$.

As in the proof of Lemma 3.2.3, to define $A(T)$ we first partition the k-subsets of $V(T)$ into five classes.

Let

$$\begin{aligned}
\mathcal{C}_1 &= \{S \subseteq V(T) : |S| = k, \text{ and } S = \overline{g}(R) \text{ for some } g \in G \text{ and } R \subseteq V(T) \\
&\qquad \text{satisfying } R \cap G^{(1)} = \{e^{(1)}\} \text{ and } |R \cap H^{(2)}| = \min\{|H|, k-1\}\}, \\
\mathcal{C}_2 &= \{S \subseteq V(T) : |S| = k, \text{ and } S = \overline{g}(R) \text{ for some } g \in G \text{ and} \\
&\qquad R \subseteq V(T) \text{ satisfying } R \cap G^{(1)} = \{e^{(1)}\} \text{ and } e^{(2)} \notin R \text{ and } R \notin \mathcal{C}_1\}, \\
\mathcal{C}_3 &= \{S \subseteq V(T) : |S| = k, \text{ and } S = \overline{g}(R) \text{ for some } g \in G \text{ and} \\
&\qquad R \subseteq V(T) \text{ satisfying } R \cap G^{(1)} = \{e^{(1)}\} \text{ and } e^{(2)} \in R \text{ and } R \notin \mathcal{C}_1\}, \\
\mathcal{C}_4 &= \{S \subseteq V(T) : |S| = k, \text{ and } 2 \leq |S \cap G^{(1)}| < k\}, \text{ and} \\
\mathcal{C}_5 &= \{S \subseteq V(T) : |S| = k, \text{ and } S \subseteq G^{(i)} \text{ for } i = 1 \text{ or } i = 2\}.
\end{aligned}$$

Notice that for each i, $S \in \mathcal{C}_i$ if and only if $\overline{g}(S) \in \mathcal{C}_i$ for all $g \in G$.

The ordering assigned to a k-subset S of $V(T)$ will depend on the class $\mathcal{C}_i$ to which S belongs; in addition the ordering will be chosen so that if $A \in A(T)$ then $\overline{g}(A) \in A(T)$ for all $\overline{g} \in \overline{G}$.

Let $\mathcal{O}$ be an orbit of $\overline{G}$ acting on $\binom{V(T)}{k}$.

If $\mathcal{O} \subseteq \mathcal{C}_1$, then we can choose $R \in \mathcal{O}$ such that $R \cap G^{(1)} = \{e^{(1)}\}$ and $|R \cap H^{(2)}| = \min\{|H|, k-1\}$. We first order R. If $R \cap G^{(2)} \subseteq H^{(2)}$, order R so that $e^{(1)}$ is in the first co-ordinate (the ordering of the remaining elements of R is arbitrary). Otherwise, $|R \cap H^{(2)}| < k-1$, and we order R so that $e^{(1)}$ is in the first co-ordinate, $h_l^{(2)}$ precedes $h_m^{(2)}$ if $l < m$, and $h_l^{(2)}$ precedes $g^{(2)}$ if $g \notin H$. In either case, let A be the resulting arc. For each k-set $\overline{g}(R) \in \mathcal{O}$, we let $\overline{g}(A)$ be the arc of T corresponding to $\overline{g}(R)$.

If $\mathcal{O} \subseteq \mathcal{C}_2$, then we can find $R \in \mathcal{O}$ such that $R \cap G^{(1)} = \{e^{(1)}\}$ and $e^{(2)} \notin R$. We order R so that $e^{(1)}$ is in the second co-ordinate (and the ordering of the remaining elements of R is arbitrary), and let A be the resulting arc. For each k-set $\overline{g}(R) \in \mathcal{O}$, we let $\overline{g}(A)$ be the arc of T corresponding to $\overline{g}(R)$.

If $\mathcal{O} \subseteq \mathcal{C}_3$, then we can find $R \in \mathcal{O}$ such that $R \cap G^{(1)} = \{e^{(1)}\}$ and $e^{(2)} \in R$. We order R so that $e^{(1)}$ is in the third co-ordinate and $e^{(2)}$ is in the second co-ordinate (and again, the ordering of the remaining element(s) of R is arbitrary). We let A be the resulting arc, and for each k-set $\overline{g}(R) \in \mathcal{O}$ we let $\overline{g}(A)$ be the arc of T corresponding to $\overline{g}(R)$.

If $\mathcal{O} \subseteq \mathcal{C}_4$, we arbitrarily select a representative k-set $R \in \mathcal{O}$, and we order R so that any elements of $G^{(1)}$ precede any elements of $G^{(2)}$. We let A be the resulting arc and let $\overline{g}(A)$ be the arc corresponding to $\overline{g}(R)$ for each k-set $\overline{g}(R) \in \mathcal{O}$.

Finally suppose $\mathcal{O} \subseteq \mathcal{C}_5$. Recall that each $R \in \mathcal{O}$ satisfies $R \subseteq G^{(i)}$, where $i = 1$ or $i = 2$. If there is some $R \in \mathcal{O}$ such that $R \subseteq H^{(i)}$, we order R so that $h_l^{(i)}$ precedes $h_m^{(i)}$ whenever $l < m$. Otherwise we arbitrarily select $R \in \mathcal{O}$ and order R arbitrarily. In either case we let A be the resulting arc, and for each k-set $\overline{g}(R) \in \mathcal{O}$ we let $\overline{g}(A)$ be the corresponding arc of T.

This defines the arc-set $A(T)$ of T. We now show that $Aut(T) \cong G$ and that there are two vertex-orbits of $Aut(T)$ acting on $V(T)$. It is clear from the definition of $A(T)$ that $\overline{G} \leq Aut(T)$; it follows that $Aut(T)$ is transitive on each of $G^{(1)}$ and $G^{(2)}$. We want to show, therefore, that $Aut(T) = \overline{G}$. We begin by showing that each of $G^{(1)}$ and $G^{(2)}$ are fixed blocks of $Aut(T)$.

Definition 3.3.2 We say that an arc $A \in A(T)$ is of *type* i if A corresponds to a k-subset belonging to $\mathcal{C}_i$, $1 \leq i \leq 5$.

Claim 3.3.3 *For any* $u, v \in G$, $deg_k(u^{(1)}) < deg_k(v^{(2)})$.

Proof. First, we observe that since $Aut(T)$ is transitive on each of $G^{(1)}$ and $G^{(2)}$, then for any $u, v \in G$, $\deg_k(u^{(1)}) = \deg_k(e^{(1)})$ and $\deg_k(v^{(2)}) = \deg_k(e^{(2)})$. Thus it suffices to show that $\deg_k(e^{(1)}) < \deg_k(e^{(2)})$.

Notice that since $A \in A(T)$ if and only if $\overline{g}(A) \in A(T)$ for every $g \in G$, and since G_L is transitive on G, then given any orbit $\mathcal{O}$ of $\overline{G}$ acting on $A(T)$, and any co-ordinate

c, the set of vertices of T which appear in co-ordinate c in some arc belonging to $\mathcal{O}$ is either $G^{(1)}$ or $G^{(2)}$. That is, either $\{A(c) : A \in \mathcal{O}\} = G^{(1)}$ or $\{A(c) : A \in \mathcal{O}\} = G^{(2)}$, $1 \leq c \leq k$.

If $k > 3$, then $A(k) \in G^{(1)}$ only if A is of type 5, and hence only if A corresponds to a k-subset S of $G^{(1)}$. Therefore, $\deg_k(e^{(1)})$ is given by the number of orbits of $\overline{G}$ acting on $A(T)$ which correspond to k-subsets of $G^{(1)}$. Thus

$$\deg_k(e^{(1)}) = \frac{1}{|G|}\binom{|G|}{k}.$$

On the other hand, $A(k) \in G^{(2)}$ only if A does *not* correspond to a k-subset of $G^{(1)}$; therefore $\deg_k(e^{(2)})$ is given by the number of orbits of $\overline{G}$ acting on $A(T)$ which do not correspond to k-subsets of $G^{(1)}$, and so

$$\deg_k(e^{(2)}) = \frac{1}{|G|}\left[\binom{2|G|}{k} - \binom{|G|}{k}\right].$$

It is now easy to see that $\deg_k(e^{(1)}) < \deg_k(e^{(2)})$.

Now let $k = 3$. In this case, $A(3) \in G^{(1)}$ only if A is of type 3 or of type 5. If A is of type 3, then A is of the form $(g^{(2)}, x^{(2)}, x^{(1)})$, where $g \neq x$. If A is of type 5 and $A(3) \in G^{(1)}$, then A corresponds to a k-subset of $G^{(1)}$. Therefore, $\deg_3(e^{(1)})$ is the sum of the number of arcs of the form $(g^{(2)}, e^{(1)}, e^{(2)})$, where $g \neq e$, and the number of orbits of k-subsets of $G^{(1)}$. Therefore,

$$\deg_3(e^{(1)}) = |G| - 1 + \frac{1}{|G|}\binom{|G|}{3}.$$

On the other hand, $A(3) \in G^{(2)}$ only if A belongs to an orbit corresponding neither to k-subsets of $G^{(1)}$ nor to elements of C_3. Thus

$$\begin{aligned}\deg_3(e^{(2)}) &= \frac{1}{|G|}\left[\binom{2|G|}{3} - \binom{|G|}{3} - |G|(|G|-1)\right] \\ &= \frac{1}{|G|}\left[\binom{2|G|}{3} - \binom{|G|}{3}\right] - |G| + 1.\end{aligned}$$

Thus $\deg_3(e^{(2)}) > \deg_3(e^{(1)})$ if and only if

$$\frac{1}{|G|}\left[\binom{2|G|}{3} - \binom{|G|}{3}\right] - |G| + 1 > |G| - 1 + \frac{1}{|G|}\binom{|G|}{3}$$

or, equivalently,

$$\binom{2|G|}{3} - 2\binom{|G|}{3} > 2|G|(|G|-1).$$

The left side of this last inequality counts the number of 3-subsets of $V(T)$ which intersect both $G^{(1)}$ and $G^{(2)}$. Since $|G| \geq k = 3$, then for any $x \in G$ there are $|G|-1$ 3-subsets of the form $\{x^{(1)}, x^{(2)}, y^{(1)}\}$ and another $|G|-1$ of the form $\{x^{(1)}, x^{(2)}, y^{(2)}\}$. This gives us $2|G|(|G|-1)$ 3-subsets of $V(T)$ which intersect both $G^{(1)}$ and $G^{(2)}$. In addition, since $|G| \geq 3$, there is at least one more 3-subset of the form $\{x^{(1)}, y^{(1)}, z^{(2)}\}$, where $x, y, z \in G$ are all different. Therefore, we have $\binom{2|G|}{3} - 2\binom{|G|}{3} > 2|G|(|G|-1)$, as desired, and consequently $\deg_3(e^{(1)}) < \deg_3(e^{(2)})$. ■

Now it is clear that if α is an automorphism of T and $\alpha(u) = v$, then $\deg_k(u) = \deg_k(v)$. It therefore follows from Claim 3.3.3 that $\alpha(G^{(i)}) = G^{(i)}$, $i = 1, 2$, for each $\alpha \in Aut(T)$. So given $\alpha \in Aut(T)$ we can write $\alpha = (\alpha_1, \alpha_2)$, where α_i is the restriction of α to $G^{(i)}$, $i = 1, 2$. We may then consider each of α_1 and α_2 as an element of S_G, the symmetric group on the elements of G.

Claim 3.3.4 *If $\alpha = (\alpha_1, \alpha_2) \in Aut(T)$ then, considering α_1 and α_2 as elements of S_G, we have $\alpha_1 = \alpha_2$.*

Proof. Suppose $\alpha_1(x) = y$, where $x, y \in G$ (equivalently, $\alpha(x^{(1)}) = y^{(1)}$). We want to show that $\alpha_2(x) = y$.

Consider the arcs of T of type 3. These are the only arcs of T which contain a unique element of $G^{(1)}$ and which have this element in the third co-ordinate. They are therefore fixed (setwise) by all automorphisms of T. In addition since $\alpha(x^{(1)}) = y^{(1)}$, the set of arcs of type 3 which contain $x^{(1)}$ is mapped onto the set of arcs of type 3 which contain $y^{(1)}$. Now all type 3 arcs containing $x^{(1)}$ have $x^{(2)}$ as their second co-ordinate, and all type 3 arcs containing $y^{(1)}$ have $y^{(2)}$ as their second co-ordinate. It therefore follows that $\alpha(x^{(2)}) = y^{(2)}$, and hence that $\alpha_1 = \alpha_2$. ■

We now know that $Aut(T) = \overline{A}$, where $G_L \leq A \leq S_G$ and $\overline{A} = \{\overline{\alpha} : \alpha \in A\}$. We therefore want to show that $G_L = A$. To do this we will use Lemma 3.1.1, so we need the following.

Claim 3.3.5 *If* $\overline{\alpha} \in Aut(T)$ *and* $\alpha(e) = e$*, then* $\alpha(h) = h$ *for each* $h \in H$*.*

Proof. Let $\overline{\alpha} \in Aut(T)$ and $\alpha(e) = e$. Then $\overline{\alpha}(e^{(1)}) = e^{(1)}$.

Consider the arcs of T of type 1. These are the only arcs of T which contain a unique element of $G^{(1)}$ and in which this element is in the first co-ordinate. They are therefore fixed setwise by $\overline{\alpha}$. Since $\overline{\alpha}(e^{(1)}) = e^{(1)}$, the set of type 1 arcs containing $e^{(1)}$ is fixed by $\overline{\alpha}$. We distinguish two cases.

Case 1: $|H| \geq k$.
Since $|H| \geq k$, any type 1 arc containing $e^{(1)}$ has its $k-1$ remaining elements taken from $H^{(2)}$.

Choose any $h \in H$. There is at least one type 1 arc of T containing both $e^{(1)}$ and $h^{(2)}$, and the image of this arc under $\overline{\alpha}$ is also of type 1, and also contains $e^{(1)}$. Thus the image of $h^{(2)}$ belongs to $H^{(2)}$, and so $\alpha(h) \in H$. Since h is arbitrary we have $\alpha(H) = H$.

Now consider $T[H^{(1)}]$, the subtournament of T induced by $H^{(1)}$ (note that this subtournament exists since $|H| \geq k$). The arcs of this subtournament are all of type 5, and are all ordered so that h_l precedes h_m if and only if $l < m$. Thus $T[H^{(1)}]$ is a transitive k-tournament. Since $\alpha(H) = H$, the restriction of α to $H^{(1)}$ is an automorphism of $T[H^{(1)}]$; the transitivity of $T[H^{(1)}]$ implies that this automorphism is the identity automorphism. Therefore $\alpha(h) = h$ for all $h \in H$.

Case 2: $|H| \leq k-1$.
Again, the arcs of type 1 which contain $e^{(1)}$ are fixed setwise by $\overline{\alpha}$. In this case each of these arcs contains $e^{(1)}$ in the first co-ordinate, and the elements of $H^{(2)}$, in their given order, in the next $|H|$ co-ordinates. It follows that each element of $H^{(2)}$ is fixed by $\overline{\alpha}$. Thus we have $\alpha(h) = h$ for each $h \in H$.

By Lemma 3.1.1, we have $Aut(T) = \overline{G}_L \cong G$. This completes the proof of the theorem. ∎

3.3.2 Regular representations

We now consider the problem of finding conditions under which there is a regular representation of a finite group G as the automorphism group of a k-tournament. Of course a necessary condition is that $\gcd(|G|, k) = 1$. We show that if in addition either G is cyclic, or $k \geq 4$ and G has a minimal generating set with at least k elements, then there is a regular representation of G as the automorphism group of a k-tournament.

Definition 3.3.6 Given a group G, we say that a k-tournament T is a *k-tournament regular representation of G*, or a *k-TRR of G*, if $Aut(T) \cong G$ and $Aut(T)$ acts regularly on $V(T)$. Equivalently, T is a k-TRR of G if $T \cong T'$ where $V(T') = G$ and $Aut(T') = G_L$.

Theorem 3.3.7 *Let $k \geq 3$ be an integer and let G be a finite cyclic group satisfying* $\gcd(|G|, k) = 1$. *Then there is a k-TRR of G.*

Proof. Let $G \cong Z_n$, where $\gcd(n, k) = 1$. We construct a k- tournament T which is a k-TRR of G; to do this we construct T so that $V(T) = G$ and $Aut(T) = G_L$. For convenience we identify G with Z_n.

Let $V(T) = G = \{0, 1, \ldots, n-1\}$, and define a linear order on the elements of G so that $0 < 1 < \cdots < n-1$. We let G_L act on the set of k-subsets of G as before, so that $g + S = \{g + s : s \in S\}$ for each $g \in G$ and $S \in \binom{G}{k}$. As usual we begin by partitioning $\binom{G}{k}$ into orbits under the action of G_L.

We now define $A(T)$. For each orbit $\mathcal{O}$, let S be the lexicographically least element of $\mathcal{O}$ (with respect to the ordering of G given above). Let $S = \{v_1, \ldots, v_k\}$, where $v_1 < \cdots < v_k$. We let the arc of T corresponding to S be $A = (v_1, \ldots, v_k)$, and we let $i + A$ be the arc corresponding to $i + S$ for each remaining element $i + S$ of $\mathcal{O}$. This defines $A(T)$.

It is clear from the definition that $G_L \leq Aut(T) \leq S_G$. To show that $Aut(T) = G_L$ we will use Lemma 3.1.1, and so we need to show that if $\alpha \in Aut(T)$ and $\alpha(0) = 0$, then $\alpha(1) = 1$.

Notice that if an arc $A = (0, v_2, \ldots, v_k)$ of T has 0 in the first co-ordinate, then A corresponds to the lexicographically least element of its orbit, and so $0 < v_2 < \cdots <$

v_n. This follows from the fact that each orbit contains exactly one arc with 0 in the first co-ordinate.

For a $(k-1)$-subset $X = \{0, x_2, \ldots, x_{k-1}\}$ of $V(T)$ with $x_2 < \cdots < x_k$, let $f(X)$ denote the number of arcs of T of the form $(0, x_2, \ldots, x_{k-1}, s)$, where $s \in V(T) \setminus X$. From the observation above, it follows that $s > x_{k-1}$ in each such arc. Also, the elements of $Aut(T)_0$ preserve the function f; that is, if $\alpha \in Aut(T)_0$ and X is as above, then $f(X) = f(\alpha(X))$. We will show that $f(\{0, \ldots, k-2\} > f(X)$ for any $X \neq \{0, \ldots, k-2\}$, and hence that $\alpha(\{0, \ldots, k-2\}) = \{0, \ldots, k-2\}$ for all $\alpha \in Aut(T)_0$.

Claim 3.3.8 *$f(\{0, \ldots k-2\}) > f(X)$ for any $X \neq \{0, \ldots, k-2\}$, where f and X are defined as above.*

Proof. First, for any $(k-1)$-subset $X = \{0, x_2, \ldots, x_{k-1}\}$ of $V(T)$ with $x_2 < \cdots < x_k$, $f(X) \leq |\{x_{k-1}+1, \ldots, n-1\}| = n - x_{k-1} - 1$, since this is the number of k-subsets of $V(T)$ containing X and one other element s satisfying $s > x_{k-1}$. Now it is easy to check that $f(\{0, \ldots, k-2\}) = n-k$, since $\{0, \ldots, k-2, k-1\}$ and $\{0, \ldots, k-2, n-1\}$ belong to the same orbit of G_L, and all other sets $\{0, \ldots, k-2, s\}$ belong to distinct orbits. Thus if $x_{k-1} > k-1$, then

$$\begin{aligned} f(X) &\leq n - x_{k-1} - 1 \\ &< n - k \\ &= f(\{0, \ldots, k-2\}). \end{aligned}$$

Now let $x_{k-1} = k-1$, so that $X = \{0, x_2, \cdots, x_{k-1} = k-1\}$, where $x_2 < \cdots < x_{k-1}$. In this case, $f(X) = n-k$ only if every set of the form $X \cup \{s\}$ with $k-1 < s < n$ is the lexicographically least element in its orbit. Since $x_{k-1} = k-1$, then $X = \{0, \ldots, k-1\} \setminus \{y\}$ for some $y \in \{1, \ldots, k-2\}$. Consequently $X \cup \{n-1\} = \{0, \ldots, y-1, y+1, \ldots, k-1, n-1\}$, and this last set is lexicographically greater than the set $\{0, \ldots, y, y+2, \ldots, k\} = 1 + (X \cup \{n-1\})$. Thus $X \cup \{n-1\}$ is not the lexicographically least element in its orbit, and so the corresponding arc does not have 0 in the first co-ordinate.

Therefore $f(X) < n - k = f(\{0, \ldots, k-2\})$. ■

By Claim 3.3.8, each $\alpha \in Aut(T)_0$ fixes $\{0, \ldots, k-2\}$ setwise. Now T contains the arc $(n-1, 0, 1, \ldots, k-2)$ (this is because the set $\{0, 1, \ldots, k-2, n-1\}$ belongs to the same orbit as the set $\{0, \ldots, k-1\}$, which is the lexicographically least set of all and so is ordered $(0, \ldots, k-1)$); and this is the only arc of T which contains $\{0, \ldots, k-2\}$ and in which 0 is not in the first co-ordinate. Therefore this arc is fixed by every element of $Aut(T)_0$. It follows from this that for any $\alpha \in Aut(T)_0$, $\alpha(i) = i$ for each $i \in \{0, 1, \ldots, k-2, n-1\}$. In particular, $\alpha(1) = 1$ for each such α.

We have shown that $\alpha \in Aut(T)_0$ implies $\alpha(1) = 1$. It now follows by an application of Lemma 3.1.1 that $Aut(T) = G_L$. ■

We now show that if G has a minimal generating set with at least k elements, and if $k \geq 4$, then G has a k-TRR. The proof of this theorem uses Lemma 2.2.14, which was proven in Chapter 2, and which states that if T is a transitive k-tournament, and $i \in \{2, \ldots, k-1\}$, then no vertex of T is the i^{th} co-ordinate of every arc of T which contains it.

Theorem 3.3.9 *Let $k \geq 4$, and let G be a finite group such that* $\gcd(|G|, k) = 1$. *Let G have a minimal generating set with at least k elements. Then there is a k-TRR of G.*

Proof. Let G and k be as in the statement of the theorem, and let H be a minimal generating set for G with $|H| \geq k$. We define a linear order on the elements of G so that $H = \{h_1 < \cdots < h_{|H|}\}$, and $G = \{e = h_0 < h_1 < \cdots < h_{|H|} < g_{|H|+1} < \cdots < g_{|G|}\}$, where the ordering of the elements of H and of $G \setminus H$ is arbitrary. Let $H^+ = H \cup \{e\}$. We let G_L act on $\binom{G}{k}$, where as before $gS = \{gs : s \in S\}$.

We now construct a k-tournament T, and later show that T is a k-TRR of G.

Let $V(T) = G$. To define $A(T)$ we first partition the set of k-subsets of $V(T)$ into orbits under the action of G_L. We then classify these orbits into two types.

Definition 3.3.10 Let $\mathcal{O}$ be an orbit of G_L acting on $\binom{V(T)}{k}$. We say that $\mathcal{O}$ is of *type 1* if $\mathcal{O}$ contains a k-subset of H^+, and of *type 2* if not.

The order assigned to a k-subset of $V(T)$ will depend on the type of the orbit to which it belongs.

Let $\mathcal{O}$ be of type 1. Then there is $S \in \mathcal{O}$ such that $S \subseteq H^+$, and by Lemma 3.1.4, S is the only k-subset of H^+ lying in $\mathcal{O}$. Let $S = \{h_{i_1}, \ldots, h_{i_k}\}$, where $0 \leq i_1 < \cdots < i_k$. We let $A = (h_{i_1}, \ldots, h_{i_k})$ be the arc of T corresponding to S, and we let gA be the arc of T corresponding to each other element gS of $\mathcal{O}$ (where $g \in G$).

Now let $\mathcal{O}$ be of type 2, and let $S \in \mathcal{O}$ be any element of $\mathcal{O}$ satisfying $e \in S$ (note that there are k elements of $\mathcal{O}$ which contain e). Let $S = \{e = s_1, s_2, \ldots, s_k\}$, where $s_1 < \cdots < s_k$ (in the linear order defined on G). Notice that e is necessarily the least element of S, and that $S \not\subseteq H^+$ implies that $s_k \notin H^+$. We want to order S so that e is in the first co-ordinate and s_k is in the third; the order of the remaining elements of S is immaterial. For definiteness we let the arc of T corresponding to S be $A = (e = s_1, s_2, s_k, s_3, s_4, \ldots, s_{k-1})$. (Recall that $k \geq 4$.) We then let gA be the arc of T corresponding to gS, for each remaining element gS of $\mathcal{O}$.

This defines $A(T)$. We begin with an important observation about the arcs of T. Since G_L acts regularly on G, then for any given orbit $\mathcal{O}$ of G_L acting on $\binom{G}{k}$, there is exactly one arc of T which corresponds to an element of $\mathcal{O}$ and has e in the first co-ordinate. Therefore, given any arc A of T which has e in the first co-ordinate, there are two possibilities. Either A corresponds to an element of an orbit of type 1, in which case every element of A belongs to H^+, or A corresponds to an element of an orbit of type 2, in which case the largest element of A is in the third co-ordinate.

We now show that T is a k-TRR of G. To do this we must show that $Aut(T) = G_L$. It is clear from the definition of $A(T)$ that $G_L \leq Aut(T) \leq S_G$. We will again use Lemma 3.1.1; therefore, we need to show that if $\alpha \in Aut(T)$ and $\alpha(e) = e$, then $\alpha(h) = h$ for each $h \in H$.

To this end, let $\alpha \in Aut(T)$ and suppose that $\alpha(e) = e$. Let $L = \alpha(H)$, and $L^+ = L \cup \{e\} = \alpha(H^+)$. From the definition of the arcs belonging to orbits of type 1, it is clear that the subtournament of T induced by H^+ is transitive, with underlying linear order $e < h_1 < \cdots < h_{|H|}$. Since $\alpha \in Aut(T)$, the subtournament $T[L^+]$ of T induced by L^+ is also transitive, and since $\alpha(e) = e$, e is least in the underlying linear order of $T[L^+]$.

We want to show that $L = H$. Towards a contradiction, suppose $L \neq H$, and let l be the largest element of L, with respect to the intial ordering of G. Then $l \notin H$. We point out that l need not be the largest element in the underlying linear order of $T[L^+]$, since the two orders bear no relation to one another.

Let $\mathcal{E}$ denote the set of arcs which contain e and which correspond to a k-subset of L^+. We transform each of these arcs into a $(k-1)$-tuple by deleting the first co-ordinate (which contains e). Thus the arc $(e, v_2, \ldots, v_k)$ would become the $(k-1)$-tuple $(v_2, \ldots, v_k)$. Let $\mathcal{E}^*$ denote the set of $(k-1)$-tuples obtained in this way. Then $|\mathcal{E}^*| = \binom{|L|}{k-1}$, and $\mathcal{E}^*$ can be viewed as the set of arcs of a $(k-1)$-tournament T^* with vertex set L. Since $T[L^+]$ is transitive then T^* is also transitive; and every subtournament of T^* is transitive.

Let S^* be a $(k-1)$-subset of $L = V(T^*)$ such that $l \in S^*$. Since l is the largest element of L (with respect to the order of the elements of G), l is also the largest element of S^*. Let A be the arc of T corresponding to the set $S = S^* \cup \{e\}$. Then $A \in \mathcal{E}^*$, and so has e in the first co-ordinate; but since $S \not\subseteq H^+$, A belongs to an orbit of type 2. Therefore A has l, the largest element of S, in the third co-ordinate. Thus the arc of T^* corresponding to S^* has l in the second co-ordinate. Since the set S^* was arbitrary, l is in the second co-ordinate of every arc of T^* in which it appears. But since $k - 1 \geq 3$, this contradicts Lemma 2.2.14.

It follows that $L = H$, and so that $\alpha(H) = H$. It remains to show that α fixes the elements of H pointwise. However this follows immediately from the fact that $T[H]$, the subtournament of T induced by H, is transitive and so has identity automorphism group. An application of Lemma 3.1.1 completes the proof of the theorem. ■

Chapter 4

Regular and almost regular k-tournaments

In this chapter we consider questions related to the degree matrix of a k-tournament. In particular, we are interested in the existence of regular and almost regular k-tournaments, and in finding explicit constructions of such k-tournaments. Section 4.1 deals with the existence of regular and almost regular k-tournaments, and in Section 4.2 we present some elementary constructions of such k-tournaments for some cases.

Recall that for a vertex v of a k-tournament T, the i^{th} degree of v in T, denoted $\deg_i(v,T)$, is the number of arcs of T in which v is the i^{th} co-ordinate, and that the degree matrix of T is the $(n \times k)$ matrix whose (v,c)-entry is $\deg_c(v,T)$. Recall also that a k-tournament T is regular if there is an integer d such that $\deg_c(v,T) = d$ for every $c \in \{1,\ldots,k\}$ and every $v \in V(T)$, and is almost regular if there is an integer d such that $\deg_c(v,T) \in \{d, d+1\}$ for every $c \in \{1,\ldots,k\}$ and every $v \in V(T)$.

Notice that every regular k-tournament is almost regular; as we will see below, it is not true that every almost regular k-tournament is regular.

4.1 The existence of regular and almost regular k-tournaments

The purpose of this section is to present necessary and sufficient conditions on integers n and k for the existence of a regular or almost regular k-tournament on n vertices.

First, let T be a k-tournament on n vertices. If T is regular, then each vertex of T appears in each of the k co-ordinates exactly d times, for some integer d. It is therefore a simple matter to calculate the value of d. Since there are in total $\binom{n}{k}$ arcs in T, and each of the n vertices appears in the first co-ordinate d times, then $nd = \binom{n}{k}$, and so $d = \frac{1}{n}\binom{n}{k}$. This immediately gives us a necessary condition for the existence of a regular k-tournament on n vertices: if there does exist a regular k-tournament on n vertices, then $\binom{n}{k} \equiv 0 \pmod{n}$.

On the other hand, suppose T is almost regular but not regular. Again we have $\binom{n}{k}$ arcs of T, and n vertices, each of which appears in the first co-ordinate of T in either d or $d+1$ arcs. It follows that $d = \lfloor \frac{1}{n}\binom{n}{k} \rfloor$, and so $d+1 = \lceil \frac{1}{n}\binom{n}{k} \rceil$. Notice that if there exists an almost regular but not regular k-tournament on n vertices, then $\binom{n}{k} \not\equiv 0 \pmod{n}$. Thus an almost regular k-tournament is regular if and only if $\binom{n}{k} \equiv 0 \pmod{n}$.

The following question was asked by E. Barbut and A. Bialostocki in [3]:

Given integers n and k such that $n \geq k \geq 2$ and $\binom{n}{k} \equiv 0 \pmod{n}$, does there exist a regular k-tournament on n vertices?

An affirmative answer was given in [5] for the case when $\gcd(n,k)$ is a prime power. In Theorem 4.1.2 below, we give an affirmative answer for all n and k satisfying the given necessary conditions. In fact we prove a more general result, namely, that an almost regular k-tournament exists for all n and k satisfying $n \geq k \geq 3$; as we observed above, this almost regular k-tournament is regular if and only if $\binom{n}{k} \equiv 0 \pmod{n}$. This extends the corresponding result for tournaments: A regular tournament on n vertices exists if and only if $n \geq 2$ and is odd, and an almost regular tournament exists for all even $n \geq 2$.

For the proof of Theorem 4.1.2, we will need some measure of how 'close' an arbitrary k-tournament T is to being regular. This is the motivation for the definition below.

Definition 4.1.1 Let T be a k-tournament on n vertices. For any vertex v of T, and any co-ordinate c, we define

$$m_c(v,T) = \begin{cases} \lfloor \frac{1}{n}\binom{n}{k} \rfloor - \deg_c(v,T), & \text{if } \deg_c(v,T) \leq \frac{1}{n}\binom{n}{k}, \\ \deg_c(v,T) - \lceil \frac{1}{n}\binom{n}{k} \rceil & \text{otherwise,} \end{cases}$$

and

$$m(T) = \sum_{v \in V(T)} \sum_{c=1}^{k} m_c(v,T).$$

Notice that $m(T) \geq 0$, with equality if and only if T is regular or almost regular, depending on n and k.

Theorem 4.1.2 *There exists an almost regular k-tournament on n vertices for all n and k satisfying $n \geq k \geq 3$. In particular, if $n \geq k \geq 3$ and $\binom{n}{k} \equiv 0 \pmod{n}$, then there exists a regular k-tournament on n vertices.*

Proof. Let n and k be integers satisfying $n \geq k \geq 3$. Our method of proof is the following. Given an arbitrary k-tournament T on n vertices, we show that if $m(T) > 0$, then there is a k-tournament T' on n vertices such that $m(T') < m(T)$. Since $m(T)$ is finite, this is enough to show the existence of a k-tournament T^* on n vertices with $m(T^*) = 0$, and hence of an almost regular k-tournament on n vertices.

Let T be an arbitrary k-tournament on n vertices, with $V(T) = \{1, \ldots, n\}$, and assume that $m(T) > 0$. Then there is some vertex $x \in V(T)$ and some co-ordinate c_0 such that $m_{c_0}(x,T) > 0$; therefore either $\deg_{c_0}(x,T) < \lfloor \frac{1}{n}\binom{n}{k} \rfloor$, or $\deg_{c_0}(x,T) > \lceil \frac{1}{n}\binom{n}{k} \rceil$.

We first assume that $\deg_{c_0}(x,T) > \lceil \frac{1}{n}\binom{n}{k} \rceil$. Since x lies in $\binom{n-1}{k-1} = k\frac{1}{n}\binom{n}{k}$ k-subsets of $\{1, \ldots, n\}$, there is also some co-ordinate $c_1 \in \{1, \ldots, k\} \setminus \{c_0\}$ such that $\deg_{c_1}(x,T) < \frac{1}{n}\binom{n}{k}$; and if $\binom{n}{k} \not\equiv 0 \pmod{n}$, then $\deg_{c_0}(x,T) \leq \lfloor \frac{1}{n}\binom{n}{k} \rfloor$.

We want to re-order some of the arcs of T so that $\deg_{c_0}(x)$ decreases, $\deg_{c_1}(x)$ increases, and wherever possible all other degrees $\deg_c(v)$ remain unchanged. The idea is to select certain arcs of T, and in each selected arc A to exchange $A(c_0)$ and

$A(c_1)$. We therefore need to select these arcs rather carefully: we would like the number of selected arcs in which x is in co-ordinate c_0 to be exactly one more than the number of selected arcs in which it is in co-ordinate c_1; and wherever possible we would like each other vertex $u \neq x$ of T to be in co-ordinate c_0 in as many selected arcs as it is in co-ordinate c_1. Of course there will necessarily be at least one other vertex $y \neq x$ such that the number of selected arcs in which y is in co-ordinate c_1 is exactly one more than the number of selected arcs in which it is in co-ordinate c_0.

In order to make this selection of arcs, we construct a bipartite graph G with vertex set $V(G) = C_0 \cup C_1$, where

$$C_i = \{(u, A) : u \in V(T),\ A \in A(T) \text{ and } u = A(c_i)\},\ i = 0, 1.$$

If we think of T as an $\binom{n}{k} \times k$ array, where each row corresponds to an arc of T and each column to a co-ordinate, then C_0 and C_1 represent the c_0^{th} and c_1^{th} columns of the array.

We let $E(G) = M \cup H$, where

$$M = \{((u, A), (v, A)) : A \in A(T),\ u = A(c_0) \text{ and } v = A(c_1)\},$$

and

$$H = \{((u, A), (u, B)) : A, B \in A(T) \text{ and } A(c_0) = u = B(c_1)\}.$$

This defines the graph G. We note that possibly $H = \emptyset$.

Notice that M is a perfect matching in G. We now construct a second matching M', which is disjoint from M. For each $u \in V(T)$ such that both $\deg_{c_0}(u) > 0$ and $\deg_{c_1}(u) > 0$, let H_u be the subgraph of G induced by the vertices $\{(u, A) : u = A(c_0) \text{ or } u = A(c_1)\}$. Then H_u is a complete bipartite graph; and if $H \neq \emptyset$, the subgraph of G induced by the edges of H is the vertex-disjoint union of the subgraphs H_u. Now let M_u be a maximum matching in H_u, for each u for which the subgraph H_u is defined, and let $M' = \bigcup_u M_u$. Again, we allow $M' = \emptyset$. Notice that M_u (and hence M') saturates $V(H_u) \cap C_i$ if and only if $0 < \deg_{c_i}(u) \leq \deg_{c_{1-i}}(u)$, $i = 0, 1$.

Consider $M \cup M'$. Since $((u, A)(v, A)) \in M$ implies that $u, v \in A$ and so are distinct, then M and M' are disjoint. The graph $M \cup M'$ is therefore a union of alternating paths and cycles (by this we mean that the edges of these paths and cycles alternate between M and M'). Since M is a perfect matching in G, every maximal alternating path begins and ends with an edge of M, and so has odd length.

Now $\deg_{c_0}(x) > \frac{1}{n}\binom{n}{k} > \deg_{c_1}(x)$ implies that there is some arc A_1 of T such that $x = A_1(c_0)$ (hence $(x, A_1) \in C_0$) and (x, A_1) is not M'-saturated. Let P be a maximal alternating path containing (x, A_1), and let (y, A_s) be the terminal vertex of this path. Since P has odd length, $(y, A_s) \in C_1$ (so that $y = A_s(c_1)$), and (y, A_s) is not M'-saturated. Thus $\deg_{c_1}(y) > \deg_{c_0}(y)$. Note that this implies that $x \neq y$.

We now use the path P to construct the new k-tournament T'. Let

$$E(P) \cap M = \{((x, A_1), (x_1, A_1)), ((x_1, A_2), (x_2, A_2)), \dots, ((x_{s-1}, A_s), (y, A_s))\}.$$

The arcs $A_1, \dots, A_s$ are the selected arcs mentioned above. We construct T' from T by replacing the arcs $A_1, \dots, A_s$ by the arcs $A'_1, \dots, A'_s$, where A'_i is obtained from A_i by exchanging $A_i(c_0)$ and $A_i(c_1)$. Thus if $A_i = (v_1, \dots, v_{c_0}, \dots, v_{c_1}, \dots, v_k)$ then $A'_i = (v_1, \dots, v_{c_1}, \dots, v_{c_0}, \dots, v_k)$; and T' is defined by

$$V(T') = V(T),$$

and

$$A(T') = (A(T) \setminus \{A_1, \dots, A_s\}) \cup \{A'_1, \dots, A'_s\}.$$

We claim that $m(T') < m(T)$. First, for $j = 1, \dots, s-1$, $\deg_{c_i}(x_j, T') = \deg_{c_i}(x_j, T)$. Second, $\deg_{c_0}(x, T') = \deg_{c_0}(x, T) - 1$ and $\deg_{c_1}(x, T') = \deg_{c_1}(x, T) + 1$, so that $m_{c_0}(x, T') < m_{c_0}(x, T)$ and $m_{c_1}(x, T') \leq m_{c_1}(x, T)$ (either $\deg_{c_1}(x) = \lfloor \frac{1}{n}\binom{n}{k} \rfloor$ and $m_{c_1}(x, T') = m_{c_1}(x, T)$, or $\deg_{c_1}(x) < \lfloor \frac{1}{n}\binom{n}{k} \rfloor$ and $m_{c_1}(x, T') = m_{c_1}(x, T) - 1$).

Finally, since $\deg_{c_0}(y, T) < \deg_{c_1}(y, T)$, it is easy to check that one of the following must hold:

$$\begin{aligned} m_{c_0}(y, T') + m_{c_1}(y, T') &= m_{c_0}(y, T) + m_{c_1}(y, T); \\ m_{c_0}(y, T') + m_{c_1}(y, T') &= m_{c_0}(y, T) + m_{c_1}(y, T) - 1; \text{ or} \\ m_{c_0}(y, T') + m_{c_1}(y, T') &= m_{c_0}(y, T) + m_{c_1}(y, T) - 2. \end{aligned}$$

Thus $m(T') < m(T)$, as we claimed.

Now suppose that $m(T) > 0$, that x and c_0 are such that $m_{c_0}(x,T) > 0$, but that $\deg_{c_0}(x) < \lfloor \frac{1}{n}\binom{n}{k} \rfloor$ (we note here that there might be no pair v, c for which $\deg_c(v) > \lceil \frac{1}{n}\binom{n}{k} \rceil$). Then proceeding as above, we can find a co-ordinate c_1 such that $\deg_{c_1}(x) \geq \lceil \frac{1}{n}\binom{n}{r} \rceil$. From this point the proof proceeds along the same lines as before, but with the roles of c_0 and c_1 reversed. Since $\deg_{c_1}(x) > \deg_{c_0}(x)$, we will be able to find a maximal alternating path P in $M \cup M'$ which begins at some vertex $(x, A) \in C_1$ and ends at some vertex $(y, B) \in C_0$, where $\deg_{c_0}(y) > \deg_{c_1}(y)$. We re-order the selected arcs (i.e., those of $P \cap M$) as before, and again $m(T') < m(T)$. ∎

The same method can be used to prove the following generalisation of Theorem 4.1.2.

Theorem 4.1.3 *Let H be a k-uniform hypergraph in which each vertex lies in qk hyperedges, for some fixed integer $q \geq 1$. Then the hyperedges of H can be oriented so that each vertex occupies each of the k positions exactly q times.*

4.2 Explicit constructions of regular and almost regular k-tournaments

In this section we provide an explicit construction of a regular or almost regular k-tournament on n vertices, for cases in which the greatest common divisor of n and k is prime. If n and k are relatively prime, the construction is very simple; for the cases in which $\gcd(n,k)$ is prime, the construction is a modification of the preceding construction. We begin, therefore, by looking at the case in which n and k are relatively prime. The following lemma is a special case of Corollary 3.1.3 from Chapter 3.

Lemma 4.2.1 *Let $n \geq k \geq 3$ be such that $\gcd(n,k) = 1$, and let G denote the cyclic group of order n. Then G_L acts semiregularly on $\binom{G}{k}$.*

Corollary 4.2.2 *Let $n \geq k \geq 3$ be such that $\gcd(n,k) = 1$. Then $\binom{n}{k} \equiv 0 \pmod{n}$.*

We now show that if $\gcd(n, k) = 1$, then, using Lemma 4.2.1, it is easy to construct a regular k-tournament on n vertices.

Proposition 4.2.3 *Let n and k be integers such that $n \geq k \geq 3$ and $gcd(n, k) = 1$. Then there is a regular k-tournament on n vertices.*

Proof. Let $G = (Z_n, +)$, the cyclic group of order n. We construct a k-tournament T. We let $V(T) = \{0, 1, \ldots, n-1\}$. To define $A(T)$, we let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of G_L acting on $\binom{V(T)}{k} = \binom{G}{k}$, where $m = \frac{1}{n}\binom{n}{k}$, and we arbitrarily select a representative $S_i \in \mathcal{O}_i$, $1 \leq i \leq m$. We arbitrarily order S_i to produce an arc A_i, $1 \leq i \leq m$, and we then let $A(T) = \{g + A_i : g \in G \text{ and } 1 \leq i \leq m\}$.

To show that T is regular, we note that since by Lemma 4.2.1 each orbit $\mathcal{O}_i$ has cardinality n, then among the arcs corresponding to an orbit $\mathcal{O}_i$ each element of $V(T)$ appears exactly once in each co-ordinate. It follows that in T, each element of $V(T)$ appears exactly m times in each of the k co-ordinates, and so T is regular. ∎

Remark 4.2.4 In Theorem 3.3.7, in Chapter 3, we constructed a k-tournament whose automorphism group is regular and is isomorphic to a given cyclic group of order relatively prime to k. This k-tournament is necessarily regular; however, since its construction is unnecessarily complicated if we are interested only in the regularity of the k-tournament constructed (rather than that of its automorphism group, as was the case in Chapter 3), we provide the construction given in Proposition 4.2.3 rather than simply quoting Theorem 3.3.7.

We now proceed to consider the construction of a regular k-tournament on n vertices if $\gcd(n, k)$ is prime. We will use the same idea as in the proof of Proposition 4.2.3; that is, we will construct a k-tournament whose vertices are the elements of the cyclic group of order n, and in defining the arc set of this k-tournament we will once again examine the orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$, where Z_n denotes the cyclic group of order n.

We begin with a lemma concerning the cardinalitites of the orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$.

Lemma 4.2.5 *Let $n \geq k \geq 3$, and let $\mathcal{O}$ be an orbit of $(Z_n)_L$ acting on $\binom{Z_n}{k}$. Let $|\mathcal{O}| = t$. Then n/t divides k and hence n/t divides $gcd(n, k)$. Moreover, each element of Z_n occurs in exactly kt/n elements of $\mathcal{O}$.*

Proof. Let n, k, $\mathcal{O}$, and t be as in the statement of the lemma. It follows immediately from the Orbit-Stabilizer Theorem (see the Introduction) that t divides n, so we let $n = qt$. Now let $S \in \mathcal{O}$, where $S = \{x_1, x_2, \ldots, x_k\}$ and $0 \leq x_1 \leq \cdots \leq x_k \leq n-1$. Since $|\mathcal{O}| = t$, then $S + t = S$, and so $\{x_1 + t, x_2 + t, \ldots, x_k + t\} = \{x_1, x_2, \ldots, x_k\}$. Therefore, $x_1 + t = x_{l+1}$ for some $l \in \{1, \ldots, k-1\}$. Since $0 \leq x_1 \leq \cdots \leq x_k$, this implies that $x_2 + t = x_{l+2}$ and that in general $x_i + t = x_{l+i}$, $2 \leq i \leq k$, where the subscripts are reduced modulo k. It follows from this that for any integer α, $x_i + \alpha t = x_{\alpha l + i}$, $1 \leq i \leq k$, again with subscripts reduced modulo k. In particular, $x_1 + n = x_1 + qt = x_{ql+1}$, and since $x_1 + n = x_1$, we then have $x_{ql+1} = x_1$. Consequently, $ql \equiv 0 \pmod{k}$. Now if $\alpha < q$, then $\alpha t < n$, and so $x_1 + \alpha t \not\equiv x_1$; consequently, $x_{\alpha l+1} \not\equiv x_1$, and so $\alpha l \not\equiv 0 \pmod{k}$. Thus we have $ql \equiv 0 \pmod{k}$, while $\alpha l \not\equiv 0 \pmod{k}$ for any $\alpha < q$. It follows that in fact $ql = k$. Therefore, $q = n/t$ satisfies $q|k$, as desired; and so n/t divides $\gcd(n, k)$.

Since $n = qt$, we can think of Z_n as consisting of q 'segments' of length t, each segment consisting of t consecutive elements of Z_n of the form $\{\alpha t + 1, \alpha t + 2, \ldots, (\alpha + 1)t - 1\}$. In addition, we know that $k = ql$, and that for each $i = 1, 2, \ldots, k$, $x_i + t = x_{l+i}$, where the subscripts are reduced modulo k. Therefore, we can also think of S as consisting of q segments of length l, where in this case each segment consists of l 'consecutive' elements of S of the form $\{x_{\alpha l+1}, x_{\alpha l+2}, \ldots, x_{(\alpha+1)l-1}\}$. In addition, we can obtain the segment $\{x_{\alpha l+1}, x_{\alpha l+2}, \ldots, x_{(\alpha+1)l-1}\}$ of S from the segment $\{x_{(\alpha-1)l+1}, x_{(\alpha-1)l+2}, \ldots, x_{\alpha l-1}\}$ by adding t to each of its elements; that is, $\{x_{\alpha l+1}, x_{\alpha l+2}, \ldots, x_{(\alpha+1)l-1}\} = \{x_{(\alpha-1)l+1} + t, x_{(\alpha-1)l+2} + t, \ldots, x_{\alpha l-1} + t\}$.

It follows from this last observation that among *any* t consecutive elements $x - t + 1, x - t + 2, \ldots, x$ of Z_n, there are exactly l elements of S. We use this to show that each element of Z_n occurs in $kt/n = l$ elements of $\mathcal{O}$.

For any fixed element x of Z_n, and any $j \in 1, \ldots, t$, we have $x \in S + j$ if and only if $x - j \in S$, where $x - j$ is reduced modulo n. Therefore,

$$\begin{aligned} |\{j : x \in S + j\}| &= |\{j : x - j \in S\}| \\ &= |S \cap \{x, x-1, \ldots, x - t + 1\}| \\ &= l. \end{aligned}$$

Thus x occurs in exactly l of the sets $S, S+1, \ldots, S+t-1$, and so in exactly l elements of $\mathcal{O}$. ∎

It follows from Lemma 4.2.5 that if n and k satisfy $\gcd(n,k) = p$, where p is prime, then the orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$ all have cardinality either n or n/p.

The following lemma shows that, for general n and k, if we have n/t orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$, each of cardinality t, then we can order the n sets belonging to these orbits so that each element of Z_n occurs in each co-ordinate just once.

Lemma 4.2.6 *Let $n \geq k \geq 3$, and suppose $\mathcal{O}_1, \ldots, \mathcal{O}_{n/t}$ are orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$, such that $|\mathcal{O}_i| = t$ for each $i = 1, \ldots, n/t$. Then each element of $\bigcup_{i=1}^{n/t} \mathcal{O}_i$ can be ordered so that, among the n resulting k-tuples, each element of Z_n occurs exactly once in each of the k co-ordinates.*

Proof. Let $n \geq k \geq 3$ and let $\mathcal{O}_1, \ldots, \mathcal{O}_{n/t}$ be as in the statement of the lemma. By Lemma 4.2.5, each element of Z_n occurs in exactly $kt/n \geq 1$ elements of $\mathcal{O}_i$, for each $i = 1, \ldots, n/t$.

We begin by ordering the n k-sets belonging to $\bigcup \mathcal{O}_i$, in a way which will not necessarily satisfy the requirements of the lemma. For each $i \in \{1, \ldots, n/t\}$, we select an element S_i of $\mathcal{O}_i$ such that S_i contains the element $(i-1)(n/t) + 1$ of Z_n. If $S_i = \{x_1, \ldots, x_k\}$, where $0 \leq x_1 \leq \cdots \leq x_k \leq n$ and $(i-1)(n/t) + 1 = x_j$, then we order S_i to produce the k-tuple $A_i = (x_j, x_{j+1}, \ldots, x_k, x_1, \ldots, x_{j-1})$. Therefore, for each $i = 1, \ldots, n/t$, A_i has the element $(i-1)(n/t) + 1$ in the first co-ordinate. Now for each remaining element $S_i + m$ $(1 \leq m \leq t-1)$ of $\mathcal{O}_i$, we let the k-tuple corresponding to $S_i + m$ be $(x_j + m, x_{j+1} + m, \ldots, x_k + m, x_1 + m, \ldots, x_{j-1} + m)$.

This defines an ordering of each k-set belonging to $\bigcup \mathcal{O}_i$, and these orderings have the property that for each i, the elements of Z_n which appear in the first co-ordinate of some element of $\mathcal{O}_i$ are exactly $(i-1)(n/t)+1, (i-1)(n/t)+2, \ldots, i(n/t)-1, i(n/t)$. It follows that among all n k-tuples, each element of Z_n occurs exactly once in the first co-ordinate.

Notice also that, as in the proof of Lemma 4.2.5, a typical k-tuple has the form

$$A = (a_1, \ldots, a_l, a_1 + t, \ldots, a_l + t, \ldots, a_1 + (n/t-1)t, \ldots, a_l + (n/t-1)t),$$

where $l = kt/n$ and the entries are reduced modulo n. It follows that an element x of Z_n occurs in co-ordinate c of A if and only if the element $x + \alpha t$ occurs in co-ordinate $c + \alpha l$ of A, $1 \leq \alpha < n/t$.

Now suppose that among the n k-tuples we defined above, some element x of Z_n occurs more than once in some co-ordinate c. From the definition of these k-tuples, it is clear that x occurs at most once in co-ordinate c in each orbit $\mathcal{O}_i$. Therefore, the k-tuples in which x occurs in co-ordinate c all belong to different orbits. Let r denote the number of k-tuples in which x occurs in co-ordinate c.

It is also clear from the definition of the k-tuples that in any particular orbit $\mathcal{O}_i$, the l co-ordinates in which x appears are consecutive modulo k (here we mean cyclically, i.e., we consider k and 1 to be consecutive). Therefore, if x appears in co-ordinate c in some k-tuple belonging to $\mathcal{O}_i$, then x appears in none of the co-ordinates $c + \alpha l$, $1 \leq \alpha < n/t$, in $\mathcal{O}_i$. Now there are n/t co-ordinates of the form $c + \alpha l$, where $0 \leq \alpha < n/t$, and n/t orbits $\mathcal{O}_i$; since x occurs in co-ordinate c in r orbits, there are at least $r-1$ co-ordinates of the form $c+\alpha l$, $1 \leq \alpha < n/t$, in which x does not appear in any of the n/t orbits. There are, therefore, at least $r-1$ co-ordinates of the form $c + \alpha l$, $1 \leq \alpha < n/t$, in which x does not appear among our n k-tuples.

Now consider any k-tuple A in which x occurs in co-ordinate c. The co-ordinates $c+l, c+2l, \ldots, c+(n/t-1)l$ of A are occupied by the elements $x+t, x+2t, \ldots, x+(n/t-1)t$, respectively. Let $c+\beta l$ be any co-ordinate in which x does not appear in any of the n k-tuples. Since x does not appear in co-ordinate $c+\beta l$, then $x+\beta t$ (the current occupant of co-ordinate $c+\beta l$ in A) does not occur in co-ordinate $c+2\beta l$, and in general $x+\alpha t$ does not occur in co-ordinate $c+(\alpha+\beta)l$, $1 \leq \alpha < n/t-1$. We therefore re-order the k-tuple A as follows. We define a new k-tuple A' by

$$A'(j) = \begin{cases} A(j), & \text{if } j \notin \{c + \alpha l : 0 \leq \alpha < n/t\}, \\ A(j - \beta l), & \text{otherwise,} \end{cases}$$

and we replace the k-tuple A with the new k-tuple A'. The effect of this replacement is to reduce the number of times the element $x + \alpha t$ occurs in co-ordinate $c + \alpha l$ from r to $r - 1$, and to increase the number of times the element $x + \alpha t$ occurs in co-ordinate $c + (\alpha + \beta)l$ from 0 to 1, for each $\alpha \in \{0, 1, \ldots, n/t - 1\}$. It is now clear that if we repeat this procedure $r - 2$ more times, each time selecting a k-tuple in which x occurs in co-ordinate c and a co-ordinate of the form $c + \beta l$ in which x does not occur, we will obtain a set of k-tuples in which $x + \alpha t$ occurs in co-ordinate $c + \gamma l$ exactly once, $0 \leq \alpha, \gamma < n/t$.

We now look for some new vertex y and some new co-ordinate d such that y occurs in co-ordinate d more than once among the current set of n k-tuples. If no such vertex and co-ordinate exist, then the current set of k-tuples has the desired property. If such a vertex and co-ordinate do exist, then either $y \notin \{x, x + t, \ldots, x + (n/t - 1)t\}$ or $d \notin \{c, c + l, \ldots, c + (n/t - 1)l\}$, or both. We now repeat the entire procedure, thereby obtaining a new set of k-tuples with the property that each element $y + \alpha t$ occurs in co-ordinate $d + \gamma l$ exactly once, $0 \leq \alpha < n/t$. Observe that this new set of k-tuples will still have the property that each element of the form $x + \alpha t$ occurs exactly once in each co-ordinate $c + \alpha l$, $0 \leq \alpha < n/t$.

Since there are only $t(l - 1)$ pairs (x, c) for which some element $x + \alpha t$ might occur more than once in a co-ordinate $c + \beta l$ in our original set of k-tuples, we will need to repeat the above procedure at most $t(l - 1)$ times in total before obtaining a set of k-tuples with the property that each element of Z_n occurs in each of the k co-ordinates exactly once. ∎

We are now ready to construct an almost regular k-tournament on n vertices for any n and k satisfying $n \geq k \geq 3$ and $\gcd(n, k) = p$, where p is prime. This gives us an alternate proof of Theorem 4.1.2 for the special case when $\gcd(n, k)$ is prime.

Theorem 4.2.7 *Let $n \geq k \geq 3$ and let $gcd(n, k) = p$, where p is prime. Then there is an almost regular k-tounament on n vertices.*

Proof. Let n and k be as in the statement of the theorem, and let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of $(Z_n)_L$ acting on $\binom{Z_n}{k}$ which have cardinality less than n. As observed following the proof of Lemma 4.2.5, $|\mathcal{O}_i| = n/p$ for each $i = 1, 2, \ldots, m$. Let $m = qp + r$, where $0 \leq r < p$. We first partition $\mathcal{O}_1, \ldots, \mathcal{O}_m$ into q classes of p orbits each and, if $r > 0$, one class of r orbits. We then apply Lemma 4.2.6 to each of the q classes containing p orbits, so that among the resulting qn k-tuples each element of Z_n occurs in each co-ordinate exactly q times. Finally, if $r > 0$ we order the k-sets belonging to the last class of orbits so that each element of Z_n occurs in each co-ordinate at most once. It is easy to see that the method given in the proof of Lemma 4.2.6 can be modified to do this.

We also apply Lemma 4.2.6 to each orbit of $(Z_n)_L$ acting on $\binom{Z_n}{k}$ of cardinality n, to produce, for each such orbit, n k-tuples in which each element of Z_n occurs in each co-ordinate exactly once.

We now define the required k-tournament T by letting $V(T) = Z_n$, and letting $A(T)$ be the set of $\binom{n}{k}$ k-tuples obtained above. If $r = 0$ then T is regular, and if $r > 0$, then T is almost regular. ■

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