COMPOSITION OF FUNCTIONS AND INVERSE FUNCTION OF A FUNCTION: MAIN IDEAS, AS PERCEIVED BY TEACHERS AND PRESERVICE TEACHERS

by

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ABSTRACT

The concept of function is one of the central concepts in mathematics; as such it has received considerable attention in mathematics education research. However, the research that focuses explicitly on the subtopics of composition of functions and the inverse function of a function is rather slim. This study investigated what teachers and prospective teachers consider to be the main ideas behind these two subtopics underlying the concept of a function and what they attend to when teaching or planning to teach these topics. The study also investigated connections between the three components of content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Furthermore, the study examined the influence of the teaching experience on content knowledge.

Ten prospective teachers and eight experienced teachers participated in the study. The data were collected through clinical interviews. In addition to this, the prospective teachers wrote a set of lesson plans for teaching the two considered topics.

The data were analyzed in detail using a theoretical framework that was modified and refined based on prior research on teacher's content knowledge related to the concept of function. The framework contained five facets: essential features and the knowledge and understanding of the mathematical concept, different representations and alternative

ways of approaching the topic, basic repertoire, knowledge of the mathematics curriculum, and knowledge of general mathematics.

The results of the study suggest that the two groups of participants were not remarkably different in terms of their subject matter knowledge and their pedagogical content knowledge. Majority of participants presented a mainly procedural approach to both topics and disregarded some essential components, such as conditions for existence of inverse. However, the experienced teachers presented a higher competency in the area of curricular knowledge. With regards to the relation between the three components of content knowledge, subject matter knowledge and pedagogical knowledge appear to be interconnected, whereas curricular knowledge appeared to be independent of the previous two.

DEDICATION

For my wife Kara, who with her good nature supported me through some of the most difficult times of this endeavour.

Pentru parintii mei, Florica si Vasilie, care in decursul intregii mele vieti au trait alaturi de mine fiecare pas pe care l-am intreprins, si care mi-au hranit imaginatia cu cele mai cutezatoare visuri.

For my parents, Florica and Vasilie, who throughout my entire life watched every step I took, and fed my imagination with the most daring dreams.

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CHAPTER 1
INTRODUCTION

1.1 The Beginning

Starting my Canadian teaching career, and being a non-native English speaker, I realized how important communication and language are. This realization guided me with my first idea: to try to compare and to parallel the learning of mathematics with the learning of a second language. I am lucky enough to be able to speak fluently, and sometimes correctly, three languages, and I am on my way to learning a fourth one. I am also quite fluent in mathematics, which itself is often referred to as a language, with its own syntax and vocabulary, and with its own alphabet, composed of symbols encompassing letters from different alphabets and graphic symbols that make sense only for the initiated person. Somebody who falls in love with mathematics will argue that mathematics is the ultimate, universal language. One can write relatively simple sentences in mathematics, such as $"2x + 3 = -2"$, or one can write complex proofs, such as the famous geometrical proof for the Nine Point Circle of any triangle, also known as Euler's Circle. The initiated in this language will be able to read the message, no matter what their mother tongue is. It can be argued, that mathematics is the most studied language on Earth. With language and mathematics in mind, I enrolled in the PhD

program at Simon Fraser University (SFU), continuing to work full time as a secondary

mathematics teacher in the British Columbia (BC) public school system.

1.2. Focus Change: Composition of Functions and the Inverse Function of a Function

Focusing on language, mathematics, and the connections between them, consider

the following sentences: "Depending on which key you press on the keyboard, you will

see a result on the computer screen" "I will take the umbrella with me today, depending

on what the weather man will say this morning!" "The relation with my boss is better

since the revenues of the company look like an increasing function of the time I spent at

work after the office hours" "In the morning, I start functioning only after my second cup

of coffee" "The study of functions is very functional in one's life. For example, one

cannot live without functions in this class, since every mark that one scores is a function

of work, talent, luck and my benevolence". From the first grader who gives his/her first

lesson in computers to a computer illiterate grandparent, to a mathematics high school

teacher who tries to send a message to the class, all of us use language, sentences, to

communicate. But, these sentences have something more in common than the simple fact

that they are written in English. For a mathematical mind they all refer to functions, either

explicitly or implicitly.

An old Romanian proverb says that only dead or crazy people do not change their

minds over time. I am still alive, and not too crazy, and consequently, my mind and my

ideas changed. I started to observe that my professional life, my personal life and almost

everything around me was a function of multiple variables. My life in the work place was

a function of a number of variables, such as my commitment to students, their commitment to be successful, the administration of the school, the parents, and the list continues with the weather on a certain day, or with the physical site of work.

Becoming more and more familiar with the BC high school curriculum, I encountered more functions: trigonometric functions, linear functions, quadratic functions, and so on. I started to feel that this word, "function", has a high frequency in our daily vocabulary, in a more or less explicit way, especially if it happens that one's occupation is that of a high school mathematics teacher. At this point, my professional interest started to be awakened and stimulated by this topic. As a result, the focus of my research shifted towards the function concept. This is a vast topic, and for that reason I concentrated my attention on two subtopics of the concept: composition of functions and the inverse function of a function.

In the following sections I present arguments for the choice I made for my research.

1.3. The Function: The Central Piece of the BC Mathematics High School Curriculum

The concept of function is the central concept in the high school mathematics curriculum, and it seems that its importance is growing, at least in Canada and in the United States. Haimes(1996) claimed: "In the teaching of algebra, a focus for reform is a change from a "generalized arithmetic" approach to one in which functions provide the medium through which the concept of variables is developed" (p. 582). On the same note,

O'Callaghan (1998) wrote that "The concept of function is essential to the algebra curriculum and is considered by many to be the most important concept in all mathematics" (p. 23). The same kind of message regarding the importance of the concept of function in mathematics comes from the work of other researchers (e.g. Beckmann, Senk, & Thompson, 1999; Knuth, 2000; Wilson & Krapfl, 1994). In the main stream of mathematics courses taught in BC, namely, the Principles of Mathematics courses, one can observe that besides being part of an intended mathematical enculturation, and leaving aside their utilitarian intended character, most of the mathematics taught is oriented to prepare the students for their encounter with calculus. As detailed in the next chapter of my dissertation, the notion of function developed at the same time as calculus, as a need of mathematicians to restrict the larger set of relations. The concept of function is central for calculus (and not just for calculus) and for that reason, a major part of British Columbia's mathematics high school curriculum is a study of different classes of functions. In the following paragraphs I present topics of the mathematics high school curriculum that use implicitly or explicitly the concept of function.

In Grade 9, the students are exposed to three basic trigonometric functions: sine, cosine and tangent. As well, computationally, they use inverse trigonometric functions: arcsine, arccosine, and arctangent, without being explicitly taught their names. They are also exposed to transformations of graphs, which can be seen as applications of composition of functions.

In Grade 10, the trigonometric functions are again present in the curriculum, and the arguments of these functions are extended to angles greater than 90 degrees. At this

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age level, the notion of function is introduced explicitly, with its definition, domain, range, different representations, as well as examples of different kinds of functions, such as absolute value functions, radical functions or quadratic functions. The function notation is introduced to the students, and relatively simple computations are performed, to familiarize students with this new notation. Furthermore in Grade 10, the students go through an extensive study of linear functions, with equations in general and standard forms, graphs, intercepts and slopes of lines.

In Grade 11, the curriculum contains an extensive and fairly in-depth study of the quadratic function, as well as the study of the quadratic equations, through a discussion of the zeroes, or x-intercepts of the function. Other topics of the Grade 11 curriculum include the extensive study of polynomial functions, radical functions and absolute value functions. These topics include discussions about the domains and ranges, general shapes of graphs, and x and y intercepts, including the theorems specific to polynomial expressions (the Factor Theorem and the Rational Root Theorem). In Grade 11, the curriculum also discusses the operations with functions: the basic arithmetic operations of adding, subtracting, multiplying and dividing, as well as the operation specific to functions, the composition of functions. In relation to this last operation, the curriculum also contains the topic of the inverse function of a function. It is fair to suggest that in the Grade 11 curriculum, more than 60% of the teaching time should be dedicated to topics related to functions. In terms of the mathematical difficulty, from my own experience, as well as from the shared experience of other colleagues who teach the Grade 11 Principles of Mathematics course, the topics treating functions are probably the most complex in

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that grade. For a successful learning/teaching experience of these topics a great deal of dedication and determination is required from both teachers and students.

In Grade 12, the curriculum includes an extensive section on exponential functions and their inverses, the logarithmic functions. It also deals, in depth, with the study of the three basic trigonometric functions, as well as the three reciprocal trigonometric functions, the secant, cosecant and the cotangent. For all these functions, the curriculum prescribes the handling of the graphs, theory of zeroes, y-intercepts, domains and ranges, asymptotes and applications. At the same time, in Grade 12, there is a section of the curriculum that treats functions in general, and deals with transformation of functions, reciprocal functions and inverse functions. The topics related to functions represent approximately 40-50% of the Grade 12 curriculum.

In the BC high school mathematics there are other topics, not mentioned above (e.g. compound interest in Grades 9, 10 or 11) that are not apparently related to functions, but they pave the students' road towards the study of different categories of functions. From this overview, it can be argued that the notion of function is the most important unifying concept of the high school mathematics curriculum in BC. This is one of the rationales for the enterprise I undertook in trying to illuminate some of the components that are part of the large topic of functions in high school mathematics.

1.4. Personal and Professional Interest

Besides the major role that the function plays in high school mathematics, my educational and professional background provided another catalyst for the start of this

study. Mathematics teacher education in Romania, the country where I completed my first teaching degree, has a stronger mathematical orientation, compared to the teaching programs in British Columbia.

In my country of origin, teacher education focuses more on providing teachers with a solid subject matter background. Some of the university courses that lead towards teacher certification actually teach future teachers *what* to teach in a certain high school mathematics course, giving multiple didactical views on a certain topic. These courses are less oriented towards the pedagogical act of teaching, rather they provide the future teachers with what Shulman (1986) called content knowledge. (As mentioned in Chapter 4, section 4.1.4, content knowledge has three components: subject matter knowledge, pedagogical content knowledge, and curricular knowledge). In BC, teacher education programs are oriented in the other direction, towards pedagogical knowledge, towards *how* to teach. In BC the pedagogy is stressed more than the content, and the "what" part of the curriculum is sometimes seen as less important. My personal mathematical experience, starting with Grade 5, was more formal than the mathematical education prescribed by the official curriculum documents in Western Canada. These facts influence my beliefs on how and what should be taught about certain topics, and in particular what should be taught about the concept of function and operations with functions.

The main teachers' resources — the textbooks used in BC and the Integrated Resource Packages (IRP) — lack the mathematical clarity and formalism needed to teach such topics as composition of functions, or the inverse function of a function. Key concepts such as relations between domain and range, one-to-one functions are

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completely missing from the textbooks. Melnick (1996) quoted Florio-Ruane and Dunn when he wrote: "teachers often teach the way they were taught" (p. 37). In my understanding, this phrase is not limited to the method of teaching, but also to what teachers teach. In the absence of reliable and comprehensive resources, mathematics teachers will just reproduce what they were taught, or, worse, what they remember from what they were taught. This is not necessarily a desirable thing, considering that teachers who teach high school mathematics in BC do not always possess in-depth postsecondary studies in mathematics.

1.5. Rationale

The rationale for my main thesis topic can be condensed into the following brief argument. The concept of function is the most important topic of the high school curriculum in BC; to teach/learn the topic of function is not an easy task for both teachers, and students; at the Grade 11 level, the mathematics becomes fairly abstract, and it requires more than just procedural knowledge; my belief is that at this age level, *what* we teach is at least as important as *how* we teach; composition of functions and the inverse function of a function are subtopics of the larger concept of function, and they are treated at the Grade 11 level, when the students are expected to possess and develop a fairly high analytical capacity; in general, the textbooks are the main working resource for teachers, and the Grade 11 textbooks used in BC high schools do not adequately treat the topics of composition of functions and the inverse function of a function.

An initial search of the existing mathematics education literature related to the topic of functions directed me to choose preservice and inservice teachers as participants in my study. As I detail in Chapter 4, most of the research in mathematics education regarding the function concept is focused on students' understanding of different aspects of this broad mathematical concept. As a consequence, I focused my research on teachers' understanding of functions, and explicitly on their approach to composition of functions and the inverse function of a function. It is important to determine whether teachers have a clear idea about what is significant when they teach this part of the high school curriculum. Without this clarity and without focusing on the relevant aspects, the process of teaching these concepts loses its mathematical value. Teaching for operational fluency should be one of the goals when teaching the composition of functions and the inverse function of a function, but it should not be the only goal. The analysis of what these processes mean and when can they be performed is at least as important as being able to manipulate algebraic expressions.

Following the above arguments, the present study investigates teachers' and preservice teachers' content knowledge with respect to the topics of composition of functions and the inverse function of a function. Further, I intend to determine how the components of the content knowledge are interrelated, and how they are influenced by the teaching experience.

In the following section I present a general overview of the study.

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1.6. Outline of the Thesis

Through this study I was mainly interested in what preservice teachers and inservice teachers consider as important mathematical knowledge regarding the topics of composition of functions and the inverse function, giving the issue of how to teach the mentioned topics a secondary importance. I was mainly interested in *what* they teach for the two considered topics. I was interested in the mathematical content the preservice and inservice teachers would consider as being important when teaching these topics. The *how* part of the teaching process was given a secondary importance in this study, and was used only to clarify some of the issues that came up during the study. It is my belief and my experience that in order to be able to teach, or to improve *how* to teach one needs to have *what* to teach, or needs to master the content to be taught. A successful Grade 11 mathematics lesson needs at least two components on the part of the teacher: the mathematical content and the methods of transmitting this content to the students. At this level of complexity of the mathematical concepts, in terms of planning the lessons, the mathematical content component might have priority over the methods of teaching. A teacher who possesses a good conceptual knowledge of the topic to be taught is free to think about how to deliver the message to the students, not being worried about any subject matter knowledge surprises. A teacher who has a limited knowledge of a topic, regardless of what kind of teaching methods he/she will use, might restrict the students' acquisition of knowledge to the boundaries that he/she is limited by.

Chapter 2 contains a succinct description of the evolutionary process of the mathematical concept of function, from antiquity to modern days. In this chapter I

emphasize the fact that the function concept developed as a mathematical concept at the same time as calculus. The mathematicians who worked on developing the relatively new branch of mathematics, calculus, struggled for a long time in their quest for a "good" definition of the function concept. This struggle continues in contemporary times, due especially to the widening of the range of mathematical fields where the function concept plays a critical role.

In Chapter 3 I make a comparison of how different mathematical curricula treat the concept of function. The comparison is made between the curricula of five different countries. The function curricula are looked upon from multiple perspectives, such as the age where the concept is formally introduced to students, the way the concept is introduced, the complexity of the language used in the teaching of the topic, and the depth at which the topics related to functions are treated.

Chapter 4 is a presentation of the mathematics education research literature related to the topic of my study. Most of the research mentioned in this chapter is related to the specific topic of the function concept in mathematics education, but it also contains references to more general research, such as the research conducted by Ma (1999) and Shulman (1986) on the importance of teachers' content knowledge in educational research.

In Chapter 5 I present in detail the participants in the study, the methods and instruments used for collecting data and the research questions that my study addresses. Also, Chapter 5 contains a detailed presentation of the theoretical framework proposed by

Even (1990) for analyzing teachers' subject matter knowledge of a certain mathematical topic, and it contains as well a refined version of this framework.

Chapter 6 details the data analysis. The field data are analyzed through a theoretical framework that is a refinement of the theoretical framework proposed by Even (1990). The analysis is conducted in parallel for the two groups of participants, the inter-group and intra-groups differences and similarities being emphasized.

As a follow up of the data analysis, the study concludes with Chapter 7 that contains a brief summary of the findings of the study, the conclusions of the study, and a set of theoretical and methodological contributions.

CHAPTER 2
A SUCCINT NARRATIVE OF THE EVOLUTION
OF THE MATHEMATICAL CONCEPT OF FUNCTION

The theory of functions, compared to other mathematical domains such as

geometry, arithmetic and algebra is a relatively new field in mathematics, but it represents

a central piece in modern mathematics and in mathematics education, through what is

called calculus and mathematical analysis. Gardiner (1982) noted that

> For many students today, calculus is the climax of their encounter with
> elementary mathematics: it exploits almost all the mathematics they have
> previously learnt, (especially algebra, trigonometry, graphs and coordinate
> geometry), and it opens the way to the analysis of a whole host of
> interesting problems (in elementary geometry, in differential geometry, in
> statics and dynamics, in physics — in fact, in any subject where we may
> reasonably assume that one quantity varies more or less smoothly with
> respect to some other quantity)(p. 3).

The most basic question is "What is a function?" Hermann Weyl answered it as

follows:

> Nobody can explain what a function is, but this is what really matters in
> mathematics: A function f is given whenever with every real number a
> there is associated a number b. b is said to be the value of the function f for
> the argument a (Gardiner, 1982, p. 254).

This answer is a little simplistic, but in present scholarship on the topic, there does

not appear to be a universally accepted definition for the concept of function.

Mathematicians are still struggling with defining mathematical functions in a rigorous

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way. The objective of this chapter is to present the evolution of the concept of function through its historical development.

2.1. The Function Concept Before Calculus

The concept of function, in its different representations, appeared in human mathematical activities in the oldest written documents existent in history museums, throughout the world. For example, Egyptian papyruses contain the volume formula of what we call today a truncated pyramid of square base. The volume V of a truncated pyramid can be considered as a function of the height h and the sides of the bases a,

b: $V(h,a,b) = \frac{h}{3}(a^2 + ab + b^2)$. Some Babylonian tablets contain what we call today tables

of values for the basic quadratic function of $f(x) = x^2$. The Greeks were the first to work with the trigonometric functions. They represented these functions as ratios of sides in right angle triangles. Furthermore, the Greeks were the first to draw the graphs of certain curves such as the spiral, the quadratics, the cissoid and the conchoid. According to Kleiner (1993), these curves were defined kinematically. There are numerous examples of this kind, but at this historical stage, the function did not exist as a mathematical concept.

According to Cajori (1938), in Medieval times, the concept of function was still not clear to mathematicians:

> Some of the mathematicians of the Middle Ages possessed some idea of a function. Oresme even attempted to graph the uniformly accelerated motion, which represented the first graphical representation of a physical

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law. But of a numeric dependance of one quantity upon another, as found in Descartes, there is no trace among them (p. 127).

2.2. The Emergence of Differential Calculus: Fermat, Newton and Leibniz

At the end of the sixteenth century and during the seventeenth century, scientists such as Kepler and Galileo, and mathematicians such as Descartes and Fermat, were "playing" with functions and their graphical representation, and with the notion of infinite processes, preparing the way for the emergence of calculus, the branch of mathematics that forced mathematicians to strive for a definition of the function concept. Due to his method of calculating tangents to curves, Laplace called Fermat "the true inventor of the differential calculus" (Boyer, 1968, p. 367), even though differentiation is a process applied to functions, and Fermat in his work, did not use the term "function".

The real breakthrough in the theory of functions was the invention of calculus by Newton and Leibnitz. Part of the object of this new branch of mathematics was represented by the study of different functions and their properties, and made it necessary to define the mathematical concept of function. Newton and Leibnitz did not explicitly define the object of their work, the functions. Newton, due to his background (physics), called the functions that he was working with "fluxions", which were rates of change, and "fluents", or continuously varying quantities (lengths, areas, volumes etc.). At this time, functions were considered to be mostly curves, classified into two major categories: geometrical and mechanical curves.

Leibniz's contribution to the development of the function concept is quite important. He practically gave the name to the differential calculus, *Calculus Differentialis* and to the integral calculus, *Calculus Summatorius* or *Integralis*, and he introduced the terms of co-ordinate and axes of co-ordinates (Cajori, 1938, p. 211). He also established the symbols that we use today for the derivative and the integral, and he "was not responsible for the modern functional notation, but it is to him that the word "function" in much of the sense as it is used today, is due" (Boyer, 1968, p. 444).

2.3 The 18th and 19th Centuries: Struggle for a "Good" Definition

The invention of calculus was followed by a prolific period of mathematical discoveries followed in the 18th century. Mathematicians such as Lagrange, the Bernoullis, L'Hospital, d'Alembert, Taylor, Rolle, and Euler were involved in research, discovery and publication activities related explicitly to calculus and implicitly to functions. At the beginning of the study of functions, in the 18th century, the functions were identified with their representing formulae, which usually were infinite series, that were relatively "well behaved" (Grabiner, 1981, p. 89). With regards to the importance of defining the function concept, Gardiner (1982) suggested that:

> The difficulty inherent in making this transition from specific examples of what we would now call "functions" to an adequately general function-concept was one of the main obstacles in the way of explaining precisely why, when and how the methods of the calculus could be trusted (p. 255).

This explained the struggle for an adequate definition for functions.

Jean Bernoulli tried different notations for functions, the closest one to the notation of our time being fx. He had a vague notion of a function, which he expressed as "a quantity composed in any manner of a variable and any constants" (Gardiner, 1982, p. 270).

The famous debate over the equation of the motion of the vibrating string (the wave equation), which involved d'Alembert, Euler and Daniel Bernoulli, pushed the mathematicians towards the need to clearly define what a function is.

In 1748, in *Introductio in Analisyn Infinitorum*, Euler defined a function of a variable quantity as "an analytic expression composed, in any manner, of that same quantity and of numbers or of constant quantities" (Grabiner, 1981, p. 51). He included polynomials, power series, and logarithmic and trigonometric expressions. He also defined a function of several variables and classified functions as algebraic functions, which involve algebraic operations, and transcendental functions, namely, the logarithmic, trigonometric, exponential of irrational powers, and some integrals. "Euler's *Introductio* was the first work in which the function concept was made primary and used as a basis for ordering the material of the two volumes" (Kline, 1972, p. 405). Sometimes, when working less formally, Euler thought of a function "as the relationship between the two coordinates of points on a curve drawn freehand in the plane" (Boyer, 1968, p. 484).

At the beginning of the 19th Century, Lagrange was convinced by Euler's work that any function could be given as an infinite power-series expansion. He defined functions as being any "*expression de calcul*" into which the variable entered in any way

(Grabiner, 1981, p. 51). In his *Theorie des Fonctions Analytic*, published in 1813,

(Lagrange defined the calculus as the "theory of analytical functions"), he defined a

function of one or more variables as being "any analytical expression useful for

calculations in which variables enter in any manner whatsoever" (Kline, 1972, p. 406).

Cauchy was the mathematician who established the rigorous foundations of

calculus. During his work, Cauchy too faced the obstacle presented by the lack of a

"good" definition of a function. Some functions that represented the solutions of

differential equations from physics and mechanics did not respect the definitions that

considered functions as analytical expressions. The old definition was not encompassing

all the possible types of functions already discovered and studied. In Chapter 1 of

Cauchy's *Cours d'Analyse* (1821) a function is defined as follows:

> When variable quantities are related in such a way that, one of them is
> given one can conclude the values of all the others, one usually thinks of
> these variables as all being expressed in terms of the one, which is called
> the independent variable, and the other quantities expressed in terms of the
> independent variable are what one calls functions of this variable
> (Gardiner, 1982, p. 263).

Even though this definition leaves the reader with the impression that functions

can be totally arbitrary relationships, it becomes clear from Cauchy's work that the

quantities are presumed to be related by some collection of algebraic expressions, which

is not different from the definitions given by Lagrange or Euler. In 1810, Lacroix

published *Traite du Calcul Differential et Integral* in which he formulated the following

definition for the concept of function:

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Any quantity whose value depends on one or on several other quantities, is called a function of these [other quantities], whether one knows or remains ignorant of the operations by which one passes from this [other quantities] to the first (Gardiner, 1982, p. 271)

2.4 Dirichlet and His Definition. Contemporary Trends

Gardiner (1982) claims that all the above definitions were driven by the credo that all functions can be expressed as Taylor series. In the 19th century, Fourier stated and tried to prove that any arbitrary function can be written as a series of sines and cosines. The problem that Fourier encountered was to calculate the coefficients of the expansion. Peter Gustav Lejeune Dirichlet was the mathematician who put some order in Fourier's theory. His first priority was to define the concept of function. When he defined the function, Dirichlet moved away from both of the intuitive images of a function (a curve, mechanical or geometrical, or an analytic expression). Against both these images, he gave

$$f(x) = \begin{cases} 1, x \in Q \\ 0, x \in \overline{Q} \end{cases}$$

the example of what we call today Dirichlet's function:

($f(x) = 1$ for any rational number, and $f(x) = 0$ for any irrational number). This function cannot be represented in the Cartesian plane of coordinates, so it is not a curve; nor can it be expressed as an analytic expression. Dirichlet gave the following definition of a function:

A function $y(x)$ is given if we have any rule which assigns a definite value y for every x in a certain set of points. It is not necessary that y be subject to the same rule as regards x throughout the interval ... indeed, one need not even be able to express the relationship through mathematical operations ... It doesn't matter if one thinks of this [correspondence] so that different parts are given by different laws or designates it [the correspondence] entirely lawless (Davis & Hersh, 1981, p. 264).

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A variation of this definition is used in most of today's BC high school textbooks. In the third chapter of this study I present a comparison between different countries' curricula with regards to the function concept. This chapter contains a few different function definitions, found in different textbooks from different countries. For example, one of the Grade 10 textbooks from BC defines the function concept as: "A function is a special relation. **A function** is a set of ordered pairs in which, for every x, there is only one y" (Knill et al., 1998, p. 232). This definition is a simpler variant of the following definition that uses more sophisticated mathematical language:

> A function f is an application of the set **A** in the set **B**, in other words it is a set of ordered pairs $(x, y) \in f$, with $x \in X \subseteq A$, $y \in Y \subseteq B$, with the property that for every $x \in X$ there is exactly one correspondent $y \in Y$. The set **X** is called the domain of the function f, and the set **Y** is called the codomain of the function (Gellert, Kustner, Hellwich, and Kastner, 1975, p. 122).

The last given example for the definition of the function concept can be called "a modern" definition. With this definition though, the controversy over the concept of functions did not cease to exist. The set of functions was divided into two main subsets: functions of a real variable and functions of a complex variable. Weierstrass ordered and systematized Cauchy's work regarding analytical functions of a complex variable, and gave a definition for the functions of a complex variable: an analytic function $f(x)$ is the totality of elements obtained from a given one by means of successive direct continuations. This definition was clarified and completed later on by Volterra and Poincare. For the functions of a real variable, the definition given by Dirichlet still plays an important role in our time, but some leading mathematicians think that Dirichlet's

definition is overly "large", not restrictive enough, or is devoid of all meaning. On the other hand, Weierstrass's definition seems for some to be too restrictive. Schnitzer and Stillwell compare the actual controversy over the definition of the function concept with the situation existent two or three centuries ago: "It thus became clear that, in our own time, the controversy about the vibrating string has been renewed in another light and with a different content" (Schnitzer & Stillwell, 2002, p. 29).

In conclusion, there still are new pages to be added to the history of the function concept. As Kleiner (1993) observes: "Textbook definitions or descriptions of function have varied with time, context and level of presentation" (p. 183). Apparently, these definitions and descriptions are still changing in the present, at least in the circle of advanced mathematics. The concept of function is central to both today's mathematics and mathematics education, and following the history of this concept, the past and present struggle for defining this concept, it seems that all the difficulties encountered by students and teachers who have to learn and teach this topic have a natural root into the evolutionary process characteristic for this mathematical concept.

CHAPTER 3
HOW ARE FUNCTIONS TREATED?
A VIEW ON HOW CURRICULA FROM DIFFERENT COUNTRIES
REGARD THE FUNCTION TOPIC

The concept of function is one of the most important concepts taught in the main

British Columbia high school curriculum, namely, the Principles of Mathematics courses.

Besides being part of an intended mathematical enculturation, and leaving aside its

intended utilitarian character, most of the mathematics taught in this sequence of high

school courses is, for the students, a preparation for the encounter with calculus.

The notion of function is central to calculus (and not just to calculus) and for that

reason, a major part of the BC mathematics high school curriculum is an actual study of

different classes of functions, such as: linear or first degree functions, trigonometric

functions, quadratic function, polynomial and rational functions, exponential and

logarithmic functions. Beside special classes of functions, the general properties and

operations with functions, such as composition of functions and finding the inverse

function of a function, are also part of the curriculum.

Taking into consideration the importance of the role played by the function

concept in BC mathematics curriculum, a question arises: Is this topic treated in similar

fashion in other countries' high school mathematics curricula? In the following sections I

compare mathematics curricula from different countries, in terms of how the function concept is treated, with special emphasis on the two topics of composition of functions and the inverse function of a function.

3.1. The Countries for Comparison

The importance played by the topics related to the notion of function in the BC mathematics high school curriculum was one of the factors that motivated this research. In this chapter, I present a comparison of how the function concept and the topics of composition of functions and the inverse function of a function are treated in several curricula around the world. The comparison is drawn between the curricula of two Easter European countries — Romania and former Yugoslavia — and two Asian countries — Singapore and South Korea. The focus will be on the way the functions are introduced to the students, and on the way the composition of functions and the inverse function of a function are treated in the different curricula.

Most of my affirmations are based on the information found in high school mathematics textbooks and on discussions that I had with colleagues who have taught in the considered countries or who are still teaching there. I also assume that the textbooks that teachers use at a certain time in their classrooms are in most cases fairly accurate reflections of the intended curricula.

The curriculum is a body of information that changes quite often. Its changes are mostly concerned with the topics to be treated in the classrooms, and less with the way in which these topics will be treated. In all the curricula I examined, the notion of function

plays a similarly important role in high school mathematics. There are some differences with regards to the proportion of the curriculum that the topics dealing with functions occupy, due mainly to the complexity and the weight of the high school mathematics curriculum in the different countries. For example, in the Romanian curriculum, functions are a relatively smaller part of the curriculum compared to BC, but that is a result of the fact that the mathematics curriculum in that country deals extensively with plane and space geometry, trigonometry and complex numbers, and some notions of linear algebra and combinatorics. Mathematics seems to have been allocated a larger part of the yearly instructional time in Romanian high schools compared to the time allocated for mathematics in BC.

3.2. The Introduction of Functions

In this section I present the time and the ways that the notion of function is introduced to the students in the considered curricula. In BC, the functions are formally introduced to students in Grade 10, when a chapter of the textbook is dedicated to the definition of the function concept and to some relatively simple calculations related to functions, such as calculating the value of the function in different points, and graphing functions. The formal definition of the function emphasizes the uniqueness character of the function, introducing the concepts of domain and range: "Another way to define a function is that, for each element in the domain (first elements), there is exactly one element in the range (second elements)" (Knill, et al., 1998, p. 232). Implicitly, different representations of a function are given: formula, table of values, graphs, arrow diagrams, word stories.

In the Yugoslavian curriculum, functions are formally introduced in Grade 9. The notion of function is introduced through examples of word stories and arrow diagrams, followed by a formal definition: "A function is a mapping of set A in set B, mapping that associates to each element from set A exactly one element from set B. The set A is called the domain of the function f and the set B is the set of the values of the function f. The mathematical notation for the function f is $f : A \rightarrow B$" (Legisa, et al., 1989, p. 98). The textbook also defines $f(A)$ as the image of the set A through the function f, and being a subset of set B it corresponds to the notion of range in the BC textbooks. The definition of a function is followed by examples of tables of values and some special functions, such as the identity function, the constant functions and the opposite function. Next, the textbook introduces some special classes of functions: injective functions, surjective functions, and bijective functions. The last class of functions is the corresponding name for one-to-one functions in North America. These classes of functions will serve as an introduction for the future study of the inversability of functions.

The Romanian curriculum introduces the notion of function at the Grade 9 level, after a brief section on set theory. The definition is very similar to the one in the Yugoslavian text: "We call a *function defined on the set A with values in set B* any law (procedure, convention etc.) f that for each element a of the set A associates an *unique* element from the set B, element called *f(a)*" (Nastasescu, Nita & Rizescu, 1993, p. 65). The textbook emphasizes the importance of the key elements of the definition: the set A called domain, the set B called the codomain, the law or procedure that associates the elements of the two sets, and the uniqueness character of the functions. The textbook

dedicates an entire section to the different ways of defining a function, encompassing all the representations illustrated in the BC texts, using a slightly higher level of mathematical language and notations. All functions are defined by introducing their domain and codomain (a set of values, which can be larger then the range): $f : A \rightarrow B$, similar to the Yugoslavian text.

In Singapore, in Grade 9, the students encounter the quadratic function and linear functions, without the formal introduction of the concept of function. The formal introduction of functions happens in Grade 10, and the way functions are introduced is similar to the way functions are introduced in BC. First, the students are acquainted to the notion of relation. With the help of relations, the textbook introduces the formal definition of functions: "A relation in which every element (member) in the domain has a unique image in the range is called a **function**" (Lee, 1999a, p. 97). Following the definition, the textbook provides some examples of functions defined through arrow diagrams and formulae, with calculations of the value of the function in different points of the domain. Compared to the previous curricula, this curriculum treats the introduction of functions in a very condensed manner, because the concept of function is reintroduced at the Grade 11 level. At this time, the function is defined as "... a relation in which **every** element in the domain has a **unique** image in the codomain" (Lee, 1999b, p. 76). A separate section of the textbook is dedicated to the range of the function, which is defined as being "the set of values $f(x)$". It is emphasized that the range is often included in the codomain, and is not necessarily the same as the codomain.

The Korean curriculum introduces functions to the students in Grade 8, as "story problems" and linear equations. The formal definition is introduced in Grade 10. At this age, the students learn about the notions of domain, codomain and image of a function. The definition and the examples use mathematical notations from the set theory, similar to the notations in the Romanian text.

3.3. Clarification of Terminology

The terminology found in the different textbooks that I took in consideration varies considerably. In this section I define the terms used in different textbooks and I clarify some of the terminology by providing examples.

1. Injective functions: "Let f be a function defined on the set **A** and taking values in the set **B**. Then f is said to be an injection (or an injective mapping or an embedding) if, whenever $f(x) = f(y)$, it must be the case that $x = y$. Equivalently, $x \neq y$ implies $f(x) \neq f(y)$" (Weisstein, 2005b).
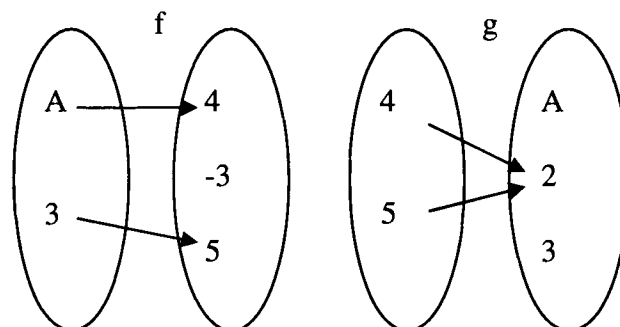
Fig. 1. Example of injective function and an example of non-injective function.

The function f is injective since all elements of the domain have different images

in the codomain of the function f, and function g is not injective, since $4 \neq 5$, but

$g(4) = g(5) = 2$. Another of example of non-injective function is the quadratic function

$f(x) = x^2$, with domain $\Re$, where for example $f(-2) = f(2) = 4$.

2. Surjective functions: "Let f be a function defined on the set **A** and taking values

in the set **B**. Then f is said to be a surjection (or a surjective map) if, for any $b \in B$, there

exists an $a \in A$ for which $b = f(a)$. A surjection is sometimes referred to as being "onto"

(Weisstein, 2005d).
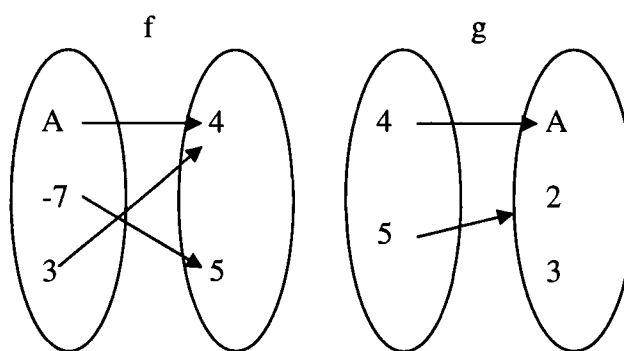
Fig. 2. Example of surjective function and an example of non-surjective function.

The function f is surjective, since all the elements of the codomain have at least

one correspondent in the domain. The codomain of the function g has one element -3 –

that is not the image of any elements of the domain.

3. Bijective functions. "A map is called bijective if it is both injective and surjective" (Weisstein, 2005a). "A and **B** are in one-to-one correspondence" is synonymous with "**A** and **B** are bijective" (Weisstein, 2005c).
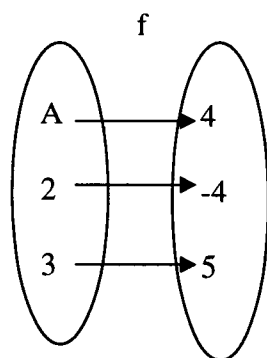
Fig. 3. The function f is bijective or one-to-one.

The function f is bijective because it is an injection and a surjection at the same time: all elements of the domain have a different correspondent in the codomain, and all elements of the codomain are images of at least on element of the domain.

The term *codomain* designates the set within which a function has its values. The North American term of *range* is equivalent with the image of the domain through a certain function. The codomain is a set that contains the range, but it can be larger then the range.

The terms of *injective, surjective* and *bijective* are terms that are used less or not at all in the North American mathematics vocabulary. For this reason, with the exception of the following section, I do not use them in this study. The term that I use is *one-to-one*, to designate bijective functions, the category of functions that have inverse functions.

3.4. The Composition of Functions and the Inverse Function of a Function

I treat these two topics together, because in some of the curricula that I considered they are regarded as one topic, or they are seen as interlinked, one being a subtopic of the other.

In BC, the topics of composition of functions and the inverse function of a function are part of the Grade 11 curriculum. The composition of functions is treated quite extensively, from a computational point of view, the examples using polynomial and rational functions, arrow diagrams, ordered pairs or radical functions. In terms of the significance of this operation with functions, and its main differences compared to previously studied operations (addition, subtraction, multiplication and division), the textbooks used in BC classrooms lack most of the information. The property of composition of function of not being commutative is presented in the textbook as an observation after an example. The emphasis is on the fact that sometimes the order of composition *does not matter*. The importance of the domain and range, for composition to be possible to be completed, is not discussed in the BC textbooks at all.

The inverse function of a function is treated as a different topic than the composition of functions, and not as a subsequent topic: "**Inverse functions** are a special class of functions that *undo* each other" (Knill et al, 1999, p. 261). The fundamental relation for the inverse functions (when composing inverse functions, the result is always the identity function) is mentioned in the text as a computational exercise of determining if two functions are inverses for each other or not.

The way the BC textbooks treat the above considered topics is in accordance with the curriculum prescribed by the BC's Ministry of Education. The Integrated Resource Packages (IRP) (BC Ministry of Education), which is the main BC Ministry of Education document that guides what students should be taught, mentions only the computational aspect of the composition of functions and the inverse function of a function. The Prescribed Learning Outcomes (PLO) for this section of the curriculum are that the students are expected *to perform* operations with functions and compositions of functions, and that the students will *determine* the inverse of a function.

In the Yugoslavian curriculum, the composition of functions and the inverse function of a function are also taught in Grade 11. Given how functions are defined in the textbook, and the definition of surjective, injective and bijective functions, the composition of functions starts with a discussion/analysis of the case when two functions can be composed, according to their domain and range. The inverse function of a function is introduced by definition, and the fundamental relation between the function and its inverse is given as a special consequence of the definition. In defining the inverse function of a function, the Yugoslavian text starts with the condition that the initial function has to be bijective (one-to-one): "If function $f : A \rightarrow B$ is a bijective function, and x is an element of the domain A, and $y = f(x)$ is an element of the codomain B, we define a function f^{-1} with domain B and codomain A, such that $f^{-1}(y) = x$. This new function is called the inverse function of function f, and it is a bijective function" (Micic, 1989, p. 56). The Yugoslavian text also proves the uniqueness of the inverse function.

The Romanian curriculum treats the composition of functions and the inverse

function in Grade 9, immediately after the definition of functions. As in the Yugoslavian

textbook, the Romanian textbook emphasizes the fact that for two functions to be

compatible for composition, there has to exist certain compatibility between their

domains and codomains. The first examples use arrow diagrams, with the intent of

stressing the importance of the domains and ranges of the two functions. Related to this

topic, the curriculum mentions two properties of the composition of functions: the

composition of functions is associative and non-commutative. The inverse function is

introduced in the following way: "A function $f : A \rightarrow B$ has an inverse if there is a

function $g : B \rightarrow A$, such that $f \circ g = 1_B$ and $g \circ f = 1_A$, where 1_A and 1_B are the

identity functions defined on the sets A and B" (Nastasescu, Nita & Rizescu, 1993, p. 64).

The uniqueness of the function g is proved in the textbook, and g is called the inverse

function of f, and is noted f^{-1}. The discussion related to the existence of the inverse

function is treated by the Romanian curriculum as a theorem: "A function f has inverse

only and only if it is bijective" (Nastasescu, Nita & Rizescu,1993, p. 65). The lesson

related to the inverse function ends with the discussion on the relation between the graphs

of inverse functions.

In Singapore, the inverse function is mentioned first in Grade 10, after the section

defining the functions. At this level, the inverse function is treated only as a

computational exercise, without regard for anything else. The text does not mention the

composition of functions as a topic preceding or following inverses. In Grade 11, the two

topics of composition and inverse function are taught again. The composition of functions

is treated through several examples of arrow diagrams and formulas, and the property of being non-commutative is mentioned by the text: "In general, the composite functions $f \circ g$ and $g \circ f$ are different functions. Composite functions are not interchangeable" (Lee, 1999b, p. 105). The inverse function is treated as a separate lesson by the text, but it is a follow up of the lesson on composition of functions. The fact that the inverse function exists only if the initial function is one-to-one is clearly stated: "f^{-1} exists as a function only when f is a one-to-one function" (Lee, 1999b, p. 109). The fundamental relation of inverse functions is stated on the next page, as the starting point for finding the inverse of a function. The topic of inverse functions ends with the discussion on the graphs of inverses.

The composition of functions and the inverse function are topics of the Grade 10 Korean curriculum. The emphasis is on the inverse function, and here again, it is clearly stated that we can talk about the existence of the inverse function only when the functions are one-to-one. Using the non-commutative character of the composition of functions, the textbook proves an interesting functional relation: $f^{-1}(g^{-1}) = (g(f))^{-1}$. Like most of the texts mentioned above, the Korean book wraps up the topic with graphs of inverse functions.

3.5. Final Remarks

Through an analysis of the ways the topic of functions are treated in the considered curricula, certain similarities become apparent, but also some differences. However, it is these differences that seem more relevant to the mathematical topics that

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the present study addresses, namely the composition of functions and the inverse function of a function. In three of the countries studied, functions are taught at the Grade 10 or 11 level. In Yugoslavia, the functions are formally introduced in Grade 9, but the topics of composition of functions and the inverse function are treated in Grade 10. In Romania, the curriculum treats the considered topics in Grade 9. All curricula give similar definitions for functions, emphasizing the uniqueness character of the function concept. None of the texts seem to stress the arbitrary character of functions (the fact that the domain and range do not have to be sets of numbers).

In relation to the composite functions and inverse functions, all texts treat the computational aspect of the topics, but in relation to the essential features of these topics there are some significant differences. The BC curriculum treats these topics primarily from a computational point of view, while the other curricula imply some aspects of analytical processes regarding the existence of composite and inverse functions. The computational character of the BC curriculum is also stressed by the number and "mathematical quality/complexity" of exercises the textbook provides as practice for students: the BC textbook provides 102 exercises for the topic of inverse functions, while the Singapore text provides 18, and the Romanian text provides 22. It seems that the BC curriculum is inclined more towards quantity than towards quality, when treating the topics of functions.

A major difference exists between the complexity of the mathematical language and notations between the Romanian, Korean, and Yugoslavian texts on one side, and the BC text on the other. The first three texts use a relatively sophisticated and exact

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language and system of mathematical symbols. The BC text is written in a lower level mathematical language. The Singapore text is somewhere in between the two groups, in terms of the language it uses.

As a conclusion, the way the textbooks are written seems to be a fairly accurate reflection of how functions are taught in the considered countries, and as an extension, how mathematics is treated in these same countries. In BC, mathematics has a more "popular" character, it tries to post an utilitarian shade, where in the other countries considered in this study, mathematics is treated more as a very exact science, taught for its own sake, using its specific language.

CHAPTER 4
FUNCTIONS IN MATHEMATICS EDUCATION RESEARCH

4.1. General View on the Research on the Topic of Function

The mathematics education research related to the topic of functions can be classified into three main categories: research that focuses on theoretical aspects of the development of the concept of function, research that investigates the students' understanding of the notion of function, and studies that examine teachers' and preservice teachers' views on different aspects related to the notion of function. There are no clear boundaries between these types of research, and many research papers have elements of two or even all three of these categories. Research papers that fall in the second category are by far more numerous than research papers from the other categories.

Schulman (1986) identified teachers' subject matter knowledge as the "missing paradigm" (p.4) in the research on teaching. In mathematics education it is relatively recently, however, that teachers' understanding of the notion of function became of interest for the research community. This tendency is illustrated by the following citations:

> Interest in teachers' subject matter knowledge has arisen in recent years (Even, 1990, p. 521).

There is a considerable and growing body of research on the nature and learning of the function concept. However, most of this research has focused on students' conceptions of functions-- little had addressed teachers' cognition or the ways in which teachers express their idiosyncratic knowledge of functions in the classroom (Norman, 1992, p. 215).

Although a rich and extensive body of research focuses on students' understanding and learning about functions, few studies investigated teachers' conceptions of functions (Loyd & Wilson, 1998, p. 251).

This is the field where my study is situated. In the following sections I present an overview of the research done in all three categories, reiterating the fact that these categories overlap.

4.2. Theoretical Perspectives for the Development of the Function Concept

In this section I present a few research papers containing theoretical orientations that guide the research on the function concept.

Sierpinska (1992), Sfard (1992) and Dubinsky & Harel (1992) present different theoretical perspectives for the development of the function concept. Sierpinska introduces the notion of epistemological obstacles, which are "common in the frame of some culture, whether present or past and thus seem to be the most objective obstacles to a new way of knowing" (Sierpinska, 1992, p. 27). She also discusses from a theoretical point of view what she calls the fundamental acts of understanding: identification, discrimination, generalization and synthesis. From these four perspectives, she explains what it means to understand a mathematical concept, with direct reference to the concept of function. Sierpinska uses the history of the concept of function in parallel with the

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study of students' difficulties in order to bring together a set of pedagogical suggestions. Along these lines, the author elaborates:

> Histories of a mathematical concept are usually presented as if the concept's development followed a smooth curve with positive gradient. Learning cannot be thus modelled. At greater cognitive depths catastrophes occur ... These acts often consist in abrupt breaking with a certain way of knowing, in overcoming an obstacle (Sierpinska, 1992, p. 58).

Sfard (1992) and Dubinsky & Harel (1992) present two theoretical notions of the mathematical universe: the notion of process and the notion of object. Sfard makes a comparison between the notions of structural conception and operational conception of mathematical entities. The structural conception views the mathematical entity as an object, as a static construct. The operational conception considers the mathematical entity as a computational process, as a dynamic construct. Sfard sees the two aspects-- process and object -- as "different faces of the same thing rather than as totally distinct, separate components of the mathematical universe" (Sfard, 1992, p. 61). Sfard considers the two modes of thinking to be "ostensibly incompatible", but complementary. The study specifically addresses the operational and structural modes of thinking regarding the notion of functions. Like Sierpinska, Sfard reviews the historical development of the concept of functions, and then she investigates psychological aspects of students' transition from an operational to a structural conception of function. In her findings, Sfard claims that at a certain moment, students develop a pseudo-structural conception of functions. An example when students possess a pseudo-structural conception of functions is the stage when they identify functions with algebraic expression. This can be an indication for operational conception — computational algorithm — but it can also be seen

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as proof of structural conception if the algebraic expression is interpreted as a static

relation between the variables. The conclusion that Sfard draws from her study is that

structural thinking can be externally stimulated, at least to a certain degree. She maintains

that there will also be students for whom the structural conception will remain out of

reach, regardless of the external stimulus.

Dubinsky & Harel (1992) limit their study to the analysis of what the concept of

function is. The researchers introduce and define what they call descriptors of a function

concept: prefunction, action, process and object. The goal of their research was to

determine how students responded to a certain instructional treatment in terms of their

advance from an action conception of function towards a process conception of function.

To determine this process of learning, Dubinsky and Harel reference students' knowledge

to four factors:

> 1. restrictions students possess about what a function is;
> 2. severity of the restriction;
> 3. ability to construct a process when none is explicit in the
> situation, and students' autonomy in such a construction;
> 4. uniqueness to the right condition, confusion with 1-1 (Dubinsky
> & Harel, 1992, p. 87).

The researchers present the analyses of four interviews out of the 13 conducted.

Some of the findings were: some of the restrictions are partially overcome by the

students; a strong process conception appears to help students to avoid the confusion

regarding the uniqueness to the right; students can describe a situation as a function if

manipulations can or cannot actually be performed.

4.3. Research Focused on Students' Understanding of the Concept of Function

This body of research is the most extensive category of research in mathematics education literature that deals with the concept of function. The research literature treats various aspects of students' understanding of functions. The summary presented below should be treated as exemplifying, rather than comprehensive.

Yerushalmy (2000) presents a study in which the techniques for function-based problem solving of two low achieving students are compared to the techniques of other students, over a period of three years. The students are part of an algebra program using a function approach, with an intensive use of graphing technology. The researcher concludes that the "choice of function as the main concept around which algebra is learned, makes the shift to inquiry natural and even not too demanding" (Yerushalmy, 2000, p. 144). This research seems to stress the importance of graphing technology for the acquiring of some of the attitudes displayed by the students: the students "dedicated time for planning, exhibited ability and tendency to speak mathematically, and they were fusing higher order thinking skills, multiple uses of representation" (Yerushalmy, 2000, p. 144).

In another research paper, the same author describes how a group of seventh graders understand functions through modelling functions of two variables. This group of students were extensively using graphing calculators in their mathematics learning, a fact that the researcher considered important in the results of his study: "two major aspects of using technology are embedded in the rationale and goals of the study, and these had an

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impact on the way the students in the study created the "things" called mathematics" (Yerushalmy, 1997, p. 464).

The impact of the graphing calculator on students' understanding of the function concept is also the object of the research conducted by Wilson and Krapfl (1994), Beckmann et al. (1999) and O'Callaghan (1998). Wilson and Krapfl (1994) conclude their article by enumerating some of the benefits of the use of the graphing calculators, as well as the problems created by this issue. They also suggest some directions for future research. The findings all these research papers suggest similar conclusions regarding the technology and students' understanding of functions: it fosters student development of richer concepts of function; improves students' attitudes and reduces their anxiety toward mathematics.

The research of Keller and Hirsch (1998) tried to determine if students have any preferences for the different representations of a function (tabular, graphical, formula), and if that is the case, what this preference is influenced by. Their conclusion is that students have preferences, which are influenced by the context of the task. In real life problems, the students prefer to use graphs or tables, and on pure mathematics problems, students use mostly formulae.

Knuth (2000) conducted a study with the objective of finding students' capacity for solving algebra problems using functions given by a formula or by a graph, according to the practicality of the solution. The problems proposed to the students had both "algebraic" (variable manipulation) and "graphical" (reading the graph) solutions, but for each problem one of the solutions was shorter and more elegant. The conclusion of the

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study is that most of the students display only a superficial connection between the graphs and the formulae of functions.

The research of Breidenbach, Dubinsky, Hawks and Nichols (1992) can be classified under the categories of both theory and student understanding. The declared goal of their study was to make two points:

> First, college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept; and second, an epistemological theory we have been developing points to an instructional treatment, using computers, that results in substantial improvements for many students (Breidenbach et al., 1992, p. 247).

The researchers noticed a significant improvement in what they called students' "process conception" of the function concept. The writers attributed this improvement to the fact that the students who took part in the study were subjects of an instructional treatment which used computer assisted learning: "We feel that there are some indications that progress occurred because of the particular instructional treatment that we used" (Breidenbach et al., 1992, p. 277). The results of this study had as consequence a revision of the epistemological theory that the researchers tried to develop.

Slavit's work (1997) has a more theoretical flavour, but it also discusses students' understanding of the concept of function. The researcher presents an "alternate perspective for utilizing the action/process/object framework when discussing student development of conception of function" (Slavit, 1997, p. 259). The theory proposed by Slavit incorporates a property-oriented perspective of functions. This orientation deals with learners' gradual awareness of specific functional properties. Students are exposed

to various learning experiences related to the concept of functions, and as a consequence they develop a concept image of a function as a related set of procedures and properties. These procedures and properties make possible the understanding of functional properties such as zeroes, concavity, and asymptotic behaviour. Slavit claims that "asserting that someone 'has' a property-oriented view of functions is not completely unambiguous" (Slavit, 1997, p. 266). The ambiguity comes from the difficulty to determine if a student is aware of a sufficient number of functional properties, and if he/she is able to use these properties to form an adequate conception of function as a mathematical object possessing these properties. The property-oriented view involves the ability of students to realize the equivalence of procedures that are performed using different representations (e.g. to solve f(x)=0 and to find x-intercepts are equivalent procedures), and it involves students' ability to generalize procedures across different classes of functions (e.g. finding zeroes of linear and quadratic functions). Slavit claims that under certain circumstances students can develop a property-oriented view of functions. The empirical data to support this claim were provided through a field study that was conducted over one year in a high school algebra course. The study comprised questionnaires, tests, and interviews.

Two studies, conducted by Zbiek (1998) and Zazkis (1992) examined how college students apply their mathematical knowledge on functions in apparently non-mathematical situations. The participants in Zbiek's (1998) study were 13 prospective secondary mathematics teachers enrolled in an elective mathematics course. The researcher explored the strategies used by the participants "to develop and validate

functions as mathematical models of real-world situations" (Zbiek, 1998, p. 184). The study concluded that most of the students did not break away completely "from total dependence on goodness-of-fit values"(Zbiek, 1998, p. 200), which means that even students familiar with different kind of functions did not use their mathematical knowledge in the given modelling, practical situation.

The study conducted by Zazkis (1992), is in some way similar to the research done by Zbiek: "the study investigates the mathematical behaviour of college students in a problem situation that requires them to find the inverse of a given compound element" (Zazkis, 1992, p. 179). The participants in the study were 30 college students, 15 mathematics majors, and 15 elementary education majors. All the participants were interviewed in order to determine their process of problem solving. The findings of this study are similar to the results of Zbiek's study. Even "mathematical literate" students did not apply their mathematical knowledge to solve the problems posed by the researcher: "the majority of the students failed to apply their previous knowledge" (Zazkis, 1992, p. 189). Most of the students used what the author called their "naïve knowledge" instead of using their formal mathematics knowledge, because they did not perceive the problem given to them as a mathematical problem. The study conducted by Zazkis addressed the capability of the students to apply their knowledge regarding the inverse function of a function in a problem-solving situation. In the next section I present explicitly research focused on students' knowledge on the composition on functions or the inverse function of a function.

4.3.1. Research Addressing Students' Knowledge of Composition of Functions and the Inverse Function of a Function

The topics of composition of functions and inverse function of a function are two subjects for which there are quite a limited number of references in the literature. One of the few research papers on students' understanding of composition of functions or the inverse function of a function is an article by Vidakovic (1996). She investigates how university students enrolled in a calculus class are able to work with the concept of inverse function, and how a computer environment might enhance students' ability to understand this concept. Vidakovic reported part of a study conducted with five individual students and five groups of students that were working together in an experimental calculus course.

Based on general theory, observations of students, and her own understanding of the inverse function of a function concept, the researcher derived a description of a construction processes for the developing schema of the inverse function of a function. The author calls the construction process for the developing schema "genetic decomposition". Based on the genetic decomposition for the concept of inverse function of a function, the goal of the study was to propose an instructional treatment.

The theoretical basis of the study was the general Piagetian theory on the development of knowledge, with its interpretation and implementation given by Dubinsky (1991), who developed a theoretical perspective on the learning of mathematics. This theoretical perspective "represents an extension and implementation of Piagetian theory

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of cognitive development to college level students and to more advanced mathematical

topics" (Vidakovic, 1996, p. 299).

In developing the instrument for data collection (she used clinical interviews),

Vidakovic used a number of assumptions, based on the fact that the students encountered

the inverse function in high school mathematics courses, as well as in the calculus course.

One of these assumption states: "the students would be able to give the definition of a

composition of two functions and the definition of an inverse function" (Vidakovic, 1996,

p. 304). As a result of the study, Vidakovic designed an instructional treatment that might

help the students "to go through the steps of reflective abstractions which appear in a

genetic decomposition of the inverse function" (Vidakovic, 1996, p. 311). The treatment

consists of a series of activities using the computer programming language ISETL, with

the hope to instil in students a better understanding of the composition of functions and of

the inverse function of a function concept.

On the same topic of inverse function, Snapper (1990) suggested that the inverse

functions should first be introduced and explained to students at a set-theoretic level. This

implies that the domains and ranges are not subsets of the real numbers set, but they are

sets containing arbitrary objects. Snapper's approach relies on the arbitrariness character

of functions and on multiple representations of this mathematical concept.

Related to students' learning of composition of functions, Ayers, Davis, Dubinsky

and Levin (1988) conducted a study on how computer activities can help students to

understand this topic. The goal of the study was to determine if teaching students

composition of functions with the help of computers would be more effective in inducing

what Piaget called *reflective abstractions*. The term refers to the cognitive process by which a physical or mental action is reconstructed and reorganized on a higher level of thought and so that it becomes understood by the learner. The authors consider the reflective abstractions as key to cognitive construction of logico-mathematical concepts.

The study was conducted on three groups of college students enrolled in a calculus course. Two groups (13 students) were taught in a computer environment, and the other group of 17 students were taught with "paper and pencil". The data collection instruments were two sets of tests: a pre-treatment test, and a post-treatment test. The pre-treatment test scores were comparable for the two groups, with a slightly higher average score for the first two groups of students. The post-treatment test showed a significant higher average score for the groups taught with the help of computers. The conclusion of the researchers was that a computer enriched learning environment is more effective in inducing reflective abstractions in constructing the concept of function and the composition of functions, compared with a traditional pen and paper learning environment.

4.4. Teachers' Subject Matter Knowledge of the Concept of Function

An important part of the mathematics education research literature focuses on determining conceptions and misconceptions among teachers and preservice teachers regarding the concept of functions. Since my research is oriented towards teachers' understanding of the function concept I review some of the work done in this direction. This section contains two subsections. The first subsection treats research conducted by

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two authors, Ma (1999) and Shulman (1986), on the importance of the subject matter

knowledge of teachers, in general, in the process of teaching/learning, without special

reference to the function concept. The second subsection contains research that addresses

specifically teachers' subject matter knowledge related to the function concept.

4.4.1. Research Literature on the Role of Teachers' General Subject Matter Knowledge

Liping Ma (1999) describes a study in which two groups of teachers, one group of

American teachers and one group of Chinese teachers, are compared in terms of their

subject matter knowledge. In order to be able to compare the groups, Ma defined what

she calls Profound Understanding of Fundamental Mathematics (PUFM) as follows: an

understanding of the terrain of fundamental mathematics that is deep, broad, and

thorough: "It is the awareness of the conceptual structure and basic attitudes of

mathematics inherent in elementary mathematics and the ability to provide a foundation

for that conceptual structure and instil those basic attitudes in students" (Ma, 1999, p.

124).

Following this definition, she explained what she understand by depth, breadth,

and thoroughness: depth is shown by the connections between a topic and more

conceptually powerful ideas of the subject; breadth appears when a topic is connected

with other topics of similar or less conceptual power; and thoroughness is the ability to

"pass through" different parts of the field, to "glue" them together. Next, Ma described

what are the main characteristics of teaching with PUFM:

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1. Connectedness — displayed by teachers that connect mathematical procedures and concepts, when the learning is not fragmented, but students will learn a unified body of mathematics knowledge.

2. Multiple perspectives — teachers can appreciate the value of different facets of an idea or the different approaches to a solution, being able to explain the advantages and disadvantages that they present; students are led to a flexible understanding of the discipline.

3. Basic ideas — teachers display attitudes and are aware of simple but powerful basic mathematical concepts and principles (e.g. arbitrariness and univalence for functions); teachers revisit and reinforce these concepts, and students are guided to conduct real mathematical activities.

4. Longitudinal coherence — teachers know and are able to build the proper foundation for the material to be studied in the next grades; they are aware of, and they review crucial concepts that students learnt in previous grades.

Shulman (1986) argued for the importance of teachers' subject matter knowledge in the general process of teaching. Shulman compared two different sets of standards that North American teachers needed to meet in order to be allowed to teach in different American states. Each of the two standards was applied at different times in the history of North American public education to certify, evaluate or review teachers' professional qualifications, and they contained different topics that teachers were tested on. The first standard was used in late 1800's and the second one was used in the 1980's.

In 1875, the California State Board examined elementary teachers in 20 areas of expertise. Some of these areas were: written arithmetic, mental arithmetic, theory and practice of teaching, algebra, written grammar, orthography, and vocal music. Except for the criterion of theory and practice of teaching, all the other criteria were related to subject matter knowledge. The test that the teachers wrote was out of 1000 points, out of which only 50 were given for questions related to the theory and practice of teaching. At that time, the most important part of teachers' preparation was subject matter knowledge.

The second standard mentioned by Shulman is made out of the categories that an American state used for teacher review and evaluation in the 1980's. The categories were: organization in preparing and presenting instructional plans, evaluation, recognition of individual differences cultural awareness, management. This standard did not contain any criteria for the teachers' subject matter knowledge. Comparing the two standards, Shulman asks: "Where did the subject matter go? What happened to the content" (Shulman, 1986, p. 5)? The two standards are fairly opposite in what they illustrate to be the essential conditions to become a teacher.

Writing about contemporary research on teaching, research that determines the policies and standards for evaluating teachers' proficiency, Shulman said:

> The emphasis is on how teachers manage their classrooms, organize activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of their questions and judge general student understanding. What we miss are questions about the *content* of the lessons taught, the questions asked and the explanations offered. (p. 8)

Shulman found the explanation for this shift in the focus of evaluating teachers "in their [education researchers'] necessary simplification of the complexities of classroom teaching" (Shulman 1986, p. 6). He said that the researchers "ignored one central aspect of classroom life: the subject matter" (Shulman, 1986, p. 6). Shulman and his collaborators referred to the absence of focus on subject matter among the various research paradigms for the study of teaching as the missing paradigm problem.

Intrigued by the suspicious cleavage between the subject matter knowledge and the pedagogical knowledge that characterized the policies that guided teachers' evaluation/training in North America in the last century, the researcher took a trip thorough the history of the European Medieval universities. His purpose was to determine if in the past, these two aspects of teaching were as far apart as they seemed to be in the last century. Shulman concluded that at their origins, European universities interwove pedagogical knowledge and subject matter knowledge: "The universities were, therefore, much like normal schools: institutions for preparing that most prestigious of professionals, the highest level of scholar, the teacher" (Shulman, 1986, p. 7). This is further illustrated by the origins of the words *master* and *doctor,* the highest university degrees, which meant teacher or professor. With this finding in mind, Shulman continued with a plea for the importance of teachers' content knowledge in the success of the teaching/learning process, without renouncing the importance of the pedagogical knowledge:

> Mere content knowledge is likely to be as useless as content-free skill. But to blend properly the two aspects of a teacher's capacities requires that we pay as much attention to the content aspects of teaching as we have

recently devoted to the elements of teaching process (Shulman, 1986, p. 8).

Shulman divided content knowledge into three categories: subject matter knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge (CK). In his description of the subject matter knowledge, the author emphasized that teachers need to know not just *it is so*, but also *why it is so*. In Shulman's view, pedagogical content knowledge does not refer to how a lesson is delivered (individual work, lecture, group work etc.). It refers to the most useful forms of representing ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations -- the ways of making the subject comprehensible to others. Implicit within the PCK is the knowledge of why a certain topic is difficult for learners. Shulman's SMK and PCK overlap two of Ma's defining attributes of teaching with PUFM: multiple perspectives and basic ideas.

Shulman defined three components of curricular knowledge: alternative curricular materials, lateral curricular connections and vertical curricular connections. The first component of the CK refers to the need for teachers to be aware of the different curricular resources available to them: software, textbooks, videos and so on. The lateral curricular connections comprise the links between a certain subject and other subjects taught in schools. The vertical connections component of the CK deals with the connections that exist "inside" a certain subject, from year to year. This last characteristic of Shulman's CK is equivalent to the first and fourth attributes of Ma's PUFM: connectedness and longitudinal coherence.

As an illustrative summary of his work, Shulman compares some old sayings: one

saying attributed to Aristotle who said that the difference between the one who knows

and the ignorant is an ability to teach; a second saying is credited to G.B. Shaw, who said

that he who knows, does, and he who cannot, but knows some teaching procedures,

teaches. As an answer to these two sayings, Shulman wrote: "Those who can, do. Those

who understand, teach" (Shulman, 1986, p. 14).

I included the work of Shulman and Ma in this part of my study, because both

researchers view the content knowledge as crucial in the process of successful teaching.

Ma specifically addressed the teaching of mathematics, and Shulman addressed teaching

in general. Ma introduced the notion of PUFM and Shulman wrote about SMK, PK, and

CK, but in reality, they talked about the same things: the need for teachers to have a good

knowledge and understanding of the subject they teach. In the next subsection, I treat

research done specifically on teachers' understanding/knowledge of the concept of

function.

4.4.2. Teachers' Subject Matter Knowledge with Regards to the Specific Topic of the Concept of Function

Norman (1992) used different terms to describe teachers' knowledge: practical

knowledge, pedagogical knowledge and content knowledge. These categories are very

similar to the categories defined by Shulman and they contain the same notions as the

attributes that defined Ma's PUFM. Norman focused his study on teachers' mathematical

knowledge of the function concept. The study included ten teachers, each working

towards a Masters degree in mathematics education. The researcher tried to determine the

concept definition and the concept image that teachers had in relation to functions. The

data were collected through individual interviews. The results of the study indicated that

"a majority of the teachers exhibited gaps, sometimes disturbing ones, in their

conceptualisations of functions" (Norman, 1992, p. 229). Some of the more striking

observations made during the analyses of the interviews are summarized below. The

teachers in Norman's study generally:

- Favour informal definitions of functions.
- Prefer graphical representations of functions to numerical or symbolic ones.
- Sometimes exhibit a single concept fixation when interpreting functions.
- Have not built strong connections between their informal definitions of functions and what they view as the formal mathematical definition.
- Are quite knowledgeable about the evolution of the function concept throughout their textbooks (Norman, 1992, p. 230).

Even (1993), tried to determine the subject matter knowledge, the pedagogical

content knowledge and the connection between the two of them for a group of preservice

mathematics teachers. Even found that many of the participants in the study lacked the

necessary subject matter knowledge. This fact had influenced their pedagogical thinking.

The researcher saw this situation as problematic for the quality of the teaching: "A

situation in which the secondary teachers at the end of the 20th century have a limited

concept image of function similar to the one from the 18th century is problematic" (Even,

1993, p, 114). Even's findings were in line with Shulman's opinion that pedagogical

knowledge needs to be tightly connected with the subject matter knowledge, in order for

the teaching to be successful.

Loyd and Wilson studied the understanding of the concept of function of a

secondary mathematics teacher, and its impact on the teaching of functions. The

researchers defined some categories to focus on when they studied the function concept in

their subject's understanding: definition and image of the function concept; repertoire of

functions in the high school curriculum; the importance and use of functions in varying

contexts; and multiple representations and connections among them. These themes are

part of the themes described by Ma and Shulman. For example, multiple representations

and connections among them are contained in Ma's connectedness and multiple

perspectives; repertoire of functions can be classified under Shulman's PCK. The

findings of the study suggested that "teachers' comprehensive and well-organized

conceptions contribute to instruction characterized by emphases on conceptual

connections, powerful representations and meaningful discussions" (Loyd and Wilson,

1998, p. 270). This conclusion is very similar to what Ma considered as teaching with

PUFM.

Another relevant study is one conducted by Stein, Baxter and Leinhardt (1990).

This study focuses on the subject matter knowledge of an elementary teacher, and the

impact of this knowledge on class instruction. The topic of instruction is functions. In

order to classify the subject matter knowledge of Mr. Gene, the teacher studied, the

researchers designed a questionnaire, and compared the answers to this questionnaire

given by the elementary teacher and the answers given by a specialist. The following

phase of the study observed and analysed some of Mr. Gene's lessons on the topic of

functions. Correlating Mr. Gene's understanding of the concept of function and his class

instruction on the topic of functions, the researchers concluded: "limited subject matter

knowledge led to the narrowing of instruction in three ways: (a) the lack of provision of

groundwork for future learning in this area, (b) overemphasis of a limited truth, and (c)

missed opportunities for fostering meaningful connections between key concepts and

representations" (Stein, Baxter & Leinhardt, 1990, p. 659). In other words, the instruction

lacked most of the attributes that Ma considered to characterize good teaching.

A very interesting piece of work that can be interpreted from the same perspective

as the above is Heaton's (2000) autobiographical book. Heaton described her own

experiences in a mathematics class where she tried to implement a new way of teaching, a

move away from the "classical" teaching. Some of her views on good teaching of

mathematics coincide with the views exposed by Ma, Norman and Stein et al.: "it

includes understanding how concepts are acquired and used ... It also includes using and

creating mathematical tools, language, and other representations to construct and

communicate understandings of particular domains and connections among them"

(Heaton, 2000, p. 4).

Reading Heaton's book with a critical, mathematical eye, one can sense that her

experiment was a failure in terms of teaching mathematics, due mostly to her lack of

SMK, PCK and CK. The author described the situation that she found herself in when

faced with the task of teaching composition of functions: "This generated more questions

for me than it answered. I knew what it meant to find an answer, but what did it mean to

investigate the composition of certain numerical functions? What was there to

investigate? What was the role of arrow diagrams ..." (Heaton, 2000, p. 20). From a

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social point of view, Heaton's lessons showed some success: students developed certain communication and social skills, were comfortable to express their opinions and were engaged in the tasks put in front of them. Heaton is to be admired for the courage to put her personal practice under critical scrutiny, but her experiences seem to reinforce what Stein et al. (1990) affirmed about the influence of subject matter knowledge on mathematics instruction in classrooms: "limited, poorly organized teacher knowledge often leads to instruction characterized by few, if any, conceptual connections, less powerful representations, and over routinized student responses. Incomplete teacher understanding may result in instruction that fails to lay the groundwork for future utilization and relationships associated with a rich concept" (p. 659).

A different kind of study that involved the scrutiny of the practice of an individual teacher is the study conducted by Haimes (1999). The researcher studied the teacher from a pedagogical point of view, and less from a subject matter knowledge point of view. Haimes' article illustrates a gap between the intended teaching approach and practice, and explains it by the fact that the teacher is unfamiliar with the "function" curriculum. The teacher in Haimes' study is lacking Shulman's CK.

4.5. Research on the Need for Rigor in Teaching Mathematics

Since my research will be discussing the need for mathematical rigor in the teaching of composition of functions and the inverse function of a function, I will now focus on the research on the need for rigor in teaching mathematics, with special reference to the teaching of the function concept.

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There seems to be no agreement among researchers on this topic. On the one hand researchers like Even and Bruckheimer (1998) who in their article "Univalence: A Critical or Non-Critical Characteristic of Functions?" argue for less mathematical rigor in teaching functions, at the benefit of a certain procedural fluency: "For example, when teaching about inverse functions one can focus on the essence of the concept - namely, undoing- and not on the unnecessary and often complicated examination whether the inverse function even exists" (Even & Bruckheimer, 1998, p. 32).

On the other hand there are researchers who discuss teachers' practices linked to their mathematical mastering of the concept of function (Loyd & Wilson, 1998; Norman, 1992; Stein et al., 1990). The work of this second group of researchers approache the problem they studied from points of views that seem to contradict Even and Bruckheimer. Most of their conclusions are based mainly on mathematical rigor. On the same note, Sierpinska (1992) quoted Byers when she wrote that there is no royal road for learning geometry and there is no painless road to learning. She claimed that the true educational task consists of managing tension, not of eliminating it. She ends her article with the phrase: "However, the only alternative to painful learning seems to be no learning at all" (Sierpinska, 1992, p. 58).

My personal opinion is that mathematical rigor has a definite and important place in the teaching of mathematics. Sometimes it seems that the task of the teacher and students would be easier if rigor is given up, but there is no royal road to knowledge, especially mathematical knowledge. The degree of mathematical rigor that one would use in teaching mathematics needs to be correlated with the age of the students to be taught,

and with the goal of the lesson to be treated. In BC's grade 11 mathematics classes the mathematical rigour has a well defined place, required by the mathematical topics prescribed by the IRP's .

CHAPTER 5
METHODOLOGY

In this chapter I present in detail the methodology for the study that I conducted, including the theoretical framework and the refined form of this framework that I used for analyzing the field data. In the first section I describe the participants in the study. In the following sections I present the settings of the study, followed by the theoretical framework and the presentation of the modified framework.

5.1 The Participants

The participants in the study were two groups of students from Simon Fraser University (SFU). The first group was composed of 10 students enrolled in the Professional Development Program (PDP). These preservice teachers were enrolled in the Education 415: Designs for Learning Secondary Mathematics course. The second group consisted of 10 practicing mathematics teachers, who were enrolled in a Masters Program in Secondary Mathematics Education program at SFU. Participation in this study was on a volunteer basis, for both of the groups, and it was facilitated thanks to the support of Dr. P. Liljedahl and Dr. S. Campbell, the professors who were teaching the two groups at the time when the data collection took place.

The data collection for the group of PDP students took place in 2003. At that time, all of the students who participated in the study had completed their "long practicum" of

three months. In order to be certified as secondary teachers by the BC College of Teachers, this group of students still had to complete one or two more courses, in which most of these students were enrolled at the time of the interviews. All the preservice teachers had a degree in sciences, some of them with a major in mathematics and some with a minor in mathematics. During their undergraduate studies, all of the participants had successfully completed undergraduate mathematics courses, including at least one Calculus course. At the same time, all the preservice teachers had taught in their practicum at least one mathematics course, and all of them expected to teach mathematics after their certification as a secondary teacher. As part of their assignments for the Designs for Learning Secondary Mathematics course, the students had to prepare a set of lesson plans oriented towards the teaching of two specified mini-units from the high school curriculum. One of the two units that the students could complete their assignment on was the teaching of the topics of composition of functions and the inverse function of a function. At the time when the task was assigned to the class, the students were instructed to select their topic, and to write the lesson plans for the selected unit, according to the specific instructions given by Dr. Liljedahl, who was teaching this class. At the time when the assignment was due, I joined the class, to introduce my research interest and myself. After the brief introduction, I asked for volunteers to participate in the study. The condition to be eligible was for the individual to be part of the group who completed the assignment related to the topics of composition of functions and the inverse function. I had eleven positive responses. Finally, only ten students were interviewed, due to scheduling conflicts between the interviews and the timetable of the

students. After finishing the interviews, I had the opportunity to come back to the class, and to discuss with the students various ideas related to the two mathematical topics.

The second group of participants consisted of practicing teachers, who were members of a Master's cohort at SFU, at the end of their second year in the Secondary Mathematics Education program. At the time of the interviews the students were close to the completion of their course work. All of them were experienced teachers, with a minimum of 5 years experience in teaching mathematics. At the time of the interviews, all of the teachers were teaching senior mathematics courses, such as Principles of Mathematics 11 and 12 or Calculus courses in different schools in BC. All but two of the teachers were working in public schools, and the exceptions were members of the staff of two different accredited private schools. Participation in the study was on a volunteer basis, without any restriction imposed by the researcher. I had 10 positive initial responses, which materialized into eight interviews. Two of the initial volunteers declined their participation due to conflict in schedule.

5.2 Purpose of the Study

According to Pirie (1998), mathematics education is "an emerging discipline, no longer in its infancy yet not fully adult" (p. 17). This relatively new research field cannot be identified with either mathematics, or with education, being in the stage when it seeks to establish an identity, as a legitimate, autonomous academic field.

The goal of the study is to find what teachers and preservice teachers see/interpret/understand/perceive as being the main ideas when teaching the topics of

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composite functions and inverse functions, and to relate these findings to their content

knowledge of the two mentioned topics. The qualitative research methodology chosen for

this study serves this goal by being "primarily concerned with human understanding,

interpretation, intersubjectivity, lived truth and so on" (Ernest, 1998, p. 33). This

methodology is also meant to use cases as illustrative and generative. The particular

studied through this methodology is expected to represent the general, with less precision

than the research of the exact sciences, but a general truth that is a more complex truth.

The aim of qualitative research is, "to see the world in a grain of sand" (Ernest, 1998, p.

34), to explain the general through the particular.

In the qualitative research: (1) qualitative researchers believe in subjective reality;

(2) the research is after the personal, constructed or socially constructed knowledge; (3)

the methods used employ mainly qualitative case studies of particular subjects and

contexts; (4) the intended outcome of qualitative studies is the illuminative subjective

understanding; (5) the researcher has an interest in understanding and making sense of the

world (Ernest, 1998).

The research questions that my doctoral study addresses are as follows:

1. What is teachers' and preservice teachers' content knowledge regarding the

   topics of composition of functions and the inverse function of a function?

2. How are the components of content knowledge — subject matter knowledge,

   pedagogical content knowledge and curricular knowledge — interrelated?

3. How are the three components of the content knowledge influenced by the

   teaching experience?

In this study I try to discover what the participants perceive to be the main issues in the teaching of the two topics mentioned above. I am also interested in why they regard these issues to be the most important ideas when teaching the two topics. The study also has the intention to discuss the connections among the subject matter knowledge, pedagogical knowledge and the curricular knowledge, and to determine if teaching experience influences any of these components of the content knowledge.

The answers given by the participants are looked upon from a mathematical perspective as well as from a didactical standpoint, probing into what Shulman (1986) called the participants' subject matter knowledge and pedagogical knowledge. The study inquires into the subjects' opinion on the usefulness of the topics for the future mathematics education of high school students, and on other links with different topics of the BC high school mathematics curricula. This inquiry tests the vertical component of Shulman's curricular knowledge. The answers I am seeking from the participants in the study are situated answers, in the sense that at least from a didactical point of view, as well as regarding the usefulness of the two topics, there is no right or wrong answer. There are, however, good answers and answers that are less adequate, depending on the approach that each individual teacher will take. Least but not last, my decision to use qualitative research methods for collecting data for my inquiry stems from the nature of my research question, where the word "perceive" is a key word. This implies that I am interested in the subjective views of teachers and preservice teachers on the considered topics. These views are influenced by the conditions created by the BC high school mathematics curriculum and the BC educational system.

5.3 Data Collection

The methods that I used to collect data are clinical, semi-structured interviews and lesson plans. In this section I present the setting of the data collection, and shortly justify why I selected these two methods as instruments for assembling the data for the study.

The interviews were conducted in the months of June and July of the years 2003 and 2004. They took place on different sites. Most of them occurred in Burnaby, BC, on the Simon Fraser University campus, but I also interviewed teachers on their high school campuses, or at their home, in order to accommodate the schedule of the participants in the study. The interviews were recorded on tape and then transcribed on paper. The length of the interviews varied between half an hour and 45-50 minutes. The interviews with preservice teachers tended to be longer compared to the interviews of the experienced teachers.

Zazkis and Hazzan (1999) claimed interviews have the "potential to reveal insights in mathematics education " (p. 430), while also paying attention to individual differences among the interviewees and their mathematical and didactical conceptions. The characteristics of the clinical interview such as " ... a particular kind of flexibility involving the interviewer as a measuring instrument", the fact that "... the interviewer has the freedom to alter tasks to promote the understanding" (Ginsburg, 1997, p. 37), and the richness of data provided, prompted me to use this method in collecting data. This technique allowed me to adapt my questions according to the answers of the interviewees, in order to clarify the subject's point of view on a certain detail. In addition to this, it also allowed the interviewees to explain their thinking in an unstructured setting. The

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interviews conducted during this study differed from one interviewee to another, not necessarily in substance, but primarily in the order that the tasks were presented to the participants. The general guiding structure of the interview is presented below:

1. What do you consider to be the necessary mathematical prerequisites for successfully teaching the topics of composition of functions and the inverse function of a function? Give a mathematical definition for the concept of function, and give some examples, including a real life example, that would illustrate the main characteristics of the function concept.

2. Give a formal mathematical definition for the composite function, or explain what you understand by the process/operation of composing functions, and what are the main points that you emphasize when teaching this topic. Make up a real life story that is an application of composition of functions, and it is a continuation of the story you used for task 1. Give at least three examples that you would use to illustrate the essential features of the composition of functions.

3. Give a formal mathematical definition for the inverse function of a function, followed by a real life story that would make the concept easier to understand for the students you teach. Give at least three examples of functions and their inverses, examples that you would use to emphasize the essential features of the inverse function of a function.

4. Do you know any examples of mathematical topics that the BC high school curriculum treats in grades 8 to 10, and that use the idea of composite function or inverse function? Can you give such examples? Explain your point of view.

5. Do you think that the topics discussed in this study are useful for the future mathematics education of the grade 11 students? Can you give some concrete examples when the students can use the two topics?

Each task of the interview is discussed separately in chapter 6.

The form of the interviews changed from case to case, depending on the answers

the participants gave. These changes occurred only in the shape of the interview, not in

the substance of the tasks. The most frequent change affected the order of the questions.

Another reason for the differences in the interviews was the complexity of the answers.

Being a teacher and a researcher at the same time, I took advantage of the fact that the

interviews are also tools for learning and understanding that work in both directions, from

the interviewee to the interviewer, and vice versa. In order to clarify the ideas some of the

participants were trying to communicate, I provided them with additional tasks to

complete. These tasks consisted of some problematic examples that highlighted essential

features of the topics of composition of functions and the inverse function. Beside the

clarification role, these supplementary tasks provided conversational stimuli, depending

on the course of the interviews. These tasks are presented below:

A. Compose the following pairs of functions:
1. $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 9}$

2.
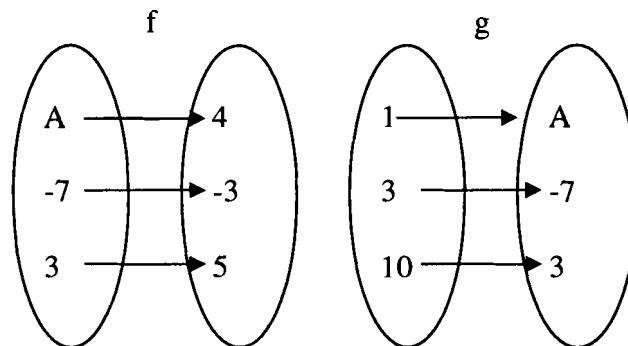
B. Find the **inverse function** of the following functions
1. $f(x) = 4, x \in \Re$ (follow the algebraic algorithm).

2.
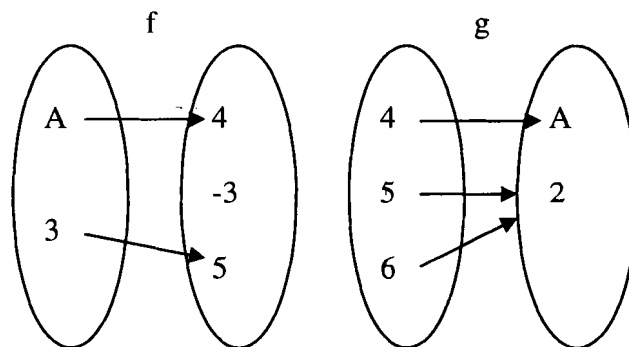
C. A student of yours calculates the inverse function of the function $f(x) = 3x - 4$ and the answer obtained is $f^{-1}(x) = -2x + 4$. The student checks his work, and when he "does" $f(x)$ with $g(x)$ he gets x. Are these two functions inverses for each other? Explain.

Fig. 4. Problematic examples used in interviews

The second set of field data, beside the interviews, consisted of lesson plans. The lesson plans are considered to be open ended tasks, and I used them only for the group of preservice teachers. The lesson plans had a role in providing data, but also in selecting the participants for the interviews. By writing lesson plans on the topics of composition of functions and the inverse function of a function, the preservice teachers had the chance to revisit and to refresh their memory with regards to the topics specified above. I considered this necessary since the student teachers probably did not have the chance to teach the topics of composition of functions and inverse function in their practicum, and some of them actually never taught these topics. The data collected through the lesson plans contributed to establish "the strength of belief ", which "helps understand whether a learner's response to a situation was an arbitrary choice or whether it is persistent and robust in his or her approaches" (Zazkis and Hazzan, 1999, p. 430).

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At the time when they wrote the lesson plans, the preservice teachers were not yet participating in the study. The tasks for the assignment specified that the students needed to design a sequence of lesson plans to teach the topics of composition of functions and the inverse function. The lesson plans asked students to mention which topic they would teach first, and to give reasons for their choice. In the assignment the students were asked to present concrete examples that they would use in class when teaching the assigned topics. These examples needed to illustrate not only the ideas that the preservice teachers wanted to teach, related to the composition of functions and the inverse function of a function, but also how these ideas were to be presented in the classroom. The student teachers also had to give their opinions, and arguments, on the necessity of the inclusion of the two topics of composite function and inverse function in the Grade 11 curriculum, and the use of these topics in the future mathematics education of the Grade 11 students. The sequences of lesson plans varied in length from 2 to 4 lessons, and were presented under varied formats.

5.4. Even's Theoretical Framework

For the data analysis I used a framework derived from the theoretical framework proposed by Even (1990). The theoretical framework is suited to investigate the subject matter knowledge as well as the pedagogical subject knowledge of teachers. Even (1990) demonstrated the building of a theoretical framework for the analysis of teachers' subject matter knowledge regarding the teaching of a specific topic in mathematics. The framework contains the following seven aspects that seem to form the main facets of teachers' subject matter knowledge about a particular mathematical topic:

1. essential features;

2. different representations;

3. alternative ways of approaching the concept;

4. the strength of the concept;

5. basic repertoire;

6. knowledge and understanding of the concept;

7. knowledge about mathematics.

Even used the proposed framework to analyze teachers' subject matter knowledge of the specific topic of function. Although Even refers just to subject matter knowledge, the criteria of her framework address some of the components of what Shulman (1986) calls content knowledge. For example, criteria 1, 2, 6 and 7 address specifically Shulman's subject matter knowledge of teachers, while criteria 3 and 5 refer to pedagogical knowledge. The curricular knowledge component of content knowledge is not addressed specifically by any individual component of Even's framework.

The choice of these aspects for the framework was based on integrated knowledge from several bodies of work: the role and importance of the topic in mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of mathematical concepts; and research and theoretical work on teachers' subject matter knowledge and its role in teaching. In the following paragraphs each individual aspect of the framework is presented in detail.

1. Essential features. The component of the framework defined as "essential features" deals with the concept image, stressing the essence of the mathematical concept.

Even claimed "teachers should have a good match between their understanding of a specific mathematical concept they teach and the "correct" mathematical concept" (Even, 1990, p. 523). Influenced by the fact that mathematically correct is content and context specific, the author argues that teachers need to be able to use analytical judgment and not just a prototypical judgment to discern if an instance belongs to a concept family or not. The pedagogical decisions that teachers make in planning a lesson, such as what questions they ask the students, what activities they design, how much they let students explore and in what direction, are influenced by how comfortable teachers feel in unfamiliar mathematical situations. That is a reason why it is not enough that teachers are able to distinguish between concept examples and non-examples when the instances match their concept image only, but they have to be able to distinguish these instances when they deal with unfamiliar situations. This is made possible if teachers possess well-structured knowledge about mathematics, or what Resnick and Ford (1984) call "correspondence" --the match of one's subjective mental picture of a specific concept with the correct mathematical concept.

2. Different representations. Mathematical concepts can be represented in different ways, each way stressing different characteristics of the concept. Complex mathematical ideas appear throughout different divisions of mathematics, under different labels and different notations. Understanding a representation of a concept does not guarantee that one understands this same object under a different representation. Teachers need to be familiar with the different representations of the concepts they teach, which in

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turn would give them an overview of the common properties of the concept, while ignoring the irrelevant characteristics imposed by the representation itself.

3. Alternative ways of approaching. This feature works in tandem with the multiple representations of a mathematical concept. A certain mathematical idea appears in different mathematical disciplines, under different labels, notations or forms. Consequently, when teaching a mathematical topic, depending on the situation, different approaches are more suitable than others. Therefore, mathematics teachers "should be familiar with the main alternative approaches and their uses" (Even 1990, p. 525).

4. The strength of the concept. This criterion measures the knowledge of the "general meaning of a topic, which captures the essence of the definition as well as a more sophisticated formal mathematical knowledge" (Even, 1990, p. 525). This kind of knowledge makes possible a comprehensive understanding of related sub-topics or sub-concepts, and opens new mathematical opportunities.

5. Basic repertoire. Teachers need to have easy access to specific examples related to the topic they teach. These examples have to illustrate important principles, properties, theorems, and so on. The basic repertoire should not be memorized without understanding, but on the contrary, in order to be used appropriately and wisely it needs to be acquired meaningfully.

6. Knowledge and understanding of a concept. This criterion deals with the ideas of conceptual knowledge versus procedural knowledge. Conceptual knowledge is the knowledge that is rich in relationships. It is constituted of a network of concepts and

relationships. To learn a new concept or relationship means to add another node or link to the existing cognitive structure. Conceptual knowledge has to be learned meaningfully. This is in some way contrary to procedural knowledge, which is made up of the formal language of mathematics and algorithms for completing mathematical tasks. This kind of knowledge can be learned without meaning.

For a thorough understanding and mastering of mathematics the two kinds of knowledge are indispensable. There is a tendency in high school mathematics to overemphasize the procedural knowledge to the detriment of conceptual knowledge (e.g. Davis, 1986; Schoenfeld, 1987). This is evident throughout the structure used in the BC high school, as explained in this study in Chapter 3. To achieve meaningful learning and understanding does not mean to ignore procedural knowledge. There are reasons behind memorizing of algorithms and procedures. The memorization of certain facts and procedures is, according to Resnick and Ford (1984), a way to extend the capacity of the working memory. When procedures and algorithms are mastered, they do not require special attention from the part of the students, who can focus on the actual problem at hand. In doing mathematics, procedural and conceptual knowledge are used dynamically. In solving a nontrivial task the focus should not be on one or the other of the two types of knowledge, but on the relationship between them. If the connection between the procedures and concepts is missing, a person may have a good intuitive feel for mathematics, but he/she cannot solve a problem, or he/she can produce an answer without understanding what he/she is doing.

7. Knowledge about mathematics. Even suggested that knowledge of a specific piece of mathematics includes more than conceptual and procedural knowledge. Mathematical knowledge also includes knowledge about the nature of mathematics. This knowledge is more general, about the discipline and its construction and use of conceptual and procedural knowledge, including means and processes by which mathematical truths are established. The nature of mathematics possesses an ever-changing character, as a result of new discoveries or inventions that happen in this field.

In the following section I present a refinement of the above framework that I used as the instrument to analyze the field data of this study.

5.5. The Refined Framework

The preliminary data analysis prompted me to modify the theoretical framework proposed by Even. For the purpose of my study, I modified Even's framework in four of its aspects. First, I decided to consider two of the aspects described by Even, the essential features and the knowledge and understanding of mathematical concepts, as one criterion. I based this decision on the fact that the two mathematical topics that I focused upon in my study, composition of functions and inverse function of a function, have what can be considered a dual character, in the sense that they can be understood as mathematical operations, or they can be considered as mathematical concepts. Both meanings are adaptable to the analysis of their features, and also to the kind of the understanding teacher should display. At the same time, both criteria address teachers' *subject matter knowledge*. In the following paragraphs I present what I consider to be the essential

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features of the function concept and more extensively the essential features of the topics of composition of functions and inverse function, and what can be considered as a display of conceptual and procedural knowledge related to these topics.

In Chapter 2 I presented a contemporary mathematical definition for the concept of function, enunciated by Gellert et al. (1975). In the BC mathematics curriculum, the notion of function is introduced formally in Grade 10, and at that level, the students are exposed to a definitions such as: "A function is a special kind of relation: for each element in the domain (first elements), there is exactly one element in the range (second elements)" (Knill et al., 1998, p. 232). The essential features of the function concept are arbitrariness and uniqueness. Since the function is not the main object of my study, I treat this issue only briefly, as it is essential for a complete understanding of the composition of functions and the inverse function topics.

The topic of composition of functions, as it is mentioned in Chapter 3, is part of the Grade 11 BC Principles of Mathematics curriculum. The mathematics of composing functions contain a number of defining features such as the definition of the composite function, the role of domains and ranges in composing functions, and the property of non-commutativity of the operation of functions. This topic can be approached from what Sfard (1992) called an *operational* perspective — a mathematical operation applied on functions — or it can be approached from a *structural* perspective — a new mathematical entity, with its own particular properties. The composition of functions is defined by Gellert et al. (1975), as follows: "If through the function G, the element a corresponds to the element b, and through the function F, the element b corresponds to the element c, we

obtain a new function through which the element a corresponds to the element c. This newly defined function is called the composite function of the two initial functions F and G. The symbolic notation for composition of functions is $(F \circ G)(a) = c$ "(p. 122).

The interpretation that is closer to the BC mathematics curriculum is to treat the composition of functions as an operation with functions. Either one of the two interpretation – operational or structural — asks the learner to deal with two aspects: the possibility of performing the operation/the existence of the composite function, and the aspect of non-commutativity/the possibility of the existence of more than one composite function when composing the same initial functions. The first aspect brings into discussion the role of domains and ranges when composing functions. The second aspect might bring into discussion the cases when composition of functions is commutative, and could be a means to consider the topic of inverse function. Since the participants in this study encountered the topic of composition of functions at the high school level mathematics as well as at the level of university or college mathematics courses, my assumption is that a display of conceptual knowledge related to the composition of functions would be demonstrated it the teachers were able to give a reasonable definition of composition of functions. This assumption is supported by other research work, such as the work of Vidakovic (1996). Procedural knowledge display resides in being able to compute correctly the composite function of two or more functions, following the algorithmic substitution, from right to left, without questioning the existence of the composite function.

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The inverse function of a function is a topic related to the subject of composition of functions, and in BC is treated formally at the Grade 11 level. This topic contains a series of important components, such as: the definition of the inverse function, the existence of the inverse function, the procedure of finding inverse functions, and the relation between the Cartesian graphs of a function and its inverse.

The mathematical definition of the inverse function is: "A function g with domain B and range A is called the inverse function of a function f with domain A and range B, if and only if, f composed with g and g composed with f have as results the identity function: $f \circ g = 1_B$, $g \circ f = 1_A$" (Gellert et al., 1975, p. 128).

The existence of the inverse function is related to the definition of a function, and demands an analysis regarding the role of the domain and range in finding the inverse function. This analysis requires the introduction of a new concept, the concept of one-to-one functions, or bijective functions. It is also important to stress the fact that the domain and the range of the function correspond, in this order, to the range and the domain of the inverse function. The connection between the Cartesian graphs of functions and their inverses brings up the "horizontal line test", which verifies if a function is one-to-one and implicitly if it has an inverse function. At the same time, the graphs of the two functions are symmetrical with regards to the line $y = x$. In the case of the inverse function, a display of conceptual knowledge would be the capacity of defining the inverse function, accompanied in the process of calculating the inverse by the questioning or testing of the existence of the inverse function. The procedural knowledge implies the mechanical computation of the inverse, following the algorithmic steps of switching x's and y's,

solving for y's and so on. Below I present two examples of functions: the first function is a one-to-one function and it has an inverse function, whereas the second function is not one-to-one and it does not have an inverse function: $f(x) = 3x - 2$ and $g(x) = 4$. Both function f and g are linear functions, but f has an inverse functions and g does not.

The second modification of Even's framework was to combine the criteria of different representations and alternative ways of approaching a topic in one single standard of analysis. I opted for this change as a consequence of the fact that the ways of approaching the two mathematical topics discussed in the study are tightly linked with the ways the function concept can be represented. Also, both these criteria relate to teachers' *pedagogical content knowledge*.

Different representations of the concept of function do not have just an informative role or just a mathematical role. These representations should be taught for more than just to emphasize the versatility of the function concept. They can play and important didactical role in designing the learning process. For example, the arbitrariness character of the function concept is easily and more palatably served to the students by using arrow diagrams, compared to using algebraic expressions or Cartesian graphs. On the same lines, relations given through formulae are harder to categorize as functions or not, compared to graphs, table of values, ordered pairs or arrow diagrams; the existence of inverse functions only for one-to-one functions is easier to grasp if arrow diagrams are used, compared to other representations; the necessity of the link between ranges and domains when composing functions is best illustrated through tables of values or arrow diagrams. I am not diminishing the importance of the algebraic expressions in

representing functions. Ultimately, the goal of the lessons treating the composition of functions and the inverse function of a function is to enable students to manipulate functions given as algebraic formulae. The point that I am trying to make is that different representations of a certain concept are from a didactical point of view more suitable than others in introducing, clarifying and emphasizing some key points of a certain mathematical topic. This new criterion of the refined framework addresses what Shulman (1986) called *pedagogical content knowledge*, or the *multiple perspectives* attribute of teaching with Ma's (1999) PUFM.

The third modification to the framework is the addition of another criterion to use in the analysis of the field data. I entitled this new criterion "the knowledge of the mathematics curriculum". I believe that teachers need to know what "came before" and what "comes after". Teachers need to be aware of the possible connections between the topic to be taught at a certain time and other topics that had been taught before, and topics that will be taught in the future. This is necessary especially for teachers working in the current BC curriculum, where high school mathematics students will meet a variety of instructors in their schooling, and where continuity in terms of teacher-student relations is limited on average to one or two mathematics courses. Without the knowledge of the curriculum, vertical and horizontal knowledge, students will be faced with a multitude of disparate topics, topics that apparently have no link to each other. This fragmentation of the high school mathematics is not the intention of the prescribed curriculum. This newly introduced criterion is related to Ma's *connectedness* and Shulman's *vertical curricular knowledge*.

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In the case of composition of functions, teachers need to be aware of the connections between the Grade 10, 11 and 12 curriculums, as well as the university or the Grade 12 calculus courses. Some of the topics that are related to composition of functions are the transformations of functions in Grade 12, the calculation of the value of a function at a certain point, and the chain rule, to mention only some of the connections.

Inverse functions are introduced in BC, informally, at the Grade 9 level, when students are asked to find acute angles when given the value of a basic trigonometric function. At this age, this is a mechanical exercise, and the notions of function and inverse function are not mentioned. The students are expected just to press the right key on the calculators and to get a numerical answer, without any conceptual understanding. Other topics that deal with the inverse function are the Grade 12 topics of trigonometric functions and the exponential and logarithmic functions. Later on in calculus, the inverse function is used in solving functional equations, and in differentiation.

The above paragraphs mentioned only a few of the connections that exist between topics treated in the BC high school mathematics. The knowledge of these connections influence the way teachers treat a certain topic of the curriculum, and makes the high school mathematics a network of concepts that can build on each other, or at least stem off each other, as opposed to a large sea of disparate mathematical entities.

The last modification of the framework is the integration of the "strength of belief" criterion into the "knowledge about mathematics" and "knowledge about the mathematics curriculum" criteria. I decided on this modification of the initial framework because the preliminary data analysis made the task of distinguishing between the two

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last criteria and the strength of belief disputable. It seemed to me that the elements given by Even as characterizing the strength of belief should be integrated with the elements of the other two criteria.

In summary, the modified framework that I use for analyzing the field data contains the following criteria:

1. Essential features, the knowledge and understanding of the mathematical concept

2. Different representations and alternative ways of approaching the topic

3. Basic repertoire

4. Knowledge of the mathematics curriculum

5. Knowledge about mathematics.

The first and fifth criteria of the new framework address the SMK of the participants in the study. The second and third criteria address the PCK of the participants, as well as the relation between the SMK and the PCK. The fourth criterion addresses specifically the CK of the participants in this study.

In the following chapter the field data are analyzed according to each of the individual criteria of the refined framework.

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CHAPTER 6
DATA ANALYSIS

As described in Chapter 5, I analyzed the collected data using a refinement of the theoretical framework proposed by Even (1990). As presented in Chapter 5, Even's framework contains seven aspects that could characterize the subject matter knowledge of teachers or preservice teachers. The refined framework that I use contains the following five aspects:

1. Essential features, knowledge and understanding of the concept.

2. Different representations and alternative ways of approaching.

3. Basic repertoire.

4. Knowledge of the mathematics curriculum.

5. Knowledge about mathematics.

All the particular aspects of the framework were detailed in Chapter 5.

In the following sections I analyze the data obtained from the interviews and the lesson plans through the criteria of my refined theoretical framework. The analysis is conducted in parallel for the group of preservice and inservice teachers, with the exception of the last criteria, the knowledge about mathematics, which is discussed in section 6.5. With regards to this criterion, the two groups of participants were not too different from each other, and I decided to treat them together.

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6.1. Essential Features, Knowledge and Understanding of the Concept

6.1.1 The Prerequisites and the Definition for the Concept of Function

The first question of the interview asked the participants: a) to list the prerequisites needed for successful learning of composing functions and inverse function of a function, and b) to give a mathematical definition for the function concept. The role of this question was mainly to introduce the participants to the topic of the interview and to open up the discussion.

6.1.1.1. The Case of Preservice Teachers

The answers to the first part of the question were similar for all preservice teachers. All the preservice teachers considered as necessary prerequisites for a successful learning of composition of functions and inverse function the following: the ability of the students to do algebraic manipulations, such as substitutions and solving equations for different variables, and the knowledge of what a function is. Also, the preservice teachers considered necessary that the students know different representations for the function concept, focusing especially on the ability of graphing functions in the Cartesian plane of coordinates. Below it is presented an excerpt from the interview with Camelia:

> Calin: So, if you have to teach the lessons of composition of functions and the inverse function of a function, what would you consider necessary for your students to know when they start this sequence of lessons?
>
> Camelia: You mean like, prerequisites?
>
> Calin: Yes, prerequisites for the two topics we are discussing.

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Camelia: Number one, they're going to have previous dealings with functions, being able to sub in vice versa, the idea of a table chart, graphing skills, basic algebraic, how you move things around ... oh, and the vertical line test.

Calin: What do you mean by previous dealings with functions?

Camelia: Hmm, I mean, they should know what a function is, to calculate what a function is, I mean for a certain number, and to graph functions.

All the answers were along the same lines. All interviewees mentioned that students should know what a function is. The preservice teachers did not mention that the students should be familiar with the notions of domain and ranges, but since they specifically mentioned the definition of a function, it is possibly that they implied the notions of domain and range as being part of the knowledge that the students will possess.

The second part of the question was a natural follow up to the first part, and provided varied answers in correctitude and completion. Below I present two excerpts from two interviews. The first excerpt is from the interview with Chen, who seemed to have a firm grasp on the definition of the function concept:

Calin: Since you considered as a prerequisite the definition of the function can you please give me a formal definition for this concept?

Chen: Mathematically, a function is similar to a relation ... but the difference between a relation and a function is that for every domain, there's only one value that appears in the range. To be clearer, every x element from the domain has only one y element from the range that corresponds to it.

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The second excerpt is from the interview with Camelia:

Calin: You mentioned among the prerequisites the fact that the students should know what a function is. Can you please give a mathematical definition for the function concept?

Camelia: Hum, my vertical goes through once, it doesn't bisect more than once, whereas if we had, say, that, then it'll pass through two points, then it's not a linear, or ...

Calin: You drew a line on the paper and you are saying that this is a function, and than you drew a horizontal parabola and you are saying that this is not a function. How about this example, say, $x = 3$?

Camelia: I don't know, um, I'll check ... I think it is not a function.

Calin: What about $x^2 - y^2 = 1$?

Camelia: I am not sure. Is there something about that it can't have any gaps, does that have anything to do with this, or no? Okay, I think I'm totally drawing a blank.

From the answers to this question, it is clear that most of the participants interviewed in this study, similar to Camelia, were not able to give a mathematical definition for the concept of function. Some of them had a "foggy" vision of what a function is not, and some of them could explain in a "laic" way what this concept is. Three out of ten students were not able to give any kind of definition for the concept of function. All three of them provided some sort of examples that illustrated what a function is not. The rest of the students were able to give examples of what a function is, and to use the machine/black box analogy to illustrate what a function is. One student was

able to give only a partial definition of a function, using the vertical line test. Three students gave a correct definition of a function using the concepts of domain and range, and three students gave definitions using the expressions "output/input" or "x/y", without mentioning the words domain or range. Two students confused the uniqueness property of functions — all elements of the domain have only one correspondent in the range — with the one-to-one property. An example of this "confused" definition is given by Stela: "... it's the relation, it explains the relationship between two variables, but it has to be one-to-one correspondence."

The definitions that were provided in this set of interviews were definitions expected from a Grade 10 or Grade 11 student. None of the preservice teachers gave a modern definition of the function concept, using ordered pairs. Without exception, all interviewees that were able to enunciate a definition for the function concept used sets of numbers, or they used a "number machine", or a "black box", where you put a number in and you get a number out. They did not seem to be aware of the arbitrariness of the function concept —functions do not deal only with numbers, but also with other symbols too — or this essential feature was not important enough for them to mention it. The lesson plan assignment did not specifically ask the students to mention the prerequisites for teaching the composition of functions and the inverse function topics or to define the function concept, and consequently, none of the students incorporated in their lesson plans definitions for the concept of function and they did not mention any prerequisites for the teaching of the two topics mentioned above.

6.1.1.2. The Case of Inservice teachers

The answers that the inservice teachers provided for the first part of the question were quite similar in content to the ones given by the preservice group. All teachers though, mentioned explicitly as prerequisites the concepts of domain and range.

The main difference between the two groups is the fact that the teachers completed their answers with statements such as the one given by Dan:

Calin:	What do you consider as essential knowledge for the students before you start teaching the composition of functions and the inverse function?
Dan:	I'd like them to have an idea about relations, equations with x and y, and that they can graph relations between x and y. That's about it before moving on to functions.
Calin:	Um, I meant to say, that you do not have to teach the whole topic of functions, but just the two subtopics. These are grade 11 topics, and the function is, supposedly, treated in grade 10.
Dan:	Okay, yeah ... Um, I'd teach it anyways, I go over the definition of the function, because very often they do not come with a very good understanding of what a function is. And so, I would reteach it before I do composition of functions. To do composition of functions they have to have a pretty good idea of what a function is.

Using different wording, all the inservice teachers were saying that they would re-teach or at least revisit the definition of a function, which reveals the importance that the teachers attributed to this definition when teaching the composition of functions and the inverse function of a function.

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For the second part of the question, all the teachers gave reasonable definitions of the function concept. None of the teachers confused the uniqueness property with the one-to-one property, and most of them used multiple representations when describing what a function is, such as formulae, graphs, arrow diagrams, and tables of values. Ken was the only one who gave an example of arrow diagrams in which the elements of the domain and range were not only numbers, fact that suggests that he was aware of the arbitrariness character of the function. He also emphasized through his examples, and explicitly, that not all the elements of the range needed to have a correspondent in the domain. All the definitions provided by the teachers were definitions that could be used for students in Grade 10 or 11, but none of the teachers gave a formal, modern, mathematical definition of the function. The closest to the modern definition of the function concept was the definition given by Mira:

> Mira: A function is a relation in which no two ordered pairs have the same x coordinate, or a function is a relation in which there is a one-to-one correspondence between the x coordinate and the y coordinate. And I mean one-to-one, in the sense that every x corresponds to only one y, and not the other way around.

This is a simplistic definition, that assumes that x and y are numbers (coordinates), but it has the merit of using ordered pairs, which brings it closer to the formal modern mathematical definition.

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6.1.1.3. Section Summary

The answers to the first question of the interview provided a wide perspective regarding knowledge of the concept of function, as well as the essential features of this concept. The preservice teachers' ability to define the function can be seen as relatively weak. Only a minority of the students displayed awareness of the uniqueness feature of the function concept. The arbitrariness feature of the functions was totally ignored by the preservice teachers.

The case of the inservice teachers was different, at least regarding the ability to provide a reasonable definition for the function concept. In their case, all teachers stressed the uniqueness property of the function, but with one exception, none of them mentioned the arbitrary character of the concept.

Regarding the prerequisites required from the high school students that would start learning the composition of functions and the inverse function of a function both groups were fairly consistent among their members, and also across the two groups. The difference came in the way that teachers stressed the importance of the details, such as the definition of the function, with the notions of domain and range, whereas the preservice teachers accentuated more the importance of algebraic manipulations.

6.1.2. The Composition of Functions

As it is mentioned in Chapter 5, the mathematics of composing functions contains a number of defining essential features such as the *definition of the composite function*,

the role of *domains and ranges* in composing functions, and *the property of non-commutativity* of the operation of functions.

The interview question regarding the composition of functions had two parts:

a. Give a mathematical definition for the composite function, or explain what you understand by composing functions, and what are the main points that you emphasize when teaching this topic. Make up a story, preferably as a continuation of the stories that you composed for illustrating the properties of the function concept.

b. Give at least three examples that illustrate the most important properties of composing functions.

In the following sections I discuss the appearance of the above-mentioned essential features in the lesson plans and in the interviews.

6.1.2.1. The Case of Preservice Teachers

The analysis of the lesson plans that referred to the composition of functions topic denoted a mechanistic approach to this lesson. The words that came up most in describing the objectives or outcomes of the lessons were words such as *perform, find, what to do, practice combining, calculating*, which are all verbs that imply a procedural approach to the topic. There wasn't any lesson plan that used words such as *analyze, inquire, see what happens;* verbs that require higher level thinking and a conceptual understanding of the topic. None of the plans included any discussion/analysis of the role of domain and range in the existence of the composite function, and the non-commutative property was not

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explicitly treated by any of the future teachers. Three lessons plans contained examples

asking the students to compose *f* with *g* as well as *g* with *f*, but they did not go further into

formalizing the property. Below are some examples extracted from some of the lesson

plans:

Alina: They will *focus on what to do* with them rather than on
the functions themselves.

Chen: **Objectives:** To introduce the composition of functions
using linear functions as examples ... Introduce the
quadratic functions and *practice combining* two
functions including quadratic functions together.

Tino: **Learning Outcomes:** *Perform* operations involving
composite functions.

Simion: By the end of this unit students will be able to: 1) *find*
the composition of 2 or more functions.

The interviews shed more light on the knowledge of this group of preservice

teachers regarding the composition of functions. Answering the first part of the question

regarding the composition of functions, none of the participants was able to enunciate a

mathematical definition for the composition of functions. All of them explained what

composing functions means, using mathematical examples or real life examples. The

feature for this topic that seemed to be important for all the preservice teachers was the

computational aspect of composing functions. Words similar in nature with the words

highlighted in the excerpts from the lesson plans (*operating, performing, doing,*

mechanical) were present in all the answers in the interviews. One of the students was not

able to complete the examples given in the interview, even though the examples were of

Grade 11 level.

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With regards to the other features of composition of functions, compared to the lesson plans, in the interviews, the situation changed somewhat. Related to the role of the domains and ranges of the functions to be composed, the answers varied from variants that specified correctly what conditions have to be met by domains and ranges, to answers that denied any role played by these two concepts in composing functions. There were also answers somewhere in between these two extremes. Some students recognized that domains and ranges play a certain role, but they were not able to pinpoint this role. Such an example is Chen's answer:

> Chen: I know why domain and range are necessary for the function per se, and given that composition of functions is one aspect of the whole concept of function ... however I cannot tell you right now why are they important for students when they are learning the concept of composition of functions.

Some of the students, although confronted with some of the problematic examples presented in Appendix 1, remained convinced that composing functions is only a procedure to be mastered, a mechanical process. An illustrative example of this is Sam's case. At the time of the interview, he did not make any connection between the domains and the ranges and the existence of the composite functions. For him, composing functions remained an exercise in algebra. Underneath I present a part from the interview with Sam.

> Calin: Can you give a mathematical definition for the composition of functions?

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Sam: I, it seems like a topic to me where um, it's quite _mechanical_, the composition of function. The main points I think to get across are, um, the actual aspect of what they're _doing_, what it means to compose a function within another function. Um, like I explained to you here [gives an example of composing two linear functions: $f(x) = x - 3$ and $g(x) = x + 2$], but what does that mean, even when I was in high school I didn't explore.

Calin: So, except for computing the composite function, do you think that there is anything else that you would teach your class?

Sam: At this point, I don't see anything that I could relate to this topic.

Calin: You said at the beginning of the interview that a prerequisite for teaching this topic is the definition of a function. How do you use this definition, or its elements, in this topic?

Sam: I do not see any relation ... I think that the definition, domain and range, are important more for the inverse function, rather then the composition of functions.

Calin: Well, lets see what happens if you try to compute the composite function of these two radical functions:
$f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 9}$.

Sam: Do you want me to combine these two?

Calin: Yes, if you don't mind, I would like you to do both, $f(g(x))$ and $g(f(x))$.

[Sam did the two required computations.]

Sam: Hmmm, we have a problem here. This one, [points to $g(f(x))$]... Did I did the calculations right?

Calin: They look right to me ...

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Sam: Well, we are getting something that does not work ... I am not sure why, but since you say that the algebraic work is good ... I don't know why this would not work. As I mentioned before, I did not explore this topic for more than only _how to do it_.

The example that Sam was asked to calculate is an example that uses radical functions. This type of functions is studied in BC in Grade 11. The radical functions usually raise problems in terms of domain and range, because in Grade 11 students are able to extract the square root from positive numbers but they cannot extract square roots from negative numbers. As a consequence of this ability, the domain of radical numbers has to be restricted so that the expression that is under the radical has a non-negative value for any element of the domain.

The two functions that Sam had to compose had restricted domains and ranges:

for $f(x)= \sqrt{4-x^2}$, $x \in [-2,2], y \in [0,2]$ and for $g(x)= \sqrt{x^2-9}$,

$x \notin (-3,3), y \in [0,+\infty)$. In the case of composing g with f, there are no overlaps between the range of the function f and the domain of the function g. For this reason, the two functions cannot be composed in this order.

After correctly calculating the composite function, Sam obtained the result $(g \circ f)(x) = \sqrt{-x^2-5}$. He rightly observed that this function cannot be defined in terms of grade 11 mathematics, since the argument of the square root is negative for any $x \in \Re$. He failed to give any explanation for his finding. The expected explanation is a direct consequence of the role played by the domains and ranges in the composition of functions. As mentioned above, for this particular case, the explanation is provided by the

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incompatibility between the range of the function f and the domain of the function g. The fact that Sam failed to provide an explanation for the impossibility of composing the two given functions denoted that Sam was not aware of the role that the domain and range play in the composition of functions.

An interesting case was presented by Camelia, who used as her first example two functions given as ordered pairs: $f(x) = \{(3,5), (-2,6), (1,2)\}$ and $g(x) = \{(2,4), (-1,7), (3,9)\}$. She gave herself a "wrong" example of functions that cannot be composed. She partially fixes the situation on the spot, by "cooking" the function $g(x)$, to $\{(2,3), (-1,7), (3,9)\}$ so that the composition "works" for the first ordered pair. After fixing the first set of ordered pairs, she did not finish the composition of the two functions, but she enunciated the domain and range property: "... well in my second function, whichever one I'm dealing with, will have, um, its x values equaling what the first function's value ... so my y value in the first one needs to be my x value in the second one" [she was clearly referring to the first and second functions to be applied to the arguments, not first and second in terms of the order she wrote them on the paper].

Camelia seemed to know that the domains and ranges of the functions need to meet certain conditions for the composition to be feasible, but she did not seem to consider this as an important part of her lesson. She did not formalize at all this requisite. She just used it at a procedural level. When the interviewer insisted on finishing her example, Camelia answered: "Well, that's why I teach them the *right* example, so that *it will work*. I would ask students to work only with one value, instead of working with the whole function". Camelia seemed to possess the knowledge of the essential features for

composing functions, but she internalized them only at a procedural level. These essential features are not conceptualized. They are just procedural conditions to be met in order for "the question to work".

The non-commutative property of composition of functions was not explicitly mentioned in any of the lesson plans. Said for example, wrote: "... in either case the idea is that the function that is "closer" to "x" acts on it directly (and first)". Simion and Tino both consider algebraic examples for which they will compute two composite functions (e.g. $f(g(x))$ and $g(f(x))$), but they do not clearly state what is the purpose of these algebraic manipulations. The other participants did not give any hints that non-commutativity will be a matter of discussion when they teach this topic.

The interviews showed more of the participants' knowledge of this feature. The following lines present parts of the interviews after the participants were asked to compose a linear function with a quadratic function and to comment on the results:

Chen: These two definitely will not be identical. You know, when I wrote this one, I actually implied that these two are actually different.

Simion: Yeah, the order definitely matters in the composition of functions.

Said: ... from $f(g(x))$ and $g(f(x))$, uh, do not work out to be the same thing, that they are different, these are different animals, although they use the same functions ... We are dealing with the commutative properties of operations, or actually, the composition of functions is not commutative.

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Only three of the students, Chen, Simion and Said, explicitly mentioned that the order in which the two functions are applied to the arguments matter, Said being the only interviewee that actually mentioned the word commutativity in the right sense.

One of the students, Alina, presented a misconception on the order in which the two functions have to be applied on the argument. She gave a personal description on how functions can be composed:

> Alina: I think I tried to encourage them [she is talking about the students she taught] to choose their own method for one thing, because you can always do it two ways. You can, sort of, I mean there's two ways of looking at it I think, maybe more: you can substitute the first function in the second or the other way around.

In this paragraph, Alina made a reference to the order the functions can be applied on the argument. She stated that the result of composition is independent of the order in which the functions are applied to the argument. Alina totally ignored the fact that commutativity is an exception for the composition of functions: most of the time, the composition of function is non-commutative. Her dual method of composing functions works for certain functions given by algebraic expressions, but it cannot be applied to functions that are given through ordered pairs, or arrow diagrams.

6.1.2.2. The Case of Inservice Teachers

Very similar to the preservice teachers, the inservice teachers did not provide a mathematical definition for the composition of functions. All of them exemplified with different types of functions what it means to compose functions. Two of the teachers

mentioned explicitly that the composition of functions has as result a new function, with certain properties. Again, words like _how to do it_, _how to combine_, or _to calculate_ were predominant in describing the teachers goals for this topic. An illustrative example is presented below:

Calin: Regarding the composition of functions, what are your main goals when you teach this topic, what do you want the students to learn?

Terry: Composition of functions ... well I'm teaching them _how to do it, how to substitute_ a function within another function ...

Calin: Okay ...

Terry: If I give you a certain function, for example $f(x) = x^2$, and then another function, $g(x) = x + 1$, so what to do with these, so that they would have to know that, well, instead of writing x^2, then here you are going to write $(x + 1)^2$.

This paragraph denotes the procedural orientation of this particular person when treating the composition of functions.

With respect to the essential features of this topic, only three teachers mentioned the role of the domain and range in composing functions, and only two of them explicitly stated what are the conditions to be met in order to be able to compute the composite function. When confronted with examples that illustrate these conditions, most of the teachers were not able to give an explanation for the non-existence of the composite function. Underneath I present an excerpt from the interview with Terry:

Calin: Well, I would like you to compose for me the following two functions: $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x^2 - 9}$.

Terry: Which way do you want them to be composed, f with g, or g with f?

Calin: I think that I prefer $g(f(x))$.

Terry: Ok [she completed the computations correctly]. Hmmm. I get something that does not make sense, $\sqrt{-x^2 - 5}$. I should slow down on my calculations ...

Calin: I think that your algebra is fine ...

Terry: Then what is the problem?

Calin: That is a question for you. The two starting functions were "grade 11 functions", we completed the procedure correctly, so, what is going on?

Terry: Um, you are right, my work is correct ... the functions are right, what is wrong? Why do I get an impossible answer? You know, if I would work with complex numbers ...

Calin: Yes, but we do not teach complex numbers, and beside that, the starting functions do not deal with complex numbers ... we can restrict their domain ...

Terry: Hmmm. You know, I don't think I can answer this question right on the spot. I did not see something like this in the textbook or in the IRP's ... I probably have to go back and think about it more.

Terry was confronted with the same problematic example as Sam, and she could not give an explanation for the "anomaly" she obtained through calculations: she started with two Grade 11 functions, she composed them correctly, and her answer was out of the boundaries of the Grade 11 curriculum. Terry tried to extend the domains and ranges of

the functions to the set of complex numbers, but did not recognize that the lack of fit of the domains and ranges of the two functions created the problem.

With regards to the non-commutative property of the composition of functions, most of the teachers seemed to be aware of this essential feature. Three of them stated the property explicitly when describing what they find important in teaching this topic, and with one exception, all the rest of the teachers mentioned this essential feature when composing the examples provided by the interviewer, or when demonstrating their own examples. Below I present an excerpt from the interview with Mira:

Mira: I guess one thing that I would illustrate, perhaps not right after I explain how to compose functions, but, just what it comes to mind, is that generally speaking $f(g(x))$ is not equal to $g(f(x))$, that the order is important. The function that you get is different, depending on which function is embedded into which function

Calin: Is this rule always true, or is it only sometimes true?

Mira: It is true most of the times, but there are some exceptions too.

It is clear that Mira knows that the composition of functions is non-commutative and she is also aware of some exceptions from the rule, but she did not mention when these exceptions occurred.

A different "feature" came up in the discussion with Dan. He tried to use Cartesian graphs to explain composition of functions. He also tried to explain what a non-rote understanding of this topic means. Below I present a part of his interview:

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Calin: So, when teaching composition of functions, what are your goals, what do the students have to learn?

Dan: Um, I'd like them to be able, uh, I'd like an understanding there, rather than just a rote procedure. An understanding of what they've done with the functions, and to be able to relate that to a graph.

Calin: Um, graphing? What do you mean by relating to a graph?

Dan: They should be able to do this [calculate the composite function], and then understand what is going to happen to the graph.

Calin: So what would happen to the graphs in the case of these two functions: $f(x) = x^2$ and $g(x) = 3x - 2$?

Dan: [after he made some calculations] Um, this should be, um, that's a good one, (laughs), I would have to do a little bit of work to come back to this and to prepare for this one, to give them a good understanding.

It is safe to assume that Dan tried to build a bridge between the topic of composition of functions and the topic of transformations of functions approached in Grade 12. The topics that are part of the transformations of functions unit can be considered as containing applications of the composition of functions. The translations, reflections, stretches and absolute value transformations are obtained by composing a linear function with another function. The reciprocal of a function transformation is a composition of the reciprocal function and a given function. This unit could be treated as an application of composition of functions, and not as the main vehicle to deliver the lesson on composition of functions. By using the transformations of functions as the delivery package in teaching the composition of functions, one reduces the breath of the topic to be taught and limits students' vision on functions: not all functions have

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Cartesian graphical representation, and not all composition of functions can be easily visualized through Cartesian graphs.

Despite his good and well-sounded pedagogical intentions, Dan displayed a weak conceptual knowledge of the topic of composition of functions. His ideas about composition of functions were limited to the direct application of this concept to a subsequent topic of the high school mathematics curriculum (the transformation of functions). There was another teacher who at the beginning tried to link composition of functions and Cartesian graphs, but renounced the idea, after considering some examples.

The interviews with the inservice teachers presented some new aspects compared with the preservice teachers, mainly as results of their experience on the job. Two teachers claimed that in their schools, due to the volume and difficulty of the Grade 11 curriculum, the composition of functions is excluded from the regular topics. The inverse function is treated in Grade 11, and is revisited in Grade 12, but the composition of functions is taught only in the calculus courses. One of these two teachers was Mira.

Most of the inservice teachers made references to the Grade 12 course, with concrete examples relying mostly on transformations of functions.

6.1.2.3. Section Summary

According to the preservice teachers who participated in this study, the composition of functions seemed to be a fairly simple and mechanical topic, containing a procedure to be mastered. The student teachers, as a group, presented a limited understanding of this topic. They were not able to define the composition of functions,

and most of the students did not display any awareness with regards to the essential features of this topic. However, all but one of the participants were able to compute the examples given through algebraic expressions that were presented to them.

In the case of inservice teachers, all the members of the group displayed procedural fluency with regards to the algebraic examples that they were confronted with. None of the teachers defined the composite function, and only some of them mentioned the essential features of this mathematical entity. They described the algorithm for obtaining the composite function through words or examples. In terms of displaying knowledge of the essential features of the composition of function, more members from the group of inservice teachers displayed one or both of these features, compared to the group of preservice teachers. Still, there were some cases where the teachers could not pinpoint any important fact about the composition of functions except for the procedure. One teacher showed a misunderstanding, or the least to say, an incomplete understanding about the role of the composition of functions and the graphs of the initial functions.

6.1.3. The Inverse Function of a Function

The topic of the inverse function of a function contains a series of essential features, such as: the definition of the inverse function, the existence of the inverse function with the introduction of a new subset of functions, one-to-one functions, the procedure of finding inverse functions, and the relation between the Cartesian graphs of a function and its inverse. All these were discussed in detail in section 5.5.

In the interviews, the topic of inverse function of a function was brought into discussion with the help of question number six which had two parts, similar to the question regarding the composition of functions:

a. Give a mathematical definition for the inverse of a function. What are the main points that you emphasize when teaching this topic?

b. Give at least three examples of functions and their inverses, examples that you would use to illustrate the main ideas of this topic.

In the following sections I analyze the collected data that relate to the inverse function of a function.

6.1.3.1. The Case of Preservice Teachers

Contrary to the composition of functions theme, in the lesson plans, a number of students provided a mathematical definition for the inverse function concept. Here are some examples from the lesson plans:

Alina: Help students come to understand that $f(g(x))= x$ and $g(f(x))= x$ must both be true in order that f and g are inverses.

Tino: ... the definition of an inverse function as a function f^{-1}, such that $f(f^{-1}(x))= x$ and $f^{-1}(f(x))= x$.

Some of the students did not treat this relation as a definition per se, but integrated it in the lessons as discovery learning activities, or as a "checking" activity. For example, Said wrote: "Finally, it is hoped that the composition of functions and their inverse

function could be investigated (i.e. $f(f^{-1}(x)) = x$) from both an algebraic and graphical perspective."

Other students used this mathematical relation to make a connection with the topic of composing functions. Along these lines, Stela wrote:

> **Learning Outcome**: Verify if two given functions are inverses of each other.
> **Rationale:** The inverse function can be connected to composite functions since two functions $f(x)$ and $g(x)$ are inverses to each other if $f(g(x)) = x$.

Stela's statement was incomplete, the second part of the relation missing ($g(f(x)) = x$). Her answer is correlated with what knowledge she exhibited in relation to the properties of composition of functions, namely the non-commutative property.

A special case is Sam, who wrote in his lesson plans: "As for the composition of functions, I cannot see how this will be important to most students", but when he referred to the inverse function, he wrote: "... provide a flow chart to indicate the relationship between a function and its inverse. Ask students what is $f(f^{-1}(x)) = ?$" From these sentences, Sam seemed to be aware of the link between the inverse function of a function and the composition of functions, although he did not really internalize the importance of the composition of functions, without which the inverse function of a function does not exist.

Out of the whole group of students, two did not mention at all in their lesson plans the defining relation of the inverse function. Jim mentioned the computational process,

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and the comparison between the graphs. He stressed the graphical aspect of the topic. The second student, Chen, did not even make the connection between the graphs, but he emphasized real life examples.

In the lesson plans, all the students described the computational algorithm for calculating the inverse function given by an algebraic expression, and all of them introduced the f^{-1} notation. Six out of the ten lesson plans contained references to the graphical relation between a function and its inverse. The students wrote that the graphs of a function and its inverse are symmetrical to each other, and the axis of symmetry is the line $y = x$.

The topics of domain and range came up in four lesson plans. Three of the lesson plans mentioned these two topics only as a completion to the algorithm, the student teachers saying that the domains and ranges "change position" in the case of the inverse function. One of the lesson plans contained a discussion of the existence of the inverse function in general, with references to the notion of one-to-one function. The other three lesson plans contained particular examples where the domain had to be restricted (e.g. $f(x) = x^2 - 3$), but the topic was not extended to all functions.

The interviews were somewhat different from the lesson plans. Some of the students expended their sharing of knowledge. Below is an excerpt from the interview with Stela, who in the lesson plans mentioned only "half" of the mathematical relation that defines the inverse function of a function:

Calin:	Talking about the inverse function now, can you please give a mathematical definition for this concept?
Stela:	Okay, um, it's a mirror image, in a sense, with respect to a certain line, so if there's a function, then you put a mirror along the line of $y = x$ and you flip it over, you make the imprint of the mirror image, like that [she shows a graph on the paper, and just flips it over the line $y = x$].
Calin:	It sounds to me that that is just a graphical explanation of how the graphs relate to each other. Not all functions are given in a graphical representation, or even more, not all functions have a graph ...
Stela:	Well, you could find the inverse as switching two variables, so for example, because this is $2x + 3$, you could inverse the position of the variables. So it becomes $x = 2y + 3$, and you could actually convince the students by saying, okay, lets just point out each coordinate of every single point, and then see what happened to the other point, the corresponding point, and they'll notice that okay, it's flipped over, every single point has got a different coordinate ... So you could switch them over in that sense, and you could rearrange everything in terms of y equals to whatever the function is. In this way you do not have to draw, but still come up with your inverse function.

Stela did not come up with the relation between the two functions, insisting on the process of calculating the inverse, but not making the connection with the composition of functions. There was an inconsistency in the information presented by Stela in the lesson plans and in the interview, in the sense that the interview revealed less information

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compared with the lesson plans. In the cases of most of the other students the situation was different. The interviews provided more information than the plans.

A separate case was Alina's, who was consistent in the interview and the lesson plans, with regards to the accuracy and the depth of the knowledge displayed. Following is a section from the interview with Alina:

Calin: So, can you please give me a mathematical definition of the inverse function?

Alina: I believe both of these have to be true [she wrote down $f(f^{-1}(x))= x$ and vice versa], which is bringing what they just learned about composition of functions into good use as well.

Calin: Can you please expand a bit on this relation?

Alina: The domain of one is the range of another when you put them together in composition. Um, the inverse needs to bring you, so to speak, from the range back to the domain. You go back to where you started, but also you need to be able to do it both ways. Well, with inverses you do eventually come up to the idea of one-to-one, and to the possibility of having an inverse, right?

As seen in the above lines, Alina's answer was fairly complete.

A special interest raised some of the mathematical vocabulary that Tino used in his interview. When he talked about the role of the range and domain for the inverse function topic, he mentioned the terms of "injection" and "surjection", terms that would lead to the idea of one-to-one correspondence. He defined what he understood by these terms, but he did not extend his sharing of information to the notion of bi-univocal

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function. He stopped short of mentioning the notion of one-to-one function. The following is a part of the interview with Tino:

> Tino: Um, every element from the domain has to produce one element in the range, but also, the y 's have to have only one corresponding in the domain. I believe this is what was called in computing sciences an injection. Also, a surjection implies that for every y there is an x, something similar to the definition of the function, but the other way around.

Another student used the word "mapping", when he described the process of obtaining the inverse function, but he did not make the connection between the composition of functions and the inverse function of a function. He mentioned the one-to-one property after having to solve examples that did not match his image about a function that does not have an inverse: "Yeah, you see, this wouldn't work, because it's unless, now you can't, well it is not a one-to-one function I guess ..."

The student teachers that did not mention in their lesson plans the mathematical relation between the function and its inverse did not mention it in the interview either. Chen, was immersed in procedural explanations: "... this is how *to operate*, or how *to convert* ... *compute*, that's right, that's a good word, how *to compute* from here to here, inverse function of function." His main emphasis was not on mathematical features of the topic, but it was on real world examples: "I always feel it's very important to be able to bring concrete examples to students". One of the examples he used was the temperature conversion problem, and I present later in this chapter this example in an excerpt from the interview with Chen.

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During the interviews, all the preservice teachers were able to perform the algorithm for calculating the inverse function of the functions given through an algebraic formula. Also, all of them had written in the lesson plans, in one way or another, that one of the goals or outcomes of the lesson treating the inverse function is for the students to be able to compute inverse functions.

With regards to the Cartesian graphing of functions and their inverses, six students mentioned correctly in their lesson plans and in the interviews the relationship between the graphs of a function and the graph of its inverse function, referring to the line $y = x$ as being the axis of symmetry. John and Chen did not write anything in the lesson plans on this topic and they did not bring it up in the interviews. Stela correctly referred to this property in the interview but did not mention it in her lesson plans.

6.1.3.2. The Case of Inservice Teachers

In the interviews with the inservice teachers, two individuals were not able to give a mathematical definition for the inverse function of a function, and in their interviews the relation between the function and its inverse was not mentioned. As substitutes for the definition of the inverse function of a function, these two teachers were using the expressions of "undoing" or "reverse the steps". An example is provided in the following lines:

Calin: What is, mathematically, the inverse function?

Robi: You mean how to find the inverse function?

Calin: No, I meant, give me a definition of the inverse function, a mathematical definition.

Robi: Well, the inverse function undoes what the first function does. So, you have to reverse the steps that the first function prescribes. For example, if f says add, for the inverse you have to subtract. You basically have to end up in the same point where you left ...

Calin: That sounds right, but can you give me a formal mathematical definition?

Robi: Well, I can give you the mathematical way of finding the inverse

Calin: You mean the procedure?

Robi: Yeah ...

Robi was not able to find a way of defining the inverse function, and the defining relation of the inverse function did not come up when she talked about composition of functions and inverse function of a function.

The other teachers gave the definition of the inverse function as a definition *per se*, or mentioned it as a property of two functions that are inverses for each other. They used the function notation, and mentioned that the relation has to be true both ways, from left to right and from right to left.

In terms of the essential features, the one-to-one concept was mentioned in the proper context only by two interviewees. The rest of the teachers mentioned the roles of the domain and range only at a procedural level, in terms of switching positions. Some teachers did state that there are functions that need restrictions of the domain in order to have an inverse function, concretely referring to the case of quadratic functions. These individuals did not extend their explanations, and did not share any information related to the one-to-one concept.

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The graphing aspect of the inverse functions was well presented by all participants, with details and explanations about symmetry and the procedure of obtaining the graph of the inverse function. The procedure of calculating the inverse function for functions given by algebraic formulae showed up in all the interviews given by the teachers, through explanations and concrete examples of functions that had different degrees of sophistication. The predominant examples were linear functions and quadratic functions. The kind of examples that the inservice teachers possessed in their basic repertoire is discussed in a subsequent section of this chapter.

6.1.3.3. Section Summary

The inverse function topic seemed to be more familiar to these two groups, compared to the topic of composition of functions. In the group of preservice teachers, there were a few people who could give a definition of the inverse function, or who at least mentioned the defining relation as a property of the inverse function. Some preservice teachers mentioned all the essential features of the topic, relating them to the analytical thinking required to determine the existence of the inverse function, not just to the procedure of finding the inverse. A number of students did not show more than procedural knowledge of this topic. All the students treated the inverse function using algebraic formulae and Cartesian graphs, but only three of them used different representations for the functions. I discuss this aspect in a following section.

The data provided by the inservice group were similar to the data collected from the group of preservice teachers. There were teachers who displayed a thorough knowledge of the topic of inverse function of a function, but in some cases teachers

showed only a procedural understanding of this topic. With one exception, all of the teachers presented examples of functions given in algebraic form, tables of values or graphical form.

6.2. Different Representations and Alternative Ways of Approaching

In mathematics, many concepts can be represented in different ways. Different representations emphasize different aspects of the concept. Being able to represent a concept in various ways extends and enhances the understanding of the concept and its pure definition that can seem sometimes narrow and abstract. "Different representations give different insights which allow a better, deeper, more powerful and more complete understanding of a concept" (Even, 1990, p. 524). Teachers need to be able to represent concepts in different ways. They need to be able to use the most appropriate representation, from a mathematical point of view as well as from a didactical point of view. Teachers also, have to be able to link the different representations between them.

When the concept of function is introduced formally in the BC high schools, the Integrated Resource Packages (IRP's) also prescribe the introduction of alternative representations for this mathematical concept. The different representations suggested by the IRP's are: word stories, arrow diagrams, ordered pairs, algebraic expressions, tables of values and Cartesian graphical representations. The introduction of the function concept through multiple representations has more than one rationale: these illustrations exemplify how a mathematical concept can be expressed in different modes, how the word stories make the link between mathematics and our daily life, and finally, how

different fashions of expressing functions have different didactical values in emphasizing

the two main features of the function concept — arbitrariness and uniqueness. For

example, the arrow diagrams are very visual in expressing the function's arbitrariness, and

they de-emphasize the numerical and the "formula" aspect of functions. The graphical

representation in the Cartesian coordinate plane is suited to introduce the so-called

vertical line test, and so on. The algebraic expressions are the representations that create

the most problems for students, because of their abstract character.

In this study, the most prevalent forms of representation for the functions were the

algebraic formulae and the Cartesian graphs. Beside this form of representation, various

students used ordered pairs, word problems, and tables of values. The Cartesian graphical

representation was mainly presented in the lesson plans treating the inverse function, and

with two exceptions, was absent in the interviews for composition of function. Only one

of the participants, a teacher, used arrow diagrams as a representation, and with the

exception of the same person, none of the other participants used more than three

different representations through their lesson plans or interviews.

6.2.1. The Composition of Functions

The topic of composition of functions is a mathematical topic that presents

various aspects: a procedural aspect dealing with the calculation of the composite

function, an existential aspect that brings into play the role of the domain and range in

composing functions, and an important property of composing functions, the non-

commutativity.

Each one of the essential features of the composition of function topic can be taught to the students using different representations for the functions. Using different representations, teachers would implicitly use different approaches to teach the topic of composition of functions.

The procedural aspect needs to be treated mainly for functions that are represented through algebraic expressions, but the students need to be able to compose functions given as sets of ordered pairs, arrow diagrams or tables of values. The existence of the composite function, or the possibility of composing functions can be approached using examples of arrow diagrams, because this representation gives explicitly and separately the domains and the ranges of the functions to be composed. This representation has the advantage of being very visual, not requiring any algebraic skills. The non-commutativity property is well illustrated through functions expressed as formulae.

6.2.1.1. The Case of Preservice Teachers

In the lesson plans referring to the composition of functions, all the preservice teachers used as examples functions expressed as algebraic formulae. Beside the algebraic representation, five students used word stories/problems to exemplify the composition of functions and seven used Cartesian graphical representations of functions. Only one person used ordered pairs to illustrate the process of composing two functions.

In the interviews, in addition to the representations mentioned in the above paragraphs, one student said he would use tables of values to demonstrate how composition of functions works.

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When presented with tasks involving arrow diagrams (see below, figure 1) most

of the students had difficulties in finding the composite functions. They did not feel

comfortable operating with functions given this way. This manner of representing

functions is didactically one of the most suitable to illustrate the role of the domain and

range in composing functions, conferring a good visual picture of the domains and ranges

for the functions to be composed. The fact that this type of representation was missing

from the students ' repertoire correlates well with the findings linked to the participants'

knowledge regarding the role of domain and range in composition of functions. Below I

present two functions represented by arrow diagrams. The interviewees were asked to

compose the two given functions, in both directions, but most of them were not able to
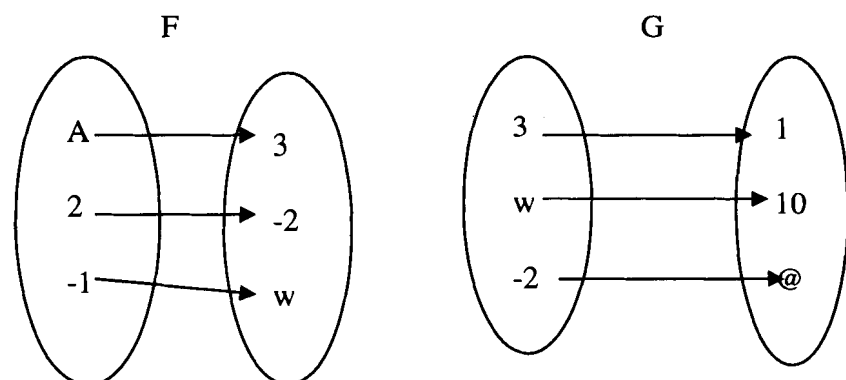
complete the tasks.

Fig. 5 Arrow diagrams for composition of functions

Without any calculations, since the range of G and the domain of F do not have

any common elements, it is not possible to compose F with G. The result of composing G

with F is a new function $H(x) = \{(A,1),(2,@),(-1,10)\}$. An example of answer to this

particular task is given below:

> Stela: Hmm! Arrow diagrams ... I don't remember doing this
>
> kind of stuff. I don't think I know how to work with this
>
> kind of functions right now, on the spot. I think I can
>
> find the inverses, but to compose them ...

The ways the preservice teachers represented the function in their lesson plans or interviews influenced their approach to teach this mini-unit. In the lesson plans, the persons using word stories used them either to introduce the topic to the students, or as examples of composition of functions after the lesson was taught. Three students used this representation at the beginning of the lesson plans as a mean to introduce the topic, and the other two used it as exercises, or applications for the topic. For example, Said wrote in his lesson plans:

> **Activities**: A] <u>Food Chain</u>: taking grass (g) as the first input; then
>
> the cow (c), acting as a function, "eats" the grass. Then tiger (t)
>
> could be the next animal (or function) that "eats" the cow [i.e.
>
> t(c(g)) with the brackets representing the stomach of the particular
>
> animal!!]

The person who utilized ordered pairs in her lesson plans introduced them at the beginning of the lesson, before starting working with formulae.

In their interviews, two students, consistent with the lesson plans, said they would approach the composition of functions from a different angle, compared with the other members of the group. The first thing they were planning to do was to make a link between Grade 10 knowledge and the topic to be taught. They started by asking their

virtual students to calculate the value of different functions given through algebraic expressions, using particular numbers. Jim wrote: "For example, students will be asked to calculate the function $f(x) = 3x + 4$ for different numbers, such as 1, 5, -3, by *substituting* the numbers in the formula. This exercise will also remind students about the function machine analogy."

Stressing the word "substitution" in their computational exercises, the student teachers went on to examples when instead of numbers they would use algebraic examples, from simpler to more complicated. Below it is presented an excerpt from the interview with Jim:

> Jim: The next step would be to ask students *to substitute* in the formula, instead of numbers, x^2. After that, we can go and use more complicated expressions, such as $2x + 1$ or other kind of functions. Hopefully by this time, the students will understand that all they have to do is basically *to substitute*.

The last step of the "teaching algorithm" was to use function notation for the algebraic expression, and to introduce the formal notation for the composition of functions. This approach had the merit of building on students' previous knowledge. It also gradually increased the difficulty of the examples used. It is a good approach towards the understanding and mastering the procedural part of the topic.

An interesting fact found in the interviews and in the lesson plans was the stubbornness with which the participants were trying to approach the composite functions

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using the Cartesian graphical representation of functions. This mode of representation of a function is not adequate to explain any of the features of composite functions. In cases that deal with functions that are a little bit more complicated than simple composition of linear functions with another kind of functions, the graphical representation of the composite function, using the two initial functions, is a tedious task that does not have any didactical value for a Grade 11 class. Giving such importance to the graphing part of this topic denotes that the preservice teachers had a narrow view on how to approach the topic of composite functions. I asked the participants who were overemphasizing the graphical aspect of composite functions how would they explain graphically the composition of two quadratic functions, and what is the benefit of trying to graph this new function. Confronted with this task, the interviewees seemed to realize the limited application of graphing for this topic.

6.2.1.2. The Case of Inservice Teachers

When talking about different representations and different ways of approaching the theme of composition of functions, the group of inservice teachers presented some particular differences compared to the group of preservice teachers. The main representation used by the teachers was the algebraic formulae, with a few exceptions. One teacher used arrow diagrams, formulae, word problems and ordered pairs through his interview. The rest of the group used formulae, and one teacher tried to explain the composition of functions through Cartesian graphs. This particular teacher, Dan, was asked by the interviewer to compose two quadratic functions, and at that point he said: "Um, this should be, um, that's a good one, I would have to do a little bit of work to come

back to this and to prepare for this one, to give them a good understanding." A more detailed excerpt from the interview with Dan is presented in section 6.1.2.2.

The inservice teachers were more oriented towards an algebraic approach when dealing with this topic. Their approaches were also less student friendly in the sense that the word problems that would relate the topic to day to day experiences were non existent, and they were less worried about motivating the students for learning this topic. The inservice teachers were more oriented towards a computational treatment of this theme. One of them, Terri, stated specifically that she teaches this topic only to practice algebraic manipulations:

> Terri: You know, when I give those kind of examples that is more practicing their algebra skills, make sure they're squaring the negatives and that kind of stuff, but conceptually, hmm, maybe just a discussion about what happens to the powers of the functions, and maybe some things about transformations added to it.

As mentioned above, the algebraic formulae are well fitted to illustrate the non-commutativity feature of the composition of functions. The fact that some of the teachers were familiar with this property corresponds to the representation that they used in presenting the topic. Most of the teachers did not deal with the domain and range role in composing functions, and that shows a correlation with the fact that they did not use representations that would illustrate well this essential feature of the topic.

Another difference between the two groups was that the teachers do not tend to use word problems as examples when dealing with this topic. They were able to

formulate word problems that illustrate the process of composing functions, but they declared that in their lessons they would use this kind of representation on a very limited basis. An example in this sense is what Mira said: "I do not use "real life" applications for this topic. I treat as applications the transformations of the functions, the Grade 12 topic, but I would talk a little bit of it in Grade 11, if I would teach composition of functions."

6.2.2. The Inverse Function

The topic of the inverse function seemed to be better understood by the participants in this study, as shown in a previous section. There were interviewees who gave complete and correct answers regarding the definition of the inverse function. They also discussed the essential features of this topic. These facts are illustrated in the ways the subjects of the study represented the functions that they were dealing with.

6.2.2.1. The Case of Preservice Teachers

The preservice teachers, as a group, represented the functions to be used in teaching the topic of inverse function in the following ways: Cartesian graphs, tables of values, ordered pairs, word problems and algebraic formulae. All of them used the Cartesian graphs and the algebraic expressions, and most of them used word problems. Some of them focused on the real world application for this concept. For example, in the excerpt below, Chen used the conversion between units of measurements to introduce the topic of the inverse function of a function:

Calin: So, when teaching the inverse function, from the aspects

we discussed earlier, what would you emphasize?

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Chen: Well, I would emphasize how to operate, or how to convert ... compute, that's right, that's a good word, how to compute from here to here, the inverse function of function. Besides this, I always feel it's very important to be able to bring concrete examples to students. I am a science major, and I like using science examples. In this case I would use converting measurements, for example from pounds to kilograms, or Celsius to Fahrenheit. I think that it is very important for the students to link the math concept with their daily lives. It motivates them.

In explaining the procedure for obtaining the graph of the inverse function, all of the students used the words "ordered pairs", but only Camelia used as an explicit example a function given through ordered pairs (her example was $f(x) = \{(2,3),(-1,4),(10,11)\}$ with the inverse $f^{-1}(x) = \{(3,2),(4,-1),(11,10)\}$). Three students mentioned that they would use examples of functions given as tables of values when showing the role of domain and range. None of the students used arrow diagrams, even after the fact that in the interviews, when discussing the composition of functions they were exposed to this kind of representation, and to its advantages in explicitly showing the domains and ranges.

6.2.2.2. The Case of Inservice Teachers

Like the previous group, the inservice teachers introduced the functions to be inverted using a multitude of representations. The main difference was that, in the case of composition of functions, the teachers did not put any emphasis on the word problems related to this topic. They were able to give examples of real life events that needed the

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use of the inverse function concept, but their main focus was on algebraic manipulations and on graphing the inverse functions in the Cartesian system of coordinates. All the teachers used at least two different representations, the common representations being the formulae and the Cartesian graphs. One of the teachers used arrow diagrams, both for introducing the inverse function as well as for illustrating the concept of one-to-one.

The fact that at the time of the study the inservice teachers were teaching senior mathematics courses, led the researcher to the expectation that these individuals would focus on formulae and Cartesian graphing representation. This assumption was made considering the content of the Grade 12 curriculum. In Grade 12, teachers are expected to teach students how to find the inverse of functions given through formulae or Cartesian graphs.

6.2.3. The Teaching Order

An important aspect when discussing the way the participants in the study plan to approach the teaching of the two topics is the order in which they considered to teach the two concepts. The way this unit is approached should be influenced by two main factors: the subject matter knowledge (the mathematical factor) and the pedagogical content knowledge (the pedagogical factor).

The mathematical factor deals with the link between the concepts, if there is a link, and with the development of these concepts. Do they develop in parallel, in a sequential manner, or do they derive from each other. In the case of the composition of functions and the inverse function, mathematically speaking, the order to teach the topics

is very definite: composition of functions comes first, and the inverse function comes second. In mathematical terms, the inverse function is a concept that is related to the composition of functions the same way as opposite numbers are related to addition, and reciprocals are related to multiplication. The inverse function is defined through the operation of composing functions, and without introducing the composition of functions the inverse function of a function does not make sense mathematically. Talking about inverse functions without knowing the composition of functions is similar to talking about opposite integers, without having any knowledge about addition of integers. That is, the introduction and the model are possible, but a complete understanding is unachievable.

Pedagogically, the opinions regarding the order in which the two topics need to be taught may vary. In the textbooks currently used in BC, the composition of functions is treated before the inverse function of a function. In the next sections I present the answers of the two groups with regards to the approach of these two topics, and their motivation for the choice they made.

6.2.3.1. The Case of Preservice Teachers

In this group, the opinions on which topic is better to be taught first were split. Two student teachers said that they would teach inverse first, and the rest of them said they would teach composition first and inverse second.

Camelia, one of the student teachers who chose to teach the inverse first was not able to give any mathematical reasons for her choice, but she gave some didactical

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reasons, both in the lesson plans and in the interview. Below is presented a part of the lesson plans that Camelia wrote:

> I chose to teach the inverse function first because I realized that students might be able to understand inverses easier, how inverses are related to the original function (with switch x/y, graphing, mirror line). Also, the computational aspect involved in inverse functions seems to me easier than the computations required for the composition of functions. In my third lesson I really wanted to tie together the idea of inverse of a function and composition of functions.

As it can be seen from above, Camelia said that the inverse function topic requires less algebraic manipulations and it is easier for students from a procedural point of view. Her arguments were sustained by a limited view on the number of different kinds of examples that a Grade 11 student might encounter in terms of inverse functions. Below I present an excerpt from the interview with Camelia. In this excerpt Camelia sustained the same point of view that she had in the lesson plans.

> Calin: In the lesson plans you wrote that you would chose to teach inverse function of a function prior to composition of functions, and you came up with a rationale for your decision. Can you elaborate a bit on this topic?
>
> Camelia: I think that the algebra involved in inverse functions is easier to deal with than the computations required in composition of functions ... The students would be able to concept or grab the concept of just switching x and y easier than plugging in and then re-plugging. ... It seems

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easier to remember to switch around and plug in ... so
then you plug x for y and y for x and you solve for y. I
just see it simpler because it's just one variable that they
are having to work with, whereas with the composition
you are going to end up with x^2 and so on. There could
be more algebraic problems or mistakes. Anyways, I
would link them in a third lesson.

In explaining why she opted for this particular order in teaching the two given

topics, Camelia took in consideration the difficulty the high school students have in

squaring binomials, and in expanding products of binomials. She did not have in mind the

difficulty of solving a quadratic equation, or a rational equation involving two variables.

Examples such as: "Find the inverse function for $f(x) = \dfrac{x+3}{x-2}$", or, "Give a maximum

domain so that the function $f(x) = x^2 - 25$ has an inverse, and find the inverse of this

function" are examples that grade 11 students are exposed to. Finding the inverse function

for rational or quadratic functions is part of the Grade 11 curriculum. The manipulations

involved in finding the inverse function of a function involve two variables, both x and y,

contrary to what Camelia said in her interview, whereas the calculations required for the

composition of functions involve only x.

The other preservice teachers gave a variety of reasons to justify why they would

teach composition first. Two of them gave a mathematical reason, saying that the inverse

function depends on composition, and also they stated that from their limited experience,

the students have less trouble with algebraic substitutions than with solving equations,

especially when there are two variables involved.

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Three of the interviewees did not provide any valid mathematical or didactical reason for their choice. Their arguments relied on the fact that composition comes first in any of the textbooks that they consulted to prepare the lesson plans. For example, in completing her argument, Stela said that she teaches composition first, as a follow up on algebraic operations with functions. She said that composition of functions is "like a kind of special multiplication for functions".

Chen based his choice on the assumption that composition of functions is more important than inverse function, and he could not think of too many real world application related to inverse function. Below is part of the interview with Chen:

> Chen: I would teach composition of functions first. The
>   composition of functions is an integral part of Function,
>   as relationships are often hierarchical or multi-folded. It
>   is more intuitive, directly relevant to our daily
>   experiences. I can easily draw examples for it and that is
>   why I think it is easier than inverse function. As well I
>   think intuitively composition of function is easier for
>   me. I don't see the inverse function of a function as
>   necessarily important or essential to introduce here.
>   However, it is surely, in my view, the least important of
>   all the concepts that I included in my lessons.

Three other preservice teachers gave only didactical reasons for their choice. Two persons argued that algebraically, composition is easier to calculate, and that students have more problems solving equations than substituting. The third person in this group approached the composition of functions as a follow up on the topic of calculating the values of functions for different numbers. He made the transition from numbers to simple

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expressions, and then to more complicated expressions and finally to the notation for the composition of functions.

The last student teacher had a split view on this issue. In the lesson plans he wrote that he would teach inverse functions first, for didactical reasons: "I would teach the inverse first. My rationale for this choice is that I can use the idea of a function machine, and it flows easily to undo something." In the interview he did not really change his idea, but he gave a mathematical reason for why someone could teach composition first: "Taking in consideration the relation between the function and its inverse, I could see someone teaching composition first. You need composition to use the relation."

6.2.3.2. The Case of Inservice Teachers

This group presented a balanced distribution in terms of the teaching order of the topics. Four teachers said that they teach inverse function of a function first. The reasons these participants gave for their choice were didactical in nature. Their choice did not seem to be influenced by the order the two topics were treated in the textbooks. One teacher related the inverse function to the Grade 9 unit on transformations of graphs, and to the Grade 12 unit on transformations of functions. She also based her selection on the assumption that inverse function is a more important topic in Grade 12 mathematics, compared with composition of functions. Another teacher gave as reason for her choice the easiness with which students grasp the idea of switching coordinates, in graphing situations as well as in algebraic examples. The other two individuals of this group stated that they would approach the inverse function topic with the help of Cartesian graphs, and

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would relate this topic to the larger topic of transformations of functions. In the following lines I present an excerpt from the interview with Robi.

Calin: When you teach composition of functions and inverse function, which of the topics would you teach first, and why do you choose that particular order?

Robi: Um, I would teach inverse function first, because I think I could teach it in a simple way by relating it to the graphs. They [the student] could visualize what's happening and they could get the idea about what's happening with inverse functions. And they could come up with their own rules as to what happens when they invert it ...

Calin: So, the reason for choosing to teach inverse first is didactical. Can you think of a mathematical reason to support your choice?

Robi: Um, no I don't. I would probably, hmmm, honestly, like a lot of times you teach out of the textbooks, right, so I'd probably have to look at what the textbook does ...

The textbook that Robi referred to in the interview treats the two topics in the opposite order compared with what Robi claimed to practice in the classroom. Robi indicated the textbook as being the main teaching resource that she uses, but on the other hand she does not seem to be aware of the content of this resource, in spite of her experience in teaching mathematics.

The other four teachers declared that they would teach composition of functions first. Two of the individuals gave as arguments for their choice the definition of the inverse function, as well as the fact that algebraic calculations involved in computing

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composite functions are in general easier for the students to perform, compared to the calculations required by the inverse function. One other teacher stated that he would do composition first because "... it gives them [his students] some practice with function, it builds their understanding of the function. With inverse function they seem to have more trouble." This particular teacher was not able to give a mathematical reason for his choice. The last teacher said that she always did composition first, but, now, thinking about it she would probably try with inverse, and then compare which way is more beneficial for students. She did not give any mathematical reasons for one way or the other, and the fact that she is willing to try both alternatives makes her argument didactical in nature.

6.2.4. Reflection and Summary

The function concept, as other mathematical concepts, can be represented in a multitude of ways. For teaching purposes, each representation has its advantages, related to the subtopic taught at a certain time. Teachers need to be aware of the different representations, not just for mathematical reasons, but in order to make the transmission and understanding of information easier for the students. Along the same lines, mathematical topics can be approached from different angles, depending of the goal of a lesson, and of the experiences of the teacher.

In terms of representations, both these groups showed a rich variety of portrayals for the function concept. The problem though is at the individual level, where both the teachers and the student teachers, with one exception, used no more then three and

usually two different representations. The algebraic formulae and the Cartesian graphs were present across the board for all the participants. Only two individuals used the arrow diagrams or the ordered pairs.

A significant difference between the two groups was the attempt of extensive use of real life situation by the group of preservice teachers, whereas teachers declared that for these two topics they ignore this kind of applications, or representation. The main emphasis in teachers' case was on algebraic and Cartesian representations.

This difference between the two groups could have a variety of explanations. One explanation could be the point where the members of the two groups were situated at the time of the study in terms of their career. The preservice teachers, at the beginning of their professional life, were enrolled in a university program, the Professional Development Program (PDP) at SFU. This program trained them according to a teaching philosophy that has as one of its central pillars the motivation of students. In high school mathematics, motivating students is often accomplished by making connections between the mathematics taught in the class and real life or daily life. From my experience and from the experience of other younger colleagues, PDP stresses the need for real life applications of high school mathematics when dealing with teenage students, and as a result of this pressure, the student teachers focus on real life examples when dealing with any topic of the mathematics curriculum. In the case of the inservice teachers, their experiences seemed to advise them differently. In the Principles of Mathematics 11 course, most of the students are over the stage of questioning the rationale for learning any mathematical topic that they encounter in the class. The experienced mathematics

teachers do not feel the need for real life connections, but rather the need for

mathematical connections with other topics of the curriculum.

Another explanation could be the fact that the preservice teachers lack first hand

experience in teaching a Grade 11 mathematics course from the beginning to the end.

Most of them had the experience of teaching part of the course during PDP's extended

practicum, but they did not complete the entire course. As a consequence, in planning

their lessons, the time factor is not really a concern. For the experienced teachers, who are

familiar with the breadth and the complexity of the topics contained in the Grade 11

mathematics curriculum, time is an issue, to the point that in some cases entire topics are

omitted. To have the time to teach all the topics of the course is probably more important

to teachers than making connections between mathematics and real life. It is safe to say

that at the time of the study, the two groups had a different point of view on the "less is

more" philosophical idea: the preservice teachers, fresh with the knowledge and ideas

inoculated through PDP, are less concerned with the time frame of teaching a topic,

whereas the inservice teachers, influenced by their experience, are more aware of the time

limitations when teaching in a real class.

For the case of the composition of functions, in the group of preservice teachers, a

very popular representation was the Cartesian graphical representation. This

representation is not suitable to be among the main illustrations of the function concept

for this topic. It is appropriate to use it as a link to other parts of the curriculum, but not to

be the main way of treating the composition of functions.

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In terms of approaching the two topics from different angles, the two groups were similar in some ways. In both groups there were individuals who gave mathematical reasons for the ways they would teach the topics, as well as didactical reasons. At the same time, there were teachers and preservice teachers that did not have a good reason for their choices. Some of the participants gave only didactical reasons for their choices, based on their opinions or experiences as teachers or as students. These didactical reasons were of two kinds: the first one was related to the complexity of the algebra involved in the topics to be taught, and the second one was based on connections with other topics of the curriculum. The latter was correlated with the use of Cartesian representation of functions, and one of the arguments presented was that this representation is more visual than others and it enhances students understanding.

To know the different representations of the function concept, and their particular didactical value is necessary for teachers in order to make informed choices regarding the kind of examples to use when trying to emphasize certain essential features of composition of functions and the inverse function. The ways of approaching a certain topic and the representations used by teachers in teaching a particular topic reflect the connections that exist between the subject matter knowledge and the pedagogical content knowledge. A strong interrelation between the two components of the content knowledge allows teachers to approach any particular lesson in the best possible way for the students they teach.

6.3. Basic Repertoire

Any topic of the high school curriculum requires the teachers to possess and have easy access to a basic repertoire of powerful and representative examples. This basic repertoire needs to include examples that illustrate the essential features and properties of the concept, principles, theorems, and so on. Having a basic repertoire makes possible a deeper understanding of the topic and creates the premises for a more meaningful teaching experience. The basic repertoire does not consist of the memorization of different kinds of examples. The examples that are part of the basic repertoire need to be understood by the teachers, and they need to be used for their specific mathematical and didactical role.

In the case of the composition of functions and the inverse function, the basic repertoire for a high school teacher should include at least the classes of functions that the Grade 11 curriculum deals with, such as: linear functions, quadratic functions, absolute value functions, radical functions and relatively simple rational functions. Beside these kinds of functions, teachers need to be able to use examples of functions that cannot be given through algebraic expressions.

The different essential features of the two topics considered in this study are better pictured if different kinds of examples are used. For example, to focus on the non-commutativity property of the composition of functions, the teacher needs to use at least two kinds of pairs of functions: examples of pairs of functions for which the general rule of non-commutativity holds, and "non-examples", pairs of functions for which the composition is commutative. The part of the basic repertoire addressing this property

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would contain examples of functions that are different from linear functions with coefficients 1 (e.g. $f(x)= 2x - 3$, $g(x)= x^2 - 2$). These examples show that the composition of functions is in general non-commutative. The non-examples are necessary in the basic repertoire and they illustrate the fact that in some particular cases the composition of functions is commutative. In this category are included pairs of first degree functions with coefficients equal to 1 (e.g. $f(x)= x + 4$, $g(x)= x - 7$).

The kind of examples of functions that are well suited didactically to emphasize the role of the domain and range in the composition of functions, are functions represented through arrow diagrams, ordered pairs, or tables of values. These kinds of examples are visual, and non-computational, and the students are able to handle them easier, since there is no algebraic manipulation involved in the process of finding the composite function. Beside these functions, the basic repertoire needs to contain examples of algebraic functions that cannot be composed because of their incompatibility in terms of domain and range. One example of each kind is presented below.

1. Pair of functions that can be composed:

$$f(x) = \{(2,3),(1,4),(5,-3)$$
$$g(x) = \{(3,1),(-3,2),(4,5)\}$$
$$(f \circ g)(x) = \{(3,4),(-3,3),(4,-3)\}$$
$$(g \circ f)(x) = \{(2,1),(1,5),(5,2)\}$$

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2. Pair of functions that cannot be composed due to non-overlapping of the domains and ranges:

$$f(x) = \begin{matrix} x & y \\ a & 3 \\ 6 & 7 \\ 4 & 2 \end{matrix} \qquad g(x) = \begin{matrix} x & y \\ 5 & b \\ 1 & c \\ 8 & 9 \end{matrix}$$

In the case of the inverse function, the basic repertoire needs to contain functions given through arrow diagrams. These functions are well suited to introduce the concept of one-to-one functions. Using these kinds of examples, the fact that functions that are not one-to-one have no inverse are visualized easily by the students, and makes easier the passage to examples of Cartesian graphs and functions given through algebraic representations. Below I present a set of examples of arrow diagrams useful for emphasizing the need of the one-to-one concept in the teaching of inverse functions.
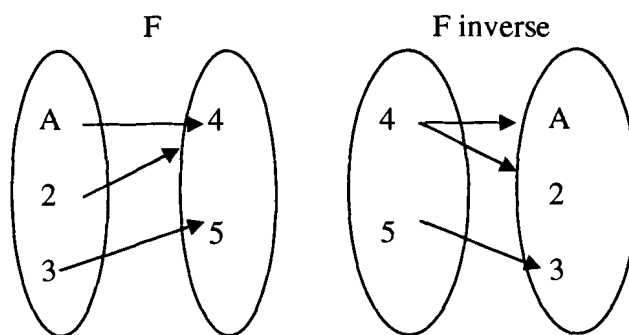

Fig. 6 Example 1 of not one-to-one function. F is not one-to-one, because both A and 2 have the same correspondent in the range. F inverse is not a function, since 4 has two correspondents in the range.

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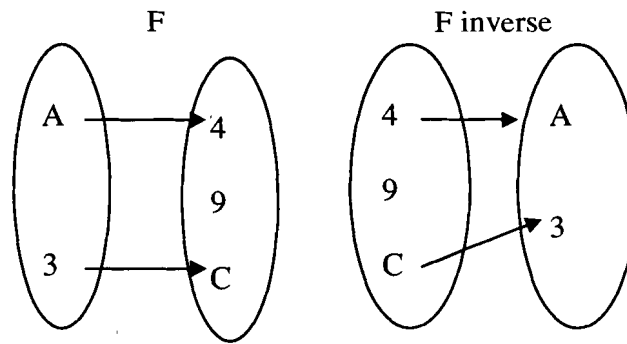
F   F inverse

Fig. 7. Example 2 of not one-to-one function. F is not one-to-one, because 9 does not have an "x" that corresponds to it. F inverse is not a function, since 9 has no image in the range.

An interesting case of linear function that should not be missing from the basic repertoire of teachers is the case of the constant functions. This is a case of functions where the algorithm of calculating the inverse function using function notation and algebraic manipulations breaks down. Constant functions are the only examples of linear functions that are not one-to-one, and consequently do not have inverses.

The richness of the basic repertoire that the participants in the study exhibited is a reflection of how broad their pedagogical content knowledge is.

In the following sections I present what the participants in the study exhibited as their basic repertoire.

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6.3.1. The Case of Preservice Teachers

As was mentioned before, the preservice teachers, besides being interviewed, provided a set of lesson plans with the objective of teaching the composition of functions and the inverse function of a function. The lesson plans asked for concrete examples that the student teachers would use in teaching their classes. These examples, combined with the examples provided in the interviews illustrated what kind of basic repertoire the preservice teachers possessed.

As a group, the preservice teachers were fairly consistent in terms of what kind of examples they used, both in the interviews and in the lesson plans. All of them used examples of linear functions and quadratic functions, for both topics of composition of functions and inverse functions.

Regarding the composition of functions, the basic repertoire exhibited matched the kind of view the preservice teachers had on the topic. There were no examples of functions that would create problems in terms of the existence of the composite function. All the algebraic examples were examples of first or second degree functions, used for mastering the procedure of composing functions. One individual proposed examples of functions given through ordered pairs, and several of them presented adequate word problems geared to this topic. The basic repertoire of several students included examples of functions represented in the Cartesian system of coordinates. As mentioned above, this kind of representation for the concept of function can be used as an example of some applications for the composition of functions, but is not adequate for the teaching of composition of functions. The ten lesson plans of the group contained fifteen examples of

linear functions, and eleven examples of non-linear functions. Below are listed all the

non-linear examples that the preservice teachers presented in their lesson plans

concerning the composition of functions:

$$y = x^2$$
$$f(x) = \sqrt{x-1}$$
$$y = x^2 + 10$$
$$g(x) = \frac{1}{x-1}$$
$$f = \{(-1,3),(1,4),(3,7)\}$$
$$g = \{(-3,6),(0,8),(4,12)\}$$
$$f(x) = \sqrt{x}$$
$$g(x) = |x|$$
$$f(x) = x^2 - 3$$
$$f(x) = 1 - x^2$$
$$G(x) = 2x^2 - 7x$$

For the inverse function, the basic repertoire was almost identical to the basic

repertoire for the previous topic. The difference is that the quadratic functions, beside a

procedural role, also played a role in the discussion regarding the fact whether the inverse

relation is a function or not. All the student teachers approached this subtopic from the

angle "the inverse of a function is a function or it is not a function": none of them

considered the existence of the inverse function. In the case of the inverse function, the

Cartesian representation of functions was used adequately, emphasizing the properties of

the domains and ranges of the function and its inverse, as well as the relation between the

two graphs. The case of the quadratic functions and the discussion on restrictions of the

domain were treated through Cartesian representation and not through algebraic

computations. For the inverse function, the individual who used ordered pairs for

composition of functions, Camelia, used them again to emphasize the switching of the

coordinates in calculating the inverse. Three other people used tables of values for the

same purpose. In the lesson plans there were twenty-three linear functions used as

examples to calculate the inverse function, and seventeen non-linear functions. Below I

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present all the non-linear examples of functions that the preservice teachers used in their lesson plans for teaching the inverse function of a function.

$$f(x) = \sqrt{x} \qquad\qquad f(x) = (x-2)^2 + 4 \qquad f(x) = \sqrt{x-1}$$

$$g(x) = \frac{x}{2x-3} \qquad\qquad h(x) = x^2 + 2 \qquad\qquad y = x^2 - 4$$

$$f = \{(-2,-8),(3,4),(4,7)\} \qquad y = x^2 + 4 \qquad\qquad y = x^2$$

$$y = \frac{1}{2x+5} \qquad\qquad g(x) = (x-7)^2 - 4 \qquad f(x) = \frac{x}{x-1}$$

$$g = \{(-1,2),(4,7),(-10,12)\} \qquad f(x) = 3x^3 \qquad\qquad y = \sqrt{x-2}$$

$$y = (x+3)^2 \qquad\qquad f(x) = \frac{x}{x-4}$$

As evident from the examples above, the examples provided in the lesson plans contain various classes of functions, besides linear functions. As specified in a previous paragraph, there are no examples of functions given through arrow diagrams or tables of values, and there is only one example of a function given through ordered pairs.

6.3.2. The Case of Inservice Teachers

The basic repertoire of inservice teachers was in some ways similar to the basic repertoire of the preservice teachers. With the exception of one teacher who also used examples of arrow diagrams and one individual who used ordered pairs, the rest of the group used algebraic formulae, mainly linear and quadratic expressions. Cartesian graphical representations were used to emphasize the essential features of the two topics considered in the study. In Figure 4 I present examples of arrow diagrams used by the Ken in the interview. He used these diagrams to illustrate the role of domains and ranges in composing functions:
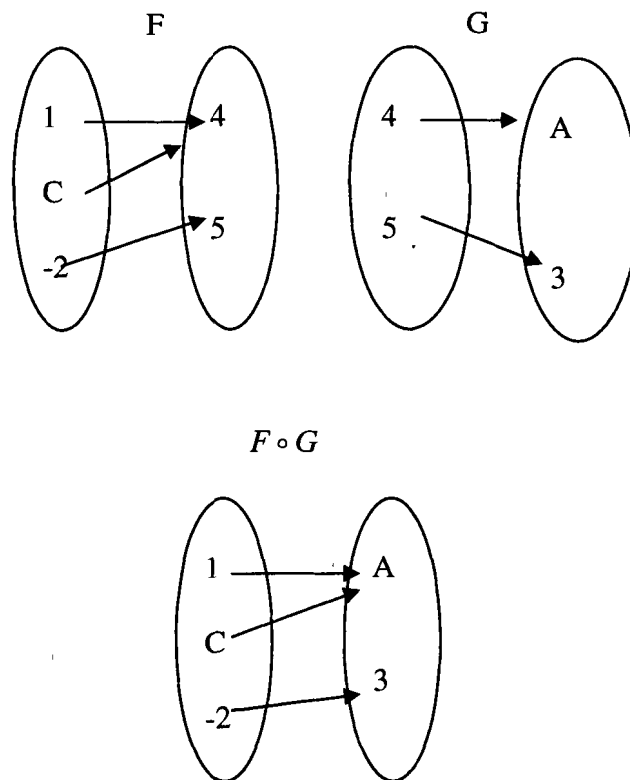
Fig. 8. The two functions used by Ken to exemplify the composition of functions and the role of the domains and ranges for this topic.

The examples of quadratic functions were used to discuss the functional character of the inverse. The inservice teachers used the Cartesian graphical representation of different quadratic functions to decide if the inverse is a function or not.

Some of the inservice teachers used some examples of relatively simple rational functions for emphasizing/practicing the procedural aspect of the inverse function.

As mentioned in previous section, the inservice teachers did not stress the importance of word problems when teaching the two considered topics. The most common examples of word problem that inservice teachers used were the *black box*,

number box, or *number machine* type of problems. With one exception, the teachers used the Cartesian representation of the functions only as applications of composition of functions, not as a main vehicle for teaching any of the essential features. This last mentioned representation was used for the inverse function, to underline the relation between the graphs of the function and its inverse.

6.3.3. Reflection and Summary

In general, both groups exhibited a limited pedagogical content knowledge, illustrated by the kinds of examples in their basic repertoire. The examples the participants showed were oriented mainly towards computational exercises. None of the participants discussed explicitly the case when composing functions is commutative, as an exception from the rule. With two exceptions, none of the participants used arrow diagrams or ordered pairs for showing any of the essential features of the two discussed topics. More than that, during the interviews, when they were presented with such examples, similar with examples found in the Grade 10 and Grade 11 textbooks, about half of the participants had trouble completing the tasks given to them.

The basic repertoire of preservice teachers contained a variety of word problems, but in the case of inservice teachers, these kinds of examples were a lot more concise and artificial. The possible reasons for this fact were presented in the discussion of the section 6.2.

None of the participants used constant functions in their explanation of the inverse function, and the radical functions, where present, were used only for computational purposes, not for their capacity of underlining some of the essential features.

To end this section, it is necessary to say that the basic repertoire displayed by both groups is in accordance to the kind of knowledge they displayed regarding the two topics. Their basic repertoire serves fairly well the mechanical, procedural knowledge shown by the participants in the study, but it is not rich enough to serve the teaching of the topics striving for students' analytical or conceptual level of understanding.

6.4. Knowledge of the High School Curriculum

This aspect of the theoretical framework addresses specifically the third component of Shulman's (1986) content knowledge, namely curricular knowledge, and it also relates to what Ma (1999) referred to as teaching with "connectedness".

Mathematics in general and the high school mathematics in particular are subjects that contain concepts and topics that are interconnected with each other. For a meaningful learning/teaching of high school mathematics, it is not enough to make connections between mathematics and the real world. It is necessary to build up a network of mathematical knowledge, with knots and synapses between concepts and sub-concepts. The learning of new mathematics topics needs to be anchored in the knowledge previously acquired. The new topics need to be connected with topics and concepts learned previously, and they need to be learned and taught with the goal of connecting them with subsequently taught concepts. Without these connections, the high school

mathematics becomes a collection of disparate units, and it loses its meaning and its beauty.

Beside the mathematical role that the composition of functions and the inverse function of a function play in the high school curriculum, these two concepts open the opportunity for developing students' analytical thinking. Through the topics it treats, a major part of the intended Grade 11 curriculum is geared towards developing students' capacity for analysis. Some examples of Grade 11 topics that are suitable to involve students' analytical thinking are: *the analysis* of the existence of roots for systems of linear equations, *the analysis* of the nature of the roots of the quadratic equations, solving absolute value equations and inequalities, the ambiguous case of the Law of Sines, and the list can continue. In order to take advantage of these new possibilities, teachers need to possess more than just a mechanical, or procedural view on the two topics of composition of functions and inverse function, they need to fully understand their essential features. It is not enough to see the inverse function as "undoing" a function. It is not enough to know that domains and ranges switch places. It is necessary to understand the role of domain and range in the existence of the composite function. It is necessary to be able to analyze if a function has an inverse or not. The one-to-one function concept needs to come into play, in order to fully understand the topic of inverse function of a function.

Teachers need to have a good knowledge of the entire high school curriculum, and not just of a certain course that they might happen to teach. Without this knowledge, their teaching will be truncated, comprising little pieces of information, meaningless to the

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students and to the education of the students. Professional high school teaching needs vertical and horizontal articulation of the topics to be taught, at least as much as it needs real life connections to the mathematics curriculum. In the next paragraphs I analyze the field data in terms of preservice and inservice teachers' curricular knowledge.

6.4.1. The Case of Preservice Teachers

In the interviews, the preservice teachers, as well as the inservice teachers, were asked their opinions on the importance of the topics of composition of functions and the inverse function of a function, and if they were aware of any high school mathematics topics where these two topics are encountered by students.

The answers to this question varied from participant to participant. Below I present an excerpt from the interview with Stela:

Calin: In your practicum you taught junior mathematics.

Stela: Yes, Grade 9.

Calin: Well, do you think that these two topics, the ones we talked about today, are they coming up somewhere, before we actually teach them to the students?

Stela: Hm, I am not sure. I did not teach the whole curriculum of Grade 9 or Grade 10 ... I know in Grade 10 we teach functions, but I am not sure if the inverse or the composition of functions are coming up. I cannot tell you exactly ... Logically, I would probably say they are not coming up before we teach them.

Calin: OK. Now, can you please tell me what do you think about the importance of these two topics in the high school mathematics?

Stela: I see these two topics quite separate from the other parts of the curriculum. I don't see how are they important if you don't do higher mathematics.

Calin: I am assuming that the students will take at least the Grade 12 mathematics course and probably they will go on and take at least one calculus course.

Stela: Oh, well, then for sure you have to know this ...

Calin: Why?

Stela: From mechanics point of view you are going to deal with a lot of functions in calculus. It is more like fundamental things that you have to have in your repertoire as a tool, rather, you may not necessarily have to understand the logics behind it.

From this interview, Stela's opinion on the two topics is clear: the topics need to be mastered from a procedural point of view. Stela was aware of the use of composition of functions in calculus, but she did not make the connection with the Grade 12 curriculum, where inverse functions such as exponential and logarithmic functions are an important part of the curriculum, together with trigonometric functions and their inverses. Also, Stela was not explicitly aware of the use of inverse functions in the lower grades (e.g. Grade 9 and 10, in solving trigonometric equations). Stela saw the two topics discussed in this study as separate pieces of knowledge, and did not integrate them with the previous acquired knowledge or the future topics to be taught.

Five preservice teachers mentioned the two topics as being useful for the students who will be oriented towards post secondary science education programs. They saw the topics as being useful in applications in physics or other sciences, but did not mention any

follow up in mathematics courses. Three student teachers mentioned the chain rule as a direct mathematical application of composite functions, and two said that the inverse function is useful in trigonometry. Two students were not able to give any uses for the topics, except for real life word problems.

In terms of previous topics that can be connected to the composition of functions and the inverse function, only four preservice teachers were able to find this kind of connection. One individual integrated the Grade 10 topic of calculating the value of a function for different numbers into the teaching the new topic of composition of functions. Relying on students' previous knowledge of calculating the value of functions at different points he developed the topic of composition of functions by gradually increasing the complexity of the substitutions required to calculate the composite function. Two student teachers said that the topic of inverse function was informally and mechanically used in Grade 9 and Grade 10 trigonometry. Another individual made a connection between the calculation of the inverse function, and the solving of linear equations, through the word "undoing":

> Chen: You kind of use the inverse function when you solve
> equations in younger grades. It is like undoing the
> equations: if it is plus you do minus, and so on. You also
> switch the order of operations. You do adding or
> subtracting first and then multiplying.

6.4.2. The Case of Inservice Teachers

In response to the usefulness of the composition of functions and the inverse function for the future mathematics education of the Grade 11 students, all the inservice teachers were able to connect these two topics with the Grade 12 curriculum, in particular

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with the unit on transformations of functions. The teachers made this connection explicitly, and also implicitly, when explaining what the essential features of the two concepts are. Regarding the inverse function, six teachers mentioned the exponential and logarithmic functions as applications of the topic, beside the transformation of functions. Below I present an excerpt from the interview with Ken that illustrates answers of the other participants:

> Calin: The next question that I want to ask you is: do you see any topics in the future mathematics education of the students where the composition of functions and inverse function of a function will be used? Are these topics useful at all, or not?
>
> Ken: They definitely will be used. The first thing that comes to my mind is the Grade 12, where the students need to know how to graph the inverse of a function. They also need to be able to calculate the inverses for different kinds of functions.
>
> Calin: Any other things that you can think of?
>
> Ken: Hmm, still in Grade 12, depending how complicated you want to teach the transformations unit, you can look at most of transformations as composition of functions. I do not do it that way, but it can be done, if you can afford the time. And, yes, inverse function is used in logs and exponentials, and in the trig chapters, when solving equations. But again, usually you do not have to much time to go in depth, mostly you teach students how to do it, how to solve the equations.
>
> Calin: That is good! Any other applications?

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Ken: Well, depending if the students will take more math
        courses.

Calin: I think that the students that take Grade 12 math will
        take at least one course of calculus, if not more.

Ken: Then, you sure have to use lots of function work. Inverse
        function ... I do not have a clear example now, but the
        composition of functions is one of the main tools when
        dealing with derivatives. I am thinking to the chain rule,
        which I think is very important in differential calculus.

Similarly to Ken, all the other teachers mentioned calculus as being the place that

the two topics will be used, and five of them mentioned explicitly the chain rule.

Regarding the past topics that relate to composition of functions, six teachers

mentioned the inverse of trigonometric functions as being a topic that relates to the

inverse functions, and four teachers alluded to the calculation of the value of a function at

a point as being a simple case of composing functions. An example along these lines is

presented below:

Calin: Can you think of places where you used the composition
        of functions or the inverse function, before you actually
        taught them?

Tasha: You mean in Grade 11?

Calin: Not necessarily.

Tasha: Well, the most frequent time is in trigonometry. I believe
        we start trig in Grade 9, or maybe 8? I did not teach in a
        while Grade 8, but I am positive we start in grade 9.

Calin: Can you be more specific?

Tasha: Well, when solving for an angle, we teach them to use
        calculators mechanically, to find degrees. It is kind of

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silly, because we do not teach them that, for example,
$\sin^{-1}$ is actually the inverse of sine, or what in calculus is
arcsin. They just know that they have to press some keys
on the calculators to get the right answer. It is not really
teaching them for understanding, but that is what the
IRP's say.

Two teachers related the composition of functions with the transformations unit

from Grade 9, considering the Grade 11 topic of composing functions an appropriate

bridge between Grade 9 and Grade 12 curriculums.

6.4.3. Discussion and Section Summary

Between the two groups of participants there was observed a considerable gap in

terms of the curricular knowledge. Prior to the interviews, as the preservice teachers

prepared lesson plans for teaching the topics of composition of functions and the inverse

function of a function, they were encouraged to consult different resources, such as

textbooks or the IRP's. In designing the lesson plans, the students were asked to provide

rationales for teaching the two topics. This could lead someone to the assumption that the

student teachers would have been able to make connections between the two topics and

other parts of the high school curriculum, despite the fact that they did not have an

extensive experience in teaching the high school curriculum. In reality, in the lesson

plans, as well as in the interviews, most of the preservice teachers were not able to relate

these topics to future or past concepts or units of the curriculum.

The teachers on the other hand, likely as a consequence of their experience in

teaching high school mathematics, were aware of some of the roles that the composition

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of functions and the inverse function of a function play in the Grade 12 curriculum. With regards to the previous topics that lead to the composition of functions and inverse functions, the inservice teachers were again more informed than the preservice teachers, most of them mentioning at least one topic from the Grade 8, 9 or 10 curricula where the inverse function of a function or the composition of functions were applied, without formal teaching. Even though, over all, the inservice teachers showed a better knowledge of the high school curricula, there were some individual cases of teachers who during the interview could not pinpoint connections between the considered topics and other mathematical topics.

As mentioned in an above paragraph, the intended Grade 11 curriculum contains several topics that have the intention to educate the students' analytical capacities. Two of these topics are the mathematical topics considered in this study. As a result of the procedural view that the majority of the of the participants in the study attributed to the composition of functions and inverse function of a function topics, the teaching of these topics loses its power and most of its significance. The two topics offer more than just opportunities for learning new mathematics and to practice algebraic manipulations. They present the educator with new didactical opportunities, opening windows for analysis and discussion regarding the existence of the composite and the inverse functions. Similarly to the unit of quadratic functions, where in order to decide the nature of the roots of an equation the students are not required to solve the equations, but they have to *analyze* the sign of the discriminant, in the case of the composite and inverse function it is not necessary to calculate the new functions, but an *analysis* of the domains and ranges of the

initial functions is required. None of the participants in the study gave any signs that they would take advantage of this new didactical opportunity offered by the two considered topics.

In the next section I analyze the general knowledge about mathematics of the participants in the study. This knowledge is part of the subject matter knowledge, one of the three components of teachers' content knowledge. This section treats together both groups of inservice and preservice teachers.

6.5. Knowledge About Mathematics

In this section the more general mathematical knowledge displayed by the participants in the study is described. Even (1990) wrote that the success of a concept in the discipline of mathematics is rooted in the new mathematical opportunities it opens. Concepts are important through their essential, unique features, which open new possibilities. Teachers should therefore be knowledgeable about the unique and powerful characteristics of a concept.

Functions in general, and the composition of functions and the inverse function of a function in particular, opened new possibilities in mathematics. The concept of function developed at the same time as calculus, and it is tightly related to the development of this branch of mathematics. The development of the function concept was treated in Chapter 2 of this study. Composition of functions is a relatively new concept, when compared with the arithmetical operations. Freudenthal (cited by Even, 1993) says that composition of functions created a wealth of new objects never known before. With the help of

composition of functions one can create functions as wild as one wants to contrive. Both composition of functions and the inverse function helped with the study of differentials and integrals.

Even (1990) claimed that knowledge about a certain piece of mathematics includes more than conceptual and procedural knowledge. It also includes knowledge about the nature of mathematics. This new kind of knowledge contains means and processes by which truths are established in mathematics, as well as the relative centrality of different concepts.

In this last category of the framework are included, for example, inductive and deductive reasoning, and the investigation methods using specific examples. Knowledge about mathematics also includes awareness of the continuous changes that take place in mathematics. Along these lines, preservice teachers and inservice teachers were expected to be able to give a modern definition for the concept of function as well as definitions of the composition of functions and the inverse function of a function. None of the participants in the study gave a function definition using sets and ordered pairs, and none of them were able to enunciate a definition for the composite function.

For the particular case of composition of functions, examples of displays of knowledge about mathematics were the cases when discussions about the commutativity of the composition of function were brought up. Below is an excerpt from the interview with Tino, a preservice teacher:

> Tino: For the most part it seems to hold [commutativity], and
>   so, and then here we notice that we can't commute these

two things ... it seems counter-intuitive, against the rules

of mathematical operations.

For Tino, the non-commutative property seemed to be counter-intuitive, and the

examples where this property is displayed seemed to be the exceptions to the rule of

commutativity. For him, the cases where commutativity holds were the cases that give the

general rule. From Tino's position, it is reasonable to affirm that his idea of mathematical

operations is limited to a commutative addition and multiplication.

Related to the commutative properties of mathematical operations, none of the

interviewees made any connection, or comparison, between the arithmetical operations

with numbers, and the composition of functions.

In the case of the inverse function of a function, a solid knowledge of mathematics

can be illustrated by the introduction of the set of one-to-one functions. The elements of

this subset of the larger set of functions are the only type of functions that have inverses.

The small number of interviewees that mentioned the one-to-one functions is discussed in

sections 6.1.3.1 and 6.1.3.2. None of the interviewees extended their explanations further

than explanations required for a Grade 11 topic. Some of the participants were not aware

of the fact that not all linear functions have inverses. An example in this sense is

presented below:

Calin: So, you teach your students the procedure to calculate

the inverse function of a function, using examples of

linear functions, to start with. Would you be able to find

for me the inverse function of this function? [$f(x) = 4$]

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Trista: Well, this function is a linear function. I change $f(x)$ to y and I get $y = 4$. I can write this as $y = 0 \cdot x + 4$ and then I will follow the procedure: I switch the variables and I get $x = 0 \cdot y + 4$. Hmm! We have a problem here ...

Calin: What do you mean?

Trista: We cannot solve for y.

Calin: Why not?

Trista: Because we will end up dividing by zero, and that is not possible. Hum, something is wrong. [She drew the graph of the initial function and then the graph of the inverse relation.] I can do it graphically. The answer is $x = 4$

Calin: So, how what is $f^{-1}(x)$?

Trista: I don't think you can do it that way. The formula doesn't have a y in it.

Calin: But you started with a linear function, and the algorithm didn't work. Why is that?

Trista: You are right ... linear function ... I am not sure why in this particular case you can't follow the normal steps.

It is clear that Trista does not think of using the concept of one-to-one function in order to solve the given example. Even after graphing the function, she did not see that the initial function is not one-to-one, and consequently, it does not have an inverse. Her reference to "normal steps" illustrates that she is influenced by a procedural approach of the topic of the inverse function.

The participants in the study had the opportunity to display knowledge about mathematics when asked about their preferred teaching order of the two topics, and the relationship between the inverse function and the composition of functions.

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Mathematically, the first topic that has to be introduced is the composition of functions, since the inverse function is defined through the composition of functions. In previous sections I discussed the arguments of participants who gave a correct mathematical reason for their choice, as well as the arguments of participants who chose to teach the inverse function first, in spite of being aware of the mathematical relation between the two topics. To illustrate the importance of the connection between the two topics, I present below an excerpt from the interview with Terry:

Calin: You chose to teach the inverse function of a function first, for didactical reasons. So, after you explain to the class that the inverse function undoes what the function does, how would you define the inverse function of a function?

Terry: Well, I would tell them that function and its inverse function would always give you x, or the point you started from. I would also start with some simpler examples and I would ask them to check them for me.

Calin: Ok. Can you think of me as your student and give me an example?

Terry: Sure. So, calculate the inverse of the function $f(x) = 2x - 3$.

Calin: Well, since the inverse has to give me x, the inverse will be $f^{-1}(x) = -x + 3$. Am I right?

Terry: No ... This is not right. They don't give you x.

Calin: Yes they do: $2x - 3 - x + 3$ give me x.

Terry: You did not follow the steps I taught you. You can't just add them.

Calin: Yeah, but I found an easier way of calculating the inverse function, and they give me x. What is wrong with it?

Terry: Hmm! You just did not follow the instructions, on switching the variables and solving for y.

Calin: I know that, but most of the times in math classes we were told that there is more than one way of doing things. Why are you saying know that my way is not right, when I get the right answer after I check the work?

Terry: I think I need to think how I would answer this question.

From this discussion, it is clear that Terry did not see where the "mistake" was: the inverse function of a function is defined in relation with the composition of functions, and not in relation with the addition of functions. She did not seem to really understand the fact that in order to define the inverse function you need to teach the composition of functions. Along the same lines, none of the participants tried to draw an analogy between the composition of functions and inverse functions on one side, and the addition and opposite numbers or multiplication and reciprocal numbers on the other side.

With regards to the relation between the composition of functions and the inverse function of a function, the teachers who gave a definition for the inverse function were expected to be aware of the role of the non-commutative property of the composition of functions. Just a few of the participants mentioned that the functions that are inverses of each other need to be composed both ways, without explaining why is this necessary.

From the data the interviews of this study provided, it seems that the teachers, preservice and inservice, showed a relatively limited knowledge of general mathematics.

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They seemed to ignore the mathematics they learned through university courses. With

some isolated exceptions, the participants treated the two discussed topics at a high

school level. They were actually intending to refer to the textbooks and to the IRP's for

clarifications on what the topics treat. A significant example in this direction is what one

of the preservice teachers said: "I will have to look in the textbook or IRP's to refresh my

memory on this stuff."

Another particularity that sustains the view that the participants in this study

seemed to ignore the mathematics they learned through university courses is the

frequency of the appearance of the Cartesian graphs in the discussion on composition of

functions. As I mentioned before, this kind of representation is not suitable for the

teaching of the mentioned topic, but a fair number of the interviewees insisted on using it.

Besides the fact that didactically the Cartesian representation of functions is not

advantageous, the usage of these kinds of examples leaves the impression that all

functions have Cartesian representations. This fact is not true, but high school

mathematics seems to perpetuate the idea that all functions can be represented in the

Cartesian plane. This kind of ideas is illustrated in Simion's lesson plans. Below I present

part of these lesson plans:

Objectives:
To have students explore Function Composition through
the use of technologies such as Graphing Calculator (or
other computer programs).
Activities:

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Graph two equations and try to predict what their
composition might look like.

Graph the composition of two functions and predict what
the original functions might have looked like.

From the activities that he proposed in the lesson plan for the composition of
functions it is clear that Simion has a very limited view on the mathematics involved in
the composition of functions. It may be 'fashionable" to use technology in exploring
mathematics, but this particular participant in the study does not seem to be able to
discern when and for what the use of technology is appropriate. It is discussed in a
previous section why the Cartesian graphical representation is not well suited for the
teaching of the composition of functions.

In conclusion, the mathematical knowledge displayed by the participants in this
study was limited. In the case of some of the interviewees this display was caused by a
lack of such knowledge. It could be argued that in other cases the limited display of
general mathematical knowledge could have been triggered by the fact that the topics
discussed were high school topics, and the participants did not consider extending their
explanations beyond the high school curriculum. However, I believe this is unlikely,
given the prompts and the opportunities provided by the clinical interviews.

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CHAPTER 7
DISCUSSION AND CONCLUSION

This study draws to an end. The last chapter of the thesis contains several

sections: in the first section I present a summary of the results of the study; the following

sections contain a discussion of these results, contributions of the study for the field of

mathematics education, and the limitations of the study.

7.1. Summary

The objective of this study was to examine the content knowledge of teachers and

preservice teachers with regards to the topics of the composition of functions and the

inverse function of a function. For a comprehensive treatment of these two concepts, it is

necessary for a teacher to have a solid general knowledge of the concept of function,

including the definition of a function, with its essential features, uniqueness and

arbitrariness. There is also a need for a clear understanding of the notion of domain and

range. The main focus of this study was on composition of functions and the inverse

function of a function, while the general concept of function played a secondary role. As

I mentioned in Chapter 6, the interview question that referred specifically to the definition

of the function concept had the role of opening up the discussion and of introducing the

notions of domain and range. In the following paragraphs I present a summary of the

findings of the study, grouped according to the theoretical framework I used for data

analysis.

1. Essential features. As I expected, all the participants in the study felt that students needed to know the definition of a function, as a prerequisite for the teaching of the composition of functions and the inverse function. Combined with "a good understanding" of what a function is, the participants mentioned algebraic skills, familiarity with the function notation, and graphing skills as being other prerequisites for teaching the considered topics. The two groups of participants were similar in their responses to this question, with the exception that some of the inservice teachers said that they would re-teach these prerequisites before starting to teach the composition of functions and the inverse function.

The answers to the question referring to the definition of the function concept are different from one group to another: four of the preservice teachers could not recall a correct definition of the function, and three of them enunciated a definition using the terms of domain and range. In the case of inservice teachers, all of them gave an acceptable definition for the function concept. It is necessary to mention that none of the participants gave a contemporary definition of the concept, using ordered pairs. All the definitions were definitions similar to the definitions found in the textbooks used in BC's high schools.

The questions concerning the composition of functions provided a variety of answers. None of the participants was able to give a definition for composing functions. All of the participants explained what they understood by composing functions, using different examples, most of them being examples of functions defined through algebraic formulae. The participants presented a good procedural knowledge for the topic of

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composition of functions, when they had to compute composition of functions given through formulae. Some of the participants, both inservice and preservice teachers, had some difficulties when they faced examples of functions given through arrow diagrams.

With respect to the essential features of composition of functions, the preservice teachers did not mention clearly what these features were. None of the students stated clearly the non-commutative property of composing functions in the lesson plans, and only a few exhibited an awareness of this property during the interview. Two of the students displayed false images about this property. The role of the domain and range, another essential feature of composition of functions, was not referred to in any of the lesson plans, and in the interviews only a few mentioned it. The group of teachers possessed a better knowledge of the essential features of this topic. Some of the teachers presented the non-commutativity property as part of their instruction, and one of them defined the role of the domain and range in composing functions.

In the case of inverse functions, both groups seemed to have a better knowledge compared to the topic of composition of functions, and they were more similar than in the case of composition of functions. Some of the interviewees were able to correctly define the inverse function, or at least they were correctly giving the relationship between the function and its inverse. With some exceptions, the participants defined the role that the domain and range play for the inverse function, and a minority mentioned one-to-one functions, the only class of functions that have inverses. All the inservice teachers and the majority of the preservice teachers were aware of the relationship between the function's graph and the graph of its inverse. Regarding the procedural knowledge, all the

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interviewees were able to compute the inverses of the elementary functions that are the object of the Grade 11 BC curriculum.

2. Different representations and alternative ways of approaching the concept. The opinions of the participants in the study regarding the ways of approaching the topics of composition of functions and the inverse function were divided: some would teach first the inverse functions and would follow up with the composition of functions, and some would treat the two topics in the opposite order. The reasons given for one or the other approach varied from individual to individual. Some of the participants gave mathematical reasons for their choice, but most of them relied on pedagogical reasons for their option.

In terms of representations of the function concept, the most common representations that the participants used were the Cartesian graphs, algebraic formulae and word problems. Only a small number of participants claimed that they would use other representations, such as tables of values, ordered pairs, or arrow diagrams. The two groups seemed to be different in the emphasize they put on the word problems: the preservice teachers seemed to focus more on this kind of representation compared to the group of inservice teachers. Some members of the second mentioned group claimed that they do not use this representation at all when they teach composition of functions and the inverse function of a function.

3. Basic repertoire. In chapter 6, a detailed list is presented with the examples used by the preservice teachers in the lesson plans. The examples mentioned in the interviews are not very different from the examples listed in Chapter 6. Most of the examples are

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examples of linear functions, and quadratic functions. There are few examples of rational functions, table of values, ordered pairs, cubic functions and radical functions. The preservice teachers gave examples of word problems that illustrate the composition of function and the inverse function concepts. Some of these word problems could not be represented through algebraic formulae, while others could be modelled by linear functions (e.g. the temperature conversion problem). Only one person presented examples of arrow diagrams.

4. The knowledge of the high school curriculum. The groups were fairly different with regards to their ability of making connections between the topics of composition of functions, the inverse function, and other segments of the high school curriculum. Although there were cases of inservice teachers who could not make these kinds of connections, overall, the group of inservice teachers presented a higher ability to connect the two considered topics to other topics of the high school mathematics curriculum. Most of the preservice teachers were not able to pinpoint any connections between the topics making the object of this study and other topics of the curriculum. Their tendency was to connect the composition of functions and inverse function to daily life experiences, rather than to make connections with other mathematical topics.

5. Knowledge about mathematics. With the exception of one inservice teacher, the display of knowledge about general mathematics was fairly limited. As written in Chapter 6, this fact could have different explanations.

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7.2. Teachers' Content Knowledge

In this section, I address the research question of my study using specific reference to Shulman's (1986) notion of content knowledge. Below I restate the research question that my study addressed:

1. What is teachers' and preservice teachers' content knowledge regarding the topics of composition of functions and the inverse function of a function?

2. How are the components of the content knowledge — subject matter knowledge, pedagogical content knowledge and curricular knowledge — interrelated?

3. How are the three components of the content knowledge influenced by the teaching experience?

As it is detailed in chapter 4, Shulman (1986) gave three components of teachers' content knowledge: subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK). To answer the first research question, the criteria of framework used for data analysis addresses the individual components of the content knowledge. The first and fifth criteria of the framework used to analyze teachers' content knowledge with regards to the composition of functions and the inverse function of a function address the SMK. From the above summary it can be concluded that both groups, teachers and preservice teachers, displayed a relatively weak SMK. With regards to this component of the content knowledge, the two groups were different in some aspects and similar in others. The inservice teachers displayed a better grasp of the

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concept of function, but on the composition of functions they did not show a higher SMK than the preservice teachers. The SMK regarding the topic of inverse function seemed to be higher than the SMK related to the composition of function, for both groups. Among the members of the groups, the variations were fairly high. For example, some participants were not able to enunciate any definition for the inverse function, where others were able to give a correct definition for this concept.

An interesting and intriguing fact related to SMK was that the preservice teachers "performed" better on the interviews then they did in the writing of their lesson plans. It is true that the interviews came after the lesson plans, but when writing their lesson plans, the students could use other available resources, along with their own "inherited" knowledge, such as high school textbooks, IRP's or university textbooks. Some of the preservice teachers actually referenced these resources in their lessons plans, but the content of the plans did not reflect an in depth use of the mentioned documents.

The minor differences between the two groups of participants, regarding the SMK of the topics of composition of functions and the inverse function, suggests a partial answer to the third research question: in the case of this study teaching experience did not enhance the SMK of teachers.

The second component of the content knowledge, the PCK, is illustrated by the second and third criteria of the theoretical framework. PKC refers to the most appropriate forms of representing ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, and it does not refer to how a lesson is delivered. Implicit within the PCK is the knowledge of why a certain topic is difficult for learners.

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My expectation was that the two components of the content knowledge, PCK and SMK, would be interrelated. In the case of the participants in this study, the above statement seems to be true. Both groups of participants showed a narrow PCK, reflected by their ways of approaching the topics of composition of functions and the inverse function, as well as by their ways of representing the functions during the process of teaching. The basic repertoire of the participants in the study was limited, but it reflected the SMK of the two mentioned topics. In the following paragraphs I present an answer to the second research question, by elaborating on the link between the SMK and PCK in the case of the participants in this study.

The basic repertoire of the participants in the study reflected their SMK. For example, most of the participants were not aware of the role of domain and range for the composition of functions. As a direct consequence, their basic repertoire did not contain any problematic examples that would emphasize the importance of domains and ranges in composing functions. Continuing with the topic of composite functions, the participants did not provide specific examples for showing the exceptions from the rule of non-commutativity. Some of the members of the two groups were aware of the non-commutative property, but they did not push the issue to the end.

Regarding the inverse function, the basic repertoire of most of the teachers and preservice teachers contained for example, quadratic functions. The role of these examples is to show that not all the "elementary" functions have inverse functions. The basic repertoire of all the participants missed the example of constant functions, where the algebraic algorithm fails to provide an answer. This gap in the basic repertoire can be

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explained by the fact that the SMK of most of the participants did not contain a clear idea about the notion of one-to-one function.

With respect to the different representations and the ways of approaching the topic, the participants presented divided ideas on the order of teaching the two topics. Since there were teachers and preservice teachers who argued for composition first, and there were participants who argued for inverse function first, it can be said that experience does not have much to do with the ways of approaching the two topics. This argument is sustained by the reasons presented by the interviewees: the same argument, algebraic "easiness", was presented as a decision factor in teaching composition first by one person, and then it was presented as a decisive factor for teaching inverse function first by a different person. There were a few participants with arguments rooted in their SMK or PCK. An example of this kind is an argument relying on the mathematical fact that there is no inverse without defining first the composition of functions.

The different representations for the functions exhibited by the members of both groups illustrated a limited PCK. For example, only one person used arrow diagrams in his explanations. In previous chapters, I mentioned the pedagogical advantages presented by the use of this representation. The most frequent representations presented in the study were the algebraic formulae and the Cartesian graphs, representations that are not necessarily the best representations to use when introducing essential features of composition of functions and the inverse function of a function. The almost exclusive use of these two representations denotes the limited capability of the participants in this study

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in using the most illustrative examples for making the mathematics comprehensible to students.

Contrary to expectations, in answering the third research question, this study shows that experience did not necessarily make a difference in the PCK displayed by the inservice and preservice teachers. The PCK seems to be influenced or related more tightly to the SMK than to the years of teaching experience that the individuals possessed.

The "knowledge of the curriculum" criteria of the framework addresses directly the CK, specifically, the vertical CK. The vertical CK deals with connections between different high school mathematics topics. Related to this criterion, the two groups were different. As shown in the summary above, the inservice teachers were able to make more connections between the topics of composition of functions and inverse function, and other topics of the curriculum. In the case of CK, this study shows that the teaching experience plays a significant role.

To summarize this section, the participants in my study displayed a relatively weak content knowledge of the topics of composition of functions and the inverse function of a function, illustrated by the weak SMK and PCK exhibited by both inservice and preservice teachers. The study shows that CMK and PCK are strongly related to each other, and, what is surprising, they are minimally influenced by experience. In terms of CK, the two groups were different: the inservice teachers showed a relatively good knowledge of the curriculum, but the preservice teachers displayed a fairly weak ability to make connections between the topics of the study and other topics of the curriculum. The

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CK is positively influenced by the teaching experience, and it seems to be relatively independent of SMK and PCK.

In the next section I present the contributions of this study to the field of mathematics education.

7.3. Contributions

7.3.1. Extension of Knowledge, Refinement of Theory

In Chapter 4, I referred to the limited amount of research in the topic area of teachers' knowledge and understanding of the concept of function. This study is part of this category of research. It looks at how teachers understand a certain mathematical topic, and it adds information on what Shulman (1986) called the missing paradigm: the absence of focus on teachers' subject matter knowledge in the study of teaching.

The research literature that looks at teachers' understanding/knowledge of the topics of composition of functions and the inverse function of a function is very limited, close to non-existent. The present study contributes to the field of mathematics education in studying the content knowledge of teachers and preservice teachers regarding the two above mathematical topics. The study suggests that in the case of the two groups of participants, the teaching experience does not have a major influence on changing the SMK and the PCK, but it extends the CK of teachers. The study found that the SMK and the PCK are interconnected to each other.

On a different note, the study produced a refined version of the theoretical

framework that Even (1990) proposed for determining teachers' subject matter

knowledge. The refined framework addresses teachers' content knowledge and contains

five criteria, compared to the seven criteria proposed by Even. The framework used for

data analysis in this study combined some of the criteria of the original framework,

although the most significant change of the framework was the addition of the criterion

"knowledge of the mathematics curriculum". This criterion makes the framework more

comprehensive: the four criteria obtained from Even's framework address the SMK and

the PCK of the participants, and the newly added criterion gives completeness to the

framework, addressing the CK of the teachers and preservice teachers.

To summarize this section, the theoretical contributions of this study consist of:

adding new knowledge on teachers' content knowledge of high school mathematics,

specifically, what is teachers' knowledge of the topics of composition of functions and

the inverse function; determining a relation between SMK, PCK, and CK; determining

the influence of the teaching experience on the components of the content knowledge; a

refinement of the theoretical framework proposed by Even, this refinement includes a

criterion for analyzing the CK of teachers.

7.3.2. Methodological Contributions

The methodological contribution that this study brings to the field of mathematics

education consists in the methods of collecting the field data. To collect field data for

investigating the inservice teachers' content knowledge of composition of functions and

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the inverse function of a function I used clinical semi-structured interviews, but in the case of the group of preservice teachers, in addition to the interviews, I collected a set of lesson plans that the preservice teachers designed for the teaching of the two topics mentioned above. The reasons for using this combination of methods for data collection were presented extensively in Chapter 5. In addition to what is contained there, I considered the lesson plans to constitute a bridge between the prescribed curriculum and the curriculum actually taught in the classroom. Teachers have the freedom to interpret and to adapt the prescribed curriculum, and their lessons are reflections of these interpretations and adaptations. The lesson plans that are designed for a certain topic are at least as complete as the actual lesson in terms of mathematical content and pedagogical content. Following this rationale, I consider that the lesson plans used as field data are a good image of what the participants in the study would have taught in a real classroom, if asked to teach the topics of composition of functions and the inverse function.

The lesson plans offered a chance to the preservice teachers to refresh their knowledge on the topics to be discussed during the interview. When writing the lesson plans, the preservice teachers had the opportunity to consult different resources, such as textbooks or Integrated Resource Packages (IRP's). The analysis of the lesson plans combined with the data from the interviews, showed that the resources used by the students did not really make a difference in deepening their SMK or PCK. This finding of the study is in line with my opinion regarding the textbooks used now in BC: they do not constitute a helpful teaching resource. The quality of the information in BC's textbooks,

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the primary teaching resource for inexperienced high school teachers, might be the main explanation for the fact that the SMK does not depend on teaching experience.

7.4. Limitations of the Study and Possible Directions for Future Research

There are several areas where this study presents certain limitations, however, these limitations can be taken as suggested directions for future research.

I start with the fact that this study did not try to correlate the formal mathematical background of the participants with the SMK and PCK they displayed. It would be interesting to find out if the SMK and the PCK differ for teachers with a major in mathematics compared with teachers with a minor in mathematics, or compared with teachers that completed mathematics courses only as requirements for their degree. It would also be useful to compare students from the professional development program (PDP) with students from SFU's professional qualification program (PQP). The last mentioned group of students are students that are certified to teach in other countries than Canada, and who try to become certified teachers in BC. Investigating this correlation could be a fruitful extension of this research.

Another possible direction for extending the findings of this study would be to collect field data through direct observation of lessons taught in classrooms. A possible question for a study like this would be to find out if there are differences between practice and the data collected through lesson plans and interviews. It needs to be mentioned that

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in the case of the present study, direct observation was impossible, because most of the preservice teachers did not teach the topics during their practicum.

Finally, this study did not probe into the effect of teaching with what Ma (1999) called profound understanding of fundamental mathematics (PUFM) would have on students' learning of the topics of composition of functions and inverse function. This study did not have as objective to determine what the effects of teaching with PUFM on high school students are. Some might argue that in the special case of composition of functions and inverse function of a function it is enough for students to possess a strong procedural knowledge, at least for the Grade 11 level. My personal opinion on this issue is different, but further studies need to decide how teaching these two topics beyond procedural fluency could affect students.

REFERENCE LIST

Ayres, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education,* 19(3), 246-259.

BC Ministry of Education.(n.d.[a]) Integrated Resource Packages , Principles of Mathematics 10. Retrieved December 10th, 2004 from the World Wide Web: http://www.bced.gov.bc.ca/irp/math1012/pm10prrf.htm

BC Ministry of Education.(n.d.[b]). Integrated Resource Packages , Principles of Mathematics 11. Retrieved December 10th, 2004 from the World Wide Web: http://www.bced.gov.bc.ca/irp/math1012/pm11prrf.htm

Beckmann, C.E., Senk, S.L., & Thompson, D.R. (1999). Assessing students' understanding of functions in a graphing calculator environment. *School Science and Mathematics,* 99(8), 451-457.

Boyer, C.B. (1968). *A History of Mathematics.* New York: John Wiley & Sons Inc.

Breidenbach, D.E., Dubinsky, E, Hawks, J. & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics,* 23, 247-285.

Cajori, F. (1938). *A History of Mathematics.* New York: The Macmillan Company.

Davis, P.J. & Hersh, R (1981). *The Mathematical Experience.* Boston: Birkhauser.

Davis, R.B. (1986). Conceptual and procedural knowledge in mathematics: A summary analysis. In J. Hiebert (Ed.). *Conceptual and Procedural Knowledge: The Case of Mathematics* (p. 265-300) New Jersey: Lawrence Erlbaum Associates Ltd.

Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L.P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience.* New York: Springer-Verlag.

Dubinsky, E. & Harel G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.). *The Concept of Function, Aspects of Epistemology and Pedagogy.* (p. 85-106) Washington, DC: Mathematical Association of America.

Ernest, P. (1998). The epistemological basis of qualitative research in mathematics education: a postmodern perspective [Monograph]. *Journal for Research in Mathematics Education, 9*, 22-39.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, 21(6), 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, 24(2), 94-116.

Even, R., & Bruckheimer M. (1998). Univalence: a critical or non-critical characteristics of functions. *For the Learning of Mathematics, 18*(3), 30-32.

Gardiner, A. (1982). *Infinite Processes, Background to Analysis*. Springer-Verlag New York Inc., New York.

Gellert, K., Kustner, J. Hellwich, A.& Kastner, H. (Eds.) (1975). *Enciclopedie Matematica* [Mathematical Encyclopaedia]. Bucharest, Romania: Editura Tehnica.

Ginsburg, H.P. (1997). *Entering the Child's Mind. The Clinical Interview in Psychological Research and Practice*. Cambridge, UK: Cambridge University Press.

Grabiner, J. V. (1981). *The Origins of Cauchy's Rigorous Calculus*. Cambridge, Massachusetts: The MIT Press.

Haimes, D.H. (1996). The implementation of a "function" approach to introductory algebra: a case study of a teacher cognitions, actions, and the intended curriculum. *Journal for Research in Mathematics Education*, 27(5), 582-603.

Harel, G., & Dubinsky E. (Eds) (1992). *The Concept of Function: Aspects of Epistemology and Pedagogy*. Washington, DC: Mathematical Association of America.

Heaton, R.,M. (2000). *Teaching Mathematics to the New Standards: Relearning the Dance*. New York, NY: Teachers College Press.

Keller, B.,A. and C.R. Hirsch (1998). Student preferences for representations of functions. *International Journal of Mathematics Education in Science & Technology, 29*(1), 1-18.

Kleiner, I. (1993). Functions: historical and pedagogical aspects. *Science and Education,* vol. 2, no. 2, p. 183-209. The Netherlands: Kluwer Academic Publishers.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press.

Knill, G. et al. (1998). *Mathpower 10*. Canada: McGraw-Hill Ryerson Ltd.

Knill, G. et al. (1999). *Mathpower 11*. Canada: McGraw-Hill Ryerson Ltd.

Knuth, E.J. (2000). Understanding connections between equations and graphs. *Mathematics Teacher*, 93(1), 48-54.

Lee, P. Y. et al. (1999a). *New Syllabus D Mathematics 3*. Singapore, Shing Lee Publishers Pte.

Lee, P. Y. et al. (1999b). *Pure Mathematics*. Singapore, Shing Lee Publishers Pte.

Legisa, P. et al. (1989). *Matematika I. Realna Stevila. Linearna Funkcija* [Mathematics I, Real Numbers, Linear Functions]. Ljubljana.

Loyd, G.M., & Wilson M. (1998). Supporting innovation: The impact of a teacher's conception of function on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, 29(3), 248-274.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics*. Mahwah, New Jersey: Lawrence Earlbaum Associates.

Melnick, S. (1996). Reforming teacher education through legislation: a case study from Florida. In K. Zeichner and all. (Eds.) *Currents of Reform in Preservice Teacher Education*. (p. 30-61) New York, NY: Teachers College, Columbia University.

Micic, V. et al. (1989). *Matematika III.* [Mathematics III]. Belgrade: Naucna Knjiga.

Nastasescu, C. Nita, C. and Rizescu, G. (1993). *Matematica. Algebra IX* [Mathematics, Algebra, Grade 9]. Bucharest, Romania: Editura Didactica si Pedagogica.

Norman, A. (1992). Teacher's mathematical knowledge of the concept of function. In G. Harel & E. Dubinsky (Eds.). *The Concept of Function, Aspects of Epistemology and Pedagogy*. (p. 215-232) Washington, DC: Mathematical Association of America.

O'Callaghan, B.R. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education,* 29(1), 21-41.

Pirie, S. (1998). Toward a definition for research [Monograph]. *Journal for Research in Mathematics Education,* 9, 17-21.

Resnick, L.B. & Ford, W.W. (1984). *The Psychology of Mathematics for Instruction*. London: Lawrence Erlbaum Associates Ltd.

T

Schnitzer, A. & Stillwell, J. (2002). *Mathematical Evolutions*. USA: The Mathematical Association of America.

Schoenfeld, A.H. (1987). *On mathematics as sense making: An informal attack on the unfortunate divorce of formal and informal mathematics*. Paper presented at the OERI/LRDC Conference on Informal Reasoning and Education, Pittsburgh, PA.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification — the case of function. In G. Harel & E.Dubinsky (Eds.). *The Concept of Function, Aspects of Epistemology and Pedagogy*. (p 59-84) Washington, DC: Mathematical Association of America.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.). *The Concept of Function, Aspects of Epistemology and Pedagogy*. (p. 25-58)Washington, DC: Mathematical Association of America.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, 33, 259-281.

Snapper, E (1990). Inverse functions and their derivatives. *The Teaching of Mathematics*. 144-147.

Stein, M.K., Baxter J.A. & Leinhardt G. (1990). Subject matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal*, 27, 639-663.

Stewart, J. (2003). *Early Transcendental Single Variable Calculus*. USA: Brooks/Cole.

Vidakovic, D. (1996). Learning the concept of inverse function. *Journal for Computers in Mathematicsand and Science Teaching*, 15(3), 295-318.

Vinner, S. (1983). Concept definition, concept image and the notion of function, *International Journal of Mathematics Education for Science and Technology*, 14(3), 293-305.

Weisstein, E. (2005a). Bijective. Math World. Retrieved February 5th 2005 from the World Wide Web: http://mathworld.wolfram.com/Bijective.html

Weisstein, E. (2005b). Injection. Math World. Retrieved February 5th 2005 from the World Wide Web: http://mathworld.wolfram.com/Injection.html

Weisstein, E. (2005c). One-to-one. Math World. Retrieved February 5th 2005 from the World Wide Web: http://mathworld.wolfram.com/One-To-One.html

Weisstein, E. (2005d). Surjection. Math World. Retrieved February 5th 2005 from the World Wide Web: http://mathworld.wolfram.com/Surjection.html

Wilson, M.R. & Krapfl C.M. (1994). The impact of graphics calculators on students' understanding of function. *Journal of Computers and Science Teaching*, 13(3), 252-264.

Yerushalmy, M. (1997). Designing representations: reasoning about functions of two variables. *Journal for Research in Mathematics Education*, 28(4), 431-466.

Yerushalmy, M. (2000). Problem solving strategies and mathematical resources: a longitudinal view on problem solving in a function based approach to algebra. *Educational Studies in Mathematics*, 43(2), 125-147.

Zazkis, R. (1992). Theorem-out-of-action: formal vs. naive knowledge in solving a graphic programming problem. *Journal of Mathematics Behavior*, 11, 179-192.

Zazkis R. & Hazzan, O. (1999). Interviewing in mathematics education research: choosing the questions. *Journal of Mathematics Behavior*, 17(4), 429-439.

Zbiek, M.R. (1998). Prospective teachers' use of computing tools to develop and validate functions as mathematical models. *Journal for Research in Mathematics Education*, 29(2), 184-201.

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