Elastic Deformations of Fibre-Reinforced Materials

by

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Abstract

An *ideal* fibre–reinforced material is defined to be one which is incompressible in bulk and inextensible in one or more directions at each point. This idealization of real materials provides an adequate model for those composites in which fibres, having a much higher extensional modulus, are bonded to or embedded in a more compliant matrix host. Materials possessing such properties cannot be deformed in a completely arbitrary manner. Every deformation to which they are subjected is restricted by the conditions of inextensibility in the fibre direction and material incompressibility. The objective of this thesis is to present a continuum theory describing these deformations.

Kinematic equations suitable for describing the motion of particles in a general fibre–reinforced body are obtained and subsequently modified to reflect the idealized constraint conditions. These equations must be satisfied by every admissible deformation. The Cauchy stress components as well as the equations of equilibrium are given in a form appropriate for an idealized solid. The material response to any deformation is assumed to be perfectly elastic. That is, a strain–energy function W, is assumed to exist and is found to be a function of three invariants of Finger's tensor and the fibre-direction tensor. Examples of simple deformations such as uniform extension and shearing of a fibre–reinforced cuboid are presented.

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Chapter 1

Introduction

A fibre-reinforced composite material is one which consists of high modulus fibres embedded in or bonded to a more compliant matrix material. In the composite both fibre and matrix retain their individual mechanical properties while the composite is endowed with properties which cannot be attained by either material acting alone. In general, composites are designed such that the fibre component carries the highest proportion of the load. The matrix serves to maintain the position and orientation of the fibres but also functions as medium through which the load is transferred. The purpose of this thesis is to describe and investigate a continuum theory appropriate to model the mechanical behaviour of these materials.

The study of fibre-reinforced materials is less than four decades old. However, materials such as fibreglass and reinforced concrete have been in use for a considerably longer time. Recent advances in science and technology have made possible the ability to manufacture high strength fibres and thus produce materials with desirable mechanical properties. Currently, fibre composites are used in a great many industrial and commercial applications. Table 1.1 is just a brief list indicating their use in such products as aircrafts, automobiles, sporting goods and boating equipment.

Typically, the most important feature one strives for in the design of composites materials is a high modulus to weight ratio. Table 1.2 compares the properties of some metallic materials with those of some modern fibre reinforced composites. Clearly, there is a potential to exploit the unique mechanical and physical characteristics which

Table 1.1: A sample of products currently manufactured with composite materials.

Application[a]	Component
Aircraft	Wing skins, fuselage, ailerons, rudder.
Automobile	Hood and door panels, radiator supports, bumper reinforcement beams, Leaf springs, drive shaft, wheels.
Sporting Goods	Tennis rackets, fishing rods, kayaks, bicycle frames, helmets, athletics shoe soles.
Marine	Boat hulls, decks, bulkheads, frames, masts, spars.

[a]Ref. [1] pp.6–11.

these materials possess. However, in order to fully realize the benefits these materials have to offer, it is necessary to develop appropriate methods to study their mechanical behaviour.

Table 1.2: Tensile Properties of Some Metallic and Structural Composite Materials.

Material[b]	Specific gravity	Modulus,[a] GPa (10^6 psi)	Ratio of Modulus to weight, 10^6 m
SAE 1010 steel	7.87	207	2.68
AL 6061–T6 aluminum alloy	2.70	68.9	2.60
Ti–6A1–4V titanium alloy	4.43	110	2.53
INCO 718 nickel alloy	8.2	207	2.57
Carbon fibre–epoxy	1.63	215	13.44
E–glass fibre–epoxy	1.85	39.3	2.16
Kevlar 49 fibre–epoxy	1.38	75.8	5.60
Boron fibre–6061 Al alloy	2.35	220	9.54

[a]The modulus for the composite materials is measured in the fibre direction.
[b]Ref. [1] pp.3–4.

There are, essentially, three distinct classes of theoretical problems in the study fibre composites. One class is concerned mainly with the mechanical interactions between the individual components. The area of interest is the region at or near the fibre–matrix interface. These problems are great importance in the design and manufacture of composite materials as well as in the study of their failure mechanisms.

In most applications, the load is applied only to the matrix. In order for the composite to perform effectively, the load must be transmitted through the matrix to the fibres by adhesion or friction at the interface. This gives rise to complex stress an strain distributions in both the fibre and matrix. Recently there has been interest in the study of fibre-bridged cracking in composites. A representative sample of current work in this area can be found in papers by Chiang *et al.* [2], Neumeister [3] and Bao and Song [4]. The theories under investigation by these, and other authors, seek to predict cracking failure in fibre–composites based on models of debonding and frictional sliding which occur during crack extension.

Another area of study concerns the relation of the properties of the composite to the individual properties of the fibre and the matrix. The predominant problem in this field, when studying elastic materials, is to obtain an expression for the *effective* or *overall* elastic moduli of a composite in terms of the moduli of the constituent materials. Among the first to study this problem were Hill [5, 6, 7], Hashin and Rosen [8] and Hashin [9]. Their accounts contain bounds and also some exact results for the overall elastic moduli of fibre composites with isotropic and transversely isotropic phases. These early results apply mainly to materials in which the fibres can be assumed to be long, continuous and perfectly aligned cylinders. Subsequent research has focused on strengthening these bounds for particular materials as well as generalizing the theory to more complicated material geometries. As an example, in a recent paper, Zhao and Weng [10] obtain expressions for the elastic moduli of a transversely isotropic composite reinforced with two-dimensional randomly-oriented elliptic cylinders. Other recent advances include the development of a three-dimensional elastic constitutive theory for application to fibre composite laminated media, (Christensen and Zywicz [11]). Shield and Costello [12] describe a model for a wire rope reinforced rubber composite plate. In this case the extension–twisting coupling of the reinforcing cord is not neglected in the formulation of the constitutive relation.

Continuum Model

The approach taken in this thesis is one in which attention is focussed mainly on the overall mechanical behaviour of the composite. The behaviour of the constituent components and their interactions are for the most part ignored. The model is strictly a continuum model; as such, no distinction is made between the particles of the fibres and those of the matrix. The main objective is to formulate equations which describe the most important features on the macroscopic scale.

The theory discussed in this thesis is tailored specifically for those materials in which the fibre is, in some way, much stronger than the matrix. Table (1.3) lists some of the more common components used in the manufacture of fibre composites. It can be seen that it is not uncommon for the fibre to have a modulus two orders of magnitude greater than that of the matrix material. We idealize this property by making the assumption that the composite is inextensible in the fibre direction. That is, the fibre does not change length in any deformation. Also, as is frequently done in solid mechanics, we will assume that the composite is incompressible. This is a good approximation for many materials but may only be valid when large deformations are considered. However, this idealization greatly simplifies some of the mathematical formulae and so may allow greater progress to be made. Also, since these two assumption are quite idealized, we refer to those materials for which the above two assumptions remain valid as *idealized* solids.

A continuum theory describing fibre-reinforced materials has been developed in a series of papers by Adkins and Rivlin [13] and Adkins [14, 15, 16]. Their work is concerned mainly with large elastic deformations of materials reinforced with inextensible cords. The basic theory for elastic materials is summarized in the book by Green and Adkins [17]. Mulhern, Rogers and Spencer [18] have proposed a continuum model for the behavior of reinforced plastic materials. This model was subsequently extended by the same authors to treat plastic–elastic materials reinforced by strong elastic fibres [19]. Pipkin and Rogers [20] were the first to discover that for certain

Table 1.3: Tensile moduli of some commercial reinforcing fibres and matrix materials.

Component[a]	Tensile Modulus, GPa (10^6 psi)
Fibres	
E–Glass	72
SAE 1010 steel	207
Kevlar 49 (DuPont)	131
SiC	400
Carbon	
GY–70 (BASF)	483
P–100 (Amoco)	758
Matrix	
Rubber	0.0005
Epoxy Resins	
Epon HPT 1072, (Shell Chemical)	3.3
Tactix 742, (DOW Chemical)	3.0
Thermoplastic Resins	
Avimid (DuPont)	3.8
Udel (Amoco)	2.5

[a]Ref. [1], pp.3,18–19,54 and 65.

types of deformations[1] the constraints imposed by incompressibility and fibre inextensibility are sufficient to determine a given deformation. That is, without providing a constitutive relation. In this case, the nature of the stress response need only be specified when one wishes to compute the surface tractions required to maintain a given deformation.

The book by Spencer [21] gives a thorough treatment of the subject, more general in nature than the previously cited works. A general account of the kinematic constraint conditions and the state of stress in idealized fibre-reinforced solids is given. As well, two chapters are dedicated to the discussion of elastic and plastic stress response. It is for this reason that we follow the development of the theory in the same

---

[1]Plane and homogeneous deformations are two examples.

spirit as found in Spencer's book.

Outline

A material which is incompressible and reinforced by inextensible fibres cannot be deformed in a completely arbitrary manner. These general properties of fibre composites are constraints that place limits upon the possible motions which a body may undergo. In Chapter 2 explicit mathematical relations are obtained which must be satisfied in every deformation. The state of stress associated with a reinforced material is discussed in Chapter 3. Here it will be found that the ability to determine the stress by way of the equations of equilibrium or motion is dictated by the number of distinct families of fibres present in the body. In order to completely determine the state of stress in a deformed medium it is usually necessary to specify a constitutive relation. In Chapter 4 we will consider the form of the constitutive relation for an elastic fibre–reinforced material. A strain–energy function W, of the deformation gradient and fibre direction is assumed to exist. It is shown that, in general, W can be expressed as a function of certain invariants of the quantities employed to describe the deformation. Further, it is found that each of the kinematic constraint conditions reduces, by one, the number of invariants upon which W depends. In the final chapter we illustrate the idealized theory by considering some example problems.

Chapter 2

Kinematics

In this chapter equations describing the motion of particles in a body are presented. We begin with a brief description and mathematical definition of a deformation. In §2.1 the presence of the fibres in a body are introduced by assuming that the fibre direction at each point can be described by a unit vector field. Other useful definitions, such as the measure of fibre density at each point and the fibre extension ratio are also introduced. The main purpose of this chapter is obtain relationships between the kinematic variables before and during a deformation as well as the time rate of change of these variables. At this point no specific material response to a deformation is imposed. As such, a more general account is given without explicit reference to elastic materials. In §2.2 the form of the kinematic equations are obtained for *ideal* fibre–reinforced materials. That is, constraints of fibre–inextensibility and incompressibility are imposed on the general kinematic equations. The simplified relations so obtained are conditions which every admissible deformation must satisfy.

Definition of Deformation

A *body* B is defined to be a compact, regular region in $\Re^3$. A point $\mathbf{p} \in B$ is called a *particle* or *material* point. We suppose that at a fixed reference time $t = t_0$, B occupies a fixed region of space $\mathcal{D}_0$ and that at some subsequent time t, it occupies a new continuous region $\mathcal{D}$, Fig. 2.1. A *deformation* (of B) is a continuous, one-to-one

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mapping from $\mathcal{D}_0$ into $\mathcal{D}$.
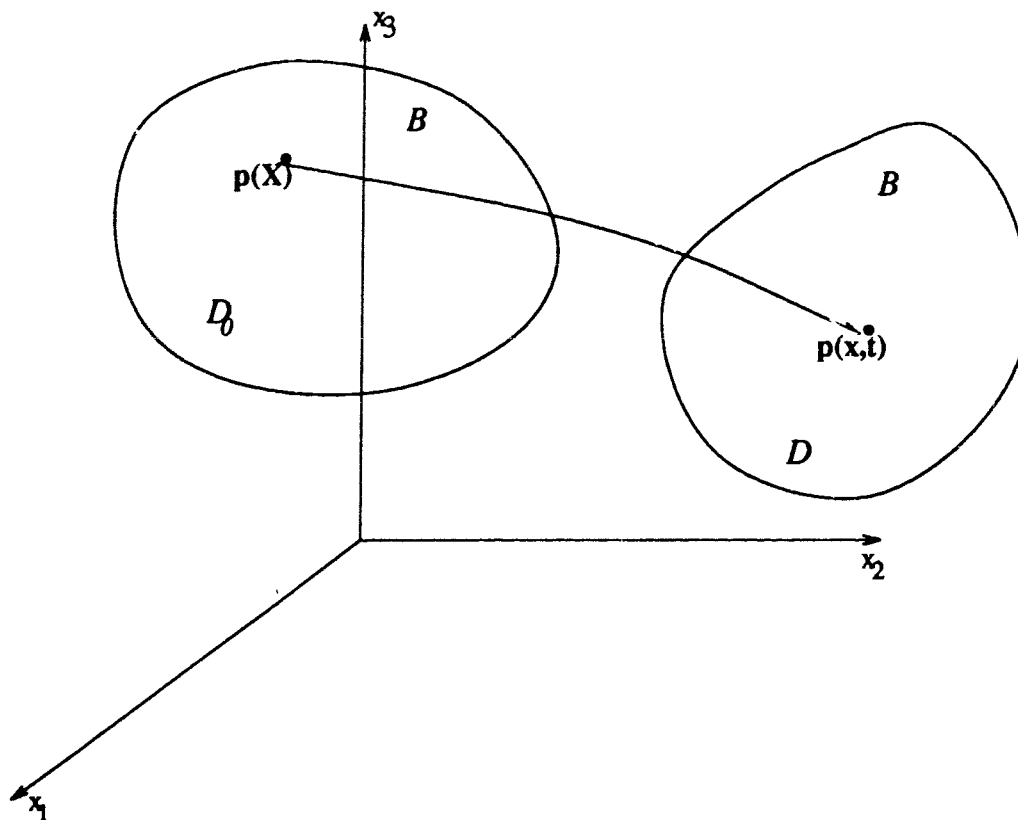
Figure 2.1: General deformation of a body.

Let a fixed rectangular cartesian coordinate system $\mathcal{O}X_1X_2X_3$ be chosen. We shall use $\mathbf{X} = (X_1, X_2, X_3)$ as the label for the *place* occupied by a material point $\mathbf{p}$ at time $t = t_0$. The configuration of the particles in $\mathcal{B}$ at $t = t_0$ is called the *reference* configuration. The vector $\mathbf{x} = (x_1, x_2, x_3)$ will be used to label the place occupied by $\mathbf{p}$ at time t. The configuration at time t is called the *current* configuration. If the motion of $\mathcal{B}$ is measured in the reference configuration then $\mathbf{X}$ serves to identify $\mathbf{p}$ for all subsequent times. That is, we assume that every particle is uniquely labeled by its position at $t = t_0$.

We describe the motion of $\mathcal{B}$ by the dependence of the positions $\mathbf{x}$, of the particles of $\mathcal{B}$ at time t, on their positions $\mathbf{X}$ in the reference configuration. The motion is written symbolically as

$$x_i = x_i(X_\alpha, t) \qquad (i, \alpha = 1, 2, 3). \tag{2.1}$$

In what follows, unless the contrary is stated, it is assumed that subscripts take the values $1, 2, 3$ and that summation over repeated indices is understood. Also, when no confusion is likely to arise, the argument t will be omitted. Thus, for example, the velocity components

$$v_i(X_\alpha, t) = \frac{\partial x_i(X_\alpha, t)}{\partial t},$$

of a particle will be denoted by v_i.

If equations (2.1) are to define a deformation they must be invertible. This condition is met if the Jacobian of the transformation is non-zero. That is,

$$\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \equiv \left| \frac{\partial x_i}{\partial X_\alpha} \right| \neq 0. \tag{2.2}$$

Since we wish to study deformations it will always be assumed that (2.2) is satisfied for all time.

2.1 General Kinematics

We consider the kinematics of materials in which a *matrix* or host material is reinforced by one or more families of strong fibres. The continuum theory is formulated by making the idealization that for a given family of fibres, a member of the family passes through every particle of the material. Since the fibres have a direction at each point, a family of fibres can be characterized by a field of unit vectors. The *fibre direction* at any material point **p** in the reference configuration will be referred to by a unit vector $\mathbf{A}(X_\alpha)$. In the continuum theory we assume that if a particle initially lies on a given fibre then it will remain on that fibre throughout a deformation. Thus, if a fibre through **p** at time t has the direction of the unit vector $\mathbf{a}(X_\alpha, t)$, then

$$\mathbf{a}(X_\alpha, t_0) = \mathbf{A}(X_\alpha). \tag{2.3}$$

The cartesian components of $\mathbf{A}(X_\alpha)$ and $\mathbf{a}(X_\alpha, t)$ will be denoted A_α and a_i respectively.

In general, a body may be reinforced by any number of families of fibres. However, to develop the kinematic equations it is sufficient to consider a body reinforced by

a single family of fibres. The extension to the general case is discussed at the end of §2.2.

Fibre Extension Ratio

Consider a line element through **p** which has the same direction as a fibre through **p** and in the reference configuration has length δL. During a deformation the particle at X_α moves to $x_i(X_\alpha)$ while the particle at $X_\alpha + A_\alpha \delta L$ moves to the $x_i(X_\alpha + A_\alpha \delta L)$. However, the coordinates of the latter point are also given by $x_i(X_\alpha) + a_i \delta l$ where δl is the length of the line element after the deformation. Expanding $x_i(X_\alpha + A_\alpha \delta L)$ in a Taylor series about X_α we obtain

$$a_i \delta l = \frac{\partial x_i}{\partial X_\alpha} A_\alpha \delta L + O(\delta L)^2 \tag{2.4}$$

so that, in the limit $\delta L \to 0$

$$a_i \frac{dl}{dL} = \frac{\partial x_i}{\partial X_\alpha} A_\alpha. \tag{2.5}$$

We denote the *fibre extension ratio* dl/dL by λ. Then, since a_i is a unit vector, we have

$$\lambda^2 = \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_i}{\partial X_\beta} A_\alpha A_\beta. \tag{2.6}$$

Further, from (2.5)

$$a_i = \lambda^{-1} \frac{\partial x_i}{\partial X_\alpha} A_\alpha. \tag{2.7}$$

We see from (2.6) that λ^2 is just the normal component of the Lagrangian strain associated with the direction **A**.

Using the above results an expression for the time rate of change of the fibre extension ratio λ, can be found. By (2.6) we have that

$$\begin{aligned} \frac{d\lambda^2}{dt} &= \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_\alpha}\frac{\partial x_i}{\partial X_\beta}A_\alpha A_\beta\right) \\ &= \frac{\partial v_i}{\partial X_\alpha}\frac{\partial x_i}{\partial X_\beta}A_\alpha A_\beta + \frac{\partial x_i}{\partial X_\alpha}\frac{\partial v_i}{\partial X_\beta}A_\alpha A_\beta. \end{aligned}$$

Substitution from (2.7) gives

$$\frac{d\lambda^2}{dt} = a_i\lambda\frac{\partial v_i}{\partial X_\alpha}A_\alpha + a_i\lambda\frac{\partial v_i}{\partial X_\beta}A_\beta.$$

Upon changing repeated indices and using the chain rule we have

$$
\begin{aligned}
\frac{d\lambda^2}{dt} &= 2\,a_i\lambda\frac{\partial v_i}{\partial X_\alpha}A_\alpha \\
&= 2\,a_i\lambda\frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_\alpha}A_\alpha \\
&= 2\,a_i\lambda\frac{\partial v_i}{\partial x_k}\left(a_k\lambda\right) \\
&= 2\,a_ia_k\lambda^2\frac{\partial v_i}{\partial x_k}.
\end{aligned}
$$

Thus,

$$
\dot{\lambda} = \frac{1}{2\,\lambda}\frac{d\lambda^2}{dt} = a_ia_k\lambda\frac{\partial v_i}{\partial x_k}. \tag{2.8}
$$

Differentiating (2.7) an expression for $\dot{a}_i$ can be obtained. We have,

$$
\begin{aligned}
\dot{a}_i &= \frac{d}{dt}\left(\lambda^{-1}\frac{\partial x_i}{\partial X_\alpha}A_\alpha\right) \\
&= \lambda^{-1}\frac{\partial v_i}{\partial X_\alpha}A_\alpha - \frac{\dot{\lambda}}{\lambda^2}\frac{\partial x_i}{\partial X_\alpha}A_\alpha \\
&= \lambda^{-1}\frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_\alpha}A_\alpha - \frac{\dot{\lambda}}{\lambda^2}\frac{\partial x_i}{\partial X_\alpha}A_\alpha.
\end{aligned}
$$

Substituting from (2.7) and (2.8) gives,

$$
\begin{aligned}
\dot{a}_i &= a_k\frac{\partial v_i}{\partial x_k} - \frac{\dot{\lambda}}{\lambda^2}\left(a_i\lambda\right) \\
&= a_k\frac{\partial v_i}{\partial x_k} - \frac{a_i}{\lambda}\left(a_ja_k\lambda\frac{\partial v_j}{\partial x_k}\right) \\
&= a_k\frac{\partial v_j}{\partial x_k}\delta_{ij} - a_ia_ja_k\frac{\partial v_j}{\partial x_k}.
\end{aligned}
$$

Therefore,

$$
\dot{a}_i = \left(\delta_{ij} - a_ia_j\right)a_k\frac{\partial v_j}{\partial x_k}. \tag{2.9}
$$

Continuity Condition

An important kinematic relation in the continuum theory is the *continuity* equation. This equation describes the behaviour of the mass density during a deformation. We

denote the mass density at the material point p at time t by $\rho = \rho(x_i, t)$. Also, define ρ_0 by

$$\rho(x_i, t_0) = \rho_0(X_\alpha).$$

Now, consider the mass m, occupying an arbitrary volume V at time t. We have

$$m = \int_V \rho(x_i, t) \, dV.$$

The law of conservation of mass requires

$$\frac{dm}{dt} = \frac{d}{dt} \left[\int_V \rho(x_i, t) \, dV \right] = \int_V \left[\frac{d\rho}{dt} + \rho \frac{\partial v_i}{\partial x_i} \right] dV = 0.$$

Therefore, the continuity equation in the Eulerian form is

$$\dot{\rho} + \rho \frac{\partial v_i}{\partial x_i} = 0, \qquad (2.10)$$

where

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + v_k \frac{\partial \rho}{\partial x_k}$$

is the material or convective derivative of ρ.

The continuity equation may also be expressed in the Lagrangian form. Let the particles which occupied the volume V_0 at time $t = t_0$, occupy the volume V at time t. Then the law of conservation of mass requires

$$\int_{V_0} \rho_0 \, dV_0 = \int_V \rho \, dV = \int_{V_0} \rho \left| \frac{\partial x_i}{\partial X_\alpha} \right| dV_0.$$

The result,

$$\rho_0 = \rho \left| \frac{\partial x_i}{\partial X_\alpha} \right| \qquad (2.11)$$

is the material form of the continuity equation.

Fibre Density

We introduce a scalar quantity, denoted $\Sigma(X_\alpha)$ in the undeformed body and $\sigma(X_\alpha, t)$ in the deformed body, which is a measure of the density of fibres at the particle p. We

can view $\sigma(X_\alpha, t)$ as the number of physical fibres per unit area intersecting a surface normal to the vector **a** at the particle **p**. Clearly,

$$\Sigma(X_\alpha) = \sigma(X_\alpha, t_0).$$

A relationship between Σ and σ can be determined by the following construction. Consider three non-collinear particles with coordinates given by X_α, $X_\alpha + dX_\alpha^{(1)}$, and $X_\alpha + dX_\alpha^{(2)}$ in the reference configuration. The coordinates of the three particles in the deformed body are x_i, $x_i + dx_i^{(1)}$, and $x_i + dx_i^{(2)}$ respectively. Let dS be the area of the triangle formed by the three points initially, and ds be the corresponding area after deformation. We denote the components of the unit normal to dS and ds by N_α and n_i respectively. Then

$$dx_i^{(\nu)} = \frac{\partial x_i}{\partial X_\alpha} dX_\alpha^{(\nu)}, \qquad (\nu = 1, 2) \tag{2.12}$$

$$N_\alpha dS = \frac{1}{2} \epsilon_{\alpha\beta\gamma} dX_\beta^{(1)} dX_\gamma^{(2)}, \tag{2.13}$$

$$\text{and} \qquad n_i ds = \frac{1}{2} \epsilon_{ijk} dx_j^{(1)} dx_k^{(2)}. \tag{2.14}$$

Substituting for $dx_j^{(1)}$ and $dx_k^{(2)}$ in (2.14) from (2.12) we have

$$n_i ds = \frac{1}{2} \epsilon_{ijk} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} dX_\beta^{(1)} dX_\gamma^{(2)}. \tag{2.15}$$

Multiplying both sides of (2.15) by $\partial x_i / \partial X_\alpha$ gives

$$\frac{\partial x_i}{\partial X_\alpha} n_i ds = \frac{1}{2} \epsilon_{ijk} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} dX_\beta^{(1)} dX_\gamma^{(2)}.$$

However,

$$\epsilon_{ijk} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} = \epsilon_{\alpha\beta\gamma} \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}. \tag{2.16}$$

Thus, using (2.13) and (2.16), we obtain the result,[2]

$$\frac{\partial x_i}{\partial X_\alpha} n_i ds = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} N_\alpha dS. \tag{2.17}$$

---

[2]Spencer [21, p. 119] refers to this as Nanson's relation.

Now, let dA be the area of the projection of dS onto the plane normal to $\mathbf{A}$ and let da be similarly defines with respect to the area of the projection of ds onto the plane normal to $\mathbf{a}$. That is,

$$dA = N_\alpha A_\alpha dS \qquad (2.18)$$

and

$$da = n_i a_i ds \qquad (2.19)$$

Since dS and ds are composed of the same particles, the surface elements dS, ds, dA, and da all intersect the same set of fibres. Consequently, a necessary condition for a consistent definition of the fibre density is that

$$\Sigma dA = \sigma da. \qquad (2.20)$$

Substituting for dA and da from (2.18) and (2.19) gives

$$N_\alpha A_\alpha \Sigma dS = n_i a_i \sigma ds. \qquad (2.21)$$

Then from (2.5), (2.10), (2.17), and (2.21) we find

$$\sigma = \lambda \frac{\rho}{\rho_0} \Sigma. \qquad (2.22)$$

Now, by differentiating (2.22) and using (2.8) and (2.11) the time rate of change of σ can be found. We have,

$$
\begin{aligned}
\dot{\sigma} &= \dot{\lambda} \frac{\rho}{\rho_0} \Sigma + \lambda \frac{\dot{\rho}}{\rho_0} \Sigma \\
&= \lambda a_i a_j \frac{\partial v_i}{\partial x_j} \frac{\rho}{\rho_0} \Sigma + \frac{\lambda \Sigma}{\rho_0} \left(-\rho \frac{\partial v_i}{\partial x_i} \right),
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\dot{\sigma} &= \left(a_i a_j \frac{\partial v_i}{\partial x_j} - \frac{\partial v_i}{\partial x_i} \right) \lambda \frac{\dot{\rho}}{\rho_0} \Sigma \\
&= (a_i a_j - \delta_{ij}) \frac{\partial v_i}{\partial x_j} \lambda \frac{\dot{\rho}}{\rho_0} \Sigma.
\end{aligned}
$$

Upon using (2.22) we see that σ satisfies

$$\dot{\sigma} - \sigma (a_i a_j - \delta_{ij}) \frac{\partial v_i}{\partial x_j} = 0. \qquad (2.23)$$

2.2 The Idealized Solid

We now consider the form of the kinematic equations when applied to an idealized solid. The assumption of fibre inextensibility results in the condition that during any deformation there is no extension of the material along a fibre direction. In terms of the defined quantities this implies $\delta L = \delta l$ or $\lambda = 1$. If the material is assumed to be incompressible in bulk then the density function must satisfy $\rho(X_\alpha, t) = \rho_0(X_\alpha)$. Applying these conditions to the general kinematic equations we obtain relations for incompressible and inextensible materials.

Equations (2.5) and (2.6) become

$$a_i = \frac{\partial x_i}{\partial X_\alpha} A_\alpha \tag{2.24}$$

and

$$\frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_i}{\partial X_\beta} A_\alpha A_\beta = 1 \tag{2.25}$$

respectively. The continuity condition, (2.10) and (2.11), implies

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{2.26}$$

and

$$\left| \frac{\partial x_i}{\partial X_\alpha} \right| = 1. \tag{2.27}$$

The remaining kinematic equations, (2.8), (2.9), (2.22) and (2.23), take the form

$$a_i a_k \frac{\partial v_i}{\partial x_k} = 0, \tag{2.28}$$

$$\dot{a}_i = a_k \frac{\partial v_i}{\partial x_k}, \tag{2.29}$$

$$\sigma = \Sigma, \tag{2.30}$$

$$\dot{\sigma} = 0, \tag{2.31}$$

respectively.

The above equations are *kinematic* constraints imposed upon all deformations of an incompressible body reinforced by a single family of inextensible fibres. That is to say, a given deformation of the body is possible only if it satisfies these constraint equations.

It is clear that a reinforcement by more than one family of fibres will, in general, be more restricting than a reinforcement by a single family. It is not difficult to extend the constraint conditions to this case. If there are, say, N fibre directions associated with each particle p then, for example, equation (2.24) becomes

$$a_i^{(\gamma)} = \frac{\partial x_i}{\partial X_\alpha} A_\alpha^{(\gamma)}, \quad \text{for each } \gamma = 1, ..., N.$$

However, the kinematic equations are not all independent (i.e. equations (2.25) and (2.28) are both statements of fibre inextensibility). Therefore, each additional fibre does not impose as many constraints as the four equations, (2.24), (2.25), (2.28) and (2.29) independently, seem to imply. As an example, in the case of elastic deformations studied in Chapter 4, only the initial and final configurations of the body are of interest. It will be seen that each family of fibres imposes only one constraint on a given deformation.

One further result may be obtained from the kinematic constraint equations. First consider equation (2.16) which, for an incompressible solid, becomes

$$\epsilon_{ijk} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} = \epsilon_{\alpha\beta\gamma}. \tag{2.32}$$

Differentiating (2.32) with respect to X_η we have

$$\epsilon_{ijk} \left(\frac{\partial^2 x_i}{\partial X_\eta \partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial x_k}{\partial X_\gamma} + \frac{\partial^2 x_j}{\partial X_\eta \partial X_\beta} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_k}{\partial X_\gamma} + \frac{\partial^2 x_k}{\partial X_\eta \partial X_\gamma} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \right) = 0.$$

Multiplying this result by

$$\frac{\partial X_\alpha}{\partial x_1} \frac{\partial X_\beta}{\partial x_2} \frac{\partial X_\gamma}{\partial x_3}$$

we obtain

$$\epsilon_{ijk} \left(\frac{\partial^2 x_i}{\partial X_\eta \partial X_\alpha} \frac{\partial X_\alpha}{\partial x_1} \delta_{j2}\delta_{k3} + \frac{\partial^2 x_j}{\partial X_\eta \partial X_\beta} \frac{\partial X_\beta}{\partial x_2} \delta_{i1}\delta_{k3} + \frac{\partial^2 x_k}{\partial X_\eta \partial X_\gamma} \frac{\partial X_\gamma}{\partial x_3} \delta_{i1}\delta_{j2} \right) = 0.$$

Summing over i, j and k gives

$$\frac{\partial^2 x_1}{\partial X_\eta \partial X_\alpha} \frac{\partial X_\alpha}{\partial x_1} + \frac{\partial^2 x_2}{\partial X_\eta \partial X_\beta} \frac{\partial X_\beta}{\partial x_2} + \frac{\partial^2 x_3}{\partial X_\eta \partial X_\gamma} \frac{\partial X_\gamma}{\partial x_3} = 0$$

or

$$\frac{\partial^2 x_i}{\partial X_\eta \partial X_\gamma} \frac{\partial X_\gamma}{\partial x_i} = 0$$

which is the same as

$$\frac{\partial}{\partial x_i} \left(\frac{\partial x_i}{\partial X_\eta} \right) = 0 \qquad (2.33)$$

Now consider the divergence of **a**. From (2.24) we see that

$$\frac{\partial a_i}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial x_i}{\partial X_\alpha} A_\alpha \right) = A_\alpha \frac{\partial}{\partial x_i} \left(\frac{\partial x_i}{\partial X_\alpha} \right) + \frac{\partial x_i}{\partial X_\alpha} \frac{\partial A_\alpha}{\partial x_i}$$

$$= A_\alpha \frac{\partial}{\partial x_i} \left(\frac{\partial x_i}{\partial X_\alpha} \right) + \frac{\partial A_\alpha}{\partial X_\alpha}.$$

Upon using (2.33) we find

$$\frac{\partial a_i}{\partial x_i} = \frac{\partial A_\alpha}{\partial X_\alpha}. \qquad (2.34)$$

That is, the divergence of the the vector **a** is constant during a deformation of an idealized solid. This result is due to Pipkin and Rogers [20].

Chapter 3

Stress

In this chapter we turn to the discussion of the state of stress associated with constrained materials. In particular, we focus here on stress in incompressible, fibre-reinforced materials. One consequence of introducing constraints into a body is that, in general, the stress field is not completely determinable. The stress in a material subject to constraints is determined only to within an arbitrary stress that does no work in any motion satisfying the constraints.[1] For example, it is well known that the state of stress associated with an incompressible body is determined by the deformation only to within an arbitrary hydrostatic pressure. Similarly, the constraint of fibre inextensibility introduces an arbitrary uniaxial tension in the fibre direction. These undetermined stresses are available to be chosen such that the equations of motion, or equilibrium, and boundary conditions are satisfied.

Without specifying any particular form for the constitutive equation we hypothesize that the *total* stress t_{ij} may be written as the sum of two parts. One part, referred to as the *reaction* stress and denoted by r_{ij}, is the reaction to the constraints. That part of t_{ij} not due to constraints will be referred to as the *extra*–stress and denoted by s_{ij}. Mathematically,

$$t_{ij} = r_{ij} + s_{ij}. \tag{3.1}$$

---

[1]Truesdell and Noll [22, p. 70] present this statement in the form of a general principle.

The form of r_{ij} for an idealized solid reinforced by one family of fibres is

$$r_{ij} = -p\delta_{ij} + Ta_i a_j \tag{3.2}$$

where p is a hydrostatic pressure and T is a uniaxial tension in the fibre direction. The total stress in this case is then

$$t_{ij} = -p\delta_{ij} + Ta_i a_j + s_{ij}. \tag{3.3}$$

Without loss of generality we may assume that

$$s_{ii} = 0 \quad \text{and} \quad a_i a_j s_{ij} = 0 \tag{3.4}$$

since these entities may be absorbed into the arbitrary functions p and T. This modification leaves four independent components of s_{ij} which are to be determined by constitutive equations.

In subsequent sections the discussion will be mainly concerned with problems involving bodies reinforced by either one or two families of fibres. However, for materials reinforced by N families of fibres t_{ij} can be written in the form

$$t_{ij} = -p\delta_{ij} + \sum_{\gamma=1}^{N} T_{(\gamma)} a_i^{(\gamma)} a_j^{(\gamma)} + s_{ij}. \tag{3.5}$$

As in the case of reinforcement by a single fibre family we assume

$$s_{ii} = 0 \tag{3.6}$$

and

$$a_i^{(\gamma)} a_j^{(\gamma)} s_{ij} = 0 \quad \text{for } \gamma = 1, ..., N \quad \text{(No sum on } \gamma.) \tag{3.7}$$

From the N conditions in (3.7) we may surmise that each family of fibres in a body reduces, by one, the number of independent extra-stress components to be determined from a constitutive relation. In essence, introducing constraints into a body increases the importance of the kinematics of the deformation while decreasing the importance of the particular mechanical behaviour of the material. In the extreme case, that being a rigid body, the stress is completely indeterminate.

Equations of Equilibrium

In Chapter 2 conditions to be satisfied by every kinematically admissible deformation have been given. However, every deformation must also satisfy the equations of motion or equilibrium. We consider here only problems of equilibrium in the absence of body forces. In this case the equations of equilibrium are

$$\frac{\partial t_{ij}}{\partial x_j} = 0. \tag{3.8}$$

Using t_{ij} given by (3.5) we see that for a material reinforced by N fibres (3.8) becomes

$$-\frac{\partial p}{\partial x_i} + \sum_{\gamma=1}^{N} \left[a_i^{(\gamma)} a_j^{(\gamma)} \frac{\partial T^{(\gamma)}}{\partial x_j} + T^{(\gamma)} \left(a_i^{(\gamma)} \frac{\partial a_j^{(\gamma)}}{\partial x_j} + a_j^{(\gamma)} \frac{\partial a_i^{(\gamma)}}{\partial x_j} \right) \right] + \frac{\partial s_{ij}}{\partial x_j} = 0. \tag{3.9}$$

A deformation for which the three equations (3.9) can be satisfied by an appropriate choice of the $N+1$ functions p and $T^{(\gamma)}$ is referred to as a *statically* admissible deformation.

With regard to the number of families of fibres present in a body, three cases are of particular interest. If $N=1$ then (3.9) are three equations for the two functions p and T which, in general, will not have a solution. That is, in this case an arbitrary kinematically admissible deformation will not be statically admissible. The equations of equilibrium then serve to place further restrictions on the deformation. When $N=2$ (3.9) are three equations to be satisfied by the three function p, $T^{(1)}$ and $T^{(2)}$. In this case, any kinematically admissible deformation is also statically admissible. Further, for a given deformation the s_{ij} are assumed to be known. Therefore, the deformation is statically admissible without regard to the form of the constitutive equation. When $N>2$ the total stress involves more than three arbitrary functions. Therefore, the equations of equilibrium can be satisfied in an infinite number of ways and the stress is statically indeterminate.

The equations of equilibrium for an idealized material reinforced by two families of fibres take the form

$$-\frac{\partial p}{\partial x_i} + a_i^{(1)} a_j^{(1)} \frac{\partial T^{(1)}}{\partial x_j} + T^{(1)} \left(a_i^{(1)} \frac{\partial a_j^{(1)}}{\partial x_j} + a_j^{(1)} \frac{\partial a_i^{(1)}}{\partial x_j} \right) +$$

$$+a_i^{(2)}a_j^{(2)}\frac{\partial T^{(2)}}{\partial x_j} + T^{(2)}\left(a_i^{(2)}\frac{\partial a_j^{(2)}}{\partial x_j} + a_j^{(2)}\frac{\partial a_i^{(2)}}{\partial x_j}\right) + \frac{\partial s_{ij}}{\partial x_j} = 0. \qquad (3.10)$$

Although the three functions p, $T^{(1)}$ and $T^{(2)}$ must be chosen to satisfy (3.10) we also require some flexibility in satisfying stress boundary conditions in a given problem. However, the homogeneous equations obtained from (3.10) by setting $s_{ij} = 0$ will have non–trivial solutions which can be superimposed on any solution of (3.10).

Integrals of a Deformation

A general result may be obtained by resolving the equations of equilibrium in the fibre direction. Multiplying (3.9) by a_i and noting that $a_i a_i = 1$ we have

$$-a_i\frac{\partial p}{\partial x_i} + a_i\frac{\partial T}{\partial x_i} + T\frac{\partial a_j}{\partial x_j} = -a_i\frac{\partial s_{ij}}{\partial x_j}.$$

But,

$$a_i\frac{\partial}{\partial x_i} = \frac{\partial}{\partial l_a},$$

where $\frac{\partial}{\partial l_a}$ is the derivative along the fibre direction. Thus, we have

$$\frac{\partial(-p+T)}{\partial l_a} + T\frac{\partial a_j}{\partial x_j} = \frac{\partial s_{ij}}{\partial x_j}.$$

However, recall from Equation (2.34) that

$$\frac{\partial a_j}{\partial x_j} = \frac{\partial A_\alpha}{\partial X_\alpha}$$

so that if the fibres are initially parallel straight lines,

$$\frac{\partial A_\alpha}{\partial X_\alpha} = 0.$$

In such a case the equilibrium equations reduces to

$$\frac{\partial(-p+T)}{\partial l_a} = \frac{\partial s_{ij}}{\partial x_j}$$

which can be integrated along the fibre direction to obtain $-p+T$. If the deformation is such that the s_{ij} are constants then the quantity $-p + T$ is an integral of the deformation.

Chapter 4

Elastic Solids

In Chapter 3 it was found that the kinematic equations are not sufficient to completely determine the state of stress in a deformed body. In order to find the extra–stress components it is necessary to specify the material response to a deformation. In this chapter we consider the constitutive equation for elastic solids, in which case the stress components are derived from a strain–energy function.

4.1   One Family of Fibres

Consider the case of an elastic material reinforced by one family of fibres. As is standard in classical elasticity theory, we assume the existence of a strain-energy function W, per unit volume which is a function of the deformation gradients. Following the formulation described by Spencer [21], we suppose that the effect of the fibres can be introduced by letting W depend also on the initial fibre direction. In the case of a single family of fibres we have,

$$W = W \left(\frac{\partial x_i}{\partial X_\alpha}, A_\alpha \right).$$

Now, W must be form invariant under rigid rotation of the deformed body. That is, an arbitrary transformation of the form

$$x_i' = \lambda_{ij} x_j, \tag{4.1}$$

where

$$\lambda_{ik}\lambda_{jk} = \lambda_{ki}\lambda_{kj} = \delta_{ij}, \quad \text{and } \det[\lambda_{ij}] = 1. \tag{4.2}$$

Thus we require

$$W\left(\frac{\partial x_i'}{\partial X_\alpha}, A_\alpha\right) = W\left(\frac{\partial x_i}{\partial X_\alpha}, A_\alpha\right) \tag{4.3}$$

for any λ_{ij} satisfying (4.1). One further point to note is that the sense of **A** is unimportant. Thus it follows[1] from (4.3) that the functional dependence of W can be written in the form

$$W = W\left(G_{\alpha\beta}, A_\alpha A_\beta\right) \tag{4.4}$$

where

$$G_{\alpha\beta} = \frac{\partial x_i}{\partial X_\alpha}\frac{\partial x_i}{\partial X_\beta} \tag{4.5}$$

If we further assume that the only anisotropic properties of the composite body are due to the presence of the fibres then W is invariant under a rigid rotation of the undeformed body. That is, W is invariant under the transformation $\mathbf{X} \to \mathbf{X'}$ where

$$X_\alpha' = \Lambda_{\alpha\beta}X_\beta \tag{4.6}$$

and the $\Lambda_{\alpha\beta}$ satisfy the same conditions as in (4.2). The transformations of A_α and $G_{\alpha\beta}$ are written as

$$A_\alpha' = \Lambda_{\alpha\beta}A_\beta \tag{4.7}$$

and

$$G_{\alpha\beta}' = \frac{\partial x_i}{\partial X_\alpha'}\frac{\partial x_i}{\partial X_\beta'} \tag{4.8}$$

respectively. It then follows from (4.4), (4.7) and (4.8) that W satisfies

$$W\left(G_{\alpha\beta}, A_\alpha A_\beta\right) = W\left(G_{\alpha\beta}', A_\alpha' A_\beta'\right). \tag{4.9}$$

Using results from the theory of algebraic invariants[2], it is possible to express W as function of the ten invariants:

$$\text{tr}(\mathbf{G}), \ \text{tr}(\mathbf{G}^2), \ \text{tr}(\mathbf{G}^3), \ \text{tr}(\mathbf{AA}), \ \text{tr}\left\{(\mathbf{AA})^2\right\}, \ \text{tr}\left\{(\mathbf{AA})^3\right\},$$

$$\text{tr}\left\{(\mathbf{AA})\mathbf{G}\right\}, \ \text{tr}\left\{(\mathbf{AA})\mathbf{G}^2\right\}, \ \text{tr}\left\{(\mathbf{AA})^2\mathbf{G}\right\}, \ \text{tr}\left\{(\mathbf{AA})^2\mathbf{G}^2\right\}, \tag{4.10}$$

---

[1]See *Thm. 1*, Green and Adkins [17, p. 7].

[2]Spencer gives a full account of the theory of invariants in [23]. Tables listing sets of invariants are also provided.

where $\mathbf{G}$ and $\mathbf{AA}$ are the matrices

$$\mathbf{G} = [G_{\alpha\beta}], \quad \mathbf{AA} = [A_\alpha A_\beta], \qquad (4.11)$$

and $\mathrm{tr}(\mathbf{G})$ denotes the trace of $\mathbf{G}$,

$$\mathrm{tr}(\mathbf{G}) = G_{\alpha\alpha}.$$

However, since $\mathbf{A}$ is a unit vector,

$$\mathrm{tr}(\mathbf{AA}) = \mathrm{tr}\left\{(\mathbf{AA})^2\right\} = \mathrm{tr}\left\{(\mathbf{AA})^3\right\} = 1,$$

and so a number of the invariants in (4.10) can be neglected, leaving

$$\mathrm{tr}(\mathbf{G}), \ \mathrm{tr}(\mathbf{G}^2), \ \mathrm{tr}(\mathbf{G}^3), \ \mathrm{tr}\left\{(\mathbf{AA})\mathbf{G}\right\}, \ \mathrm{tr}\left\{(\mathbf{AA})\mathbf{G}^2\right\}. \qquad (4.12)$$

Using the Cayley–Hamilton equation,

$$\mathbf{G}^3 - \mathrm{tr}(\mathbf{G})\mathbf{G}^2 + \frac{1}{2}\left\{\mathrm{tr}(\mathbf{G})^2 - \mathrm{tr}(\mathbf{G}^2)\right\}\mathbf{G} - \det(\mathbf{G})\mathcal{I} = 0,$$

where $\mathcal{I}$ is the identity matrix, we find

$$\det(\mathbf{G}) = \frac{1}{3}\mathrm{tr}(\mathbf{G}^3) + \frac{1}{2}\left\{\frac{1}{3}\mathrm{tr}(\mathbf{G})^2 - \mathrm{tr}(\mathbf{G}^2)\right\}\mathrm{tr}(\mathbf{G}).$$

Thus, the set of invariants in (4.12) may be replaced by the following equivalent but more convenient set:

$$J_1 = \mathrm{tr}(\mathbf{G}), \quad J_2 = \tfrac{1}{2}\left\{\mathrm{tr}(\mathbf{G})^2 - \mathrm{tr}(\mathbf{G}^2)\right\}, \quad J_3 = \mathrm{tr}\left\{(\mathbf{AA})\mathbf{G}^2\right\}$$
$$K_1 = \det(\mathbf{G}), \qquad K_2 = \mathrm{tr}\left\{(\mathbf{AA})\mathbf{G}\right\}. \qquad (4.13)$$

The reason that the invariants in (4.13) are more convenient than those in (4.12) can be seen at once from the constraint conditions (2.26) and (2.25). They imply, simply, $K_1 = 1$ and $K_2 = 1$. Therefore, $\mathcal{W}$ may be regarded as a function of J_1, J_2 and J_3 only.

Introducing the Finger strain tensor defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_\alpha}\frac{\partial x_j}{\partial X_\alpha}, \qquad (4.14)$$

the expressions in (4.13) may be determined in terms of the deformed body. Using, (2.24), (4.5) and (4.14) the strain invariants become

$$J_1 = \text{tr}(\mathbf{g}), \quad J_2 = \tfrac{1}{2}\left\{\text{tr}(\mathbf{g})^2 - \text{tr}(\mathbf{g}^2)\right\}, \quad J_3 = \text{tr}\left\{(\mathbf{aa})\mathbf{g}\right\}$$

$$K_1 = \det(\mathbf{g}) = 1, \quad K_2 = \text{tr}(\mathbf{aa}) = 1, \tag{4.15}$$

where **aa** is the matrix defined by

$$\mathbf{aa} = [a_i a_j].$$

The stress components for an unconstrained elastic body is given in terms of the displacement gradients by [17, p.26],

$$t_{ij} = \frac{1}{\sqrt{K_1}} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \left(\frac{\partial W}{\partial G_{\alpha\beta}} + \frac{\partial W}{\partial G_{\beta\alpha}}\right). \tag{4.16}$$

The constraint conditions may be taken into account by an appropriate modification of the strain–energy function. We may introduce p and T as two Lagrange multipliers and replace W with

$$W - \frac{1}{2}p(K_1 - 1) + \frac{1}{2}T(K_2 - 1).$$

Then, (4.16) written out in terms of the invariants becomes,

$$
\begin{aligned}
t_{ij} &= \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \left\{ W_1\left(\frac{\partial J_1}{\partial G_{\alpha\beta}} + \frac{\partial J_1}{\partial G_{\beta\alpha}}\right) + W_2\left(\frac{\partial J_2}{\partial G_{\alpha\beta}} + \frac{\partial J_2}{\partial G_{\beta\alpha}}\right) \right.\\
&\quad + W_3\left(\frac{\partial J_3}{\partial G_{\alpha\beta}} + \frac{\partial J_3}{\partial G_{\beta\alpha}}\right) - \frac{1}{2}p\left(\frac{\partial K_1}{\partial G_{\alpha\beta}} + \frac{\partial K_1}{\partial G_{\beta\alpha}}\right)\\
&\quad \left. + \frac{1}{2}T\left(\frac{\partial K_2}{\partial G_{\alpha\beta}} + \frac{\partial K_2}{\partial G_{\beta\alpha}}\right)\right\},
\end{aligned}
\tag{4.17}
$$

where

$$W_i = \frac{\partial W}{\partial J_i}.$$

Using (4.13), the derivatives of the invariants are found to be

$$\frac{\partial J_1}{\partial G_{\alpha\beta}} = \delta_{\alpha\beta}, \quad \frac{\partial J_2}{\partial G_{\alpha\beta}} = J_1\delta_{\alpha\beta} - G_{\alpha\beta}, \quad \frac{\partial J_3}{\partial G_{\alpha\beta}} = A_\alpha G_{\beta\gamma} A_\gamma + A_\beta G_{\alpha\gamma} A_\gamma,$$

$$\frac{\partial K_1}{\partial G_{\alpha\beta}} = J_2\delta_{\alpha\beta} - J_1 G_{\alpha\beta} + G_{\alpha\gamma}G_{\gamma\beta}, \quad \frac{\partial K_2}{\partial G_{\alpha\beta}} = A_\alpha A_\beta. \tag{4.18}$$

The terms in (4.17) may be simplified using (4.5), (4.14), and (4.15). For example, using the fourth equation of (4.18) we have

$$
\begin{aligned}
\frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \frac{\partial K_1}{\partial G_{\alpha\beta}} &= \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} \left(J_2 \delta_{\alpha\beta} - J_1 G_{\alpha\beta} + G_{\alpha\gamma} G_{\gamma\beta} \right) \\
&= J_2 g_{ij} - J_1 g_{ir} g_{rj} + g_{ir} g_{rs} g_{sj} \\
&= \det(\mathbf{g}) \delta_{ij} = \delta_{ij},
\end{aligned}
$$

the last two equations following directly from the Cayley–Hamilton equation and the fact that $\det(\mathbf{g}) = 1$. The remaining terms in (4.17) may be similarly simplified. The details are omitted and only the final form is given here as,

$$
t_{ij} = 2\left(\mathcal{W}_1 + J_1 \mathcal{W}_2 \right) g_{ij} - 2 \mathcal{W}_2 g_{ik} g_{jk} + 2 \mathcal{W}_3 \left(a_j g_{ik} a_k + a_j g_{ik} a_k \right) - p \delta_{ij} + T a_i a_j. \quad (4.19)
$$

As was stated in Chapter 3, we find that it is not necessary to postulate the existence of reaction stresses for an elastic material. In this case p and T occur naturally as Lagrange multipliers in the strain–energy function. Spencer [21] has found that this is possible because the strain–energy function is a potential function for the stress. Further, it does not happen if the constitutive equation is such that the stress is not derived from a potential. In the absence of a potential function the presence of the reaction stress has to be postulated.

One further modification can be made in order to express (4.19) in the form (3.3), that is

$$
t_{ij} = -p \delta_{ij} + T a_i a_j + s_{ij}
$$

where the extra–stress components s_{ij} satisfy (3.4),

$$
s_{ii} = 0 \quad \text{and} \quad a_i a_j s_{ij} = 0.
$$

This may be done by absorbing all terms found by evaluating t_{ii} and $a_i a_j t_{ij}$ into $-p$ and T respectively. To begin, let

$$
\hat{s}_{ij} = 2\left(\mathcal{W}_1 + J_1 \mathcal{W}_2 \right) g_{ij} - 2 \mathcal{W}_2 g_{ik} g_{jk} + 2 \mathcal{W}_3 \left(a_j g_{ik} a_k + a_j g_{ik} a_k \right),
$$

then

$$
\begin{aligned}
\hat{s}_{ii} &= 2\left(W_1 + J_1 W_2\right) g_{ii} - 2W_2 g_{ik} g_{ik} + 2W_3 \left(a_i g_{ik} a_k + a_i g_{ik} a_k\right) \\
&= 2\left(W_1 + J_1 W_2\right) J_1 - 2W_2 \left(J_1^2 - 2J_2\right) + 4W_3 J_3 - 3p + T \qquad (4.20) \\
&= 2W_1 J_1 + 4W_2 J_2 + 4W_3 J_3,
\end{aligned}
$$

and

$$
\begin{aligned}
a_i a_j \hat{s}_{ij} &= 2\left(W_1 + J_1 W_2\right) a_i a_j g_{ij} - 2W_2 a_i a_j g_{ik} g_{jk} + 2W_3 a_i a_j \left(a_j g_{ik} a_k + a_j g_{ik} a_k\right) \\
&= 2\left(W_1 + J_1 W_2\right) J_3 - 2W_2 a_i a_j g_{ik} g_{jk} + 4W_3 J_3 \qquad (4.21) \\
&= 2\left(W_1 J_3 + W_2 J_2 + 2W_3 J_3 - W_2\right).
\end{aligned}
$$

To determine s_{ij} we first notice that the components m_{ij} defined by,

$$
m_{ij} = \frac{1}{2}\left(\delta_{ij} - a_i a_j\right)
$$

satisfy

$$
a_i a_j m_{ij} = 0 \quad \text{and} \quad m_{ii} = 1.
$$

Also, the components n_{ij} defined by

$$
n_{ij} = \frac{1}{2}\left(3a_i a_j - \delta_{ij}\right)
$$

satisfy

$$
a_i a_j n_{ij} = 1 \quad \text{and} \quad n_{ii} = 0.
$$

Now, s_{ij} may be obtained by multiplying (4.20) by m_{ij} and (4.21) by n_{ij} and subtracting the results from $\hat{s}_{ij}$. That is,

$$
\begin{aligned}
s_{ij} &= \hat{s}_{ij} - \hat{s}_{pp} m_{ij} - a_p a_q \hat{s}_{pq} n_{ij} \\
&= 2\left(W_1 + J_1 W_2\right) g_{ij} - 2W_2 g_{ik} g_{jk} + 2W_3 \left(a_j g_{ik} a_k + a_j g_{ik} a_k\right) \\
&\quad - \left(W_1 J_1 + 2W_2 J_2 + 2W_3 J_3\right)\left(\delta_{ij} - a_i a_j\right) \\
&\quad - \left(W_1 J_3 + W_2 J_2 + 2W_3 J_3 - W_2\right)\left(3a_i a_j - \delta_{ij}\right).
\end{aligned}
$$

Simplifying the right hand side we obtain,

$$
\begin{aligned}
s_{ij} \;=\; & 2\left(\mathcal{W}_1 + J_1\mathcal{W}_2\right)g_{ij} - 2\mathcal{W}_2 g_{ik}g_{jk} + 2\mathcal{W}_3\left(a_j g_{ik}a_k + a_j g_{ik}a_k\right)\\
& - \left\{(J_1 - J_3)\mathcal{W}_1 + (J_2 + 1)\mathcal{W}_2\right\}\delta_{ij} \\
& + \left\{(J_1 - 3J_3)\mathcal{W}_1 - (J_2 - 3)\mathcal{W}_2 - 4J_3\mathcal{W}_3\right\}a_i a_j.
\end{aligned}
\tag{4.22}
$$

We note here that it is possible to obtain results due to Adkins and Rivlin [13], Adkins [14, 15, 16] and Green and Adkins [17] regarding large elastic deformations of constrained isotropic elastic materials. In the case of materials reinforced by a single family of fibres, that theory is obtained if it is assumed that $\mathcal{W}$ depends only on J_1 and J_2 and is independent of J_3. Thus, by setting $\mathcal{W}_3 = 0$ wherever it appears, the results for constrained isotropic materials can be obtained. The equation for the stress components reduces to:

$$
t_{ij} = 2\left\{(\mathcal{W}_1 + J_1\mathcal{W}_2)\delta_{jk} - \mathcal{W}_2 g_{jk}\right\}g_{ik} - p\delta_{ij} + Ta_i a_j.
$$

Spencer [21] suggests that such a theory may be appropriate in the case in which the fibres are sparsely distributed or, equivalently, are regarded as having infinitesimal thickness.

4.2 Two Families of Fibres

Following the same argument as in §4.1, we assume the existence of a strain–energy function $\mathcal{W}$, which depends on the deformation gradients as well as two initial fibre directions **A** and **B**. We assume also that the angle subtended between the two fibre families at each point in the reference configuration is 2Φ, where, in general Φ is a function of position. Then,

$$
\mathcal{W}\left(G_{\alpha\beta}, A_\alpha, B_\alpha\right) = \mathcal{W}\left(G'_{\alpha\beta}, A'_\alpha, B'_\alpha\right),
$$

where

$$
B'_\alpha = \Lambda_{\alpha\beta}B_\beta.
$$

Taking into account the fact that **A** and **B** are unit vectors, it follows from the tables of invariants [23] that $\mathcal{W}$ can be expressed as a function of

$$J_1, \ J_2, \ J_3, \ J_4 = \operatorname{tr}\left\{(\mathbf{BB})\mathbf{G}^2\right\} = \operatorname{tr}\left\{(\mathbf{bb})\mathbf{g}\right\},$$

$$J_5 = \operatorname{tr}\left\{\mathbf{G}(\mathbf{AA})(\mathbf{BB})\right\} = \cos(2\Phi)\operatorname{tr}(\mathbf{ab}), \qquad (4.23)$$

$$J_6 = \operatorname{tr}\left\{\mathbf{G}^2(\mathbf{AA})(\mathbf{BB})\right\} = \cos(2\Phi)\operatorname{tr}\left\{(\mathbf{ab})\mathbf{g}\right\},$$

and $\cos^2(2\Phi)$, where the matrices **ab** and **AB** are defined by

$$\mathbf{ab} = [a_i b_j] \quad \text{and} \quad \mathbf{AB} = [A_\alpha B_\beta]$$

and

$$\cos(2\Phi) = A_\alpha B_\alpha = \operatorname{tr}\left\{(\mathbf{ab})\mathbf{g}^2\right\} - J_1 \operatorname{tr}\left\{(\mathbf{ab})\mathbf{g}\right\} + J_2 \operatorname{tr}(\mathbf{ab}).$$

The last equality above follows from the Cayley–Hamilton equation. The kinematic constraints in this case imply,

$$K_1 = 1, \quad K_2 = 1, \quad K_3 = \operatorname{tr}\left\{(\mathbf{BB})\mathbf{G}\right\} = \operatorname{tr}(\mathbf{bb}) = 1.$$

In the same manner as in §4.1 the stress components are found to be,

$$
\begin{aligned}
t_{ij} =\ & 2\left(\mathcal{W}_1 + J_1 \mathcal{W}_2\right) g_{ij} - 2\mathcal{W}_2 g_{ik} g_{jk} + 2\mathcal{W}_3\left(a_j g_{ik} a_k + a_j g_{ik} a_k\right) \\
& + 2\mathcal{W}_4\left(b_j g_{ik} b_k + b_j g_{ik} b_k\right) + \mathcal{W}_5 \cos(2\Phi)\left(a_i b_j + a_j b_i\right) \\
& + \mathcal{W}_6 \cos(2\Phi)\left(a_i g_{jk} b_k + a_j g_{ik} b_k + b_i g_{jk} b_k + b_j g_{ik} a_k\right) \\
& - p\delta_{ij} + T_a a_i a_j + T_b b_i b_j,
\end{aligned}
\qquad (4.24)
$$

where T_a and T_b denote arbitrary tension in the two fibre directions. Equation (4.24) may take a simpler form in a number of instances. If the two fibres are initially orthogonal then $A_\alpha B_\alpha = \cos(2\Phi) = 0$. Thus, the invariants J_5 and J_6 as well as $\cos^2(2\Phi)$ can be omitted and $\mathcal{W}$ is a function of J_1, J_2, J_3 and J_4. In this case the stress components are,

$$
\begin{aligned}
t_{ij} =\ & 2\left(\mathcal{W}_1 + J_1 \mathcal{W}_2\right) g_{ij} - 2\mathcal{W}_2 g_{ik} g_{jk} + 2\mathcal{W}_3\left(a_j g_{ik} a_k + a_j g_{ik} a_k\right) \\
& + 2\mathcal{W}_4\left(b_j g_{ik} b_k + b_j g_{ik} b_k\right) - p\delta_{ij} + T_a a_i a_j + T_b b_i b_j.
\end{aligned}
$$

This form of W is equivalent to that given by Green and Adkins [17, p.14] for materials with rhombic symmetry. That is, the material is orthotropic. A further simplification arises if the fibres are indistinguishable except for their directions. If this is the case then W is symmetric in J_3 and J_4. It follows from the theory of invariants [23] that W is a function of J_1, J_2 and $J_3 + J_4$.

If the fibres are sparsely distributed or may be considered to have negligible thickness then the strain–energy function depends only on J_1 and J_2. For materials of this type the stress components are,

$$t_{ij} = 2\left\{(W_1 + J_1 W_2)\delta_{jk} - W_2 g_{jk}\right\} g_{ik} - p\delta_{ij} + T_a a_i a_j + T_b b_i b_j,$$

which is the same as the result found at the end of §4.1 except for the term involving T_b.

4.3 Functional Form of W

Equations (4.19) and (4.24) are general expressions for the stress components for materials reinforced by one or two families of fibres. Our goal is to provide a mathematical description of a body's behaviour in reaction to any specified deformation. The problem is solved if the strain–energy function can be expressed as a known function of the invariants. However, one cannot expect all composites to exhibit the same reaction to a given deformation. That is, each different fibre–matrix pair will have its own characteristic strain–energy function. Further, since the form of W is not known *a priori*, it must be determined experimentally.

Green and Adkins [17, Ch.10] discuss methods for determining an empirical form of the strain–energy function. The predominant method of finding W is to assume that it can be approximated by a multivariate polynomial in the strain invariants. The coefficients are determined with the use of (4.17) and experiments which involve subjecting materials to special deformations. These experiments may include deformations such as pure homogeneous strain of a thin sheet, pure shear of cuboid and torsion of a cylindrical rod. However, the degree of the assumed polynomial form is generally limited by the number of independent experiments which can be performed.

These authors point out that any approximation so obtained is likely to be valid only for a limited range of deformations if the degree of the polynomial must be restricted.

Pipkin [24] suggest that for moderate deformations it is reasonable to approximate the W_i by constants. This form of the strain–energy function can be seen as a modification of the Mooney[3] form,

$$W = C_1(J_1 - 3) + C_2(J_2 - 3),$$

with an additional term to reflect the presence of the fibres. That is, W is given by,

$$W = C_1(J_1 - 3) + C_2(J_2 - 3) + C_3(J_3 - 1). \qquad (4.25)$$

When the fibres are of negligible thickness or are sparsely distributed, then $C_3 = 0$ and the form suitable for isotropic materials is regained.

---

[3]Citation for the original paper is given by Green and Adkins [17, p.26]. M. Mooney, *J. Appl. Phys.* **11** *(1940), 582*

Chapter 5

Example Deformations

In this chapter examples of some simple deformations are presented to illustrate the theory developed in Chapters 2–4.

5.1   Homogeneous Deformations

5.1.1   One family of fibres

Consider a body in the shape of a rectangular parallelepiped, reinforced by one family of fibres. We assume that in the reference configuration the fibres are straight, parallel and lie in planes normal to the X_3 axis. Further, assume that the fibres are initially inclined at an angle Φ to the positive X_1 axis as shown in Fig. 5.1. The initial fibre direction is

$$\mathbf{A} = (\cos\Phi, \sin\Phi, 0). \tag{5.1}$$

Consider first, simple extension deformations of the form

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 \tag{5.2}$$

where λ_1, λ_2, λ_3 are constants. The incompressibility condition, (2.26) requires

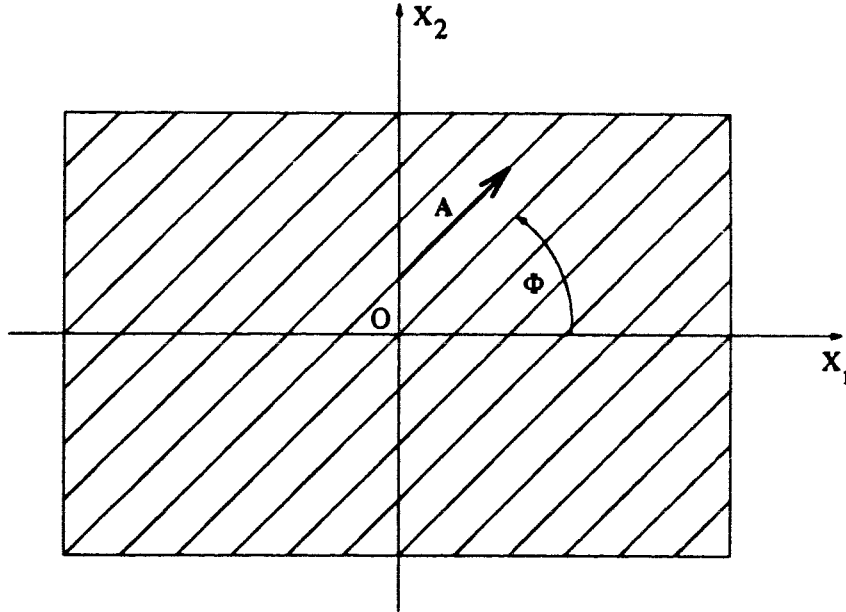$$\lambda_1 \lambda_2 \lambda_3 = 1, \tag{5.3}$$

Figure 5.1: Rectangular block reinforced by one family of fibres

while the fibre-inextensibility condition, (2.25) gives

$$\lambda_1^2 \cos^2 \Phi + \lambda_2^2 \sin^2 \Phi = 1. \tag{5.4}$$

Thus, any deformation of the form (5.2) can be completely determined from (5.3) and (5.4) once one of λ_1, λ_2, λ_3 is specified. Further, the constraint conditions also provide bounds on admissible values of the λ_i. We see from (5.4) that every deformation must satisfy

$$\lambda_1 \leq |\sec \Phi| \quad \text{and} \quad \lambda_2 \leq |\csc \Phi| .$$

Also, since

$$0 \leq (\lambda_1 \cos \Phi - \lambda_2 \sin \Phi)^2 ,$$

we have

$$2\lambda_1 \lambda_2 \cos \Phi \sin \Phi \leq \lambda_1^2 \cos^2 \Phi + \lambda_2^2 \sin^2 \Phi = 1. \tag{5.5}$$

Substituting for $\lambda_1 \lambda_2$ from (5.3), we obtain the bound for λ_3,

$$\lambda_3 \geq \sin 2\Phi$$

when $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Thus the constraint conditions only restrict the amount of contraction in the X_3 direction as they provide no upper bound for λ_3. If the fibres

are initially aligned so that $\Phi = \pi/4$, then $\lambda_3 \geq 1$. In this case the material can be extended but not contracted in the X_3 direction.

Using (2.24), the fibre direction in the deformed body is found to be

$$\mathbf{a} = (\lambda_1 \cos \Phi, \lambda_2 \sin \Phi, 0) = (\cos \phi, \sin \phi, 0). \tag{5.6}$$

It is seen that the fibres remain straight and parallel and only rotate during a deformation of this type.

The most general homogeneous deformation possible under which (5.3), (5.4) and (5.6) remain valid is given by:

$$x_1 = \lambda_1 X_1 + \alpha_{13} X_3, \quad x_2 = \lambda_2 X_2 + \alpha_{23} X_3, \quad x_3 = \lambda_3 X_3 \tag{5.7}$$

where α_{13} and α_{23} are constants. This deformation allows for shear in any direction on the planes $X_3 = $ constant. The components of the Finger strain tensor for this deformation is

$$[g_{ij}] = \left[\frac{\partial x_i}{\partial X_\alpha}\frac{\partial x_j}{\partial X_\alpha}\right] = \begin{bmatrix} \lambda_1^2 + \alpha_{13}^2 & \alpha_{13}\alpha_{23} & \alpha_{13}\lambda_3 \\ \alpha_{13}\alpha_{23} & \lambda_2^2 + \alpha_{23}^2 & \alpha_{23}\lambda_3 \\ \alpha_{13}\lambda_3 & \alpha_{23}\lambda_3 & \lambda_3^2 \end{bmatrix},$$

The strain invariants are calculated from (4.15) to be:

$$\begin{aligned} J_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \alpha_{13}^2 + \alpha_{23}^2, \\ J_2 &= \lambda_1^2\lambda_2^2 + \lambda_1^2\left(\lambda_3^2 + \alpha_{13}^2\right) + \lambda_2^2\left(\lambda_3^2 + \alpha_{23}^2\right), \\ J_3 &= \lambda_1^2 \cos^2 \phi + \lambda_2^2 \sin^2 \phi + (\alpha_{13}\cos(\phi) + \alpha_{23}\sin \phi)^2 \\ &= \lambda_1^4 \cos^2 \Phi + \lambda_2^4 \sin^2 \Phi + (\alpha_{13}\lambda_1 \cos(\Phi) + \alpha_{23}\lambda_2 \sin \Phi)^2. \end{aligned} \tag{5.8}$$

The stress components are

$$\begin{aligned} t_{11} &= -p + T\lambda_1^2 \cos^2 \Phi + s_{11}, \\ t_{22} &= -p + T\lambda_2^2 \sin^2 \Phi + s_{22}, \\ t_{33} &= -p + s_{33}, \\ t_{12} &= \lambda_1 \lambda_2 \sin \Phi \cos \Phi T + s_{12}, \\ t_{13} &= s_{13}, \ t_{23} = s_{23}. \end{aligned} \tag{5.9}$$

where the extra stress, s_{ij}, are given by

$$
\begin{aligned}
s_{11} &= 2\lambda_1^2 W_2 \left(\lambda_3^2 + \alpha_{23}^2\right) + 2\left(\lambda_1^2 + \alpha_{13}^2\right)\left(W_1 + \lambda_2^2 W_2 + 2\lambda_1^2 W_3 \cos^2 \Phi\right) \\
&\quad + 2\lambda_1\lambda_2\alpha_{13}\alpha_{23}W_3 \sin 2\Phi, \\
s_{22} &= 2\lambda_2^2 W_2 \left(\lambda_3^2 + \alpha_{13}^2\right) + 2\left(\lambda_2^2 + \alpha_{23}^2\right)\left(W_1 + \lambda_1^2 W_2 + 2\lambda_2^2 W_3 \sin^2 \Phi\right) \\
&\quad + 2\lambda_1\lambda_2\alpha_{13}\alpha_{23}W_3 \sin 2\Phi, \\
s_{33} &= 2\lambda_3^2 \left(W_1 + \left(\lambda_1^2 + \lambda_2^2\right) W_2\right), \\
s_{12} &= 2\alpha_{13}\alpha_{23}\left(W_1 + W_3\right) + \lambda_1\lambda_2 W_3 \sin 2\Phi \left(\lambda_1^2 + \lambda_2^2 + \alpha_{13}^2 + \alpha_{23}^2\right) \\
s_{13} &= \alpha_{23} \sin 2\Phi + 2\alpha_{13}\lambda_3 \left(W_1 + \lambda_2^2 W_2 + \lambda_1^2 W_3 \cos^2 \Phi\right) \\
s_{23} &= \alpha_{13} \sin 2\Phi + 2\alpha_{23}\lambda_3 \left(W_1 + \lambda_1^2 W_2 + \lambda_2^2 W_3 \sin^2 \Phi\right)
\end{aligned}
$$

Deformations under which the body is not subjected to shear have been discussed by Spencer [21]. A non–zero shear deformation requires $\alpha_{13} = \alpha_{23} = 0$, in which case the extra-stress components reduce to:

$$
\begin{aligned}
s_{11} &= 2\lambda_1^2 \left(W_1 + \left(\lambda_2^2 + \lambda_3^2\right) W_2 + 2\lambda_1^2 \cos^2 \Phi W_3\right), \\
s_{22} &= 2\lambda_2^2 \left(W_1 + \left(\lambda_1^2 + \lambda_3^2\right) W_2 + 2\lambda_2^2 \sin^2 \Phi W_3\right), \\
s_{33} &= 2\lambda_3^2 \left(W_1 + \left(\lambda_1^2 + \lambda_2^2\right) W_2\right), \\
s_{12} &= \lambda_1\lambda_2 \left(\lambda_1^2 + \lambda_2^2\right) \sin 2\Phi W_3, \\
s_{23} &= s_{13} = 0.
\end{aligned}
$$

If W can be assumed to be of the form given by (4.25) then the W_i are constants and the equilibrium equations reduce to,

$$
-\frac{\partial p}{\partial x_i} + a_i a_j \frac{\partial T}{\partial x_j} = 0. \tag{5.10}
$$

Therefore, assigning any constant values to p and T produces a statically admissible solution. In particular, if any two of t_{11}, t_{22}, t_{33}, t_{12} are specified by choosing appropriate values for p and T, then the remaining stress components can be determined.

Uniform Extension

As an example, suppose that $t_{11} = 0$ and $t_{22} = 0$ are specified. From (5.9), p and T are found to be

$$p = \frac{s_{22} \cos^2 \phi - s_{11} \sin^2 \phi}{\cos 2\phi}$$

and

$$T = \frac{s_{22} - s_{11}}{\cos 2\phi}.$$

The remaining stress components are obtained by substituting for p and T in (5.9). They are,

$$
\begin{aligned}
t_{33} &= s_{33} - \frac{s_{22} \cos^2 \phi - s_{11} \sin^2 \phi}{\cos 2\phi}, \\
t_{12} &= s_{12} - 2 \tan (2\phi) (s_{11} - s_{22}), \\
t_{13} &= s_{13}, \\
t_{23} &= s_{23}.
\end{aligned}
\qquad (5.11)
$$

We see that, in general, $t_{12} \neq 0$. Thus, uniform extension cannot be produced by applying uniaxial tension only. It will be seen in §5.1.2 that a uniform extension can be produced by uniaxial tension alone if the material is reinforced by two families of fibres. Another feature to note from (5.11) is that $t_{33} \to \infty$ as $\phi \to \frac{1}{4}\pi$ unless $s_{11} \to 0$ and $s_{22} \to 0$ as well. It was noted in §5.1.2 that no further contraction in the X_3 direction is possible when $\phi = \frac{1}{4}\pi$.

5.1.2 Two families of fibres

We consider in this section is the same material as in §5.1.1 but now reinforced by two families of fibres. The fibres are assumed to be initially straight, parallel and lie in planes normal to the X_3 axis. The angle between the two fibre families is 2Φ. Further, the X_1 and X_2 axes are chosen to bisect the angle between the fibres as shown in Fig. 5.2. The initial fibre directions are then

$$\mathbf{A} = (\cos \Phi, \sin \Phi, 0) \quad \text{and} \quad \mathbf{B} = (\cos \Phi, -\sin \Phi, 0).$$
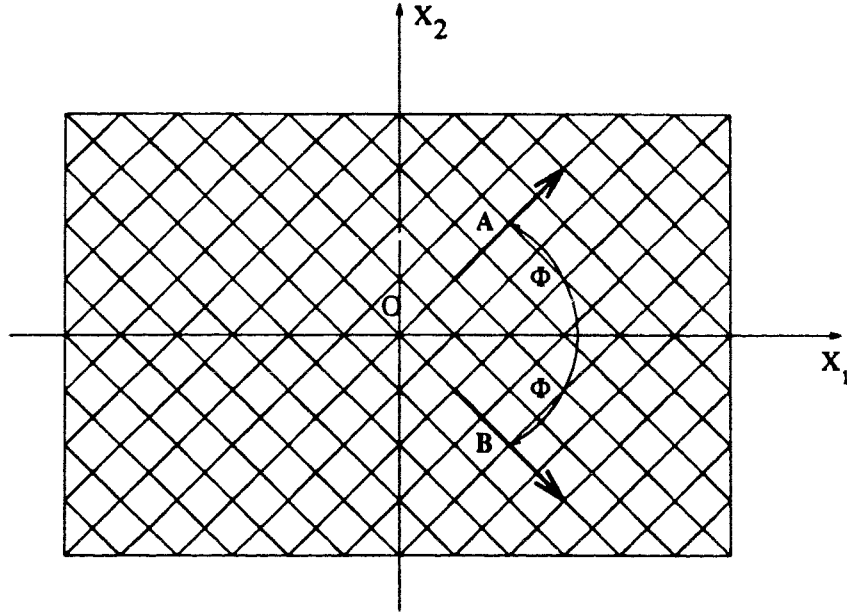
Figure 5.2: Rectangular block reinforced by two families of fibres.

Consider the same class of deformation studied in §5.1.2, namely,

$$x_1 = \lambda_1 X_1 + \alpha_{13} X_3, \quad x_2 = \lambda_2 X_2 + \alpha_{23} X_3, \quad x_3 = \lambda_3 X_3. \tag{5.12}$$

From (2.24), the fibre directions after deformation are

$$\mathbf{a} = (\cos\phi, \sin\phi, 0) = (\lambda_1 \cos\Phi, \lambda_2 \sin\Phi, 0)$$

and

$$\mathbf{b} = (\cos\phi, -\sin\phi, 0) = (\lambda_1 \cos\Phi, -\lambda_2 \sin\Phi, 0).$$

The fibre-inextensibility and incompressibility conditions yield the same constraint equations as in the case of reinforcement by one fibre family, (5.3) and (5.4). In addition to the strain invariants J_1, J_2 and J_3 given in (5.8), the following invariants are defined:

$$
\begin{aligned}
J_4 &= \lambda_1^4 \cos^2\Phi + \lambda_2^4 \sin^2\Phi + (\alpha_{13}\lambda_1 \cos(\Phi) - \alpha_{23}\lambda_2 \sin\Phi)^2 \\
J_5 &= \left(\lambda_1^2 \cos^2\Phi - \lambda_2^2 \sin^2\Phi\right)\cos(2\Phi) \\
J_6 &= \left\{\lambda_1^2 \cos^2\Phi \left(\lambda_1^2 + \alpha_{13}^2\right) - \lambda_2^2 \sin^2\Phi \left(\lambda_2^2 + \alpha_{23}^2\right)\right\}\cos(2\Phi).
\end{aligned}
\tag{5.13}
$$

The stress components are

$$
\begin{aligned}
t_{11} &= -p + (T_a + T_b)\,\lambda_1^2\cos^2\Phi + s_{11}, \\
t_{22} &= -p + (T_a + T_b)\,\lambda_2^2\sin^2\Phi + s_{22}, \\
t_{33} &= -p + s_{33}, \\
t_{12} &= \lambda_1\lambda_2\sin\Phi\cos\Phi\,(T_a - T_b) + s_{12}, \\
t_{13} &= s_{13},\quad t_{23} = s_{23}.
\end{aligned}
\tag{5.14}
$$

where the extra stress, s_{ij}, are given by

$$
\begin{aligned}
s_{11} &= 2\left[\mathcal{W}_1 + \mathcal{W}_2\left(J_1 - \lambda_1^2 - \alpha_{13}^2\right)\right]\left(\lambda_1^2 + \alpha_{13}^2\right) - 2\,\mathcal{W}_2\alpha_{13}^2\left(\lambda_3^2 + \alpha_{23}^2\right) \\
&\quad + 2\,\lambda_1\lambda_2\alpha_{13}\alpha_{23}\sin(2\Phi)\,(\mathcal{W}_3 - \mathcal{W}_4) + 2\,\lambda_1^2\cos(2\Phi)\cos^2\Phi\mathcal{W}_5 \\
&\quad + 4\,\lambda_1^2\cos^2\Phi\,[\mathcal{W}_3 + \mathcal{W}_4 + \cos(2\Phi)\mathcal{W}_6] \\[4pt]
s_{22} &= 2\left[\mathcal{W}_1 + \mathcal{W}_2\left(J_1 - \lambda_2^2 - \alpha_{23}^2\right)\right]\left(\lambda_2^2 + \alpha_{23}^2\right) - 2\,\mathcal{W}_2\alpha_{23}^2\left(\lambda_3^2 + \alpha_{13}^2\right) \\
&\quad + 2\,\lambda_1\lambda_2\alpha_{13}\alpha_{23}\sin(2\Phi)\,(\mathcal{W}_3 - \mathcal{W}_4) - 2\,\lambda_1^2\cos(2\Phi)\sin^2\Phi\mathcal{W}_5 \\
&\quad + 4\,\lambda_1^2\sin^2\Phi\,[\mathcal{W}_3 + \mathcal{W}_4 - \cos(2\Phi)\mathcal{W}_6] \\[4pt]
s_{33} &= 2\lambda_3^2\left[\mathcal{W}_1 + \left(\lambda_1^2 + \lambda_2^2\right)\mathcal{W}_2\right] \\[4pt]
s_{12} &= 2\,\alpha_{13}\alpha_{23}\left[\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_4 + \cos(2\Phi)\left(\lambda_1^2\cos^2\Phi - \lambda_2^2\sin^2\Phi\right)\mathcal{W}_6\right] \\
&\quad + \lambda_1\lambda_2\sin(2\Phi)\left(J_1 - \lambda_3^2\right)(\mathcal{W}_3 - \mathcal{W}_4) \\[4pt]
s_{13} &= 2\,\alpha_{13}\lambda_3\left[\mathcal{W}_1 + \lambda_2^2\mathcal{W}_2 + \lambda_1^2\cos^2\Phi\,(\mathcal{W}_3 + \mathcal{W}_4 + \cos(2\Phi)\mathcal{W}_6)\right] \\
&\quad + \alpha_{23}\sin(2\Phi)\,(\mathcal{W}_3 - \mathcal{W}_4) \\[4pt]
s_{23} &= 2\,\alpha_{23}\lambda_3\left[\mathcal{W}_1 + \lambda_1^2\mathcal{W}_2 + \lambda_2^2\sin^2\Phi\,(\mathcal{W}_3 + \mathcal{W}_4 - \cos(2\Phi)\mathcal{W}_6)\right] \\
&\quad + \alpha_{13}\sin(2\Phi)\,(\mathcal{W}_3 - \mathcal{W}_4)
\end{aligned}
$$

If it is possible to assume $\mathcal{W}$ is of the form given by (4.25) then the $\mathcal{W}_i$ are constants and the equilibrium equations reduce to

$$
-\frac{\partial p}{\partial x_i} + a_i a_j \frac{\partial T_a}{\partial x_j} + b_i b_j \frac{\partial T_b}{\partial x_j} = 0.
\tag{5.15}
$$

Therefore, any constant p, T_a and T_b will produce a statically admissible solution. As in §(5.1.1), it is possible to assign values to three of t_{11}, t_{22}, t_{33}, t_{12}. However, it is not

possible to specify all three of t_{11}, t_{22}, t_{33} since the resulting system of equations for p, T_a and T_b has no solution.

Uniform Extension

Consider again the example discussed in §(5.1.1). That is, $t_{11} = 0$ and $t_{22} = 0$ are specified. Now, though, a third arbitrary function is available for use in fixing the stress components. The condition $t_{12} = 0$ may be given. Using (5.9), p, T_a and T_b are found to be

$$p = \frac{s_{22} \cos^2 \phi - s_{11} \sin^2 \phi}{\cos 2\phi},$$

$$T_a = \frac{2 \sin (2\phi) (s_{22} - s_{11}) - 4 \cos (2\phi) s_{12}}{\sin 4\phi},$$

and

$$T_b = \frac{2 \sin (2\phi) (s_{22} - s_{11}) + 4 \cos (2\phi) s_{12}}{\sin 4\phi}.$$

The remaining stress components are obtained by substituting for p, T_a and T_b in (5.9). They are, as previously found,

$$
\begin{aligned}
t_{33} &= -\frac{s_{22} \cos^2 \phi - s_{11} \sin^2 \phi}{\cos^2 \phi - \sin^2 \phi} + s_{33} \\
t_{13} &= s_{13} \\
t_{23} &= s_{23}.
\end{aligned}
\tag{5.16}
$$

Thus it is possible to produce a uniform extension in the material by applying a uniaxial tension alone.

5.2 Cylindrically Symmetric Deformations

One method of manufacturing fibre–reinforced composites is to wind fibres onto a revolving drum. This produces a composite with cylindrical symmetry in its undeformed state. In this section we investigate deformations of a circular cylinder reinforced by two families of fibres which preserve this symmetry.

Refer the position of a generic particle to a cylindrical polar coordinate system. A particle which has coordinates (R, Θ, Z) in the undeformed body moves during a deformation to the location (r, θ, z) where,

$$X_1 = R \cos \Theta, \quad X_2 = R \sin \Theta, \quad X_3 = Z$$

and

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z.$$

Assume that the two families of fibres initially lie in the surface $R = $ constant and are inclined at an angles Φ and $-\Phi$ to the Z axis, Fig. 5.3. Then the components of the initial fibre directions referred to the cylindrical polar coordinate system are,

$$\tilde{\mathbf{A}} = (0, \sin \Phi, \cos \Phi), \quad \tilde{\mathbf{B}} = (0, -\sin \Phi, \cos \Phi).$$

We consider deformations of the form

$$r = r(R), \quad \theta = \Theta + Z\Psi(R), \quad z = Zw(R). \tag{5.17}$$

The function $r(R)$ represents an inflation or contraction, $\Psi(R)$ a torsion of the cylindrical surface with initial radius R and $w(R)$ an axial extension or contraction of the sheet with initial radius R. After deformation the fibres lie in the surface $r = $ constant so we may take the final fibre directions to be

$$\tilde{\mathbf{a}} = (0, \sin \phi, \cos \phi), \quad \tilde{\mathbf{b}} = (0, -\sin \phi, \cos \phi).$$

In order to discuss deformations of this type it is necessary to express the relevant quantities in terms of cylindrical polar coordinates. If we denote

$$\mathbf{T} = [t_{ij}], \quad \mathbf{F} = [\partial x_i / \partial X_\alpha]$$

then

$$\mathbf{g} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{G} = \mathbf{F}^T\mathbf{F}.$$

Also let

$$\tilde{\mathbf{T}} = \begin{bmatrix} t_r & t_{r\theta} & t_{rz} \\ t_{r\theta} & t_\theta & t_{\theta z} \\ t_{rz} & t_{\theta z} & t_z \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
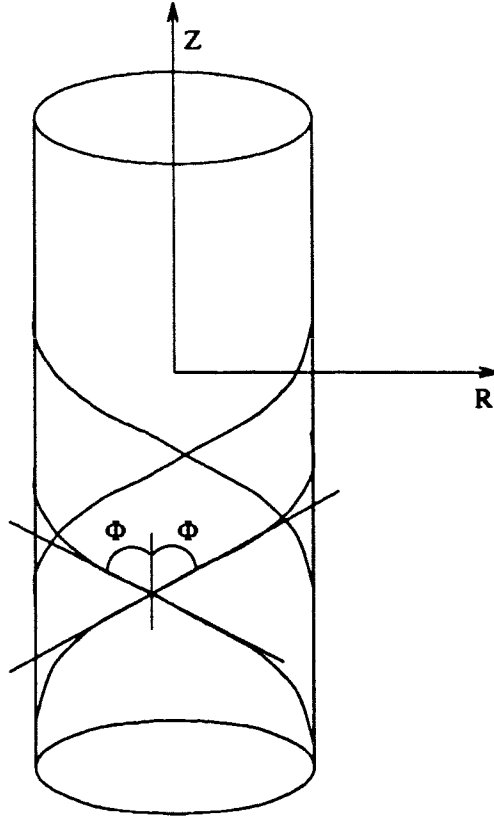
Figure 5.3: Cylinder reinforced by two families of fibres.

$$\mathbf{Q} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the rule for the transformation of stress components is

$$\tilde{\mathbf{T}} = \mathbf{q}\mathbf{T}\mathbf{q}^T.$$

Also note that $\mathbf{q}$ and $\mathbf{Q}$ are orthogonal matrices, so that

$$\mathbf{q}\mathbf{q}^T = \mathbf{q}^T\mathbf{q} = \mathbf{I}, \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$

If we further define

$$\tilde{\mathbf{F}} = \mathbf{q}\mathbf{F}\mathbf{Q}^T = \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix},$$

then components of Finger's strain tensor are transformed as

$$\tilde{g} = qgq^T = qFF^Tq^T = qFQ^TQF^Tq^T = \tilde{F}\tilde{F}^T.$$

The transformation of the fibre direction can be similarly obtained. We have,

$$\tilde{a} = qa, \quad (\tilde{a}\tilde{a}) = q(aa)q^T. \tag{5.18}$$

The constitutive equation can be transformed into cylindrical polar coordinates by multiplying Equation (4.24) on the left by q and on the right by q^T. We obtain,

$$
\begin{aligned}
\tilde{T} = {} & 2(W_1 + J_1 W_2)\tilde{g} - 2W_2\tilde{g}^2 + 2W_3\left\{(\tilde{a}\tilde{a})\tilde{g} + \tilde{g}(\tilde{a}\tilde{a})\right\} \\
& + 2W_4\left\{(\tilde{b}\tilde{b})\tilde{g} + \tilde{g}(\tilde{b}\tilde{b})\right\} + W_5\cos(2\Phi)\left\{(\tilde{a}\tilde{b}) + (\tilde{b}\tilde{a})\right\} \\
& + W_6\cos(2\Phi)\left\{(\tilde{a}\tilde{b})\tilde{g} + (\tilde{b}\tilde{a})\tilde{g} + \tilde{g}(\tilde{a}\tilde{b}) + \tilde{g}(\tilde{b}\tilde{a})\right\} \\
& - p\mathbf{I} + T_a(\tilde{a}\tilde{a}) + T_b(\tilde{b}\tilde{b}),
\end{aligned}
$$

where

$$J_1 = \text{tr}(\tilde{g}), \quad J_2 = \tfrac{1}{2}\left\{(\text{tr}(\tilde{g}))^2 - \text{tr}(\tilde{g}^2)\right\}, \quad J_3 = \text{tr}(\tilde{a}\tilde{a})\tilde{g}$$

$$J_4 = \text{tr}(\tilde{b}\tilde{b})\tilde{g}, \quad J_5 = \cos(2\Phi)\text{tr}(\tilde{a}\tilde{b}), \quad J_6 = \cos(2\Phi)\text{tr}(\tilde{a}\tilde{b})\tilde{g}$$

Now, referring to the deformation (5.17) the constraint conditions imply that $r(R)$, $\Psi(R)$ and $w(R)$ must satisfy

$$w\tfrac{r}{R}\tfrac{dr}{dR} = 1,$$

$$w^2\cos^2\Phi + \tfrac{r^2}{R^2}(\sin\Phi + R\Psi\cos\Phi)^2 = 1, \tag{5.19}$$

$$w^2\cos^2\Phi + \tfrac{r^2}{R^2}(-\sin\Phi + R\Psi\cos\Phi)^2 = 1.$$

The second and third equations of (5.19) are compatible only if $\sin\Phi = 0$ or $\cos\Phi = 0$, in which case both families of fibres coincide, or if $\Psi = 0$. Thus if the fibres are assumed to be distinct then $\Psi = 0$ and torsion is not possible. The deformation then has axial, and not just cylindrical symmetry. The constraint conditions then simplify to

$$w\frac{r}{R}\frac{dr}{dR} = 1, \quad \frac{r^2}{R^2}\sin^2\Phi + w^2\cos^2\Phi = 1,$$

in which case w may be eliminated between these two equations to give

$$\frac{r}{R}\frac{dr}{dR} = \frac{\cos\Phi}{\left\{1 - \left(\frac{r}{R}\right)^2 \sin^2\Phi\right\}^{\frac{1}{2}}}. \tag{5.20}$$

Now, (5.20) may be used to determine r once Φ has been specified. Further, using (5.18) we find that the fibre directions in the deformed body are

$$\tilde{\mathbf{a}} = (0, \sin\phi, \cos\phi) = (0, (r\sin\Phi)/R, w\cos\Phi)$$

and

$$\tilde{\mathbf{b}} = (0, -\sin\phi, \cos\phi) = (0, -(r\sin\Phi)/R, w\cos\Phi),$$

and so ϕ and w may also be obtained once r is found.

The deformation gradient matrix $\tilde{\mathbf{F}}$, for this deformation is

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{dr}{dR} & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ Z\frac{dw}{dR} & 0 & w \end{bmatrix}.$$

Thus, the components of $\tilde{\mathbf{g}}$ are calculated to be

$$\tilde{\mathbf{g}} = \begin{bmatrix} \left(\frac{dr}{dR}\right)^2 & 0 & Z\frac{dr}{dR}\frac{dw}{dR} \\ 0 & \frac{r^2}{R^2} & 0 \\ Z\frac{dr}{dR}\frac{dw}{dR} & 0 & Z^2\left(\frac{dw}{dR}\right)^2 + w^2 \end{bmatrix}.$$

Using these quantities it is a straightforward matter to calculate the strain invariant J_1 to J_6. We find,

$$J_1 = \left(\frac{dr}{dR}\right)^2 + \left(\frac{r}{R}\right) + Z^2\left(\frac{dw}{dR}\right)^2 + w^2$$

$$J_2 = 1 + \left(\frac{dr}{dR}\right)^2 w^2 + \left(\frac{r}{R}\right)^2 \left(Z^2\left(\frac{dw}{dR}\right)^2 + w^2\right)$$

$$J_3 = J_4 = \left(\frac{r}{R}\right)^4 \sin^2\Phi + w^2\cos^2\Phi\left(Z^2\left(\frac{dw}{dR}\right)^2 + w^2\right)$$

$$J_5 = \left\{w^2\cos^2\Phi - \left(\frac{r}{R}\right)^2\sin^2\Phi\right\}\cos(2\Phi)$$

$$J_6 = \left\{w^2\cos^2\Phi\left(Z^2\left(\frac{dw}{dR}\right)^2 + w^2\right) - \left(\frac{r}{R}\right)^4\sin^2\Phi\right\}\cos(2\Phi)$$

The stress components may be obtained in a manner similar to that which was employed in Sections 5.1.1 and 5.1.2.

Chapter 6

Conclusions

Fibre–reinforced composites have become increasingly important in the manufacture
of modern products. Their success is due mainly to their low weight and high modulus
or strength. The possibility of incorporating these materials into applications exists
wherever there is a potential for weight savings. However, their development and
use must be based on a firm understanding of their unique mechanical and physical
characteristics. Therefore, it is necessary to develop appropriate methods of stress
and strain analysis.

In this thesis a continuum theory for fibre–reinforced elastic materials has been
studied. The two assumptions central to the development of this theory are that the
fibres are inextensible and the composite is incompressible in bulk. The inextensibility
condition can be seen as an idealized model for materials in which a more compliant
matrix is reinforced with very strong fibres. The incompressibility assumption is one
which is frequently made in continuum mechanics and is a good approximation for
many materials. However, it is probably more appropriate for those composites for
which finite deformation theories apply.

The conditions of fibre inextensibility and incompressibility are kinematic con-
straints which have the effect of restricting the range of possible deformations. The
kinematic relations obtained in this thesis are sufficiently general and so may be ap-
plied to any material type. However, in order to completely determine the state
of stress in a deformed body it is necessary to specify the form of the constitutive

equation. In this regard, we have considered materials which are perfectly elastic in response to a deformation. The stress components are given in terms of a strain–energy function which depends on the deformation gradients and also the fibre directions in the undeformed body. Using results from the theory of algebraic invariants, the strain–energy is written as a function of certain invariants of the deformation as well as the fibre direction. The result is an equation for the stress components for any elastic deformation in terms of the quantities employed to describe the deformation. Once the functional form of the strain–energy function is specified for a material the problem of finding the associated stress is solved.

Areas for further study

- We have noted that the ideal approximation of material incompressibility may not provide adequate results if the deformations are linear. The ideal theory may be better suited to study small deformations if the incompressibility condition is relaxed to permit small changes in the density.

- In this thesis, only elastic stress response has been considered. Thusfar there has been little research concerning any other material behaviour other than elastic–perfectly plastic, and plastic. An investigation of fibre–reinforced visco–elastic materials and fibre–reinforced fluids may yield useful results.

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