Regression tests of fit and some comparisons

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REGRESSION TESTS OF FIT AND SOME COMPARISONS

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Abstract

Probability plots have a long history in statistics. Such plots should, if the observations are from a proposed distribution, appear to be close to a straight line. Nevertheless, actual tests, based on the use of polynomial regression models to detect non-linearity in probability plots have until now been studied for the normal distribution only. Some empirical results were given by LaBrecque(1977). The asymptotic theory of the normal case was given by Stephens(1975) but the asymptotic results are related to asymptotic properties of the covariance matrix of the normal order statistics and therefore are not applicable to other distributions. A unified approach to the theory for goodness-of-fit based on the use of polynomial regression models from probability plots is presented and asymptotic results are given, valid for any sufficiently regular distribution in the location-scale family. In particular, two methods for estimating the parameters in the model are compared, namely, generalized and ordinary least squares, and test statistics are proposed. As an illustration of the techniques, asymptotic results are obtained for the logistic distribution. A test of fit for the Gumbel distribution, based on polynomial regression, is also developed, with special attention to the more practical aspects of the test. An empirical power study to assess the performance of the proposed tests is presented.

Within the same general approach, a method to construct tests of fit for distributions which depend on an unknown threshold parameter, such as the three-parameter Weibull and the three-parameter log-normal is proposed. The technique is also applied to construct a test of bivariate normality.

Finally, tests of fit based on the empirical distribution function (EDF) statistics are developed for two cases of practical importance: The three-parameter Frechet (or type II) distribution, and the Gumbel distribution when parameters are estimated from a type II censored sample.

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1.1 Preliminaries

Let $Y_1, \ldots, Y_n$ be a random sample from an absolutely continuous distribution $F_0(y)$ and let $Y_{(1)}, \ldots, Y_{(n)}$ denote the corresponding order statistics.

Here, F_0 is assumed to be of the form $F(x)$ where $x = (y - \alpha)/\beta$. Hence if $X_1, \ldots, X_n$ is a random sample from $F(x)$ and $X_{(1)}, \ldots, X_{(n)}$ are the order statistics, we can express

$$E(Y_{(i)}) = \alpha + \beta\, m_i \text{ where } m_i = E(X_{(i)}) \tag{1.1}$$

If we are interested in testing the null hypothesis that $Y_1, \ldots, Y_n$ is a random sample from $F_0(y)$, a common approach is to test how well the data fit the model (1.1). Goodness of fit tests of this type have been developed for several distributions and the reader is referred to Stephens [17, Chapter 5].

The model (1.1) can be extended to:

$$E(Y_{(i)}) = \alpha_0\psi_0(m_i) + \alpha_1\psi_1(m_i) + \alpha_2\psi_2(m_i) + \cdots + \alpha_p\psi_p(m_i) \tag{1.2}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_p$ are constants and $\psi_0(m_i), \psi_1(m_i), \ldots, \psi_p(m_i)$ are certain functions of m_i. If the null hypothesis is true, we choose $\psi_0(m_i)$ to be a constant function and $\psi_1(m_i)$

must be a linear function of m_i. The functions $\psi_2(.)$, $\psi_3(.), \ldots$ are chosen to be certain non linear functions for which the extended model is expected to provide a better fit than (1.1) under departures from the null hypothesis.

Suppose that V_X is the covariance matrix of vector $\mathbf{X}^\top = \left(X_{(1)}, \ldots, X_{(n)}\right)$, let V_Y be the covariance matrix of vector $\mathbf{Y}^\top = \left(Y_{(1)}, \ldots, Y_{(n)}\right)$. Then, under the null hypothesis, $V_Y = \beta^2 V_X$.

The estimates $\hat{\alpha}_j$ of the constants in the model can then be obtained and a test of fit becomes a test of

$$H_0 : \alpha_2 = \alpha_3 = \cdots = 0$$

An easy choice of $\psi_j(m_i)$ is a polynomial of degree j, which is also the usual way for testing linearity of the model. LaBrecque [32] has shown that, for a test of the normal distribution, models containing polynomials up to order three provided a better fit under several alternatives considered in his work. LaBrecque [32] pointed out that the optimal choice of the functions $\psi_j(.)$ for the normal or any other distribution remains an open question; however, here we adopt the above approach and take $\psi_j(.)$ to be a polynomial of degree j.

1.2 Generalized least squares estimators

Let

$$\boldsymbol{\alpha}^\top = (\alpha_0, \ldots, \alpha_p)$$

$$\boldsymbol{\psi}_j{}^\top = (\psi_j(m_1), \ldots, \psi_j(m_n))$$

and

$$\Psi = [\boldsymbol{\psi}_0\ , \boldsymbol{\psi}_1\ , \ldots, \boldsymbol{\psi}_p\ ]$$

and write the model (1.2) as

$$E(\mathbf{Y}) = \Psi \boldsymbol{\alpha} \tag{1.3}$$

The generalized least squares (GLS) estimate of $\boldsymbol{\alpha}$ is given by:

$$\hat{\boldsymbol{\alpha}} = \left[\Psi^\top V_X^{-1} \Psi\right]^{-1} \Psi^\top V_X^{-1} \mathbf{Y} \tag{1.4}$$

with variance

$$Var(\hat{\boldsymbol{\alpha}}) = \beta^2 \left[\Psi^\top V_X^{-1} \Psi\right]^{-1} \tag{1.5}$$

Definition 1.1 *Let A be a n by n symmetric matrix. The two vectors ψ_i and ψ_j will be said to be* A-orthogonal *if*

$$\psi_i{}^\top A \psi_j = 0.$$

The estimates of the constants in the model (1.2) will be uncorrelated if the polynomials ψ_j are chosen to be V_X^{-1}-orthogonal.

Suppose

$$\psi_j(m_i) = m_i^j + a_{j(j-1)} m_i^{j-1} + a_{j(j-2)} m_i^{j-2} + \cdots + a_{j(0)} \qquad j = 0, 1, 2, \ldots$$

and $\mathbf{m}^r = (m_1^r, \ldots, m_n^r)^\top$.

Note that here we take $a_{j(j)} = 1$.

Starting with $\psi_0(m_i) = a_{00} = 1$, we find the coefficients $a_{j(r)}$ $(r = 0, 1, \ldots, j-1)$ of the j-th polynomial $\psi_j(.)$ by simultaneously solving the system of j $(j \geq 1)$ equations:

$$\sum_{r=0}^{j} \sum_{s=0}^{k} a_{j(r)} a_{k(s)} (\mathbf{m}^r) V_X^{-1} \mathbf{m}^s = 0 \qquad \text{for } k = 0, \ldots, j-1 . \tag{1.6}$$

The generalized least squares estimator of α_j is then

$$\hat{\alpha}_j = \frac{\psi_j{}^\top V_X^{-1} \mathbf{Y}}{\psi_j^\top V_X^{-1} \psi_j} \tag{1.7}$$

From (1.7), it follows that, under H_0,

$$E(\hat{\alpha}_j) = \frac{\psi_j{}^\top V_X^{-1} \mathbf{m}}{\psi_j{}^\top V_X^{-1} \psi_j} = 0 \text{ for } j \geq 2$$

$$Var(\hat{\alpha}_j) = \frac{\beta^2}{\psi_j{}^\top V_X^{-1} \psi_j}$$

If we define

$$c_{ij} = \lim_{n \to \infty} \frac{\psi_i{}^\top V_X^{-1} \psi_j}{n} \tag{1.8}$$

we have, asymptotically,

$$\hat{\alpha}_j \approx \frac{\psi_j{}^\top V_X^{-1} \mathbf{Y}}{n c_{jj}}$$

$$Var(\hat{\alpha}_j) \approx \frac{\beta^2}{n c_{jj}}$$

A test of $\alpha_j = 0$ can be based on the quantity:

$$T_j = \frac{\left(\psi_j^\top V_X^{-1} \mathbf{Y}\right)^2}{\psi_j{}^\top V_X^{-1} \psi_j \; \beta^2} \tag{1.9}$$

When the value of β is unknown, it must be estimated from the sample in order to be able to compute the test statistic defined above.

If $Var(X) = \sigma_X^2$, then $Var(Y) = \beta^2 \sigma_X^2$. Hence, β^2 can be estimated unbiasedly using

$$\widehat{\beta^2} = S_Y^2 / \sigma_X^2$$

where

$$S_Y^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$$

The test statistic then becomes

$$\hat{T}_j = \frac{\sigma_X^2 \left(\psi_j^\top V_X^{-1} \mathbf{Y}\right)^2}{\psi_j{}^\top V_X^{-1} \psi_j \; S_Y^2} \tag{1.10}$$

Under certain conditions, use of a result by Stigler [66] (Theorem 9.6.1 in David [20] , see also Chernoff, Gastwirth and Johns [14]) shows that, if

$$0 < \lim_{n \to \infty} n \, Var(\hat{\alpha}_j) < \infty$$

then

$$\frac{\sqrt{n}\hat{\alpha}_j}{\beta} \xrightarrow{\mathcal{D}} \mathcal{N}(0, c_{jj}^{-1}) \text{ for } j \geq 2$$

It follows that

$$T_j \xrightarrow{\mathcal{D}} \chi^2_{(1)} \; , j \geq 2$$

Moreover, since $\widehat{\beta^2} \xrightarrow{P} \beta^2$, by Slutsky's theorem

$$\frac{\sigma_X \sqrt{n}\hat{\alpha}_j}{S_Y} \xrightarrow{\mathcal{D}} \mathcal{N}(0, c_{jj}^{-1}).$$

This gives

$$\hat{T}_j \xrightarrow{\mathcal{D}} \chi^2_{(1)}.$$

A test of $\alpha_2 = \cdots = \alpha_k = 0$ can be based on the quantity

$$T_{23\cdots k} = T_2 + T_3 + \cdots + T_k$$

or

$$\hat{T}_{23\cdots k} = \hat{T}_2 + \hat{T}_3 + \cdots + \hat{T}_k$$

and its sample value compared with critical points from a $\chi^2_{(k-1)}$ distribution. This follows from the asymptotic independence of the terms involved in each of the above sums.

1.3 Ordinary least squares estimators

The calculation of the covariance matrix V_X of the order statistics $X_{(1)}, \ldots, X_{(n)}$ is often difficult.

A simpler approach to derive test statistics is to estimate the constants in the model (1.2) by ordinary least squares. The ordinary least squares (OLS) estimator of $\boldsymbol{\alpha}$ in the model (1.3) is

$$\hat{\boldsymbol{\alpha}} = \left(\Psi^{\top}\Psi\right)^{-1}\Psi^{\top}\mathbf{Y}.$$

The estimators of α_j are easier to work with if the matrix $\Psi^{\top}\Psi$ is diagonal. This will be the case when the polynomial functions are chosen to be I-orthogonal (or simply orthogonal), that is

$$\boldsymbol{\psi}_i{}^{\top}\boldsymbol{\psi}_j = 0 \text{ for } i \neq j$$

Thus, the coefficients of the j-th polynomial must satisfy

$$\sum_{r=0}^{j}\sum_{s=0}^{k} a_{j(r)}a_{k(s)}(\mathbf{m}^r)^{\top}\mathbf{m}^s = 0 \;\;, k = 0, 1, \ldots, j-1 \tag{1.11}$$

The ordinary least squares estimator of α_j can then be written as

$$\hat{\alpha}_j = \frac{\boldsymbol{\psi}_j{}^{\top}\mathbf{Y}}{\boldsymbol{\psi}_j{}^{\top}\boldsymbol{\psi}_j}$$

Then,

$$E(\hat{\alpha}_j) = 0 \text{ for } j \geq 2$$

and,

$$Var\left(\hat{\alpha}_j\right) = \frac{\beta^2 \boldsymbol{\psi}_j^\top V_X \boldsymbol{\psi}_j}{\left(\boldsymbol{\psi}_j^\top \boldsymbol{\psi}_j\right)^2}.$$

By defining

$$c^*_{ij} = \lim_{n\to\infty} \frac{\boldsymbol{\psi}_i^\top V_X \boldsymbol{\psi}_j}{n} \tag{1.12}$$

and

$$d_{ij} = \lim_{n\to\infty} \frac{\boldsymbol{\psi}_i^\top \boldsymbol{\psi}_j}{n} \tag{1.13}$$

we can write, asymptotically,

$$\hat{\alpha}_j \approx \frac{\boldsymbol{\psi}_j^\top \mathbf{Y}}{n d_{jj}}.$$

$$Var\left(\hat{\alpha}\right) \approx \frac{\beta^2 c^*_{jj}}{n d^2_{jj}}.$$

In this case, a test of $\alpha_j = 0$ is based on

$$T^*_j = \frac{\left(\boldsymbol{\psi}_j^\top \mathbf{Y}\right)^2}{\boldsymbol{\psi}_j^\top \boldsymbol{\psi}_j \ \beta^2}$$

By assuming some regularity conditions,

$$\xi_j = \frac{\sqrt{\boldsymbol{\psi}_j^\top \boldsymbol{\psi}_j}\ \hat{\alpha}_j}{\beta} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, c^*_{jj}/d_{jj}\right).$$

Therefore, the test statistic

$$T^*_j = \xi^2_j = \frac{\left(\boldsymbol{\psi}_j^\top \mathbf{Y}\right)^2}{\boldsymbol{\psi}_j^\top \boldsymbol{\psi}_j \ \beta^2} \xrightarrow{\mathcal{D}} w_j \chi^2_{(1)}$$

where

$$w_j = c^*_{jj}/d_{jj}.$$

Since $\widehat{\beta^2} \xrightarrow{P} \beta^2$, we also have

$$\hat{T}^*_j = \frac{\sigma^2_X \left(\boldsymbol{\psi}_j^\top \mathbf{Y}\right)^2}{\boldsymbol{\psi}_j^\top \boldsymbol{\psi}_j \ S^2_Y} \xrightarrow{\mathcal{D}} w_j \chi^2_{(1)}$$

The estimators of the parameters in the model will be, in general, correlated, and so will the tests for α_j $(j = 2, 3, \ldots)$ In fact,

$$Cov(\hat{\alpha}_i, \hat{\alpha}_j) \propto \boldsymbol{\psi}_i{}^{\top} V_X \boldsymbol{\psi}_j \ .$$

The right hand side of the above expression is not necessarly zero because the polynomials were not chosen to be V_X-orthogonal.

More precisely, the asymptotic covariance matrix of $\boldsymbol{\alpha}$ is

$$\Sigma = (\sigma_{ij}) \text{ where } \sigma_{ij} = \frac{\beta^2 c^*_{ij}}{n d_{ii} d_{jj}}.$$

The limiting covariance matrix of $\mathbf{T}^* = (\xi_2, \ldots, \xi_k)^{\top}$ is

$$\Omega = \begin{bmatrix} \frac{c^*_{22}}{d_{22}} & \frac{c^*_{23}}{\sqrt{d_{22}d_{33}}} & \cdots & \frac{c^*_{2k}}{\sqrt{d_{22}d_{kk}}} \\ \vdots & \frac{c^*_{33}}{d_{33}} & \cdots & \frac{c^*_{3k}}{\sqrt{d_{33}d_{kk}}} \\ & & & \frac{c^*_{kk}}{d_{kk}} \end{bmatrix} \tag{1.14}$$

A natural statistic for testing $\alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$ is then

$$T^*_{23\cdots k} = T^*_2 + T^*_3 + \cdots + T^*_k .$$

To find the limiting distribution of $T^*_{23\cdots k}$ we require the following:

Proposition 1.1 *Under the conditions described above,* $\mathbf{T}^* \xrightarrow{\mathcal{D}} \mathcal{N}_{k-1}(\mathbf{0}, \Omega)$.

Proof:

For any non-null (k-1)-dimensional vector $\mathbf{a}$, $W = \boldsymbol{a}^{\top}\mathbf{T}^*$ can be written as a linear combination of the order statistics, namely

$$W = \sum_{j=1}^{n} b_j X_{(j)}$$

with zero mean and limiting variance

$$\sigma^2_W = \boldsymbol{a}^{\top}\Omega\boldsymbol{a} \ . \tag{1.15}$$

Since Ω is positive definite and (1.15) is a finite linear combination of finite quantities, it follows that $0 < \sigma^2_W < \infty$.

Application of theorem 9.6.1 in David [20], now gives:

$$W \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_W^2). \tag{1.16}$$

Since (1.16) is true for any $\boldsymbol{a} \neq \mathbf{o}$, it follows, via the Cramér Wold device, that the limiting distribution of the vector $\boldsymbol{T}^*$ is multivariate normal •

The limiting distribution of $T^*_{23\cdots k}$ is then that of a quadratic form in normal variables with zero mean and covariance matrix Ω.

It follows (Scheffé [48]) that

$$T^*_{23\cdots k} \xrightarrow{\mathcal{D}} \sum_{j=1}^{k-1} \lambda_j \chi^2_{(1)}$$

where $\lambda_1, \ldots, \lambda_{k-1}$ are the eigenvalues of the matrix Ω.

When β is unknown, we use the quantity

$$\hat{T}^*_{23\cdots k} = \hat{T}^*_2 + \hat{T}^*_3 + \cdots + \hat{T}^*_k.$$

which, by Slutsky's theorem, has the same limiting distribution.

1.4 OLS Test for symmetric distributions

If the distribution is symmetric, the matrix Ω will be diagonal. This comes from the conditions:

$$\boldsymbol{\psi}_j^{\mathsf{T}} \mathbf{1} = 0 \ , \ \sum_{i=1}^{n} m_i^r = 0 \ , \text{ for r odd}$$

For j even, $\psi_j(.)$ will be a polynomial in even powers of m_i only. Therefore, due also to the double symmetry of V_X, it follows that, for a symmetric distribution, $\lambda_{j-1} = w_j$.

1.5 Asymptotic calculations

As we have seen in sections 1.2 and 1.3 we require the calculation of quantities of the form:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n m_i^k}{n} \tag{1.17}$$

$$\lim_{n\to\infty} \frac{(\boldsymbol{m}^r)^\top V_X^{-1} \boldsymbol{m}^s}{n} \tag{1.18}$$

$$\lim_{n\to\infty} \frac{(\boldsymbol{m}^r)^\top V_X \boldsymbol{m}^s}{n} \tag{1.19}$$

In order to compute these limits, we will make use of the well known asymptotic results (see Cramer [16, p.369])

$$F(m_i) \approx \frac{i}{n+1} \tag{1.20}$$

$$v_{ij} = Cov(X_{(i)}, X_{(j)}) \approx \frac{F(m_i)[1 - F(m_j)]}{(n+2)f(m_i)f(m_j)} \text{ ,for } i \leq j \tag{1.21}$$

We now examine the limits required in (1.17)- (1.19).

Consider the limit in (1.17).

$$\frac{\sum_{i=1}^n m_i^k}{n} \approx \sum_{i=1}^n \left[F^{-1}(t_i)\right]^k \Delta t_i \text{ ,where } t_i = \frac{i}{n+1} \text{ and } \Delta t_i = t_{i+1} - t_i \tag{1.22}$$

Equation (1.22) has the form of a Riemann sum. Hence, letting $n \to \infty$ and replacing the sum by an integral, we obtain

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n m_i^k}{n} = \int_0^1 F^{-1}(t)dt = \int_{-\infty}^{\infty} x^k f(x)dx = E(X^k). \tag{1.23}$$

For limits of the form (1.18) we use the following result :

Proposition 1.2 (Stephens [64]) *Let* $\boldsymbol{u}$ *be a n-dimensional vector with elements* $u_i = g(m_i)$ *i=1,...,n and let* $\boldsymbol{v}$ *be the vector whose elements are* $v_i = h(m_i)$*. Also define*

$$u^* = gf \text{ , } v^* = vf$$

If

$$\lim_{x\to-\infty} v^* = \lim_{x\to\infty} v^* = 0$$
$$\lim_{x\to-\infty} u^* = \lim_{x\to\infty} u^* = 0$$

Then

$$\lim_{n\to\infty} \frac{\boldsymbol{u}^\top V_X^{-1} \boldsymbol{v}}{n} = \int_{-\infty}^{\infty} \frac{du^*}{dx}\frac{dv^*}{dx}\frac{1}{f(x)}dx \quad \bullet \tag{1.24}$$

In particular, for $g(m_i) = m_i^r$ and $h(m_i) = m_i^s$

$$\lim_{n\to\infty} \frac{(\boldsymbol{m}^r)^\top V_X^{-1} \boldsymbol{m}^s}{n} = rsE\left[X^{r+s-2}\right] + (r+s)E\left[X^{r+s-1}\frac{d\ln f(x)}{dx}\right] + E\left[X^{r+s}\left(\frac{d\ln f(x)}{dx}\right)^2\right] \tag{1.25}$$

Stephens also shows

Proposition 1.3

$$\lim_{n\to\infty} \frac{\boldsymbol{u}^\top V_X^{-1} V_X^{-1} \boldsymbol{v}}{n} = \int_{-\infty}^{\infty} \frac{du_1}{dx}\frac{dv_1}{dx} f(x)dx$$

where

$$u_1 = \frac{1}{f}\frac{du^*}{dx} \text{ and } v_1 = \frac{1}{f}\frac{dv^*}{dx} \bullet$$

Finally, the limits appearing in (1.19) are found using the following:

Proposition 1.4 *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be two n-dimensional vectors with components* $u_i = g(m_i)$ *and* $v_i = h(m_i)$ *respectively, for* $i = 1, \ldots, n$ *and let us define*

$$G(x) = \int g(x)dx \ , \ H(x) = \int h(x)dx$$

If

$$\lim_{x\to-\infty} G(x)F(x) = \lim_{x\to\infty} G(x)\left[1 - F(x)\right] = 0$$
$$\lim_{x\to-\infty} H(x)F(x) = \lim_{x\to\infty} H(x)\left[1 - F(x)\right] = 0$$

Then

$$\lim_{n\to\infty} \frac{\boldsymbol{u}^\top V_X \boldsymbol{v}^\top}{n} = Cov\left[G(X), H(X)\right] \tag{1.26}$$

Proof:

$$\frac{\boldsymbol{u}^\top V_X \boldsymbol{v}}{n} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} g(m_i)h(m_j)v_{ij}$$

Using (1.20) and (1.21), asymptotically,

$$\approx \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{g(m_i)h(m_j)\{min\,[F(m_i),F(m_j)] - F(m_i)F(m_j)\}}{f(m_i)f(m_j)} \Delta F(m_i)\Delta F(m_j)$$

If we put $x = m_i$, $y = m_j$, let $n \to \infty$ and replace the sums by integrals, we obtain:

$$\lim_{n\to\infty} \frac{\boldsymbol{u}^\top V_X \boldsymbol{v}}{n} = \int_{-\infty}^{\infty}\int_{x}^{\infty} g(x)h(y)F(x)\,[1 - F(y)]\,dydx + \int_{-\infty}^{\infty}\int_{y}^{\infty} g(x)h(y)F(y)\,[1 - F(x)]\,dydx. \tag{1.27}$$

We now must show that the above sum of integrals equals the right hand side of (1.26).

In fact, let us consider

$$\int_{x}^{\infty} H(y)f(y)dy$$

Using two different expressions for $\int f(x)dx$, namely $F(x)$ and $-[1 - F(x)]$, integration by parts gives:

$$\int_{x}^{\infty} H(y)f(y)dy = H(x)\,[1 - F(x)] + \int_{x}^{\infty} h(y)\,[1 - F(y)]\,dy \tag{1.28}$$

Similarly,

$$\int_{-\infty}^{x} H(y)f(y)dy = H(x)F(x) - \int_{-\infty}^{x} h(y)F(y)dy \tag{1.29}$$

Multiplying (1.29) and (1.29) by $E[G(x)]$ expressed as an integral, we obtain:

$$\int_{-\infty}^{\infty} G(x)f(x)dx \int_{x}^{\infty} H(y)f(y)dy = \int_{-\infty}^{\infty} G(x)H(x)[1 - F(x)]f(x)dx + \int_{-\infty}^{\infty} G(x)f(x)dx \int_{x}^{\infty} h(y)[1 - F(y)]dy$$

$$\int_{-\infty}^{\infty} G(x)f(x)dx \int_{-\infty}^{x} H(y)f(y)dy = \int_{-\infty}^{\infty} G(x)H(x)F(x)f(x)dx - \int_{-\infty}^{\infty} G(x)f(x)dx \int_{-\infty}^{x} h(y)F(y)dy$$

Adding the above two equations we have that

$$E[G(X)]E[H(X)] = E[G(X)H(X)] + \int_{-\infty}^{\infty} G(x)f(x)dx \int_{x}^{\infty} h(y)[1 - F(y)]dy - \int_{-\infty}^{\infty} G(x)f(x)dx \int_{-\infty}^{x} h(y)F(y)dy \tag{1.30}$$

Now, changing the order of integration, the two double integrals above are, respectively, equal to

$$\int_{-\infty}^{\infty} h(y)G(y)F(y)[1-F(y)]dy - \int_{-\infty}^{\infty}\int_{x}^{\infty} g(x)h(y)F(x)[1-F(y)]dxdy \qquad (1.31)$$

and

$$\int_{-\infty}^{\infty} h(y)G(y)F(y)[1-F(y)]dy + \int_{-\infty}^{\infty}\int_{y}^{\infty} g(x)h(y)F(y)[1-F(x)]dxdy \qquad (1.32)$$

Finally, substitution of (1.31) and (1.32) into (1.30) gives the right hand side of (1.26) •

In particular, when $g(m_i) = m_i^r$, $h(m_i) = m_i^s$, equation (1.26) gives:

$$\lim_{n\to\infty} \frac{(\boldsymbol{m}^r)^{\top} V_X \boldsymbol{m}^s}{n} = \frac{\mu'_{r+s+2} - \mu'_{r+1}\mu'_{s+1}}{(r+1)(s+1)} \qquad (1.33)$$

where μ'_j denotes the j-th non-central moment of the random variable X.

The above expression is an extension for second-order moments of results given by Burr [11].

The asymptotic coefficients of the j-th polynomial in the GLS fit are obtained by solving:

$$\sum_{r=0}^{j}\sum_{s=0}^{k} a_{j(r)}a_{k(s)} \lim_{n\to\infty} \frac{(\boldsymbol{m}^r)^{\top} V_X^{-1} \boldsymbol{m}^s}{n} = 0 \;\; , k = 0, 1, \ldots, j-1. \qquad (1.34)$$

Similarly, for the OLS fit, the asymptotic coefficients will be the solution of

$$\sum_{r=0}^{j}\sum_{s=0}^{k} a_{j(r)}a_{k(s)} \lim_{n\to\infty} \frac{\sum_{i=1}^{n} m_i^{r+s}}{n} = 0 \;\; , k = 0, 1, \ldots, j-1. \qquad (1.35)$$

1.6 Tests for the Logistic distribution

In this section, the theory is illustrated by deriving asymptotic results of the tests for the Logistic distribution

$$F(x) = \frac{e^x}{1+e^x} \quad -\infty < x < \infty$$

Here, quadratic and cubic polynomials are taken as alternative models.

1.6.1 Test based on GLS

Using the change of variable $w = e^x$, the limits in (1.24) can be expressed as

$$\begin{aligned}
\lim_{n\to\infty}\frac{(\boldsymbol{m}^r)^\top V_X^{-1}\boldsymbol{m}^s}{n} &= rs\int_0^\infty \frac{\ln(w)^{r+s-2}}{(1+w)^4}dw + 2\,rs\int_0^\infty \frac{\ln(w)^{r+s-2}w}{(1+w)^4}dw \\
&+r\int_0^\infty \frac{\ln(w)^{r+s-1}}{(1+w)^4}dw + rs\int_0^\infty \frac{\ln(w)^{r+s-2}w^2}{(1+w)^4}dw \\
&-r\int_0^\infty \frac{\ln(w)^{r+s-1}w^2}{(1+w)^4}dw + s\int_0^\infty \frac{\ln(w)^{r+s-1}}{(1+w)^4}dw \\
&+\int_0^\infty \frac{\ln(w)^{r+s}}{(1+w)^4}dw - 2\int_0^\infty \frac{\ln(w)^{r+s}w}{(1+w)^4}dw \\
&-s\int_0^\infty \frac{\ln(w)^{r+s-1}w^2}{(1+w)^4}dw + \int_0^\infty \frac{\ln(w)^{r+s}w^2}{(1+w)^4}dw
\end{aligned}$$

Hence, we have the following results:

$$\begin{aligned}
\lim_{n\to\infty}\frac{\boldsymbol{1}^\top V_X^{-1}\boldsymbol{1}}{n} &= 1/3 \\
\lim_{n\to\infty}\frac{\boldsymbol{1}^\top V_X^{-1}\boldsymbol{m}^2}{n} &= \frac{\pi^2}{9} - 2/3 \\
\lim_{n\to\infty}\frac{\boldsymbol{m}^\top V_X^{-1}\boldsymbol{m}}{n} &= 1/3 + \frac{\pi^2}{9} \\
\lim_{n\to\infty}\frac{\boldsymbol{m}^\top V_X^{-1}\boldsymbol{m}^3}{n} &= \frac{7\,\pi^4}{45} - \frac{\pi^2}{3} \\
\lim_{n\to\infty}\frac{(\boldsymbol{m}^2)^\top V_X^{-1}\boldsymbol{m}^2}{n} &= \frac{7\,\pi^4}{45} \\
\lim_{n\to\infty}\frac{(\boldsymbol{m}^3)^\top V_X^{-1}\boldsymbol{m}^3}{n} &= \frac{31\,\pi^6}{63} - \frac{7\,\pi^4}{15}
\end{aligned}$$

In general, due to the symmetry of the distribution, the limit equals zero whenever $r + s$ is odd.

The coefficients of the polynomials were obtained by solving (1.34). The results are:

$$\begin{aligned}
\psi_0(m_i) &= 1 \\
\psi_1(m_i) &= m_i \\
\psi_2(m_i) &= m_i^2 - 1.2899 \\
\psi_3(m_i) &= m_i^3 - 8.2958\,m_i
\end{aligned}$$

We also find from (1.8), $c_{11} = 1.43$, $c_{22} = 14.5979$ and $c_{33} = 329.1965$.

1.6.2 Test based on OLS

The first three non-zero moments about the origin required to solve (1.35) are:

$$
\begin{aligned}
E(X^2) &= \pi^2/3 \\
E(X^4) &= 7\pi^4/15 \\
E(X^6) &= 31\pi^6/21
\end{aligned}
$$

The asymptotic coefficients for the OLS fit then give:

$$
\begin{aligned}
\psi_0(m_i) &= 1 \\
\psi_1(m_i) &= m_i \\
\psi_2(m_i) &= m_i^2 - 3.2899 \\
\psi_3(m_i) &= m_i^3 - 13.8174\, m_i
\end{aligned}
$$

Also, using (1.33), we find:

$$
\begin{aligned}
\frac{\mathbf{1}^{\mathsf{T}} V_X \mathbf{1}}{n} &= \pi^2/3 \\
\lim_{n\to\infty} \frac{\mathbf{1}^{\mathsf{T}} V_X \boldsymbol{m}^2}{n} &= 7\pi^4/45 \\
\lim_{n\to\infty} \frac{\boldsymbol{m}^{\mathsf{T}} V_X \boldsymbol{m}}{n} &= 4\pi^4/45 \\
\lim_{n\to\infty} \frac{\boldsymbol{m}^{\mathsf{T}} V_X \boldsymbol{m}^3}{n} &= 52\pi^6/315 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^2)^{\mathsf{T}} V_X \boldsymbol{m}^2}{n} &= 31\pi^6/189 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^3)^{\mathsf{T}} V_X \boldsymbol{m}^3}{n} &= 116\pi^8/225
\end{aligned}
$$

from which the quantities (1.12) and (1.13) are $c_{11}^* = 8.659$, $c_{22}^* = 93.5956$, $c_{33}^* = 2159.1679$; $d_{11} = 3.2899$, $d_{22} = 34.6343$, $d_{33} = 791.086$.

The matrix Ω defined in (1.14) is

$$
\Omega = \begin{bmatrix} 2.702 & 0 \\ 0 & 2.729 \end{bmatrix}
$$

1.6.3 Asymptotic relative efficiency of OLS-GLS estimators

We can now find the asymptotic relative efficiency of the OLS estimators with respect to those obtained by GLS. This is

$$ARE(\hat{\alpha}_j) = \frac{d_{jj}^2}{c_{jj}c_{jj}^*}$$

Thus, we have:

$$\begin{aligned} ARE(\alpha_1) &\approx 87.4\% \\ ARE(\alpha_2) &\approx 87.8\% \\ ARE(\alpha_3) &\approx 88.0\% \end{aligned}$$

In all the three cases above, the asymptotic relative efficiency of the OLS to the GLS estimators is greater than 80%. Considering that the calculation of OLS estimators does not require the calculation of V_X, these results show that in many situations the simpler procedure may be preferred without too much loss of efficiency.

1.6.4 Final remarks

As early as 1952, Gupta [26] proposed the simplified approach of taking $V_X = \mathbf{I}$; that is, to estimate the location and scale parameters in the model (1.1) by ordinary least squares. (cf. David [20]) For the normal case, the simplified approach was shown to give surprisingly good results in estimating σ, the scale parameter. Shapiro and Francia [50] later used this fact and proposed a simplified test statistic for normality. In his 1975 paper, Stephens [61] provides a theoretical explanation of this phenomenon by showing that the j-th asymptotic eigenvector of V_X is a Hermite polynomial of degree j in m_i, and that therefore $\psi_i{}^T V_X^{-1} \psi_j = 0$. So, for the normal distribution, the estimators obtained by ordinary and generalized least squares are asymptotically equivalent.

For other distributions which are sufficiently regular, the asymptotic relative efficiencies can be calculated following the procedure described above, and a choice between the OLS or GLS-based tests can be made.

Chapter 2

Polynomial tests for the Gumbel distribution

The importance of the Gumbel distribution is due to its many practical applications, where it has been used as a parent distribution as well as an asymptotic model in different situations.

References to applications are given in Castillo [13]. They include: extreme wind speeds (Thom [68], [69] ; Simiu et al [52], [53], [54], [55], [56], [57]; Grigoriu [23]), sea wave heights (Longuet-Higgins [37], [38] ; Dattatri [19]; Borgman [9], [10] ; Battjes [6]), floods (Shane and Lynn [49]; Benson [7]; Reich [47]; Todorovic [71], [72], [73], [74]; North [42]), rainfall (Hershfield [29]; Reich [47]), age at death (Gumbel and Goldstein [25]), minimum temperature, rainfall during droughts, electrical strength of materials, air pollution problems (Singpurwalla [58]; Barlow and Singpurwalla [5]), geological problems, naval engineering, etc.

In this chapter, tests of fit for the Gumbel distribution

$$F(x;\alpha,\beta) = 1 - exp(-exp(\frac{x-\alpha}{\beta})) \quad , \quad -\infty < x < \infty \tag{2.1}$$

are given.

The tests are based on the technique described in chapter 1. Here, we consider the fit of second and third-order polynomial regression models by both generalized and ordinary least squares estimation methods.

The motivation of the regression tests comes from the technique of probability plots, which has been used by statisticians for many years to judge a distribution. For location and scale parameters unknown, the regression line should be a straight line, and this was often judged by eye. In order to illustrate the polynomial tests given here, we give Q-Q plots for Weibull distributions against Gumbel quantiles. Clearly, if the data were Gumbel, we should expect a straight line when the ordered data are plotted against Gumbel quantiles. However, the Q-Q plot $W(3)$, for example, shows the type of curve that we might expect when the true distribution is Weibull with shape parameter 3. We can see that close approximations to these curves could be given by polynomials, and this motivates the test procedure to decide if a polynomial fit is better than a straight line. The Q-Q plots are given in figure 2.1. Note that the quantiles were standardized by subtracting the median and dividing by the interquartile range.

Plot for a data set . Figure 2.2, shows a plot of the order statistics of the data set 4 (given in appendix A) against Gumbel quantiles. Again, the figure shows that a polynomial model could give a better approximation to describe the curve.

2.1 Tests based on GLS estimators

Taking $w = e^x$, we can now express the limits (1.24) as

$$\begin{aligned}\lim_{n\to\infty}\frac{(\boldsymbol{m}^r)^\top V_X^{-1}\boldsymbol{m}^s}{n} &= rs\int_0^\infty \ln(w)^{r+s-2}e^{-w}dw + r\int_0^\infty \ln(w)^{r+s-1}e^{-w}dw\\ &\quad -r\int_0^\infty \ln(w)^{r+s-1}e^{-w}wdw + s\int_0^\infty \ln(w)^{r+s-1}e^{-w}dw\\ &\quad +\int_0^\infty \ln(w)^{r+s}e^{-w}dw - 2\int_0^\infty \ln(w)^{r+s}e^{-w}wdw - s\\ &\quad \int_0^\infty \ln(w)^{r+s-1}e^{-w}wdw + \int_0^\infty \ln(w)^{r+s}e^{-w}w^2dw\end{aligned}$$

The above integrals can be evaluated in terms of the Γ function and its derivatives, since

$$\int_0^\infty [\ln(w)]^k w^t e^{-w}dw = \Gamma^{[k]}(t+1)$$

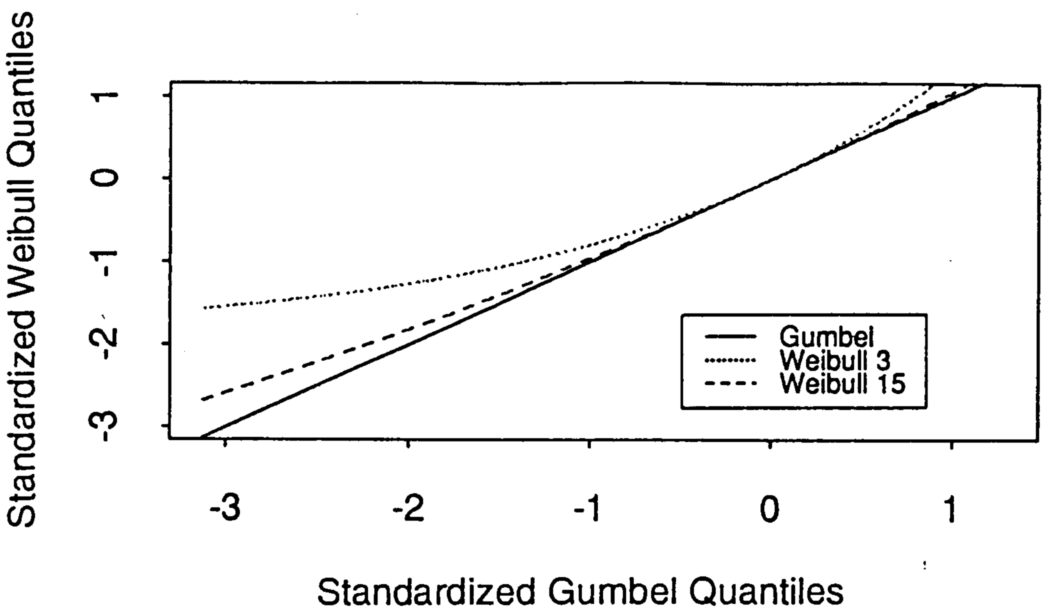

Figure 2.1: Q-Q plots of the Gumbel against Weibull for different values of the shape parameter

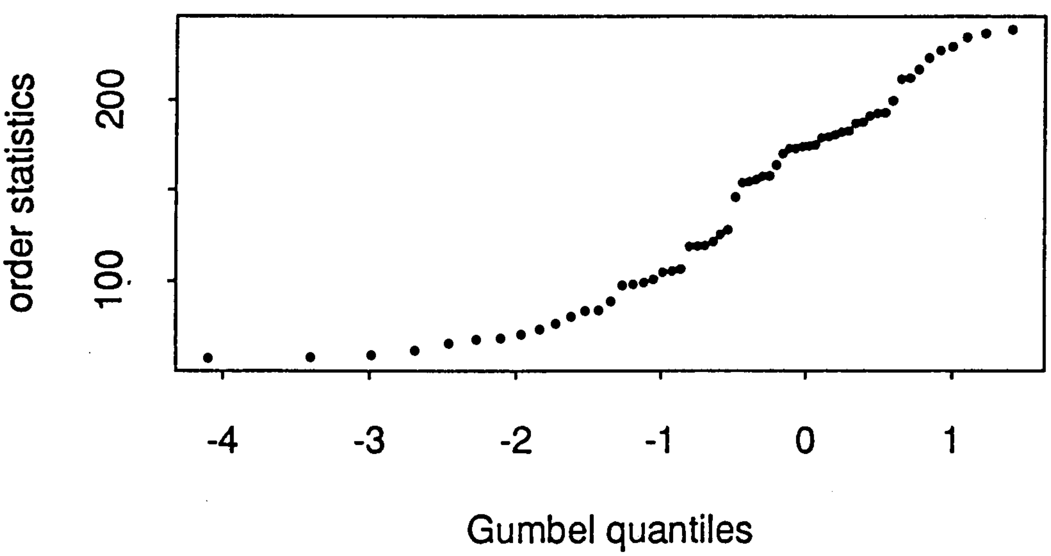

Figure 2.2: Gumbel probability plot for data set 4

where $\Gamma^{[k]}(t+1)$ denotes the k-th derivative of the Γ function evaluated at $(t+1)$.

We then find:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X^{-1}\mathbf{1}}{n} &= 1 \\
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X^{-1}\boldsymbol{m}}{n} &= 1-\gamma \approx 0.4228 \\
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X^{-1}\boldsymbol{m}^2}{n} &= \frac{\pi^2}{6} - 2\gamma + \gamma^2 \approx 0.8237 \\
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X^{-1}\boldsymbol{m}^3}{n} &= \frac{\pi^2}{2} - 2\zeta(3) - \frac{\pi^2\gamma}{2} + 3\gamma^2 - \gamma^3 \approx 0.4895 \\
\lim_{n\to\infty} \frac{\boldsymbol{m}^\top V_X^{-1}\boldsymbol{m}}{n} &= 1 - 2\gamma + \frac{\pi^2}{6} + \gamma^2 \approx 1.8237 \\
\lim_{n\to\infty} \frac{\boldsymbol{m}^\top V_X^{-1}\boldsymbol{m}^2}{n} &= 3\gamma^2 - \gamma^3 + \left(-2 - \frac{\pi^2}{2}\right)\gamma + \frac{\pi^2}{2} - 2\zeta(3) \approx -0.6650 \\
\lim_{n\to\infty} \frac{\boldsymbol{m}^\top V_X^{-1}\boldsymbol{m}^3}{n} &= \gamma^4 - 4\gamma^3 + \left(3+\pi^2\right)\gamma^2 + \left(8\zeta(3) - 2\pi^2\right)\gamma \\
&\quad + \frac{\pi^2}{2} + \frac{3\pi^4}{20} - 8\zeta(3) \approx 7.7163 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^2)^\top V_X^{-1}\boldsymbol{m}^2}{n} &= \gamma^4 - 4\gamma^3 + \left(\pi^2+4\right)\gamma^2 + \left(8\zeta(3) - 2\pi^2\right)\gamma \\
&\quad + \frac{2\pi^2}{3} + \frac{3\pi^4}{20} - 8\zeta(3) \approx 9.6944 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^2)^\top V_X^{-1}\boldsymbol{m}^3}{n} &= 5\gamma^4 - \gamma^5 + \left(-6 - \frac{5\pi^2}{3}\right)\gamma^3 + \left(5\pi^2 - 20\zeta(3)\right)\gamma^2 \\
&\quad + \left(40\zeta(3) - 3\pi^2 - \frac{3\pi^4}{4}\right)\gamma + \frac{3\pi^4}{4} \\
&\quad - 12\zeta(3) - \frac{10\zeta(3)\pi^2}{3} - 24\zeta(5) \approx -32.7013 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^3)^\top V_X^{-1}\boldsymbol{m}^3}{n} &= \gamma^6 - 6\gamma^5 + \left(9 + \frac{5\pi^2}{2}\right)\gamma^4 + \left(40\zeta(3) - 10\pi^2\right)\gamma^3 \\
&\quad + \left(\frac{9\pi^4}{4} - 120\zeta(3) + 9\pi^2\right)\gamma^2 \\
&\quad + \left(144\zeta(5) + 20\zeta(3)\pi^2 + 72\zeta(3) - \frac{9\pi^4}{2}\right)\gamma \\
&\quad - 20\zeta(3)\pi^2 + 40\zeta(3)^2 + \frac{61\pi^6}{168}
\end{aligned}
$$

$$-144\,\zeta(5) + \frac{27\,\pi^4}{20} \approx 220.0842$$

where $\gamma \approx 0.5772$ is Euler's constant and $\zeta(n)$ denotes the Riemann's Zeta function. For a definition of these quantities, see, for example, Abramowitz and Stegun [1].

If we take

$$\psi_0(m_i) = 1$$

the solution of (1.34) gives, asymptotically,

$$\begin{aligned}
\psi_1(m_i) &= m_i - 0.4228 \\
\psi_2(m_i) &= m_i^2 + 0.6160\,m_i - 1.0841 \\
\psi_3(m_i) &= m_i^3 + 3.3936\,m_i^2 - 2.4748\,m_i - 2.2384
\end{aligned}$$

We also find from (1.8), $c_{11} = 1.6449$, $c_{22} = 8.3918$ and $c_{33} = 88.9159$.

2.2 Tests based on OLS estimators

The first eight non-central moments for this distribution, required to compute the limits in (1.17) and (1.19), are:

$$\begin{aligned}
E[X] &= -\gamma = -0.5772 \\
E[X^2] &= \frac{\pi^2}{6} + \gamma^2 = 1.9781 \\
E[X^3] &= -2\,\zeta(3) - \frac{\pi^2\gamma}{2} - \gamma^3 = -5.4449 \\
E[X^4] &= \frac{3\,\pi^4}{20} + 8\,\zeta(3)\gamma + \pi^2\gamma^2 + \gamma^4 = 23.5615 \\
E[X^5] &= -24\,\zeta(5) - \frac{3\,\pi^4\gamma}{4} - \frac{10\,\zeta(3)\pi^2}{3} - 20\,\zeta(3)\gamma^2 - \frac{5\,\pi^2\gamma^3}{3} - \gamma^5 = -117.8394 \\
E[X^6] &= \gamma^6 + \frac{5\,\pi^2\gamma^4}{2} + 40\,\zeta(3)\gamma^3 + \frac{9\,\pi^4\gamma^2}{4} + \left(144\,\zeta(5) + 20\,\zeta(3)\pi^2\right)\gamma + \\
&\quad \frac{61\,\pi^6}{168} + 40\,\zeta(3)^2 = 715.0674 \\
E[X^7] &= -\gamma^7 - \frac{7\,\pi^2\gamma^5}{2} - 70\,\zeta(3)\gamma^4 - \frac{21\,\pi^4\gamma^3}{4} + \left(-504\,\zeta(5) - 70\,\zeta(3)\pi^2\right)\gamma^2 \\
&\quad + \left(-\frac{61\,\pi^6}{24} - 280\,\zeta(3)^2\right)\gamma - \frac{21\,\zeta(3)\pi^4}{2} - 720\,\zeta(7) - 84\,\zeta(5)\pi^2 = -5019.8489
\end{aligned}$$

$$\begin{aligned}
E[X^8] &= \gamma^8 + \frac{14\,\pi^2\gamma^6}{3} + 112\,\zeta(3)\gamma^5 + \frac{21\,\pi^4\gamma^4}{2} + \left(1344\,\zeta(5) + \frac{560\,\zeta(3)\pi^2}{3}\right)\gamma^3 \\
&+ \left(1120\,\zeta(3)^2 + \frac{61\,\pi^6}{6}\right)\gamma^2 \\
&+ \left(5760\,\zeta(7) + 84\,\zeta(3)\pi^4 + 672\,\zeta(5)\pi^2\right)\gamma \\
&+ \frac{1261\,\pi^8}{720} + \frac{560\,\zeta(3)^2\pi^2}{3} + 2688\,\zeta(5)\zeta(3) = 40243.6216
\end{aligned}$$

From which we obtain:

$$\begin{aligned}
\frac{\mathbf{1}^\top V_X \mathbf{1}}{n} &= \frac{\pi^2}{6} = 1.6449 \\
\lim_{n\to\infty} \frac{(\mathbf{1})^\top V_X \boldsymbol{m}}{n} &= -\frac{\pi^2\gamma}{6} - \zeta(3) = -2.1515 \\
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X \boldsymbol{m}^2}{n} &= \frac{\pi^2\gamma^2}{6} + 2\,\zeta(3)\gamma + \frac{\pi^4}{20} = 6.8062 \\
\lim_{n\to\infty} \frac{\mathbf{1}^\top V_X \boldsymbol{m}^3}{n} &= -\frac{\pi^2\gamma^3}{6} - 3\,\zeta(3)\gamma^2 - \frac{3\,\pi^4\gamma}{20} - 6\,\zeta(5) - \frac{5\,\zeta(3)\pi^2}{6} = -26.0598 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m})^\top V_X \boldsymbol{m}}{n} &= \frac{\pi^2\gamma^2}{6} + 2\,\zeta(3)\gamma + \frac{11\,\pi^4}{360} = 4.9121 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m})^\top V_X \boldsymbol{m}^2}{n} &= -\frac{\pi^2\gamma^3}{6} - 3\,\zeta(3)\gamma^2 - \frac{\pi^4\gamma}{9} - 4\,\zeta(5) - \frac{\zeta(3)\pi^2}{2} = -17.8448 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m})^\top V_X \boldsymbol{m}^3}{n} &= \frac{\pi^2\gamma^4}{6} + 4\,\zeta(3)\gamma^3 + \frac{29\,\pi^4\gamma^2}{120} + \left(18\,\zeta(5) + \frac{7\,\zeta(3)\pi^2}{3}\right)\gamma \\
&+ \frac{71\,\pi^6}{1680} + 5\,\zeta(3)^2 = 83.5575 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^2)^\top V_X \boldsymbol{m}^2}{n} &= \frac{\pi^2\gamma^4}{6} + 4\,\zeta(3)\gamma^3 + \frac{2\,\pi^4\gamma^2}{9} + \left(16\,\zeta(5) + 2\,\zeta(3)\pi^2\right)\gamma \\
&+ \frac{61\,\pi^6}{1512} + 4\,\zeta(3)^2 = 76.1579 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^2)^\top V_X \boldsymbol{m}^3}{n} &= -\frac{\pi^2\gamma^5}{6} - 5\,\zeta(3)\gamma^4 - \frac{23\,\pi^4\gamma^3}{60} + \left(-42\,\zeta(5) - \frac{16\,\zeta(3)\pi^2}{3}\right)\gamma^2 \\
&+ \left(-\frac{37\,\pi^6}{180} - 22\,\zeta(3)^2\right)\gamma - 7\,\zeta(5)\pi^2 \\
&-60\,\zeta(7) - \frac{17\,\pi^4\zeta(3)}{20} = -407.6300 \\
\lim_{n\to\infty} \frac{(\boldsymbol{m}^3)^\top V_X \boldsymbol{m}^3}{n} &= \frac{\pi^2\gamma^6}{6} + 6\,\zeta(3)\gamma^5 + \frac{23\,\pi^4\gamma^4}{40} + \left(\frac{32\,\zeta(3)\pi^2}{3} + 84\,\zeta(5)\right)\gamma^3
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{37\,\pi^6}{60}+66\,\zeta(3)^2\right)\gamma^2 \\
&+\left(360\,\zeta(7)+42\,\zeta(5)\pi^2+\frac{51\,\pi^4\zeta(3)}{10}\right)\gamma+168\,\zeta(5)\zeta(3) \\
&+\frac{389\,\pi^8}{3600}+\frac{35\,\zeta(3)^2\pi^2}{3}=2480.5299
\end{aligned}
$$

The solution of(1.35) yields:

$$
\begin{aligned}
\psi_0(m_i) &= 1 \\
\psi_1(m_i) &= m_i+0.5772 \\
\psi_2(m_i) &= m_i^2+2.6160\,m_i-0.4681 \\
\psi_3(m_i) &= m_i^3+6.3936\,m_i^2+4.3124\,m_i-4.7132
\end{aligned}
$$

The required quantities defined by (1.12) and (1.13) are now,

$$c^*_{11}=2.976 \quad c^*_{22}=15.67 \quad c^*_{33}=168.71 \quad c^*_{23}=-27.88$$

$$d_{11}=1.6449 \quad d_{22}=8.39 \quad d_{33}=88.92$$

In this case, matrix defined in (1.14) is then

$$\Omega=\begin{bmatrix} 1.868 & -1.021 \\ -1.021 & 1.897 \end{bmatrix}$$

whose eigenvalues are: $\lambda_1=2.9036$ and $\lambda_2=0.8614$.

2.3 Asymptotic relative efficiency

The asymptotic relative efficiency of the OLS estimators with respect to those obtained by GLS is

$$ARE(\alpha_j)=\frac{d^2_{jj}}{c_{jj}c^*_{jj}}$$

Thus, we have:

$$
\begin{aligned}
ARE(\alpha_1) &\approx 55.3\% \\
ARE(\alpha_2) &\approx 53.5\% \\
ARE(\alpha_3) &\approx 52.7\%
\end{aligned}
$$

Roughly, twice the sample size is required in order for the variance of the OLS estimators to be comparable to those from the GLS fit.

2.4 Numerical calculations

The calculation of the test statistics involves the values of the mean vector and covariance matrix of the order statistics. For the Gumbel distribution, expected values and variances for $n = 1(1)20$ are given (for instance) in White [76]. Tables for covariances have been given by Lieblein and Zelen [33] for $n \leq 6$, White [75] for $n \leq 20$ and by Mann [39] for $n \leq 25$.

In this thesis, the expected values and covariances of the order statistics were obtained by numerical integration using subroutines DQDAGI and DTWODQ from IMSL and by using a Taylor's series expansion of the inverse cumulative distribution function (David and Johnson [21] to order $(n+2)^{-3}$) as an approximation.

It is worthwhile to examine the accuracy of the approximation. The following comparative results are given for sample sizes 5 and 10. Here, $\boldsymbol{m}$ and V_X denote the parameters computed by numerical integration whereas $\hat{\boldsymbol{m}}$ and $\hat{V}_X$ denote the corresponding approximations.

For $n = 5$, the results were:

$$\mathbf{m}^{\mathsf{T}} = [-2.1866, -1.0709, -0.4255, 0.1069, 0.6902]$$

$$\hat{\mathbf{m}}^{\mathsf{T}} = [-2.1871, -1.0699, -0.4251, 0.1071, 0.6912]$$

$$V_X = \begin{bmatrix} 1.6449 & 0.5899 & 0.3172 & 0.1897 & 0.1090 \\ & 0.6491 & 0.3527 & 0.2123 & 0.1226 \\ & & 0.4060 & 0.2468 & 0.1436 \\ & & & 0.3085 & 0.1820 \\ & & & & 0.2849 \end{bmatrix}$$

$$\hat{V}_X = \begin{bmatrix} 1.5747 & 0.5754 & 0.3113 & 0.1874 & 0.1081 \\ & 0.6342 & 0.3472 & 0.2099 & 0.1216 \\ & & 0.4002 & 0.2442 & 0.1425 \\ & & & 0.3058 & 0.1809 \\ & & & & 0.2855 \end{bmatrix}$$

$n = 10$

$$\mathbf{m}^\top = [-2.8798, -1.8262, -1.2672, -0.8681, -0.5436, -0.2574, 0.0120, 0.2837, 0.5846, 0.9899]$$

$$\hat{\mathbf{m}}^\top = [-2.8840, -1.8261, -1.2670, -0.8680, -0.5435, -0.2574, 0.0121, 0.2837, 0.5847, 0.9808]$$

$$V_X = \begin{bmatrix} 1.6449 & 0.6187 & 0.3592 & 0.2426 & 0.1761 & 0.1328 & 0.1018 & 0.0780 & 0.0582 & 0.0396 \\ & 0.6459 & 0.3765 & 0.2549 & 0.1854 & 0.1399 & 0.1074 & 0.0823 & 0.0615 & 0.0418 \\ & & 0.3970 & 0.2695 & 0.1964 & 0.1484 & 0.1140 & 0.0875 & 0.0654 & 0.0445 \\ & & & 0.2874 & 0.2099 & 0.1589 & 0.1222 & 0.0939 & 0.0702 & 0.0479 \\ & & & & 0.2269 & 0.1721 & 0.1326 & 0.1020 & 0.0764 & 0.0521 \\ & & & & & 0.1896 & 0.1464 & 0.1128 & 0.0846 & 0.0578 \\ & & & & & & 0.1658 & 0.1281 & 0.0963 & 0.0660 \\ & & & & & & & 0.1519 & 0.1147 & 0.0789 \\ & & & & & & & & 0.1488 & 0.1032 \\ & & & & & & & & & 0.1714 \end{bmatrix}$$

$$\hat{V}_X = \begin{bmatrix} 1.6180 & 0.6122 & 0.3567 & 0.2414 & 0.1754 & 0.1324 & 0.1016 & 0.0779 & 0.0581 & 0.0396 \\ & 0.6393 & 0.3740 & 0.2536 & 0.1846 & 0.1395 & 0.1071 & 0.0821 & 0.0614 & 0.0417 \\ & & 0.3945 & 0.2683 & 0.1956 & 0.1480 & 0.1137 & 0.0873 & 0.0653 & 0.0444 \\ & & & 0.2861 & 0.2091 & 0.1584 & 0.1220 & 0.0937 & 0.0701 & 0.0477 \\ & & & & 0.2261 & 0.1716 & 0.1323 & 0.1018 & 0.0763 & 0.0520 \\ & & & & & 0.1891 & 0.1461 & 0.1126 & 0.0845 & 0.0577 \\ & & & & & & 0.1655 & 0.1279 & 0.0962 & 0.0659 \\ & & & & & & & 0.1517 & 0.1146 & 0.0788 \\ & & & & & & & & 0.1488 & 0.1031 \\ & & & & & & & & & 0.1732 \end{bmatrix}$$

The above results show that the approximation appears to produce reasonable results even for values of n as low as 5.

2.5 Test procedures

The following subsections describe the procedure for testing the null hypothesis, H_0, that the random sample $Y_1, \ldots, Y_n$, with order statistics $Y_{(1)}, \ldots, Y_{(n)}$, came from the distribution (2.1).

2.5.1 GLS-based Tests

For a given value of n, the coefficients of the second and third-order polynomials are found by solving (1.6). The test statistics

$$\hat{T}_2 = \frac{\sigma_X^2 \left(\boldsymbol{\psi}_2^{\top} V_X^{-1} \mathbf{Y}\right)^2}{\boldsymbol{\psi}_2{}^{\top} V_X^{-1} \boldsymbol{\psi}_2 \; S_Y^2}$$

$$\hat{T}_3 = \frac{\sigma_X^2 \left(\boldsymbol{\psi}_3^{\top} V_X^{-1} \mathbf{Y}\right)^2}{\boldsymbol{\psi}_3{}^{\top} V_X^{-1} \boldsymbol{\psi}_3 \; S_Y^2}$$

$$\hat{T}_{23} = \hat{T}_2 + \hat{T}_3$$

are then calculated. Note that $\sigma_X^2 = \pi^2/6$.

For values of $n = 10(10)100$ 10,000 samples were generated and the empirical percentage points obtained. The smoothed points are given in tables (2.2) to (2.4).

From these tables it can be seen that the empirical percentage points converge quickly to their asymptotic values, so that the $\chi^2_{(1)}$ and $\chi^2_{(2)}$ approximations can be used with good accuracy for small n.

An approximation to the standard errors of the estimated quantile $1-\alpha$, say $\xi_{1-\alpha}$, can be obtained by using the asymptotic expression:

$$\widehat{SE}\,(\xi_{1-\alpha}) \approx \sqrt{\frac{\alpha(1-\alpha)}{n}}\,\frac{1}{f(\xi_{1-\alpha})}$$

where $f(\xi_{1-\alpha})$ can be estimated by approximating the derivative of the CDF using two quantiles adjacent to $f(\xi_{1-\alpha})$.

For example, from table 2.2 the approximate standard errors of the quantiles are about 0.02 for $\alpha = 0.25$, 0.03 for $\alpha = 0.15$ and about 0.12 for $\alpha = 0.01$. These approximations correspond to typical values for a given significance level, since little change was observed for different values of n.

2.5.2 OLS-based Tests

In this case, the coefficients of the polynomials were found by solving (1.11) for a given value of n, and the following test statistics calculated:

$$\hat{T}_2^* = \frac{\sigma_X^2\left(\psi_2{}^{\mathsf{T}}\mathbf{Y}\right)^2}{\psi_2{}^{\mathsf{T}}\psi_2\; S_Y^2}$$

$$\hat{T}_3^* = \frac{\sigma_X^2\left(\psi_3{}^{\mathsf{T}}\mathbf{Y}\right)^2}{\psi_3{}^{\mathsf{T}}\psi_3\; S_Y^2}$$

$$\hat{T}_{23}^* = \hat{T}_2^* + \hat{T}_3^*$$

The empirical percentage points were also found using 10,000 simulated samples. The results showed that the convergence of these points to their asymptotic values was extremely slow. Even for values of n as large as $n = 1000$, the difference between the empirical and the asymptotic percentage points was still considerable.

The problem appeared to be the slow rate of convergence to their limits, of the quantites

$$c^*_{ij}(n) = \frac{(\boldsymbol{m}^r)^\top V_X \boldsymbol{m}^s}{n}.$$

In the next section, a practical solution to overcome this problem is presented.

2.5.3 Modified OLS-based tests

Consider the quantity,

$$\xi_j = \frac{\sigma_X \left(\boldsymbol{\psi}_j{}^\top \mathbf{Y}\right)}{S_Y \left[\boldsymbol{\psi}_j{}^\top V_X \boldsymbol{\psi}_j\right]^{\frac{1}{2}}} \tag{2.2}$$

Under the null hypothesis,

$$\xi_j^2 \xrightarrow{\mathcal{D}} \chi^2_{(1)}.$$

Note that ξ_j^2 depends on V_X only through the value of the quadratic form $c^*_{jj}(n)$.

Thus, if an approximation $h_{jj}(n)$ of $c^*_{jj}(n)$ can be found, we can define the modified test statistics:

$$\hat{t}_j = \frac{\sigma_X^2 \left(\boldsymbol{\psi}_j{}^\top \mathbf{Y}\right)^2}{n S_Y^2 h_{jj}(n)} \tag{2.3}$$

for $j = 2, 3$ and $\hat{t}_{23} = \hat{t}_2 + \hat{t}_3$.

Note that (2.3) corresponds to the square of the quantity defined in (2.2) with the only difference that $\boldsymbol{\psi}_j{}^\top V_X \boldsymbol{\psi}_j$ has now been replaced by the approximation $n h_{jj}(n)$.

Exponential fits of $c^*_{jj}(n)$ were considered by regressing $\ln\left(\frac{c^*_{jj}(n)}{c^*_{jj}}\right)$ against $n^{1/j}$ for values of $n = 10(5)200$.

For $j = 2, 3$ the following approximations were obtained.

$$h_{22}(n) = c^*_{22} \exp\left(-8.824/\sqrt{n}\right)$$

$$h_{33}(n) = c^*_{33} \exp\left(-12/\sqrt[3]{n}\right)$$

In both cases, the fit was found to be very accurate.

The limiting distribution of the vector $\boldsymbol{\xi}^\top = (\xi_2, \xi_3)$ is then bivariate normal with mean vector zero and covariance matrix

$$\Lambda = \begin{bmatrix} 1 & -0.542 \\ -0.542 & 1 \end{bmatrix}$$

It follows that the limiting distribution of the test statistic $\hat{t}_{23} = \hat{t}_2 + \hat{t}_3$, is that of a weighted sum of two independent $\chi^2_{(1)}$ variables, where the weights correspond to the eigenvalues of Λ.

The critical points of the distribution of $\hat{t}_{23}$ were calculated using Imhof's [30] method.

The simulation study was now carried out for the modified test statistics. The empirical percentage points are given in tables (2.5) to (2.7). These tables show a quick convergence of the empirical percentage points to their asymptotic values.

2.6 Empirical power study

A simulation study was conducted to estimate the power of the tests proposed against some selected alternatives: Weibull with shape parameter θ ($\mathcal{W}(\theta)$); Normal ($\mathcal{N}$); Logistic ($\mathcal{L}$); Uniform ($\mathcal{U}$); Double exponential ($\mathcal{DE}$) and $\chi^2_{(10)}$. Table 2.1 shows the percentages of rejections from 1000 simulations for each test. The tests were done using a significance level of 10%.

For comparison purposes, available tests for the Gumbel distribution, based on the following statistics were also included:

a) The correlation coefficient (Stephens [17, section 5.14]).

$$Z(\mathbf{Y}, \mathbf{H}) = n\left\{1 - R^2(\mathbf{Y}, \mathbf{H})\right\}$$

where $\mathbf{Y}$ denotes the vector of order statistics, and $\mathbf{H}$ is an approximation to the vector of expected values of the order statistics in a sample from a standard Gumbel distribution whose i-th component is $H_i = \ln\{-\ln[1 - i/(n+1)]\}$.

b) Statistics based on the empirical distribution function.

Anderson-Darling:

$$A^2 = -n - (1/n)\sum_{i=1}^{n}(2i-1)\left[\ln Z_{(i)} + \ln\left\{1 - Z_{(n-i+1)}\right\}\right]$$

Cramér-von Mises:

$$W^2 = \sum_{i=1}^{n}\left\{Z_{(i)} - (2i-1)/(2n)\right\}^2 + 1/(12n)$$

The Watson statistic:

$$U^2 = W^2 - n(\bar{Z} - 0.5)^2$$

where $Z_{(i)} = F(Y_{(i)}; \hat{\alpha}, \hat{\beta})$ and $\hat{\alpha}$, $\hat{\beta}$ denote the maximum likelihood estimators of the location and scale parameters respectively. Tables of percentage points of the above EDF statistics for testing goodness of fit for the Gumbel distribution have been given by Stephens [63].

Table 2.1: Empirical power (in percent) of tests based on polynomial regression; size of the test=0.10

F	n	$\hat{T}_2$	$\hat{T}_3$	$\hat{T}_{23}$	$\hat{t}_2$	$\hat{t}_3$	$\hat{t}_{23}$	$Z(\mathbf{Y},\mathbf{H})$	W^2	A^2	U^2
$\mathcal{W}(3)$	20	56	16	45	44	9	26	23	39	39	38
	40	86	21	79	78	11	58	52	62	66	59
	80	99	30	99	99	12	94	87	88	94	85
$\mathcal{W}(6)$	20	25	10	19	17	8	10	9	18	16	17
	40	42	11	33	28	6	14	10	25	26	24
	80	67	11	51	50	5	28	7	38	44	37
$\mathcal{W}(9)$	20	17	10	14	12	8	7	6	14	13	14
	40	26	8	18	16	6	9	6	15	16	15
	80	41	9	32	25	5	14	6	24	27	23
$\mathcal{N}$	20	50	18	40	32	11	20	18	34	33	33
	40	78	24	69	60	14	42	37	55	58	52
	60	90	30	84	76	17	62	52	70	75	68
$\mathcal{L}$	20	54	31	50	34	22	31	29	42	42	41
	40	80	48	76	50	30	49	45	64	68	63
	60	91	65	90	59	45	65	59	82	84	82
$\mathcal{U}$	20	28	30	35	50	13	36	28	40	41	40
	40	49	58	85	89	15	89	68	72	77	70
	60	65	79	99	99	14	99	91	88	94	86
$\mathcal{DE}$	20	61	52	62	32	42	44	45	59	59	59
	40	80	74	84	38	61	63	65	85	86	85
	60	89	88	94	47	69	72	75	93	93	93
$\chi^2_{(10)}$	20	88	38	81	85	30	71	70	70	72	67
	40	99	57	99	99	48	97	96	96	96	93

The results indicate that the statistic $\hat{T}_2$ based on the second-order polynomial is in general the most powerful (but note the Uniform exception). This can be explained by the fact that the kind of departure from linearity in the Gumbel probability plot, when the true distribution is Weibull or Frechet, is approximately a quadratic pattern (see, for example Castillo [13, section 6.2]).

2.7 Conclusions

1. The test statistic $\hat{T}_2$ is recommended when Weibull alternatives are of interest. This is, for instance, the case in extreme value problems where the non-rejection of the Gumbel model, when the true parent distribution is Weibull, is considered the most serious error and therefore the most powerful test available should be used.

2. In general, the test statistic $\hat{T}_{23}$ can be used as an overall test statistic. According to the simulation results obtained here, the power of this test statistic appears to be greater than any of the EDF statistics W^2, A^2 and U^2. It must be remarked, however, that more extensive comparative studies are needed.

3. The test based on the correlation statistic $Z(\mathbf{Y}, \mathbf{H})$, appears to be a biased test, in the sense that its power is less than the significance level of the test, for Weibull alternatives with a large value of the shape parameter. This also appears to be the case for the regression statistics which involve a third order polynomial. Thus, use of these statistics is not recommended.

Table 2.2: Empirical percentage points of $\hat{T}_2$. Gumbel distribution

Significance level

n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	0.52	1.42	2.20	2.84	4.15	5.76	8.21	10.29
20	0.47	1.34	2.08	2.69	3.89	5.27	7.28	8.99
30	0.46	1.32	2.07	2.68	3.85	5.16	7.04	8.60
40	0.46	1.32	2.06	2.68	3.84	5.12	6.92	8.41
50	0.46	1.32	2.06	2.68	3.84	5.10	6.86	8.30
60	0.46	1.32	2.06	2.68	3.84	5.08	6.82	8.23
70	0.46	1.32	2.06	2.69	3.84	5.07	6.79	8.18
80	0.46	1.32	2.07	2.69	3.84	5.06	6.77	8.14
90	0.46	1.32	2.07	2.69	3.84	5.05	6.76	8.11
100	0.46	1.32	2.07	2.69	3.84	5.05	6.74	8.09
∞	0.455	1.323	2.072	2.706	3.841	5.024	6.635	7.879

Table 2.3: Empirical percentage points of $\hat{T}_3$. Gumbel distribution.

Significance level

n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	0.52	1.45	2.12	2.69	3.55	4.60	6.26	8.00
20	0.46	1.28	1.95	2.51	3.49	4.63	6.36	7.90
30	0.45	1.28	1.96	2.53	3.56	4.72	6.44	7.89
40	0.45	1.28	1.97	2.56	3.61	4.78	6.48	7.88
50	0.45	1.28	1.99	2.58	3.65	4.82	6.51	7.88
60	0.45	1.29	2.00	2.60	3.68	4.85	6.53	7.88
70	0.45	1.29	2.01	2.61	3.70	4.87	6.54	7.88
80	0.45	1.29	2.01	2.62	3.71	4.89	6.55	7.88
90	0.45	1.30	2.02	2.63	3.73	4.90	6.56	7.88
100	0.45	1.30	2.02	2.64	3.74	4.91	6.57	7.88
∞	0.455	1.323	2.072	2.706	3.841	5.024	6.635	7.879

Table 2.4: Empirical percentage points of $\hat{T}_{23}$. Gumbel distribution.

	Significance level							
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	1.50	2.76	3.62	4.42	6.08	8.25	11.86	15.56
20	1.39	2.64	3.52	4.29	5.81	7.66	10.46	12.95
30	1.38	2.65	3.57	4.35	5.82	7.53	10.02	12.13
40	1.37	2.67	3.61	4.40	5.84	7.48	9.81	11.74
50	1.37	2.69	3.64	4.43	5.86	7.45	9.69	11.50
60	1.38	2.70	3.66	4.45	5.88	7.44	9.61	11.35
70	1.38	2.71	3.68	4.47	5.89	7.43	9.55	11.24
80	1.38	2.71	3.69	4.49	5.90	7.42	9.51	11.16
90	1.38	2.72	3.70	4.50	5.91	7.41	9.47	11.10
100	1.38	2.72	3.71	4.51	5.92	7.41	9.45	11.05
∞	1.386	2.773	3.794	4.605	5.991	7.378	9.210	10.597

Table 2.5: Empirical percentage points of $\hat{t}_2$. Gumbel distribution.

	Significance level							
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	0.46	1.33	2.08	2.70	3.79	4.89	6.36	7.43
20	0.46	1.32	2.07	2.70	3.82	4.98	6.56	7.77
30	0.46	1.32	2.07	2.70	3.83	5.00	6.60	7.83
40	0.46	1.32	2.07	2.70	3.83	5.01	6.61	7.85
50	0.46	1.32	2.07	2.70	3.83	5.01	6.62	7.86
60	0.46	1.32	2.07	2.70	3.84	5.02	6.62	7.87
70	0.46	1.32	2.07	2.70	3.84	5.02	6.63	7.87
80	0.46	1.32	2.07	2.70	3.84	5.02	6.63	7.87
90	0.46	1.32	2.07	2.70	3.84	5.02	6.63	7.87
100	0.46	1.32	2.07	2.70	3.84	5.02	6.63	7.88
∞	0.455	1.323	2.072	2.706	3.841	5.024	6.635	7.879

Table 2.6: Empirical percentage points of $\hat{t}_3$. Gumbel distribution.

	Significance level							
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	0.47	1.36	2.13	2.77	3.93	5.05	6.59	7.78
20	0.46	1.33	2.08	2.72	3.86	5.04	6.67	7.93
30	0.46	1.33	2.08	2.71	3.85	5.04	6.67	7.93
40	0.46	1.32	2.07	2.71	3.85	5.04	6.66	7.93
50	0.46	1.32	2.07	2.71	3.84	5.03	6.66	7.92
60	0.46	1.32	2.07	2.71	3.84	5.03	6.66	7.92
70	0.46	1.32	2.07	2.71	3.84	5.03	6.66	7.912
80	0.46	1.32	2.07	2.71	3.84	5.03	6.65	7.91
90	0.46	1.32	2.07	2.71	3.84	5.03	6.65	7.91
100	0.46	1.32	2.07	2.71	3.84	5.03	6.65	7.90
∞	0.455	1.323	2.072	2.706	3.841	5.024	6.635	7.879

Table 2.7: Empirical percentage points of $\hat{t}_{23}$. Gumbel distribution.

	Significance level							
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
10	1.18	2.57	3.69	4.63	6.28	7.93	10.39	12.48
20	1.21	2.60	3.74	4.68	6.35	8.08	10.68	12.88
30	1.22	2.62	3.76	4.71	6.38	8.13	10.75	12.98
40	1.23	2.63	3.77	4.72	6.40	8.15	10.78	13.01
50	1.23	2.64	3.78	4.73	6.41	8.17	10.80	13.03
60	1.23	2.64	3.78	4.74	6.42	8.18	10.80	13.05
70	1.24	2.64	3.78	4.74	6.42	8.18	10.81	13.05
80	1.24	2.64	3.79	4.75	6.43	8.19	10.82	13.06
90	1.24	2.65	3.79	4.75	6.43	8.19	10.82	13.06
100	1.24	2.65	3.79	4.75	6.43	8.19	10.83	13.07
∞	1.246	2.662	3.809	4.773	6.459	8.220	10.843	13.096

Chapter 3

Maximum-correlation estimates and tests of fit

3.1 The three-parameter Weibull

A random variable Y is said to have a Weibull distribution if its cumulative distribution function is given by

$$F_Y(y) = 1 - \exp\left\{-\left(\frac{y-\alpha}{\beta}\right)^{\theta}\right\}, \; y \geq \alpha \tag{3.1}$$

where α, $\beta > 0$ and $\theta > 0$ are parameters.

When α is known the distribution (3.1) is known as the *two-parameter Weibull distribution* and $Z = \ln(Y - \alpha)$ has the Gumbel distribution

$$F_Z(z) = 1 - e^{-e^{\left(\frac{z-\alpha_G}{\beta_G}\right)}}, -\infty < z < \infty \tag{3.2}$$

It is also well known that (3.2) is the limiting form of (3.1) as θ tends to infinity (see, for example, Johnson and Kotz [31]).

Hence, the null hypothesis that the random sample $Y_1, \ldots, Y_n$ comes from (3.1) may be made by testing the null hypothesis that the transformed observations $Z_1, \ldots, Z_n$ were drawn from the population (3.2). When the parameter α is not known, estimates of the parameters are difficult to obtain and many problems arise.

In maximum-likelihood estimation, for $\theta < 1$, the likelihood function can be made infinite when $\hat{\alpha} = Y_{(1)}$. For this reason and due to the fact that some regularity problems occur for $1 < \theta < 2$, the maximum-likelihood method is difficult to apply for Weibull populations with $\theta < 2$. (See Castillo [13] and Lockhart and Stephens [35], [36])

In this chapter an estimation method is presented, based on a regression technique, which appears to avoid many of the difficulties arising when MLE is used.

The method is illustrated using some examples taken from the existing literature and goodness-of-fit procedures are given.

3.1.1 Parameter estimation: Method A

Suppose that $Y_1, \ldots, Y_n$ is a random sample from the distribution (3.1). We can write,

$$E\left[Y_{(i)}\right] = \alpha + \beta m_i(\theta) \tag{3.3}$$

where $m_i(\theta)$ denotes the expected value of the i-th order statistic, in a sample of size n, from a standard ($\alpha = 0$, $\beta = 1$) Weibull distribution with parameter θ.

The *correlation coefficient* $R(\mathbf{X}, \mathbf{Y})$ between two vectors $\mathbf{X}$ and $\mathbf{Y}$, where $\mathbf{X}^\top = (X_1, \ldots, X_n)$ and $\mathbf{Y}^\top = (Y_1, \ldots, Y_n)$ is defined as

$$R(\mathbf{X}, \mathbf{Y}) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}} \tag{3.4}$$

In (3.3), different choices of θ will produce different degrees of linear relationship between $\mathbf{Y}$ and $\mathbf{m}^\top(\theta) = (m_1(\theta), \ldots, m_n(\theta))$.

A "good" estimate should then produce a high degree of linear association as measured by $R^2(\mathbf{Y}, \mathbf{m}(\theta))$. This consideration leads us to the following

Definition 3.1 *The* Maximum Correlation Estimator *(MCE) of θ is the value of θ , $\hat{\theta}$, which maximizes*

$$R^2(\theta) = R^2(\mathbf{Y}, \mathbf{m}(\theta)) \tag{3.5}$$

Estimates of α and β will then be obtained by fitting the model (3.3) with $m_i(\theta)$ replaced by $m_i(\hat{\theta})$. This method of estimation will be referred to as method A.

3.1.2 Parameter estimation: Method B

A similar approach for estimating the parameters when α is known, is based on the fact that the substitution of $Z = \ln(Y - \alpha)$ in (3.1) gives the Gumbel distribution (3.2). Thus, estimates of the parameters can be obtained by fitting the model:

$$E\left[\ln(Y_{(i)} - \alpha)\right] = \alpha_G + \beta_G m_i \tag{3.6}$$

where $\mathbf{m}^\top = (m_1, \ldots, m_n)$ are the expected values of the order statistics from a standard Gumbel distribution.

The m_i now do not depend on an unknown parameter, and estimates of the parameters in the Gumbel fit may be obtained, for example, by ordinary least squares (OLS). Then we can make use of the relations:

$$\alpha_G = \ln\beta \tag{3.7}$$
$$\beta_G = \frac{1}{\theta} \tag{3.8}$$

to compute estimates of the parameters of the Weibull distribution (3.1).

The same procedure can be used when α is not known, provided that a good estimate of this parameter is used in transforming the original observations. Let $\mathbf{Z}^\top = (Z_1, \ldots, Z_n)$ where $Z_i = \ln(Y_i - \alpha)$ and let us define

$$R^2(\alpha) \equiv R^2(\mathbf{Z}, \mathbf{m}) \tag{3.9}$$

Definition 3.2 *The* maximum correlation estimate *of α is defined as the value of α, $\hat{\alpha}$, which maximizes the correlation coefficient (3.9).*

This method will be referred to as method B. In this case, estimates of α_G and β_G are now obtained by fitting the model (3.6) with α replaced by $\hat{\alpha}$, and then estimates of β and θ follow from (3.7) and (3.8). These will be called maximum correlation estimates (MCE).

3.1.3 Remarks

Remarks on the asymptotic distribution of the maximum-correlation statistic: Method B

Consider the model

$$E(Y_{(i)}) = \alpha + \beta m_i \tag{3.10}$$

When the parameters α and β are unknown, the correlation statistic $T_n = n\left[1 - R^2(\mathbf{Y}, \mathbf{m})\right]$ is equivalent to

$$W_n(\mathbf{Y}, \mathbf{m}) = \frac{n \sum_{i=1}^{n} \left(Y_{(i)} - \hat{\alpha} - \hat{\beta} m_i\right)^2}{\sum_{i=1}^{n} \left(Y_{(i)} - \bar{Y}\right)^2} \tag{3.11}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the ordinary least squares estimators of the parameters α and β in the model (3.10).

When some of the parameters are specified by the null hypothesis, the correlation statistic takes the more general form

$$W_n^*(\mathbf{Y}, \mathbf{m}) = \frac{\sum_{i=1}^{n} \left(Y_{(i)} - \alpha^* - \beta^* m_i\right)^2}{\tilde{\beta^*}^2}$$

where α^*, β^* and $\tilde{\beta^*}$ denote suitable consistent estimators of the parameters, or their values specified by the null hypothesis.

Asymptotic normality of the correlation test statistic from a Gumbel parent: If the sample comes from a Gumbel distribution, it has been shown in McLaren [40] and McLaren and Lockhart [41] that

$$T_n(\alpha, \beta) = \frac{W_n(\mathbf{Y}, \mathbf{m})/n - \log n}{2\sqrt{\log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \tag{3.12}$$

McLaren [40] also showed that the above result holds regardless of which parameters in (3.10) are estimated.

On the asymptotic normality of the maximum-correlation test statistic for the Weibull distribution: In order to simplify the notation, let us define:

$$W_n(\alpha) = n\left[1 - R^2(\alpha)\right]$$

and

$$T_n(\alpha) = \frac{n\left[1 - R^2(\alpha)\right] - \log n}{2\sqrt{\log n}}$$

where $R^2(\alpha)$ is defined as in (3.9).

In this new notation, test statistic based on method B is then $W_n(\hat{\alpha})$.

By writing

$$T_n(\hat{\alpha}) = T_n(\alpha) + (4\log n)^{-\frac{1}{2}}\, n\left[R^2(\alpha) - R^2(\hat{\alpha})\right]$$

we can show that

$$T_n(\hat{\alpha}) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

by showing that the second term of the right hand side of the above equation is $o_p(1)$, so that the result will follow by application of Slutsky's theorem.

From a practical point of view, the proof of the asymptotic normality of the correlation test statistic becomes less important due to the fact that the rate of convergence to normality of this type of statistic is extremely slow; this makes the approximation inappropriate for practical use. In the exponential case, Lockhart [34] shows that the normal approximation with $n = e^{16} \approx 9 \times 10^6$, gives $P(R^2 > 1) \approx 0.023$.

In this work, the percentage points of the test statistic were obtained by simulation based on 10,000 samples, and are given in table 3.8.

On the asymptotic means and variances

In order to investigate the properties of the estimators obtained by maximum correlation method B, 1000 samples from a Weibull distribution with $\alpha = 0$, $\beta = 1$ and different values of θ were simulated, the location parameter estimated by method B, and then estimates of the parameters in the Gumbel model (3.6) calculated by ordinary least squares. The estimates of the parameters of the Weibull distribution were then found by direct use of the relations (3.7) and (3.8). It was found that, for values of $\theta < 2$, $\hat{\alpha}$ was very close to its true value and therefore the variance of the estimators was well approximated by the variance of these estimators when the the location parameter α is known.

The ordinary least squares estimators of α_G and β_G in (3.6) are:

$$\hat{\beta}_G = \frac{\sum_{i=1}^{n} Z_i(m_i + \gamma)}{\sum_{i=1}^{n}(m_i + \gamma)^2}$$
$$\hat{\alpha}_G = \bar{Z} + \gamma\hat{\beta}_G$$

where $Z_i = \ln(Y_i - \alpha)$ and $\gamma \approx 0.5772$ is Euler's constant.

Using the results from chapter 2, we find the following asymptotic expressions for the variances of the OLS estimators in the Gumbel model (3.6).

$$Var\left(\hat{\beta}_G\right) \approx \frac{1.1}{n\theta^2}$$
$$Cov\left(\bar{Z}, \hat{\beta}_G\right) \approx -\frac{6\zeta(3)}{n\pi^2\theta^2} = -\frac{0.7308}{n\theta^2}$$
$$Var(\hat{\alpha}_G) \approx \frac{1.1678}{n\theta^2}$$

From (3.7) and (3.8) we have:

$$\hat{\beta} = e^{\hat{\alpha}_G} \text{ and } \hat{\theta} = \hat{\beta}_G^{-1}$$

Expansion in Taylor series of the above expressions gives:

$$E\left(\hat{\beta}\right) \approx \beta\left[1 + \frac{0.5839}{n\theta^2}\right] \text{ , } Var\left(\hat{\beta}\right) \approx \frac{0.5839\beta^2}{n\theta^2}$$

and

$$E\left(\hat{\theta}\right) \approx \theta\left(1 + \frac{1.1}{n}\right) \text{ , } Var\left(\hat{\theta}\right) \approx \frac{1.1\theta^2}{n}$$

3.1.4 Examples

In this section, the methods described above are illustrated by analyzing some particular data sets, given in the appendix. These have been discussed by Lockhart and Stephens [36] in connection with Maximum Likelihood estimation (MLE) of the parameters. The MLE results are given also when available. Plots of the trial values of the parameter versus the correlation coefficient, for methods A and B, are shown in figures (3.1) to (3.16).

a) Data set 1, from Cox and Oakes [15, table 1.3] consists of 10 values of the number of cycles to failure when springs are subjected to various stress levels. For these data, the stress level is $950 N/mm^2$ and the values are in units of 1000 cycles.

Table 3.1: Results for Data set 1

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	61.39	84.99	99.02
$\hat{\beta}$	120.08	95.39	78.23
$\hat{\theta}$	3.056	2.638	2.38
A^2	0.24	0.21	0.26

For this data set both methods give similar estimates producing nearly the same fit as measured by the Anderson-Darling A^2 statistic. The value of A^2 is here used as a measure of goodness of fit relative to the parameters used and a significance level is not associated with test statistic as its distribution will depend upon the method of estimation used.

b) Data set 2 consists of 15 times to failure of air conditioning equipment in aircraft. These values were taken from a table given in Proschan [46, table 1].

In this case, method B appears to be clearly a better option as it gives a smaller value of A^2 . In addition, the estimates of the parameters are consistent with the MLE results.

c) Data set 3 was artificially constructed by Lockhart and Stephens [36] to illustrate some problems in MLE estimation. For this example, no MLE exists and they recommend fitting a Gumbel distribution. This data set has 20 observations.

Table 3.2: Results for Data set 2

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	2.11	11.15	9.30
$\hat{\beta}$	105.48	87.67	93.50
$\hat{\theta}$	0.629	0.759	0.763
A^2	0.92	0.54	0.54

Table 3.3: Results for Data set 3

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	-9.10	-0.35	N/A
$\hat{\beta}$	10.44	1.65	N/A
$\hat{\theta}$	25.14	3.52	N/A
A^2	0.50	0.71	N/A

The estimates of the parameters obtained by method A, suggest that a Gumbel fit would be appropriate as it yields a large value of $\hat{\theta}$. Method B however produces estimates of the parameters which appear to be more consistent with a Weibull fit. Even though the value of A^2 is smaller when the parameters obtained by method A are used, method B might be preferred if a Weibull fit is the primary choice.

d) Data set 4 is taken from Castillo [13]. It consists of 60 distances in miles to a nuclear power plant of the most recent 8 earthquakes of intensity larger than a given value. The data are needed to evaluate the risk associated with earthquakes ocurring close to a central site. It is known that a fault is the main cause of earthquakes in the area and the closest point of the fault is 50 miles from the plant.

This data set must be more carefully dealt with since there are some practical considerations which have to be taken into account. Method B appears to give the best results since the location parameter is associated with the nearest distance to the fault. Method A produces a negative estimate of the location parameter, which might have some geological explanation.

Table 3.4: Results for Data set 4

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	-18.89	51.24	38.02
$\hat{\beta}$	182.49	104.73	119.97
$\hat{\theta}$	3.18	1.42	2.047
A^2	0.83	1.33	1.33

e) Data set 5, from Castillo [13], consists of fatigue strengths to failure of 35 specimens of wire. The aim of the study was to find a design for fatigue stress.

Table 3.5: Results for Data set 5

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	40855.27	36465.41	38852.25
$\hat{\beta}$	62895.14	68714.65	63966.98
$\hat{\theta}$	1.13	1.32	1.24
A^2	N/A	0.15	0.2023

This example shows that method A requires the use of a restricted maximization procedure as it can give unacceptable solutions in the sense that the estimate of the location parameter is sometimes greater than the minimum sample value: In this case, the minimum value is equal to 39611, and $\hat{\alpha}$ is greater than this number. Method B gives a good solution.

f) Data set 6 consists of 41 failure times of an electronic module. Analysis of these data appeared in Ansell and Phillips [4].

Again, the solution given by method A is not acceptable as the minimum failure time is

Table 3.6: Results for Data set 6

Estimate	Method A	Method B	MLE
$\hat{\alpha}$	6.39	-0.023	1.19
$\hat{\beta}$	34.48	45.86	43.51
$\hat{\theta}$	0.902	1.27	1.15
A^2	N/A	0.23	0.22

equal to 1.4 and $\hat{\alpha}$ is greater than this number.

g) Data set 7 consists of strengths of glass fibers of length 15 cm. from the National Physical Laboratory in England.

h) Data set 8 consists of strengths of glass fibers of length 15 cm. from the National Physical Laboratory in England.

Data sets 7 and 8 have been used by Smith and Naylor [60] to illustrate problems in MLE estimation. In their paper, they suggest the use of Bayesian methods for handling likelihoods of unusual shape. The problem of specifying an appropriate prior distribution could be assessed by studying the sensitivity of the analysis to the choice of the prior.

For these data sets, no maximum can be found as the value of $R^2(\alpha)$ can be made as close as we wish to the value of R^2 obtained for the model (3.14) , by making α tend to $-\infty$. For these two situations, the maximum correlation method indicates that the best fit will be obtained for the Gumbel distribution (3.2). Problems of this type are discussed in the next section.

From practical considerations, method B appears to be better than method A for two major reasons: (a) it implies the use of an *unrestricted* maximization procedure (the constraint that the location parameter is less than the minimum sample value is recognized when the interval for α, for the search, is given) and (b): it does not require the calculation of expected values of order statistics for every value of the shape parameter, as is the case for method A. The expected values of order statistics of the Gumbel distribution for a sample of size n are more easily calculated and can be reused for different values of the trial parameter. Also, the analysis of the data sets suggests that method B produces better estimates which are consistent with results obtained by MLE .

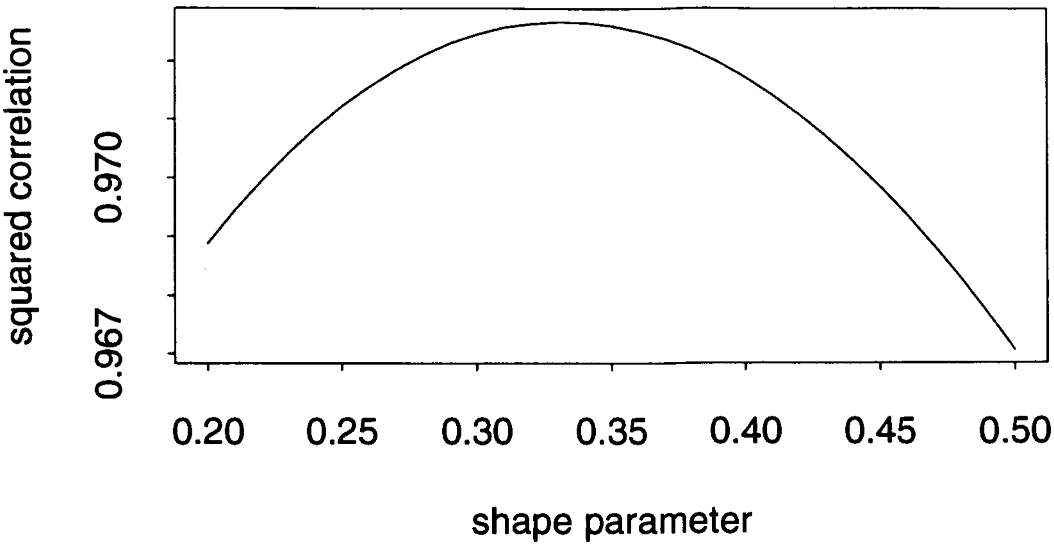

Figure 3.1: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 1, Method A

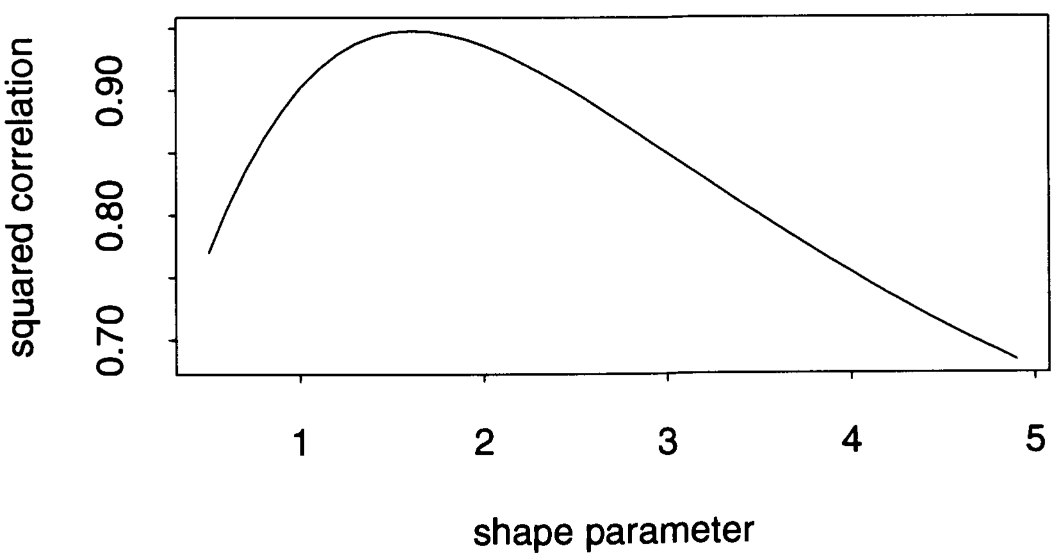

Figure 3.2: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 2, Method A

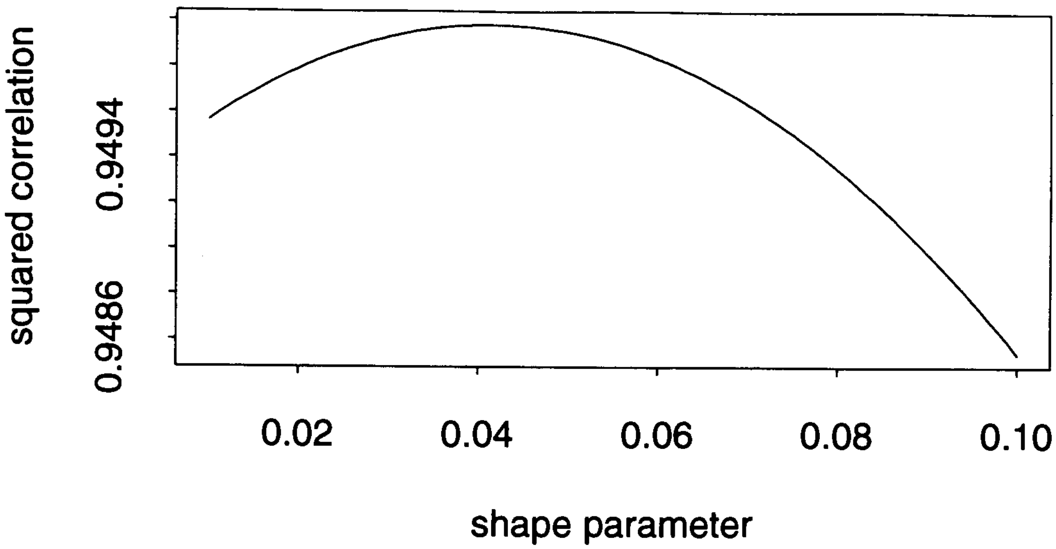

Figure 3.3: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 3, Method A

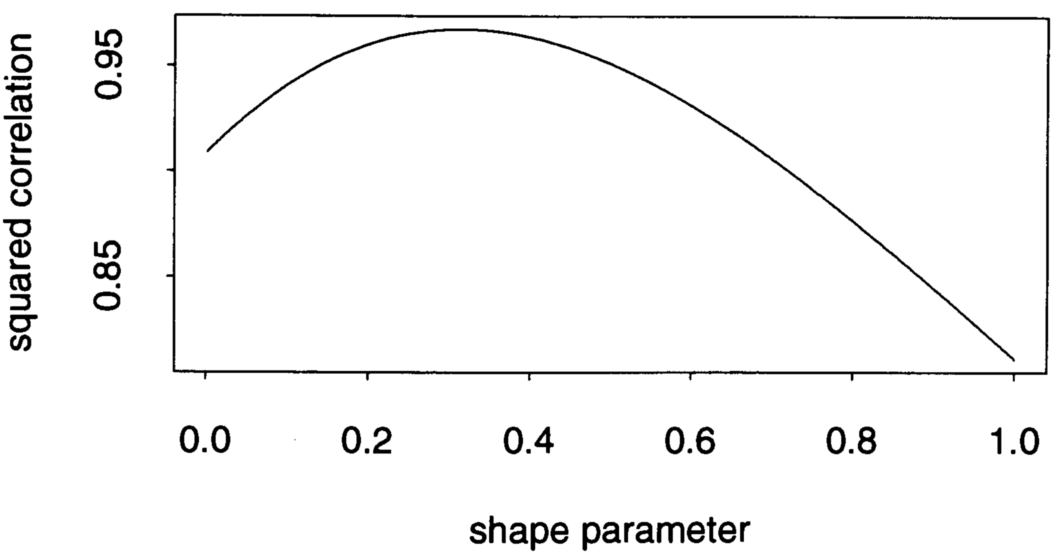

Figure 3.4: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 4, Method A

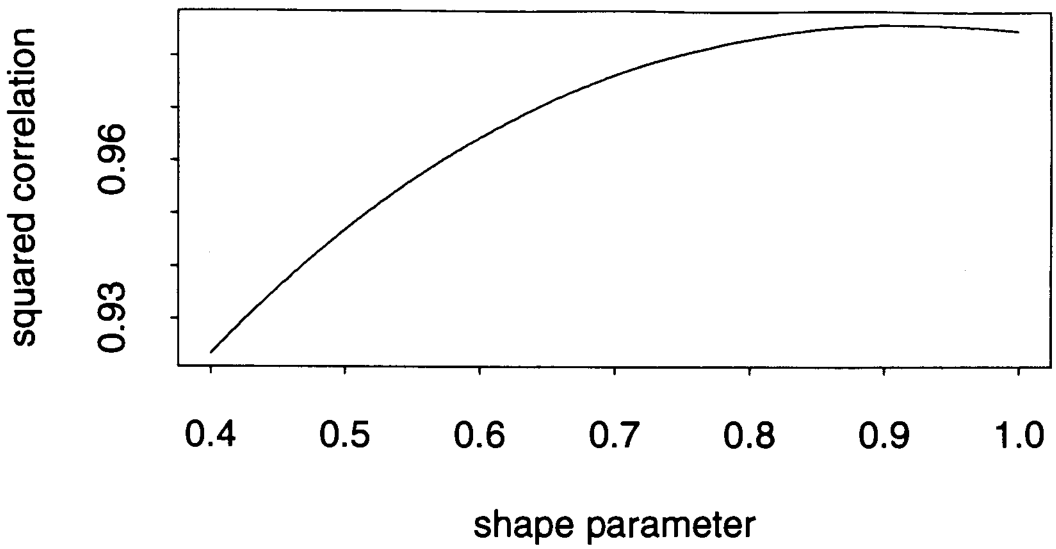

Figure 3.5: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 5, Method A

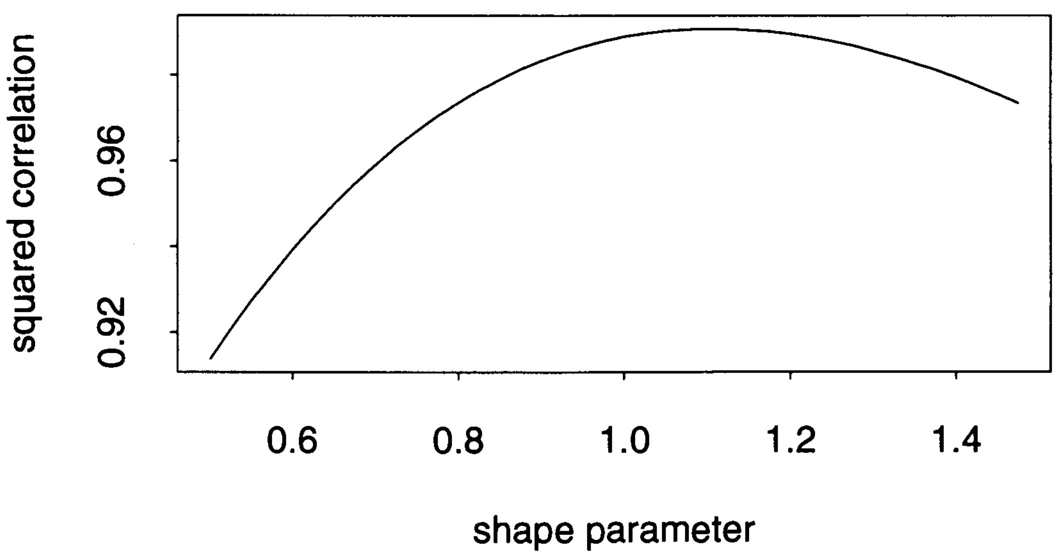

Figure 3.6: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 6, Method A

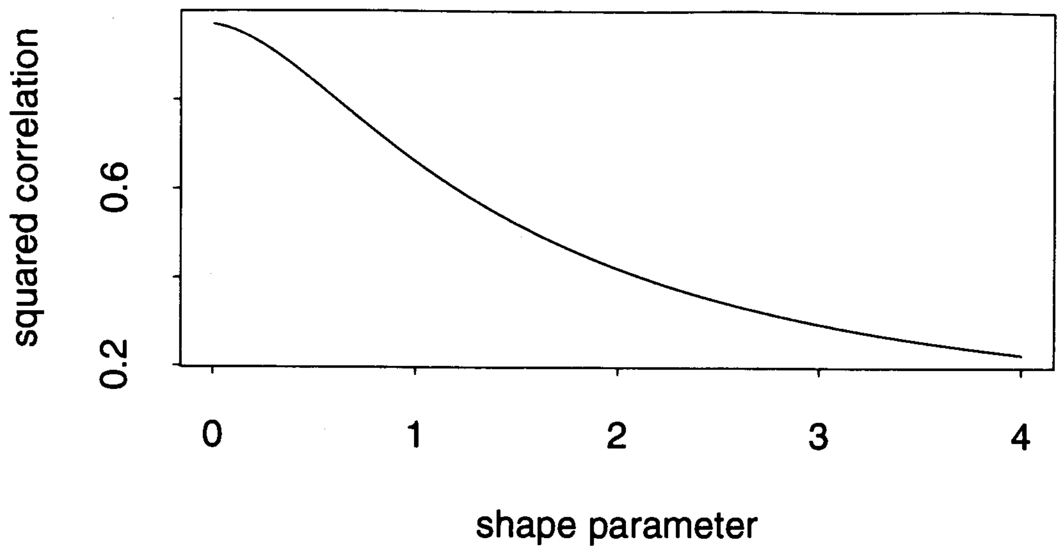

Figure 3.7: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 7, Method A

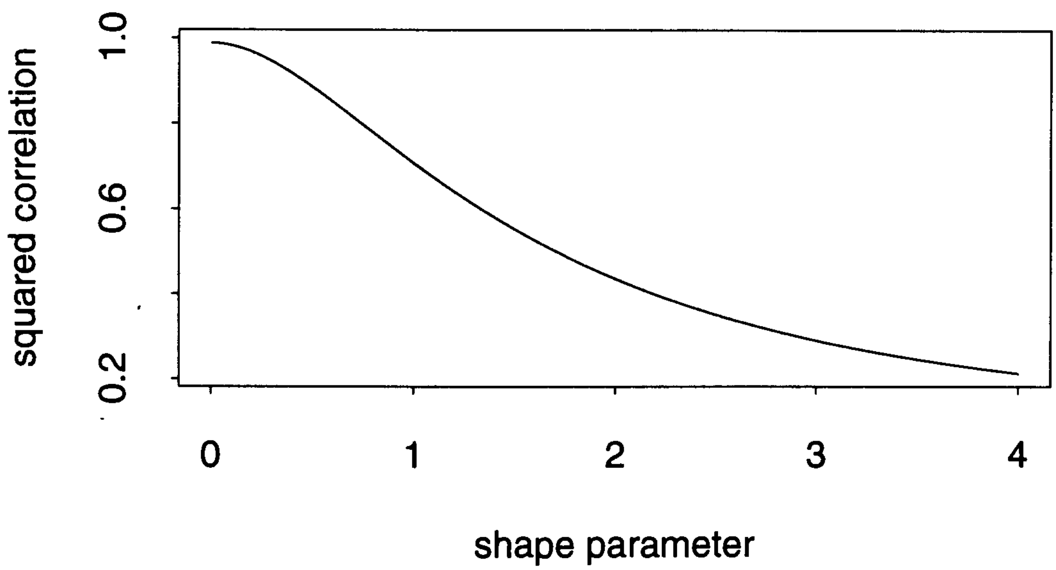

Figure 3.8: Plot of $R^2(\theta)$ vs θ^{-1}. Data set 8, Method A

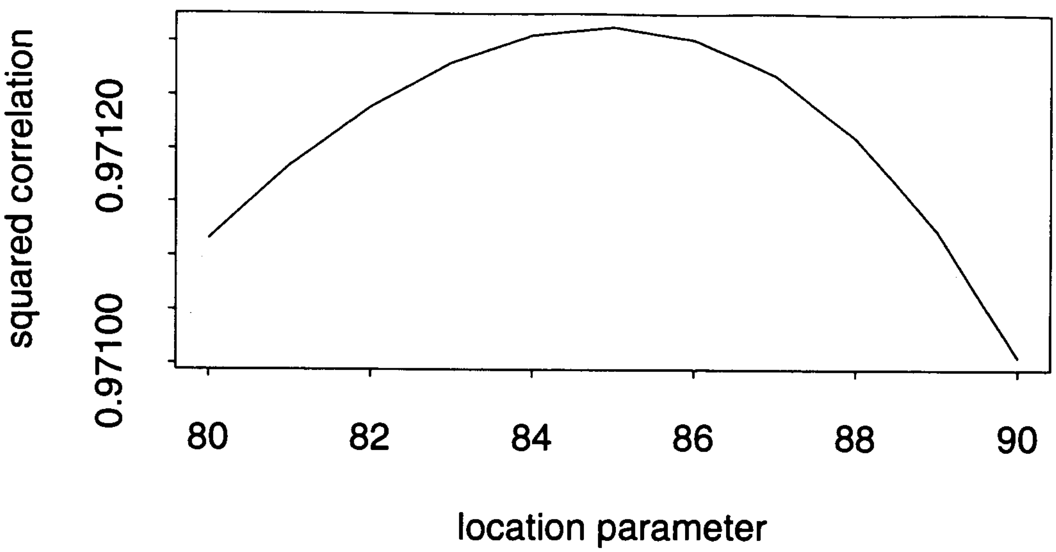

Figure 3.9: Plot of $R^2(\alpha)$ vs α . Data set 1, Method B

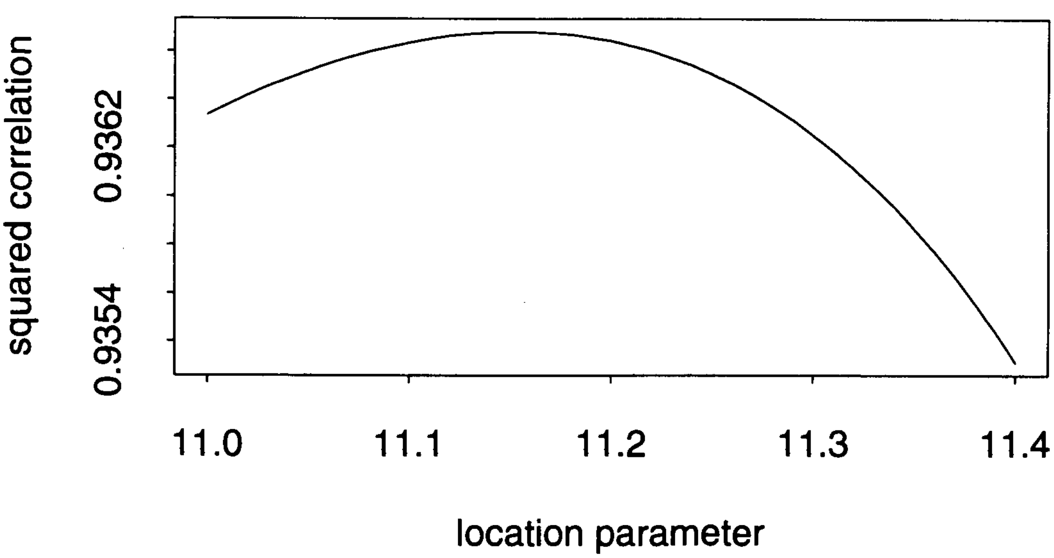

Figure 3.10: Plot of $R^2(\alpha)$ vs α . Data set 2, Method B

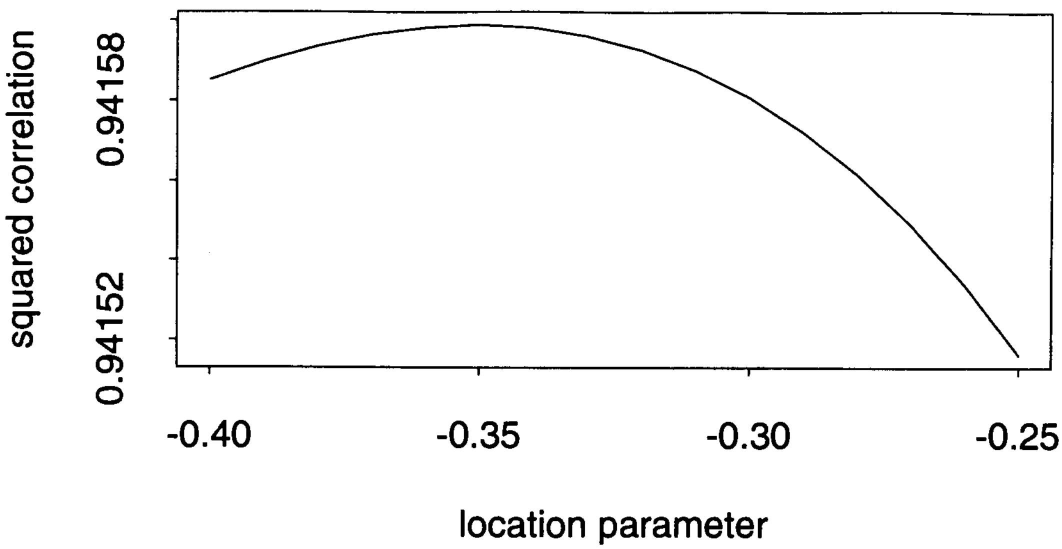

Figure 3.11: Plot of $R^2(\alpha)$ vs α . Data set 3, Method B

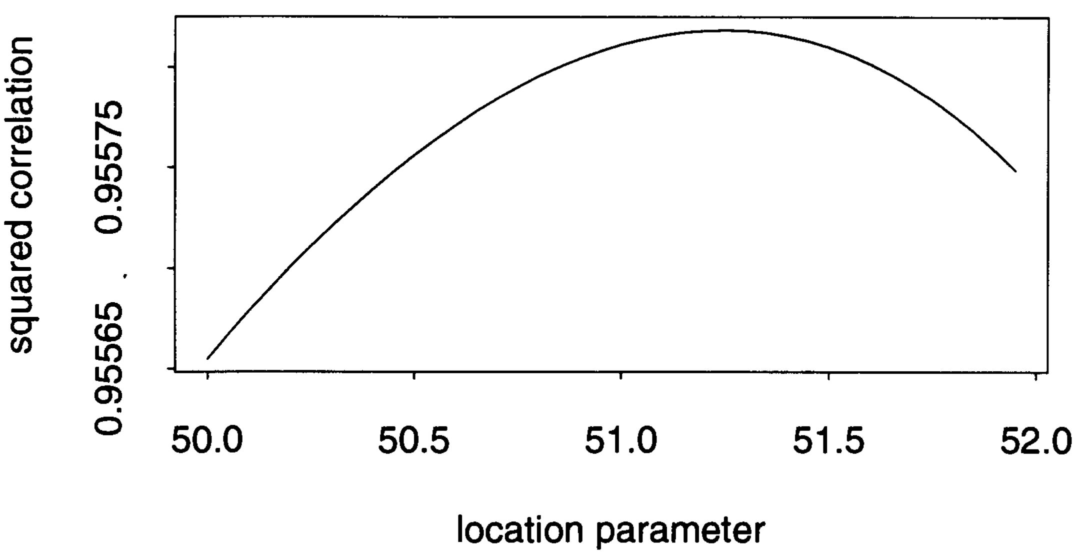

Figure 3.12: Plot of $R^2(\alpha)$ vs α . Data set 4, Method B

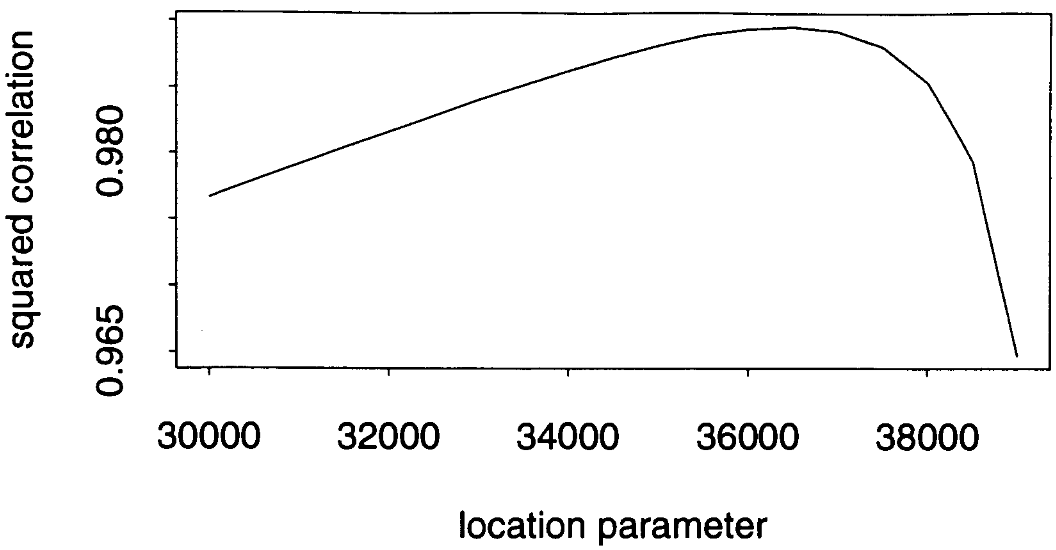

Figure 3.13: Plot of $R^2(\alpha)$ vs α . Data set 5, Method B

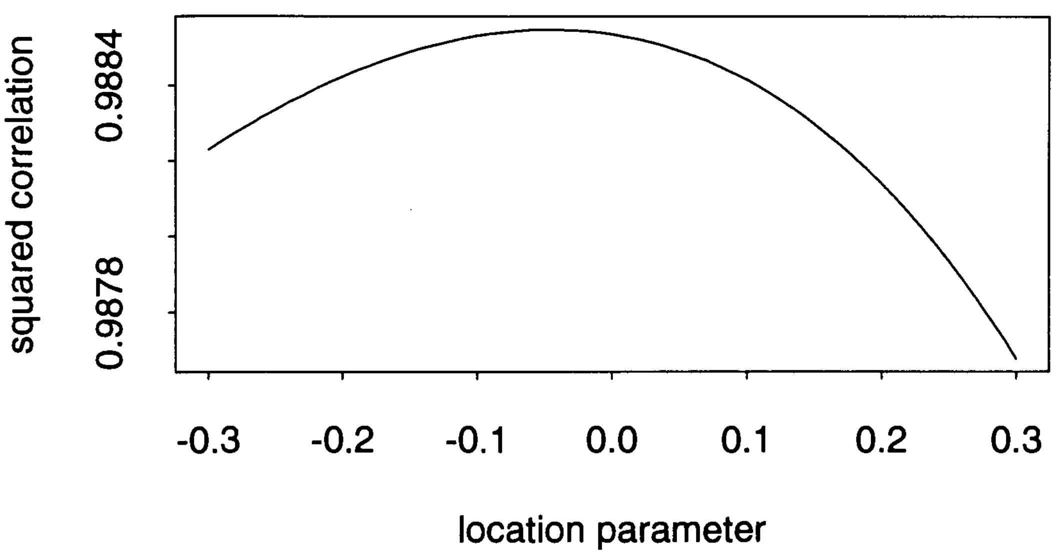

Figure 3.14: Plot of $R^2(\alpha)$ vs α . Data set 6, Method B

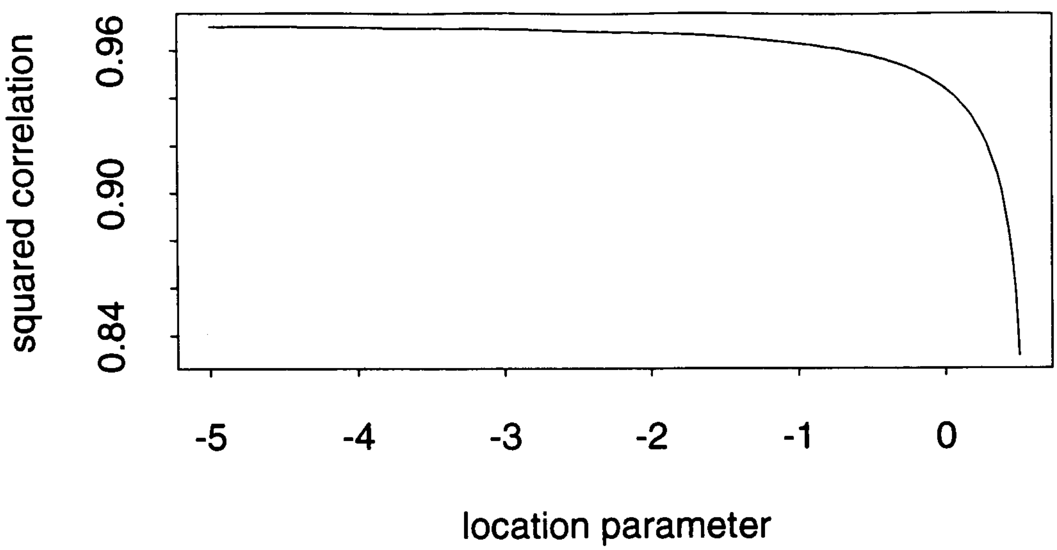

Figure 3.15: Plot of $R^2(\alpha)$ vs α . Data set 7, Method B

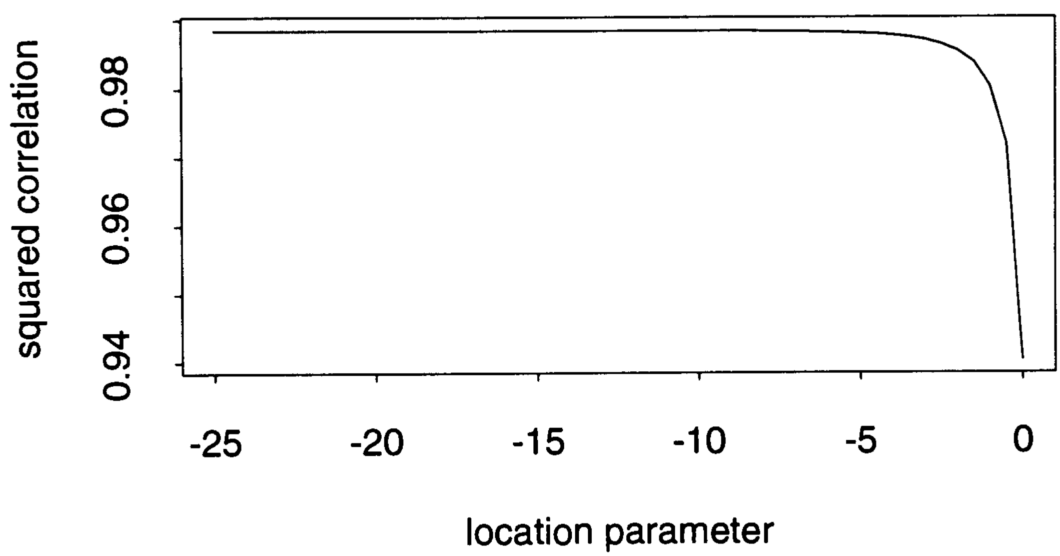

Figure 3.16: Plot of $R^2(\alpha)$ vs α . Data set 8, Method B

3.1.5 Practical problems for estimation

When $\theta < 1$, the maximum correlation method produces estimates without any apparent difficulty. However, if $\theta > 1$, simulations show that there are samples for which the methods (either A or B) find no maximum (see table 3.7) . This can be regarded as a shortcoming of the method; the same problem appears when the maximum likelihood method is used.

For large values of θ, the Weibull distribution becomes close to the Gumbel distribution (in a well defined mathematical sense). Suppose then that we approximate the expected value of the i-th order statistic from the Weibull distribution (3.1), $m_i(\theta)$, by the quantity

$$H_i(\theta) = \left(-\ln\left(1 - \frac{i}{n+1}\right)\right)^{\frac{1}{\theta}} \tag{3.13}$$

and consider its Taylor's expansion about $\left(\frac{1}{\theta}\right) = 0$. Then, approximately

$$E(Y_i) \approx (\alpha + \beta) + \frac{\beta}{\theta} H_i \tag{3.14}$$

where $H_i = \ln\left(-\ln\left(1 - \frac{i}{n+1}\right)\right)$ is an approximation to the expected value of the i-th order statistic in a sample from the a standard Gumbel distribution.

Hence, for large θ, a Gumbel distribution with parameters $(\alpha + \beta)$ and $\frac{\beta}{\theta}$ will provide a good fit to the data.

The frequency with which this situation occurs is summarized in table 3.7 , based on a Monte Carlo study using 1000 samples. It can be conjectured that, for fixed n, the percentage of "Gumbel-type" samples will approach fifty percent as $1/\theta$ tends to zero, the reason being that the two distributions will be indistinguishable from one another. On the other hand, for fixed θ, the frequency with which these samples appear will approach zero (slowly for large θ) as n tends to infinity.

3.1.6 Existence of a maximum

Here we describe a procedure that can be used to detect the ocurrence of figures 3.15 and 3.16, produced by data sets 7 and 8, in which $R^2(\alpha)$ increases by letting $\alpha \to -\infty$.

First, we compute the quantity

$$D = \left[\sum_{i=1}^{n}(m_i - \bar{m})Y_{(i)}\right]^2 \left[\sum_{i=1}^{n}\left(Y_{(i)} - \bar{Y}\right)Y_{(i)}^2\right] - \tag{3.15}$$

$$\left[\sum_{i=1}^{n}(m_i - \bar{m})Y_{(i)}\right]\left[\sum_{i=1}^{n}(m_i - \bar{m})Y_{(i)}^2\right]\left[\sum_{i=1}^{n}\left(Y_{(i)} - \bar{Y}\right)^2\right] \tag{3.16}$$

For a given value of α_t, D is proportional to the derivative of the function $R^2(\alpha)$ at that point. Thus, a negative value of D will indicate the ocurrence of a plot similar to 3.15 and 3.16. In this case, a Gumbel fit for the original observations is the solution given by the maximum correlation method B.

The derivation of the quantity D, is described in the next section.

3.1.7 Derivation of D

The approach used to obtain D consists of examining the behaviour of the function

$$G(\alpha) = \frac{d}{d\alpha}R^2(\alpha)$$

As $\alpha \to -\infty$.

For a large negative value of α, we use the approximation

$$\ln\left(Y_{(i)} - \alpha\right) \approx \ln(-\alpha) - \left(\frac{Y_{(i)}}{\alpha} + \frac{Y_{(i)}^2}{2\alpha^2} + \frac{Y_{(i)}^3}{3\alpha^3}\right)$$

Thus, if we let $R_k = \sum_{i=1}^{n}(m_i - \bar{m})Y_{(i)}^k$, we obtain the following approximations:

$$\sum_{i=1}^{n}(m_i - \bar{m})\ln\left(Y_{(i)} - \alpha\right) \approx -\left(\frac{R_1}{\alpha} + \frac{R_2}{2\alpha^2} + \frac{R_3}{3\alpha^3}\right).$$

$$\left[\sum_{i=1}^{n}(m_i - \bar{m})\ln\left(Y_{(i)} - \alpha\right)\right]^2 \approx \frac{R_1^2}{\alpha^2} + \frac{R_1R_2}{\alpha^3}$$

Since

$$\sum_{i=1}^{n}\ln^2\left(Y_{(i)} - \alpha\right) - \frac{\left(\sum_{i=1}^{n}\ln(Y_{(i)} - \alpha)\right)^2}{n} = \sum_{i=1}^{n}\ln^2\left(1 - \frac{Y_{(i)}}{\alpha}\right) - \frac{\left[\sum_{i=1}^{n}\ln\left(1 - \frac{Y_{(i)}}{\alpha}\right)\right]^2}{n}$$

We will also make use of

$$\begin{aligned} \sum_{i=1}^{n} \ln^2\left(1-\frac{Y_{(i)}}{\alpha}\right) &= \frac{S_2}{\alpha}+\frac{S_3}{\alpha^3} \\ \left[\sum_{i=1}^{n} \ln\left(1-\frac{Y_{(i)}}{\alpha}\right)\right]^2 &= \frac{S_1^2}{\alpha^2}+\frac{S_1 S_2}{\alpha^3} \end{aligned}$$

where $S_k = \sum_{i=1}^{n} Y_{(i)}^k$.

Using the above approximations, as $\alpha \to -\infty$, we can write

$$\begin{aligned} R^2(\alpha) &= \frac{R_1^2}{\sum_{i=1}^{n}(m_i-\bar{m})^2\left[\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})^2+\alpha^{-1}\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})Y_{(i)}^2\right]} \\ &+ \frac{R_1R_2}{\sum_{i=1}^{n}(m_i-\bar{m})^2\left[\alpha\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})^2+\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})Y_{(i)}^2\right]} \end{aligned}$$

and consequently,

$$G(\alpha) = \frac{R_1^2\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})Y_{(i)}^2 - R_1R_2\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})^2}{\sum_{i=1}^{n}(m_i-\bar{m})^2\left[\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})^2+\alpha^{-1}\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})Y_{(i)}^2\right]^2}$$

Clearly, a negative value fo the derivative is obtained when the top term in the above expression is negative, that is, when

$$\begin{aligned} &\left[\sum_{i=1}^{n}(m_i-\bar{m})Y_{(i)}\right]^2\left[\sum_{i=1}^{n}\left(Y_{(i)}-\bar{Y}\right)Y_{(i)}^2\right] \\ -&\left[\sum_{i=1}^{n}(m_i-\bar{m})Y_{(i)}\right]\left[\sum_{i=1}^{n}(m_i-\bar{m})Y_{(i)}^2\right]\left[\sum_{i=1}^{n}\left(Y_{(i)}-\bar{Y}\right)^2\right] < 0 \end{aligned}$$

as in equation (3.16).

From equation (3.17) , it also follows that

$$\lim_{\alpha\to-\infty} R^2(\alpha) = \frac{\left[\sum_{i=1}^{n}(m_i-\bar{m})Y_{(i)}\right]^2}{\sum_{i=1}^{n}(m_i-\bar{m})^2\sum_{i=1}^{n}(Y_{(i)}-\bar{Y})^2}$$

that is, the correlation coefficient between the vector $\mathbf{m}$ and the vector $\mathbf{Y}$, of the original observations.

On the other hand, as $\alpha \to Y_{(1)}$, we have

$$\lim_{\alpha \to Y_{(1)}} R^2(\alpha) = \frac{n(m_1 - \bar{m})^2}{(n-1)\sum_{i=1}^{n}(m_i - \bar{m})^2} \tag{3.17}$$

In fact, for $\alpha = Y_{(1)} - \epsilon$ ($\epsilon > 0$),

$$\sum_{i=1}^{n}(m_i - \bar{m})\ln\left(Y_{(i)} - \alpha\right) = (m_1 - \bar{m})\ln(\epsilon) + \sum_{j=1}^{n}(m_j - \bar{m})\ln\left[Y_{(j)} - Y_{(1)} + \epsilon\right]$$

$$\begin{aligned}\sum_{i=1}^{n}\ln^2\left(Y_{(i)} - \alpha\right) - \frac{\left(\sum_{i=1}^{n}\ln(Y_{(i)} - \alpha)\right)^2}{n} = \\ & \ln^2(\epsilon)\frac{n-1}{n} - \frac{2}{n}\ln(\epsilon)\sum_{j=2}^{n}\ln\left[Y_{(j)} - Y_{(1)} + \epsilon\right] \\ & + \sum_{j=2}^{n}\ln^2\left[Y_{(j)} - Y_{(1)} + \epsilon\right] \\ & - \frac{1}{n}\left\{\sum_{j=2}^{n}\ln\left[Y_{(j)} - Y_{(1)} + \epsilon\right]\right\}^2\end{aligned}$$

The result (3.17) is then obtained by substituting the above expressions into the definition of $R^2(\alpha)$ and taking the limit as ϵ tends to zero.

Table 3.7: Percentage of samples for which $(1/\hat{\theta}) = 0$. Method B

θ \ n	10	20	30	40	50	60	70	80	90	100
1	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	3.2	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	12.3	1.4	0.8	0.2	0.0	0.0	0.0	0.0	0.0	0.0
4	18.9	5.7	2.3	0.9	0.3	0.1	0.0	0.0	0.0	0.0
5	19.9	11.2	6.3	3.7	1.7	1.6	1.0	0.6	0.3	0.3
6	25.9	14.6	11.6	8.2	5.8	3.1	3.1	1.4	1.8	0.4
7	28.5	21.0	16.7	13.0	8.6	6.9	6.4	3.8	3.3	2.9
8	29.0	24.8	18.8	15.2	14.5	11.5	8.0	7.8	7.3	4.9
9	31.5	24.9	22.1	17.9	15.5	12.8	10.1	11.7	9.9	7.2
10	31.1	28.4	22.8	21.2	20.1	16.5	14.8	13.0	12.6	11.5

3.1.8 Test procedures

We now move to tests of fit. The test will be based on using method B to estimate the parameters.

a) First, find $\hat{\alpha}$ using method B, using H_i in (3.14) as an approximation to m_i.

b) Compute the test statistic:

$$W_n(\hat{\alpha}) = n\left[1 - R^2(\hat{\alpha})\right]$$

c) Refer to table 3.8 for the appropriate sample size. If the value of the test statistic is greater than the value given for level p, the null hypothesis is rejected at level p.

Example: Data set 1.

For data set 1, we find $R^2(\hat{\alpha}) = 0.9713$ producing a value for the test statistic of $T = 0.287$. The p-value is > 0.50, so the fit is good.

Example: Data set 2.

In this case, $R^2(\hat{\alpha}) = 0.9367$ and the value of the test statistic is $T = 0.95$. Linear interpolation from table 3.8 gives $0.05 <$p-value< 0.10 .

Example: Data set 3.

For the third data set we obtain $R^2(\hat{\alpha}) = 0.9416$, $T = 1.168$. For $n = 20$ we find $0.025 <$p-value< 0.05 which suggests the rejection of the Weibull model. This agrees with the analysis of this data set in Lockhart and Stephens [36].

Example: Data set 4.

$R^2(\hat{\alpha}) = 0.9558$ and $T = 2.65$. Since the p-value is < 0.01 the Weibull fit should be rejected. As an illustration, figures 3.17 and 3.18 show the probability plots obtained by maximum correlation for this data set, using methods A and B. From these plots, it can be seen that, even after selecting the best value of the parameters, based on maximizing the correlation coefficient, the probability plots still show departures from linearity in the lower and upper tails.

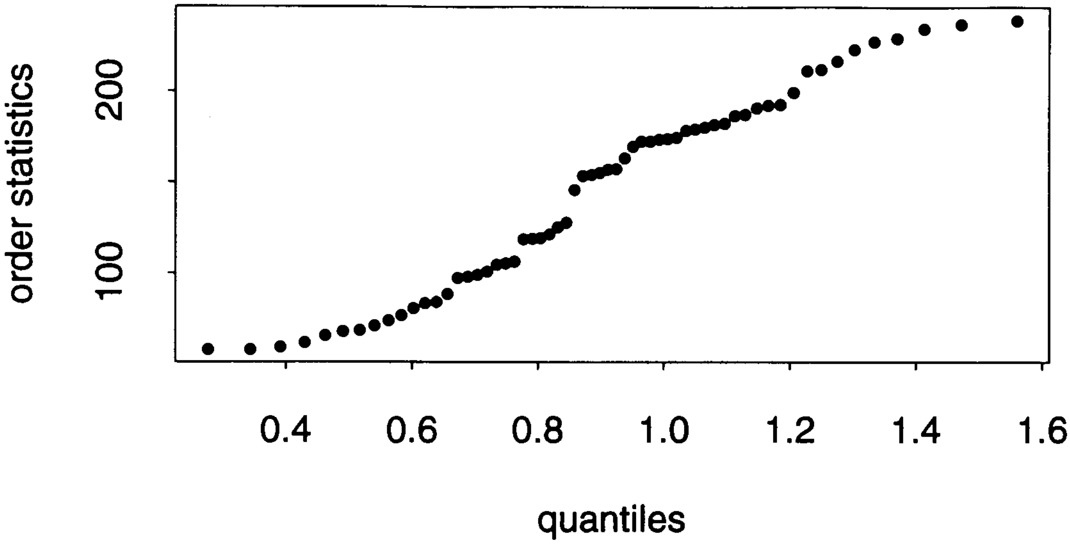

Figure 3.17: Weibull probability plot, method A, for data set 4. Weibull quantiles found using maximum correlation estimates

Example: Data sets 5 and 6.

These data sets both give a value of $R^2(\hat{\alpha}) = 0.9997$ indicating almost a perfect fit. The test statistic is not significant at level 0.50.

Data sets 7 and 8 produce values of R^2 for the model (3.14) respectively equal to 0.9995 and 0.9997, indicating a very good fit to the Gumbel distribution. A test of fit can then be carried out using the method given in Lockhart and Stephens [36]. Figure 3.19 shows the corresponding Gumbel probability plot for data set 8.

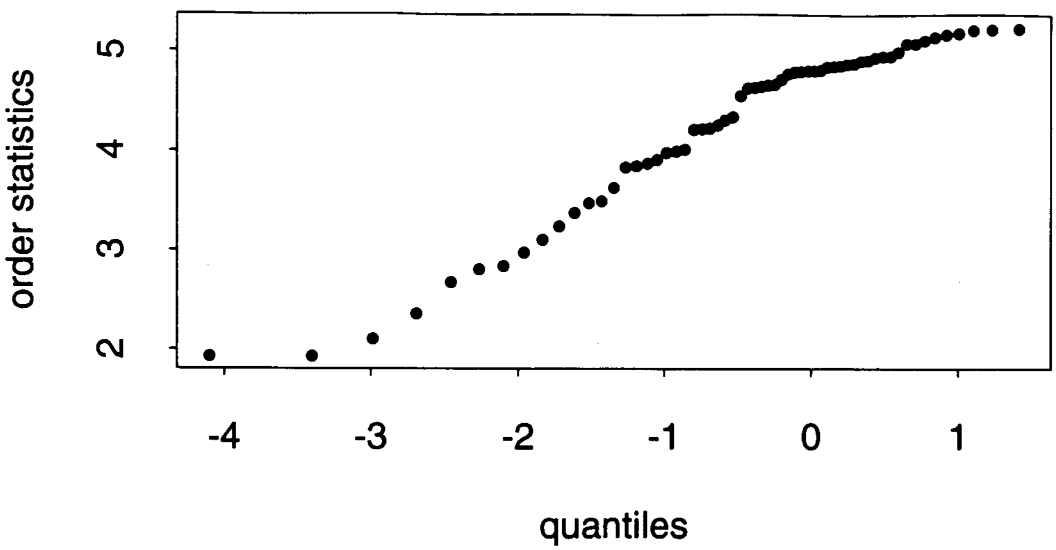

Figure 3.18: Gumbel probability plot, method B, for data set 4. Observations transformed using maximum correlation estimates

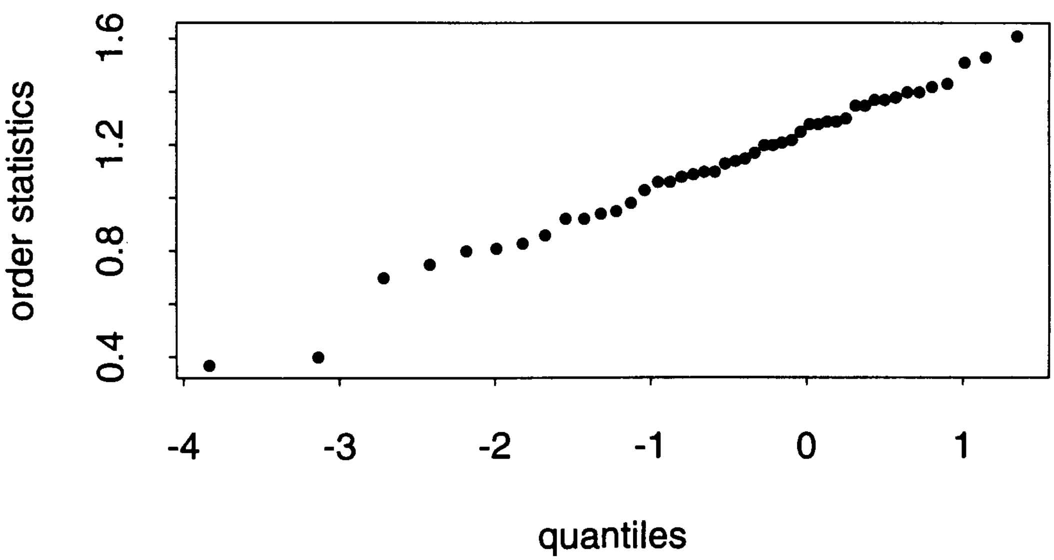

Figure 3.19: Gumbel probability plot for original observations. Data set 8

Table 3.8: Empirical percentage points of the correlation statistic

Significance level

n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
10	0.32	0.48	0.59	0.66	0.78	0.94	1.15
20	0.50	0.70	0.85	0.97	1.15	1.34	1.60
30	0.60	0.84	1.01	1.14	1.36	1.61	1.90
40	0.68	0.95	1.14	1.29	1.55	1.82	2.17
50	0.74	1.04	1.25	1.40	1.68	1.97	2.32
60	0.79	1.11	1.33	1.50	1.80	2.07	2.45
70	0.83	1.16	1.38	1.56	1.84	2.15	2.60
80	0.88	1.24	1.48	1.68	2.01	2.32	2.70
90	0.91	1.27	1.52	1.73	2.09	2.41	2.83
100	0.95	1.30	1.55	1.75	2.07	2.41	2.93

3.2 The three-parameter lognormal distribution

Let X be a standard normal random variable and let

$$Y = \theta + \exp(\alpha + \beta X)$$

Y is then said to have a lognormal distribution with parameters α, β and θ.

When θ is known, the distribution is called the *two-parameter log normal.*

The null hypothesis that a random sample $Y_1, \ldots, Y_n$ comes from a two-parameter lognormal distribution can be made by testing that the transformed observations $Z_i = \ln(Y_i - \theta)$ is a sample from a normal distribution. A review of several procedures for testing normality can be found in D'Agostino and Stephens [17, chapter 9].

If θ is unknown, the distribution is called the *three-parameter lognormal.* In this case, the estimation procedures present considerable difficulty. As pointed out in Johnson and Kotz [31], accuracy in the estimation of θ is not important in tests of fit. This is because large variations in the value of θ lead to small changes in the percentage points of the distribution when in its standard form (i.e., α and β are chosen so that the distribution has mean zero and unit variance).

The three-parameter log normal is then another distribution for which the maximum correlation method gives a relatively simple approach for constructing a test of fit.

If $Y_{(1)}, \ldots, Y_{(n)}$ are the order statistics and $Z_i = \ln(Y_{(i)} - \theta)$, then

$$E(Z_{(i)}) = \alpha + \beta m_i$$

where m_i denotes the expected value of the i-th order statistic in a sample of size n from a standard normal distribution.

Here, the maximum correlation estimator of θ is the value $\hat{\theta}$ which maximizes

$$R^2(\theta) = R^2(\mathbf{Z}, \mathbf{m}).$$

As in the case of the three-parameter Weibull distribution, it is possible to find samples for which the maximum value of $R^2(\theta)$ is obtained by making θ tend to $-\infty$. However, a Monte Carlo study based on 1000 simulations revealed that this situation appear to occur only for $n < 30$. In fact, the results of the study using $\alpha = 0$ and $\beta = 1$ produced 6% of such samples for $n = 10$ and 0.3% for $n = 20$. A plot of θ vs $R^2(\theta)$ in this situation will look like figure 3.15.

3.2.1 Test of fit for the lognormal distribution

A test of fit can be based on the quantity

$$T = n\left[1 - R^2(\hat{\theta})\right]$$

Empirical percentage points of the test statistic T were obtained empirically from 10,000 simulations. The results are shown in table 3.9

3.3 A test of multivariate normality

Roy's union-intersection principle can be applied to construct a test of multivariate normality based on any test statistic used for the univariate case. In this section, the maximum correlation test statistic is shown to be useful for testing multivariate normality. The technique is illustrated by considering a test for bivariate normality.

Table 3.9: Empirical percentage points of T. Log-normal distribution.

	Significance level						
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
10	0.36	0.52	0.62	0.70	0.85	1.00	1.20
20	0.51	0.72	0.87	0.98	1.18	1.41	1.74
30	0.60	0.85	1.01	1.14	1.37	1.63	2.02
40	0.66	0.92	1.11	1.25	1.52	1.80	2.20
50	0.71	0.98	1.17	1.32	1.59	1.86	2.30
60	0.74	1.03	1.24	1.40	1.70	2.03	2.50
70	0.76	1.05	1.25	1.42	1.72	2.02	2.49
80	0.79	1.09	1.29	1.47	1.78	2.10	2.59
90	0.81	1.11	1.32	1.48	1.78	2.10	2.58
100	0.84	1.14	1.36	1.54	1.84	2.13	2.56

3.3.1 Test statistic

Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be a p-variate random sample from a normal population with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Also, define $Z_i = \mathbf{c}^\top \mathbf{X_i}$, where c is a p-dimensional non-random vector.

Then, if $Z_{(1)}, \ldots, Z_{(n)}$ are the ordered transformed observations we have, for all non null vectors $\mathbf{c}$,

$$E(Z_{(i)}) = \mathbf{c}^\top \boldsymbol{\mu} + \left\{\mathbf{c}^\top \Sigma \mathbf{c}\right\}^{\frac{1}{2}} m_i$$

where $\mathbf{m}^\top = (m_1, \ldots, m_n)$ is the vector of expected values of the order statistics from a standard normal distribution. Note that the mean value of the m_i is zero.
Then, define $R(\mathbf{c})$ to be the correlation coefficient between $\mathbf{Z}$, the vector of order statistics corresponding to the transformed observations, and $\mathbf{m}$. Thus

$$R^2(\mathbf{c}) = \frac{\left(\sum_{i=1}^n Z_{(i)} m_i\right)^2}{\sum_{i=1}^n m_i^2 \sum_{i=1}^n (Z_i - \bar{Z})^2} \tag{3.18}$$

The correlation $R(\mathbf{c})$ is then independent of the parameters of the distribution, and Roy's union intersection principle can be applied to this situation by finding the vector $\mathbf{c}^*$ so that $R^2(\mathbf{c}^*)$ is a minimum.

The null hypothesis of multivariate normality will then be rejected for small values of the correlation or, equivalently, large values of the test statistic

$$T = n\left[1 - R^2(\mathbf{c}^*)\right] \tag{3.19}$$

3.3.2 Test of bivariate normality

We illustrate the use of $R(\mathbf{c})$ for dimension two. For this case, the minimization process reduces to a simple unidimensional search of the normalized vector $\mathbf{c}^*$. Using polar coordinates we can write the normalized vector c as

$$c^{\mathsf{T}} = (\sin\theta, \cos\theta)$$

Hence the problem reduces to find the value θ^* which minimizes (3.18).

Empirical percentage points of the distribution of the test statistic

$$T = n[1 - R^2(\theta^*)]$$

were obtained empirically from ten thousand simulations and are shown in table 3.10.

In the following four examples, simulated data sets are analyzed. In each case, a search in the interval $[0°, 180°)$ was conducted using increments of one degree. This magnitude for the increment produces a precision greater than 0.01 in the calculated value of θ^*.

a) The first data set consists of 20 observations from a Normal distributions with zero mean vector and identity covariance matrix. A plot of $R^2(\theta)$ vs θ is shown in figure 3.20. In this case, the minimum value of the correlation is 0.8856.

b) In the second simulated data set the first component of the vector was sampled from a chi-squared distribution with one degree of freedom. A plot of $R^2(\theta)$ vs θ is shown in figure 3.21. The minimum value of the correlation for this sample is 0.7048.

c) Figure 3.22 shows the plot for the third simulated data set consisting of 20 vectors, in which both components of the vector were sampled from a chi-squared distribution with one degree of freedom. Here, the minimum value of the correlation was 0.7070

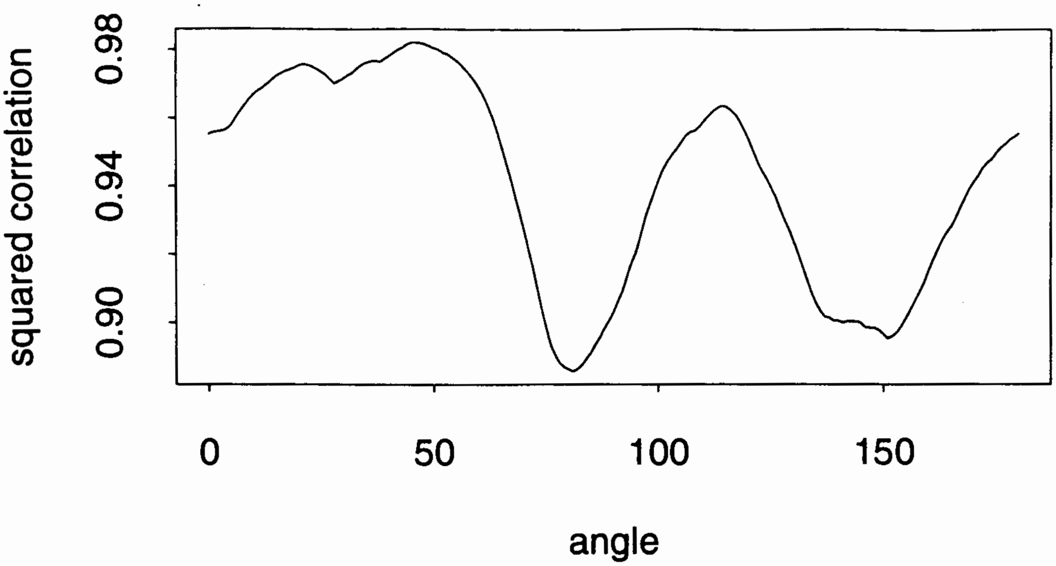

Figure 3.20: Plot of $R^2(\theta)$ vs θ. Simulated data set 1

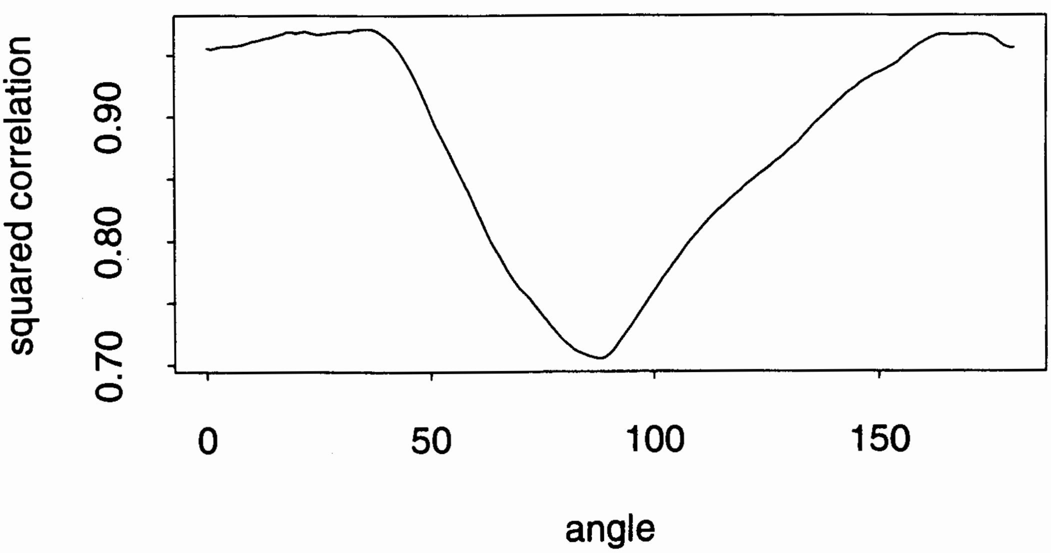

Figure 3.21: Plot of $R^2(\theta)$ vs θ. Simulated data set 2

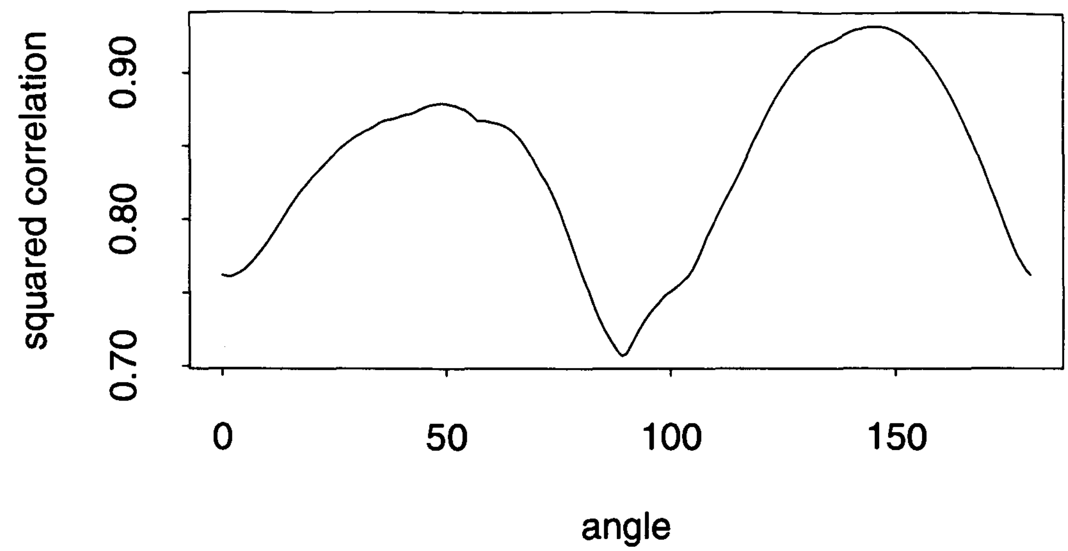

Figure 3.22: Plot of $R^2(\theta)$ vs θ. Simulated data set 3

d) The fourth data set was constructed to illustrate the situation in which the components of the random vector are marginally normal but not jointly normal. Two independent standard normal values u and v were first generated. The second component of the vector was set equal to v whereas the first component was set equal to u, if $uv > 0$ or $-u$ if $uv < 0$. A sample of twenty vectors was simulated. The corresponding plot is shown in figure 3.23. The minimum correlation for this data set was found to be 0.8653.

The type of plot illustrated in figures 3.20 to 3.23 is a useful tool to summarize relevant information about the type of departure. For example, non normality in the first component translates into a significant drop of $R^2(\theta)$ in the neighbourhood of 90° ; non-normality of both components will produce significant drops in the neighbourhoods of 0° and 90°, etc.

Hence, the test based on the maximum correlation method combines simultaneously the properties of graphical and formal statistical methods.

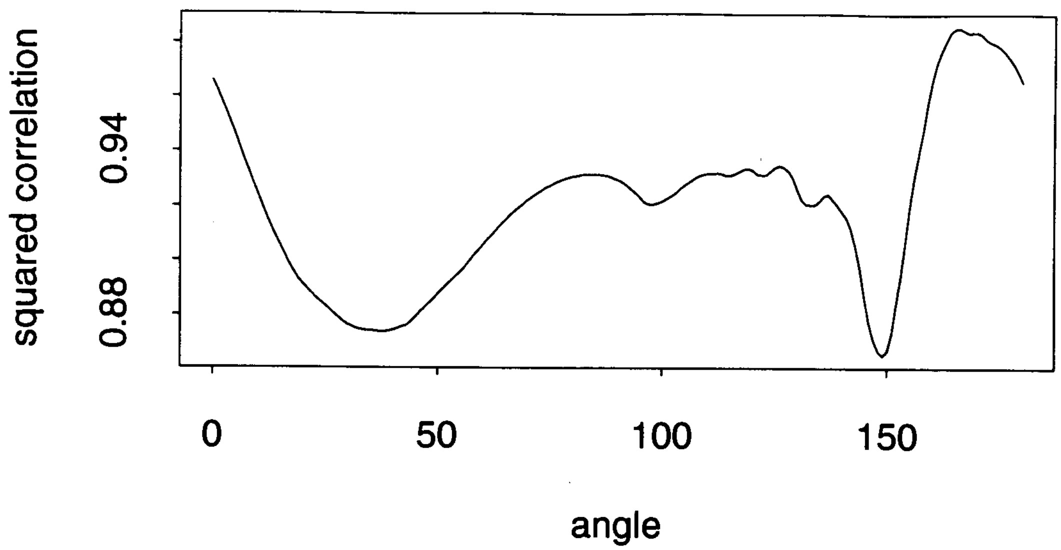

Figure 3.23: Plot of $R^2(\theta)$ vs θ. Simulated data set 4

Table 3.10: Empirical percentage points of T for bivariate normality.

	Significance level						
n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
10	1.34	1.73	2.00	2.20	2.54	2.82	3.24
20	1.57	2.04	2.38	2.60	3.03	3.44	4.00
30	1.72	2.23	2.58	2.87	3.34	3.86	4.53
40	1.82	2.33	2.67	2.96	3.46	3.92	4.55
50	1.88	2.41	2.79	3.05	3.51	4.00	4.68
60	1.94	2.48	2.82	3.13	3.63	4.15	4.72
70	1.98	2.52	2.89	3.17	3.66	4.20	4.78
80	2.00	2.55	2.90	3.19	3.71	4.24	5.00
90	2.02	2.57	2.96	3.26	3.74	4.22	4.90
100	2.01	2.63	3.02	3.29	3.82	4.38	4.99

Chapter 4

EDF Tests for the Frechet distribution

4.1 Introduction

Three important distributions arise as limiting distributions for extreme values. These are the Gumbel, Weibull and Frechet distributions. Tests of fit based on the empirical distribution function (EDF) statistics W^2, A^2 and U^2 have been given for the first two distributions by Stephens [63] and Lockhart and Stephens [35] when the location parameter is unknown.

The Frechet distribution (or Type II extreme value distribution) has gained importance in practice and it has been applied, for instance, to sea waves and wind speeds Thom [67], [68], [69], [70]. In this chapter, goodness-of-fit tests for this distribution are presented.

Suppose that $X_1, \ldots, X_n$ is a random sample; $X_{(1)}, \ldots, X_{(n)}$ are the corresponding order statistics and that we are interested in testing the null hypothesis H_0 : that the sample has been drawn from the distribution

$$F(x) = 1 - e^{-\left(\frac{\beta}{\alpha - x}\right)^m} \quad , \alpha > x \tag{4.1}$$

where α, $\beta > 0$ and $m > 0$ are parameters. The parameter α is known as the location parameter whereas β and m are referred to as the scale and shape parameters respectively. Plots of the standard form of the density ($\alpha = 0$, $\beta = 1$) are shown in figure 4.1 for selected values of the shape parameter m. Note that the mode decreases as m increases. The tests

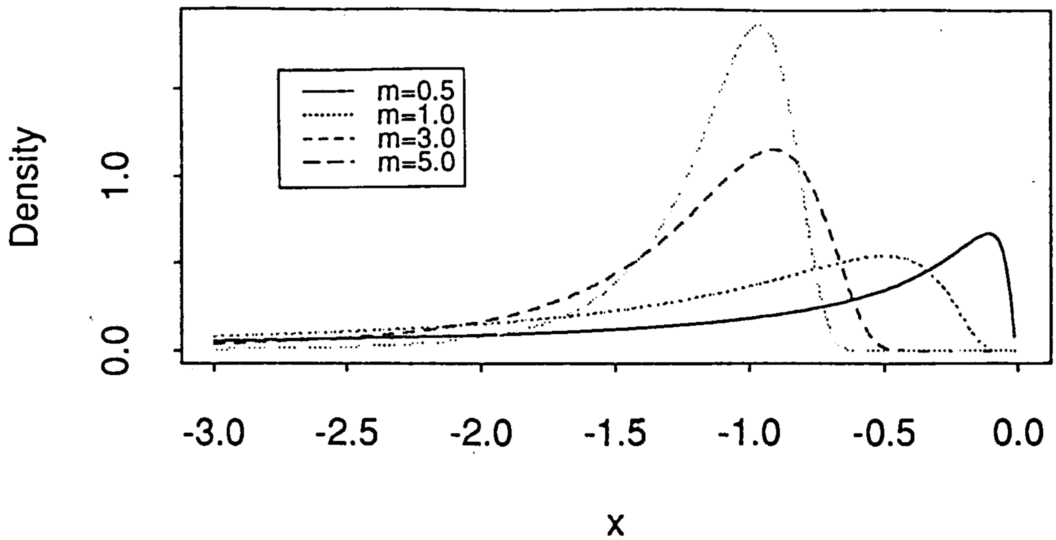

Figure 4.1: Frechet densities with parameters m=0.5, 1, 3, 5.

of fit are based on the EDF of the quantities $z_i = F(x_i; \alpha, \beta, m)$, the Probability Integral Transformation, (PIT) where estimated values of the unknown parameters are inserted into $F(.)$ if they are not known. Using the same notation as in Lockhart and Stephens [34], [35] we will distinguish eight cases for the test of H_0 according to which parameters must be estimated in (4.1). These are: Case 0: none; Case 1: α ; Case 2: β ; Case 3 α, β ; Case 4: m ; Case 5: α, m ; Case 6: β, m ; Case 7: α, β, m.

For cases 2, 4 and 6, where α is known, the transformation $Y = -\log(\alpha - X)$ can be made and the y-sample can be tested to come from a Gumbel distribution with location and scale parameters α_G and β_G respectively. The relationship between the parameters of the Gumbel and the Frechet distribution are : $\alpha_G = -\log(\beta)$, $\beta_G = 1/m$. Tests for this situation are described in Stephens [62] or Stephens[16, section 4.10]. For case 0, the reader is referred to Stephens [16, section 4.4].

Thus, in this chapter we consider tests for cases 1, 3, 5 and 7. The estimation will be done by maximum likelihood.

4.2 Maximum likelihood estimation

The log-likelihood for the sample $x_1, \ldots, x_n$ is:

$$\lambda = nm\ln(\beta) - (m+1)\sum_{i=1}^{n}\ln(\alpha - x_i) + n\ln(m) - \sum_{i=1}^{n}\frac{\beta^m}{(\alpha - x_i)^m}$$

from which the likelihood equations are:

$$\frac{\partial\lambda}{\partial\alpha} = -(m+1)\sum_{i=1}^{n}(\alpha - x_i)^{-1} + m\beta^m\sum_{i=1}^{n}(\alpha - x_i)^{-(m+1)} = 0 \quad (4.2)$$

$$\frac{\partial\lambda}{\partial\beta} = n - \beta^m\sum_{i=1}^{n}(\alpha - x_i)^{-m} = 0 \quad (4.3)$$

$$\frac{\partial\lambda}{\partial m} = n\ln(\beta) - \sum_{i=1}^{n}\ln(\alpha - x_i) + \frac{n}{m} + \beta^m\sum_{i=1}^{n}\left[\frac{\ln(\alpha - x_i) - \ln(\beta)}{(\alpha - x_i)^m}\right] = 0. \quad (4.4)$$

Depending on which parameters are unknown, the above equations are solved to obtain the MLE estimates for any given case.

For case 1, $\hat{\alpha}$ is obtained from (4.2). In case 3, (4.3) gives

$$\beta = \left[\frac{n}{\sum_{i=1}^{n}(\alpha - x_i)^{-m}}\right]^{1/m} \quad (4.5)$$

and this is substituted into (4.2). For cases 5 and 7, equations (4.2) and (4.4) are solved using the known value of β or by making use of (4.5) to eliminate β from these equations when it is unknown. For case 7, (4.5) is substituted into (4.2) and (4.4) to give

$$\frac{\partial\lambda}{\partial\alpha} = \frac{m+1}{m}\sum_{i=1}^{n}(\alpha - x_i)^{-1} - \frac{n\sum_{i=1}^{n}(\alpha - x_i)^{-(m+1)}}{\sum_{i=1}^{n}(\alpha - x_i)^{-m}} = 0 \quad (4.6)$$

$$\frac{\partial\lambda}{\partial m} = \frac{1}{m} + \frac{\sum_{i=1}^{n}(\alpha - x_i)^{-m}\ln(\alpha - x_i)}{\sum_{i=1}^{n}(\alpha - x_i)^{-m}} - \frac{\sum_{i=1}^{n}\ln(\alpha - x_i)}{n} = 0 \quad (4.7)$$

When α is unknown, support of the probability density function depends on an unknown parameter. In such cases, the classical regularity conditions for maximum likelihood estimation (MLE) are not satisfied. However, we make use of the results by Smith [59] which show, in particular, that for the Frechet distribution, the solutions of the likelihood equations produce estimators for which the classical properties of the MLE's are still valid.

4.2.1 The profile-likelihood approach

This approach has been proposed by Lockhart and Stephens [36] for the three-parameter Weibull distribution.

The *profile-likelihood* $L^*(\alpha_t)$ is the likelihood L maximized with respect to β and m, for a fixed value $\alpha = \alpha_t$. Hence,

$$L^*(\alpha_t) = L(\alpha_t, \beta(\alpha_t), m(\alpha_t))$$

where $\beta(\alpha_t)$ and $m(\alpha_t)$ denote either a known value of the parameter or the solution of the corresponding likelihood equation. For computational purposes, it is often convenient to work with

$$\ell(\alpha_t) = log L^*(\alpha_t). \tag{4.8}$$

In case 7, for a particular value α_t of α, equation (4.7) can be solved iteratively to obtain $m(\alpha_t)$ and then obtain $\beta(\alpha_t)$ from (4.5). The MLE of α is then the value which maximizes (4.8). There appear to be only two possible shapes of $\ell = \ell(\alpha_t)$ which are shown in figures 4.2 to 4.3.

Figure 4.2 corresponds to data set 9, from Castillo [13], and are yearly maximum wind speeds in miles per hour registered at a given location during fifty years. In this case, there is a MLE solution.

The plot shown in figure 4.3 was constructed using data set 10, which consist of maximum wave heights in a certain location, taken from Castillo [13]. For this situation, the likelihood can be increased by letting α tend to ∞. Here, no MLE solution exists and a Gumbel fit of the original observations is recommended.

4.2.2 Detection of figure 4.3

In order to find a maximum for ℓ, we must solve equations (4.6) and (4.7). Let $m_\alpha(\alpha_t)$ be the solution of (4.6) and $m_m(\alpha_t)$ the solution of (4.7). Sometimes they will be abreviated as m_α and m_m. When $m_m = m_\alpha$ for a given α_t, we know that we have a solution of the likelihood equations. Corresponding to the log-likelihood plots, there will be two plots when m_α and m_m are plotted against α_t. They are shown in figures 4.2 and refprof3.

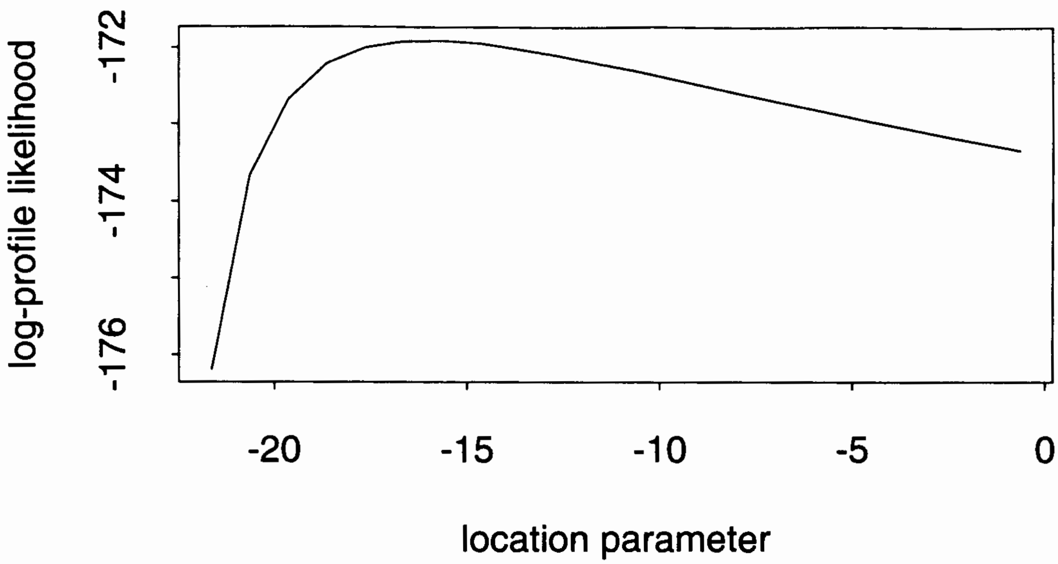

Figure 4.2: log-profile likelihood ℓ, for wind data

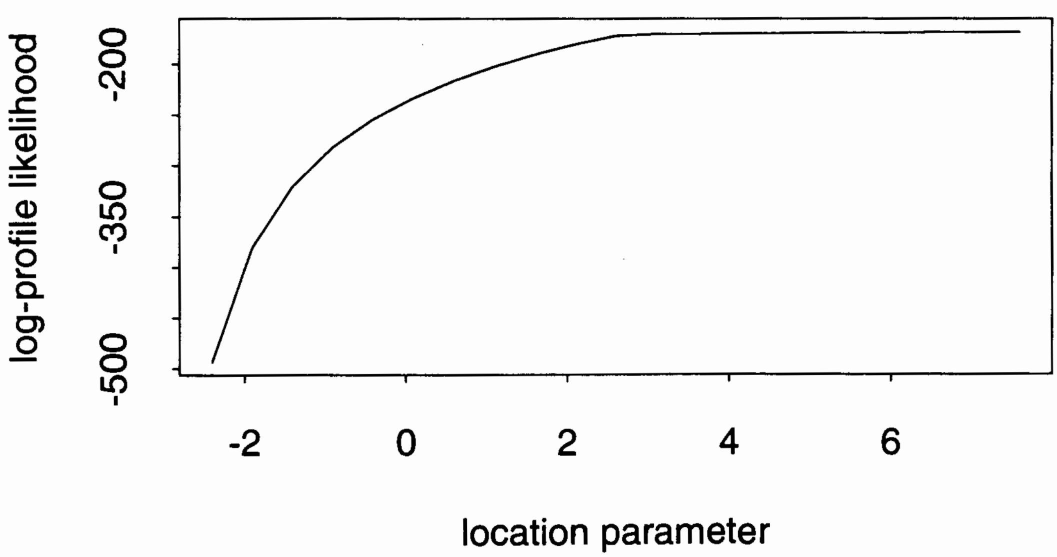

Figure 4.3: log-profile likelihood ℓ, for wave data

From figure 4.4 we can see that a MLE solution exists since the line m_α intersects the line m_m. Data set 9, from Castillo [13], are yearly maximum wind speeds in miles per hour registered at a given location during fifty years. Here, a test of fit for the distribution of the maximum is required. The test is then based on the transformed observations $X_i' = -X_i$.

The maximum likelihood estimates are $\hat{\alpha} = -16.04$, $\hat{\beta} = 12.06$ and $\hat{m} = 2.38$ which produce the following values of the test statistics: $W^2 = 0.0244$, $A^2 = 0.16$, $U^2 = 0.024$.

None of the test statistics is significant at 50% level, so the fit is considered very good.

Figure 4.5 shows that, for the case of the wave data, these two lines appear to be parallel (with $m_\alpha(\alpha_t) < m_m(\alpha_t)$) and no MLE exists. One can interpret the figure as suggesting that the MLE of m will be infinite: then the limiting Frechet, namely the Gumbel distribution, should be fitted to the data. Note that in these figures the lines tend to parallel lines as $\alpha_t \to \infty$. This appear to be always the case. We can therefore discriminate between the two plots by using the asymptotic gap Δ between the lines.

Let us define,

$$\Delta = \lim_{\alpha_t \to \infty} (m_\alpha(\alpha_t) - m_m(\alpha_t))$$

to be the limiting gap between these lines. The value of Δ is found as follows, using a procedure proposed by Lockhart and Stephens [36] to address a similar situation arising for the case of the three-parameter Weibull distribution. The derivation of the procedure is described in the next section.

Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$, $s = n^{-1} \sum_{i=1}^{n} x_i^2$ and $T_r = \sum_{i=1}^{n} x_i^r \exp(\gamma x_i)$, where γ is the solution of

$$\frac{1}{\gamma} = \left(\frac{T_1}{T_0} - \bar{x} \right) \tag{4.9}$$

The value of γ can be obtained by iteration using an initial approximation γ_0 in the right-hand side of the above equation.

The value of γ is the limiting slope of the two lines and

$$\Delta = \frac{\frac{\gamma}{2}(T_2 - sT_0) - \bar{x}T_0}{\bar{x}T_0 - \gamma(T_2 - \bar{x}T_1)}$$

Thus a negative value of Δ implies the occurrence of figure 4.3.

Note that the quantity,

$$-\Delta = \frac{\bar{x}T_0 + \frac{\gamma}{2}(sT_0 - T_2)}{\bar{x}T_0 - \gamma(T_2 - \bar{x}T_1)}$$

when $\gamma < 0$, corresponds to the expression obtained by Lockhart and Stephens [36] for the case of the Weibull distribution. If we denote by $\Delta(\gamma)$ the value of Δ for a given value of γ, obtained by solving (4.9), we have that the corresponding value of Δ, for the Weibull case, is $-\Delta(-\gamma)$.

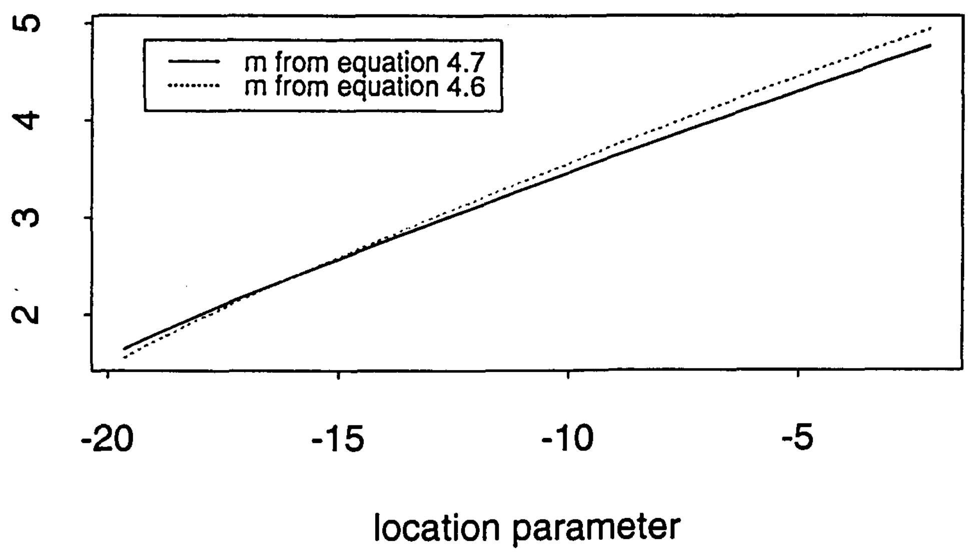

Figure 4.4: Lines m_α and m_m for wind data.

4.2.3 Derivation of Δ

Suppose that as $\alpha \to \infty$, the solution of the equations (4.6) and (4.7) is of the form $m = \gamma\alpha + b_0 + b_1/\alpha$. The coefficients γ, b_0 and b_1 will now be obtained for m_α and m_m, the solutions corresponding to (4.6) and (4.7), respectively.

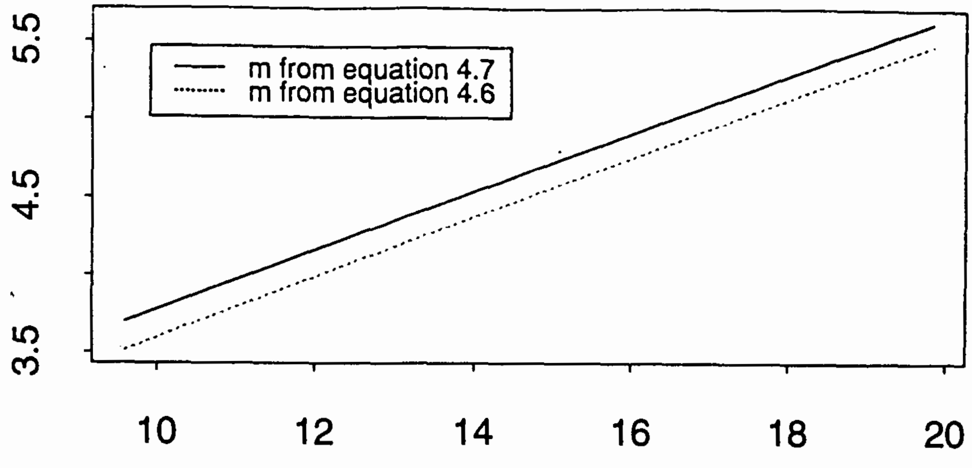

Figure 4.5: Lines m_α and m_m for wave data.

Since $(\alpha - x_i)^{-m} = \alpha^{-m}\left(1 - \frac{x_i}{\alpha}\right)^{-m}$, we want to examine the quantity $\left(1 - \frac{x_i}{\alpha}\right)^{-m}$ as $\alpha \to \infty$. To do this, we write $\left(1 - \frac{x_i}{\alpha}\right)^{-m}$ as $\exp\left(-m \ln\left(1 - x_i/\alpha\right)\right)$. Hence, up to terms of order three,

$$\ln\left(1 - \frac{x_i}{\alpha}\right) = -\left(\frac{x_i}{\alpha} + \frac{x_i^2}{2\alpha^2} + \frac{x_i^3}{3\alpha^3}\right)$$

Thus,

$$\left(1 - \frac{x_i}{\alpha}\right)^{-m} \sim \exp\left\{\left[\gamma\alpha + b_0 + \frac{b_1}{\alpha}\right]\left[\frac{x_i}{\alpha} + \frac{x_i^2}{2\alpha^2} + \frac{x_i^3}{3\alpha^3}\right]\right\}.$$

Retaining terms up to order 2, we can approximate it by

$$\left(1 - \frac{x_i}{\alpha}\right)^{-m} \sim e^{\gamma x_i} \exp\left[\frac{\gamma x_i^2/2 + b_0 x_i}{\alpha} + \frac{\gamma x_i^3/3 + b_0 x_i^2/2 + b_1 x_i}{\alpha^2}\right] \tag{4.10}$$

If we put

$$V = \frac{\gamma x_i^2/2 + b_0 x_i}{\alpha} + \frac{\gamma x_i^3/3 + b_0 x_i^2/2 + b_1 x_i}{\alpha^2}$$

the second term in the right-hand side of (4.10) is e^V, which can be expanded about zero to obtain, as a new approximation,

$$\left(1 - \frac{x_i}{\alpha}\right)^{-m} \sim e^{\gamma x_i}\left[1 + V + \frac{V^2}{2}\right]$$

which yields

$$\left(1 - \frac{x_i}{\alpha}\right)^{-m} \sim e^{\gamma x_i}\left[1 + \frac{\gamma x_i^2/2 + b_0 x_i}{\alpha} + \frac{\gamma x_i^3/3 + b_0 x_i^2/2 + b_1 x_i + \gamma^2 x_i^4/8 + \gamma b_0 x_i^3/2 + b_0^2 x_i^2/2}{\alpha^2}\right]$$

Using the above approximation, and defining $T_r = \sum_{i=1}^n x_i^r e^{\gamma x_i}$, we then have:

$$\sum_{i=1}^n \left(1 - \frac{x_i}{\alpha}\right)^{-m} \sim T_0 + \frac{A_2}{\alpha} + \frac{B_2}{\alpha^2}$$

where

$$\begin{aligned} A_2 &= \frac{\gamma}{2}T_2 + b_0 T_1 \\ B_2 &= \frac{\gamma}{3}T_3 + \frac{b_0}{2}T_2 + b_1 T_1 + \frac{\gamma^2}{8}T_4 + \frac{\gamma b_0}{2}T_3 + \frac{b_0^2}{2}T_2 \end{aligned}$$

Similarly

$$\sum_{i=1}^n \left(1 - \frac{x_i}{\alpha}\right)^{-(m+1)} = T_0 + \frac{A_3}{\alpha} + \frac{B_3}{\alpha^2}$$

where

$$\begin{aligned} A_3 &= \frac{\gamma}{2}T_2 + (b_0+1)T_1 = A_2 + T_1 \\ B_3 &= \frac{\gamma}{3}T_3 + \frac{(b_0+1)}{2}T_2 + b_1 T_1 + \frac{\gamma^2}{8}T_4 + \frac{\gamma(b_0+1)}{2}T_3 + \frac{(b_0+1)^2}{2}T_2 \\ &= B_2 + \frac{\gamma}{2}T_3 + b_0 T_2 + T_2 \end{aligned}$$

Also,

$$\left(1 - \frac{x_i}{\alpha}\right)^{-1} \sim 1 + \frac{x_i}{\alpha} + \frac{x_i^2}{\alpha^2}.$$

We now use the above results in equation (4.6), which can be written as

$$\frac{m+1}{n}\sum_{i=1}^n \left(1 - \frac{x_i}{\alpha}\right)^{-1}\left(1 - \frac{x_i}{\alpha}\right)^{-m} = m\left(1 - \frac{x_i}{\alpha}\right)^{-(m+1)}$$

Thus, we have:

$$\left(\gamma\alpha + b_0 + \frac{b_1}{\alpha} + 1\right)\left(1 + \frac{\bar{x}}{\alpha} + \frac{s}{\alpha^2}\right)\left(T_0 + \frac{A_2}{\alpha} + \frac{B_2}{\alpha^2}\right) =$$

$$\left(\gamma\alpha + b_0 + \frac{b_1}{\alpha}\right)\left(T_0 + \frac{A_2}{\alpha} + \frac{T_1}{\alpha} + \frac{B_2}{\alpha^2} + \frac{\gamma T_3}{2\alpha^2} + \frac{b_0 T_2}{\alpha^2} + \frac{T_2}{\alpha^2}\right)$$

where $\bar{x} = n^{-1}\sum_{i=1}^n x_i$ and $s = n^{-1}\sum_{i=1}^n x_i^2$.

Equating the coefficients of the constant term, we obtain the following equation for the slope γ,

$$\frac{1}{\gamma} = \left(\frac{T_1}{T_0} - \bar{x}\right) \tag{4.11}$$

Equating now the coefficients of $1/\alpha$, we have

$$b_0(\bar{x}T_0 - \gamma T_2 + \gamma\bar{x}T_1) - \frac{\gamma^2}{2}(T_3 - \bar{x}T_2) + \gamma(sT_0 - T_2/2) + \bar{x}T_0 = 0 \qquad (4.12)$$

The solution of (4.11) for γ and (4.12) for b_0 which here will be denoted by δ_α, give the coefficients of the asymptote $m_\alpha = \gamma\alpha + \delta_\alpha$.

We now write equation (4.7) as

$$\sum_{i=1}^{n}\left(1-\frac{x_i}{\alpha}\right)^{-m} = m\left\{\left[\frac{1}{n}\sum_{i=1}^{n}\ln\left(1-\frac{x_i}{\alpha}\right)\right]\left[\sum_{i=1}^{n}\left(1-\frac{x_i}{\alpha}\right)^{-m}\right] - \sum_{i=1}^{n}\ln\left(1-\frac{x_i}{\alpha}\right)\left(1-\frac{x_i}{\alpha}\right)^{-m}\right\}. \qquad (4.13)$$

Expanding, we have:

$$-\sum_{i=1}^{n}\ln\left(1-\frac{x_i}{\alpha}\right)\left(1-\frac{x_i}{\alpha}\right)^{-m} \sim \frac{T_1}{\alpha} + \frac{\gamma T_3/2 + b_0T_2}{\alpha^2} + \frac{T_2/2}{\alpha^2}.$$

Substitution of the above expression into (4.13) gives

$$\begin{aligned} T_0 + \frac{A_2}{\alpha} + \frac{B_2}{\alpha^2} &= -(\gamma\alpha + b_0)\left\{\left(\frac{\bar{x}}{\alpha} + \frac{s}{2\alpha^2}\right)\left[T_0 + \frac{A_2}{\alpha} + \frac{B_2}{\alpha^2}\right]\right. \\ &\quad \left. -\frac{T_1}{\alpha} - \frac{\gamma T_3/2 + b_0T_2}{\alpha^2} - \frac{T_2/2}{\alpha^2}\right\} \end{aligned}$$

Equating the constant terms, we arrive at the same equation in γ obtained previously, that is,

$$\frac{1}{\gamma} = \left(\frac{T_1}{T_0} - \bar{x}\right) \qquad (4.14)$$

so that the two lines will be parallel. After some algebra, we also obtain

$$b_0(\gamma T_2 - \gamma\bar{x}T_1 - \bar{x}T_0) + \frac{\gamma^2}{2}(T_3 - \bar{x}T_2) - \frac{\gamma}{2}sT_0 = 0 \qquad (4.15)$$

The solution of (4.15) for b_0 now gives δ_m, the constant term in the asymptote $m_m = \gamma\alpha + \delta_m$. Finally, using (4.12) and (4.15) we obtain the following equation for $\Delta = \delta_\alpha - \delta_m$, namely,

$$(\delta_\alpha - \delta_m)[\bar{x}T_0 - \gamma T_2 + \gamma\bar{x}T_1] = \frac{\gamma}{2}(T_2 - sT_0) - \bar{x}T_0\ ,$$

which then gives Δ as in the above section.

4.3 Test procedures

Let θ denote the vector of unknown parameters and $\hat{\theta}$ its MLE estimate. The test procedure can be summarized as follows:

a) First, the MLE of θ must be found as described above.

b) Compute $z_i = F(x_i; \hat{\theta})$ for $i = 1, \ldots, n$.

c) Calculate the EDF statistics:

$$
\begin{aligned}
A^2 &= -n - (1/n)\sum_{i=1}^{n}(2i-1)\left\{\ln(z_{(i)}) + \ln(1 - z_{(n-i+1)})\right\} \\
W^2 &= \sum_{i=1}^{n}\left\{z_{(i)} - \frac{2i-1}{2n}\right\}^2 + \frac{1}{12n} \\
U^2 &= W^2 - n(\bar{z} - 0.5)^2 \quad \text{where } \bar{z} = \frac{\sum_{i=1}^{n} z_i}{n}
\end{aligned}
$$

d) Using the value of m or $\hat{m}$ refer to tables 4.6- 4.17 for the appropriate case and test statistic. If the value of the test statistic is greater than the value given for level p, the null hypothesis is rejected at level p.

4.4 Asymptotic theory

In this section, the process of finding the asymptotic distributions of the EDF statistics is briefly summarized. For a more detailed treatment of the subject, the reader is referred to Durbin [22] and Stephens [62].

Let $F_n(x)$ stand for the empirical distribution function calculated from the sample and denote by $\hat{\theta}$ the MLE of the vector of p parameters with estimates where necessary.

Following Durbin [22] the process

$$\sqrt{n}\left\{F_n(x) - F(x; \hat{\theta})\right\} \text{ evaluated at } \hat{t} = F(x; \hat{\theta})$$

converges weakly to a Gaussian process $\{Z(t) : t \in (0,1)\}$ which has zero mean and covariance function:

$$\rho(s,t) = min(s,t) - st - \mathbf{g}^{\top}(t)\mathbf{I}^{-1}(\theta)\mathbf{g}(t) \tag{4.16}$$

where $\mathbf{I}(\theta)$ is Fisher's information matrix (per unit sample) and $\mathbf{g}(t)$ is the vector

$$\mathbf{g}^{\top}(t) = \left(\frac{\partial F(x;\theta)}{\partial \theta_1}, \ldots, \frac{\partial F(x;\theta)}{\partial \theta_p}\right) \tag{4.17}$$

evaluated at the point $t = F(x;\theta)$.

The three statistics W^2, A^2 and U^2 are functionals of this process.

Let $\rho_k(s,t)$ be the covariance function for case k. Then

$$W^2 \xrightarrow{\mathcal{D}} \int_0^1 Z^2(t)dt$$

and(see for example Durbin [22])

$$W^2 \xrightarrow{\mathcal{D}} \sum_{i=1}^{\infty} \lambda_i \nu_i$$

where $\nu_1, \nu_2, \ldots$ are independent $\chi^2_{(1)}$ variables and $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the integral equation

$$\int_0^1 \rho_k(s,t) f_i(s) ds = \lambda_i f_i(t) \tag{4.18}$$

Similarly,

$$A^2 \xrightarrow{\mathcal{D}} \int_0^1 a^2(t)dt$$

where

$$a(t) = \frac{Z(t)}{\sqrt{t(1-t)}}$$

and

$$A^2 \xrightarrow{\mathcal{D}} \sum_{i=1}^{\infty} \lambda_i^* \nu_i.$$

Here, $\lambda_1^*, \lambda_2^*, \ldots$ are the solutions of the integral equation (4.18) with $\rho_k(s,t)$ replaced with

$$\rho_k^*(s,t) = \frac{\rho_k(s,t)}{\sqrt{s(1-s)t(1-t)}} \tag{4.19}$$

Finally, U^2 is asymptotically the integral over $(0,1)$ of the square of the process

$$u(t) = Z(t) - \int_0^1 Z(s)ds$$

which has, as covariance function,

$$\rho_k^{**}(s,t) = \rho_k(s,t) + \int_0^1 \int_0^1 \rho_k(s,t)dsdt - \int_0^1 \rho_k(s,t)ds - \int_0^1 \rho_k(s,t)dt \qquad (4.20)$$

The solutions of equation (4.18) with $\rho_k^{**}(s,t)$ replacing $\rho_k(s,t)$ are, in this case, denoted by $\lambda_1^{**}, \lambda_2^{**}, \ldots$

4.4.1 Case 7

Consider, as an illustration, the most complicated situation where all the parameters are unknown: $\theta = (\alpha, \beta, m)$.

The elements of vector (4.17) are:

$$\begin{aligned}
g_1(s) &= -\frac{m(1-s)\left[-\ln(1-s)\right]^{\frac{m+1}{m}}}{\beta} \\
g_2(s) &= -\frac{m(1-s)\ln(1-s)}{\beta} \\
g_3(s) &= -\frac{(1-s)\ln(1-s)\ln(-\ln(1-s))}{m}
\end{aligned}$$

Using the transformation $Y = -\ln(\alpha - X)$, we also obtain:

$$\begin{aligned}
\frac{\partial^2 \ln f(x)}{\partial \alpha^2} &= \frac{(m+1)}{(\alpha-x)^2} - \frac{m(m+1)\beta^m}{(\alpha-x)^{m+2}} \\
&= (m+1)e^{2Y} - m(m+1)\beta^m e^{(m+2)Y}. \\
\frac{\partial^2 \ln f(x)}{\partial \alpha \partial \beta} &= \frac{m^2\beta^{m-1}}{(\alpha-x)^{m+1}} \\
&= m^2\beta^{m-1}e^{(m+1)Y}. \\
\frac{\partial^2 \ln f(x)}{\partial \alpha \partial m} &= \frac{m\beta^m \ln(\beta) - m\beta^m \ln(\alpha-x) + \beta^m}{(\alpha-x)^{m+1}} - \frac{1}{(\alpha-x)} \\
&= (m\beta^m \ln\beta + \beta^m)e^{(m+1)Y} + m\beta^m Y e^{(m+1)Y} - e^Y. \\
\frac{\partial^2 \ln f(x)}{\partial \beta^2} &= -\frac{m(m-1)\beta^{m-2}}{(\alpha-x)^m} - \frac{m}{\beta^2} \\
&= -m(m-1)\beta^{m-2}e^{mY} - \frac{m}{\beta^2}.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \ln f(x)}{\partial\beta\partial m} &= \beta^{-1} + \frac{m\beta^{m-1}\ln(\alpha-x) - m\beta^{m-1}\ln\beta - \beta^{m-1}}{(\alpha-x)^m} \\
&= \beta^{-1} - m\beta^{m-1}Ye^{mY} - \left[m\beta^{m-1}\ln\beta + \beta^{m-1}\right]e^{mY}. \\
\frac{\partial^2 \ln f(x)}{\partial m^2} &= \frac{2\beta^m\ln(\beta)\ln(\alpha-x) - \beta^m(\ln\beta)^2 - \beta^m\left[\ln(\alpha-x)\right]^2}{(\alpha-x)^m} - \frac{1}{m^2} \\
&= -\frac{1}{m^2} - \beta^m(\ln\beta)^2e^{mY} - 2\beta^m\ln\beta Ye^{mY} - \beta^mY^2e^{mY}.
\end{aligned}
$$

Using the fact that Y has a Gumbel distribution with location and scale parameters respectively equal to $-\ln\beta$ and $1/m$, whose moment generating function (MGF) is given by

$$M_Y(t) = \frac{1}{\beta^t}\Gamma(1+t/m)$$

with first and second derivatives at t

$$
\begin{aligned}
M_Y'(t) &= -\frac{\ln\beta}{\beta^t}\Gamma(1+t/m) + \frac{1}{m\beta^t}\Gamma'(1+t/m) \\
M_Y''(t) &= \frac{(\ln\beta)^2}{\beta^t}\Gamma(1+t/m) - 2\frac{\ln\beta}{m\beta^t}\Gamma'(1+t/m) + \frac{1}{m^2\beta^t}\Gamma''(1+t/m),
\end{aligned}
$$

the Fisher's Information matrix is found to be

$$
\mathbf{I}(\theta) = \begin{pmatrix}
\frac{2(m+1)^2}{m\beta^2}\Gamma(2/m) & -\frac{m+1}{\beta^2}\Gamma(1/m) & -\frac{1}{\beta}\left[\frac{1}{m^2}\Gamma(1/m) + \Gamma'(2+1/m)\right] \\
 & \frac{m^2}{\beta^2} & \frac{1}{\beta}\left[\Gamma(2) + \Gamma'(2) - 1\right] \\
 & & \frac{1}{m^2}\left[1 + \Gamma''(2)\right]
\end{pmatrix}
$$

where Γ denotes the Gamma function and Γ', Γ'' its first and second derivatives. A definition of these functions can be found, for instance, in Abramowitz and Stegun [1].

For W^2, the above expressions are inserted into (4.16) to obtain $\rho_k(s,t)$ and use of (4.19) gives the covariance function for A^2.

Finally, for U^2, (4.20) takes the form:

$$
\begin{aligned}
\rho_k^{**}(s,t) &= min(s,t) - st + 1/12 - 1/2[t(1-t) + s(1-s)] \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3} g_i(s)\delta^{ij} g_j(t) - \sum_{i=1}^{3}\sum_{j=1}^{3} \int_0^1 g_i(s)ds\, \delta^{ij} \int_0^1 g_j(t)dt \\
&\quad + \sum_{i=1}^{3}\sum_{j=1}^{3} g_i(s)\, \delta^{ij} \int_0^1 g_j(t)dt + \sum_{i=1}^{3}\sum_{j=1}^{3} \int_0^1 g_i(s)ds\, \delta^{ij} g_j(t).
\end{aligned}
$$

where δ^{ij} denotes the (i,j)-th element of $\mathbf{I}^{-1}(\theta)$, and $g_1(t)$, $g_2(t)$ and $g_3(t)$ are the components of the vector $\mathbf{g}(t)$.

The integrals involved in calculating $\rho_k^{**}(s,t)$ are:

$$
\begin{aligned}
\int_0^1 g_1(t)dt &= -\frac{m}{4\beta}\frac{\Gamma(2+1/m)}{2^{1/m}} \\
\int_0^1 g_2(t)dt &= \frac{m}{4\beta} \\
\int_0^1 g_3(t)dt &= \frac{1-\ln(2)-\gamma}{4m}
\end{aligned}
$$

4.5 Calculation of asymptotic percentage points

Equation (4.18) is difficult to solve analytically and the following technique was employed. A grid of 50 points equally spaced in $(0,1)$ was used for values of s and t, as the integrals were approximated numerically. Also in the 50 by 50 grid, the covariance functions $\rho_k(s,t)$, $\rho_k^*(s,t)$, $\rho_k^{**}(s,t)$ were evaluated, and the eigenvalue problem solved for several values of the shape parameter m. After the eigenvalues λ_i, λ_i^* and λ_i^{**} were obtained the asymptotic percentage points were calculated using Imhof's method [30]. The above procedure was repeated using 100 equally spaced points in the unit interval and the results compared. The tables obtained were almost identical except for a few discrepancies of less than two units in the third decimal figure. The percentage points given are those obtained using the larger grid. From the tables of asymptotic percentage points of the EDF statistics considered here, some equivalences between limiting cases of the Frechet and Gumbel tests can be deduced.

Using the notation in Stephens [17, section 4.10], consider the following cases of a test of fit for the Gumbel distribution, according to which parameters must be estimated. Case 0: none; case 1: α ; case 2: β ; case 3: α, β.

Based on the asymptotic percentage points of the EDF statistics for the Frechet distribution in the limiting cases $m = 0$ and $m = \infty$, we can establish a correspondence between the different cases of the Frechet and the Gumbel tests. They are summarized in table 4.1. An heuristic explanation for these equivalences can be given as follows:

Table 4.1: Equivalent cases of the Frechet and Gumbel tests

m	Frechet case	Gumbel case
0	1	0
	3	1
	5	2
	7	3
∞	1	1
	3	3
	5	3
	7	3

a) $m \longrightarrow \infty$.

For large m, it is known that the Frechet distribution is close (in a well defined mathematical sense) to a Gumbel distribution with parameters $\alpha_G = (\alpha - \beta)$ and $\beta_G = \beta/m$. Depending on which parameters of the Frechet distribution are unknown, these relations between the parameters can be used to deduce the corresponding case of the Gumbel test.

b) $m \longrightarrow 0$

When the shape parameter m is close to zero, the mode of the distribution becomes very close to the value of α, producing a MLE estimate whose variance decreases with m. When this is the case, the transformed observations can be regarded as if they were calculated using the true value of the location. Thus, the null hypothesis that the sample came from a Frechet distribution, will be equivalent to testing that the transformed observations $Z_i = -\ln(\alpha - Y_i)$, are a random sample from a Gumbel distribution with parameters $\alpha_G = -\ln\beta$ and $\beta_G = 1/m$. From these relations, the corresponding Gumbel case can be inferred from a given case for the Frechet test.

4.6 Small sample distributions

In order to investigate the speed of convergence of the empirical percentage points to their corresponding asymptotic limits, a simulation study was conducted for case 7. One thousand samples from a standard Frechet distribution were simulated for some selected values of m and n, and the percentage points were recorded. The results are shown in tables 4.3, 4.4 and 4.5. These results indicate that the distribution of the EDF statistics converge very quickly to their asymptotic distributions and can be used with good accuracy for $n \geq 20$. The same remarks on accuracy of the nominal α apply as were made in section 2.5.1.

4.6.1 Empirical significance level of the tests

A question that arises is whether the significance level of the tests are mantained when $\hat{m}$, rather than m, is used to obtain the percentage points from the tables for a given size of the test. It can be seen from tables that, for $m > 2$, the asymptotic percentage points vary little with m and it can be conjectured that the use of $\hat{m}$ should not affect the significance level of the tests. A Monte Carlo experiment was done to verify this.

For selected small values of m, and for $n = 40$, 1000 samples were simulated and the tests carried out at significance levels of 5% and 10%. For each sample simulated, the percentage points for estimated $\hat{m}$ were obtained from the asymptotic tables by using linear interpolation in $1/m$. The overall proportion $\hat{p}$ of rejected samples was then recorded, for nominal levels 0.05 and 0.10. The results are given in table 4.2. It can be seen that the significance levels of the tests are close to p; however, the tests appear to be conservative, with $\hat{p}$ less than p. This may be explained by the fact that the asymptotic percentage points were used in the simulation process.

4.7 Conclusions

Tests of fit for the three-parameter Frechet distribution, based on the EDF statistics W^2,A^2 and U^2 have been presented and the asymptotic distributions of the test statistics have been found. Simulation results obtained for the case when the three parameters are known, indicate that the usual property of fast convergence of these test statistics to their asymptotic distribution, observed for distributions in the location-scale family, is preserved. Even

Table 4.2: Empirical significance level of EDF tests. n=40. Case 7

m	p	W^2	A^2	U^2
0.5	0.05	0.046	0.048	0.045
	0.10	0.097	0.089	0.094
0.8	0.05	0.042	0.039	0.040
	0.10	0.089	0.086	0.085
1.0	0.05	0.044	0.041	0.046
	0.10	0.089	0.087	0.092
1.5	0.05	0.042	0.039	0.043
	0.10	0.090	0.087	0.093

Table 4.3: Empirical percentage points of W^2: Case 7

		Significance level						
m	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.5	20	0.047	0.066	0.079	0.090	0.113	0.139	0.169
	40	0.046	0.065	0.081	0.090	0.107	0.131	0.150
0.8	20	0.044	0.062	0.077	0.086	0.105	0.119	0.147
	40	0.043	0.061	0.074	0.084	0.103	0.123	0.150
1.0	20	0.042	0.059	0.072	0.082	0.098	0.114	0.136
	40	0.043	0.060	0.073	0.083	0.101	0.118	0.140
1.5	20	0.041	0.059	0.070	0.080	0.101	0.120	0.139
	40	0.043	0.059	0.074	0.080	0.096	0.116	0.141
2.0	20	0.042	0.060	0.069	0.078	0.097	0.109	0.125
	40	0.042	0.060	0.071	0.085	0.101	0.116	0.140

though the asymptotic distribution of the statistics depends on the value of the shape parameter m, the empirical study shows that the use of the estimate of the shape parameter to carry out the test, produces little effect on the significance level when interpolating in $1/\hat{m}$ so that this dependence does not represent an objection for their use. This is of importance due to the fact that many studies have indicated that the tests based on the the EDF statistics, specially the test based on A^2, are powerful against a wide range of alternatives and therefore their use is recommended.

Table 4.4: Empirical percentage points of A^2: Case 7

		Significance level						
m	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.5	20	0.307	0.411	0.482	0.536	0.665	0.781	0.963
	40	0.300	0.418	0.492	0.556	0.662	0.779	0.952
0.8	20	0.290	0.387	0.456	0.510	0.610	0.692	0.786
	40	0.283	0.379	0.454	0.508	0.616	0.733	0.864
1.0	20	0.275	0.373	0.431	0.481	0.584	0.658	0.782
	40	0.283	0.378	0.451	0.500	0.586	0.684	0.803
1.5	20	0.272	0.366	0.429	0.485	0.593	0.682	0.799
	40	0.283	0.373	0.432	0.478	0.557	0.629	0.692
2.0	20	0.278	0.370	0.425	0.465	0.563	0.625	0.702
	40	0.280	0.371	0.429	0.486	0.552	0.642	0.775

Table 4.5: Empirical percentage points of U^2: Case 7

		Significance level						
m	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.5	20	0.045	0.064	0.075	0.089	0.107	0.133	0.159
	40	0.044	0.063	0.076	0.087	0.105	0.120	0.148
0.8	20	0.042	0.060	0.074	0.083	0.101	0.112	0.137
	40	0.042	0.058	0.071	0.081	0.098	0.117	0.149
1.0	20	0.041	0.058	0.069	0.079	0.097	0.110	0.134
	40	0.042	0.058	0.071	0.081	0.095	0.111	0.133
1.5	20	0.040	0.058	0.069	0.078	0.098	0.117	0.138
	40	0.043	0.058	0.070	0.076	0.091	0.105	0.122
2.0	20	0.042	0.059	0.068	0.076	0.095	0.106	0.122
	40	0.042	0.058	0.069	0.078	0.094	0.108	0.124

Table 4.6: Asymptotic percentage points of W^2: Case 1

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.119	0.209	0.284	0.347	0.461	0.581	0.744	0.870
0.4	0.118	0.209	0.283	0.347	0.461	0.580	0.742	0.869
0.6	0.116	0.205	0.279	0.342	0.455	0.574	0.733	0.860
0.8	0.113	0.200	0.272	0.334	0.444	0.560	0.715	0.840
1.0	0.109	0.193	0.263	0.323	0.430	0.542	0.693	0.813
2.0	0.097	0.167	0.225	0.275	0.364	0.457	0.578	0.683
4.0	0.086	0.144	0.190	0.230	0.301	0.375	0.476	0.554
6.0	0.082	0.135	0.177	0.212	0.275	0.341	0.431	0.501
8.0	0.080	0.130	0.169	0.202	0.261	0.323	0.408	0.472
10.0	0.079	0.127	0.165	0.197	0.253	0.313	0.394	0.446
15.0	0.077	0.123	0.159	0.189	0.243	0.298	0.375	0.430
20.0	0.076	0.121	0.156	0.185	0.237	0.291	0.365	0.423

Table 4.7: Asymptotic percentage points of W^2: Case 3

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.074	0.115	0.148	0.174	0.221	0.270	0.337	0.389
0.4	0.069	0.108	0.138	0.162	0.206	0.251	0.313	0.361
0.6	0.064	0.098	0.124	0.145	0.183	0.222	0.275	0.316
0.8	0.060	0.090	0.113	0.131	0.164	0.198	0.244	0.279
1.0	0.057	0.084	0.105	0.122	0.151	0.181	0.222	0.253
2.0	0.051	0.074	0.090	0.104	0.127	0.150	0.181	0.205
4.0	0.049	0.071	0.087	0.099	0.120	0.142	0.170	0.192
6.0	0.049	0.071	0.087	0.099	0.120	0.141	0.170	0.192
8.0	0.049	0.071	0.087	0.099	0.120	0.142	0.170	0.192
10.0	0.049	0.071	0.087	0.100	0.121	0.142	0.171	0.193
15.0	0.050	0.072	0.088	0.100	0.122	0.143	0.172	0.194
20.0	0.050	0.072	0.088	0.100	0.122	0.144	0.173	0.194

Table 4.8: Asymptotic percentage points of W^2: Case 5

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.101	0.185	0.258	0.320	0.431	0.547	0.705	0.827
0.4	0.098	0.180	0.252	0.313	0.423	0.537	0.693	0.814
0.6	0.093	0.170	0.237	0.295	0.398	0.505	0.651	0.764
0.8	0.086	0.154	0.213	0.262	0.352	0.446	0.573	0.672
1.0	0.079	0.136	0.185	0.226	0.300	0.377	0.481	0.564
2.0	0.060	0.092	0.115	0.134	0.168	0.204	0.252	0.289
4.0	0.053	0.078	0.095	0.109	0.134	0.158	0.191	0.216
6.0	0.052	0.075	0.092	0.105	0.128	0.151	0.182	0.206
8.0	0.051	0.074	0.091	0.104	0.126	0.149	0.179	0.202
10.0	0.051	0.074	0.090	0.103	0.125	0.148	0.178	0.200
15.0	0.051	0.073	0.090	0.103	0.125	0.147	0.176	0.199
20.0	0.050	0.073	0.089	0.102	0.124	0.146	0.176	0.198

Table 4.9: Asymptotic percentage points of W^2: Case 7

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.050	0.073	0.089	0.102	0.123	0.145	0.175	0.197
0.4	0.048	0.069	0.084	0.096	0.117	0.138	0.165	0.187
0.6	0.046	0.066	0.080	0.091	0.110	0.129	0.155	0.175
0.8	0.044	0.063	0.077	0.087	0.106	0.124	0.149	0.167
1.0	0.044	0.062	0.075	0.085	0.103	0.121	0.145	0.163
2.0	0.042	0.060	0.073	0.083	0.100	0.117	0.139	0.157
4.0	0.043	0.060	0.073	0.083	0.100	0.117	0.140	0.157
6.0	0.043	0.061	0.074	0.084	0.101	0.118	0.141	0.158
8.0	0.043	0.061	0.074	0.084	0.101	0.118	0.141	0.159
10.0	0.043	0.061	0.074	0.084	0.101	0.119	0.142	0.159
15.0	0.043	0.061	0.074	0.085	0.102	0.119	0.143	0.160
20.0	0.043	0.062	0.075	0.085	0.102	0.120	0.143	0.161

Table 4.10: Asymptotic percentage points of A^2: Case 1

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.774	1.247	1.621	1.932	2.492	3.077	3.878	4.489
0.4	0.760	1.231	1.604	1.915	2.473	3.057	3.856	4.484
0.6	0.732	1.192	1.558	1.864	2.415	2.991	3.779	4.389
0.8	0.704	1.147	1.501	1.798	2.333	2.894	3.660	4.251
1.0	0.680	1.104	1.445	1.732	2.248	2.789	3.528	4.100
2.0	0.607	0.964	1.250	1.490	1.924	2.377	2.998	3.476
4.0	0.555	0.858	1.095	1.293	1.649	2.021	2.531	2.922
6.0	0.536	0.818	1.035	1.216	1.541	1.880	2.344	2.704
8.0	0.526	0.797	1.005	1.177	1.485	1.806	2.247	2.588
10.0	0.520	0.784	0.986	1.153	1.451	1.762	2.188	2.518
15.0	0.512	0.768	0.962	1.122	1.406	1.703	2.109	2.424
20.0	0.508	0.759	0.950	1.106	1.384	1.674	2.070	2.378

Table 4.11: Asymptotic percentage points of A^2: Case 3

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.494	0.733	0.912	1.058	1.316	1.584	1.951	2.236
0.4	0.468	0.694	0.862	1.000	1.245	1.499	1.846	2.115
0.6	0.437	0.640	0.791	0.915	1.133	1.360	1.670	1.910
0.8	0.414	0.598	0.735	0.846	1.041	1.243	1.519	1.733
1.0	0.397	0.569	0.694	0.795	0.973	1.157	1.407	1.601
2.0	0.362	0.505	0.606	0.687	0.827	0.970	1.162	1.311
4.0	0.346	0.478	0.571	0.645	0.770	0.897	1.067	1.198
6.0	0.343	0.473	0.564	0.636	0.759	0.883	1.048	1.175
8.0	0.342	0.471	0.562	0.633	0.755	0.878	1.042	1.167
10.0	0.342	0.471	0.561	0.632	0.754	0.876	1.039	1.164
15.0	0.342	0.470	0.561	0.631	0.753	0.875	1.037	1.161
20.0	0.342	0.470	0.561	0.632	0.753	0.874	1.037	1.161

Table 4.12: Asymptotic percentage points of A^2: Case 5

m	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.631	1.056	1.415	1.721	2.273	2.849	3.646	4.243
0.4	0.604	1.020	1.374	1.675	2.219	2.786	3.558	4.159
0.6	0.567	0.952	1.279	1.558	2.061	2.587	3.310	3.857
0.8	0.524	0.856	1.135	1.372	1.801	2.249	2.860	3.333
1.0	0.483	0.760	0.986	1.178	1.525	1.888	2.385	2.771
2.0	0.383	0.544	0.661	0.755	0.920	1.090	1.322	1.502
4.0	0.352	0.488	0.583	0.659	0.788	0.919	1.094	1.228
6.0	0.347	0.479	0.571	0.644	0.769	0.895	1.062	1.191
8.0	0.345	0.476	0.567	0.639	0.763	0.887	1.052	1.178
10.0	0.344	0.474	0.566	0.637	0.760	0.883	1.047	1.173
15.0	0.343	0.473	0.564	0.635	0.757	0.879	1.043	1.168
20.0	0.343	0.472	0.563	0.634	0.756	0.878	1.041	1.166

Table 4.13: Asymptotic percentage points of A^2: Case 7

m	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.341	0.469	0.559	0.630	0.751	0.873	1.035	1.159
0.4	0.322	0.443	0.527	0.593	0.707	0.821	0.973	1.089
0.6	0.307	0.419	0.496	0.557	0.661	0.766	0.905	1.011
0.8	0.298	0.404	0.477	0.535	0.633	0.731	0.862	0.962
1.0	0.293	0.395	0.467	0.522	0.617	0.711	0.837	0.933
2.0	0.286	0.384	0.452	0.505	0.595	0.684	0.804	0.894
4.0	0.286	0.385	0.453	0.506	0.596	0.686	0.806	0.896
6.0	0.288	0.387	0.456	0.509	0.600	0.691	0.811	0.902
8.0	0.289	0.389	0.458	0.511	0.603	0.694	0.815	0.907
10.0	0.289	0.390	0.459	0.513	0.605	0.696	0.817	0.910
15.0	0.290	0.391	0.461	0.515	0.607	0.699	0.822	0.914
20.0	0.291	0.392	0.462	0.517	0.609	0.701	0.824	0.917

Table 4.14: Asymptotic percentage points of U^2: Case 1

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.070	0.105	0.130	0.151	0.185	0.220	0.265	0.300
0.4	0.069	0.104	0.130	0.150	0.184	0.219	0.264	0.299
0.6	0.068	0.103	0.128	0.148	0.181	0.215	0.260	0.293
0.8	0.067	0.101	0.125	0.144	0.177	0.210	0.254	0.287
1.0	0.066	0.099	0.123	0.142	0.174	0.206	0.248	0.281
2.0	0.063	0.094	0.116	0.134	0.164	0.195	0.236	0.267
4.0	0.061	0.091	0.113	0.130	0.160	0.190	0.231	0.262
6.0	0.061	0.091	0.112	0.129	0.159	0.189	0.230	0.261
8.0	0.061	0.090	0.112	0.129	0.159	0.189	0.229	0.260
10.0	0.060	0.090	0.112	0.129	0.159	0.189	0.229	0.260
15.0	0.060	0.090	0.112	0.129	0.158	0.189	0.229	0.260
20.0	0.060	0.090	0.111	0.129	0.158	0.189	0.229	0.260

Table 4.15: Asymptotic percentage points of U^2: Case 3

	Significance level							
m	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.060	0.090	0.111	0.128	0.158	0.188	0.229	0.260
0.4	0.059	0.088	0.109	0.126	0.155	0.185	0.225	0.255
0.6	0.056	0.084	0.104	0.120	0.148	0.177	0.215	0.245
0.8	0.054	0.081	0.100	0.115	0.141	0.169	0.205	0.234
1.0	0.053	0.078	0.096	0.111	0.136	0.162	0.197	0.224
2.0	0.050	0.072	0.089	0.101	0.124	0.147	0.177	0.200
4.0	0.048	0.070	0.085	0.098	0.119	0.140	0.168	0.190
6.0	0.048	0.070	0.085	0.097	0.118	0.139	0.167	0.188
8.0	0.048	0.069	0.085	0.097	0.117	0.138	0.166	0.187
10.0	0.048	0.069	0.085	0.097	0.117	0.138	0.166	0.187
15.0	0.048	0.069	0.085	0.097	0.117	0.138	0.166	0.187
20.0	0.048	0.069	0.085	0.097	0.117	0.138	0.166	0.187

Table 4.16: Asymptotic percentage points of U^2: Case 5

m	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.057	0.085	0.105	0.121	0.149	0.178	0.217	0.247
0.4	0.055	0.082	0.101	0.117	0.145	0.173	0.212	0.242
0.6	0.055	0.081	0.100	0.116	0.143	0.172	0.211	0.241
0.8	0.055	0.081	0.100	0.116	0.143	0.172	0.211	0.241
1.0	0.054	0.080	0.100	0.115	0.142	0.170	0.208	0.238
2.0	0.051	0.075	0.092	0.106	0.129	0.153	0.185	0.210
4.0	0.049	0.071	0.087	0.100	0.121	0.143	0.171	0.193
6.0	0.049	0.070	0.086	0.098	0.119	0.140	0.168	0.190
8.0	0.049	0.070	0.086	0.098	0.119	0.140	0.167	0.188
10.0	0.048	0.070	0.085	0.097	0.118	0.139	0.167	0.188
15.0	0.048	0.070	0.085	0.097	0.118	0.139	0.166	0.187
20.0	0.048	0.070	0.085	0.097	0.118	0.139	0.166	0.187

Table 4.17: Asymptotic percentage points of U^2: Case 7

m	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.2	0.048	0.069	0.085	0.097	0.117	0.138	0.165	0.186
0.4	0.046	0.066	0.080	0.092	0.111	0.130	0.156	0.176
0.6	0.044	0.063	0.076	0.087	0.105	0.123	0.147	0.165
0.8	0.043	0.061	0.074	0.084	0.101	0.118	0.141	0.159
1.0	0.042	0.060	0.072	0.082	0.099	0.116	0.139	0.156
2.0	0.042	0.059	0.071	0.081	0.098	0.114	0.136	0.153
4.0	0.042	0.060	0.072	0.082	0.099	0.116	0.138	0.155
6.0	0.042	0.060	0.073	0.083	0.100	0.117	0.139	0.156
8.0	0.043	0.060	0.073	0.083	0.100	0.117	0.140	0.157
10.0	0.043	0.061	0.073	0.083	0.100	0.118	0.140	0.158
15.0	0.043	0.061	0.074	0.084	0.101	0.118	0.141	0.159
20.0	0.043	0.061	0.074	0.084	0.101	0.119	0.142	0.159

Chapter 5

EDF tests for the Gumbel distribution

5.1 Introduction

Tests of fit based on the empirical distribution function (EDF) statistics A^2, W^2 and U^2 are developed for the problem of testing goodness-of-fit for the Gumbel distribution when the parameters are estimated from a sample censored at the right. The type of censoring considered here is known as *type II*, which corresponds to the situation in which, for fixed r, the $(n-r)$ largest observations are missing.

Such a problem has some importance due to its applications in extreme-value problems where it is often required to test goodness of fit for the tails of the distribution. In particular, the Anderson-Darling statistic is known to be a powerful statistic for detecting departures in the tails from the hypothesized distribution, which makes it a natural choice to be applied in this situation.

The distribution of a modified form of the Anderson-Darling and Cramér-von Mises statistics A^2 and W^2 has been empirically investigated in papers by Wozniak and Li [77] and by Aho, Bain and Engelhardt [2], [3] in connection with tests of fit for the two-parameter Weibull distribution. In the last two papers referenced above, the formula used to compute W^2 is that for the uncensored case and its use was proposed as a simplified form of the statistic. Tables obtained by simulation are given for some selected significance levels and censoring rates. In Wozniak and Li [77], a table of empirical percentage points of A^2 was given for selected censoring rates and a significance level 5% only. Although the formula for A^2 used in the simulations was not given, we believe (due to similarities in the results reported) that the formula for the uncensored case was also used.

5.2 Estimation and Test procedures

Let $x_{(1)}, \ldots, x_{(n)}$ denote the order statistics in a random sample of size n and suppose that we want to test the null hypothesis that the random sample was drawn from the distribution:

$$F(x;\alpha,\beta) = 1 - \exp(-\exp(\frac{x-\alpha}{\beta})) \quad , -\infty < x < \infty. \tag{5.1}$$

based on the r smallest order statistics $x_{(1)}, \ldots, x_{(r)}$ only.

The distribution (5.1) is known as the Gumbel or Type I extreme-value distribution for the minimum.

The test of fit will be based on the quantities $z_i = F\left(x_{(i)}, \hat{\alpha}, \hat{\beta}\right)$, the probability integral transformation with the parameters estimated by Maximum Likelihood. For the right-censored sample, the log-likelihood is given by

$$\lambda = -r \ln \beta - \sum_{i=1}^{r} \left[\frac{\alpha - x_{(i)}}{\beta} + \exp\left(\frac{\alpha - x_{(i)}}{\beta}\right)\right] - (n-r)\exp\left(\frac{\alpha - x_{(r)}}{\beta}\right) \tag{5.2}$$

The maximum likelihood estimators (MLE) are then the solutions of

$$\alpha - \beta \ln \left[\frac{\sum_{i=1}^{r} \exp(x_{(i)}/\beta) + (n-r)\exp(x_{(r)}/\beta)}{r}\right] = 0 \tag{5.3}$$

$$\frac{\sum_{i=1}^{r} x_{(i)} \exp(x_{(i)}/\beta) + (n-r)x_{(n-r)} \exp(x_{(r)}/\beta)}{\sum_{i=1}^{r} \exp(x_{(i)}/\beta) + (n-r)\exp(x_{(r)}/\beta)} - \beta - \frac{\sum_{i=1}^{r} x_{(i)}}{r} = 0 \tag{5.4}$$

Equation (5.4) does not depend on α and it is solved, usually by iteration, to obtain $\hat{\beta}$. The estimator of α is then obtained by substituting $\hat{\beta}$ into (5.3).

Once the parameters have been estimated, a test of fit can be carried out as follows:
a). Compute $z_i = F\left(x_{(i)}, \hat{\alpha}, \hat{\beta}\right)$

b). Calculate the EDF statistics in their version for a type II right-censored sample (see Stephens [17, Section 4.7]), namely

$$\begin{aligned}
A^2_{r,n} &= -\frac{1}{n}\sum_{i=1}^{r}(2i-1)\left[\ln z_{(i)} - \ln\{1-z_{(i)}\}\right] \\
&\quad -2\sum_{i=1}^{r}\ln\{1-z_{(i)}\} - \frac{1}{n}\left[(r-n)^2\ln\{1-z_r\} - r^2\ln z_r + n^2 z_r\right] \\
W^2_{r,n} &= \sum_{i=1}^{r}\left(z_{(i)} - \frac{2i-1}{2n}\right)^2 + \frac{r}{12n^2} + \frac{n}{3}\left(z_{(r)} - \frac{r}{n}\right)^3 \\
U^2_{r,n} &= W^2_{r,n} - nz_{(r)}\left[\frac{r}{n} - \frac{z_{(r)}}{2} - \frac{r\bar{z}}{nz_{(r)}}\right]^2
\end{aligned}$$

c). Using the value $q = 1 - r/n$, the percentage of right-censoring, refer to the tables given. If the value of the test statistic exceeds the value from the table, reject the null hypothesis at the corresponding significance level.

5.3 Asymptotic distributions

The asymptotic theory of the EDF-based statistics A^2, U^2, W^2 for doubly censored samples with known parameters (case 0) has been given by Pettitt and Stephens [43]. Pettitt [44] modified the theory given in Durbin [22] for testing normality from censored samples with parameters estimated by maximum likelihood. Here, the same procedure can be applied to find the asymptotic distribution of the test statistics for the Gumbel distribution.

In order to derive the asymptotic distribution of the test statistic it will be assumed that $q = 1 - p$ (with $p = r/n$), the proportion censored, remains constant as n tends to infinity.

For a singly right-censored sample, the process :

$$\sqrt{n}\left\{F_n(x) - F(x;\hat{\alpha},\hat{\beta})\right\}$$

evaluated at $\hat{t} = F(x;\hat{\alpha},\hat{\beta})$, converges weakly to a Gaussian process $\{Y(t) : t \in (0,p)\}$ whose covariance function is given by:

$$\rho(s,t) = min(s,t) - st - \mathbf{g}^{\top}(s)V\mathbf{g}(t) \quad \text{for} \quad 0 \le s,t \le p.$$

where $V = n\mathbf{I}^{-1}(\alpha,\beta)$ denotes Fisher's information matrix for the censored case, and $\mathbf{g}$ is the vector with components:

$$\begin{aligned} g_1(t) &= \frac{\partial F(x;\alpha,\beta)}{\partial\alpha} \\ g_2(t) &= \frac{\partial F(x;\alpha,\beta)}{\partial\beta} \end{aligned}$$

evaluated at $t = F(x;\alpha,\beta)$.

For the Gumbel distribution (5.1), the following quantities A_{11},A_{12},A_{21} and A_{22} were given in Harter and Moore [28].

$$\begin{aligned} A_{11} \equiv \lim_{n\to\infty} -\frac{\beta^2}{n}E\left[\frac{\partial^2\lambda}{\partial\alpha^2}\right] &= 1-q \\ A_{12} = A_{21} \equiv \lim_{n\to\infty} -\frac{\beta^2}{n}E\left[\frac{\partial^2\lambda}{\partial\alpha\partial\beta}\right] &= \Gamma'_{-\ln q}(2) - q\ln q\ln(-\ln q) \\ A_{22} \equiv \lim_{n\to\infty} -\frac{\beta^2}{n}E\left[\frac{\partial^2\lambda}{\partial\beta^2}\right] &= q - q\ln q\ln(-\ln q)\left[2+\ln(-\ln q)\right] \\ &\quad +2\left[\Gamma'_{-\ln q}(2) - \Gamma'_{-\ln q}(1)\right] + \Gamma''_{-\ln q}(2) - 1 \end{aligned}$$

where

$$\Gamma_w(z) = \int_0^w y^{z-1}e^{-y}dy$$

and Γ'_w, Γ''_w denote the first and second derivatives of $\Gamma_w(z)$ with respect to z.

If we let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

then the asymptotic covariance matrix of the maximum likelihood estimators is $\frac{\beta^2}{n}A^{-1}$.

We also have:

$$g_1(t) = \frac{(1-t)\ln(1-t)}{\beta}$$

$$g_2(t) = \frac{(1-t)\ln(1-t)\ln(-\ln(1-t))}{\beta}$$

Since $\rho(s,t)$ will not depend on the parameters, we take $\beta = 1$ to obtain:

$$V = A^{-1}.$$

The statistics $W^2_{r,n}$, $A^2_{r,n}$ and $U^2_{r,n}$ are, asymptotically, functions of the process $\{Y(t)\}$, that is,

$$W^2_{r,n} \xrightarrow{\mathcal{D}} \int_0^p Y^2(t)dt$$
$$A^2_{r,n} \xrightarrow{\mathcal{D}} \int_0^p a^2(t)dt$$
$$U^2_{r,n} \xrightarrow{\mathcal{D}} \int_0^p u^2(t)dt,$$

where

$$a(t) = \frac{Y(t)}{\sqrt{t(1-t)}}$$

and

$$u(t) = Y(t) - \frac{1}{p}\int_0^p Y(z)dz$$

are Gaussian processes defined in $(0,p)$ with zero mean and covariance functions respectively equal to

$$\rho_a(s,t) = \frac{\rho(s,t)}{\sqrt{(s-s^2)(t-t^2)}} \quad 0 \le s,t \le p.$$

$$\rho_u(s,t) = \rho(s,t) + \frac{1}{p^2}\int_0^p\int_0^p \rho(s,t)dsdt - \frac{1}{p}\int_0^p \rho(s,t)ds - \frac{1}{p}\int_0^p \rho(s,t)dt \quad 0 \le s,t \le p.$$

Let $\rho^*(s,t)$ denote the covariance function of the asymptotic process for a given statistic. The limiting distribution of the test statistic (see for example Durbin [22]) is that of

$$\sum_{i=1}^{\infty} \lambda_i \nu_i$$

where $\nu_1, \nu_2, \ldots$ are independent $\chi^2_{(1)}$ variables and $\lambda^*_1, \lambda^*_2, \ldots$ are the eigenvalues of the integral equation

$$\int_0^p \rho^*(s,t) f_i(s)ds = \lambda^*_i f_i(t) \tag{5.5}$$

5.4 Calculation of asymptotic percentage points

The percentage points of the asymptotic distribution of the test statistics were found numerically using 50 points in $(0, p)$ to approximate the integral and solve (5.5). In the 50 by 50 grid, the appropriate covariance function was evaluated and the eigenvalue problem solved for values of $p = 0.05(0.05)0.95$. Once the eigenvalues were obtained, the asymptotic percentage points were calculated using Imhof's method [30]. The results are shown in tables 5.1 to 5.3. As a check of accuracy, the procedure was repeated using a 100×100 grid and the results compared. The maximum difference encountered was less than one unit in the fourth decimal place, except for a few values corresponding to $p > 0.9$.

Table 5.1: Asymptotic percentage points of $A^2_{r,n}$, for values $q = 1 - r/n$

	Significance level							
q	0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.00	0.342	0.472	0.563	0.634	0.756	0.878	1.041	1.165
0.05	0.297	0.415	0.498	0.564	0.676	0.790	0.941	1.057
0.10	0.267	0.376	0.453	0.513	0.617	0.722	0.863	0.970
0.15	0.242	0.343	0.414	0.470	0.567	0.664	0.795	0.895
0.20	0.221	0.313	0.379	0.431	0.521	0.612	0.733	0.826
0.25	0.201	0.287	0.347	0.396	0.479	0.563	0.676	0.763
0.30	0.183	0.262	0.318	0.362	0.439	0.517	0.622	0.702
0.35	0.166	0.238	0.290	0.331	0.402	0.474	0.570	0.644
0.40	0.150	0.216	0.263	0.301	0.366	0.432	0.520	0.589
0.45	0.135	0.195	0.238	0.272	0.331	0.391	0.472	0.535
0.50	0.121	0.174	0.213	0.244	0.297	0.352	0.425	0.482
0.55	0.107	0.155	0.189	0.217	0.265	0.314	0.380	0.430
0.60	0.093	0.136	0.166	0.190	0.233	0.276	0.335	0.380
0.65	0.080	0.117	0.144	0.165	0.202	0.240	0.291	0.330
0.70	0.068	0.099	0.122	0.140	0.171	0.204	0.247	0.281

Table 5.2: Asymptotic percentage points of $U^2_{r,n}$, for values $q = 1 - r/n$

q	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.00	0.0482	0.0696	0.0849	0.0969	0.1176	0.1382	0.1657	0.1866
0.05	0.0451	0.0652	0.0796	0.0909	0.1103	0.1298	0.1557	0.1754
0.10	0.0416	0.0602	0.0736	0.0841	0.1022	0.1203	0.1445	0.1628
0.15	0.0379	0.0550	0.0673	0.0770	0.0936	0.1103	0.1325	0.1494
0.20	0.0341	0.0497	0.0608	0.0696	0.0848	0.0999	0.1201	0.1355
0.25	0.0304	0.0444	0.0544	0.0623	0.0759	0.0896	0.1077	0.1216
0.30	0.0268	0.0392	0.0481	0.0552	0.0673	0.0794	0.0956	0.1079
0.35	0.0234	0.0343	0.0421	0.0483	0.0589	0.0696	0.0838	0.0946
0.40	0.0201	0.0295	0.0363	0.0417	0.0509	0.0602	0.0725	0.0818
0.45	0.0171	0.0251	0.0309	0.0354	0.0433	0.0512	0.0618	0.0698
0.50	0.0142	0.0209	0.0258	0.0296	0.0362	0.0429	0.0517	0.0584
0.55	0.0116	0.0171	0.0211	0.0242	0.0297	0.0351	0.0424	0.0480
0.60	0.0092	0.0136	0.0168	0.0193	0.0237	0.0281	0.0339	0.0384
0.65	0.0071	0.0105	0.0130	0.0149	0.0183	0.0217	0.0262	0.0297
0.70	0.0052	0.0078	0.0096	0.0111	0.0136	0.0161	0.0195	0.0221

Table 5.3: Asymptotic percentage points of $W^2_{r,n}$, for values $q = 1 - r/n$

q	Significance level 0.500	0.250	0.150	0.100	0.050	0.025	0.010	0.005
0.00	0.0503	0.0729	0.0891	0.1018	0.1238	0.1458	0.1750	0.1973
0.05	0.0474	0.0688	0.0841	0.0963	0.1172	0.1383	0.1664	0.1878
0.10	0.0440	0.0642	0.0786	0.0902	0.1099	0.1299	0.1566	0.1770
0.15	0.0405	0.0592	0.0728	0.0835	0.1020	0.1207	0.1458	0.1650
0.20	0.0368	0.0541	0.0666	0.0766	0.0937	0.1111	0.1345	0.1524
0.25	0.0331	0.0489	0.0604	0.0695	0.0852	0.1012	0.1227	0.1392
0.30	0.0295	0.0438	0.0541	0.0624	0.0767	0.0913	0.1109	0.1259
0.35	0.0260	0.0387	0.0479	0.0553	0.0682	0.0813	0.0990	0.1126
0.40	0.0225	0.0337	0.0419	0.0485	0.0599	0.0715	0.0872	0.0993
0.45	0.0193	0.0290	0.0361	0.0418	0.0517	0.0619	0.0757	0.0863
0.50	0.0162	0.0244	0.0305	0.0354	0.0440	0.0527	0.0646	0.0737
0.55	0.0133	0.0202	0.0253	0.0294	0.0366	0.0439	0.0539	0.0616
0.60	0.0107	0.0163	0.0204	0.0238	0.0296	0.0357	0.0439	0.0502
0.65	0.0083	0.0127	0.0160	0.0186	0.0232	0.0280	0.0346	0.0396
0.70	0.0062	0.0095	0.0120	0.0140	0.0175	0.0211	0.0261	0.0300

5.5 Small sample distributions

In order to investigate the speed of convergence to the asymptotic points, a simulation study was conducted using 10,000 samples. The results showed that the speed of convergence decreases as the rate of censoring increases. From tables 5.4 - 5.6, it appears that when the percentage of censoring is not too large (say, $q < 0.7$) the asymptotic tables can be used with good accuracy for $n > 20$. For heavily censored samples ($q \geq 0.7$) larger sample sizes will be required, but of course one would expect difficulties in testing for a distribution with over 70% of the observations censored.

Table 5.4: Empirical percentage points of $W^2_{r,n}$

		Significance level						
q	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.10	20	0.045	0.064	0.078	0.088	0.107	0.126	0.153
	40	0.044	0.064	0.078	0.089	0.110	0.130	0.157
	60	0.044	0.064	0.078	0.089	0.108	0.128	0.153
	80	0.044	0.064	0.079	0.090	0.110	0.131	0.160
	∞	0.044	0.064	0.079	0.090	0.110	0.130	0.157
0.30	20	0.031	0.044	0.053	0.061	0.075	0.088	0.103
	40	0.030	0.044	0.054	0.063	0.078	0.092	0.110
	60	0.030	0.044	0.054	0.062	0.076	0.091	0.109
	80	0.030	0.044	0.053	0.062	0.076	0.090	0.111
	∞	0.030	0.044	0.054	0.062	0.076	0.091	0.111
0.50	20	0.014	0.021	0.026	0.030	0.036	0.042	0.052
	40	0.015	0.023	0.028	0.033	0.040	0.048	0.058
	60	0.016	0.023	0.029	0.033	0.042	0.050	0.062
	80	0.016	0.024	0.030	0.034	0.042	0.050	0.062
	∞	0.016	0.024	0.030	0.035	0.044	0.053	0.065

Table 5.5: Empirical percentage points of $A^2_{r,n}$

		Significance level						
q	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.10	20	0.273	0.377	0.450	0.506	0.600	0.707	0.837
	40	0.268	0.373	0.446	0.508	0.618	0.713	0.847
	60	0.269	0.376	0.4531	0.513	0.611	0.721	0.866
	80	0.267	0.377	0.454	0.513	0.623	0.722	0.878
	∞	0.267	0.376	0.453	0.513	0.617	0.722	0.863
0.30	20	0.188	0.265	0.316	0.360	0.433	0.509	0.595
	40	0.186	0.263	0.323	0.372	0.448	0.523	0.614
	60	0.183	0.262	0.316	0.362	0.438	0.513	0.613
	80	0.183	0.259	0.319	0.361	0.438	0.518	0.626
	∞	0.183	0.262	0.318	0.362	0.439	0.517	0.622
0.50	20	0.112	0.159	0.192	0.220	0.264	0.306	0.358
	40	0.115	0.167	0.203	0.232	0.286	0.331	0.394
	60	0.117	0.168	0.205	0.236	0.289	0.341	0.414
	80	0.120	0.172	0.209	0.238	0.290	0.344	0.415
	∞	0.121	0.174	0.213	0.244	0.297	0.352	0.425

Table 5.6: Empirical percentage points of $U^2_{r,n}$

		Significance level						
q	n	0.500	0.250	0.150	0.100	0.050	0.025	0.010
0.10	20	0.042	0.060	0.072	0.081	0.098	0.117	0.139
	40	0.042	0.060	0.072	0.083	0.101	0.119	0.143
	60	0.042	0.060	0.073	0.084	0.100	0.119	0.142
	80	0.042	0.060	0.074	0.084	0.102	0.120	0.145
	∞	0.042	0.060	0.074	0.084	0.102	0.120	0.144
0.30	20	0.028	0.040	0.048	0.054	0.066	0.076	0.091
	40	0.027	0.040	0.048	0.056	0.068	0.080	0.094
	60	0.027	0.039	0.048	0.055	0.067	0.079	0.096
	80	0.027	0.039	0.048	0.054	0.066	0.079	0.094
	∞	0.027	0.039	0.048	0.055	0.067	0.079	0.096
0.50	20	0.013	0.018	0.022	0.025	0.030	0.035	0.040
	40	0.013	0.020	0.024	0.027	0.033	0.039	0.046
	60	0.014	0.020	0.024	0.028	0.034	0.041	0.049
	80	0.014	0.020	0.025	0.028	0.034	0.041	0.050
	∞	0.014	0.021	0.026	0.030	0.036	0.043	0.052

Appendix A

Data sets

Table A.1: Data set 1: Cycles to failure of springs

117.00	135.00	135.00	162.00	162.00
171.00	189.00	189.00	198.00	225.00

Table A.2: Data set 2: Times to failure of air conditioning equipment

12.00	21.00	26.00	27.00	29.00
29.00	48.00	57.00	59.00	70.00
74.00	153.00	326.00	386.00	502.00

Table A.3: Data set 3: Artificial

0.27	0.47	0.50	0.54	0.62
0.80	0.93	1.03	1.29	1.29
1.29	1.38	1.40	1.41	1.42
1.50	1.52	1.54	1.68	1.74

Table A.4: Data set 4: Distances to a nuclear power plant

58.20	58.20	59.50	61.80	65.80
67.80	68.50	70.90	73.70	77.00
80.80	83.70	84.30	89.00	97.60
98.30	99.60	101.40	105.10	105.80
106.70	119.10	119.50	119.90	121.90
125.70	128.40	146.10	153.90	154.60
155.80	157.40	157.70	163.70	170.10
172.70	173.00	173.90	174.20	175.10
178.70	179.50	180.70	182.10	182.70
186.70	187.50	191.00	192.60	193.00
199.40	211.60	212.10	216.80	222.90
227.30	229.40	234.50	236.80	238.90

Table A.5: Data set 5: Fatigue strengths of wire

39611.00	44132.00	44209.00	45898.00	50139.00
54625.00	58970.00	64703.00	64950.00	66508.00
70208.00	72098.00	75001.00	80393.00	81868.00
82202.00	82447.00	89268.00	90021.00	96136.00
96723.00	101610.00	101833.00	106055.00	112833.00
119154.00	122366.00	134511.00	135220.00	136395.00
138378.00	153790.00	184916.00	216370.00	240316.00

Table A.6: Data set 6: Times to failure of an electronic module

1.40	3.30	5.10	9.20	10.60
10.60	12.40	13.50	16.10	16.70
17.00	17.50	17.90	19.20	20.60
22.60	26.70	27.60	33.10	34.50
34.90	35.00	36.10	36.90	37.20
37.30	41.50	44.70	50.70	51.80
59.90	60.50	63.20	68.80	72.60
89.10	90.30	98.90	107.20	123.30
173.50				

Table A.7: Data set 7: Strengths of glass fiber (15 cm.)

0.55	0.74	0.77	0.81	0.84
0.93	1.04	1.11	1.13	1.24
1.25	1.27	1.28	1.29	1.30
1.36	1.39	1.42	1.48	1.48
1.49	1.49	1.50	1.50	1.51
1.52	1.52	1.54	1.55	1.55
1.58	1.59	1.60	1.61	1.61
1.61	1.61	1.62	1.62	1.63
1.64	1.66	1.66	1.66	1.67
1.68	1.68	1.69	1.70	1.70
1.73	1.76	1.76	1.77	1.78
1.81	1.82	1.84	1.84	1.89
2.00	2.01	2.24		

Table A.8: Data set 8: Strengths of glass fiber (1.5 cm.)

0.37	0.40	0.70	0.75	0.80
0.81	0.83	0.86	0.92	0.92
0.94	0.95	0.98	1.03	1.06
1.06	1.08	1.09	1.10	1.10
1.13	1.14	1.15	1.17	1.20
1.20	1.21	1.22	1.25	1.28
1.28	1.29	1.29	1.30	1.35
1.35	1.37	1.37	1.38	1.40
1.40	1.42	1.43	1.51	1.53
1.61				

Table A.9: Data set 9: Wind speeds

22.64	22.80	23.75	24.01	24.04
24.24	24.74	25.45	25.55	25.66
25.99	26.63	26.69	26.88	26.89
27.12	27.43	27.69	27.71	28.12
28.58	28.88	29.12	29.45	29.48
30.18	31.31	31.55	31.57	32.54
32.98	33.83	33.86	34.64	35.21
36.82	37.23	38.09	38.26	38.82
38.90	38.96	42.99	43.66	44.61
45.24	47.91	54.75	69.40	98.19

Table A.10: Data set 10: Sea waves

2.91	6.93	9.17	10.28	11.82
13.88	14.86	17.36	21.06	24.75
3.74	7.21	9.50	10.45	12.27
13.98	15.03	18.68	21.13	25.45
4.09	7.92	9.62	10.77	12.68
14.32	15.30	18.72	21.53	28.13
5.88	8.26	10.00	11.65	13.28
14.38	16.07	19.44	21.80	29.95
6.42	8.79	10.14	11.65	13.46
14.46	16.23	20.09	23.15	37.19

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