Numbers Externalities

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Abstract

Interaction in the presence of externalities is what makes our science a *social* science. Outcomes in such situations rarely have the desirable efficiency properties economists have come to use for normative purposes. Pareto optimality often just a fluke or not possible at all.

One particular kind of externality is of interest in this thesis: a "numbers" or "network" externality that is propagated by individuals engaging in a similar activity. Modeling behaviour in the presence of numbers externalities is made difficult because, cast as games, such situations tend to have many equilibria, and discerning the most probable is difficult. This thesis documents the nature of this difficulty in a variety of settings, and discusses the properties of equilibria and mechanisms by which they can be reached.

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Contents

List of Tables

List of Figures

Chapter 1

Methodological Preliminaries

1.1 Modeling Social Institutions.

In the modeling of the nature and evolution of a social institution it is natural that at times we refer to the behaviour of what are actually conceptual entities, although the word "behaviour" is, in some sense, inappropriate. To be clear, we might talk about the prevalence of the use of the English language through time, or we might talk about the use of contractions in speech or the onset of regional accents, without immediate reference to the fact that we are really describing the manifestation of the activities of many individual people. It would be silly if we were to refer to "the English language" directly as a sentient being, with a mind and will of its own, with its own anthropomorphic desires, and use such a notion to explain the widespread use of the English language today. Less obviously, but more importantly, it may be inappropriate to model social institutions even *as-if* they were sentient beings: if "the English language" were a conscious entity, with its goal being to facilitate communication in the most efficient manner, then it would surely choose for itself some incarnation other than the unwieldy and irregular version we see today.

1.1.1 'As-if' theorising.

The kind of "as-if" theorising I am concerned with here takes the form of the supposition that the composition of many individuals' activities can be more conveniently treated as the single decision of some metaphysical aggregate entity. Endowing the entity with appropriate goals, or an "objective function," allows us to use a convenient expository device, in that we need only refer to the entity and its single choice, and not to the sum of the choices of many individuals. We hope, in using such a theoretical manoeuver, that the details we have omitted are unimportant—or "averaged out." A price index summarises the movements in many prices, and is indeed a convenient summary of those movements. All the descriptive statistics economists are so familiar with exemplify this process of summarisation when large amounts of data are to be referred to without simply reproducing that data.

The most famous example in economics of as-if theorising concerns the invocation of a "social planner" in general equilibrium models. The social institution in question is market-mediated exchange, which is a truly amazing feat of coordination by which individuals specialise in production, and then exchange their specific products for a variety of other products, produced by others. The as-if theory suggests that if a social planner were to force individuals to produce particular products in particular amounts so as to maximise a (carefully chosen) social welfare function, and in particular the social planner were to make sure that the final pattern of production and exchange were to obey the dictates of Pareto Optimality, then the result would be identical to that which would obtain if the many fold individuals were to make these decisions independently in the pursuit of their own personal welfare. This is Adam Smith's "beatific vision" by which the uncoordinated attempts of private individuals coordinate themselves "as-if guided by some invisible hand."

Naturally, if we are to be sure that the metaphysical entity's choices exactly mimic what the sum of many individuals' choices would have been, we must have a clear idea of what the motives and capabilities of those individuals actually are. Satisfied that the as-if analogy is appropriate, the idea of the entity can be safely used, making a presentation of a theory less cumbersome, just as using a price index makes references to changes in the cost of living less cumbersome than it would be if we were to refer to every individual price movement. Similarly, we talk of capital "moving" from one industry to another, so that "capital" can equalise "its" rate of return across industries, even though nobody has visions of disgruntled machinery trekking across the land in search of its fortune, it is just a manner of speech[1]. If we are not careful, however, and we use an as-if entity analogy without being sure of its appropriateness as a heuristic device, we may find ourselves unwittingly making arguments which require such an absurd literal translation.

The focus on individuals has always been thought to be an important part of economic methodology, a feature termed *methodological individualism.* The main proponent of individualism has been Popper, as stated in his criticism of Marxian method in Popper [1976]. Despite this lip service to individualism, students of economics often find themselves studying the "theory of the firm" and read of 'firms' maximising profit. To some extent this is a manner of speaking[2], but Harvey Leibenstein sees it as a serious defect. For example, in Leibenstein [1976] we find the following passage[3]:

> "Suppose a world in which all households and firms are made up of single individuals. [...] In their capacity as consumers (that is, households), individuals maximize utility, and in their capacity as firms they maximize profits. Since firms are single individuals no labour as such is bought and sold. As a consequence, the evaluation of labour's productivity within the firm does not enter explicitly as a problem. Now suppose we try to generalize the results and shift to multi-person households and firms, and we *assume* that they behave the same way, in terms of basic motivations and responses to motivations, they would behave [as] if they were individuals. We then proceed to develop the implication of such a theory. It seems to me that one can speak of a logical leap in this area in the sense that

[1] We also talk of "demand" and "supply" as "shifting" in response to exogenous parametric changes—amd use geometric entities to depict this—all as shorthand for the aggregated responses of an underlying body of pertinent individuals.

[2] Sherman [1982] argues that this was true of Marx too, despite Popper's criticisms.

[3] This is on pp. 4-5. The word "as" enclosed in square brackets does not appear in the original text.

> we do not examine in detail whether such a shift from single-person to multi-person units can be made without involving possible internal contradictions, and under what circumstances this transition can be carried out. By and large this question has not been examined: it has been silently passed over, as it were."

In this regard, Leibenstein's work is sometimes described as 'micro-micro-theory.' Leibenstein sometimes describes it as 'atomistic' economics, and contrasts it with the more aggregate 'molecular' economics.

1.2 Sealing-Off

When as-if theorising is warranted, we effectively 'seal-off' our inquiry from underlying details. This general method is discussed a little in Hofstadter [1989, p. 305], and the notion (and phrase) belong to Simon [1969]. Hofstadter puts it this way:

> "[...] a nuclear physicist can proceed with theories of nuclei that are based on protons and neutrons, and ignore quark theories and their rivals. The nuclear physicist has a *chunked* picture of protons and neutrons—a description derived from lower-level theories but which does not require understanding the lower-level theories. Likewise, an atomic physicist has a chunked picture of an atomic nucleus derived from nuclear theory. Then a chemist has a chunked picture of the electrons and their orbits, and builds theories of small molecules, theories which can be taken over in a chunked way by the molecular biologist, who has an intuition for how small molecules hang together, but who's technical expertise is in the field of extremely large molecules and how they interact. Then the cell biologist has a chunked picture of the units which the molecular biologist pores over, and tries to use them to account for the ways that cells interact. The point is clear. Each level is, in some sense, "sealed off" from the levels below it."

In economics it has been traditionally deemed appropriate to view a firm as a decision-making entity, sealing-off from the underlying social structure. Using 'representative agents' it is also possible to think of the behaviour of 'market demand.' However, the observation that firms have a good deal of important and interesting internal structure has led to a fruitful line of enquiry, and sealing-off the interbal machinations of a firm by positing a profit-maximising box has obscured these issues, and can be shown to be counterfactual in the case in which externalities between workers cause them to fall into a Pareto inferior equilibrium akin to the one that occurs in the Prisoner's dilemma game.

1.3 The Invisible-Hand Analogy

It is well known that the social-planner analogy breaks down in the presence of "externalities." If one individual's pursuit of self-interest impinges upon another's ability to do the same, then the social planner could, under almost all circumstances, make both better-off than they would be acting independently. And this would imply that the social planner's decision does not mimic precisely that of the individuals. A nice analogy is to a multi-variate optimisation problem. Suppose we believe the unfettered market system leads to individual decisions that maximise a social welfare function, as-if guided by an invisible hand. If we, rather abstractly, suppose that each individual chooses some anonymous choice variable, x_i, where i runs from 1 to n, the number of individuals. Let $\mathbf{x}$ be a vector of the x_i. We postulate a social-welfare function, W, that has enough desirable properties to justify what follows. A state of the world in this scenario is a particular realisation of $\mathbf{x}$. Individuals have preferences over possible states of the world representable by utility functions, $U_i(\mathbf{x})$. The social-welfare function will be of the form $W(U_1, \ldots U_n)$. The invisible hand would solve the following problem:

$$\frac{\partial W}{\partial U_i}\frac{\partial U_i}{\partial x_i} + \sum_{j \neq i} \frac{\partial W}{\partial U_j}\frac{\partial U_j}{\partial x_i} = 0 \qquad i = 1, \ldots n.$$

Individuals, left alone, would solve the following:

$$\frac{\partial U_i}{\partial x_i} = 0 \qquad i = 1, \ldots n.$$

The difference between these two sets of n equations lies in the fact that individuals acting in their own self-interest do not account for externalities, the term $\sum_{j\neq i} \frac{\partial W}{\partial U_j}\frac{\partial U_j}{\partial x_i}$, hence there is not much reason to suppose independent individuals will act like an invisible hand.[4]

1.4 Numbers Externalities

The presence of an externality implies that if there were some mechanism for coordinating the activities of individuals so that they can all be made better-off, then such a mechanism is desirable. This mechanism, of course, is the social institution of "government." One of the chief reasons representative government can be justified is that individual rationality does not imply social rationality, and a central coordinating mechanism can rectify this—a *visible* hand.

Under certain circumstances in the presence of externalities, the potential for mutually beneficial coordination occurs without the central direction of the representatives of some agency. Whether all opportunities for mutual gain are exhausted is not entirely clear, as we shall see.

Previously it was argued that "as-if" theorising should only be treated as a convenient expository device—"a manner of speaking"—and that we should be careful about the appropriateness of this manner of speech. In this section I turn to a more specific analysis of a class of social phenomena which may not lend themselves to as-if theorising.

A Working Definition

A 'numbers' or 'network' externality exists when the utility, if you will, of engaging in an activity (or one of a range of activities), depends upon the number of other people engaging in that (or one of the alternative) activities. For example, the enjoyment I experience from visiting a local park will be dependent in part upon the number of

[4]The generic failure of the invisible hand in simple games is explored in Eaton and Eswaran [1993]. There is a book that documents the differences in the two first-order conditions in this section in a variety of contexts, Shandler [1992].

others choosing to do so at the same time. An often mentioned example of such a network externality concerns the use of telephones: if you are the only person with one it is totally useless, but as the number of individuals with 'phones increases, the usefulness of your own 'phone increases (well, up to a point anyway). In the presence of such numbers externalities we have observed that there may occur an apparent mass coordination of usually disassociated persons without some "announcement" that such coordination is in anyone's or everyone's benefit. Such is the case with language (within certain confines); individuals speaking English to one another have apparently mastered a feat of coordination without consciously recognising it. When such coordination appears stable then we have what I have been referring to as a "social institution." Of course, the same applies to any group of individuals speaking the same language, not just English.

Many social interactions are only worthwhile if they are undertaken with the consent, or at least the acknowledgement of others. Quite often this is all that is required. Perhaps the *importance* of many events is established by consent, and nothing more. As an example, consider sporting events. As something of an outsider I have been able to be an impassive witness to the fanaticism displayed by Canadians for the professional baseball team the 'Toronto Blue Jays' which won the coveted 'World Series' in 1992 and 1993. We can ask: what is the source of this emotional tie to a sports team? Such a question can be answered on different levels. We can argue that individuals gain vicarious pleasure from the activities of the players, and that since winning the world series is an important event, each spectator can feel for or with the players as they win. Such an argument is false however, since there would be no reason for sports fans to 'support' one team or another. Moreover, the immense joy experienced by the players is derived from the fact that winning the world series is important—important to the vast majority that do not participate in professional baseball. So we are still left with the question of 'importance.' My friends tell me that there is some 'national' pride in the success of a Canadian team, which must compete with many American teams. The Toronto Blue Jays, in fact, have just one Canadian player (who plays only a very minor role), and none of the senior coaching staff are Canadian. It just happens that the team plays in Canada. Under such circumstances it is hard to understand any nationalism. At least on one level. It is my opinion that the importance attached to professional sports teams is a particular kind of 'bootstrapping equilibrium.' This particular team acts as a focus for Canadians, just as the 'Minnesota Twins' act as a coordinating device in Minneapolis and so on. This suggests that individuals like to have similar mood swings, and one way of solving such a coordination game is focus attention on a professional sports team. The fact that individuals living in the same region tend to support the same team is perhaps just an indication that the local team is a *focal point*—it is a manifest option for those engaged in a coordination game[5]. The desire to be similar to others *per se* is a theme explored in Jones [1984]. Similar work on 'herd' beahviour and 'crowd psychology' exists in the psychology literature.

1.5 Bootstrapping Equilibria

Social institutions apparently engendered by numbers externalities often have the interesting "bootstrapping" property, in that we often find ourselves arguing in the circuitous fashion; "the reason this many people do X is because this many people do X ..." For example, many popular bars are popular because, well, because they are popular. Along similar lines, the reason so many people choose to learn English as a second language (in contemporary times) is precisely because so many people have in the past and continue to do so.

Large public lotteries often have prizes proportional to the number of tickets sold, so that an individual purchasing a ticket confers two externalities upon other lottery participants: on the one hand he or she has increased the size of the jackpot, and on the other hand he or she has increased the chances that the jackpot is shared. While the effect of the former is greater than that of the latter the number of participants will increase, but this seems unlikely to be the case that this is true with a small number of participants. If this is allowed, then the large number of tickets sold probably reflects a self-fulfilling prophecy that a large number will be sold, and as such we have

[5]The notion of a focal point, which will recur in what follows, is, of course, due to Schelling [1960]

another example of a bootstrapping equilibrium.[6] There is a distinction to be made between examples of equilibria potentially generated by some evolutionary, dynamic process in which individuals gradually organise themselves into a coordinated mass, and equilibria which occur due to the kind of self-fulfilling prophecy referred to above.

1.6 Bandwagon Effects

The kind of coordination I have been referring to is sometimes called a *bandwagon effect* and is well documented. There is a brief summary in Hargreaves Heap *et al* [1992, pp. 291-294].[7] Leibenstein [1950, 1976, chapter 4] also discusses these kind of effects. In the latter work Leibenstein identifies three similar effects in demand behaviour. A *bandwagon effect* as one, in the context of consumer demand, one individual will demand more (or less) simply because others demand more (or less). Leibenstein includes a careful graphical study of the analysis of comparative statics under such circumstances. The effect is purest if their are no externalities between consumers other than those caused by the *knowledge* that others are making purchases. A *snob* effect is the opposite of a bandwagon effect, and in the same context this effect takes the form of a negative correlation between personal demand and market demand. We can also talk of a *Veblen* effect, after Veblen's theories of conspicuous consumption.[8] Such an effect occurs when preferences over bundles of goods are not independent of prices, and in general an increase in price has three effects; income, substitution, and Veblen effects.

Of these effects, the Veblen effect is not my concern here. The other two effects provide some insight into the kind of coordination problem and multiple equilibria that can occur with numbers externalities, a theme of this thesis. Let us examine a market characterised by these effects (this differs considerably from the analysis in

[6] These lotteries have been around for hundreds of years. They were very popular in France during the 17th and 18th centuries, which may partly explain the superb tradition of french probability theorists! I came across references to these lotteries in Keynes [1921].

[7] The summary there is very similar to discussions in Schelling [1978], I have amended both expositions a little to integrate with what follows.

[8] Veblen was certainly not the first to elaborate such theories, see Leibenstein [1976, pp. 48–49].

Leibenstein [1975, 1976]). The bandwagon effect relates an individual's demand to his perception of market demand. Consider the market for a commodity in which n consumers participate. Each of then conjectures a market demand curve, excluding their own, of the form $D(p)$. A convenient assumption is that each individual wishes to purchase an "average" amount, so that is the market price is p, each consumer, on the assumption that they have conjectured the same market demand function, will desire to purchase $D(p)/n$ units of X. This is indeed a rational expectations equilibrium, whatever the conjectured $D(p)$.[9] There is a curious relationship between consumers under such circumstances, in that their consumption decision is based entirely on that of others. To see the coordination problem, consider these consumers as involved in a game, in which their action sets are individual demand functions, say $d_i(p)$, and utility is 1 if:

$$d_i(p) = \sum_j d_j(p)/n$$

and zero otherwise. This game has an infinite number of Nash equilibria, all those in which $d_i(p) = d_j(p)$, none of which are demonstrably more likely to occur than any others, hence the coordination problem.

In everyday speech the term 'bandwagon' has a somewhat different meaning to the one detailed above. The word itself can be traced back to 1855 according to the *Websters* electronic dictionary, when it referred to an ornate wagon used for the musicians taking part in a circus parade. Presumably this was the centre of attention during the parade, and I suppose the general public liked to join in the revelry by jumping on the wagon itself. The word is now used to represent (to quote from the dictionary) "a party, faction, or cause that attracts adherents or amasses power by its timeliness, showmanship, or momentum." The thesaurus offers the alternatives "fad, craze, rage." To model demand for a product subject to these effects requires something a little different from Leibenstein's characterisation, I think. This meaning of bandwagon seems to suggest that individuals will engage in consumption of a

[9] I am not sure of the choice theoretic implications of such formulations are, but representing preferences and deriving associated maximum value functions would be an interesting enterprise. It may well have been undertaken, but I have not come across anything since research for this thesis has been along different lines.

product (to one degree or another) is there is ***sufficient*** demand from others. This notion of a *critical mass* is important in the kind of games I will be considering.

The snob effect is subtly different. A pure bandwagon effect obtains in response to a critical mass. A snob effect seems more like a coordination device in which one group of individuals try to distinguish themselves from another. Let us again think of individuals in a market who are snobs, in the sense defined by Leibenstein.[10] One characterisation, that is convenient here, is that individuals will buy one unit of a good, providing there are *less than* a certain number of others doing so.[11]. If there are n identical individuals, and if the critical number is c, then there are ${}_nC_c$ Nash equilibria, and again the assumed anonymity in this game does not allow us to distinguish between Nash equilibria, and more importantly nor can the players. A Nash equilibrium is a collection of strategies, so it would be inappropriate to say that there is *one* Nash equilibrium, that in which c consumers buy the product and $(n-c)$ do not. In buying one unit of an indivisible good, and nothing thereafter, it makes little sense to speak of mixed-strategies. In this way, high ticket items, purchased by snobs, allow a small group of snobs to enjoy their mutual appreciation by excluding low income individuals from a market. Hence, there is an exclusivity associated with certain items that are expensive, and this is independent of any intrinsic qualities of those items, it is just a coordination device.[12] This grouping by income by purchasing hight price items seems to underly what we have referred to above as the 'Veblen Effect' as the implication is that price and demand are related.

1.6.1 Functional and Non-Functional Demand

Leibenstein refers to two components of demand when dealing with bandwagon effects (and the others), *functional demand* is that for a good for its own intrinsic properties, whereas *non-functional demand* is that for a good because of the fact that others do

[10] Again, the dictionary definition, and common usage, differs somewhat from this characterisation. A 'snob' is typically someone with an artificial or assumed air of superiority in matters of knowledge or taste.

[11] This differs from the Leibenstein exposition

[12] A related notion can be found in Eaton and White [1992] in which agents wish to interact with particular types of others, and find convenient coordination devices to do so.

or do not buy it. I would like to pose a novel thought experiment for distinguishing between the two. Let an individual's demand function be given by $d_i(p; D(p))$. Now $D(p)$ is made up of the decision of others, $n-1$ of them. If this individual is told that a subset of these others, say persons j, k, l, are not human beings at all, but are robots that have been programmed to implement a particular individual demand function. If $d_i(p; D(p)$ *changes* when this information is revealed, then the change can only have occurred because i, j, k are not human beings, and hence identifying with their social group, or relying on their reverence for your purchasing power or somesuch, is meaningless: some consumption externalities are engendered by an affinity between human beings. This is to be contrasted with the type of externality that exists between purchasers of a product who find the usefulness of what they purchased increases with the amount purchased in total becuase there are improved complementary products that emerge to take advantage, which occurs in many consumer electronics industries.

1.7 Sub-Optimality and Path Dependence

The fact that, for example, a *particular* bar is popular, or that a particular language is adopted so frequently, may have little to do with (and may be *in spite of*) the characteristics peculiar to that bar or that language. Thus it would be fallacious to argue that the popularity of English is derived from any intrinsic suitability of that language in communication. Hence if one were to model language acquisition *as-if* individuals chose the most efficient means of communicating a given idea, one would be making a mistake. It seems that the apparent trend towards English as a common language from among the range of possibilities, exhibits a fairly obtrusive path dependency (England's domination of world trade over 100 years ago), and that the result—the overriding popularity of English—has few "optimality" properties; presumably most people would agree that if the world populous were given the option of speaking one language, and the transition to its universal use would be costless, then it seems unlikely that the chosen language would be this rather irregular version called "English".

The idea of "path dependency" has several connotations, but in this context I infer it to mean that if a numbers externality exists between individuals choosing one from a range of alternative actions, and there are many distributions of the numbers of people engaging in each of the possible actions that are "equilibria," and that which of these particular equilibria emerge will depend upon many things—many of which are *exogenous* to the analysis. As a simple example, consider a small community of twenty individuals deciding upon which of two bars to patronise one evening. If all of these individuals prefer the bar with more people in, then there are two "equilibria;" one bar with the twenty people in (and the other empty) or vice versa. *Which* of the two bars attracts the band of revelers will depend upon assumptions regarding customer movements between the bars, and an initial distribution of customers. In simple models assumptions regarding out-of-equilibrium dynamics may be a meaningful extension of the analysis, in more complicated scenarios it may not.[13]

In particular, if there simply is no identifiable real-time process in the context, using some theoretical analogy to one in order to explain which of several equilibria may obtain is inappropriate.

Perhaps the best documented form of this bootstrapping equilibrium in economics is the use of intrinsically useless media of exchange: "money." The acceptance of paper money in exchange for real goods and services is an act undertaken on the understanding that the paper money will, at some future point, be accepted by some other individual in return for other goods and services: money is universally acceptable because money is universally accepted. This raises some interesting questions about the characteristics of the objects used as media of exchange. We might all agree that the desirable characteristics include portability, durability, scarcity, that the object should be difficult to reproduce and so on. That is, we may all have the same notion of an *ideal* medium of exchange. Having said this, though, there is no reason to

[13]This kind of analysis is Schelling's concern in Schelling [1978]. The examples contained in this eminently readable book serve to highlight the multiplicity and (in many cases) sub-optimality of equilibria in the presence of numbers externalities of one form or another. Usually Schelling also considers the process by which equilibrium is reached by taking note of plausible rules which govern out of equilibrium behaviour (chapters 4 and 5 for example), and does not concentrate on the self-fulfilling prophecies that seem to characterise something like lotteries. It will become obvious that Schelling's work has influenced this thesis greatly. Schelling, in general, makes little reference to efficiency, however.

certainty that this ideal will emerge: the one overriding characteristic of money, if it is to be money, is that it is universally acceptable. This allows the possibility that a "substandard" entity—one which is not perfect in the way it embodies the characteristics described above—could emerge in widespread use if a critical mass of individuals had already begun to use it successfully as a medium of exchange (that mere fact can being sufficient for others to adopt it). If traditional histories of the development of paper money are reliable, then the objects that initially became acceptable were paper IOUs representing claims on individuals' gold held in storage. This initial point in the "path" of the evolution of money—on which the subsequent path is dependent—has led to the institution of money as we know it today. Whether this final outcome could have been improved upon, or could be altered beneficially now seems to be an interesting issue. In Kiyotaki and Wright [1989], for example, an economy in which there are three potential commodities that may form the medium of exchange is analysed. Each commodity has a different storage cost. The ideal commodity to serve the transactions role would be the one with the smallest storage cost, but it is shown that there are (Pareto inferior) equilibria in which the other commodities are used as money.[14] In Jones [1976] there is an equilibrium in which there is a natural tendency for one commodity to enter into all transactions. The intuition is that individuals will accept a commodity as an intermediate trade if they feel there is sufficient numbers of people will ing to accept it in later exchanges. In deciding which commodity to so accept, individuals choose the one which is in greatest demand, and (barring flukes) there will be just one such commodity. However, this has the effect of increasing the proportion of individuals who will accept this commodity, thus further validating its use. This shows that in the development of a medium of exchange the notion of a 'critical mass' is a relative one, and one commodity will always have sufficient demand to warrant its use (in the Jones framework, anyway). However, it might be suggested that such a framework endows agents with a good deal of information, and that it is not clear that agents can *learn* to use money as they feel around for information. This is taken up somewhat in Marimon *et al* [1990] in which a

[14] These authors' most recent articulation on the subject uses a search-theoretic model, see Kiyotaki and Wright [1993].

genetic algorithm[15] is used to operate on artificially adaptive agents making decisions via *classifier systems*—string representations representing behaviour for given input—in models developed by Kiyotaki and Wright.[16] They find a tendency towards the "fundamental" equilibrium in which the small storage cost commodity emerges.[17]

It may be that in order that a critical mass to use a medium of exchange initially it would indeed need to be preeminent in its embodiment of the desirable characteristics. This leaves open the question of whether the medium of exchange should also perform the other functions of money usually attributed to it (a store of value and a unit of account). It might also be argued that the ideal medium of exchange would be a focal point in initial attempts to use something as a medium of exchange, but this still leaves the same puzzle: if something ideal as a medium of exchange is used by a critical mass of individuals for some other purpose, then this, *due to a numbers externality*, may be used in these subsidiary roles. It is quite possible that there may be some other device, more suited to this subsidiary role, which is not in use because it fails as a medium of exchange. For example, the fact that money can be used as a store of value detracts from its role as a medium of exchange (since it is out of circulation). Many ways of storing wealth have evolved so that, under normal circumstances, this has not been particular problem, with the exeption of well documeted hyper-inflations.

[15] Holland [1975]

[16] This paper, and several recent ones, for example Arifovic [1992a, 1992b], are written by authors who have visited the Santa Fe Institute. The most prominent computer scientist there is John Holland, who did enough work on genetic algorithms to give the field a life of its own. It is no surprise, then, then genetic algorithms are used, but this is somewhat peculiar because these techniques were never intended as models of human learning, but of natural evolution.

[17] The intuition is that artificially adaptive agents act essentially randonly at the beginning of a simulation. This implies an approximate symmetry in their payoffs, accept for the good which they use as a medium of exchange. Those players who, early on, use the low storage cost good can be expected to be 'fitter' than the others, and hence their strategies emerge from this early confusion to form the basis of future populations in the Darwinian world of the genetic algorithm. Once there is an asymmetry in numbers of different player types the numbers externality dictates that (as in the Jones model) those players adopting the most popular strategy (good they will accept) are fitter by virtue of this fact alone, hence the fundamental equilibrium emerges.

	X	Y
X	1	0
Y	0	1

Figure 1.1: A game of coordination.

1.8 Games of Coordination

1.8.1 Pre-Play Communication

The games I have discussed so far have a multiplicity of (Nash) equilibria. Games such as these are said to possess a "coordination problem" in that rational agents wish to engage equilibrium strategies, but are uncertain as to which equilibrium will be played out. In such games there are sometimes external devices that can aid in coordination. One such device is pre-play communication, which in itself is not payoff relevant, but can be relevant as a strategy component. Such communication is termed *cheap talk*, after Crawford and Sobel [1982], and Farrell [1987].[18]. For example, in a game of coordination like that in Figure 1.1 it is fairly clear that some pre-play communication will help, nothing else seems reasonable. However, even in a game such as the "battle-of-the-sexes," in which players rank outcomes differently (Figure 1.2), Farrell [1987] shows that many rounds of pre-play communication can yield outcomes preferable to the mixed-strategy equilibrium. Curiously, many animal species face similar coordination problems. For example, certain coral reef fish, and almost all sea-slugs are hermaphrodite. The sea-slug is particularly interesting, as mating seems to play out a mixed strategy equilibrium in a repeated discoordination

[18] We will develop a model of language use similar to Farrell [1987] in chapter 4.

	X	Y
X	(2,1)	(-1,-1)
Y	(-5,-5)	(1,2)

Figure 1.2: The "battle of the sexes," see Rasmussen [1990, pp. 34–35].

game. When sea-slugs meet with mating in mind they make repeated attempts at successful copulation. The only choice they can make is whether to transmit sperm or an ovum in the direction of the other. It has been shown that producing and transmitting an egg is more costly that transmitting sperm, in the sense that it takes more energy. However, species that only transmitted sperm would not survive, but an individual slug that produced only ova would be less fit that others that randomise. In the absense of a coordinating mechanism it seems the slugs play out a mixed-strategy equilibrium—or at least evolution has evolved such a species.[19]

Human beings have come up with one rather direct solution to coordination problems: the notion of the *time of day*, *day of the week* and so on are blatant coordination devices that we nowadays take for granted.

1.8.2 Mixed-Strategies

The idea of a mixed-strategy itself, due to von Neumann and Morgenstern [1944], is a way of suggesting equilibria in games with many or no Nash equilibria. However, as a means of solving a coordination problem it really only applies to games in which being predictable can only be detrimental, or in which interatctions are sparse and

[19] I am grateful to Larry Dill for pointing out the coordination problem inherent in hermaphrodite species.

anonymous so that players are unable to communicate sufficient information to achieve Pareto superior pure strategy plays. Under other circumstances the use of a non-random means of coordination make more sense since they achieve repeated plays of Nash equilibrium moves. For example, in the repeated game of "Rock-Scissors-Paper" (*very* popular amongst the kids I see on the bus most days), there is no Nash equilibrium. If we assign a payoff of -1 to a loss, 0 to a draw, and 1 to a win, there is a mixed strategy equilibrium of playing each alternative with probability $\frac{1}{3}$, and this probability should be adopted unconditionally so that a players' history of moves yields no information about future moves. The mixed-strategy equilibrium is indeed the most appealing prediction of this game, which is zero-sum. However, a game like either of the two illustrated previously is unlikely to be played out as a mixed-strategy equilibrium, precisely because they are not zero-sum. In playing the coordination game in Figure 1.1, players would rather engage in a few rounds of play in order to communicate *via observed moves* which equilibrium they prefer, providing there are enough rounds of play to warrant any early asymmetric plays in the name of communication. This early play may well be random however. A strategy that appeals to me is the following, which must necessarily dominate a strictly mixed-strategy play: play X or Y with probability $\frac{1}{2}$ until there has been a play of (X, X) or (Y, Y), thereafter play X or Y exclusively. To see this, we can consider the strategy combination of "play X with probability p until there has been a combination of identical moves, and thereafter play these moves." Presumably, the choice of p will be so that the expected time taken until a 'match' occurs is minimised. Let there be two players, trying to decide upon p_1 and p_2, their respective probabilities of playing X in the communication pre-play. Clearly, $(p_1, p_2) = (1, 1)$ and $(p_1, p_2) = (0, 0)$ are best-responses, but this is precisely the coordination problem we are trying to solve by adopting strictly mixed strategies. If an event occurs with probability p the expected value of the number of independent trials before the event occurs is $\frac{1}{p}$. For a particular pair (p_1, p_2) in this game the expected number of rounds before a 'match' in moves is:

$$\frac{1}{p_1 p_2 + (1 - p_1)(1 - p_2)}$$

Players are attempting to minimise this magnitude. Differentiating this with respect to p_1 yields:

$$\frac{1-2p_2}{(1-p_1-p_2+2P_1p_2)^2}$$

which is positive if $p_2 > 1/2$, negative if $p_2 < 1/2$ and zero if $p_2 = 1/2$. Hence we obtain the strictly mixed strategy equilibrium (by symmetry) of $p_1 = p_2 = 1/2$. It is clear that these strategies—play X with probability 1/2 until a match—are best-responses to one another, forming one Nash equilibrium in this repeated game. This equilibrium is Pareto dominated by the simple (X, X) and (Y, Y), of course, but the fact that there are two such pure strategy equilibria was the starting point for this discussion.

This game is slightly different to the 'battle-of-the-sea-slug-sexes' I briefly mentioned earlier. We will return to the slugs when discussion communication directly in chapter 4.

1.8.3 Focal Points

Schelling [1960, p.57] offers the following example of what he terms a *focal point*:

> "You are to meet somebody in New York City. You have not been instructed where to meet; you have no prior understanding with the person on where to meet; and you cannot communicate with each other. You are simply told that you will have to guess where to meet and that he is being told the same thing and that you will just have to try to make your guesses coincide. Furthermore the two of you must guess the exact time you will appear at the meeting place that you elected."

Games in which the players are not in conflict like this are often referred to as *pure coordination games*. Showing these instructions to a group of players, more than half of them chose Grand Central Station's information booth as a location, and all but a couple chose 12 noon as a meeting time.[20]

[20] Thereby taking advantage of the ubiquitous temporal cooordination device we mentioned: "time of day."

Schelling is arguing that in coordination games some moves (or strategies) have a significance that cannot be explained by their payoff relevance, but players have a common feeling that they are likely to be chosen because of this extrinsic significance. Schelling [1960 pp. 54–58] suggests other games in which focal points may exist, such as naming 'heads' or 'tails' (many people choose heads, 86% of Schelling's sample, perhaps because they are always asked "heads or tails?" and not "tails or heads?"), writing down some positive number (many people choose 7 if asked to choose a number between 1 and 10, presumably because Western societies consider 7 to be a lucky number. Many Eastern societies attach a similar significance to the number 4. Schelling just asked that the players choose a positive number, and in his sample 40% chose '1'). There are several other examples.

1.8.4 Evolution of Conventions

Sugden [1986] is a fairly extensive study of how coordination in these kind of games is facilitated in a dynamic process. The emergent behaviour surrounds the choice of *conventions*, which might perhaps be viewed as strategies that have become focal due to past success—implying a repetitious dynamic structure to the game. I will develop a simple example in chapter 4, but we can see the basic idea here.[21] The essential idea is that in games involving many players (unlike those of the previous section), it is very difficult for any kind of communication to take place between a "critical" mass of individuals. The players must use their own behaviours to communicate which strategy should be used in a coordination game, in the knowledge that their own behaviour may not be directly observed. Such a game has an enormous strategy space, since players must decide upon which information to use in determining future actions.[22] The minimum information they will have will concern their own choices in the past, perhaps subject to memory restrictions, and their own payoffs in the past,

[21] Several computer simulations have been conducted in the literature in similar settings to those looked at by Sugden. See, for example, the discussion in Harrald [1994], and the work in Fogel and Harrald [1994], Marks [1989], and the discussion in Arthur [1992] and the references there. Sugden's analysis uses the concept of an *evolutionary stable strategy*, which is discussed in chapter 4.

[22] They could use all of the previous history of the game, or just some gross classifications. This actually amounts to an artificial coarsening of information sets by the players.

which may reveal information abou the moves employed (but not the strategies of) other players. In an attempt to formalise some of the notions of *procedural* or *bounded* rationality, it is possible to simulate the evolution of player behaviour in many games, of which coordination games are an example.[23] By using agents endowed essentially inductive methods it is easily shown how equilibria can emerge in which coordination problems are solved. For example, consider a large population meeting in random pair-wise encounters to play the game depicted in Figure 1.1. If these agents adopted a practice of playing the strategy that the *majority* of players had played against them in the past, then under any reasonable dynamics there would be quick convergence to all playing X or Y.[24] Dynamics are usually much more interesting in games such as the Prisoner's dilemma that involve direct conflict between the players.

1.8.5 Analogies

One reason a strategy may have some intuitive appeal among several possible equilibria (the payoffs in which are equal for all players) is because it forms an *analogy* to some other similar game.[25] For example, players who must play one game in which moves might meaningfully be described as 'left' and 'right' might look for some analogous game in which the coordination problem has been solved. Christine Tremblay [personal communication] indicates that when individuals need to move to one side in a corridor, it is common for them to use the same side of the corridor as they drive on, for example.

[23] A very useful discussion of the notion of less than full rationality due to uncertainty is in Dosi and Egidi [1991].

[24] If players update their behaviour after they have all played one another, the convergence would take place immediately, except in the case where their was a tie in the distribution of X and Y as the players' first moves.

[25] Sugden [1989] calls such strategies social-conventions. For example, in matters of day-to-day politeness, men typically defer to women, the young to the old, and the healthy to the sick. If a novel situation arises in which coordination in some piece of petty politeness emerges, these analogous results carry over.

Chapter 2

Numbers Externalities

2.1 On Numbers Externalities in General

The notion of a "numbers externality" or "network externality" is often central in social science. If human beings did not experience externalities in general, then, of course, there would be no 'social' science. In this chapter I discuss the notions of numbers and network externalities, and add a brief exposition of the externality involved in public lotteries and in a model after the well known work of Tiebout on local public expenditures. The idea is to further examine the nature of these games by way of simple manageable examples.

2.2 Numbers and Network Externalities.

The notion of a 'network' has been most thoroughly analysed by systems scientists, for example Flood and Carson [1988], including some work by the economist Kenneth Boulding. In the abstract a network consists of a set of *nodes*, indexed by $i \in \{1, \ldots, n\}$, and a set of *links* that connect pairs of nodes. The set of links is conveniently represented by a set of pairs, $\{(j, k)\}$ where $j, k \in \{1, \ldots, n\}$, indicating a connection between nodes j and k exists. If the notion of a one-way connection is appropriate then the pairs are ordered. Any node can have between zero and $n-1$ connections (which may or may not be two-way). In the case in which it is $n-1$, the total number of connections in a network is n^{n-1}. This explosion in the number of connections as n grows is at the heart of the complexity of networks, and hence the foundation of the notion of a network externality. However, it is *not* this effect with which I concern myself in this chapter (which is in no way intended to undermine its importance). Rather, I am concerned with a particular class of 'games' in which the feature of importance (for the modeler, and the players) is the *number* of players engaging in one of a range of alternative actions.

There is a degree of player anonymity implied in this loose classification that makes such games prone to severe coordination problems, similar to those in our discussion in the previous chapter on 'bandwagon' and other effects in demand. Generally, equilibria are of the form that N_x players should engage in activity x, where x indexes the range of actions. However, it cannot be specified *whom* should comprise each of the N_x.

The most severe coordination problems emerge when there are many equilibria, none of them Pareto optimal (among the set of equilibria), and none of them 'focal' in the sense of Schelling [1960], and there is no obvious means of distinguishing one player from another.

In general, then, a numbers externality exists between N individuals playing a game in which each has action set $a_i \in A = \{a^1, \ldots, a^n\}$ if the payoff function can be written as $U_i(a_i \in A; N_1, \ldots, N_n)$. The players are presumed to have the same action sets to preserve the symmetry and anonymity of their predicament.

There is an extension of this definition that I would like to introduce. While the identity of other players is irrelevant for an individual's preferences, this does not mean that players cannot be differentiated in any other respect. I therefore define the following. A numbers externality game is *undifferentiated* if:

$$U_i(a_k^i; N_1, \ldots, M_m) = U_j(a_k^j; N_1, \ldots, M_m) \qquad (2.1)$$

This asserts that players have identical preferences, so that the assessment of what is welfare improving reallocations of individuals across actions is the same for everybody, for all possible reallocations. If this is not the case then we have a *differentiated* numbers externality. As an example consider a simplification of Katz and Schapiro

[1985]. N agents must choose between purchasing one of two varieties of video format, which we'll call V and B in period 1. In period 2 they must purchase videos to play on machines endowed with these formats. Because of the complementarity between machines and video tapes, the quality of the tapes (perhaps simply measured by the number of them around) will vary directly with the number of owners of the corresponding video formats. In the first period each agent has a choice of two activities, $A = \{V, B\}$, indicating chosen format. If these formats are practically identical, then utility for an individual could be happily written as $U_i(a_i; N_V, N_B)$. We could also safely assert that:

$$\text{If } a_i = V, \quad U_i(V; N_V^1, N - N_V^1) > U_i(V; N_V^2, N - N_V^2) \text{ iff } N_V^1 > N_V^2 \qquad (2.2)$$

without worrying about who i is, or who makes up the rest of the N_V and the N_B. The final part of these expression, indicating that utility is increasing in the number of others choosing the same format, is the numbers externality, and it is to be viewed a little differently from the bandwagon effect of chapter 1, since we have in our mind that this is functional demand, and that if robots participated in this game as rational players, then preferences are unaltered. This then, fits into the category of an undifferentiated numbers externality. Imposing simultaneous decision-making, we can identify two Nash equilibria quite readily, those in which $(N_V, N_B) = (N, 0)$ or $(0, N)$. If individuals make sequential decisions (one after another, with a knowledge of previous choices) then a sub-game perfect Nash equilibrium emerges in which they all buy the same format as the first decision-maker.

We do not require that the two formats be identical to get an undifferentiated model. It could be that the two products be vertically differentiated. We must be a little careful in defining this in the current context. The most natural definition, I think, is the following: Choice V is vertically-preferred to B in the sense that:

$$U_i(V; x, y) > U_i(B; y, x) \qquad \forall i (2.3)$$

This corresponds to the traditional distinction between vertically and horizontally fifferentiated products—the term "vertcal" indicating a concensus opinion as to the better product. Under such circumstances the only sub-game perfect equilibrium in the (unlikely) sequential game is $N_V = N$, moreover this equilibrium (we still maintain (2.1) and (2.2)) Pareto dominates—not just any other equilibria, but all possible outcomes in the game. I have not yet explored the full implications of (2.3), but may in the future. This definition allows us to distinguish between absolute preference for an activity *in itself*, and preferences over outcomes in games in which the numbers externality has a role to play. The importance of this distinction is mentioned throughout, and we refer to it because we can make efficiency comparisons of equilibria, just as in the Kiyotaki–Wright result that an "inefficient" medium of exchange can be an equilibrium, because the use as a meduim of exchange outweighs any other consideration.

2.3 Public Lotteries.

As a further example of the forgoing we now to focus on the kind of coordination problem involved when individuals confer numbers externalities on one another by looking at a social phenomenon that is becoming more and more common; public lotteries. The reason these events are interesting is because individuals participating tend to confer two types of externalities on others: when a ticket is purchased the size of the prizes increases, but typically the probability that those prizes are shared increases. Let's look at some simple examples of lotteries and examine properties like those we have been discussing. In particular, a brief analysis of public lotteries will help clarify the notion of "bootstrapping" equilibria, and demonstrate the implicit coordination problem. Lotteries have the requisite anonymity properties we have been suggesting, although differing attitudes to risk may make these differentiated numbers games.

2.3.1 A Simple Lottery.

Consider a very simple lottery: a ticket can be purchased for \$1. The owner of the ticket has a $\frac{1}{n}$ of winning $\$n$, where n is the number of tickets sold. Of course, the expected change in wealth involved in the purchase of any number of tickets is 0. We can conclude that risk-neutral individuals are indifferent between buying any number of tickets and any other number, including zero, while risk-averse individuals will never purchase tickets, and risk-inclined individuals will buy as many as they can. From now on, mainly for convenience, I presume that an individual is restricted to buying at most one ticket.

I'll "close" the simple lottery in the following way. Everyone has exactly $1 of wealth, and they have the same utility function to describe their preferences over alternative probability distributions over levels of wealth:

$$U(W) = W^k$$

except for the parameter k, which may vary from individual to individual. If this parameter does not vary, this is an undifferentiated numbers externality game, otherwise it is differentiated. The simple Lottery above offers an individual the option of two states of the world—two 'prospects.' On the assumption that $(n - 1)$ 'other' tickets are purchased, an individual will buy a ticket if the prospect $(\frac{1}{n}, \frac{n-1}{n}; n, 0)$ is preferred to the prospect $(1; 1)$. The expected value of the first prospect is \$ 1, the second is the same of course. A risk-neutral individual (with $k = 1$) is indifferent between these two prospects, a risk-averse individual ($k < 1$) prefers to ignore the lottery, and a risk-inclined individual ($k > 1$) prefers to buy a ticket than not. None of this relies on n. So, although there is an externality (among those risk-inclined), because expected-utility is increasing in n, it has no qualitative affect on behaviour. Such an equilibrium does not exhibit the "bootstrapping" property I have discussed, and I do not think it describes the nature of actual lotteries: I think that participation of many is the primary reason so many participate.

Since the expected change in wealth involved in purchasing a ticket is zero no matter how many tickets are sold, we would expect to see all the risk-inclined individuals buying lottery tickets, in the absence of some other more attractive gamble that is. Although there is an externality between lottery ticket purchasers, it does not affect behaviour. It is worth taking quick look at the externality, however. Consider a risk-inclined individual who has spent all of his wealth on lottery tickets, \$ 1 on 1 ticket. This individual is waiting for the draw to take place, and assumes there are currently n total tickets having been purchased. Such an individual's expected utility—ignoring the probability of future ticket buyers—can be schematically written as:[1]

$$\mathbf{EU}(n) = \mathbf{P}rob\mathbf{W}in\mathbf{L}ottery(n)U(\mathbf{P}rize(n))+$$
$$[1 - \mathbf{P}rob\mathbf{W}in\mathbf{L}ottery(n)]U(0)$$

We have the following specifications:

$$\mathbf{PWL}(n) = \frac{1}{n}$$
$$\mathbf{P}(n) = n$$
$$U(\mathbf{P}(n)) = \mathbf{P}(n)^k$$
$$U(0) = 0$$

Although tickets are purchased in discrete amounts, it is useful to look at the derivative of $\mathbf{EU}$ with respect to n. We have:

$$\frac{d\mathbf{EU}}{dn}(n) = \frac{d\mathbf{PWL}}{dn}U(\mathbf{P}(n)) + \mathbf{PWL}(n)\frac{dU}{d\mathbf{P}}\frac{d\mathbf{P}}{dn} - \frac{d\mathbf{PWL}}{dn}U(0)$$

illustrating the two basic externalities involved in public lotteries. Substituting for the exact expressions gives:

$$\begin{aligned}\frac{d\mathbf{EU}}{dn}(n) &= \frac{-1}{n^2}(n)^k + \frac{1}{n}kn^{k-1}(1) - \frac{-1}{n^2}(0)\\ &= -n^{k-2} + kn^{k-2}\\ &= n^{k-2}(k-1)\end{aligned}$$

which is positive. Hence the numbers externality. The first term in the last but one expression, $-n^{k-2}$, is the change in expected utility attributable to a decrease in the probability of winning the lottery, the second, kn^{k-2} is the change in expected utility associated with the increase in the size of the prize in the lottery.

[1] In Cook and Clotfelter [1993] it is shown that in actual lotteries the expected *value* of a \$1 bet on a lottery ticket depends upon the number of others playing, using a formulation similar to what follows. Their paper suggests that individuals are mislead (not necessarily intentionally) by the size of jackpots (by not being clear on the required probabilities). Cross-section and time-series data are used to estimate 'jackpot' elasticities of demand. Similar data can be found in Gulley and Scott [1983]. That decisions in Lotto games do not adhere to expected utility maximisation, apart from being suggested in Cook and Clotfelter, can be found in Camerer and Kunreuther [1989] and Shapira and Itzhak [1990], and in more general contexts the essential ideas are in Kahneman and Tversky [1984]

2.3.2 A More Than Fair Lottery.

Consider now a rather generous lottery in which there is a fixed prize fund in place of $\$P$, which is added to any revenue from tickets. . In this case, if an individual thinks that a total of $n-1$ tickets will be sold in addition to his own purchase, he will buy one ticket if:

$$\left(\frac{1}{n}\right)\mathbf{EU}(P+n)+\left(\frac{n-1}{n}\right)\mathbf{EU}(0)$$

In a more complete model, n would be treated as a random variable too, and the prospects become quite complicated. For a second consider the case in which all potential lottery ticket buyers have the same preferences.

Now, an *interior* equilibrium will consist of a number n^* of tickets being purchased by n^* individuals such that[2]:

$$\mathbf{EU}(n^*)\geq\mathbf{EU}(1) \text{ and } \mathbf{EU}(n^*+1)<\mathbf{EU}(1)$$

Substituting in for exact expressions gives:

$$\left(\frac{1}{n^*}\right)(n^*+P)^k\geq 1 \text{ and } \left(\frac{1}{n^*+1}\right)(n^*+P+1)^k<1$$

Notice that an interior equilibrium does not exist if $k>1$, since this lottery is better than a fair gamble. Let us continue under the assumption that $k<1$. The value of n which solves $\frac{(n^*+P)^k}{n^*}=1$ will be rough approximation to n^*, if n^* exists. It is given implicitly by the equation $n^*(P,k)^{\frac{1}{k}}-n^*(P,k)=P$. In the convenient case in which $k=1/2$ this yields the two solutions:

$$n^*(P,1/2)=\frac{1+\sqrt{1+4P}}{2},\qquad n^*(P,1/2)=\frac{1-\sqrt{1+4P}}{2}$$

only the first of which is positive. Note that the expected utility of a representative ticket holder is *decreasing* in n:

$$\frac{d\mathbf{EU}(n)}{dn}=\frac{(k\,n-n-P)\,(n+P)^{k-1}}{n^2}<0 \text{ since } k<1$$

so that $\mathbf{EU}(\frac{1+\sqrt{1+4P}}{2}+1)<1$, meaning that the truncated integer value of $\frac{1+\sqrt{1+4P}}{2}$ is the number we are after. The relationship is graphed in Figure 2.6. If we suppose that

[2]Recall the assumption that individuals can purchase at most one ticket

for a particular value of P, n^* is less than the population of potential buyers, then we have our "equilibrium" solution. The problem is, naturally, that a value of n^* does not specify *whom* it is that buys the tickets—our fundamental coordination problem. Before I consider this further, we will look at a"mirror image" of this lottery—one in which a chunk of the revenue is appropriated by some agency, and only the remaining fraction is paid out.[3]

2.3.3 A Less Than Fair Lottery

Consider an alternative lottery, rather more realistic. We will suppose that the 'jack-pot' is only $x(100)\%$, $0 < x < 1$, of the revenue accrued from ticket sales. Again, all potential ticket buyers have identical utility functions, this time characterised by a parameter $k > 1$, since this lottery is never a fair gamble.[4] Now, the equilibrium conditions become:

$$\left(\frac{1}{n^*}\right)(n^*\,x)^k \geq 1 \text{ and } \left(\frac{1}{n^*+1}\right)((n^*+1)\,x)^k < 1$$

Consider the value of n that solves $\left(\frac{1}{n}\right)(n\,x)^k = 1$, given by:

$$n = x^{\frac{-k}{k-1}}$$

and also note that:

$$\frac{d\mathbf{EU}(n)}{d\,n} = \frac{(n\,x)^k\,(k-1)}{n^2} > 0 \text{ since } k > 1.$$

Here we have a coordination problem again: if *at least* $n = x^{\frac{-k}{k-1}}$ lottery tickets are purchased, then all the risk-inclined individuals will rationally purchase a ticket. If less than $n = x^{\frac{-k}{k-1}}$ tickets are purchased, it is not rational for any individual to purchase a ticket. If $n = x^{\frac{-k}{k-1}}$ is less than the population of potential buyers of tickets, then there is one equilibrium in which n^*=0. Otherwise, there is an equilibrium in which again

[3]In a popular Canadian lottery, "Lotto 6/49," only 45% of revenue is returned in prize money, the rest is used by the Government. Properly modeling the purchase of a lottery ticket (or tickets) would account for this—the money is not simply 'lost.'

[4]Basically, I am ignoring the behaviour of risk-averse or -neutral individuals since they will never buy a ticket.

$n^* = 0$, and another one in which $n^* = N$, provided N is larger than the truncated integer of $x^{\frac{-k}{k-1}}$. Critical values for the case in which $k = 2$ is shown in Figure 2.1.

2.3.4 A Note on the Differences in Coordination Problems

Notice the subtle difference between the coordination problems in the lotteries that are not actuarily fair. (Recall, there was no coordination problem in the fair lottery, although there was a numbers externality.) We have seen that in a world populated by risk-averse individuals, there was a maximum number of tickets that would be purchased that was (potentially) less than the population of prospective buyers. This was because the expected utility of a ticket purchase was decreasing in the number of (other) tickets purchased. However, since the expected utility associated with being the *only* buyer was necessarily greater than unity (the certainty prospect), we obtained an 'interior' solution.

In the case of the unfair lottery in a world populated by risk-lovers we have a slightly different problem. It is not that the buyers have to coordinate themselves so that an exact number buy a ticket, just that *at least* a certain number of them buy tickets. We can think of a game being played amongst the potential buyers. For concreteness assume there are a total of N of them, each with action set $\{$*buy*, *don't buy*$\}$. This game has precisely two Nash equilibria in pure strategies: one in which all players choose '*don't buy*' and one in which all players choose '*buy*' if N is large enough. The second equilibrium Pareto dominates the first, and is potentially, therefore, a focal point. Once this equilibrium is established it is hard to see why it would not characterise all future repetitions of the lottery.

The more than fair lottery is more interesting. The first observation is that in equilibrium the only thing preventing the players from being indifferent between buying a ticket and not is an integer constraint: in equilibrium just enough tickets are purchased to discourage the marginal $(n^* + 1)$th buyer. In this case, it seems sensible to look for a mixed-strategy Nash equilibrium. Such an equilibrium will consist of N equal probabilities, say p, which represent the probability of buying a ticket for each player. These probabilities will have the property that the players are indifferent between a) choosing to buy a ticket with probability p, b) not buying a ticket, and c) buying a ticket.[5] An individual buying a ticket faces the following prospect:

$$(P+i;\ (1/(i+1))((N-1)!/i!(N-1-i)!)p^i(1-p)^{N-1-i}) \qquad i=0,\dots,N-1$$

An individual not buying a ticket has expected utility of 1. So, we can state that, in equilibrium:

$$p\left(\sum_{i=0}^{N-1}(1/(i+1))((N-1)!/i!(N-1-i)!)(P+i)^k p^i(1-p)^{N-1-i}\right)+(1-p)=1,$$

or, what amounts to the same thing,

$$\sum_{i=0}^{N-1}(1/(i+1))((N-1)!/i!(N-1-i)!)(P+i)^k p^i(1-p)^{N-1-i}=1.$$

This is an Nth-order polynomial, and will have N solutions, although not all of them will be equilibria (or real for that matter!). The solutions themselves, if derived by a symbolic mathematics package would be ugly for $N \geq 3$, not to mention quite time consuming. This is pursued no further at this juncture.

The Actual Externality

The actual numbers externality involved in public lotteries is slightly different from the examples we have discussed. When a ticket is purchased this does (usually) increase the size of the jackpot (and subsidiary prizes). It also increases the probability that a ticket holder must share the jackpot (and some other prizes). This makes the gamble quite complicated. For example, many public lotteries sell tickets which specify a small subset of integers in some range. As the name implies, *Lotto 6/49*, a popular canadian lottery, sells tickets with six integers between 1 and 49 printed on. Each Wednesday and Saturday a televised draw of six numbers takes place (at 6:49 p.m.). Prizes are awarded for tickets that match three or more of the drawn numbers. The manner in which the prizes are determined is quite complicated, too complicated to warrant a close study here. The basic property is that the prizes tend to be pools of money (a particular fraction of revenue) that are shared among the winners.

[5] Again, I am ignoring the integer constraint.

The actual externality involved is quite interesting. Consider a simpler version of a lottery with similar characteristics to *6/49*. In this formulation a ticket buyer pays \$1 for a ticket, and selects an integer between 1 and 10. When the draw takes place, all those ticket holders whose numbers match the (randomly) drawn ticket share the revenue generated from ticket sales. If there are n tickets sold to n individuals, the probability of any individual winning is $\frac{1}{10}$. The probability that x other players chose the same number as any given individual is:

$$\left(\frac{(n-1)!}{x!(n-1-x)!}\right)\left(\frac{1}{10}\right)^x\left(\frac{9}{10}\right)^{n-1-x}$$

Thus we can calculate the expected value of a ticket, $\mathbf{EV}(n)$, as:

$$\mathbf{EV}(n) = (-1)\frac{9}{10} + \sum_{x=0}^{n-1}\left(\frac{n}{x+1}\right)\frac{1}{10}\left(\frac{(n-1)!}{x!(n-1-x)!}\right)\left(\frac{9}{10}\right)^{n-1-x}\left(\frac{1}{10}\right)^x$$

This value is graphed in Figure 2.2. Notice how the expected change in wealth is monotonic in n and tends to zero as n approaches infinity (I have included non-integer values of n for clarity). The expression for expected utility is, of course:

$$\mathbf{EU}(n) = \sum_{x=0}^{n-1}\left(\frac{n}{x+1}\right)^k\frac{1}{10}\left(\frac{(n-1)!}{x!(n-1-x)!}\right)\left(\frac{9}{10}\right)^{n-1-x}\left(\frac{1}{10}\right)^x$$

The case for $k = 2$ is graphed in Figure 2.3. These graphs are quite interesting. As $n \to \infty$ the gamble is approaching the fair bet $(-1, 10\,; \frac{9}{10}, \frac{1}{10})$. Hence, the expected change in wealth is approaching zero, and expected utility is approaching 99. The expected change in wealth cannot be zero because there is always a chance that nobody wins the lottery (this probability being $\left(\frac{9}{10}\right)^n$). Concentrating on the risk-inclined, we see the positive numbers externality they can confer on one another—expected utility is increasing in n. This is not because $k > 1$ however, as Figure 2.3 indicates, if we were to force the risk-neutral to participate they would also experience an increase in expected utility as n grows. Now, expected utility (for $k = 2$) is less than unity for $n = 3$ and greater than unity if $n = 4$. Thus, in this fictitious lottery, with $k = 2$, there two Nash equilibria. One in which nobody buys a ticket, and one in which all potential buyers actually do. Again, the second Pareto dominates the first, and can probably be expected to obtain. We can refer to $n = 3$ as a "critical mass" in this lottery. If potential buyers think at least 3 others will buy tickets, then all potential buyers will buy tickets.

2.3.5 Indivisibilities

When individuals buy lottery tickets it does not, necessarily, seem like a gamble in the same sense as individuals in a casino. It seems that it is the massive prizes on offer that attract the lottery ticket buyer. Perhapsthe massive prize will allow the purchase of a good or goods that create a discrete increase in welfare, a different regime of expected utility. Consider an individual with the following utility function;

$$\begin{array}{ll} U(W) = W^{1/2} & W < 10 \\ U(W) = (W+101)^{1/2} & \text{otherwise} \end{array}$$

shown in Figure 2.4.

One reason for such a utility function is a strong complementarity in consumption of a good that can only be acquired if wealth is at least 10 units. Such an individual is risk averse over prospects defined on wealth levels in [0,10) and in [10,∞), but could be observed to be risk-inclined in gambles with payoffs that span these intervals. For example, the prospect $(5, 10\,;\, 1/2, 1/2)$ is preferred to the prospect $(7\frac{1}{2}, 1)$.[6] We can use our initial formulation of the lottery to make the point here. Consider a population of potential ticket buyers who, if n tickets are sold, have a $\frac{1}{n}$ chance of winning n units of wealth. They have the discontinuous utility function above. The expected utility of a ticket buyer, on the assumption of initial wealth of one unit, is given below:

$$\begin{array}{ll} \mathbf{EU}(n) = \frac{n^{1/2}}{n} & n < 10 \\ \mathbf{EU}(n) = \frac{(n+101)^{1/2}}{n} & \text{otherwise} \end{array}$$

shown in Figure 2.5.

Now, this lottery has, again, two Nash equilibria (in pure strategies), one in which nobody buys a ticket, and one in which exactly ten people buy tickets. It would be

[6]This, of course, is the idea of state-dependent expected utility, which has been used to rationalise the apparent simultaneity of risk-aversion and -inclination. There is a strong complementarity between one's enjoyment of many things in life, the most obvious being one's health. The popularity of public lotteries indicates that high price items have a similar complementarity.

inappropriate to treat the number ten as a 'critical mass' however, since a purchase of ten tickets does not precipitate a further avalanche of ticket buyers. This equilibrium is probably better termed a 'knife edge' equilibrium. Notice that it is the *negative* numbers externality that causes this knife-edge phenomenon.

There is *no* strictly mixed strategy Nash equilibrium in this game. To see this note that expected utility is less than one if the number of tickets actually purchased is 9 or less, or 11 or more. Expected utility is only slightly greater than 1 if exactly 10 tickets are sold. Hence, any variance around 10 will generate expected utility less than one unless the probability of exactly 10 tickets sold is *very* close to one.[7] How close can the probability that the number of tickets sold be to one, yet still remain unequal to 1? The expression for the probability that exactly ten tickets are sold is:

$$\frac{N!}{10!(N-10!)}p^{10}(1-p)^{N-10}$$

Naturally, we consider only values of N greater than ten. This probability is maximised when $N = 11$ and p is $\frac{10}{11}$. The maximum value is less than 0.4. The discontinuity of the payoff function is what takes this game out of the realms in which mixed-strategy equilibria can be proven to exist, even as in Dasgupta and Maskin [1986].

2.4 The Tiebout Hypothesis

The beginning point of the Tiebout hypothesis[8] is Samuelson [1954], which itself is based upon the framework of Bowen [1943]. The result in Samuelson is that public goods ("non-rivalrous" goods) cannot be efficiently supplied. For Tiebout, the essential insight appears to be that if public goods are produced with a U-shaped average cost curve, and if taxes are proportional to average cost in local jurisdictions, then consumers will arrange themselves to minimise average cost, thus effecting the efficient production of public goods (subject to integer constraints, and divisor constraints). I do not wish to examine Tiebout's hypothesis itself here, but to develop a model of

[7] This is a loose statement of lack of existence, but seems to get the point across.

[8] Tiebout [1956].

location choice by consumers of public goods to illustrate that suggesting that because equilibria exist we can be sure optimality is reached is a fallacious argument if we cannot specify realistic dynamics by which an equilibrium is reached (which was, it might be remembered, the point made in the first chapter).

2.4.1 Numbers Externalities and a Characteristics Space

I now begin to recast the Tiebout scenario to emphasise the numbers externality I have been considering in general previously. I then introduce a construct—differentiated public goods—to provide the link to a second basic concept, the characteristics space.

2.4.2 An Initial Framework

Now consider a simple variant of a Tiebout model to illustrate some of the observations we have made so far on numbers externalities. There are n distinct 'regions,' each of which has its own distinct, differentiated public good. Regions are indexed by $j = 1, \dots n$. Variants of the public good are indexed by an element of the unit interval, so that we can say that region j produces variant $p_j \in [0,1]$. We use q_j to indicate the number of units of the public good provided by region j (of variant p_j).

Living in region j are a number, N_j, individuals, each of whom is charged a per head tax of t_j. If $C(q_j, p_j)$ is the cost function associated with producing q_j units of variant p_j, then it is presumed that all tax revenue is devoted to production of the public good. So:

$$C(q_j;p_j) = t_j N_j$$

Is a region's budget constraint. There are m 'consumers' dispersed among the regions ($\sum_j N_j = m$), indexed by $i = 1, \dots, m$. We denote consumer i's 'residence' (where they actually live) by $l_i \in \{1, \dots, n\}$.

Consumers have preferences defined over bundles of quantities and types of the public good, and a composite commodity, that are representable by a utility function. Consumers have diverse preferences, however, since they disagree on the *ideal* variant of the public good. Denote agent i's ideal point as $m_i \in [0,1]$, rendering this game differentiated.

At any moment an agent will be living in some region, enjoying consumption of a particular quantity and type of pubic good, and will be taxed at a region specific rate. With the income they have left after tax the agents consume a composite commodity. All agents have the same income of Z. For convenience costs are assumed to be very simple: one unit of any variant of the public good costs \$1 to produce;

$$q_j = t_j N_j.$$

Consumers locate themselves in the most-preferred region, using, I assume, Nash conjectures. Alternatively, I use the notion of Nash equilibrium to close this model. Formally, if l_i^* is the location that maximises consumer i's utility (given the N_j), then a Nash equilibrium obtains when $l_i^* = l_i \ \forall i$. Notice that, from the point of view of a consumer, the statistics of interest in a region are (p_j, N_j, t_j), which makes utility most conveniently written as $U_i(p_j, N_j, t_j)$. This gives a definition of Nash equilibrium: the vector of residences $l = (l_1, \ldots, l_m)$ forms a Nash equilibrium if, $\forall i$:

$$U_i(p_{l_i}, t_{l_i}, N_{l_i}) \geq U_i(p_k, t_k, N_k + 1) \ \forall k \neq l_i.$$

2.4.3 An Exact Model

Consider the following specifications:

$$m = n = 3$$

$$U_i(p_j, N_j, t_j) = (1 - x \mid m_i - p_j \mid)(q_j)(Z - t_j) = (1 - x \mid m_i - p_j \mid)(t_j N_j)(Z - t_j)$$

$$(m_1, m_2, m_3) = (0, \frac{1}{2}, 1)$$

$$(p_1, p_2, p_3) = (0, \frac{1}{2}, 1)$$

$$(t_1, t_2, t_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

This Cobb-Douglas form utility function includes a parameter, $x > 0$, which represents the decreases in utility associated with the consumption of goods of increasing difference from a consumer's ideal. Whatever the magnitude if this difference, they are still 'goods' and not 'bads' however.

There are 27 possible combinations of residences, i.e.:

$$\begin{aligned} l \in \{ \; & (1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),(1,2,3),(1,3,1),(1,3,2),(1,3,3), \\ & (2,1,1),(2,1,2),(2,1,3),(2,2,1),(2,2,2),(2,2,3),(2,3,1),(2,3,2),(2,3,3), \\ & (3,1,1),(3,1,2),(3,1,3),(3,2,1),(3,2,2),(3,2,3),(3,3,1),(3,3,2),(3,3,3)\} \\ & \equiv \{l^{01}, \ldots l^{27}\}. \end{aligned}$$

Since the tax rates and public good varieties are specified, and since the quantity of public good provided by a region is determined by amount of tax revenue accrued—which is itself determined by the number of residents, I will introduce a new notation for utility; $U_i(l^k)$, indicating the utility index associated by agent i is the consumers distribute themselves according to l^k. Let us begin with the following question: what must be true if l^{06} is to be a Nash equilibrium? We have $l^{06} = (1,2,3)$. Consumer 1 could, by relocating, transform l^6 into l^{15} or l^{24} by relocating to region 2 or region 3 respectively. Consumer 2 could engender l^{03} or l^{09} by moving to region 1 or region 3 respectively, while consumer 3 could engender l^{05} or l^{04} by moving to regions 1 or 2 respectively. Applying the definition of Nash equilibrium we have the following requirements:

$$U_1(l^{06}) \geq U_1(l^{15}), U_1(l^{24})$$
$$U_2(l^{06}) \geq U_2(l^{03}), U_2(l^{09})$$
$$U_3(l^{06}) \geq U_3(l^{05}), U_3(l^{04})$$

It can be shown that these requirements boil down to $x \geq 1$.[9] The intuition is as follows. For l^{06} not to be a Nash equilibrium, a consumer's utility must increase by relocating to a region which is identical in all respects to the one left, except that it has a less preferred public good, but will have 1/2 a unit of that good extra when relocation is completed. It is easy to see that a consumer considering relocation, and causing a consequent disturbance from l^{06}, will go to the region with the public good variant closer to that consumer's ideal (consumer 1 would relocate to region 2, not 3, if at all, consumer 2 is indifferent between regions 1 and 3, consumer 3 will relocate to

[9] I presume that $Z > 1/2$.

region 2, not region 1). In relocating in this manner we can separate out two effects on utility; the first an increase associated with increased consumption of the public good, the second a decrease in utility associated with the consumption of a less preferred variant. The total change, written to reflect this decomposition, is:

$$\Delta U = \left((Z-\frac{1}{2})-(\frac{1}{2}Z-\frac{1}{4})\right)-\frac{1}{2}x(Z-\frac{1}{2})$$

Rewriting gives:

$$\Delta U = \frac{1}{2}(Z-\frac{1}{2})(1-x)$$

and so, if l^{06} is a Nash equilibrium, $\Delta U \leq 0$, implying $x \geq 1$.

In fact, we can readily specify the structure of equilibria in this example with some *a priori* reasoning. For any vector of residences to be an equilibrium, there cannot be a region that a) has a public good preferred by an agent than the one provided by the residence of that agent *and* b) has at least as many residents as the region the agent lives in. Formally, if l^* is a Nash equilibrium vector of residences:

$$\text{If } N_r \geq N_{l_i} \text{ then } \mid m_i - p_r \mid > \mid m_i - p_{l_i} \mid \quad \forall r \neq l_i \in l^*.$$

Applying this theorem reduces our original 27 possible patterns of residences to 8:

$$l^{01} = (1,1,1) \qquad l^{03} = (1,1,3) \qquad l^{05} = (1,2,2) \qquad l^{06} = (1,2,3)$$

$$l^{09} = (1,3,3) \qquad l^{14} = (2,2,2) \qquad l^{15} = (2,2,3) \qquad l^{27} = (3,3,3).$$

Due to symmetry, we can say that: if l^{01} is a Nash equilibrium, then so is l^{27}, the same applies to l^{03} and l^{09}, l^{05} and l^{15}.Playing with the appropriate inequalities shows that it is possible for some, but not all, of these to be equilibria for particular values of x. Notice that l^{03} can be formed from l^{06} by one consumer relocating (consumer 2), and hence these two cannot be strict Nash equilibria for the same value of x. Both l^{03} and l^{06} can be formed from l^{09}, again by a relocation of consumer 2, and hence all three of these cannot simultaneously be strict Nash equilibria, nor any pair of them. The same applies to the pair l^{01} and l^{03}, the pair l^{05} and l^{14}, the pair l^{14} and l^{15}, the pair l^{05} and l^{06}, the pair l^{06} and l^{15}, the pair l^{09} and l^{27}, and the pair l^{14} and l^{15}, and so on (the symmetric pattern can be substituted for any one of these). Again, due to symmetry, I restrict attention to the following:

$$(1,1,1)\;(1,1,3)\;(1,2,2)\;(1,2,3)\;(2,2,2)$$

or l^{01}, l^{03}, l^{05}, l^{06}, and l^{14}. Examining the required inequalities yields the following, where **L** represents a set of Nash equilibria:

$$\mathbf{L}=\left\{l^{01}, l^{14}\ :\ 0 \le x \le \tfrac{2}{3}\right\}$$
$$\mathbf{L}=\left\{l^{03}, l^{14}\ :\ \tfrac{2}{3} \le x \le 1\right\}$$
$$\mathbf{L}=\left\{l^{06}, l^{14}\ :\ 1 \le x \le \tfrac{4}{3}\right\}$$
$$\mathbf{L}=\left\{l^{06}\ :\ \tfrac{4}{3} \le x\right\}.$$

We can add some intuition to these results. If we define a Hirfindhal index as the sum of the squares of the proportion of the population residing in each region, then this index can take on 3 values, $\frac{3}{9}$, $\frac{5}{9}$, and 1, depending on whether the split is 1 consumer each region, 2 consumers in one region and 1 in another, or all three in one region respectively. The degree of concentration of the consumers represents the advantages garnered by consuming more of the public good, and is only tempered by the degree to which public goods are differentiated *in the eyes of the consumers*, which in our case is given by the parameter x. If x is large, there is no incentive to agglomerate in one region. In our example, the smallest Hirfindhal index Nash equilibrium appears only as $x \ge 1$, and is the only Nash equilibrium if $x \ge \frac{4}{3}$. The ubiquity of l^{14} for $x \le \frac{4}{3}$ is explained by the fact that it represents, for both consumer 1 and consumer 3, only a mild deviation from there ideal public good, and represents all the possible 'quantity' advantages from agglomeration. If x is small enough, the advantages from clustering outweigh the disadvantages associated with less preferred public goods, and we see l^{03} appear as an equilibrium. In this case the move to a region with a public good similar to one's ideal is warranted (providing someone already lives there), but the move to one with a significantly differentiated public good from one's own is not warranted (no matter how many residents there are there). Finally, for small values of x the incentive to cluster is so great that it is always warranted, so that (other than (2,2,2)), the high Hirfindhal index equilibria emerge.

Notice that (1,2,2), l^{05} that is, cannot be a Nash equilibrium. The reason for this is that if it is in consumer 3's interest to move to region 2 when there is only one resident, it must be in consumer 1's interest to move there when there is two. Hence, from our original 27 possible distributions of the consumers, we have eliminated 21 (or 78%) from the set of possible Nash equilibria.

What about efficiency? If there is to be a configuration of residences in which at least one consumer is better-off and none worse-off than the Nash equilibria, then it must be one that is formed by either two or three of the consumers relocating. If it involves a consumer relocating to a region with a less preferred public good than the one provided by their residence, then this candidate for a Pareto improvement must also involve a greater number of consumers at the new location than in Nash equilibrium. More formally, if $l = (l_1, l_2, l_3)$ is a Pareto improvement over $l' = (l'_1, l'_2, l'_3)$ then we can say the following:

$$\text{If } l_i = l'_i \text{ then } l_k \neq l'_k \text{ and } l_r \neq l'_r$$

$$\text{Where } i = 1,2,3 \;\; k = 2,3,1 \;\; r = 3,1,2$$

And, if we let N_j be the populations implied by l and N'_j be those implied by l', then we have:

$$\text{If } N_{l_i} \leq N_{l'_i} \text{ then } \mid m_i - p_{l_i} \mid < \mid m_i - p_{l'_i} \mid$$

This restricts the comparisons we have to make. For example, in looking at (2,2,2), the candidates for Pareto improvements are *none*, since it is impossible to change anything without making consumer 2 worse-off. The same applies to (1,1,1) of course. Things are more complicated in the case of (1,1,3). Candidates for Pareto improvements are (2,2,2) and (3,3,3), recall that (1,2,3) and (1,3,3) are ruled out by definition of Nash equilibrium. We need to see if consumers 1 and 3 are no worse off in (2,2,2) than in (1,1,3), or if consumers 1 and 2 are no worse off in (3,3,3) than in (1,1,3). We have:

$$U_1(1,1,3) = Z - 1/2 \qquad U_1(2,2,2) = (1 - \frac{1}{2}x)(3/2)(Z - 1/2)$$

so that $U_1(1,1,3) \geq U_1(2,2,2)$ if $x \geq \frac{1}{6}$. Also:

$$U_3(1,1,3) = (1/2)(Z - 1/2) \qquad U_3(2,2,2) = (1 - \frac{1}{2}x)(3/2)(Z - 1/2)$$

so that $U_3(1,1,3) \geq U_3(2,2,2)$ if $x \geq \frac{2}{3}$. We know that $x \geq \frac{2}{3}$ is necessary for (1,1,3) to be a Nash equilibrium, hence when (1,1,3) is a Nash equilibrium it is Pareto optimal. Finally, we can consider (1,2,3). The only candidates for a Pareto improvement are (1,1,1), (2,2,2) and (3,3,3). If consumer 3 is no worse of in (1,1,1) than in (1,2,3) we know consumer 2 is no worse-off, and if consumer 3 is no worse-off in (2,2,2) than in (1,2,3) then we know consumer 1 is no worse-off too. If (1,1,1) is a Pareto improvement over (1,2,3) then so is (3,3,3). Moreover, if (1,1,1) is a Pareto improvement, then so is (2,2,2) since consumer 3 is better-off in (2,2,2) than in (1,1,1). Hence we need the following:

$$U_3(1,2,3) = 1/2(Z - 1/2) \qquad U_3(1,1,1) = (1-x)(3/2)(Z - 1/2)$$

from which it can be deduced that $U_3(1,1,1) \leq U_3(1,2,3)$ if $x \geq \frac{2}{3}$, which is included in the restriction on x required for (1,2,3) to be an equilibrium. Hence, all Nash equilibria are Pareto optimal in this simple scenario.

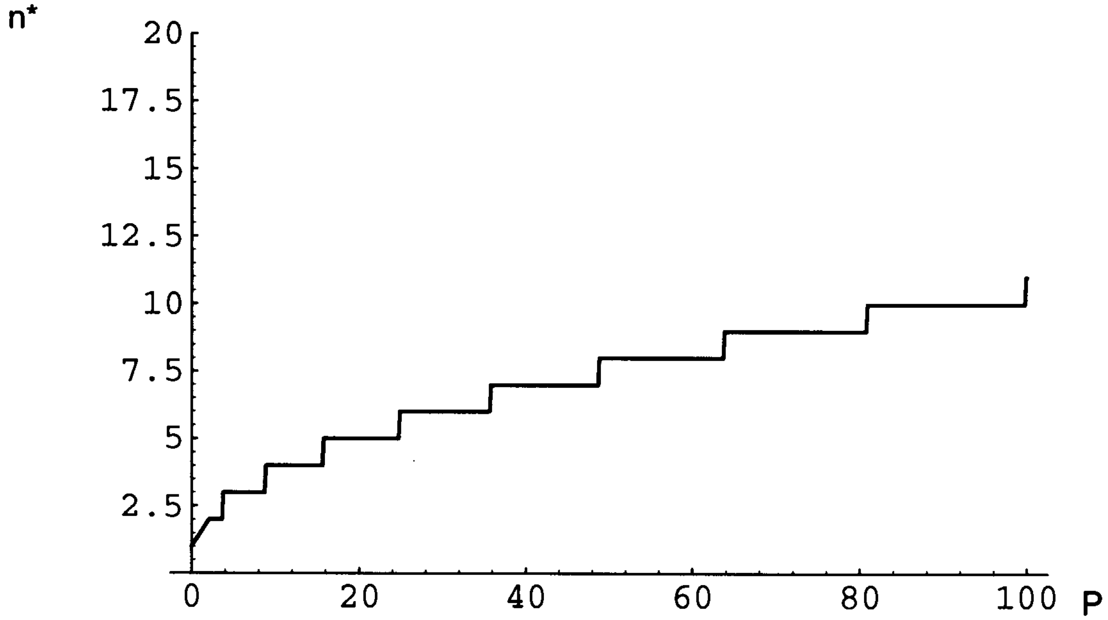

Figure 2.1: Equilibrium ticket purchases in the "more than fair lottery" in which all agents are risk-averse.

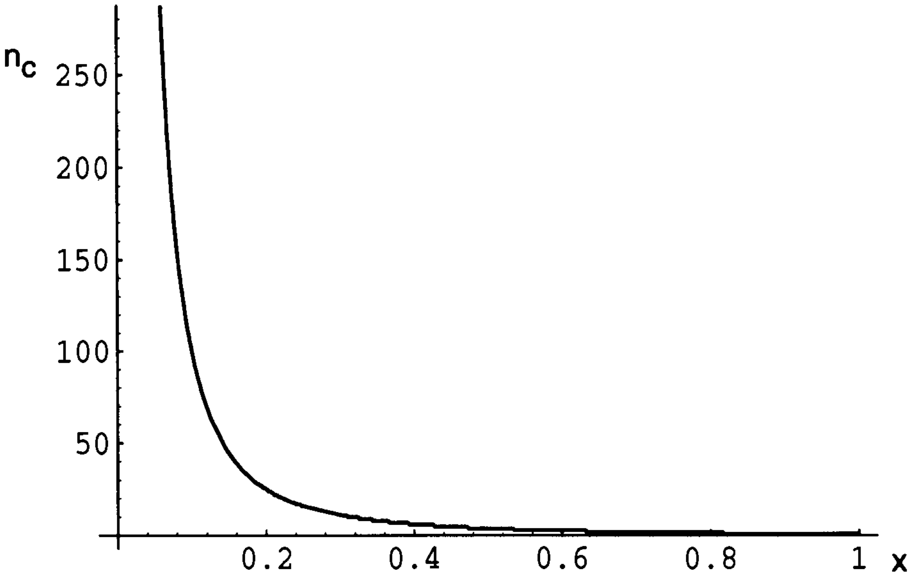

Figure 2.2: Critical values of n in the "less than fair lottery."

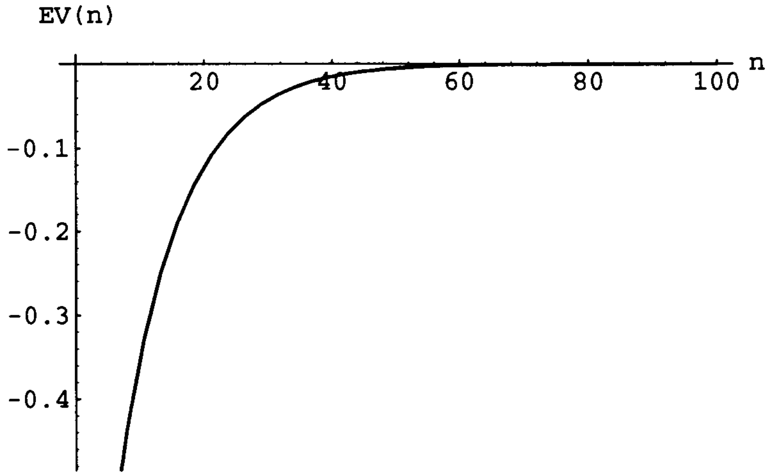

Figure 2.3: Expected value of one ticket in the asymptotically fair lottery.

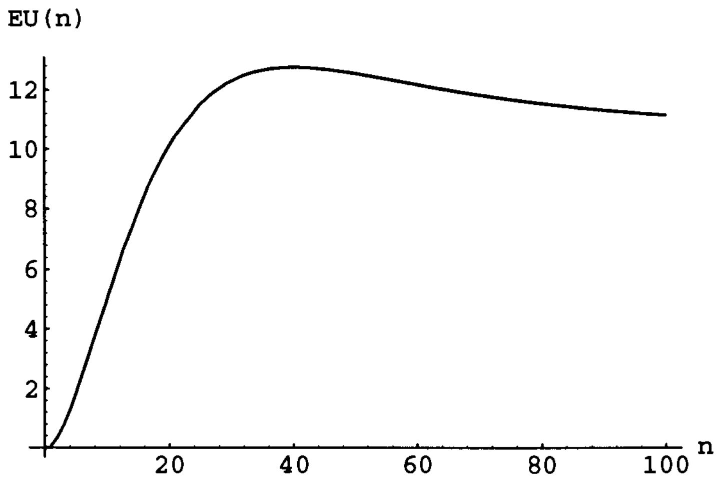

Figure 2.4: Expected utility in the asymptotically fair lottery.

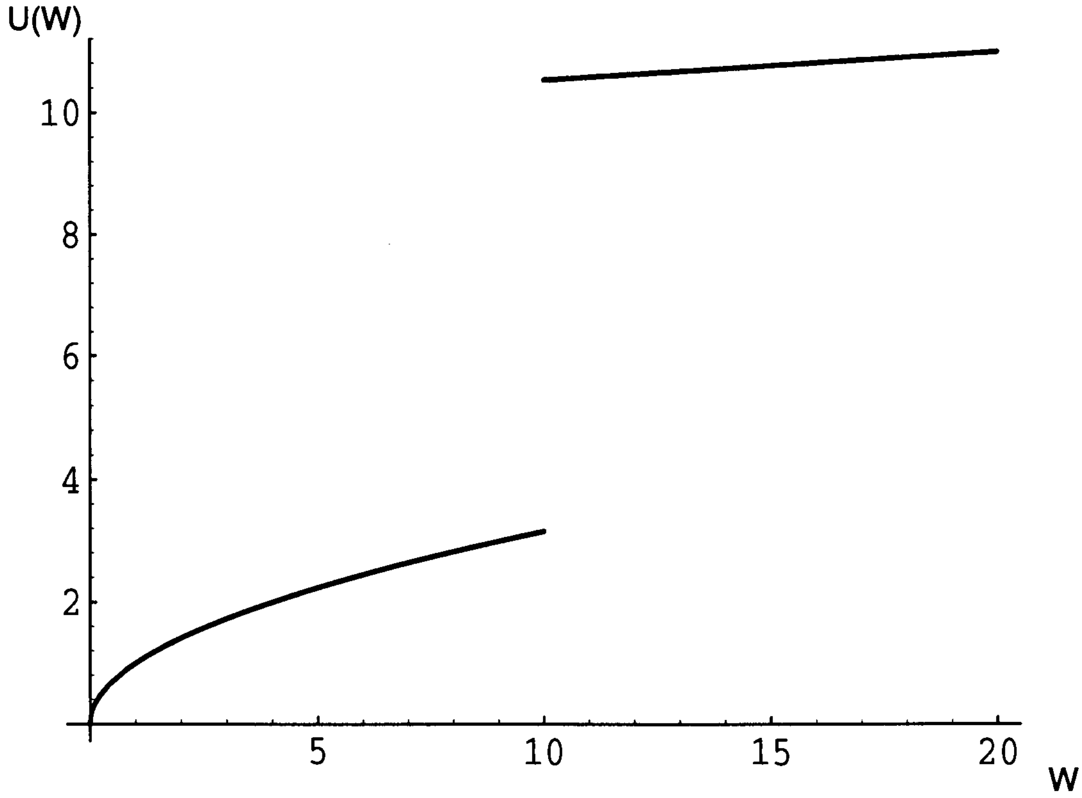

Figure 2.5: Discontinuous expected utility at a critical wealth level.

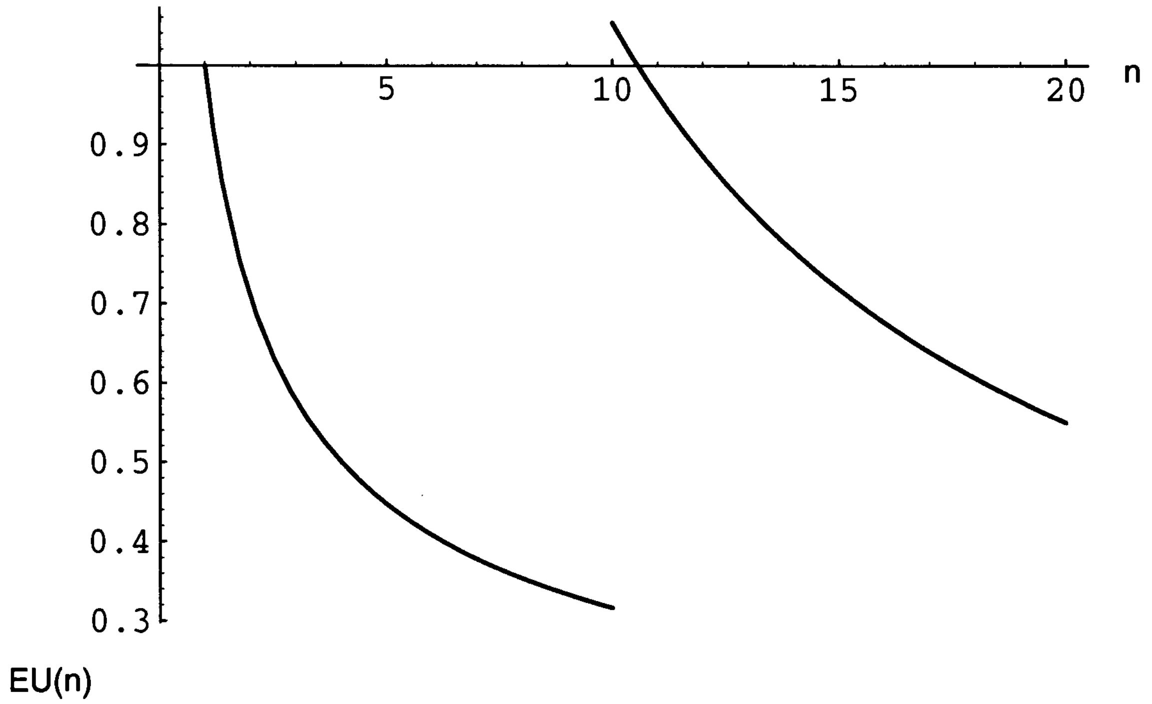

Figure 2.6: Expected utility of a single ticket, with n-1 other purchases.

Chapter 3

A Spatial Voting Model

One of the purest numbers externality is that involved in voting. Most of the other examples we have detailed involve 'impurities' of one form or another in the sense that the welfare impinging description of an activity is actually quite rich. For example, my enjoyment of an evening out will depend not only on the number of others in the establishment I choose, but also their general demeanour—which is itself a coverall term for a very complicated concept. When buying a lottery ticket I am concerned not with the number of others doing the same, but with the number of tickets they buy. And so on. Thus far I have been trying to isolate the idea of a numbers externality in these different contexts, and in doing so I have ignored many of the details of human interaction as an exercise in abstraction. I will now analyse the behaviour of voters in mass elections, and in particular the nature of the externality voters impose on one another.

The immediate observation is that the act of voting itself does not impose an externality on another, it is the choice made in the act of voting. Since voting is an anonymous and secret act, no one cares about the identity of voters, just whom they vote for. As far as anyone is concerned, all outcomes in elections are equivalent if they imply the same distribution of *votes* across candidates. In this respect we have an exact example of the social numbers externality.

This part of the thesis, its main contribution I think, tries to argue that the general policy positions of successful candidates in mass elections can be thought of as attractors in an address space of all possible candidate policy positions. The curious nature of the numbers externality involved in voting engenders the familiar coordination problem I have suggested is to be found under many circumstances involving numbers externalities. The institution of a *political party* has evolved to facilitate coordination, it is argued, and I will suggest that we have no reason to suppose that the successful attractors have any other desirable properties.

3.1 An Observation on Democratic Representation.

A feature common to all democracies is the presence of political *parties.* Candidates associate themselves with the name of some party, thereby associating themselves with a particular set of policy stances common to all other candidates thus associated. The number of such parties tends to quite small. In the United States there are only two political parties which are ever successful in garnering significant support. In the United Kingdom there are three major parties (although in Scotland and Northern Ireland others rival these three). In some countries there are more, but nowhere is there more than, say, ten political parties with any real chance of forming a government. A further feature of these democracies is that not only is there a small number of successful parties, but for long periods of time it is the *same* small number of parties that do relatively well. Political scientists have, of course, pondered the reason behind these features at length, and in the final section of this chapter some of their insights are presented.

The purpose of this chapter is to suggest an aspect of the mass electoral process which may help us understand these, and some other persistent features. The analysis is far from complete, but it is hoped that an intuition concerning the nature of party politics may be conveyed in what follows. The central question is this: why should there be so few political parties that have any chance of winning the election? In suggesting one possible contributory factor (there are perhaps many) we will come across certain other features of interest concerning the nature of those parties that do emerge as successful, and hence some preliminary insight the efficacy of the democratic process itself. However, all of what follows is tentative, and derived via superficial and artificial models of actual electoral processes. This chapter is intended to be merely suggestive, and not a formal analysis of any actual electoral system. And importantly, the model developed in this chapter is designed to highlight the similarities between the externality that drives agglomeration in the space of possible policy positions, and the others we have discussed so far.

3.2 The Hotelling-Downs tradition.

In this section I describe a simple rendition of a model of party politics due to Anthony Downs, Downs [1957], which was itself based on Hotelling [1929]. Downs' work has spawned the field known as "spatial voting theory," a sophisticated and popular branch of political science. What follows does scanty justice to the work of spatial voting theorists, and to both Hotelling and Downs. It is my own naïve construction used as a convenient expository device. The reader is referred to the final section for a brief summary of similar models from spatial voting theory.

Imagine an upcoming election in which there is but one issue of concern to the voting public, and that issue is the proportion of government revenue expended on welfare programs. A candidate's "platform" is a policy which will be implemented should that candidate be elected. Under plurality, the candidate obtaining the most votes is elected, and it is assumed subsequently implements his or her stated policy. The description of a candidate's policy position in this scenario is represented conveniently as a number between 0 and 1. We will further imagine that each of the voters has a particular policy stance they like the most, which may or may not correspond to one of the platforms adopted by the collection of candidates. Hence, a voter's most preferred policy stance can also be represented by a number between 0 and 1.

The spatial analogy adopted by Hotelling and Downs is to suppose that when voters compare two candidates, any two that is, the one they would prefer to win the election is the one whose stated policy stance is "closest" to the voter's most preferred policy stance. Although we can be finicky about the meaning of "closest," the most obvious interpretation of the spatial analogy is that voters' preferences can be represented by utilising the absolute value of the difference between a candidate's policy stance and the voter's most preferred policy stance. For example, let a voter have most preferred policy stance $\alpha \in [0,1]$, and let a candidate "A" have policy $a \in [0,1]$, and a candidate "B" have policy $b \in [0,1]$. Viewed as real numbers the distance candidate A's policy is from the voter's ideal is $d(\alpha, a) = ((\alpha - a)^2)^{1/2}$, and that of candidate B's is $d(\alpha, b) = ((\alpha - b)^2)^{1/2}$. The essence of the spatial model is to suppose that the voter in question here would prefer candidate A to win the election if, and only if, $d(\alpha, a) < d(\alpha, b)$, and would be indifferent between the two if $d(\alpha, a) = d(\alpha, b)$.[1]

There is the irresistible analogy to physical space in this formulation; candidates can be thought of as "locating" themselves on a line of unit length, and voters can be thought of as located on that line at a position reflecting their most preferred policies. In this spirit, we will use the word "address" to describe a candidate's announced policy, and the phrase "most-preferred address," abbreviated to *mpa*, to describe a voter's most preferred policy. Of course, the reason I say "irresistable analogy" is because of the original Hotelling formulation.

Suppose that the goal of the candidates is to maximise the number of votes they will get come election time, and they do so by choosing their respective addresses. For their part, the voters choose to vote for the candidate with address closest to their *mpas*. The propriety of this behaviour is examined carefully in the next section.

It is assumed that voters' *mpas* are evenly distributed on [0, 1], and that there are a total of D voters—all of whom will vote. We will need to be careful about the precise meaning of "evenly distributed" shortly.

The predictions, or 'explanations,' of this model are based on the assumption that the candidates' chosen addresses will form a Nash equilibrium; the candidates will choose addresses so that each of them could not obtain more votes by choosing another address, given the address of the other.

It is immediate that for two candidates, any combination of addresses that are different do not form a Nash equilibrium: either candidate could attract more votes

[1]Since d is a metric we have $d(x, y) = d(y, x)$, which may not be warranted.

simply by choosing an address nearer to the other. This reasoning allows us to assert that in a Nash equilibrium there must be zero distance between the addresses of the two candidates. However we can state a little more. If the candidates choose the same address, and that address is greater than 1/2, then either could obtain more votes by choosing an address slightly smaller. A similar argument rules out the possibility of a Nash equilibrium in which both candidates have the same address, and that address is less than 1/2. This leaves only one possibility for Nash equilibrium; both candidates choose "1/2".

Matters become more complicated in the case of three or more candidates.[2] In fact there is no Nash equilibrium in the case of three candidates: the candidates always wish to locate as near as possible to one another without choosing an identical address, in which case one of them is in the "middle" and gets no votes. It must be preferable to choose some other address for such a candidate, making another the candidate in the middle, so to speak. Since there is always a candidate that can do better by choosing another location, there is no Nash equilibrium. There are Nash equilibria with larger numbers of parties. What there is *not* is any natural extension of the model which *endogenises* the number of parties, and hence we have no insights into the observations of the opening paragraphs.[3] It has also been noted that many of the results of the Hotelling-Downs framework are extremely fragile when there is a stochastic element in voter's choices—see Anderson, Kats and Thisse [1992]. At a more fundamental level, we can demonstrate that the characterisation of voter behaviour can be faulted So, without further ado, we shall begin a reformulation.

3.3 Representing voters

If voters are to have an interest in voting, they must consider themselves to have "mass"—that is they must believe their vote to contribute non-negligibly to subsequent events. In this light we must be careful in our interpretation of the Hotelling

[2]See Eaton and Lipsey [1975] for the details of what follows (in the context of product differentiation).

[3]There is some work on "entry-deterrence" in the Hotelling-Downs framework. See Enelow and Hinich [1990], ch. 2.

-Downs assumption that "voters are evenly distributed ..." The only sensible view appears to be that, come election time as far as the candidate's can tell, voters' *mpas* will be *drawn from* a particular probability distribution, and that there are a finite number of these voters. In this way we allow the possibility of voters having an importance to justify their concern with elections at all.

In this regard, we will develop an alternative diagrammatic technique to the one typically used, retaining the "Hotelling line" to represent a policy space, and indicate candidate's platforms as points on this line. A voter will be represented by a box, and if a particular voter chooses to support some candidate we will draw the box above the address of that candidate. In the interior of the box is indicated the voter's *mpa.* The boxes are stacked if more than one voter is offering support for any one candidate. For example Figure 3.1 illustrates the case in which two candidate's, candidate *1* and candidate *2*, have platforms located at 1/4 and 3/4 respectively. There are a total of eight voters, all behaving as suggested by the Hotelling-Downs assumptions: the five voters choosing to support candidate *1* have *mpas* closer to 1/4 than to 3/4, those three supporting candidate *2* have *mpas* closer to 3/4 than to 1/4. A three-candidate, five-voter scenario is also illustrated. The candidates, *1*, *2*, and *3*, have addresses 1/4, 1/2, and 3/4 respectively. A quick look at the *mpas* of the voters indicates they are behaving according to Hotelling-Downs assumptions.

3.4 The Rationality of Voting

Whenever one talks of "rational" behaviour in economics, it seems something quite specific is intended. Rational decisions follow an algorithmic procedure:

(1) A specification of available actions (in the context).

(2) Reasoning from each available actions to an outcome.

(3) An expression of preference over the possible outcomes.

(4) Choice of action which engenders the most preferred outcome.

As a change of emphasis let us consider not the rational behaviour of candidates in the Hotelling-Downs model, but that of the *voters.*

Consider first those voters depicted in Figure 3.1a, and in particular any one of the voters supporting candidate *2*. Each of these has (or had) *three* available actions: to vote for candidate *1*, to vote for candidate *2*, or, we must allow, *not to vote at all.* The third alternative, abstention, is an important omission of our discussion so far. If we consider that, from the point of view of each of the voters taken individually, given the actions of the other seven voters, all three of these actions lead to the same outcome – a win for candidate *1* – then if we are to accept this result in a model as an analogue of real world events, then our explanation is somewhat empty. It seems unreasonable to suppose that voting is a random act, devoid of purpose, a choice made on arbitrary grounds since all choices are equivalent in the way they impinge upon one's personal well-being. To be specific, each voter in this figure is being rational in the sense that given the choices of the others, they can do *no better* than vote as illustrated. They can also do no worse; in fact given all of their alternative ways of voting, they are equally well off. Consider the voter with *mpa* 0.8 in Figure 3.1b, and how he or she would consider her alternatives. Voting for candidate *2* implies a win for candidate *1*, so does a vote for candidate *1*, and so does abstention. This is true for each voter. So, certainly Figure 3.1b can represent an equilibrium, but there are many others possible, all involving voters indifferent between voting and not voting, and indifferent between voting for their chosen candidate or the other one. Surely, when we invoke equilibrium as an explanation, we have something more in mind?

Another way of saying this is that if, in the configuration illustrated by Figure 3.1a, there were even a *minuscule* shoe-leather cost involved in the act of voting (which there is), each of these voters would prefer to abstain than to vote as indicated. We certainly observe somewhat asymmetric outcomes in real elections, and we must conclude that voting is either intended to do more than choose winning candidates (which it is) or that if we want to build a model in which voting is designed solely to affect the outcome of an election, something will need to be added.[4]

Along similar lines, consider now the voters shown in Figure 3.1b, and specifically the voter choosing to vote for candidate 1. A moment's thought tells us that this voter must, necessarily, be able to improve his welfare (if you will) by choosing candidate

[4]If there is a shoe-leather cost involved in voting, the only equilibria in the Hotelling model, if they exist, have differences of at most one voter between any two candidates.

2[5], however we choose to resolve the case in which two parties are tied. If, for the purposes of argument we assume that if candidate *2* and candidate *3* accrue the same number of votes this can be viewed as them having an equal chance of winning the election, then the voter in question *must* prefer a 50-50 chance each of candidates *2* and *3* winning the election to a certain win for candidate *3*. Hence, given our definition of "rational," voting for the candidate with platform address closest to one's *mpa* can, strictly speaking, be irrational. Once again, however, we do observe asymmetric voting patterns similar to (at least the relative distribution of voters in) Figure 3.1.

Finally, consider the consequences of the traditional Hotelling equilibrium in which two candidates locate at the median of the distribution of voters *mpas*. Since all voters are indifferent between the two, no voter has any incentive to vote at all.

3.4.1 Stochastic voting—preview.

It may already be clear that we can resolve the apparent irrationality in the behaviour of voters depicted in these figures by suggesting that the behaviour was based on imperfect information. For example, the voters may have been unsure as to the exact addresses of the candidates (a possibility that has been explored in the spatial voting literature, see Enelow and Hinich [1984], ch. 7.). Another possibility is that voters are uncertain about the behaviour of other voters, and are consequently unaware of the possibility of improving their own welfare. For example, if the voter voting for candidate *1* in Figure 3.1b were unaware of the support for candidate *2*, or attaches a somewhat low probability that there is any support for candidate *2*, then we can understand his decision to vote for candidate *1*.

3.5 Voters' "Alignments" – an Addition to the Framework.

An important substantive distinction between the Hotelling-Downs formulation and the one at hand is introduced at this stage. Prior to actual elections we imagine a

[5]That is, engage in what is known as "strategic voting."

preliminary period in which the members of the electorate choose what is termed their *alignments.* At this stage we can view a voter's alignment as the candidate which that voter *intends* to vote for. The distinction between this intention and the actual act of voting is that the voter's alignment can be changed during this preamble to election day. A voter can also choose to (intend to) abstain.

In choosing alignments, voters take into consideration the way in which their decisions, combined with their perceptions of the intentions of other voters, affect what they can only discern as the *probability* that each candidate will win the upcoming election. The important point being that for the vast majority of possible voter alignments, these probabilities are nondegenerate. A useful way to think about this idea is as follows: consider an election in which candidates *physically* locate themselves to form a row. A candidate's position in the row is an indication of his or her platform. The voting public then form queues, standing behind their chosen candidate; the choice of queue corresponds to a choice of alignment. The voters are allowed at any time to "balk" – to change the queue or column they are standing in, and they can register the intention to abstain in some equally obvious way. It is presumed that that the length of the queue behind the candidate's bears some relationship to the probability that the candidate will win the upcoming election, but this information is insufficient to determine with confidence which candidate *will* win. We assume the probabilities are purely subjective evaluations. That is, throughout what follows the randomness involved in the electoral process is a property of the way voters think. This is important and should be borne in mind.

3.6 Voter Preferences: Expected Utility.

Step (4) of the schematic rational decision-making process of the previous section required the "expression of preference" over outcomes engendered by actions. With this in mind, let us denote N_j as the number of voters aligned with candidate j. We write $\mathbf{N} = (N_1, ..., N_n)$, in the case of n candidates, and $\mathbf{N}_{-j}$ denotes the vector $\mathbf{N}$ with the jth component removed. For any given distribution of the voters' alignments it is posited that candidate j has a *perceived* probability π_j of winning the (upcoming) election. This probability is given by $\pi(N_j; \mathbf{N}_{-j})$. For convenience, assume that these rules for generating probabilities are common across voters, and the function $\pi(N_j; \mathbf{N}_{-j})$ will be referred to as a *Political Probability Calculus*, or *PPC.*[6], The situation facing any particular voter can be fully described by a lottery of the form $(\pi_1, \ldots, \pi_n; 1, \ldots, n)$, which is to be interpreted as the lottery in which candidate 1 has probability π_1 of winning, and so on.[7] Just to be clear, consider a voter currently aligned with candidate 1, who considers aligning with candidate 2. This alters $\mathbf{N}$ from, say $(N_1, N_2, \ldots, N_n)$ to $(N_1 - 1, N_2 + 1, \ldots, N_n)$, and has a consequent affect on the π_j. If this voter, currently aligned with candidate 1, were to choose to register the intention to abstain, then $(N_1, \ldots, N_n)$ would become $(N_1 - 1, \ldots, N_n)$, again with a supposed impact upon the probabilities, and the relevant lottery.

In this reformulation we have retained the spatial analogy between preference and distance in the space of policy alternatives, and have assumed that the only reason to vote is to influence the chances that particular candidates win the election. Therefore, it is not the identity of a candidate that is important, but his or her platform. And a particular voter's perception of a candidate can be summarised, in the spatial framework, as the distance between that candidate's platform and the voter's *mpa.* Let candidate *j*'s platform be described by $p_j \in \Re^x$ and let voter *i*'s *mpa* be denoted $m_i \in \Re^x$. Then voter *i*'s perception of candidate j can be summarised via some metric $d(p_j, m_i)$. That is to say, the only relevant feature of candidate j, as far as voter i is concerned, is $d(p_j, m_i)$.

It is clear, then, that the appropriate representation of the lotteries faced by voters is $(\pi_1, \ldots, \pi_n; d(p_1, m_i), \ldots, d(p_n, m))$. We have now to complete our representation of voter preferences by postulating the existence of a von Neumann-Morgenstern expected utility function, common across voters, $u(d(p_j, m_i))$. We can therefore state

[6]One way to look at his notion of subjective probabilities and alignments is to consider the way one might evaluate the information conveyed in the many pre-election poll results. If it is announced that a particular party has attracted the support of, say, 38% of a sample of voters in a poll, while another has 32% and a third 30%, then it seems reasonable to say that this implies that the first party has *a good chance* of winning a plurality, but also that the other two have *some* chance.

[7]I have assumed that there can be only one winner.

that, for a given vector of candidate platforms, voter i will prefer one configuration of voter alignments, $\mathbf{N}$, to another configuration, say $\mathbf{N}^1$, if:

$$\sum_{j=1}^{n} \pi(N_j; \mathbf{N}_{-j}) u(d(p_j, m_i)) > \sum_{j=1}^{n} \pi(N_j^1; \mathbf{N}_{-j}^1) u(d(p_j, m_i)).$$

It can be seen that, for a given array of candidate addresses, and since $u(\cdot)$ is common across voters, we can reduce the arguments of the utility function to the N_j, or simply $\mathbf{N}$. So, in what follows we will write:

$$U(\mathbf{N}; m_i) = \sum_{j=1}^{n} \pi(N_j; \mathbf{N}_{-j}) u(d(p_j, m_i))$$

3.6.1 The PPC

There are some properties of a *PPC* that I have in mind that it is worthwile stating. I am considering scenarios in which there can be only one winner in an election. So, we assume that:

(A1)

$$\sum_{j=1}^{n} \pi(N_j; \mathbf{N}_{-j}) = 1.$$

I am also presuming that the only thing that differs across voters is their *mpas*. This implies that the identity of the voters that make up a particular N_j is irrelevant (as is j). Hence if we let $\mathbf{N}_{-j}^1$ be a permutation of $\mathbf{N}_{-j}$, and let N_k be a component of both of them:

(A2)

$$\pi(N_j; \mathbf{N}_{-j}) = \pi(N_j, \mathbf{N}_{-j}^1),$$

$$\text{if } N_j = N_k \text{ then } \pi(N_j; \mathbf{N}_{-j}) = \pi(N_k; \mathbf{N}_{-k}).$$

Intuition imposes some other restrictions, I think. If a candidate gains the support of a voter who was previously registering the intention to abstain, then it seems reasonable that the candidate has a better chance of winning the election, and everyone else a worse chance:

(A3)

$$\text{Let } \mathbf{N} = (\ldots N_j \ldots) \text{ and } \mathbf{N}^1 = (\ldots N_j + 1 \ldots), \text{ then :}$$
$$\pi(N_j; \mathbf{N}_{-j}) < \pi(N_j + 1; \mathbf{N}^1_{-j}), \text{ and,}$$
$$\pi(N_k; \mathbf{N}_{-k}) > \pi(N_k; \mathbf{N}^1_{-k}), \text{ for } k \neq j.$$

Now consider the effect of a voter realigning with j who was previously aligned with k. This should have the same net effect as the sum of the changes in π_j and π_k that would occur if this voter registered the intention to abstain (having previously been aligned with k) and then chose to align with j. This yields the obvious conclusion that π_j increases and π_k decreases. But what about the others? I can see no *a priori* reason to say anything about them individually. We can say that the sum of the changes in the other probabilities equals $\pi_j - \pi_k$, but not much else. Moreover, there seems no reason to suppose, in general, that $\pi_j - \pi_k = 0$. This is an interesting observation we shall return to.

Finally I make the assumption that a candidate withth no voter support has a probability of winning the election that is independent of the distribution of the actively aligned voters' alignments, this is **(A4)**. One way to justify such an assumption is by saying that if a candidate has no voter support the only way they can expect to win an election is if nobody votes at all, and if this happens we assume all candidates have an equal chance of forming government.

3.6.2 Interpreting the PPC

The PPC is a difficult concept. What voters actually choose to do is *vote* in the final analysis, not to 'align.' However, as I have stated before, voting is a totally anonymous act, and there is absolutely no information about other voters (in a particular region) available at the moment the voting decision is made. In this regard, I chose the construct of an 'alignment' to illustrate the idea of a numbers externality, or 'bandwagon effect', in this context. An alternative formulation would be to consider successive elections, and model voters with a particular expectations structure that would map from a history of election results into an expectation of vote distributions in the future. This makes the game played among voters quite complicated, and as yet I have not considered such a model.

There is one thing that must be said about the *PPC* and that concerns the way voters imagine the impact of their own decisions. Consider a voter who has changed his mind: he no longer intends to vote for candidate i, and has decided to vote for j. If the election were now to take place, this voter will indeed vote for j, I presume. Before entering the poll booth, there would be an assessment of π_i and π_j on which the alignment decision was made, but now it is time to vote: is it still rational to vote for j? The problem is that the voter now knows voter j will receive his vote for *certain*, and for the alignment decision and the voting decision to be consonant, it must be assumed that alignment decisions are considered permanent when they are assessed, and no account of any future realignments considered. Perhaps the clearest way to think of a *PPC* is along the following lines: each voter chooses an alignment and makes a guess at the N_j. Since information is imperfect, the voters must conjecture a distribution that the N_j's have, not point estimates. Describe the distributions by a c.d.f, say F_j, so that if we define V_j as the number of votes actually cast for candidate j then:

$$\textbf{Prob}(V_j \geq x) = F_j(x)$$

One way to think about these estimates in the following way: in the run up to the election the voters sample a small number of fellow voters (or observe poll information). The sample information is given by $\mathbf{s}= (s_1, \ldots, s_j)$, where s_i is an estimate of N_i. From these samples a distribution over the actual N_j can be constructed, with support $[s_j, m - \sum_{i \neq j} s_j]$. This distribution is f_j, from which F_j can be estimated. Naturally these estimates will be voter specific, which undermines some of the assumptions I have made.

To avoid (for the present) these difficulties, I prefer the following kind characterisations: the N_j are known by all, but there are two types of voter: those who when asked about their intentions tell the truth, and those when asked do not, or there are two types of responses in polls: responses from those who will vote, and express their exact intention, and those who express preference for a candidate but who probably will not vote at all.[8]. Given these characterisations, if the propensity to lie or to be

[8]This final characterisation is explored fully in later sections

uninformed is not correlated with expressed intentions, the characterisation of the *PPC* of the previous section is warranted almost in whole, save that voters who know they have told the truth in a poll will suppose the amount of votes cast for their nominated candidate has a floor of 1.

3.7 Equilibrium.

A Nash equilibrium obtains when the voters arrange themselves in such a way that none of them have anything to gain (could increase utility) by either choosing to align with another candidate (if they are aligned with any), choosing to register abstention if they have not, or choosing to align with any candidate if currently registering abstention.

To be a little more formal, a set of voter alignments and abstentions represented by $\mathbf{N}$, forms a Nash equilibrium if:

For all voters, i, aligned with a candidate j

$$U(\mathbf{N}; m_i) \geq U(\ldots N_j - 1 \ldots; m_i) \text{ and}$$

$$U(\mathbf{N}; m_i) \geq U(\ldots N_j - 1 \ldots, N_k + 1 \ldots; m_i) \text{ for all } k \neq j,$$

and, for all voters z registering abstention,

$$U(\mathbf{N}; m_z) \geq U(\ldots N_x + 1 \ldots; m_z) \text{ for all } x.$$

This Nash equilibrium definition has the property that in equilibrium, if voters know the actions of others (or basically just the N_j), they can do no better than choose their own action. A Nash equilibrium would also obtain if, in the full knowledge of the N_j voters continually adjust their decisions to the most preferred ones, on the basis that their own movements will not induce reactions from others, until there are no preferred choices. There is no need to view Nash equilibrium as a result of such a process, of course. In fact another feature of Nash equilibria is that such a set of choices would obtain if it were *expected* to occur.

3.8 Randomness and meaningful equilibrium

Consider again Figure 3.1a, and let us determine circumstances in which these voters can rationally participate as shown. The diagram is now interpreted as depicting alignments, not actual votes cast (although the two could be the same thing *ex post*). We have the following specifications:

Voter *1, 2, 3, 4, 5, 6, 7, 8* has *mpa*,

$$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (0.0, 0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 0.8)$$

Candidate *1* has address, $p_1 = 1/4$, candidate *2* has address $p_2 = 3/4$.

Policy space is [0,1].

Now suppose the following:

$$\pi_j(N_j; \mathbf{N}) = N_j / \sum_{x=1}^{n} N_x$$

$$u(d(\cdot)) = -d(p_j, m_i) = -((m_i - p_j)^2)^{1/2}$$

These specifications are made, at this point to show that randomness *per se* can render such an equilibrium more appealing as a prediction of actual behaviour than the noiseless version of previously. We will have a deal more to say about the nature of the *PPC* shortly.

We can now state what is required for the distribution of voters and lack of abstention in Figure 3.1a to be a Nash equilibrium. Consider first voter *1*. Representing the shoe-leather cost of voting by a utility penalty of s, voter 1's expected utility is given by:

$$U(5,3;0) = -[(5/8)(1/4) + (3/8)(3/4)] - s$$
$$= -7/16 - s.$$

If voter *1* were to abstain, the new level of expected utility is,

$$U(4,3;0) = -[(4/7)(1/4) + (3/7)(3/4)]$$

$$= -13/28.$$

And if voter *1* were to realign with candidate *2*, we have,

$$U(4,4;0) = -[(4/8)(1/4) + (4/8)(3/4)] - s$$
$$= -1/2 - s.$$

In fact, it is possible to show that if this voter chooses to vote at all it will only be for candidate *1*, so I do not calculate expected utility involved with this voter choosing the other candidates. So, if voter *1* is in equilibrium (neither prefers to abstain nor vote for candidate *2*), then we have the following inequality:

$$s \leq 3/112$$

Similar calculations indicate that if $s \leq 0.107$, the voters could rationally participate as shown (it is voter *5* that makes this constraint binding, interestingly enough).

This is intended to show that the mere presence of randomness can justify distributions of voters across candidates like those in Figures 3.1a and 3.1b. In configurations of *alignments* in which there is a difference of more than one voter between parties, there can be *some chance* that each voter can make a difference. Voters do not consider themselves as necessarily pivotal, as they would in a typical Hotelling-Downs equilibrium with a shoe-leather cost, nor do they consider themselves as engaging in a meaningless act, as they would in the vast array of Hotelling-Downs equilibria without shoe-leather cost and one party having at least two more votes than the rest. Rather, voters align themselves with a candidate on the understanding that they *may* make a difference, and that potential influence on the future course of events motivates them to act.

3.9 Agglomeration—a 'Convexity' in the PPC?

With the specified function for generating the π_j in the previous example, it is, in fact, always rational to vote for the nearest candidate if a voter chooses to vote at all.

To see this, consider again Figure 3.1b, the three candidate scenario. We have:

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (0.2, 0.4, 0.5, 0.7, 0.8, 0.9)$$
$$(p_1, p_2, p_3) = (0.25, 0.50, 0.75)$$
$$\mathbf{N} = (1, 2, 3)$$
$$\pi_1 = \pi(1; 2, 3)$$
$$\pi_2 = \pi(2; 1, 3)$$
$$\pi_3 = \pi(3; 1, 2) = 1 - \pi(1; 2, 3) - \pi(2; 1, 3)$$
$$U(1, 2, 3; 0.2) = \pi(1; 2, 3)u(d(0.2, 0.25)) + \pi(2; 1, 3)u(d(0.2, 0.5))$$
$$+(1 - \pi(1; 2, 3) - \pi(2; 1, 3))u(d(0.2, 0.75)).$$

It is suggestive to ask the following question: what must be true if voter *1* were to prefer to align with candidate *2* than to align with candidate *1*?[9]

There is a decrease in the probability that candidate *1* wins and an increase in the probability that candidate *2* wins. If voter 1's utility is to unambiguously increase, there must be a decrease in the probability that candidate *3* wins. That is,

$$U(1; 2, 3) < U(0; 3, 3) \Rightarrow \pi_3(3; 1, 2) < \pi_3(3; 0, 3), \text{ or,}$$
$$U(1; 2, 3) < U(0; 3, 3) \Rightarrow (\pi_1(1; 2, 3) + \pi_2(2; 1, 3)) - (\pi_1(0; 3, 3) + \pi_2(3; 0, 3)) < 0.$$

In the previous example, the specification of the π_j does not satisfy this condition. Since $\pi_j = \frac{N_i}{N}$ in that example, it is as though one voter is drawn at random and asked who he or she would like to win the election. Naturally, they would choose their most-preferred candidate. To be more precise, the realignment of a voter simply increases one probability and decreases another by the same amount, all other probabilities remain the same. The intuition of this chapter is that, even if there were many candidates to choose from in an election, candidates with addresses as diverse even as voters' preferences, there is a natural tendency for voters to pool their support. Clearly

[9] I am ignoring the possibility of abstention at this point.

this implies that the manner in which voters reason from alignments to probabilities is somewhat different to the version of the previous section. The number of candidates that could attract voter support is not limited by the behaviour of voters under such circumstances, and is a large as the number of voters in the absence of other considerations. Note that randomness of the type suggested in this section would imply that if one party had the support of say, 1 million voters, prior to the election, and ten parties had the support of 100,000 voters, then the probability that the larger party wins the election is the same as the probability it does not. Introspection seems to rule out this as a realistic property. The feature of the randomness of our current example that will be of interest is that a voter realigning from one candidate to another, say from i to j, there is no change in the probability that one of i or j win the election, and no change in any other probability at all. We will refer to a probability calculus with this property as a *constant sum calculus*, or *CSC*. Formally, a *PPC* is a *CSC* (sorry), if it satisfies **(A1)**, **(A2)**, **(A3)**, **(A4)** and:

(A5)

$$\text{If } \exists N_j \geq N_k > 0 \text{ and } N_l > 0, \text{ where } l \neq k, k \neq j, l \neq j \text{ and let,}$$
$$\mathbf{N} = (\ldots N_j \ldots N_i \ldots N_l \ldots) \text{ and } \mathbf{N}^1 = (\ldots N_j + 1 \ldots N_i - 1 \ldots N_l \ldots), \text{ then:}$$
$$\pi(N_j + 1; \mathbf{N}^1) + \pi(N_k - 1; \mathbf{N}^1) = \pi(N_j; \mathbf{N}) + \pi(N_k; \mathbf{N}), \text{ and,}$$
$$\pi(N_l; \mathbf{N}) = \pi(N_l; \mathbf{N}^1).$$

3.10 Another example, with agglomeration.

We now presume that the probabilities that each candidate wins the election are given by the following process (or, more properly, the voters perceive randomness in a way that is identical to the following process): if a voter i is aligned with a candidate j, then there is a probability x that voter i will indeed vote for candidate j, but a probability $(1 - x)$ that voter i *does not vote at all.* Perhaps this is due to some particularly hazardous route to the polling booth. In this fictional electoral process, if there is a tie for the most votes cast for two or more candidates, some unbiased random mechanism decides the winner (so if there are h candidates tied, each can be viewed as having a $1/h$ probability of winning the election). Even in spite of the dangers of the trip to vote, these voters never *choose* to abstain (!)

Notice that the number of votes actually cast for a candidate is a binomial random variable. So, if we denote by c_j the number of votes cast for candidate j we have:

$$\Pr(c_j = r : N_j) = \frac{N_j!}{r!(N-r)!}x^r(1-x)^{N_j-r} \qquad \text{if } r \leq N_j$$
$$= 0 \qquad \text{if } r > N_j$$

There are to be five voters in this new scenario, and five candidates. Perhaps the candidates are the voters, and if they were we might imagine them announcing their respective *mpas* as addresses (this is an assumption, not a result). Specifically we have the following:

$$(m_1, m_2, m_3, m_4, m_5) = (p_1, p_2, p_3, p_4, p_5) = (0, 0.25, 0.5, 0.75, 1)$$

The advent of ties makes the calculation of the *PPC* a little cumbersome, however a little patience yields the following expressions for it; *in order of magnitude*:[10]

$\pi(5; 0,0,0,0) = 1/5 + 4x - 8x^2 + 8x^3 - 4x^4 + (4/5)x^5$

$\pi(4; 1,0,0,0) = 1/5 + 3x - 6x^2 + 8x^3 - 6x^4 + (9/5)x^5$

$\pi(3; 1,1,0,0) = 1/5 + 2x - 4x^2 + 6x^3 - 4x^4 + (4/5)x^5$

$\pi(3; 2,0,0,0) = 1/5 + 2x - 4x^2 + 5x^3 - (7/2)x^4 + (13/10)x^5$

$\pi(2; 1,1,1,0) = 1/5 + x - 2x^2 + 3x^3 - (3/2)x^4 + (3/10)x^5$

$\pi(2; 2,1,0,0) = 1/5 + x - 2x^2 + (7/3)x^3 - (7/6)x^4 + (2/15)x^5$

$\pi(2; 3,0,0,0) = 1/5 + x - 2x^2 + x^3 + (1/2)x^4 - (7/10)x^5$

$\pi(1; 2,1,1,0) = 1/5 - (1/3)x^3 + (1/6)x^4 - (1/30)x^5$

$\pi(1; 2,2,0,0) = 1/5 - (2/3)x^3 + (1/3)x^4 + (2/15)x^5$

$\pi(1; 3,1,0,0) = 1/5 - x^3 + x^4 - (1/5)x^5$

$\pi(1; 4,0,0,0) = 1/5 - 2x^3 + 3x^4 - (6/5)x^5$

$\pi(0; \cdot) = 1/5 - x + 2x^2 - 2x^3 + x^4 - (1/5)x^5$

[10]Many of these can be written as " $= 1 - \ldots$," this is avoided so that an independent verification of each was possible by checking that appropriate combinations add up to one.

Although $\pi(2;2,1,0,0) > \pi(1;1,1,1,1) = 1/5 > \pi(1;2,1,1,0), \pi(1;1,1,1,1)$ can be less than, equal to, or greater than $\pi(2;3,0,0,0)$ depending on whether x is less than, equal to, or greater than (approximately) 0.810558. All thirteen of these probabilities are graphed in Figure 3.2.

Note that the probability a candidate with no voter support wins this election is independent of the distribution of the other five alignments (since the only way such a candidate can win the election is if no voter votes, regardless of alignment). There are 3125 (5^5) possible configurations of the five voters' alignments. In a simulation each of these was checked to see if it was a Nash equilibrium (abstention was not considered), and this for values of x between 0 and 1 in increments of 0.01. The results of this rather coarse analysis are presented below.[11] Figures 3.3 through 3.10 illustrate these results.

In the case where $x = 1$ we are essentially back to the Hotelling-Downs model.[12] Incidentally, it is not the chosen policy space that causes these type of equilibria, but a combination of the real-valued function $d(\cdot)$ and the differences in the addresses of candidates, along with the conjectured randomness. For example, Figure 3.11, shows an equilibrium with $x = 0.6$, the same utility function, but

$$(m_1, m_2, m_3, m_4, m_5) = (1, 1, 1, 2, 2)$$

and

$$(p_1, p_2, p_3, p_4, p_4) = (1, 2, 2.1, 3, 3)$$

There are many striking features of these equilibria, some of which will be mentioned shortly. The most important feature is that for a sufficiently large value of x agglomeration takes place. As x increases there is a propensity for voters to pool their support, in an orderly or systematic manner however. In fact, if we were to use the equivalent of a Hirfindhal index to measure the degree of concentration in these equilibria (the sum of the squares of the proportion of all voters aligned with each candidate), then the higher index equilibria appear in order as x increases. It is this aspect of the equilibria that constitutes the basic intuition of this chapter: there is a property of

[11] A copy of the QuickBASIC program is reproduced in an appendix.

[12] It is a relatively simple matter to deduce equilibria in this case.

the way voters perceive randomness in voting which dictates that they suppress the details of their beliefs and pool their voting power around a candidate (or party) who represents, in a broad sense, the basic tenets of their beliefs[13] .

Recall that with a *CSC* it is always rational to align with the "nearest" candidate, yet in this section we can observe the agglomeration which might, it is suggested, engender the institution of the "political party."[14] It has been argued that if a voter is to realign with a candidate when a preferred one is available, then it must be that there has been a decrease in some other, less preferred candidate's or candidates' probability of winning the election. Moreover, if such moves are to increase the degree of concentration of voters, then this effect must be valid only if the candidate who gains a voter had at least as many voters aligned with him (prior to the realignment) as did the candidate who loses the voter. In the current example this, for the most part, is the case. In fact, the only distributions of voters' alignments in which it is not the case are those in which only two candidates have any voter support. Let us piece this together.

Consider only distributions of voters in which at least three candidates have some voter support. In this case a voter is aligned with a candidate who has either 1, 2 or 3 voters aligned with him. If it is it 3, none of these voters can consider realigning with a candidate who currently has more voter support. If it is it 2, one of these voters could consider realigning to a candidate also with two voters aligned. A voter who is the sole supporter of a candidate could consider aligning with a candidate with it 1, 2 or 3 voters currently aligned. It can be shown that the realignment has an effect on the probabilities of winning consistent with discussions of the previous paragraph. In fact, a voter who realigns from a candidate k to a candidate l, where $N_k \leq N_l$ increases the probability that candidate l *or* candidate k wins the election, and decreases the probability that each of the other candidates win the election –

[13]This point may not be obvious. On inspection it can be seen that the groups of voters do not differ considerably in their *mpas*—we never observe voter *1* and voter *5* aligned with the same party for example.

[14]If it is not accepted that the major reason elections are dominated by parties is because of the nature of the randomness in voting, then at least it might be accepted that if there are many candidates, only a few will garner significant support—for reasons outlined in this chapter. I will argue that "parties" act as focal points in a coordination game between voters.

except for those with no voter support whose probability of winning the election is constant as we stated previously. That is:

For $0 < x < 1$

$$
\begin{array}{ll}
(a) & \pi(2;1,1,1,0) + \pi(0;2,1,1,1) - \pi(1;1,1,1,1) - \pi(1;1,1,1,1) > 0 \\
(b) & \pi(3;1,1,0,0) + \pi(0;3,1,1,0) - \pi(2;1,1,1,0) - \pi(1;2,1,1,0) > 0 \\
(c) & \pi(2;2,1,0,0) + \pi(0;2,2,1,0) - \pi(1;2,1,1,0) - \pi(1;2,1,1,0) > 0 \\
(d) & \pi(3;1,1,0,0) + \pi(1;3,1,0,0) - \pi(2;2,1,0,0) - \pi(2;2,1,0,0) > 0 \\
(e) & \pi(3;2,0,0,0) + \pi(0;3,2,0,0) - \pi(2;2,1,0,0) - \pi(1;2,2,0,0) > 0 \\
(f) & \pi(4;1,0,0,0) + \pi(0;4,1,0,0) - \pi(3;1,1,0,0) - \pi(1;3,1,0,0) > 0 \\
(g) & \pi(2;3,0,0,0) + \pi(0;3,2,0,0) - \pi(1;3,1,0,0) - \pi(1;3,1,0,0) > 0
\end{array}
$$

$$
\begin{array}{ll}
(h) & \pi(1;1,1,1,1) - \pi(1;2,1,1,0) > 0 \\
(i) & \pi(1;2,1,1,0) - \pi(1;2,2,0,0) > 0 \\
(j) & \pi(1;2,2,0,0) - \pi(1;3,1,0,0) > 0 \\
(k) & \pi(1;2,1,1,0) - \pi(1;3,1,0,0) > 0 \\
(l) & \pi(1;3,1,0,0) - \pi(1;4,0,0,0) > 0 \\
(m) & \pi(2;1,1,1,0) - \pi(2;2,1,0,0) > 0 \\
(n) & \pi(3;1,1,0,0) - \pi(3;2,0,0,0) > 0
\end{array}
$$

These inequalities are graphed in Figures 3.12 and 3.13. It is perhaps worth reiterating what they represent. They demonstrate that in all scenarios in which three or more candidates have at least one voter aligned with them, the conjectured randomness implies that a voter realigning from one candidate, say k, to another, say l, increases the probability that *either* k *or* l wins the election, and decreases the probability that at least one other candidate wins the election, if $N_k \leq N_l$. We will refer to a *PPC* with this property as an *increasing sum calculus*, or *ISC*. Even if the voter in question prefers k to l, there is a possible increase in expected utility involved in this realignment, which also has the effect of increasing the concentration of voters' alignments across candidates. Formally, an ISC satisfies:

(A6)

$$\text{If } \exists N_j \geq N_k > 0 \text{ and } N_l > 0, \text{ where } l \neq k, k \neq j, l \neq j \text{ and let,}$$

$$\mathbf{N} = (\ldots N_j \ldots N_i \ldots N_l \ldots) \text{ and } \mathbf{N}^1 = (\ldots N_j + 1 \ldots N_i - 1 \ldots N_l \ldots), \text{ then:}$$

$$\pi(N_j + 1; \mathbf{N}^1) + \pi(N_k - 1; \mathbf{N}^1) > \pi(N_j; \mathbf{N}) + \pi(N_k; \mathbf{N}), \text{ and,}$$

$$\pi(N_l; \mathbf{N}) < \pi(N_l; \mathbf{N}^1).$$

This property is as much the intuition of this chapter as is anything else. It is certainly not the only way of expressing the intuition, unfortunately.

3.10.1 The agglomeration effect at its barest.

The type of convexity exhibited by an *ISC* has, it is thought, a tendency to cause voters to pool their support. Stepping aside from the spatial context for a moment it is possible to see this effect in a simpler setup. Consider the possibility that there is a single issue which dominates an election—an issue on which there are only two possible policy positions. There are, we will say, four candidates in this election, two of which adopt one policy stance ("Yes" or whatever), and two of which adopt the other position ("No"). There m_1 voters who prefer the first two candidate's policy stance to the second, and m_2 voters of the other persuasion. If candidate *j*'s chance of winning the election is given by a *CSC*, say $N_j/(m_1 + m_2)$, then it would not matter how the voters distributed their alignments across the two candidates adopting a particular policy position, since the probability that one or the other wins is always (when voters act rationally, and ignoring abstentions), $m_1/(m_1+m_2)$ or $m_2/(m_1+m_2)$ depending on which policy stance we are talking about. Should the relationship between alignments and probabilities constitute an *ISC*, so that a voter currently aligned with candidate j can increase the probability that either candidate j or some candidate k wins the election, where $N_k \geq N_j$, by aligning with k, then there will be but two candidates with any voter support in equilibrium. If we presume some initial initial distribution of alignments, the continual realignment of voters will generate two increasing masses of voters offering support for two lucky candidates.

This extends easily to the case of many candidates of each persuasion: there is an unambiguous increase in utility associated with a realignment to a candidate with at least as much support as the one a voter is currently aligned with (providing, of course, that the candidate is of the appropriate policy type) since this increases the probability that the winner of the election is of the preferred type. The reason for this is straightforward: the realignment, say from k to j, increases the probability that either k or j win the election. Call this increase Δy. There is a decrease in the probability that any one of the other candidates win summing to $-\Delta y$, but only part of this is accounted for by those candidates of the same type as k and j, and hence the probability that such a type wins has increased.[15]

3.11 Focal points and political parties.

The previous section indicates that there may be forces at work which limit the number of successful candidates in an election. In particular, the *ISC* dictates that in an election in which there is a dominant dichotomous issue, there may be only one successful candidate of each of the two policy positions. This would certainly be the case if there were many issues at stake but voter preferences were lexicographic.[16] This does not in any way imply that in successive elections it will be the same candidate. Nor does it imply that if the election took place in various regions within a nation that a candidate who associates himself with a wider organisational name will be any more successful than an independent with an identical policy stance. It is suggested, however, that political parties act as *focal points*: they are a means of one candidate distinguishing him or herself from other identical (or similar) ones, and becomes the

[15]Notice that it is important to assume that the probability of winning for *each* individual candidate, not involved in the realignment, decreases.

[16]If there were a dichotomous issue and a policy described by an element of the unit interval, then a voter with lexicographic preferences would prefer all candidates adopting his or her preferred stance on the dichotomous issue to all candidate disagreeing, but among those agreeing, those with continuous address closer to the voter's *mpa* would be preferred to those further away.

centre of voter attention, at least among those voters of a particular group or type. This effect would be most pronounced if voters are aware that the relationship between alignments and probabilities does indeed exhibit the properties of an *ISC*, but they are unsure of the N_j. The question becomes: how can one be certain that there has been a coordinated effort on the part of like minded voters to maximise the probability that a candidate of their mutually preferred type wins? It is commonly accepted that in situations of this sort there is a tendency for individuals to solve the coordination problem by use of focal points: strategic options that are manifestly distinguishable from alternatives in a manner that may be extrinsic to the choice-theoretic problem at hand.[17]

One such way a candidate can distinguish himself is to associate with a larger organisation that has sufficient resources to promote a name in association with a policy stance. Presumably this is an option open to other candidates, but it also seems reasonable to assume that once a name has been established, there is a natural barrier to entry, in that there can be only one focal point, and once established as such, a party has another distinguishing feature (previous success) that reinforces the effect.

Extending all of this to the spatial framework suggests that the democratic process may lead to unpredictable outcomes, perhaps suboptimal[18]. By analogy consider an externality similar to the one present in electoral processes, a tale hinted at in Chapter 1. This is based on a true story, ... In a town in England there are just a few pubs for the local inhabitants to enjoy. For convenience let's say it is two (quite a bit else will be said purely for convenience). People in the town do not communicate or organise themselves very well, but do act in their own self-interest. Each night they must decide which of the two pubs to go to, and there are two influences affecting their decision: they enjoy the company of others, and always prefer the same bar with more people in; but they also enjoy a nicely decorated pub. However their preferences are

[17] As we have had occasion to mention previously, an idea introduced by Schelling [1960].

[18] It is difficult to define optimality in a model of voting. If define the outcome of the democratic process as the location of successful political parties, then if we can discover another equilibrium set of successful parties, and the expected utility of all voters is higher (as defined in this chapter), then we have a Pareto comparison. See the next paragraphs.

actually lexicographic—given a choice they always prefer the bar with more people in, regardless of the quality of decoration, but between two bars with the same number of people attending, they prefer the better decorated. Initially the bars are equally physically attractive, and a Nash equilibrium would emerge in which only one bar is patronised.

If, over time, the wealthy management of the popular pub were to allow the place to deteriorate, while the starving owner of the other tries to woo patrons with ever fancier adornment, the Nash equilibrium we would observe (of the two possible) would remain as is—a Pareto inferior one, unless some kind of coordination takes place. Providing the pub patrons behave non-cooperatively they become entrenched and immobile in the lower quality establishment.[19]

Could it be true that the parties we vote for are successful because they are successful, and so are some distasteful alternatives? I think our example suggests this possibility at least. Consider a slight variant on the five voter simulation in which the randomness is of the same sort, but voters' *mpas* are given by

$$(m_1, m_2, m_3, m_4, m_5) = (1, 1, 1, 2, 2)$$

and candidates' addresses by

$$(p_1, p_2, p_3, p_4, p_5) = (1, 2, 2.1, 3, 3).$$

If $x = 0.6$, there is an equilibrium illustrated in Figure 3.11. If we were to remove candidates *3*, *4*, and *5*, the only Nash equilibrium is one in which *all* voters are better off. If we were to remove candidates *2* and *4*, the only Nash equilibrium is a possibility when candidates *2* and *4* were present. Hence, their successful entry is barred.[20]

3.11.1 Differentiation

I considered the role of an ISC in ensuring just one successful candidate in the case of a dichtomous issue. We can use a pseudo (δ, ϵ) argument to extend this to the spatial

[19]This, of course, is the reason for drinks specials and opening day promotions and the like. It might be possible to think of credible analogies for political parties.

[20]At least, their simultaneous entry.

framework. In particular I prove for a three candidate scenario that in equilibrium the candidates cannot have the same address and positive voter support. Use the notation EDA to refer to an equilirium distribution of parties.

Let there be three parties with locations l_1, l_2, l_3.

The number of voters aligned with party i is $N_i > 1$.

The probability party 1 wins the election is $\pi_1(N_1; N_2, N_3)$

Assume that $\pi_i(N_i; N_j, N_k) + \pi_j(N_j; N_i, N_k) < \pi_i(N_i + 1; N_j - 1, N_k) + \pi_j(N_j - 1; N_i + 1, N_k)$ if $N_i \geq N_j$.

Proposition: In any EDA the parties have different addresses.

Proof: Let $N_2 \geq N_3$ by choice of subscripts.

Let a voter currently aligned with party 3 have *mpa* α.

Let $U(\alpha, P_i)$ be this voter's expected utility function.

Expected utility is given by:

$$\pi_1 U(\alpha, p_1) + \pi_2 U(\alpha, p_2) + \pi_3 U(\alpha, p_3)$$

Let $p_2 = p_3$.

Expected utility becomes:

$$EU_1 = (1 - \pi)U(\alpha, p_1) + \pi U(\alpha, p_2)$$

where $\pi = \pi_2 + \pi_3$.

If this voter realigns with party 2, expected utility becomes:

$$EU_2 = (1 - \pi - \Delta\pi)U(\alpha, p_1) + (\pi + \Delta\pi)U(\alpha, p_2)$$

If this voter is in equilibrium, $EU_1 \geq EU_2$, which implies $U(\alpha, p_1) < U(\alpha, p_1)$ since $\Delta\pi > 0$.

If this voter realigns with party 1, expected utility becomes:

$$EU_3 = (1 - \pi + \Delta_1\pi)U(\alpha, p_1) + (\pi - \Delta_1\pi)U(\alpha, p_2)$$

If this voter is in equilibrium, $EU_1 \geq EU_3$, which implies that $U(\alpha, p_1) > U(\alpha, p_2)$ since $\Delta_1\pi > 0$.

This is a contradiction, *Q.E.D.*

Since $\pi_x(0, \bullet)$ is independent of the distribution of non-abstainers, we can have as many such candidates as we want and the proof follows. Since there is a minimum distance between parties, if the policy space is bounded there is a maximum number of distinct parties that can exist in equilibrium.

3.12 Concluding Comments.

This chapter represents work which is most definitely "in progress," I think. It has been established that strategic voting only takes place if randomness is not of the *CSC* type. Other than this we cannot come up with a sufficient condition without specifying preferences. Future research will establish more precisely what we can and cannot say about equilibria. I can speculate however. It seems reasonable to presume that in equilibrium there will be a finite difference in the address of candidates, at least in the presence of effects like those in the *ISC* (a "local" *CSC* would allow coincident addresses). There are also a lot of features in the equilibria of the simulation that may persist in more general frameworks. Notice that we can choose any two voters and the one with the smaller *mpa* always aligns with a candidate with an address no larger than the other voter. This , I suspect may be very general. If it is true then equilibria will appear as if they were generated by Hotelling-Downs behaviour (or a *CSC*) if the addresses of candidates with no voter support are ignored.

No voter ever aligns with the candidate he or she considers the "worst" of the available alternatives, and hence there is always at least two candidates with some support. This is quite general. There is an obvious symmetry property which is, of course, very general. There are theorems that can be proven, and I have some that are yet to be proven. The point of this chapter is *not* to provide a formal analysis of the *ISC*, however, that is left to more able others should they have the time, and future research of my own.

The efficiency properties of equilibria will also need to be examined, and perhaps an interpretation of alternative institutional structures. Notice that in the examples of equilibria the median voter, many times, is not best served by a plurality system, which is suggestive. In some senses the five voter simulation is not well chosen for examining efficiency properties, but we do have our one example of a suboptimal equilibrium in Figure 3.11.

Finally, it might be feasible to consider a marriage of models in which candidates are strategic and so are voters, although at this stage it might be an ambitious undertaking. At present I can only claim to have pointed to a specific representation of the obvious externality between voters, and that, as usual, the presence of the externality in a non-cooperative situation may yield unpredictable and sub-optimal results.

3.12.1 Bibliographic notes.

This chapter is intended to be "self-contained," not requiring a knowledge of any previous literature, and I have avoided an excessive reference to other works. However, there has been a *considerable* amount of work surrounding the issues I have discussed, and I can only point to some of the major works.[21]

The impact of stochastic voting on the Hotelling-Downs model, as extended in Eaton and Lipsey [1975], has been analysed most recently in Anderson, Kats and Thisse [1992]. In that paper the number of candidates is exogenously given, and voters vote in some specified random manner. The analysis, then, almost necessarily focuses on the rational behaviour of candidates, who are presumed to be interested in the absolute number of votes cast for them, or the number of votes cast for them relative to other candidates. Osborne [1991] is another fairly recent examination of candidate strategies. The present work takes a different view. The number of candidates who are *successful*, and their policy positions, are presumed to be determined by sophisticated voter behaviour. The emphasis, then, is on the nature of possible outcomes, and a suggestive condition on the nature of uncertainty in the voting process which may explain why there are such small numbers of successful candidates in mass elections. In this way political parties are viewed as social institutions, not sentient beings – the sentient beings are the candidates.

[21]It might be remembered that an initial version of this paper, which contained all of the simulation results and t comments on focal points, was presented in the seminar series at Simon Fraser University in 1988, in ignorance of the more recent works that have made it into the published literature.

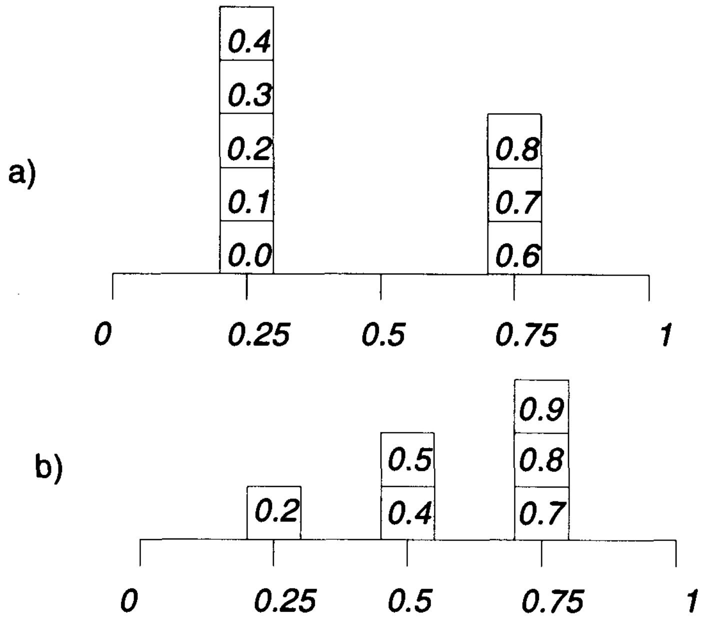

Figure 3.1: Illustrating "voters with mass."

In Feddersen, Sened, and Wright [1990] a model of candidate competition is developed in which voters choose among candidates "insincerely," which means they vote rationally in the sense of this chapter (although without any stochastic influence). In that paper it is shown that there are equilibria in which several candidates enter at the median of the distribution of voters' *mpas*. An entrant (or deviant) attracts the support of all voters that strictly prefer the entrant, and the remaining vote for exactly one of the candidates at the median. The authors comment that "a disturbing feature of the equilibrium set is that it depends upon implicit coordination (or cooperation) among voters."[22] I hope to have shed a little light on this and some other matters.

[22]Feddersen, Sened and Wright [1990], p.1014.

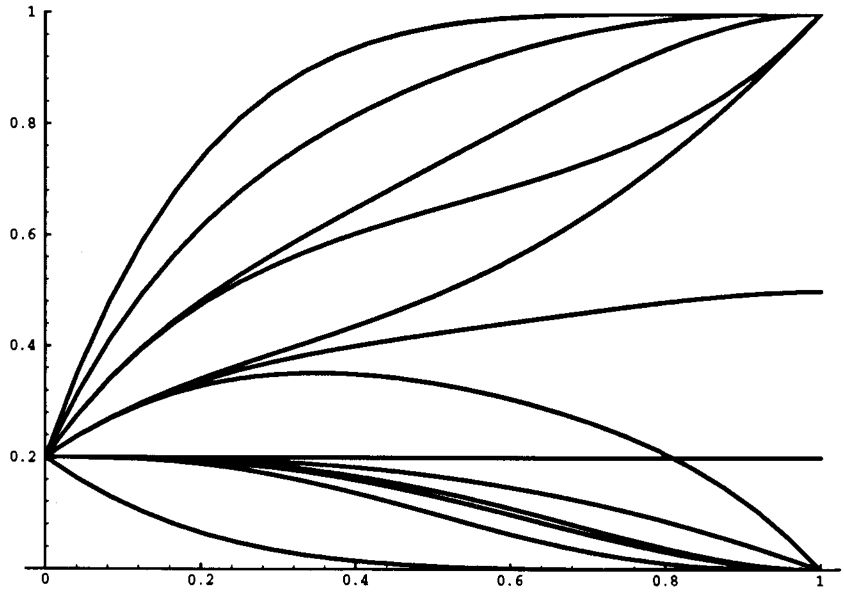

Figure 3.2: The thirteen winning probabilities in the five voter simulation.

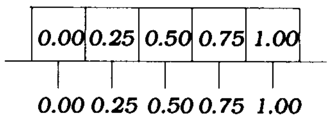

Figure 3.3: Equilibrium in the five voter simulation for $x < 0.49$

Figure 3.4: Equilibrium in the five voter simulation for $x = 0.50$

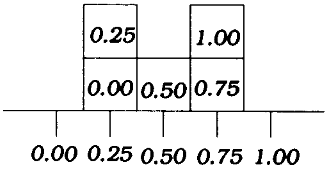

Figure 3.5: Equilibria in the five voter simulation for $x = 0.51$ and $x = 0.52$

Figure 3.6: Equilibria in the five voter simulation for $x = 0.53$ to 0.57

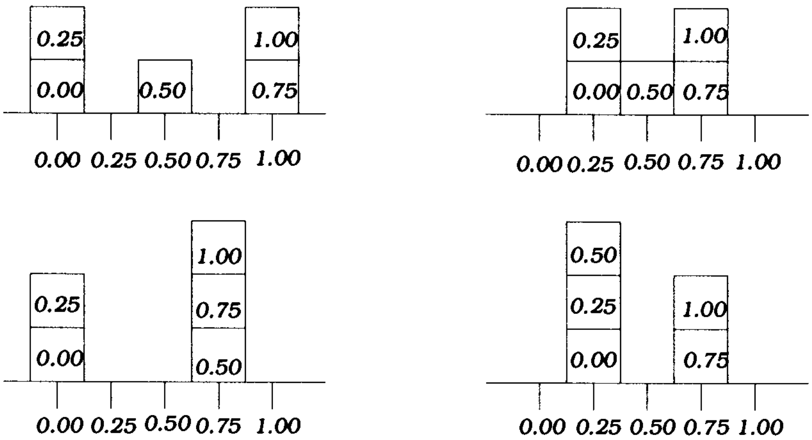

Figure 3.7: Equilibria in the five voter simulation for $x = 0.58$ to 0.63

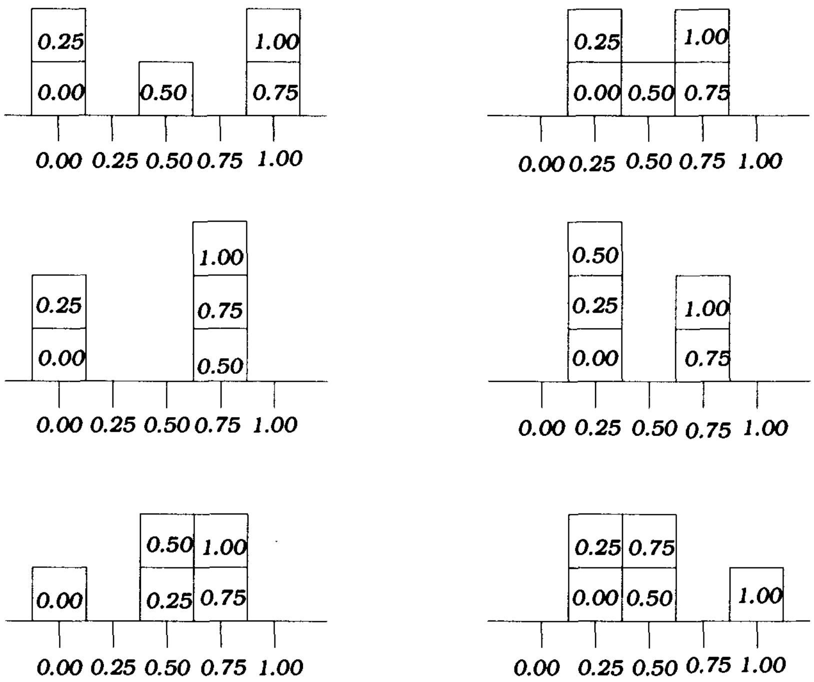

Figure 3.8: Equilibria in the five voter simulation for $x = 0.64$ to 0.66

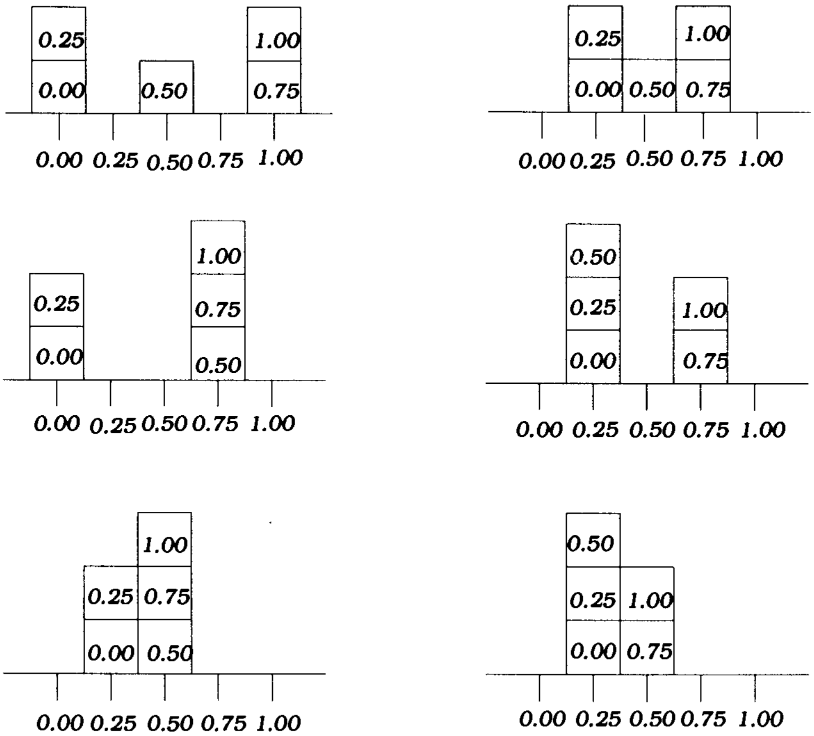

Figure 3.9: Equilibria in the five voter simulation for $x = 0.67$ to 0.71

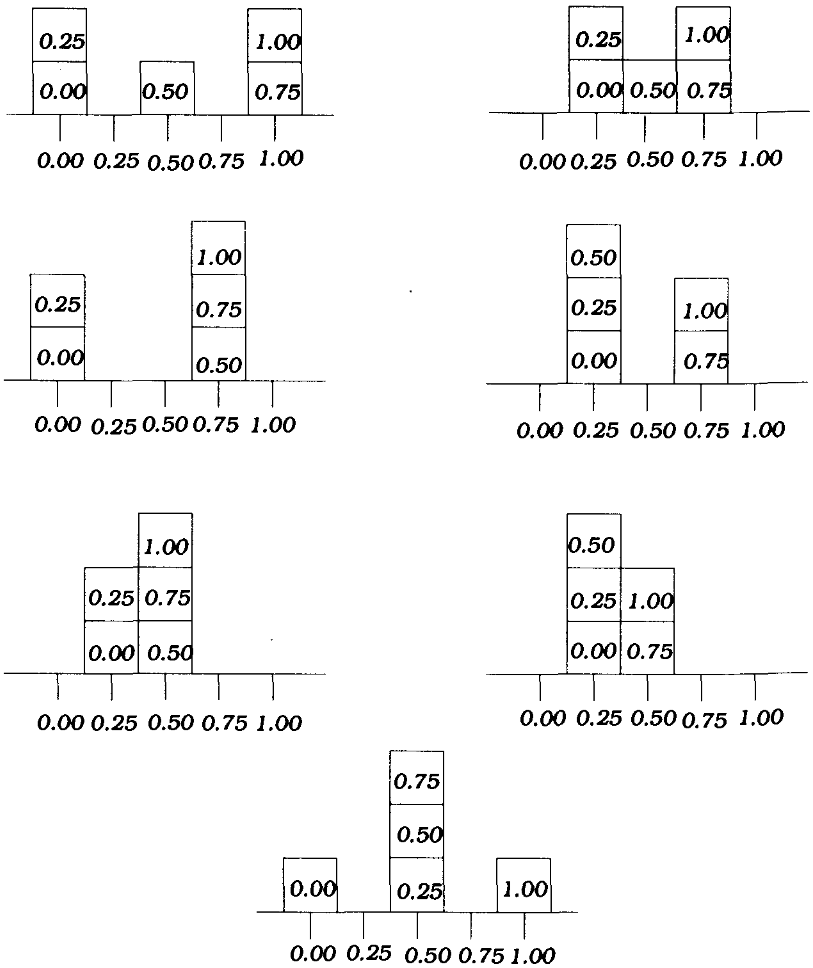

Figure 3.10: Equilibria in the five voter simulation for $x = 0.72$ to 0.99

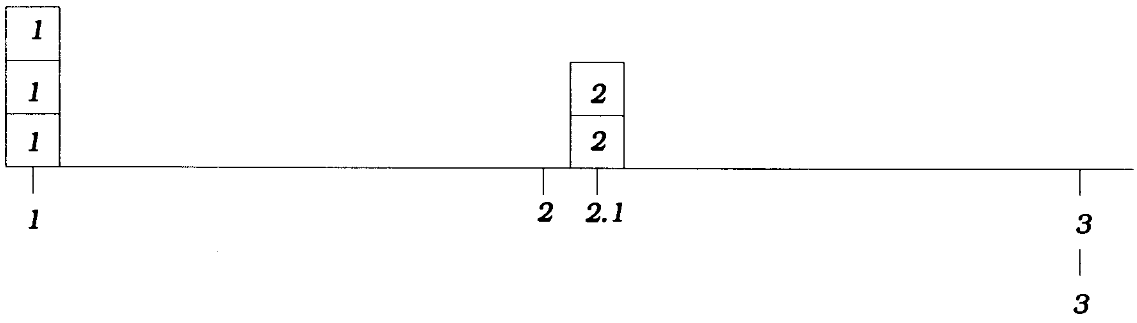

Figure 3.11: An unambiguously suboptimal equilibrium.

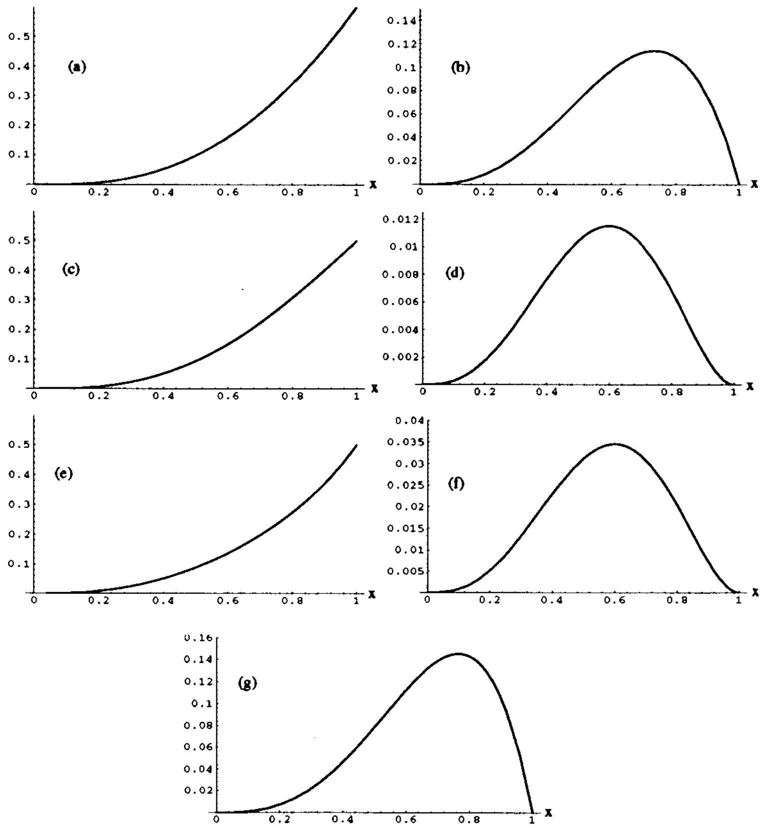

Figure 3.12: Inequalities (a)-(g) defining an *ISC* in the simulation.

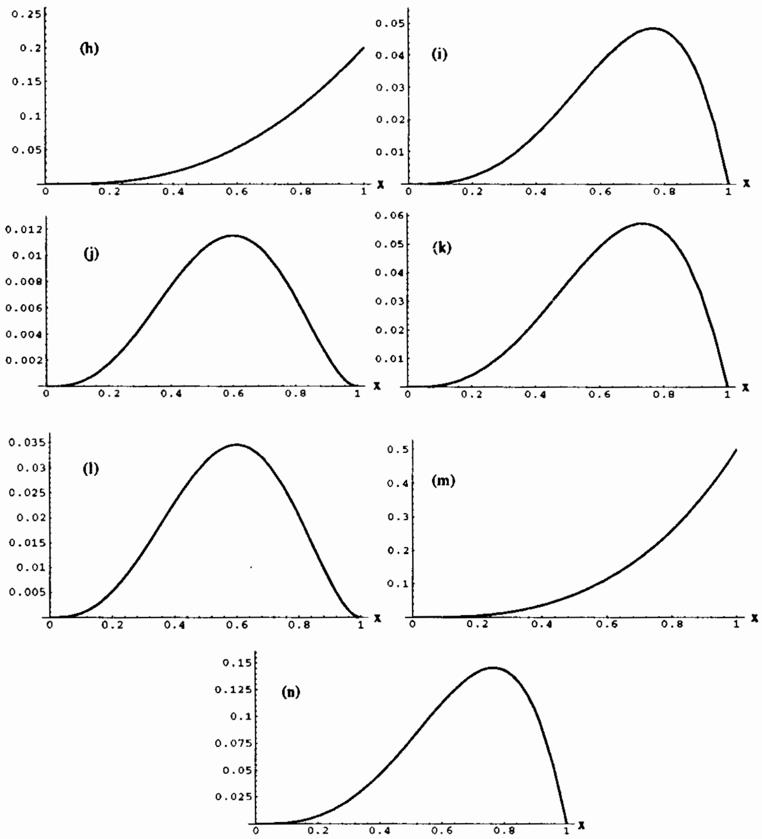

Figure 3.13: Inequalities (h)-(n) defining an *ISC* in the simulation.

Appendix: Source Code.

```
' FIVES
' Five voter simulation.
'
'
'
' By Paul Harrald
'
' Version 1.2
' In MS QuickBasic ("look no REMs or line numbers").
' Very un-QB though, GOSUBS all over the place. No SUBS. Did not
' use labels for lines. This is so it ports without too much editing.
'
' March 1st 1992.
'
'
' This program searches for equilibrium distributions of
' alignments in a five voter simulation. It is not fast,
' much of it written for clarity, not efficiency. SUBs and so on
' for speed, and CASE SELECTS in the subroutine at 3000.
' This version prints to screen.
'
' TODO: allow abstention. Choice of screen, printer, file for
'       output. Write speedy version for use, ignoring
```

```
'       impossible equilibria as per theorems.
'
' LATER: n-voters. Um...



   DEFDBL A-H, K-Z
   DEFINT I-J

'  INITIALISATION. CHANGE TO A USER INPUT ROUTINE?

   FOR I=1 TO 5
     READ Y,Z
     M(I)=X : L(I)=Y
   NEXT I
   DATA 0,0,0.15,0.25,0.5,0.5,0.75,0.75,1,1

10  CLS
    INPUT "VALUE OF K :"; K

'   CALCULATE EXPECTED UTILITY FUNCTIONS

    FOR I = 1 TO 5
      FOR J = 1 TO 5
        U(I,J)=-(ABS(M(I)-L(J))^K)
      NEXT J,I

'   MAIN LOOP PARAMETERS

    INPUT "MINIMUM VALUE OF X :"; XMIN
    INPUT "MAXIMUM VALUE OF X :"; XMAX
```

```
    INPUT "INTERVAL :"; XSTEP
    CLS : ' PART OF MAIN DISPLAY
    LOCATE 1: PRINT "****************************"
    LOCATE 4: PRINT "****************************"

'   MAIN LOOP

    FOR X=XMIN TO XMAX STEP XSTEP

'     CALCULATE ALL THE REQUIRED PROBABILITIES
      GOSUB 1000

      LOCATE 2: PRINT USING "*      K=###.###         *"; K
      LOCATE 3: PRINT USING "*      X=#.######        *"; X

'   BASIC CHECKING ROUTINE. NOT OPTIMISED!ALSO SHOULD USE SUBS, BUT WON'T PORT.
'   HUMPH! CAN'T USE ARRAY ELEMENTS AS INDEXES :(
      FOR I1 = 1 TO 5
        FOR I2 = 1 TO 5
          FOR I3 = 1 TO 5
            FOR I4 = 1 TO 5
              FOR I5 = 1 TO 5
                 LOCATE 12,20: PRINT "CHECKING "; I1; I2; I3; I4; I5

'                LOCATE VOTERS IN ALIGNMENTS, IPOS(I)

                 IPOS(1) = I1
                 IPOS(2) = I2
                 IPOS(3) = I3
                 IPOS(4) = I4
                 IPOS(5) = I5
```

```
'                 THEN FOR EACH VOTER....

                  FOR II = 1 TO 5

'                    FIND WINNING PROBABILITIES IN CURRENT POSITIONS...

                     GOSUB 1000

'                    FIND CURRENT UTILITY...

                     GOSUB 2000
                     A = UTIL

'                    NOW, FOR VOTER II, CHECK OTHER POSSIBLE ALIGNMENTS...

                     FOR INPOS = 1 TO 5
                       IF INPOS = IPOS(II) THEN NEXT
                       OLD = IPOS(II)
                       IPOS(II) = INPOS
                       GOSUB 1000
                       IF A < UTIL THEN

'                        NOT AN EQUILIBRIUM

                         IPOS(II) = OLD
                         GOTO 400
                       ENDIF
                     NEXT INPOS
'                  NEXT II
```

```
'                   IF WE REACHED HERE, MUST BE AN EQUILIBIRUM, SO SAY SO!

                    FOR IX = 1 TO 15
                       SOUND 40, 1
                       LOCATE 9, 25: PRINT I1, I2, I3, I4, I5
                       SOUND 50, 1
                       LOCATE 9, 25: PRINT "                         "
                    NEXT IX

400               NEXT I5, I4, I3, I2, I1
     NEXT X

     LOCATE 20, 25: PRINT "ANOTHER VALUE OF K? (Y/N)"
450  A$=INKEY$
     IF A$="" THEN 450
     IF A$="y" OR A$="Y" THEN 10
     IF A$="n" OR A$="N" THEN END ELSE 450

1000 ' SUBROUTINE TO CALCULATE ALL REQUIRED PROBABILITIES.

     X2 = X ^ 2 ' ONLY DO THIS ONCE PER LOOP, :)
     X3 = X ^ 3
     X4 = X ^ 4
     X5 = X ^ 5

     P50000 = 0.2 + 4 * X - 8 * X2 + 8 * X3 - 4 * X4 + (4/5) * X5
     P41000 = 0.2 + 3 * X - 6 * X2 + 8 * X3 - 6 * X4 + (9/5) * X5
     P31100 = 0.2 + 2 * X - 4 * X2 + 6 * X3 - 4 * X4 + (4/5) * X5
     P32000 = 0.2 + 2 * X - 4 * X2 + 5 * X3 - 3.5 * X4 + (13/10) * X5
     P21110 = 0.2 + X - 2 * X2 + 3 * X3 - (3/2) * X4 + (3/10) * X5
     P22100 = 0.2 + X - 2 * X2 + (7/3) * X3 - (7/6) * X4 + (2/15) * X5
```

```
    P23000 = 0.2 + X - 2 * X2 + X3 + (1/2) * X4 - (7/10) * X5
    P12110 = 0.2 - (1/3) * X3 + (1/6) * X4 - (1/30) * X5
    P12200 = 0.2 - (2/3) * X3 + (1/3) * X4 + (2/15) * X5
    P13100 = 0.2 - X3 + X4 - 0.2 * X5
    P14000 = 0.2 - 2 * X3 + 3 * X4 - (6/5) * X5
    P11111 = 0.2
    P0 = 0.2 - X + 2 * X2 - 2 * X3 + X4 - X5
    RETURN

2000 ' CALCULATES CURRENT EXPECTED UTILITY OF A VOTER, RETURNS IT AS UTIL

    GOSUB 3000: ' CALCULATE RELEVANT PROBABILITIES
    FOR J = 1 TO 5
      UTIL = UTIL + P(J) * U(II, J)
    NEXT J
    RETURN

3000 ' ASSIGNS APPROPRIATE PROBABILITIES. VERY UGLY AT PRESENT.

     FOR J1 = 1 TO 5
        IZ(J1) = 0
        FOR J2 = 1 TO 5
          IF IPOS(J2) = J1 THEN IZ(J1) = IZ(J1) + 1
        NEXT J2
     NEXT J1

     FOR J1 = 1 TO 5
       IN(J1) = 0
       FOR J2 = 1 TO 5
         IF IZ(J2) = J1 THEN IN(J1) = IN(J1) + 1
       NEXT J2
```

```
    NEXT J1

'   I *HATE* THESE NEXT LINES, BUT, WELL.... ACTUALLY 'CASE SELECT' ALL
'   THROUGH HERE, BUT WON'T PORT.

    IF IN(5) = 1 THEN
      FOR J = 1 TO 5
        IF IZ(J0 = 5 THEN P(J) = P50000 ELSE P(J) = P0
      NEXT J
      GOTO 3010: ' CAN'T REMEMBER IF I CAN RETURN FROM HERE OR NOT.
    END IF

    IF IN(4) = 1 THEN
      FOR J = 1 TO 5
        IF IZ(J) = 4 THEN P(J) = P41000
        IF IZ(J) = 1 THEN P(J) = P14000
        IF IZ(J) = 0 THEN P(J) = P0
      NEXT J
      GOTO 3010
    END IF

    IF IN(3) = 1 AND IN(1) = 2 THEN
      FOR J = 1 TO 5
        IF IZ(J) = 3 THEN P(J) = P32000
        IF IZ(J) = 2 THEN P(J) = P23000
        IF IZ(J) = 0 THEN P(J) = P0
      NEXT J
      GOTO 3010
    END IF

    IF IN(3) = 1 AND IN(1) = 2: ' A BIT REDUNDANT, BUT KEEP IT IN.
```

```
    FOR J = 1 TO 5
      IF IZ(J) = 3 THEN P(J) = P31100
      IF IZ(J) = 1 THEN P(J) = P13100
      IF IZ(J) = 0 THEN P(J) = P0
    NEXT J
    GOTO 3010
  END IF

  IF IN(2) = 2 THEN
    FOR J = 1 TO 5
      IF IZ(J) = 2 THEN P(J) = P22100
      IF IZ(J) = 1 THEN P(J) = P12100
      IF IZ(J) = 0 THEN P(J) = P0
    NEXT J
    GOTO 3010
  END IF

  IF IN(2) = 1 AND IN(1) = 3 THEN
    FOR J = 1 TO 5
      IF IZ(J) = 2 THEN P(J) = P21110
      IF IZ(J) = 1 THEN P(J) = P12110
      IF IZ(J) = 0 THEN P(J0 = P0
      GOTO 3010
    NEXT J
  ENDIF

  FOR J = 1 TO 5
    P(J0 = 1 / 5
  NEXT J

3010  RETURN
```

Chapter 4

The Acquisition of Human Language

Human language is, if we think for a second, a remarkable achievement of human ingenuity. In the following sections I try to develop an experimental design that epitomises the suggestions made in the previous part of this essay, although the experimental design itself will need to be expanded if a serious attempt at modeling language acquisition is to come of it. First some background materials are presented, including a simple evolutionary model, followed by the design itself and some mildly suggestive results.

4.1 Languages and Numbers Externalities

Much of the stability in the structure of language (a structure we will at least demonstrate in the next few sections) is due to two factors. One is a switching cost, the other a numbers externality. The fact that languages remain fairly stable is for a similar reason that the QWERTY keyboard is still popular, it is very difficult to retrain people if they have already learned some technique or other. It is difficult because for these people with an expertise, the gain in efficiency is not worth the switching cost.[1] However, this view must be complemented by an observation that the utility of a language is *first and foremost* dictated by the number of others that are also proficient in it. This is the essential externality of this thesis.

The English language does change over time, and perhaps becomes more efficient too. The presence of acronyms certainly reduces the time taken to communicate ideas, as do the contractions "don't," "I'll," "o'clock," "goodbye" and so on. The change in a language is necessarily slow, as we will see in an amusing letter to the *Economist* shortly, and usually makes slight alterations in accepted forms. If keyboards are to alter, then presumably it will be an equally slow process.[2]

It can be seen that the idea of a switching cost cannot be separated from the numbers externality when we look at language, because a proficiency in a language is only useful in so far as it allows communication with a large group of people.

4.2 Language and Speech.

Every normal human being has the same speech apparatus, and operates it in the same way. Nevertheless, man has evolved myriad manners of conveying his thoughts and wishes to others. It need not have been this way. An alternative world might have consisted of separate civilisations each developing its own language, and all eventually adopting the same one without any conscious act of coordination. Is this a monstrous impossibility, even less likely than one of the hundred chimpanzees typing a Shakespeare sonnet by chance? Well, I don't suppose anyone would suggest that separate civilisations would come up with the same sound to represent any particular object. The sound "dog" doesn't seem an inevitable verbal insinuation of the four-legged canine. Of course, the languages of civilisations that come into contact with one another do tend to converge slightly, usually just by adopting similar words for

[1] Although David [1985] indicate that if companies retrained secretarial staff to use a superior keyboard layout (CLIO), the costs would be recovered within a year. Also, staff that learned a new keyboard layout are unlikely to forget QWERTY, so they have little to lose too.

[2] The switching cost involved in changing keyboard design is not limited to those that use them for typing. Modern keyboards have dedicated cpu's that send specific signals to other computer parts. These would all have to be altered, or else some intervening software developed, if the basic keyboard design changed.

common objects or ideas. We would also expect that all languages would adopt easy words for ideas that must be conveyed a lot, like "I" and "me" for example. Can the same be said for grammatical structure? Is there some obvious structure that human beings naturally adopt? I cannot answer this question. However, one important point is that whatever sounds we choose to make in order to represent a particular idea, the more of us that agree on what sounds represent which ideas the better—our final example of the central externality I am discussing.

Naturally, it is not my purpose to survey the gargantuan literature on human language, but to point to areas that are of interest for this thesis. Eventually I will propose a very simple model of the evolution of languages, and most of what follows is background for that. If we all have the same speech apparatus, we must look for some environmental explanation for language diversity, and also for the evolution of languages after they come into conflict. First some general observations to make about languages and linguistics.

4.3 Human Languages

Conservative estimates put the number of languages currently in use at over 4000, and some estimate the number at 8000, which is quite remarkable. Of these, there are just twenty-nine that are spoken by more than thirty-million speakers.[3] We might immediately ask the question: when is one language different from another? This is quite interesting. Linguists adopt a fairly simple approach to this problem, which is quite natural; one language is considered to be different to another if speakers of the two languages cannot understand one another: *mutual intelligibility.* Such a definition

[3]This data was obtained from the Center for Applied Linguistics, in Washington D.C., so thanks to them. The languages and the number of speakers (defined as the language usually spoken at home) are as follows: Mandarin Chinese (720 million), English (305 million), Spanish (240 million), Arabic (150 million), Bengali (150 million), Russian (145 million), Portuguese (140 million), Hindi with Urdu (140 million), Japanese (120 million), German (105 million), Wu (Shanghai) Chinese (75 million), Javanese (66 million), Cantonese (66 million), Italian (65 million), Panjabi (63 million), Korean (61 million), French (60 million), Telugu (57 million), Marathi (56 million), Tamil (55 million), Vietnamese (50 million), Eastern Hindi (49 million), Ukranian (47 million), Bhojpuri (46 million), Amoy-Swatow Chinese (46 million), Turkish (42 million), Thai with Lao (41 million), Polish (40 million), Gujarati (34 million). Figures above 100 million are rounded to the nearest 5 million.

makes the distinction between dialect and language awkward since it does not make reference to structure at all, just phonology and morphology (the study of word sounds and word structure). It is not a definition that appeals to certain factions either: Serbs and Croatians speak a mutually intelligible language that linguists term 'Serbo-Croatian,' yet these two groups consider themselves to speak a different language.[4] In contrast, many people are heard to speak of the 'Chinese language,' even though the population of China speaks many mutually unintelligible languages, each of them with a number of distict dialects.

For my purpose, the most interesting problem with the mutual intelligibility definition is that it is not complete. Mutual intelligibility is a binary relation on the set of existing languages: two languages are either mutually intelligible or they are not. However, if language A is mutually intelligible with language B, and language B is mutually intelligible with language C, it does *not* follow that languages A and C are mutually intelligible.[5] For example, in the border area between Holland and Germany Dutch and German are mutually intelligible, but the Dutch of Amsterdam and the German of Munich are not. The same applies to Palestinian Arabic and Syrian Arabic—they are mutually intelligible, but Moroccan Arabic and Saudi Arabian Arabic are not.[6] The reason I find this interesting is because it suggests that languages can be 'merged' and something intermediate produced.

4.3.1 Invented Languages and Script.

It may be that language in general did evolve as an evolutionary process[7], but this is not true of all languages specifically. In fact there are numerous incidents of languages being designed from scratch, and written representations of languages too. Moreover, there are also attempts to manipulate current languages in order to simplify them. The Cherokee chief Sequoia devised a script for his tribe's use that consisted of symbols

[4]See Steinbergs [1987].

[5]See the evidence in Voegelin and Voegelin [1977].

[6]Steinbergs [1987], p.228.

[7]That is not to say that human language is a charactersitic that has evolved via natural selection in *homo sapiens*, just that languages themselves (here is one of those anthropomorphisms) undergo change in an evolutionary manner.

based on letters of the English alphabet and newly devised symbols.[8] It remained in use until 1903 when it was effectively replaced by English. In the nineteenth century the missionary J. Evans invented the syllabic script used by the Cree indians, and this script is still in use today across Canada. Curiously, the same symbols are used by the Inuit of Baffin Island to represent their own language, which is unrelated to Cree. Korean was once written as Chinese is today[9]. Chinese was not entirely suitable to represent Korean suffixes however, and in the middle of the fifteenth century the then King Sejong commissioned a suitable script, which is, for all intents and purposes, still in use today.

Changing the pictographic representation of a language is by necessity a slow process. This is very nicely illustrated in a letter to *The Economist* from one M.J. Shields:

> For example, in Year 1 that useless letter 'c' would be dropped to be replased by either 'k' or 's', and likewise 'x' would no longer be part of the alphabet. The only kase in which 'c' would be retained would be the 'ch' formation, which will be dealt with later. Year 2 might reform the 'w' spelling, so that 'which' and 'one' would take the same konsonant, wile Year 3 might well abolish 'y' replasing it with 'i' and Iear 4 might fiks the 'g-j' anomali wonse and for all.
>
> Jenrally, then, the improvement would kontinue iear by iear with Iear 5 doing awai with useless double konsonants, and Iers 6–12 or so modifaiing vowlz and the rimeining voist and unvoist konsonants. Bai Ier 15 or sou, it wud fainali be posible tu meik ius ov thi ridandant leterz 'c', 'y' and 'x'—bai now jast a memori in the maindz of ould doderers—to replais 'ch' 'sh' and 'th' respektivli.
>
> Fainali, xen, after sam 20 iers ov orxogrephkl riform, we wud hev a lojikl, kohirnt speling in ius xrewawt xe Ingly spiking werld...

[8] Anderson [1987].
[9] Chinese having been introduced in Korea in the fourteenth century

Of course, the point is that learning a new method of communicating does not help those already acquainted with the one in current use, but may help others in the future.

I cannot resist reporting the results from two studies which highlight the degree of structure of the English language, which we are essentially unaware of, that makes alterations of the kind suggested in the letter above difficult. Bennett [1977] used the text of *Hamlet* III i to estimate the relative frequency (unconditional) of the letters of the alphabet, and used these estimates to derive the following (no account was taken of the case of letters):

> AAOOAAORH ONNNDGELC TEFSISO VTALIDA POESDHEMHIESWON ...

... not much meaning. However, Bennett then made the probability that a symbol appear contingent upon the previous four characters that have appeared, generating:

> A GO THIS BABE AND JUDGMENT OF TIMEDIOUS RETCH AND RETCH AND NOT LORD WHAT IF THE EASELVES AND DO AND MAKE AND BASE GATHEM I AY BEATELLOUS WE PLAY MEANS HOLY FOOL MOUR WORK FROM INMOST BED BE CONFOULD HAVE MANY JUDGMENT ...

A follow up study, Hayes [1983], used eighth-order transition probabilities, taking account of case and puntuation, to generate:

> Did he receive them. No, not I; I never gave you well, well, well. Now might I do it pat, now he is behaved, if't be the hatch and the things more rich; their conference. If she find him not, nor stands it safe with us to let his madness, keeps aloof, when he is drunk asleep, or in that should not have better commerce than with him; and am I then revenged, to take arms against a sea of troubles, and both neglected love, the glass of fashion...

This demonstrates "that language is highly predictable" (Fogel [1992, p. 473]), and that as language has evolved it has become highly structured. Mozart wrote a pamphlet explaining how to compose as many German waltzes as one wants by throwing dice. Bennett suggests that artistic genius may simply be the inherent characteristic of having the requisite higher-order correlations.

It comes as no surprise, then, that is the invention of a verbal *language* is unlikely to be successful, although I suspect this is more due to our numbers externality than anything else. The best known example, of course is *Esperanto* which is still not in widespread use. A rough estimate indicates that there are about 4000 speakers of *esperanto*, almost all are linguists.... My central point here is that much of language has *evolved*, it does not form part of a deductive process by which we have decided upon what structure should characterise a language. We see this becuase the structure we have just uncovered comes as a novel surprise (or at least to me).

4.3.2 Languages in General

Most of the analysis of this Chapter applies to all modes of communicating information. A striking example of this is the attempts of computer users to pass information from one to another: compatibility. Consider the problem of passing on a screen image. Such images are stored in one kind of format or another, which means that a file containing just ASCII characters, and maybe a few more, summarises the information regarding what colour (or shade of grey) a particular pixel is.[10] A piece of software I happen to use shows no less than 23 different formats for such pictures, and an image viewer designed for just one of these could not in any way read a file in one of the other formats, although any one format is compatible with many different platforms.[11] The same applies to programming languages. These are all just alternative ways of mapping from fundamental binary code, or machine code, into a more readable code a user can input. If all programs were written to solve the same or a

[10] Of course, the ASCII (American Standard Code for Information Interchange) is itself a convention for passing on information at a more fundamental level. The numbers externality has been sufficiently effective that ASCII has become (thankfully) quite standard for most day-to-day file transfers. ASCII, though, is not adequate for transferring complicated objects such as compiled code for example, and for such the less readable binary format is used.

[11] The software in question is a NeXT application written by Lennart Lövstrand, *ImageViewer*, which is free. I happen to have version 0.9e.

very similar task, then only one would need to exist, and this would be optimal, since all programming languages are fundamentally designed to perform the same act of translation. If more than one such language did exist, it might stay in use for as long as the current generation of users used it—presuming that learning an alternative is sufficiently costly. Since there is a familiar numbers externality involved we would expect new programmers to use the language (they perceive to be) in greatest use. Hence there is a natural tendency for one program to emerge.

The co-existence of many programming languages indicates the variety of uses for computer programs.[12] However, all computer users have one task in common (pretty much) and that is printing documents. A task that is almost as common is incorporating diagrams and figures into these documents. This problem is one of making marks on a page, and one language has become dominant in storing these instructions to be passed on to a printer: *Postscript* and a variant *Encapsulated PostScript.* These two formats contain printing instructions in the aforementioned ASCII format, and have successfully taken advantage of the established popularity of that convention.

4.4 A Model of Language Use

This section attempts to show how verbal communication may have emerged to solve interactions in which coordination makes agents better-off due to complementarities in human interactions. It is an example of an undifferentiated numbers externality game, and gives us a look at a dynamic process by which the complicated coordination problem in a game of language acquisition might emerge.

[12]Software packages like *Mathematica*, *SHAZAM*, TEX and so on are all programming languages, specifically designed to solve a common kind of task.

4.4.1 Description of the Model.

An infinite population of players meet in random pairwise encounters to play a 2×2 game[13] with payoffs as given in Figure 4.1—a 'discoordination' game.[14]

Prior to playing this game, the two players have the option of making a sound from a finite vocabulary, $V = \{w_0, w_1\}$. Only one player makes a sound, and this player is randomly chosen by Nature to do so.

Players are characterised by the interpretations they place on each word, where an interpretation is as an intention to play either X or Y. Hence, a player's language is a function $l : V \to \{X, Y\}$. There are four possible languages:

$$
\begin{array}{ll}
l_1(w_0) = X & l_1(w_1) = X \\
l_2(w_0) = Y & l_2(w_1) = Y \\
l_3(w_0) = X & l_3(w_1) = Y \\
l_4(w_0) = Y & l_4(w_1) = X
\end{array}
$$

If a player is selected by Nature to speak, then that player utters a word from V at random, and subsequently chooses from $\{X, Y\}$ according to the interpretation placed on the sound made. If a player is not the one to speak, then that player listens to the word uttered by the other, interprets it according to their own language, and then makes a best-response to the interpreted move.

4.4.2 Nash Equilibrium.

There are several ways to define equilibrium. Since an agent's language fully determines play, we can look for Nash equilibria in languages. There are two pure strategy Nash equilibria, $< l_1, l_2 >$ and $< l_3, l_4 >$, which correspond to the two Nash equilibria in the end-game. However, the reason games like this attract interest is because they pose a coordination problem for the players. Asserting that either of these Nash

[13] Maynard-Smith [1982] deals with a similar construct in his 'Hawk-Dove' game.

[14] As mentioned in chapter 1, this is similar to Farrell [1987] in which 'cheap talk' prior to a 'battle-of-the-sexes' game solves a different type of coordination problem (this is in the context of market entry).

equilibria are likely to predict the outcome of the game requires some thought. There is a symmetric mixed strategy Nash equilibrium, which amounts to playing X with probability 1/2 and Y with probability 1/2, for an expected payoff of 1.

4.4.3 Evolutionary Stable Strategy.

An evolutionary stable strategy, or ESS [15], is one which, if adopted by the vast majority of a population, cannot be 'invaded' by another, 'mutant' strategy. That is, if $p \approx 1$ is the proportion of players adopting a strategy I, then I forms an ESS if, for any other strategy J, comprising the negligible minority of $(1-p) \approx 0$:

$$P(I,I) > P(J,I)$$

or, if $P(I,I) = P(J,I)$,

$$P(I,J) > P(J,J)$$

where $P(a,b)$ is the one-shot payoff accruing to a type a player against a type b player. We can establish the following:

Theorem 1. *Neither l_1 nor l_2 is an ESS.*

Theorem 2. *Both l_3 and l_4 are an ESS.*

These are easy to establish. A population of type 1 players can be invaded by any strategy that ever plays Y. A population of type 2 players can be invaded by any strategy that ever plays X. A population of type 3 players accrue a payoff of 1 when playing one another, as do a population of type 4 players, and naturally this cannot be improved upon. Also:

Theorem 3. *The symmetric Nash equilibrium mixed-strategy in the endgame (which ignores speech) is not an ESS.*

To see this, note that any strategy playing against a population of players adopting this mixed strategy expects to accrue a payoff of 1/2 per interaction against them.

[15] *sensu* Maynard-Smith [1982], after the work in Maynard-Smith [1974] and elsewhere. Applications of this idea to economics are suggested in Friedman [1991], along with some discussion of the concept itself.

However, a population of type 3 or type 4 players can invade such a population since they accrue a payoff of 1 each time they interact with one another. Of course, this scenario allows only one mutant type at a time.

Hence, only those strategies that actively use words to choose actions form an impenetrable population in the sense of an ESS.
Equilibrium Population Proportions.

Rather than consider a population dominated by a particular strategy type, we might try to discover if there are equilibrium populations containing a non-negligible proportion of different strategies.

4.4.4 Equilibrium Population Proportions

Consider only populations consisting of the four pure-strategy types types we have considered. Let there be a proportion π_i of each type, $i \in \{1,2,3,4\}$. Let $E(i)$ be the expected payoff to a player of type i in an encounter, so we have:

$$E(i) = \sum_{j=1}^{4} \pi_j P(i,j).$$

Let us define an equilibrium point as a collection of four proportions, $(\pi_1^*, \ldots, \pi_4^*)$, such that:

$$\pi_j \geq 0 \qquad \sum_{j=1}^{4} \pi_j^* = 1$$

and

$$E(i) = E(j) \;\; \forall i,j \in \{1,2,3,4\}.$$

The condition on the payoffs amounts to the following;

$$\pi_1(0) + \pi_2(2) + \pi_3(1) + \pi_4(1) = k$$

$$\pi_1(2) + \pi_2(0) + \pi_3(1) + \pi_4(1) = k$$

$$\pi_1(1) + \pi_2(1) + \pi_3(2) + \pi_4(0) = k$$

$$\pi_1(1) + \pi_2(1) + \pi_3(0) + \pi_4(2) = k$$

The first two imply that $\pi_1^* = \pi_2^*$, the second two that $\pi_3^* = \pi_4^*$, first and last together that $\pi_1^* = \pi_2^* = \pi_3^* = \pi_4^* = 1/4$. We can say a fair bit about stability without deriving any complicated dynamics. For example, if we suppose that the π_i reflect relative performance, a small tremble from this equilibrium that increased π_3 would increase $E(3)$ relative to the other three, which would reinforce the effect. The same applies to an increase in $E(4)$. Increases in $E(1)$ and $E(2)$ have the reverse effect, since they increase the relative fitness of the 'other' three strategies. We know that $\pi_1 = 1$ and $\pi_2 = 1$ are not stable, since l_1 and l_2 do not form ESSs. In a population of virtually all l_1 players and a minority of l_3 players, π_3 would *inexorably* increase. The same applies to a small minority of l_4 players invading a population of l_2 players or l_1 players of course.

We can also consider a population of l_4 and l_5 players. There is an unstable equilibrium at $\pi_3 = \pi_4 = 1/2$, in any other combination there is a tendency for either proportion to increase, and continue to increase. We cannot say that the proportion of one type is tending towards unity, of course, without some explicit dynamic rules.

4.5 Evolution and Numbers Externalities

At this point I think it might be worthwhile to more properly make the link between evolution and numbers externalities. The use of the probability notation above may have obscured this link. When we write π_4 and so on above we mean $\pi_4 = N_4/N$ in a hopefully obvious notation. Since we have restricted th strategy space, players payoffs can be written solely as a function of the number of others adopting one strategy or another—the basic form of a game I have discussed. There is a distinction to be made, I think, between the kind of evolution by which individual memebers of a species have characteristics that would make them fitter regardless of the behaviour of others. For example, improved wing shapes can be expected to emerge as evolution continues in an obvious manner. However, something like flocking behaviour cannot be described in this fashion, since the payoff to a bird adopting one form of behaviour or another will depend on the behaviour of others. We now explore this distinction in the context of language.[16]

4.6 Modeling the Evolution of Language

There have been just a limited number of attempts to model the evolution of language—either of a language itself, or of the evolution of verbal communication. Fogel [1992] considers the problem of communicating the 26 letters of the alphabet using the Morse code. To do so, each letter is represented by a unique sequence of a dot, a dash or a space. A dot is presumed to be one third the length of a dash, and the same length as a space. Each letter is presumed to occur with the frequency that has been estimated in studies of the English language[17]. Morse himself used "Huffman Coding" in which most frequent letters are assigned shorter codes in inverse proportion to their frequency. However, the Morse code does not apply strict proportionality, and hence there is room for improvement. Fogel used a technique known as *evolutionary programming*[18] to evolve coding. Each coding was represented as a sequence of 26 numbers representing the length coding for each possible letter of the alphabet. Each sequence was evaluated by the weighted average of the durations of the coding for each letter of the alphabet (the weights being the frequencies with which the letters occur). A population of 50 such trial solutions was maintained, with successive generations being formed by selecting 25 parents and randomly changing two letters' codings. The 25 parents were selected in a competitive manner by which each is compared to 10 randomly selected codings, and should they have a smaller average duration they receive a "win." The 25 parents with the most wins form half the next population, the other 25 made up by mutations of them. After 1000 offspring were produced, the program had produced a more efficient code than Morse's, and arrived at the a most

[16] The traditional process of evolution does not explicitly recognise this distiction. See Lewontin [1974] or Atmar [1992].

[17] *e* is the most common, occurring almost twice as often as the next most frequent, *a* which occurs 7.81 times per hundred characters. The least common letter is *z* occurring 0.09 times per hundred characters, at least in a 1956 study. No doubt studies of American writings would show a higher proportion of *z*'s nowadays. The most frequent consonant was *t*, with a frequency of 9.02. These statistics can be found in full in Gaines [1956].

[18] Fogel, Owens and Walsh [1966], and Fogel [1992].

efficient solution after 4000 offspring.

This view of language has a disticnt engineering bent. There is similar work to be found in Chambers [1985] and Hamming [1990]. The stated problem in language or code design is to communicate a fixed set of symbols in terms of some other. The number of possibilities is enormous in such problems, and sometimes the objective function will vary dramatically from application to application (Fogel [1992, p.1]). However, these engineering views, and even Fogel's evolutionary approach, do not account for the numbers externality I have detailed.[19] That is, in looking at communicative efficiency it is taken for granted that the listener and the speaker have already agreed upon a code. In designing an information transmission mechanism (such as a "file transfer protocol") this is perfectly sound, since the communicating software is yet to be written. In the development of natural language this is not necessarily the case. An inefficient language (in the engineering sense) may emerge if sufficient individual use it, and there are benefits to be accrued from using it. This is a similar argument to the one which suggests that the item that emerges as money need not be best suited to the job, as we mentioned in reference to Kiyotaki and Wright's work in the first chapter.

Natural languages do undergo evolutionary changes, once they are sufficiently well known that the process is not disruptive—that is languages can only change for the better (in the engineering sense) if the change does not disrupt the agreement amongst the many as to what is being communicated (as in the the cited *Economist* letter of previously).[20]

[19]This is *not* to say that once codes have been designed, they are not subject to the externality. For example, there are many ways of inputting "code" into a computer, via an operating system. However, the switching cost and numbers externality that have surrounded DOS have ensured its enormous success to date, However, once more convenient operating systems are available that run DOS-based software, we will see (and are seeing) its demise.

[20]See Harnad, Steklis and Lancaster (eds.) [1976] for a description of the evolutionary aspects. A study of the change in the statistical structure is in Herdan [1956].

4.6.1 Is Language a Heritable Trait?

There is some debate as to the origins of verbal communication. The model of the ESS earlier in this chapter, by its terminology at least, suggests a phyletic aspect of language. However, the definition of an ESS need not so apply: we can think of the evolutionary game as one in which agents use feedback from their encounters with others to determine their behviour in the future, in a way that mimics some kind of Bayesian analysis. It is not difficult to see that in any reasonable model agents will adopt strategies that are currently yielding greater payoffs (or would do if adopted). A crude dynamic equilibrium can be defined as one in which there is no tendency for players to adopt strategies different from their own, which will be a function of the number of agents adopting one of a range of alternatives, just as we defined in the section on equilibrium population proportions previously (and the part of the model that links it to those other numbers externalities we have examined). That is, we can think of that analysis as one of *learning* a behaviour, rather than as a model of natural adaptation. However, it is not clear what aspects of language acquisition are the result of natural selection, and what are learned behaviours that are passed on by teaching offspring rather than passed on by genetic recombination. The development of vocal communication is presumably a mixture of both. The ability to make distinct sounds is itself a by-product of an evolutionary innovation in the larynx that enables animals to divert food and liquids from the oesophagus (the epiglottis covers the glottis during swallowing, and itself is partly comprised by the vocal chords. The part of the human anatomy that works in concert to regulate swallowing and speech is the larynx.).

I think the notion of a numbers externality may help somewhat in understanding what may have happened, although this is pure speculation. The first observation is that the physical ability to produce a varied phonetic system may not have been an evolutionary innovation in the traditional sense: one which endows the individual with greater Darwinian fitness *per se*, but is, as we have said, a consequence of another innovation that certainly does. The reason for this is the same reason given for the uselessness of having one phone (other than as an ornament or a door stop)—the numbers externality. A creature that discovers the ability to make a varied assortment of sounds might be able to please itself by babbling away, but is unlikely to get anything but curious sidelong glances from his dumb compatriots. However, if these others begin to correlate the noise-maker's ramblings with behaviour then it is possible that they will learn to understand him, and this might be benficial to all. Only if this happens will the 'speaker' be fitter than others, raising the possibility that language use might evolve naturally, aa those with the ability to make sounds become fitter than others. This raises a new problem. If a significant portion of a population have the ability to make sounds, then in order that their individual behaviours become predictable then those who wish to predict might need to be able to translate the utterences of each individual, as was the case with the babblers that tried to build the tower of Babel (the origin of the word 'babble'). Naturally, this is infeasible, and is one effect that suggests that at some point in our evolution this serious coordination problem—perfectly analogous to that facing a community trying to evolve a medium of exchange—was circumvented.[21] It seems to me that the propensity to mimic the behaviour of others was a contributory factor. If we suppose that there is an inherent tendency to ape the behaviour of others, perhaps purposelessly, then this in combination with a propensity to correlate behaviour by any means available to solve coordination problems, may be a begining in understanding the evolution of language use. The fact that languages—as a sytem of understood sequences of sounds—are quite stable is (I suppose) due to the incredible attention paid by children to their parents' speech, and the children's delight in the approval of their parents when speech is correctly used.[22]

In summary, I would suggest that the ability to make a fairly varied array of noises is an epiphenomenon, but the ability to use the correlative ability of these sounds is a trait that might be genetic. We can view the use of speech as something like a correlated strategy equilibrium in which sounds are used as extrinsic coordinating devices. There is a notion, that was once very popular, of *punctuated equilibrium* in evolutionary biology. According to this view natural evolution is not a gradual

[21]Not just in *homo sapiens* of course, but many species coordinate behaviour based on vocalisations: everything from warnings to mating calls (alerts and flirts one might say...)

[22]Sometimes the parents get this the other way around, of course, by reciprocating the infantile burbles and delighting in the approval of their offspring...

procedure, but is characterised by discrete periods of adaptation, presumably to the sudden cumulative effect of neutral mutations which innocuous in and of themselves, have profound effects in concert. The coincidence of a genetic change that endows creatures with a propensity to take notice of sounds with another mutation that endows creatures with the ability to make such sounds, seems provide sufficient grounds for such

4.7 A Simulation

In what follows I describe a basic computer simulation in which agents, represented as very simple neural networks, learn to communicate a small set of ideas to one another. The results of the simulation are presented for the reader's interest, but this model, as complex as it is, is only suggestive of what might be involved in simulating agents in the numbers externality ridden environment of language.

4.7.1 Some Technical Details

GALE was written by myself and Peter Dudey in LISP, the language most common in artificial intelligence programming. In particular, it was compiled by Bruno Haible and Michael Stoll's CLISP, which they have generously made available for nothing. The interpreter, for various platforms, is available via anonymous FTP from `ma2s2.mathematik.uni-karlsruhe.de`. The source code for GALE is in an **Appendix**.

As GALE runs it saves quite a bit of data (optionally). For example, a run of 2001 iterations saves around 10 mb of data in the form of plain ASCII files if data are saved as often as possible. Each data point is separated by a blank space, and a carriage return separates one set of observations. GALE has been made public domain, and can be obtained by e-mail from the author `harrald@sfu.ca`.

4.7.2 An Overview of GALE

GALE stands for "Genetic Acquisition of a Language of Exchange," The idea of GALE is to demonstrate the effect of the numbers externality of language in an essentially economic environment.

When running (on a vt100 monitor) GALE displays a grid, of up to 80 x 25. Each element of the grid can contain either a "resource" or a "Critter." There are three types of resource, they are represented by the non-alphanumeric objects of a comma (,), a bar (_), or a dot (.). A "Critter" is our whimsical name for the main actors in GALE. Critters are small neural networks. They are represented by alphabetic characters, either upper or lower case. The Critters move randomly to adjacent grid points. If the grid point contains a resource, the Critter picks it up. If there is another Critter adjacent, at any point in time they attempt to communicate. the reason for communication is that the individual resources are of no value in themselves, but a combination of a comma, bar and dot can be used to produce one 'widget' (what else?). A widget endows a Critter with 100 units of 'lifespan'—they can exist for 100 more moves than they could without having consumed the widget. The resources are 'manna from heaven' in that they appear randomly in unoccupied grid points from time to time.

When two Critters meet they attempt to communicate by uttering a sound from a fixed vocabulary. The sound they utter is intended to convey two things: what they have that is available and what they want in exchange. They listen to one another, try to guess the intention of the other, and then offer what they think the other wants—literally present the item.[23] If they have successfully interpreted one another a trade takes place, otherwise it does not.

As the critters move around they are trying to associate a word utterance with

[23]It might be complained that the Critters can simply see what the other has, and hence a possible trade is manifest: no need for language. We have in mind that carry around the resources is very arduous, and that they are hidden in a secret archive known only to the Critters. They then wander around, bumping into the others. Once they guess at the meaning of the other's utterance, they go to their respective archives and retrieve what they think is appropriate (if they think there is the possibility of trade). When they reconvene, they then expectantly present what they have, and are either pleased or disappointed by what the other has.

an idea. GALE represents the links between the various utterances and the ideas as integer valued links between them. There are six basic "ideas" the Critters would like to convey;

"I want a bar I will give you a comma" "I want a bar I will give you a dot"
"I want a comma I will give you a bar" "I want a comma I will give you a dot"
"I want a dot I will give you a bar" "I want a dot I will give you a comma"

There are six sounds that a Critter can associate with each idea:[24]

"Fnord" "Flugblogs" "Bar"
"Baz" "Foo" "Quux"

Each critter assigns 36 integer-valued weights between the six ideas and the six words. When the critter wishes to convey a certain idea, it utters the word with greatest weight. Similarly, when a Critter hears a particular word, it associates an idea on the basis of greatest weight. The Critters alter the weights as experience dictates.[25]

Periodically two Critters, when adjacent will mate, producing a genetic crossbreed Critter that is located no more than ten grid points from either of its parents.

GALE keeps track of many statistics to illustrate the evolution of common interpretations of words due to a numbers externality. These are reported in the **Results** section later on. I now give a detailed description of the various operations of GALE.

4.7.3 Representing the Agents

The agents in GALE are comprised of two parts: a 'brain' and a 'gene.' The brain is stored as an array, the 36 weights between the ideas and words. The brain can 'learn' by increasing weights in response to correct interpretation, and decreasing weights in response to an incorrect interpretation.

Initially, and at birth, an agent's brains and it's genes are identical. Over time,

[24]Programming can be very tedious, hours of looking at error messages or staring at a screen watching critters do nothing and so on. Please forgive the sounds we chose.

[25]There are strong similarities between the activity of neural networks and Bayesian updating.

the brain learns via interaction, but the genes (naturally) remain the same. When two agents with more than 100 units of lifespan meet, they mate. A child is produced with genes and brain that are formed from the genes of the parents. Notice that a child does not inherit any learned attributes of its parents, just their initial structure—the babies' genes and brain are identical. By default GALE randomly selects elements from each parent's brain to form a child.[26]

Initially, all critters brains are generated randomly, and identical to their respective genes.

4.7.4 Movement and interaction.

The initial grid is randomly seeded with critters and resources, randomly placed in the "world." Each critter moves to one of the adjacent grid points: if the critter is located at grid point (x, y) it will move to grid points , $(x, y+1)$, $(x+1, y)$, $(x-1, y)$, $(x, y-1)$, with equal probability. If any of these new positions are not in the "world," or are occupied by another critter, the critter moves to one of the available adjacent positions with equal probability. One of three things can happen when a critter takes up a new position: it can find a resource, it can find nothing, or it can become adjacent to another critter. Naturally, the first two are mutually exclusive events. A critter who finds a resource that enables it to produce a widget does so immediately.

When two critters become adjacent, they "interact." The nature of an interaction depends upon the critters' current stores of resources. A critter with two identical resources (and nothing else) will want to trade one of them for either of the two resources it does not have in its possession. This applies to a critter with a resource of one type and two of another, in which case, however, it will want to acquire the third type. Since critters always manufacture (and consume) widgets immediately, they can have at most two resource types in their possession. There is no limit on the quantity of resources the critters can hoard.[27] Generally, then, a critter has a

[26] A procedure known as 'uniform crossover' in the genetic algorithm literature.

[27] Strictly speaking, there are restrictions imposed by the hardware on the amount of any type of information that can be stored. GALE is unlikely (in the extreme) to cause such problems, at least not because of critter hoarding.

particular quantity of one resource, and a particular quantity of some other. If either of these quantities is greater than two, the critter would like to trade.

4.7.5 Attempts to trade

It is most important to note that trade can only take place if there is successful communication. In some ways I find this unsatisfactory, but it is still a retained feature of GALE at present. When two critters meet, and if they want to make a trade, they must choose a word to represent their desires. An idea is represented as a *dotted pair*, of the form "(*resource1.resource2*)" where *resource1* and *resource2* are both $\in$ {dot,comma,bar}. The notation '(*resource1.resource2*)" indicates a desire to swap a unit of *resource1* for a unit of *resource2*, this constitutes an *idea*. There is no quibbling about prices.[28] A critter who does not wish to trade simply says nothing. Should either critter do this, that is the end of the interaction. This leaves the case in which both critters have uttered a word. Recall, the word that is uttered is the one with greatest link to the idea that the critter wishes to convey. Having exchanged this initial utterance, the critters then must decide on an interpretation of what the other said—again based on which idea has greatest weight associated with the word the other chose. If a critter interprets the other's utterance as the converse of its own (that is, it thinks they can trade), it will make the same utterance again. If the critter thinks (given its interpretation of the other) that the other does not want to trade, then a new offer is made—a new word, if the critter has a secondary trade it would like to make. If both critters now feel they agree on the trade, *and they are correct*, a trade takes place. Otherwise there is no trade. This is a bit involved, so here's a sample meeting between critter `A` and critter `B`:

We'll suppose critter `A` has a hoard of $(comma, comma, bar, bar, bar)$, while critter `B` has a hoard of $(dot, dot, bar, bar, bar)$. Critter `A` initially wants to offer a *comma* in exchange for a *dot*, and suppose this maps into the word "*foo*." Critter `B` initially wants to swap a *bar* for a *comma*, and suppose this maps into the word "*foo*" also.

[28] In a general equilibrium, the resources would not necessarily have equal prices, since their relative abundance might differ.

Although they disagree on the meaning of *foo*, the critters maintain the assumption that they have correctly understood one another—in which case it has become apparent that they have made incompatible offers. They now go to secondary trades, which is feasible for these two critters because they have more than two units of the two resource types in their possession. Critter `A` wants to convey the idea that is represented by the dotted pair (*comma.dot*), while critter `B` wants to convey the idea (*dot.comma*). They do not have the double-coincidence of wants problem, but now they have to communicate too. Only if they both use the same word to convey *both* these ideas will a trade take place. Under such circumstances, the weights between the chosen words and the ideas conveyed *remain the same.* Neural networks only learn when they get things wrong. Under all other circumstances the weights between ideas and words that were employed in the conversation are decreased, all other weights increased.

Double-Quadruple-coincidence

It can be seen that trades will be quite a rare occurrence. A prerequisite is the incidence of double-coincidence of wants in the traditional sense. With only three goods, this is not all that unlikely. Notice that there can be only one feasible trade between any two critters with two resources, but if one critter has more than one unit of just one resource, then there are two possible trades with another critter with more than one unit of the other two resources. In order that a trade take place the critters must then use the same words to indicate the two ideas to be conveyed. A further coincidence is required: the critters must be adjacent before anything can happen. Given this, we would not expect to see communication adapt very quickly. If GALE illustrates one thing, it is that if language is an emergent phenomenon, it certainly was not very rapid in its emergence. Yet, by evolutionary standards, we have developed an incredibly complex means of verbal communication in very short order. This still remains a puzzle for me.

4.7.6 Screen output of GALE and internal operations.

In this section I show some sample screens from GALE as it is running to help explain what is possible when two critters bump into one another, and also discuss several other features of GALE. Consider the following:

```
   _           UJ   .          ,Q
       N ,  ,I        , _
   S   .E           .
     .         ,          ,     Y
  .,        _       ,
      .
 ,BW     Z   M          D
               ,   ,        A_
     F                        C
_     .         O_  R_ _    _T _ K
   G .         _         V  ,X  ,
    _              ,
   .  .     _H         .  .        _
                ,  _        P
       L           ,  ,         ,

Command? [QOTHNDFEL]
```

This is the start-up screen, and will be the same for every GALE run. The capital letters represent critters, which have been randomly positioned. In this run, there are 26 critters represented by the letters of the alphabet. Resources have been randomly positioned in free grid-points. The crude menu, "`Command? [QOTHNDFEL]`" offers the user the the following options: '`Q`'= Quit, '`O`'= one epoch, '`T`'= one thousand epochs, '`H`'= one hundred epochs, '`D`'= calculation of population diversity, '`F`'= an approximate diversity calculation, `F` for 'fast diversity,' '`L`' = a list of statistics indicating the following:

```
                              STATISTICS

Word:      (./,)    (./_)    (,/.)    (,/_)    (_/.)    (_/,)
Foo        0/0      0/0      0/0      0/0      0/0      0/0
Bar        0/0      0/0      0/0      0/0      0/0      0/0
Quux       0/0      0/0      0/0      0/0      0/0      0/0
Baz        0/0      0/0      0/0      0/0      0/0      0/0
Fnord      0/0      0/0      0/0      0/0      0/0      0/0
Flugblogs  0/0      0/0      0/0      0/0      0/0      0/0


Number of times used/correctly interpreted


(Hit any key to continue)
```

The root menu, [QOTHNDFEL], appears after each menu choice is completed. An 'epoch' consists of one move per critter, and of course, any associated interactions. If two critters meet, and neither of them wishes to trade, they make no attempt at speech. This happens a lot early on as resource acquisitions are limited. As the critters acquire resources, the potential for trade increases, as does the complexity of interactions. Consider two critters, one of which wishes to acquire a COMMA, the other a DOT. GALE shows one possible interaction below:[29]

```
Critter 3 (C) at (6 . 6) moving.  Lifespan: 538  Goodies: (DOT BAR BAR BAR BAR)
C: Bar? (meaning (BAR . COMMA), interpreted as (COMMA . DOT))
S: Baz? (meaning (BAR . COMMA), interpreted as (BAR . DOT))

S: Baz. (meaning (DOT . COMMA), interpreted as (BAR . DOT))
```

[29]These messages appear just below the display of critters and resources, which updates itself every time a critter moves. It has been suppressed in this example display.

It worthwhile decomposing these events, I think. Critter `C` has the hoard `(DOT BAR BAR BAR BAR)`, and naturally wishes to acquire a `COMMA`, and would like to swap a `BAR` for a `COMMA`.[30] Although not shown, critter `S` also wants to acquire a `COMMA`, and is also going to offer a `BAR`. These critters do not actually have a trade to make, but they are not sure of this. To convey the idea "`(BAR . COMMA)`" critter `C` has selected the word "`BAR`" which critter `S` has interpreted as "`(COMMA . DOT)`." Critter `S` does not want a `DOT`, and makes an offer to swap its `BAR` for `A`'s `COMMA`, which of course `A` doesn't have. After the first attempt at speech, the two critters have different ideas about what has happened. Critter `A` thinks the following sequence of events has occurred: 1) I offered him my `BAR` for his `COMMA`. 2) His response was to offer me his `BAR` for my `DOT`. Critter `A` then waits for `S` to make a second offer, which may differ from the first offer. All that happens is that `S` makes the same offer again (in `A`'s estimation) Critter `A` ends this interaction by saying nothing, meaning there is no trade to take place. Now let us see these events as critter `S` interpreted them. The first event was 1) The other critter offers me a `COMMA` in exchange for a `DOT`, then 2) I offer `BAR` in exchange for a `COMMA`, 3) The other critter did not make an attempt to trade, so I make the new offer, 4) I offer, as an alternative, a `DOT`, in exchange for a `COMMA`, 5) nothing doing, apparently.

It may seem odd that critter `S` uses the same word, `Baz`, to convey two different ideas. This, however, will happen from time to time. It so happens that, for critter `S`, the meanings `(BAR . COMMA)` and `(DOT . COMMA)` are both expressed the same way. Every critter (at any moment) has exactly one word to express a given meaning, and exactly one interpretation of a given word it hears, but the converses are not true. A word may be used to express several different meanings, and an idea may be the interpretation of several different (heard) words. Note that neither critter can be sure that the other has misunderstood any utterance in this interaction. In this regard, the interaction illustrated does *not* alter the brains of the critters. Consider a more fruitful encounter (again, I have suppressed the "world"):

[30] If critter `C` thought it would be very easy to acquire a `DOT` in the future, then it might consider trading its `DOT` for a `COMMA`. This kind of reasoning is not yet embodied in GALE.

```
Critter 12 (L) at (18 . 12) moving.  Lifespan: 990  Goodies: (COMMA...
L: Flugblogs? (meaning (BAR . DOT), interpreted as (DOT . COMMA))
F: Foo? (meaning (COMMA . BAR), interpreted as (DOT . BAR))
L: Flugblogs. (meaning (BAR . DOT), interpreted as (DOT . COMMA))
F: Foo. (meaning (DOT . BAR), interpreted as (DOT . BAR))
L trades BAR for F's DOT
```

In this encounter, only the final statement of F is properly understood, and it happens to correspond to a feasible trade, which takes place. Two critters could make a mistake in expecting they have a compatible trade to make, only to subsequently discover they had misinterpreted each other. This has the obvious effects on the weights between ideas and words.

The option E in the root menu allows the user to observe, and edit if desired, a critter's brain. For example, in the results to be presented later, critter A's brain looks like this initially:

Critter A

Word:	(./,)	(./_)	(,/.)	(,/_)	(_/.)	(_/,)
Foo	-4	-9	9	-2	-6	10
Bar	5	-8	-3	5	-2	-5
Quux	-1	9	0	-2	8	2
Baz	-1	9	-5	-2	1	-2
Fnord	8	0	-9	4	2	1
Flugblogs	5	-10	3	-1	-9	-6

Option? [QCS]

The sub-menu here allows the user to go back to the main screen, Q, perhaps to quit altogether or to carry on with some more epochs, to "clear" this critter's brains and genes (reset all links to zero), or to set any one of the links displayed. The links (initially) range from 10 to -10.

The option `D` calculates an overall "diversity" calculation. For GALE diversity is based on which links are operational, not (as yet) the size of links. It is an average of the 'diversity' between the brains of all critters compared pairwise (but not with themselves). The critters can react to 6 types of 'input:' each one of the ideas. By "react" I mean generate an interpretation. There are also the converse words used by critters to use to indicate each of the 6 ideas. In this respect, critters can either agree or disagree on 12 different counts, the 6 interpretations associated with the six words, and the six words associated with the 6 ideas. This number is calculated for each pair of critters, and 1 added to it to ensure safe division. This calculation is then averaged for all critters with more than 100 units of lifespan (or some other number if desired) so that children don't create spurious jumps in the diversity calculation. "`F`" makes a 'fast' diversity population by sampling the population randomly.

The option `N` creates a new world populated by a fresh new-born population. There are some other details. The initial population of 26 critters are given the names `A` through `Z`, and are indexed by 1 through 26. Children are given the names `a` through `z` sequentially, and after that they always appear as `@`'s. However, indexes are incremented sequentially.

In some runs of GALE there are many births. When two critters meet if both their lifespans are "large" they give birth with a probability equal to a specifiable constant, `*fecundity*`. If two critters breed, they still interact linguistically, and perhaps make a trade.

If the user is interested only in the development of neural networks, it is possible to set `*fecundity*` equal to zero, and ensure initial lifespans are large enough. However, it may be of interest to watch a genetic algorithm-like evolution too, in which case birth is desirable. It can be tricky to set lifespans and `*fecundity*` in a particular relationship that maintains a reasonable sized population, and overcrowding can be a problem. This simulation is not designed to mimic adaptation to life-threatening situations, although clearly it could be adapted to do so. We have found that a method of 'stark-fist' removal helps. Each epoch a grid-point is randomly selected. If a critter is unfortunate enough to be residing in that grid point then that critter loses a certain amount of lifespan, which may be fatal.

Variables and constants in Lisp are declared at the top of the code for gale. Some of the ones a user might want to alter are:

`*save-critter-files*`: GALE can save statistics related to each critter, described in detail in the next section. If `*save-critter-files*=T` then these statistics are saved, if `*save-critter-files*=nil` they are not.[31]

`*world-width*` and `*world-height*` are the dimensions of the grid inhabited by the critters. The example here show a 35×15 grid.

`*learning-rate*` is the magnitude of the change in the links between words and ideas when there is any change. It is currently set at 3.

`*initial-resources*` is, as you might guess, the number of initial resources placed in the world. Whatever this number is, the choice of resource type is random when a resource is placed in a grid-point. Each epoch a number `*resources-per-epoch*` of randomly selected resources are placed in the grid, if there is room. If there is not, GALE will hang!

`*fecundity*`, as we have mentioned, is the probability of a birth when two 'fit' critters meet.

`*delay*` is the amount of time, in seconds (approximately) that GALE will pause after it displays the details of an interaction.

`*smite*` is the number of units of lifespan removed by the stark fist. The stronger is the stark fist, the more important will be the invisible hand...

File output.

In its current form GALE will save many statistics related to its current run, some of which are reported for a particular run in the next section. Always stored is a file called "`stats`" are contains 110 variables. It is useful to index words and ideas when describing these files, as shown in Tables 4.1 and 4.2. The file `stats` then stores the data as shown in Table 4.3.

In addition to the stats file, GALE will (optionally) save data pertaining to each individual critter. These files are named `critter1.stats`, `critter2.stats` and so

[31] CLisp is not case-sensitive, so `t` and `NIL` will do fine too

Words	
Foo	Word 1
Bar	Word 2
Quux	Word 3
Baz	Word 4
Fnord	Word 5
Flugblogs	Word6

Table 4.1: Words indexed.

Ideas	
`(dot . comma)`	Idea 1
`(dot . bar)`	Idea 2
`(comma . dot)`	Idea 3
`(comma . bar)`	Idea 4
`(bar . dot)`	Idea 5
`(bar . comma)`	Idea 6

Table 4.2: Ideas indexed.

Columns and Data Stored.	
1	Epoch number
2	Population diversity
3	Number of times Word 1 used as Idea 1
4	Number of times Word 1 used as idea 2
⋮	⋮
9	Number of times Word 1 used as Idea 6
⋮	⋮
38	Number of times Word 6 used as Idea 6
39	Number of times Word 1 correctly interpreted as Idea 6
⋮	⋮
75	Proportion of critters using Word 1 as Idea 1
⋮	⋮
110	Proportion of critters using Word 6 as Idea 6

Table 4.3: The `stats` file.

on. In Table 4.4 is the description of the `critter3.stats` file, all other files are analogous. Notice that individual diversities and distances are not recorded for the critter and itself. The actual number of columns will depend upon the *maximum* number of critters that have ever lived.

4.8 Results and Suggested Data Analysis from a 7380 Epoch run

GALE is quite versatile, and many scenarios can be modeled. In this thesis I report some results from a run of 7380 epochs.[32] For this run `*fecundity*` was zero, so that no birth takes place, in keeping with the observation that language is unlikely to be genetic. Initial lifespans were 1000, and `*smite*` was zero. Although there was the potential for death, it didn't actually happen, and so there were 26 critters from start

[32]This run took 69 hours on a Sun 4 with virtually nothing else going on. The run ended when someone decided to turn off my computer for no apparent reason.

Columns and Data Stored.	
1	Epoch number
2	Current lifespan
3	Number of Interactions critter3 has ever had.
4	Number of trades of Idea 1 type
5	Number of trades of Idea 2 type
⋮	⋮
9	Number of trades of Idea 6 type
10	Current (Manhattan) distance from critter 1
11	Current diversity with critter 1
12	Current (Manhattan) distance from critter 2
13	Current diversity with critter 2
14	Current (Manhattan) distance from critter 4
15	Current diversity with critter 4
⋮	⋮

Table 4.4: The `critter3.stats` file.

to finish. All the data files were kept. In the following sub-sections, figures for some of the data are plotted.

4.9 Actual Data

The raw results contained in the `stats` file are shown here, with some minor observations. These are presented for the reader to browse through so that a feel for what is going on in GALE can be acquired.

4.9.1 Population Diversity

Figure 4.2 indicates the population diversity calculation for this run. While certainly not monotonic, we do see a steady decline as the critters begin to coordinate themselves. There is no precipitous drop however. This is not surprising as the rate of interactions per epoch is roughly constant, an example of the cumulative number of interactions for critter H is shown in Figure 4.3 which demonstrates that as soon as the critters have a significant hoard to talk about interactions remain steady. I strongly suspect that in a version of GALE in which agents can either choose to remain together and chat, or move around in a correlated fashion, we would observe a far more dramatic decrease in diversity, punctuated by periodic statsis after possible trades are exhausted. This is, in fact, somewhat similar to actual experience.

4.9.2 Word Usage

Recall that it is perfectly possible for critters to use words to convey two or more different ideas. In Figures 4.4 through 4.9 the proportion of critters using an individual word for each of the six possible intentions are graphed. What is surprising is the strong tendency towards synonyms. Word 1, in fact, is used, by the end of the run, by most critters to convey four different ideas, and word 2 to convey the other two. Words 3, 4, 5, and 6 are barely used at all by the end of the run, although word 3 had a periods of use for *each* idea at different points in the run, which is quite interesting. This, I think, harks back to our notion of path dependence, and may indicate something of a 'butterfly' effect in that there are probably key events occuring, presumably successful trades, to cause the rise and fall in a word's popularity. These events correspond to critters solving the double-quadruple coincidence of wants problem, so that they have complementary trades to make, and have learned to communicate such. I might draw a parallel between this and the kind of economic path dependency hinted at in the first chapter of this thesis as an explanation of English as a dominant mode of communication. A glance at the evolution of word 3 usage shows that there is nothing permanent in such a regime.

I do find it intersting that the critters economise on the number of words in circulation. However, if the reader runs GALE for long enough he will notice that (using the same parameter settings that I did) eventually the critters are only interested in one trade, and that many intentions are never vocalised, although interpretations of others still need to be made. If critters who perpetually need to make just one trade use the same word to mean different things, then unless they meet a critter with a compatible trade, they never learn to restrict the use of the word.

4.9.3 Word Interpretations

While a critter may use the same word to mean different things, they can only have one interpretation of a given word. In Figures 4.10 through 4.13 the proportion of critters associating each of the six words with a given idea are displayed. It can be a bit tricky to remember what is going on here. These diagrams are made up from the ones described in the previous section. Each idea is taken in turn, and graphed are the proportion of the critters who, when hearing one of the six words, interpret it as each of the ideas. Since there can only be one interpretation for each word, these series should sum to unity. In some ways these explain a little more about the dynamics of languages in GALE, since we can observe critical moments in the run. In the last third of the run we can see that, apparently, certain batches of critters successfully used and interpreted words 1 and 2, as ideas 1 and 2 respectively. There are similar features for the other ideas, with idea 3 being successfully interpreted by some critters again in the last third of the run, idea 4 has no settled translation, idea 5 almost exclusively associated with word 2.

4.9.4 Task-Specific Languages

The fact that the same word, word 1, is being used and interpreted in two different ways indicates, to me, that the critters are segregating themselves into groups for whom compatible trades are possible, and developing their own languages. When there is no compatible trade to be made, there is no potential for learning a successful sequence of words, this is only faciltated by trade. It is clear that physical proximity is not *sufficient* for successful communication in the GALE environment. This is a similar feature to the first model of this chapter. It is hoped in future that the `critter.stats` files will be of use, but at this juncture I do not feel a study of them is particularly warranted, and is neglected.

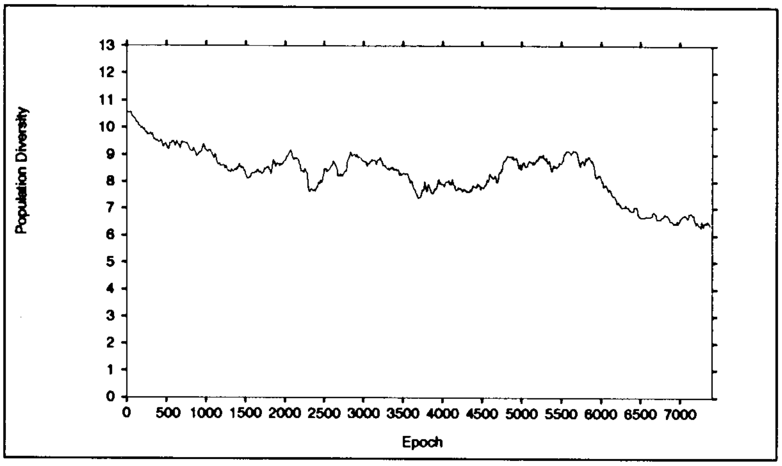

Figure 4.1: Population diversity in the GALE run.

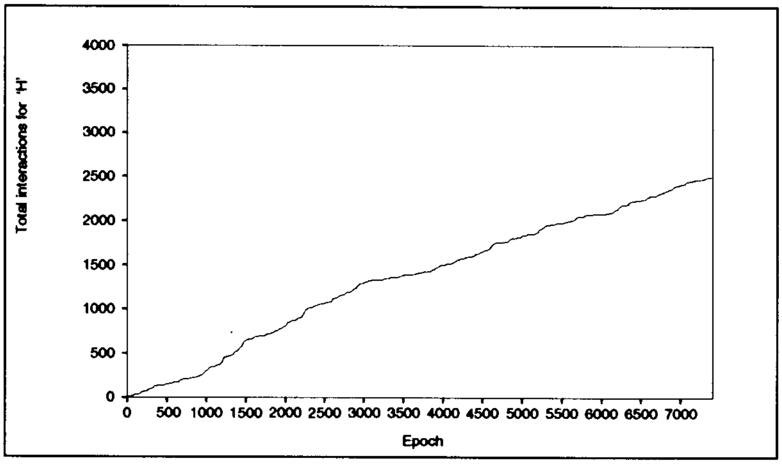

Figure 4.2: Cumulative interactions for critter `H`.

	X	Y
X	0	1
Y	1	0

Figure 4.3: The discoordination game for language use.

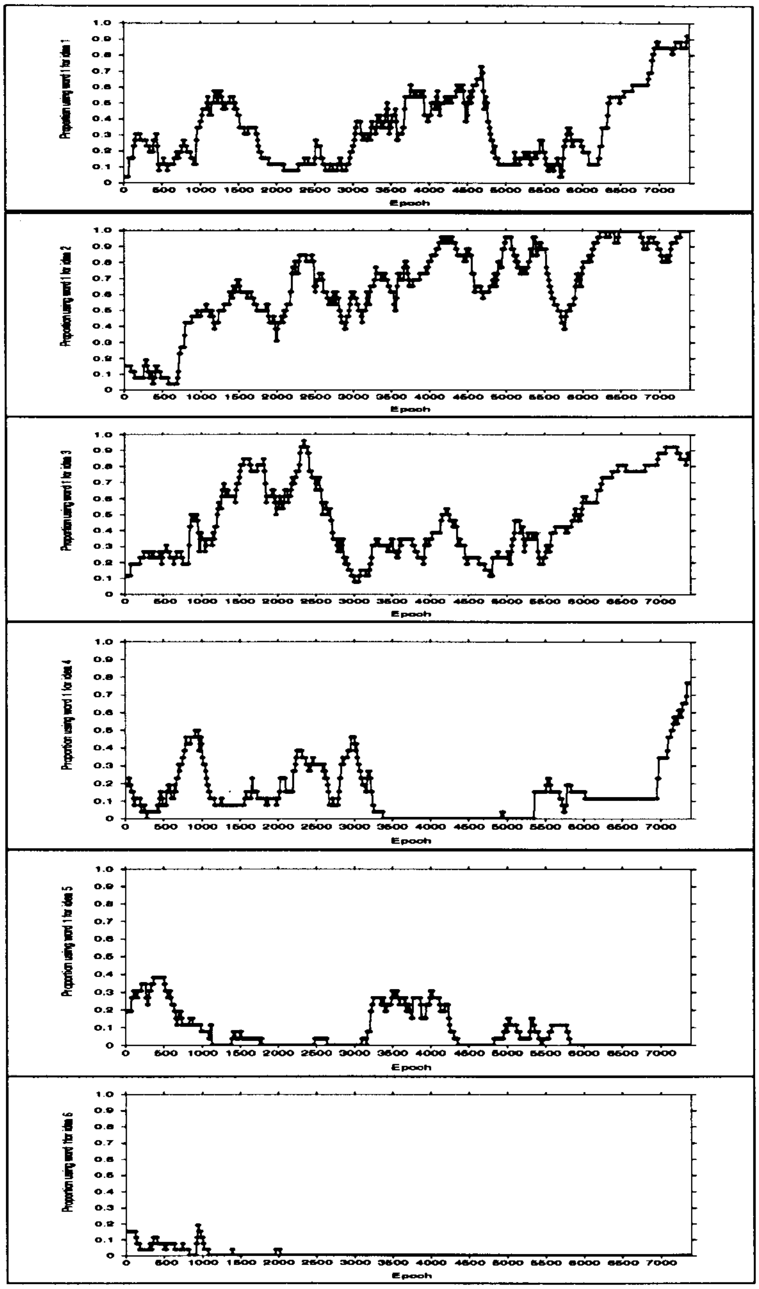

Figure 4.4: Word 1 usage in the GALE run.

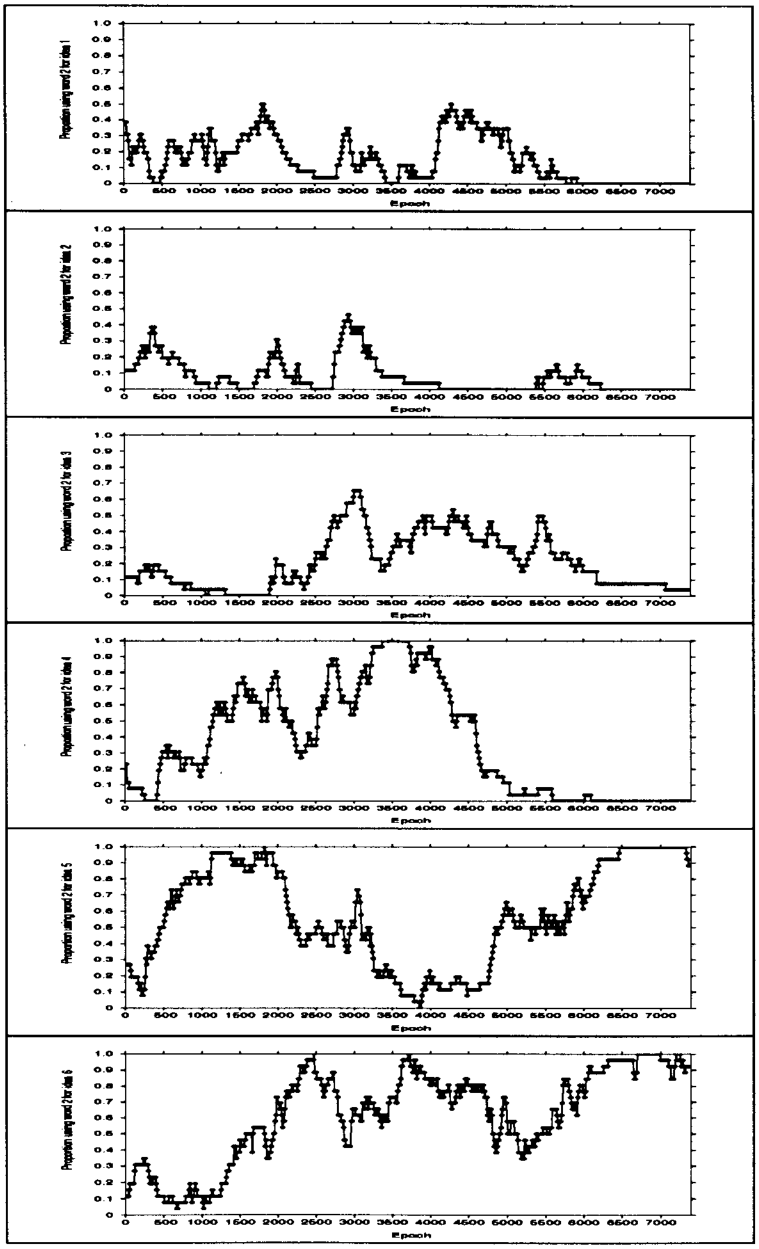

Figure 4.5: Word 2 usage in the GALE run.

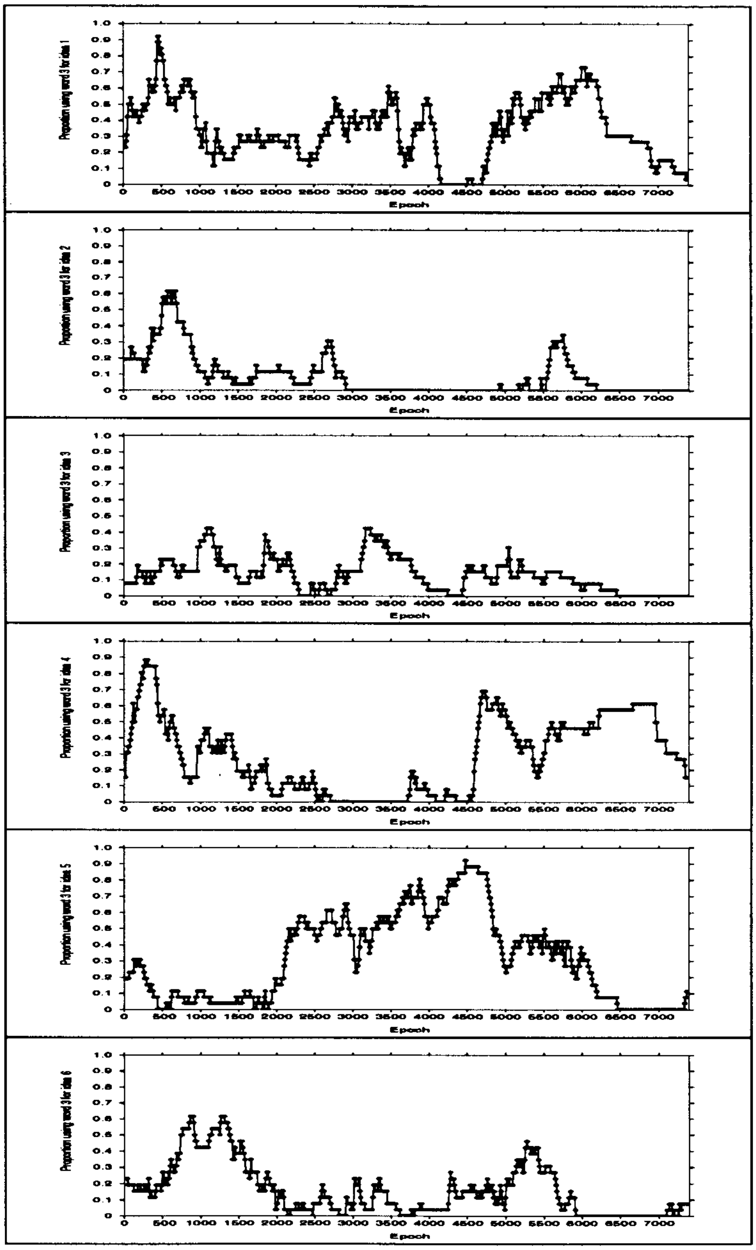

Figure 4.6: Word 3 usage in the GALE run.

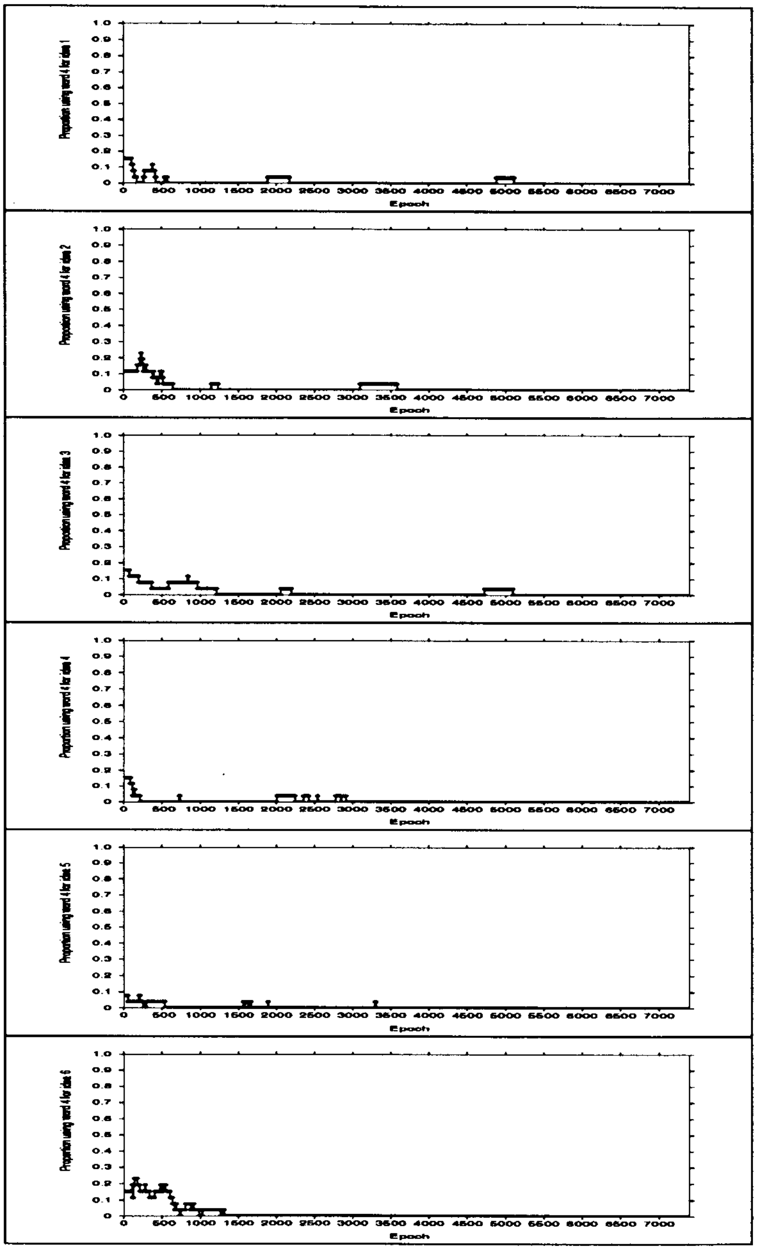

Figure 4.7: Word 4 usage in the GALE run.

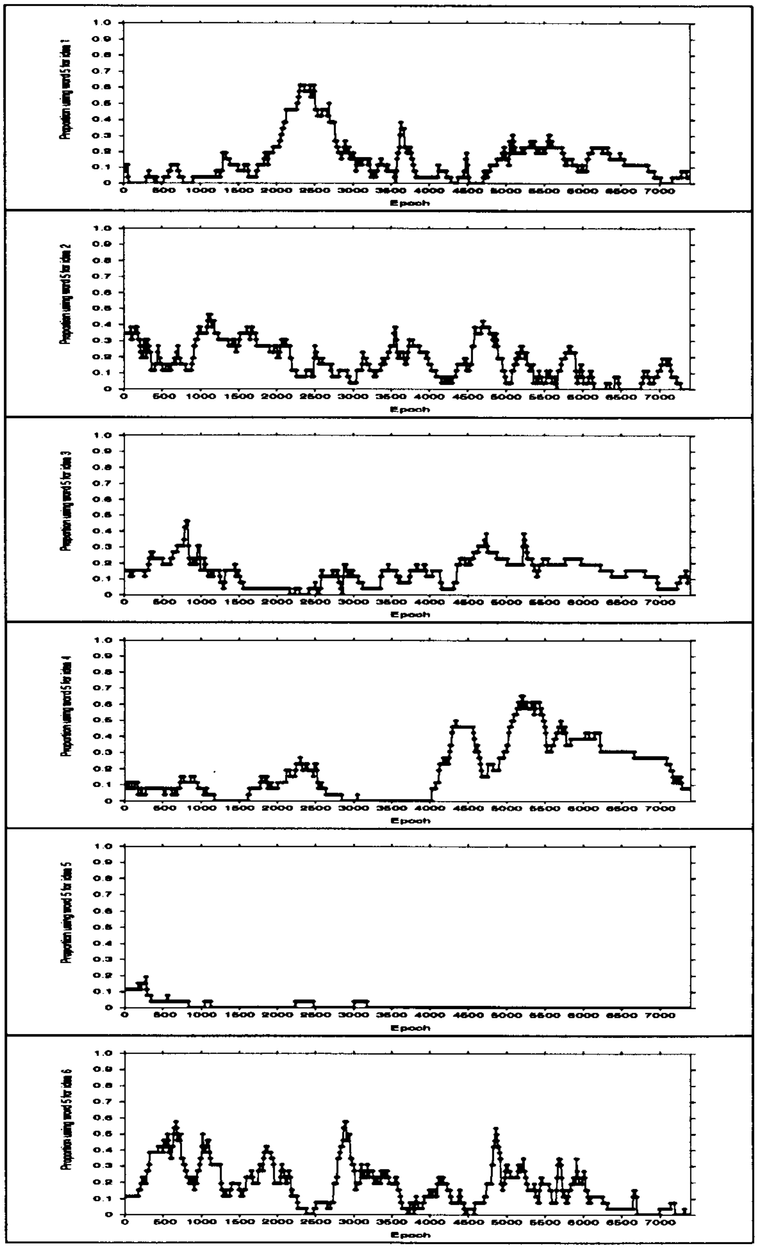

Figure 4.8: Word 5 usage in the GALE run.

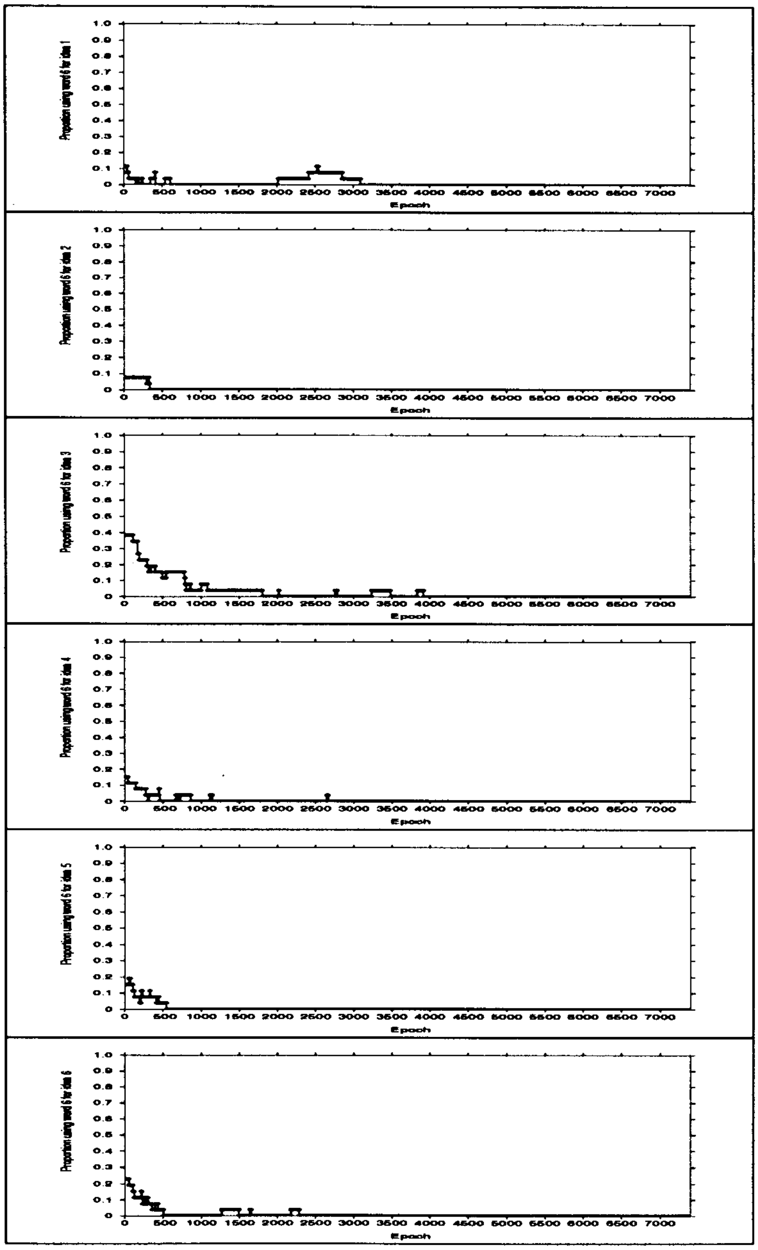

Figure 4.9: Word 6 usage in the GALE run.

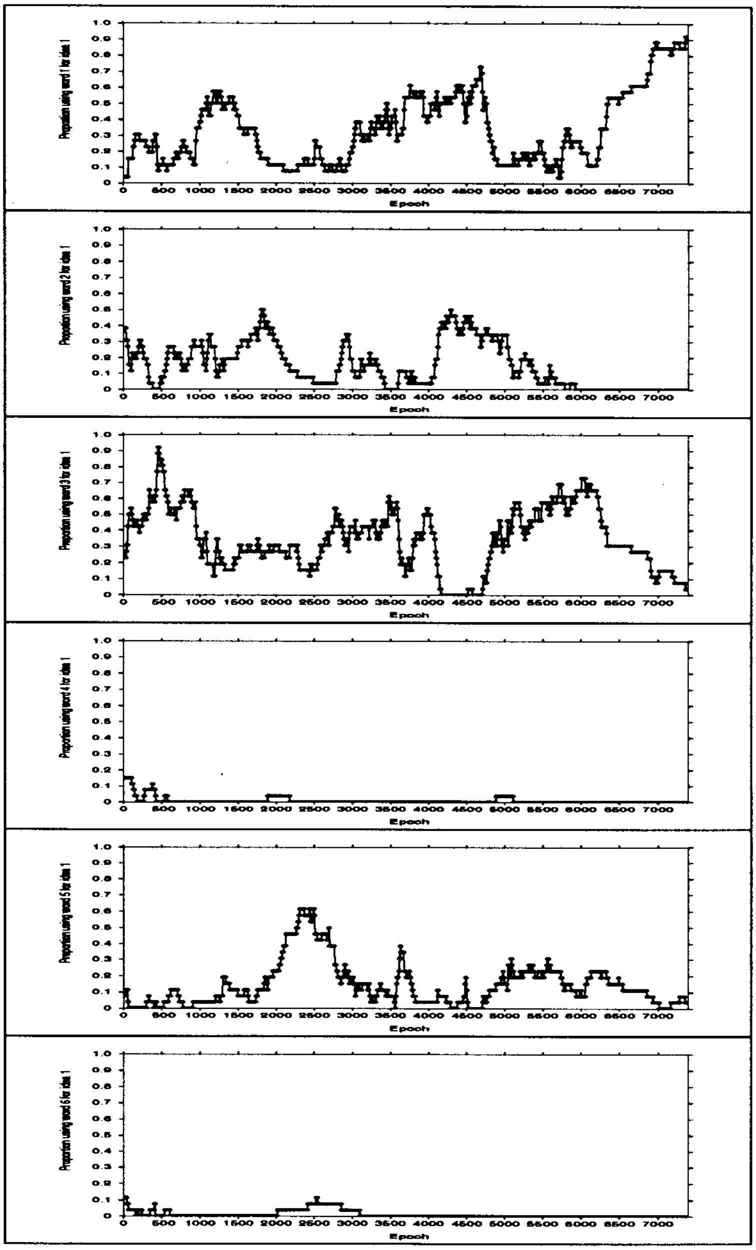

Figure 4.10: Idea 1 translation in the GALE run.

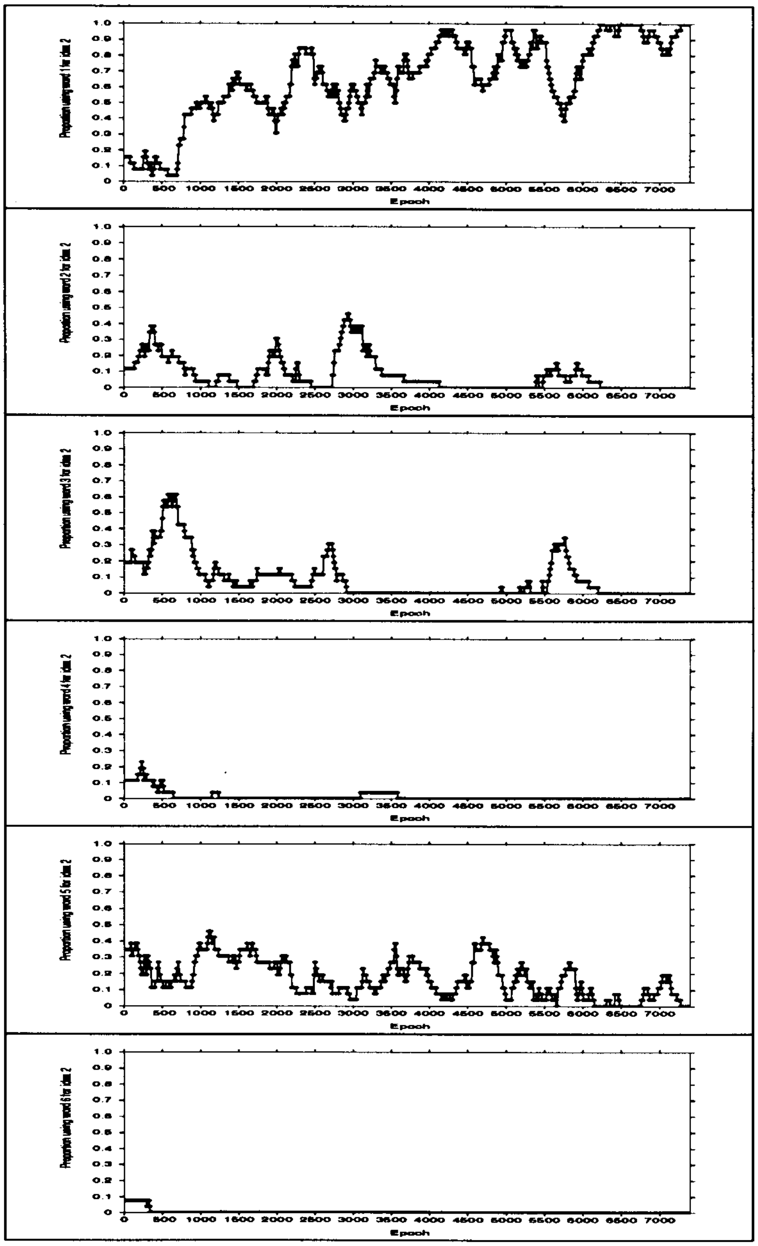

Figure 4.11: Idea 2 translation in the GALE run.

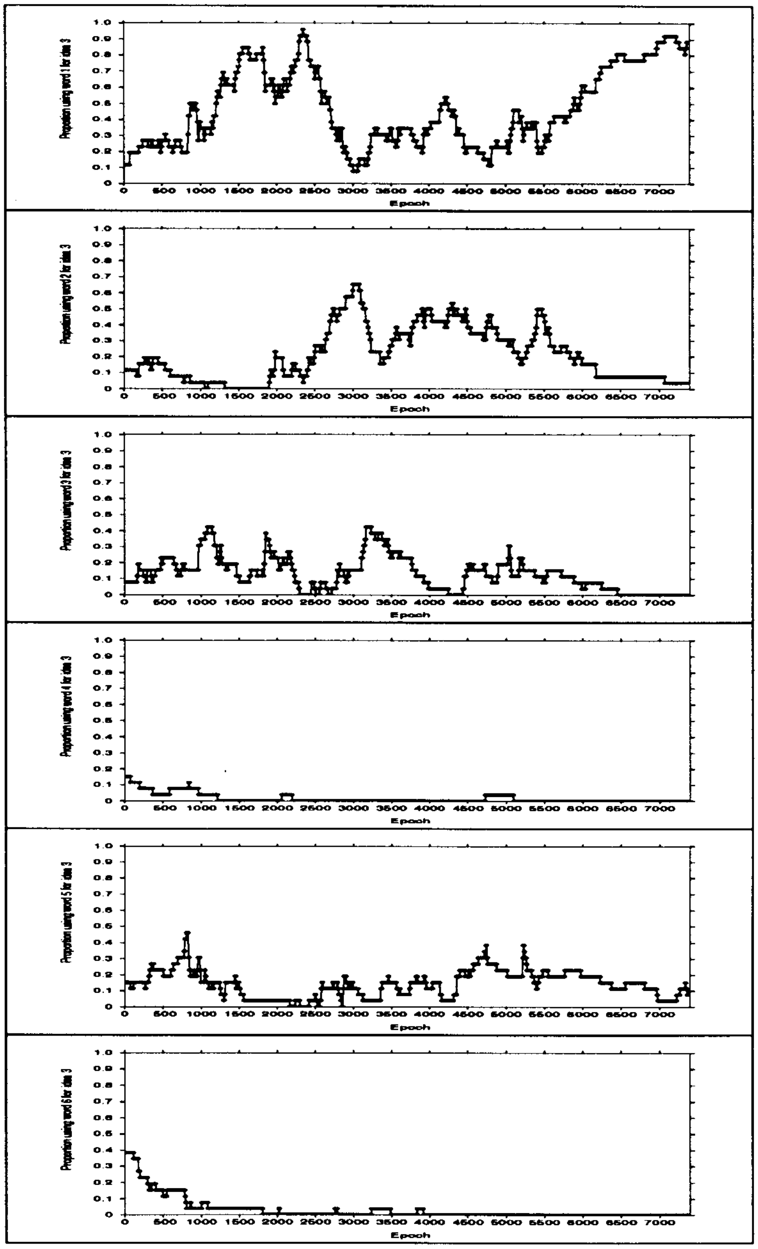

Figure 4.12: Idea 3 translation in the GALE run.

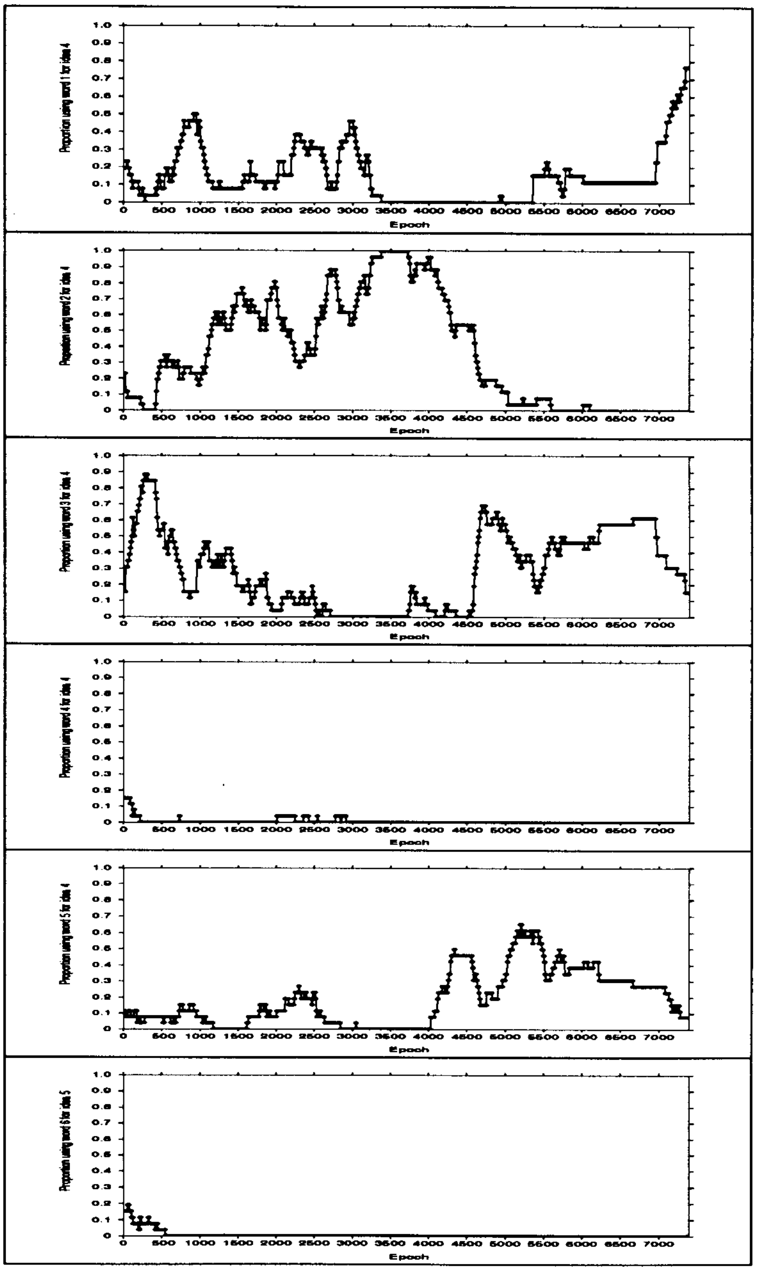

Figure 4.13: Idea 4 translation in the GALE run.

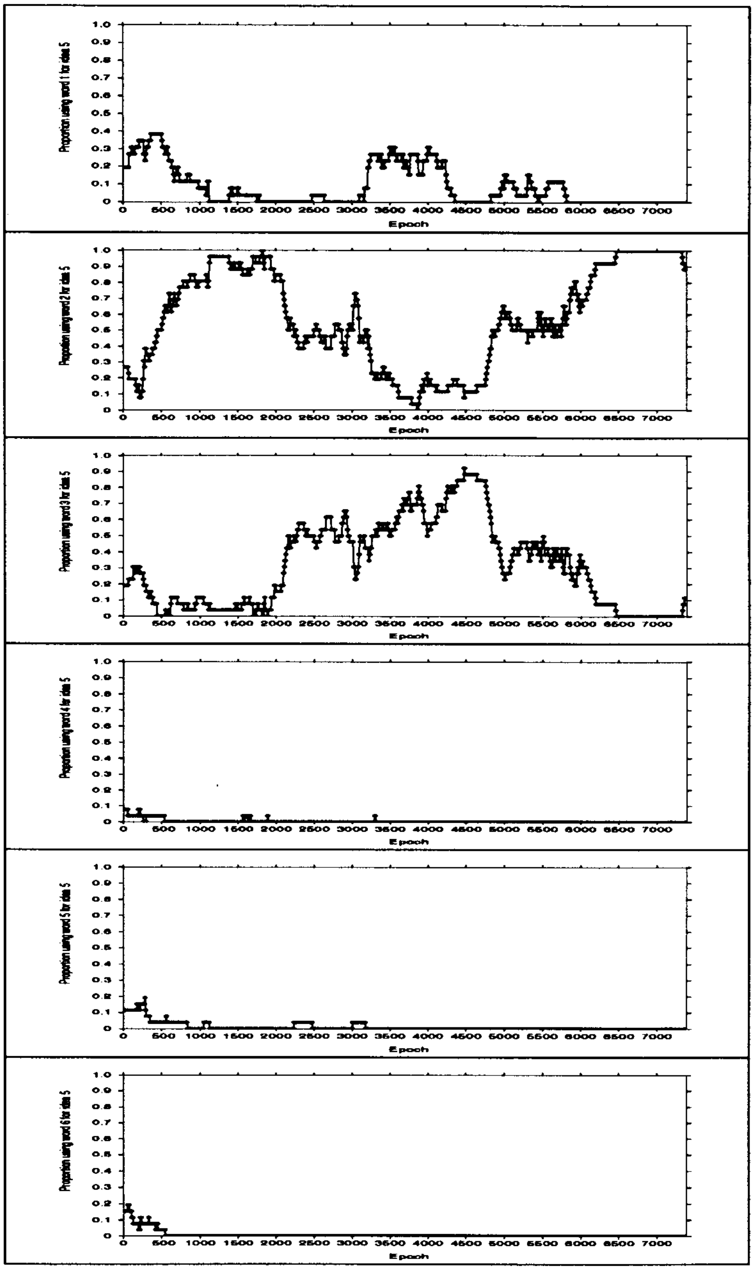

Figure 4.14: Idea 5 translation in the GALE run.

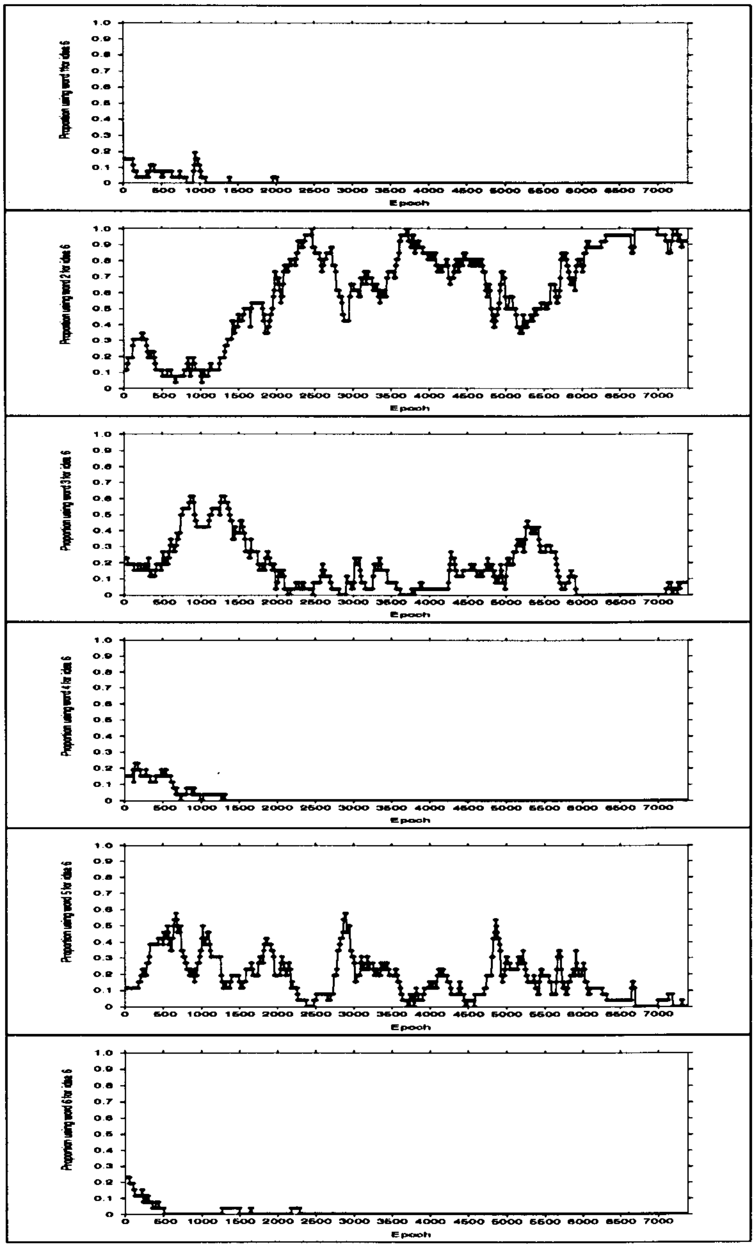

Figure 4.15: Idea 6 translation in the GALE run.

Appendix: Source Code

This code is heavily commented, so that features not mentioned in the text can easily be discovered.

```
;;;; GALE
;;;; (Genetic Acquisition of a Language of Exchange)
;;;
;;; by Peter Dudey and Paul Harrald
;;; Design by Paul Harrald
;;; Version 1.9
;;; 7 June 1993
;;;


;;; Load in a few utilities

(load "utils")


;;; Declare constants and global variables
```

```
(defconstant *stats-file* "stats") ; Filename, or NIL to not save
(defconstant *stats-rarity* 100) ; Saves 'em every n epochs
(defconstant *save-critter-files* T) ; T saves 'em, NIL does not
(defconstant *world-width* 35)
(defconstant *smite* 0)                ; Stark fist removal,.
(defconstant *world-height* 15)
(defconstant *learning-rate* 3)
(defconstant *initial-resources* 50) ; Initial # of resources
(defconstant *resources-per-epoch* 5) ; New resources that appear each epoch
(defconstant *fecundity* 0) ; # of interactions that produce babies
(defconstant *delay* 0.0) ; Pause after important message
(defvar *epoch*) ; The current epoch number
(defvar *next-critter-number*) ; The number assigned to the next critter
(defvar *world*)
(defvar *living*)                      ; The list of living critters
(defvar *dead*)                        ; Names available for new critters
(defvar *uses*) ; Number of times each word has been
;  used to mean each message
(defvar *correct-interpretations*) ; # of times each word was correctly
;  interpreted for each meaning
(defvar *idea-list* ; The ideas to be communicated
        '((dot . comma) (dot . bar)
  (comma . dot) (comma . bar)
  (bar . dot) (bar . comma)))
(defvar *word-list* ; The 'sounds' the critters can make
        '("Foo" "Bar" "Quux" "Baz" "Fnord" "Flugblogs"))


;;; Structures

(defstruct critter
```

```
;;
;; The basic critter structure
;;
 (genes                                    ; The critter's brain at birth
  (make-array '(6 6)
   :initial-element 0
   :element-type 'fixnum)
  :type (simple-array t (6 6)))
 (brain                                    ; The critter's current language
  (make-array '(6 6)
   :initial-element 0
   :element-type 'fixnum)
  :type (simple-array t (6 6)))
 (name                                     ; The critter's initial
  #\C
  :type character)
 (number ; The critters ID number
  (let ((n (incf *next-critter-number*)))
   (if *save-critter-files*
       (with-open-file (f (concatenate 'string "critter" (format nil "~a" n)   "
                       :direction :output
                       :if-exists :supersede
                       :if-does-not-exist :create)))
   n)
  :type fixnum)
 (location                                 ; Where the critter is
  (cons 0 0)
  :type cons)
 (lifespan                                 ; How many cycles the critter has
  1000 ; left to live
  :type fixnum)
```

```
 (interactions ; How many times this critter has interacted
  0
  :type fixnum)
 (trades ; How many of each of the possible trades
  (make-list 6   ; the critter has had, parallel to *idea-list*
   :initial-element 0)
  :type list)
 (goodies                                ; What the critter is carrying
  nil
  :type list))


;;; Functions

(defun draw-world (w)
;;
;; Draws the world in the window w.
;;
 (system::clear-window w)
 (system::set-window-cursor-position w 0 0)
 (dotimes (y *world-height*)
  (dotimes (x *world-width*)
   (format w "~a"
    (let ((here (aref *world* x y)))
     (cond
      ((not here) #\SPACE)
      ((equal here 'dot) #\.)
      ((equal here 'comma) #\,)
      ((equal here 'bar) #\_)
      (t (critter-name here))))))
  (format w "~%")))
```

```
(defun setup ()
;;
;; Makes the global *world* array, and fills it with critters.
;;
;; Also initializes other global variables:  *baby-names*.
;;
 ;; Initialize variables
 (setq *epoch* 0)
 (setq *next-critter-number* 0)
 (setq *dead*
       (let ((result nil))
        (do ((name #\z (code-char (1- (char-code name)))))
            ((< (char-code name) (char-code #\a)))
         (push name result))
        result))
 (setq *living* nil)
 (setq *uses*
       (make-array '(6 6) :initial-element 0 :element-type 'fixnum))
 (setq *correct-interpretations*
       (make-array '(6 6) :initial-element 0 :element-type 'fixnum))
 ;; Clear the stats file
 (if *stats-file*
     (with-open-file (f *stats-file* :direction :output
                                     :if-exists :supersede
     :if-does-not-exist :create)))
  ;; Create the world array
 (setq *world* (make-array `(,*world-width* ,*world-height*)))
 ;; Create the original critters
 (do ((name #\A (code-char (1+ (char-code name)))))
```

```
    ((> (char-code name) (char-code #\Z)))
  (do ((done nil)
       (x (random *world-width*) (random *world-width*))
       (y (random *world-height*) (random *world-height*)))
      (done)
   ;; Unless this point is occupied...
   (unless (aref *world* x y)
    ;; Create a critter
    (setf (aref *world* x y) (make-critter :name name
                                           :location (cons x y)))
    ;; Set the critter's genes and brain to random values
    (let ((critter (aref *world* x y)))
     (dotimes (i 6)
      (dotimes (j 6)
       (setf (aref (critter-genes critter) i j)
             (- (random 21) 10)
             (aref (critter-brain critter) i j)
             (aref (critter-genes critter) i j))))
     ;; Add the critter to the list of living critters
     (push critter *living*))
    (setq done t)))))


(defun draw-cell (x y w)
;;
;; Draws whatever is at [x, y] at that point in window w.
;;
 (system::set-window-cursor-position w y x)
 (format w "~a"
  (let ((here (aref *world* x y)))
   (cond
```

```
    ((not here) #\SPACE)
    ((equal here 'dot) #\.)
    ((equal here 'comma) #\,)
    ((equal here 'bar) #\_)
    (t (critter-name here))))))


(defun create-resource (w)
;;
;; Picks a random point.  If it's empty, a resource is dropped there.  If there's
;; critter there, the Stark Fist of Removal takes 50 epochs off the poor sucker's
;; lifespan, which may be fatal.
;;
 (let* ((x (random *world-width*))
        (y (random *world-height*))
        (here (aref *world* x y)))
  (cond
   ;; If the spot is empty, create a resource
   ((not here)
    (setf (aref *world* x y)
          (case (random 3)
           (0 'dot)
           (1 'comma)
           (2 'bar)))
    (draw-cell x y w))
   ;; If there's a resource here, do nothing.
   ((member here '(dot comma bar) :test 'equal)
    nil)
   ;; If there's a critter here, smite it
   (t
    (setf (critter-lifespan (aref *world* x y))
```

```
          (- (critter-lifespan (aref *world* x y)) *smite*))
    (if (< (critter-lifespan (aref *world* x y)) 1)
        (kill (aref *world* x y) w))))))


(defun kill (c w)
;;
;; Kills the critter c, in window w.
;;
 (let ((x (car (critter-location c)))
       (y (cdr (critter-location c))))
  (setf (aref *world* x y) nil) ; Set the old point to nil
  (draw-cell x y w) ; Draw it
  (setf *living* (delete c *living*)) ; Remove c from the living
  (push (critter-name c) *dead*))) ; and add it to the dead!


(defun message (w line message)
;;
;; Puts a message at the bottom of the screen, at "line" lines below the
;; world, and clears the rest of the message area.
;;
 (system::set-window-cursor-position w (+ *world-height* line) 0)
 (format w message)
 (if (> line 0)
     (format t "~a" #\BELL))
 (if (< (+ line *world-height*) 23)
     (sleep (* line *delay*)))
 (system::clear-window-to-eot w))
```

```
(defun move (c w)
;;
;; Moves the critter c in window w.  If the critter steps on a resource, that
;; resource is returned.
;;
 (message w 0 (format nil "Epoch ~a.  Critter ~a (~a) at ~a moving.  Lifespan: ~a
                           Goodies: ~a"
          *epoch*
  (critter-number c)
                          (critter-name c)
                          (critter-location c)
  (critter-lifespan c)
      (critter-goodies c)))
 (let* ((x (car (critter-location c)))
        (y (cdr (critter-location c)))
        (newx x)
        (newy y)
        (direction (random 4))
        (find nil))
  ;; Pick an adjacent point
  (case direction
   (0 (setq newy (1- newy)))
   (1 (setq newy (1+ newy)))
   (2 (setq newx (1- newx)))
   (3 (setq newx (1+ newx))))
  ;; Unless it's off the edge of the world or occupied
  (unless (or (>= newy *world-height*)
              (< newy 0)
              (>= newx *world-width*)
              (< newx 0))
   (unless (not (member (aref *world* newx newy)
```

```
                    '(nil dot comma bar)
                    :test 'equal))
    (setq find (aref *world* newx newy)) ; Note what was stepped on
    (setf (critter-location c) (cons newx newy)) ; Tell the critter it's moved
    (setf (aref *world* newx newy) c) ; Tell the world where it is
    (setf (aref *world* x y) nil) ; Clear the old location
    (draw-cell x y w) ; Erase the old critter
    (draw-cell newx newy w))) ; Draw the new one
  find))


(defun breed (mom dad w)
;;
;; There is a chance equal to *fecundity* that a child will be produced and
;; dropped in window w.
;;
 (when (and (> (critter-lifespan mom) 99)
            (> (critter-lifespan dad) 99)
            (> *fecundity* (/ (random 1000) 1000)))
  (message w 1 "BREEDING!")
  ;; Pop any excess @s off of *dead*;  we want new babies to have letter
  ;; names, if possible
  (do ()
      ((or (not (cdr *dead*)) (not (equal #\@ (car *dead*)))))
   (pop *dead*))
  ;; Make a baby . . .
  (let ((baby (make-critter :name (pop *dead*))))
  ;; If there are no names left, add a @
   (unless *dead* (push #\@ *dead*))
   ;; Pick a location for the birth
   (do ((x (random *world-width*))
```

```
        (y (random *world-height*))
        (done nil))
       (done)
    (unless (aref *world* x y)
     (setf (aref *world* x y) baby)
     (setf (critter-location baby) (cons x y))
     (draw-cell x y w)
     (push baby *living*))
    (setq done t))
   ;; Set up the baby's genes: a cross between those of the parents.  The
   ;; baby's brain is, initially, equal to its genes.
   ;; For each point in the genome . . .
   (dotimes (i 6)
    (dotimes (j 6)
     ;; Pick a chromosome from one parent
     (let ((chromosome
            (aref (critter-genes (if (= 0 (random 2)) mom dad)) i j)))
      ;; Put that in the baby's genes
      (setf (aref (critter-genes baby) i j) chromosome)
      ;; And in the baby's brain
      (setf (aref (critter-brain baby) i j) chromosome))))))


(defun desires (critter)
;;
;; Creates a list of trades the critter would want to make.  Each possible
;; trade is a dotted pair.  (dot . comma) means "Trade a dot for a comma."
;;
 (let ((result nil)
       (goodies (critter-goodies critter)))
  ;; For each possible trade
```

```
  (dolist (give '(dot comma bar))
   (dolist (receive '(dot comma bar))
    (unless (equal give receive)
    ;; If you have more than one of what you would have to give, and you
    ;; don't have any of what you would receive, add this trade to the list
    ;; of desired trades.
     (if (and (> (count give goodies :test 'equal) 1)
              (= (count receive goodies :test 'equal) 0))
        (push (cons give receive) result)))))
  result))


(defun statement (desire critter)
;;
;; Produces a string which the specified critter would "say" to communicate
;; the indicated desire.
;;
 (let ((result
        nil)
       (best-rating
        most-negative-fixnum)
       (x (position desire *idea-list* :test 'equal)))
 (if x
      (dotimes (y 6)
       (when (> (aref (critter-brain critter) x y) best-rating)
        (setf best-rating (aref (critter-brain critter) x y))
        (setf result (nth y *word-list*)))))
  result))


(defun interpret (offer critter)
```

```
;;
;; The opposite of "statement":  The critter's idea of the desire conveyed by
;; "offer" is conveyed.
;;
 (let ((result
        nil)
       (best-rating
        most-negative-fixnum)
       (y (position offer *word-list* :test 'string=)))
 (if y
      (dotimes (x 6)
       (when (> (aref (critter-brain critter) x y) best-rating)
        (setf best-rating (aref (critter-brain critter) x y))
        (setf result (nth x *idea-list*)))))
  result))


(defmacro react (desires interpretation)
;;
;; "Interpretation" is the offer critter thought the other one made.  If it
;; thinks this is the converse of the offer it made, it does nothing.  Other-
;; wise, the top offer is popped from "desires", which should leave either
;; something converse to the offer, or nil.
;;
 `(unless (and (equal (car ,interpretation) (cdar ,desires))
               (equal (cdr ,interpretation) (caar ,desires)))
   (pop ,desires)))


(defun learn (critter word correct)
;;
```

```
;; Increases the chance of critter interpreting word as correct, and decrease
;; the chance of critter interpreting word as whatever word would previously
;; have beenn interpreted as.
;;
 (let ((guess (interpret word critter)))
  (unless (or (equal guess correct) (not word))
   (let ((y (position word *word-list* :test 'string=))
         (badx (position guess *idea-list* :test 'equal))
         (goodx (position correct *idea-list* :test 'equal)))
  (setf (aref (critter-brain critter) badx y)
         (- (aref (critter-brain critter) badx y) *learning-rate*))
   (setf (aref (critter-brain critter) goodx y)
         (+ (aref (critter-brain critter) goodx y) *learning-rate*))))))


(defun note (idea word interpretation)
;;
;; Updates *uses* and *correct-interpretations*.
;;
 (if idea
  (let ((x (position word *word-list* :test 'string=))
        (y (position idea *idea-list* :test 'equal)))
   (setf (aref *uses* x y)
         (1+ (aref *uses* x y)))
   (if (equal idea interpretation)
       (setf (aref *correct-interpretations* x y)
             (1+ (aref *correct-interpretations* x y)))))))


(defun interact (c1 c2 w)
;;
```

```
;; Linguistic and economic interaction and learning between c1 and c2.
;; Messages are posted to window w.
;;
 ;; Increment the interaction totals for each critter
 (incf (critter-interactions c1))
 (incf (critter-interactions c2))
 ;; Generate a list of desires for each critter, and make initial offers
 (let* ((desires1 (desires c1))
        (desires2 (desires c2))
        (offer1 (statement (car desires1) c1))
        (offer2 (statement (car desires2) c2))
(interpretation1 (interpret offer2 c1))
(interpretation2 (interpret offer1 c2)))
  ;; Note how each word was used an interpreted
  (note (car desires1) offer1 interpretation2)
  (note (car desires2) offer2 interpretation1)
  ;; Write the first "round" of communication to the screen
  (if offer1
      (message w 1 (format nil "~a: ~a (meaning ~a, interpreted as ~a)~%"
                           (critter-name c1)
   (concatenate 'string offer1 "?")
   (car desires1)
   interpretation2)))
  (if offer2
      (message w 2 (format nil "~a: ~a (meaning ~a, interpreted as ~a)~%"
                           (critter-name c2)
   (concatenate 'string offer2 "?")
   (car desires2)
   interpretation1)))
  ;; Have each critter interpret and react to what the other said
  (react desires1 interpretation1)
```

```
(react desires2 interpretation2)
(setq offer1 (statement (car desires1) c1))
(setq offer2 (statement (car desires2) c2))
(setq interpretation1 (interpret offer2 c1))
(setq interpretation2 (interpret offer1 c2))
;; Note how each word was used an interpreted
(note (car desires1) offer1 interpretation2)
(note (car desires2) offer2 interpretation1)
;; Write out the second round of communication
(if offer1
    (message w 3 (format nil "~a: ~a (meaning ~a, interpreted as ~a)~%"
                         (critter-name c1)
 (concatenate 'string offer1 ".")
 (car desires1)
 interpretation2)))
 (if offer2
    (message w 4 (format nil "~a: ~a (meaning ~a, interpreted as ~a)~%"
                         (critter-name c2)
 (concatenate 'string offer2 ".")
 (car desires2)
 interpretation1)))
;; If there is a trade, do it
(when (and (car desires1) (car desires2))
 (when (and (equal (caar desires1) (cdar desires2))
            (equal (cdar desires1) (caar desires2)))
  ;; Swap the actual goodies
  (setf (critter-goodies c1)
        (cons (cdar desires1)
              (delete (caar desires1) (critter-goodies c1) :count 1)))
  (setf (critter-goodies c2)
        (cons (cdar desires2)
```

```
                (delete (caar desires2) (critter-goodies c2) :count 1)))
    ;; Note the trade in each critter's list of successful trades
    (incf (nth (position (car desires1) *idea-list* :test 'equal)
               (critter-trades c1)))
    (incf (nth (position (car desires2) *idea-list* :test 'equal)
               (critter-trades c2)))
    ;; Announce the transaction
    (message w 5 (format nil "~a trades ~a for ~a's ~a"
                          (critter-name c1)
                          (caar desires1)
                          (critter-name c2)
                          (cdar desires1)))
    (sleep 5)))
  ;; Learn
  (learn c1 offer2 (car desires2))
  (learn c2 offer1 (car desires1))))


(defun critter-epoch (c w)
;;
;; One time cycle for the critter c, in window w.  In this time, the critter:
;;
;; - Loses one time-cycle of life, and dies if it has 0 or less
;; - Moves at random.  If the way is blocked by another critter or the
;; edge of the world, the critter does not move.
;; - If the critter walks onto a resource, it picks it up.
;; - If the critter has one of each kind of resource, it consumes
;; the three of them, and gains 50 more time-cycles of life.
;; - If the critter is next to another critter, it may:
;; - Breed
;; - Converse, trade, and learn
```

```
;;
 ;; Age
 (setf (critter-lifespan c) (1- (critter-lifespan c)))
 (cond
  ((= 0 (critter-lifespan c))
   (kill c w))
  ;; If it lives...
  (t
   ;; Move, and pick up a resource if there is one
   (let ((find (move c w)))
    (if find
        (push find (critter-goodies c))))
   ;; If you c has one of each resource, consume them, and add 50 to lifespan
   (when (and (member 'dot (critter-goodies c) :test 'equal)
              (member 'comma (critter-goodies c) :test 'equal)
              (member 'bar (critter-goodies c) :test 'equal))
    (setf (critter-goodies c)
          (delete 'dot
                  (delete 'comma
          (delete 'bar
                  (critter-goodies c)
  :count 1)
          :count 1)
  :count 1))
    (setf (critter-lifespan c) (+ 50 (critter-lifespan c))))
   ;; For each adjacent point
   (let ((x (car (critter-location c)))
         (y (cdr (critter-location c))))
    (dolist (location (list (cons x (1+ y))
                            (cons x (1- y))
                            (cons (1+ x) y)
```

```
                              (cons (1- x) y)))
     (let ((i (car location))
           (j (cdr location)))
      ;; If it's not off the edge of the world...
      (unless (or (< i 0)
                  (>= i *world-width*)
                  (< j 0)
                  (>= j *world-height*))
       ;; ...and there's a critter there...
       (unless (member (aref *world* i j) '(nil dot comma bar) :test 'equal)
        ;; Breed with them
        (breed c (aref *world* i j) w)
        ;; And interact with them
        (interact c (aref *world* i j) w)))))))))


(defun uses-of (word)
;;
;; Returns a list of six floats.  Each one is the number of critters in *living*
;; that interpret the word in question as the corresponding idea in *idea-list*,
;; by the number of living critters.
;;
 (let ((result (make-list 6 :initial-element 0))
       (living-critters (length *living*)))
  (dolist (c *living*)
   (incf (nth (position (interpret word c) *idea-list*) result)))
  (dotimes (i 6)
   (setf (nth i result) (float (/ (nth i result) living-critters))))
 result))
```

```
(defun adultp (x)
;;
;; True iff x > 100
;;
 (> x 100))


(defun % (x y)
;;
;; Safe division for "Adult-diversity".  Divides, but returns 13 (the largest po
;; diversity number) if y is 0.
;;
 (if (zerop y)
     13
     (/ x y)))


(defun adult-diversity ()
;;
;; "Diversity" among individuals with a lifespan of more than 100.
;;
 (let ((adults (remove-if-not 'adultp *living* :key 'critter-lifespan)))
  (float (% (let ((tally 0))
              (dolist (a adults)
               (dolist (b adults)
                (setq tally (+ tally (difference a b)))))
              tally)
             (* (length adults) (length adults))))))
```

```
(defun manhattan-distance (a b)
;;
;; Determines the Manhattan distance between the locations of the critters a and
;;
 (+ (abs (- (car (critter-location a)) (car (critter-location b))))
    (abs (- (cdr (critter-location a)) (cdr (critter-location b))))))



(defun local-diversities (c)
;;
;; Returns a list of numbers.  Each two numbers correspond to another critter in
;; The first number is the Manhattan distance between the two, and the second is
;; diversity, as per "difference".
;;
 (let ((result nil))
  (dolist (other (remove c *living*))
   (setq result (cons (manhattan-distance c other)
                      (cons (difference c other)
                            result))))
  result))


(defun dump-stats-to-file ()
;;
;; Appends the stats that print-stats uses to the file *stats-file*.
;;
;; Lisp's FORMAT command makes C look like a high-level language.  I'm sure
;; there's a more elegant way to do this, but I don't know what it is.
;;
 ;; Dump stuff to *stats-file*
```

```
(let ((living-critters (length *living*)))
 (with-open-file (f *stats-file* :direction :output
                                 :if-exists :append
    :if-does-not-exist :create)
  (apply 'format f (let ((r "~%"))
                     (dotimes (i 110)
              (setq r (concatenate 'string "~a " r)))
             r)
                 *epoch*
 (adult-diversity)
        (append
  (let ((uses nil))
   (dotimes (x 6)
    (dotimes (y 6)
     (setq uses (append uses
                           (list (float (/ (aref *uses* x y)
                          living-critters))))))
   uses)
  (let ((correct nil))
   (dotimes (x 6)
    (dotimes (y 6)
     (setq correct
           (append correct
                   (list (float (/ (aref *correct-interpretations* x y)
                     living-critters))))))
   correct)
  (let ((freq nil))
                   (dotimes (x 6)
    (setq freq (append freq (uses-of (nth x *word-list*)))))
   freq)))))
;; Dump stuff to critter-files
```

```
 (if *save-critter-files*
     (dolist (c *living*)
      (with-open-file (f (concatenate 'string
                                      "critter"
                                    (format nil "~a" (critter-number c))
          ".stats")
                         :direction :output
                         :if-exists :append)
      (apply 'format f (let ((r "~%"))
                        (dotimes (i (+ 9 (* 2 (1- (length *living*)))))
                 (setq r (concatenate 'string "~a " r)))
                r)
      *epoch*
      (critter-lifespan c)
      (critter-interactions c)
      (append
           (critter-trades c)
          (local-diversities c)))))))


(defun epoch (w)
;;
;; Performs one time-cycle, wherein a resource might be created and each
;; undergoes a critter-epoch.
;;
 ;; Reset the *uses* and *correct-interpretations* tallies
 (setq *uses*
       (make-array '(6 6) :initial-element 0 :element-type 'fixnum))
 (setq *correct-interpretations*
       (make-array '(6 6) :initial-element 0 :element-type 'fixnum))
 (incf *epoch*)
```

```
 (dotimes (i *resources-per-epoch*)
  (create-resource w))
 (dolist (critter *living*)
  (critter-epoch critter w))
 (if (and *stats-file* (integerp (/ *epoch* *stats-rarity*)))
  (dump-stats-to-file)))


(defun difference (c1 c2)
;;
;; The difference between the brains of c1 and c2.  Difference is 1, plus
;; 1 for each case where c1 and c2 don't produce the same output for the
;; same input (either way).  Maximum difference, therefore, is 13.
;;
 (let ((tally 1))
  (dolist (word *word-list*)
   (unless (equal (interpret word c1) (interpret word c2))
    (setq tally (1+ tally))))
  (dolist (idea *idea-list*)
   (unless (equal (statement idea c1) (statement idea c2))
    (setq tally (1+ tally))))
  tally))


(defun diversity (w)
;;
;; Returns a string indicating the BRAIN diversity of the population.
;;
 (format nil "Diversity:  ~a (on a scale of 1 to 13)"
  (float (/ (let ((tally 0))
             (dolist (a *living*)
```

```
              (dolist (b *living*)
                (message w 0 (format nil "Tally of numerator: ~a" tally))
                (setq tally (+ tally (difference a b)))))
             tally)
            (* (length *living*) (length *living*))))))


(defun fast-diversity (w)
;;
;; Returns a string indicating a rough estimate of diversity.
;;
 (format nil "Fast Diversity:  ~a (on a scale of 1 to 13)"
  (float (/ (let ((tally 0))
             (dotimes (i 100)
               (message w 0 (format nil "Tally of numerator: ~a" tally))
               (setq tally
                     (+ tally (difference (nth (random (length *living*))
                                                    *living*)
                                               (nth (random (length *living*))
                                                    *living*)))))
             tally)
            100))))


(defun print-stats (w)
;;
;; Clears the window w, and prints out the information in *uses* and
;; *correct-interpretations*.
;;
 (system::clear-window w)
 (system::set-window-cursor-position w 0 35)
```

```
 (format w "STATISTICS~2%")
 (format w "Word:    (./,)     (./_)     (,/.)     (,/_)      ~
           (_/.)     (_/,)~%")
 (format w "Foo~%Bar~%Quux~%Baz~%Fnord~%Flugblogs")
 (dotimes (x 6)
  (dotimes (y 6)
   (system::set-window-cursor-position w (+ 3 x) (* 10 (1+ y)))
   (format w "~a/~a" (aref *uses* x y) (aref *correct-interpretations* x y))))
 (message w 5 "Number of times used/correctly interpreted")
 (message w 8 "(Hit any key to continue)")
 (with-keyboard (read-char *keyboard-input*)))


(defun draw-critter (c w)
;;
;; Draws the brain of critter c in window w.
;;
 (system::clear-window w)
 (system::set-window-cursor-position w 0 0)
 (format w "Critter ~a~2%" (critter-name c))
 (format w "Word:    (./,)     (./_)     (,/.)     (,/_)      ~
           (_/.)     (_/,)~%")
 (format w "Foo~%Bar~%Quux~%Baz~%Fnord~%Flugblogs")
 (dotimes (x 6)
  (dotimes (y 6)
   (system::set-window-cursor-position w (+ 3 x) (* 10 (1+ y)))
   (format w "~a" (aref (critter-brain c) x y)))))


(defun edit-critter (w)
;;
```

```
;; Allows the user to edit a critter, in window w
;;
 (system::clear-window w)
 (format w "CRITTER EDITOR~2%Critter name? ")
 (let ((c (find (with-keyboard (read-char *keyboard-input*))
               *living*
:key 'critter-name)))
  (declare (special c))
  (draw-critter c w)
  (message w 8 (format nil "Option? [QCS]"))
  (do ((option
       (with-keyboard (read-char *keyboard-input*))
       (with-keyboard (read-char *keyboard-input*))))
      ((member option '(#\q #\Q)) nil)
   (case option
    ((#\c #\C) ; Clear brain & genes
      (dotimes (x 6)
       (dotimes (y 6)
        (setf (aref (critter-genes c) x y) 0)
(setf (aref (critter-brain c) x y) 0))))
    ((#\s #\S) ; Set a link
      (message w 0 "SET LINK")
      (system::set-window-cursor-position w (1+ *world-height*) 0)
      (format w "Word? ")
      (let ((r (read-line))
       (format w "Idea? ")
       (let ((i (read)))
        (format w "Value? ")
(let ((v (read))
      (x (position r *word-list* :test 'string=))
      (y (position i *idea-list* :test 'equal)))
```

```
 (when (and x y)
  (setf (aref (critter-genes c) x y) v)
  (setf (aref (critter-brain c) x y) v)))))))
   (draw-critter c w)
   (message w 8 (format nil "Option? [QCS]")))))



;;; The program proper:  calls to the main functions

(setup)


(with-window
 (draw-world *window*)
 (dotimes (i *initial-resources*)
  (create-resource *window*))
 (message *window* 8 "Command? [QOTHKNDFEL]")
 (do ((command
       (with-keyboard (read-char *keyboard-input*))
       (with-keyboard (read-char *keyboard-input*))))
     ((member command '(#\q #\Q)) nil)
  (case command
   ((#\o #\O) (epoch *window*)) ; One epoch
   ((#\t #\T) (dotimes (i 10) (epoch *window*))) ; Ten epochs
   ((#\h #\H) (dotimes (i 100) (epoch *window*))) ; Hundred epochs
   ((#\k #\K) (dotimes (i 10000) (epoch *window*))) ; TenThousand epochs
   ((#\n #\N) ; New world
     (dotimes (i (length *living*))
      (pop *living*))
```

```
    (setup)
    (draw-world *window*)
    (dotimes (i *initial-resources*)
     (create-resource *window*)))
  ((#\d #\D) ; Diversity rating
   (message *window* 0 (diversity *window*)))
  ((#\f #\F) ; Fast diversity
   (message *window* 0 (fast-diversity *window*)))
  ((#\e #\E) ; Edit a critter
   (edit-critter *window*)
   (system::clear-window *window*)
   (draw-world *window*))
  ((#\l #\L) ; Language stats
   (print-stats *window*)
   (system::clear-window *window*)
   (draw-world *window*)))
 (message *window* 8 (format nil "Epoch ~a  Command? [QOTHKNDFEL]" *epoch*))))
```

Chapter 5

A Summing Up

The games described in this thesis have a common structure, which we outlined in chapter 2. That structure can be summarised as follows. There are N individuals playing a game. Each individual must choose one of a range of activities, $(a_1, \ldots, a_n)$. Each players' payoff can be written as $P(x; \theta, N_1, \ldots, N_n)$, where x is the player's action, N_i the number of player's adopting action i when strategies are played out, and θ a vector of person-specific parameters describing the individual. If θ is irrelevant we talk of an "undifferentiated" numbers externality, otherwise a "differentiated" one. The crucial part of the payoff function is that it can be written as with simply the N_i as arguments. Thus we can think of equilibria, in fact *must* think of equilibria as a collection of strategies implying a particular vector of the N_i, and herein lies the focus of this thesis.

We have seen various different circumstances. In our simple rendition of the complementarity issue in demand we noted that in equilibrium only one of several undifferentiated products will be purchased if the utility of any one of them increases with the total number purchased. We have seen the demise of the BetaMax format for video cassettes as an example. This equilibrium did not emerge immediately, but evolved. This equilibrium is *perhaps* Pareto optimal, providing that BetaMax is *not* an intrinsically superior technology. In a similar fashion languages have evolved and media of exchange. The important feature of these equilibria is the efficacy of a certain strategy if *primarily* determined by a numbers externality, and the intrinsic properties of a language or a medium of exchange are of secondary importance. In my opinion such equilibria can be thought of as naturally emergent, and perhaps a good approximation to out of equilibrium behaviour would be a simple dynamic process in which, in a repeated game, individuals adjust to previous period's activity. To be specific, consider a game in which the equilibrium consists of a vector of the N_i that contains $(n-1)$ zeroes and one N—that is in equilibrium it doesn't matter which a_i the players choose, just that they all choose the same one. Let the game be played in each of successive time periods, $t = 1, \ldots$. A simple adjustment mechanism would have the players randomise in period 1, and then choose the action chosen by most players in the last period as time progresses. If several activities are tied for this lead, then another simple randomising device is employed so that players choose each of the tied activities with equal probability. Such a process yields the equilibrium in a finite amount of time with probability one, and presumably any other reasonable dynamic will do the same. For players in such games the single goal is coordination, which is to the benefit of all.

The Tiebout scenario and the spatial voting model are not like this. These are differentiated games in which players may disagree on the most desirable equilibrium. These games have a structure that is strongly reminiscent of the *Gravity* models of human geography.[1] In the simplest case gravity models predict the concentration of individuals by invoking a two-dimensional model suggesting Newtonian mechanics. Individuals are attracted to other groups with a force proportional to the mass of a group, and inversely proportional to the distance (or the square of the distance) to the group. The crucial point is that individuals are differentiated by their physical location, just as they were in the Tiebout model. In the spatial voting model it was preferences over platforms that distinguished people from one another. In these models the possible equilibria are many, and differ in significant qualitative characteristics.[2]

[1] Isard [1960], chapter 11.

[2] Our Tiebout model was not developed sufficiently to illustrate this, but the reader can perhaps see the potential.

Chapter 6

References

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