TREES, ORDERS AND FORCING

by

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ABSTRACT

Results are presented regarding trees, orders, and forcing. To begin, the rich play between trees and orders is observed. A number of theorems on properties that transmit between trees and orders are spread out. The question as to which trees have branches added by forcing with a stationary subset of ω_1 is asked. A theorem stating that trees whose levels are of size less than $2^{\aleph_0}$ get no new branches gives partial information. The development of an interior operator to describe this situation in more detail is presented, followed by a useful monotonicity theorem. Finally, a result concerning universal elements for certain classes of trees and orders is documented; for example, it is shown that there are no universal separable linear orders of size $2^{\aleph_0}$.

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INTRODUCTION

Trees furnish the natural setting for investigation of any number of mathematical questions. Revealing questions concerning the real line have been tackled with much success through the analysis of different species of trees. The Souslin problem springs to mind immediately [DeJ]. Two cardinal theorems in model theory often depend on the knowledge of whether or not a given tree exists [ChK]. Questions in Boolean algebra and linear order theory can be echoed quite precisely through the terminology of trees — see Chapter 1.

Chapter 1 offers a tying together of various mathematical structures closely akin to the tree. An idea of what sort of formal properties transmit between the tree and the considered structure is given. For example, a result on the $\aleph_1$-Baire property is supplied.

Chapter 2 asks which trees have no new branches added when shooting a club through a stationary subset of ω_1? Some results are displayed in answer to this; namely, it is shown that trees that cannot support the cardinal arithmetic necessary to add a new branch will not have any branch added.

Chapter 3 contains examples which illustrate the role played by the $\aleph_1$-Baire property when dealing with trees that have branches added by shooting a club through a stationary set.

Chapter 4 describes an interior operator, constructed as an attempt to analyse the situation in Chapters 1 and 2 in more detail. Algebraic facts about the interior operator are spread out and a reasonable monotonicity result is documented. This section is also populated with elementary facts about stationary sets.

Chapter 5 seeks, using the relationships established in Chapter 1, to demonstrate that certain classes of linear orders do not have a universal element. The main theorem relies on an idea located in [HVa].

CHAPTER 1

TREES AND ORDERS

Connections between trees and linear orders have been noted for quite some time [Kup]. The play between the two is rich enough to offer translations of many problems from the language of the one to the language of the other. Other mathematical structures provide yet another setting in which to discuss trees. This chapter explores what can be developed along this line.

Most of what is said is confirmed in [Tor1,Tor2], but a fuller picture is described albeit in quicker fashion. In addition some observations are placed as to what kind of properties are preserved when shifting between different structures; in particular, a generalization of an implicit result of [Tor1] is presented alongside some easy results of similar nature. The usual notation of ZFC set theory is adopted and is like that found in [Tor1]; or for that matter, any standard text in set theory or topology.

A tree is a poset $\langle T, > \rangle$ such that $b_t = \{s \in T : s > t\}$ is well-ordered by $>$ for every $t \in T$. To be consistent with the convention for posets, trees will go downward. $h_t = tp(b_t, >)$ is the height of t. The α'th level of T is the set $R_\alpha T = \{t \in T : h_t = \alpha\}$, and the height of T is $h_T = \min\{\alpha : R_\alpha T = \emptyset\}$. If $h_T = \alpha$ then T is called an α-tree; however, a short $\aleph_1$-tree refers to an ω_1-tree that has cardinality $\aleph_1$ and has no uncountable branches. Note that the usual definition of $\aleph_1$-tree has the further restriction that all levels are countable. Furthermore, assume all trees are pruned to the extent that they satisfy $\forall\alpha < \beta < h_T \ \ \forall x \in R_\alpha T \ \ \exists y, z \in R_\beta T$ s.t. $y < x$, $z < x$, and $y \perp z$ where $y \perp z$ says that x and y have no common extension; of course, distinct elements on a given level have no common extension.

A filter S on ω_1 is a collection of subsets of ω_1 closed under finite intersection such that if $A \in S$ and $A \subseteq B \subseteq \omega_1$ then $B \in S$. To avoid trivialities $\emptyset \notin S$. A filter A on ω_1 is σ-complete if A is closed under countable intersections; A is normal if A is closed under diagonal intersections — see [Ku]. The dual to the notion of a filter on ω_1 is an ideal on ω_1.

A subset S of ω_1 is said to be closed and unbounded (club) if S is unbounded in ω_1 and for any countable $T \subseteq S$, $\sup(T) \in S$. A subset $S \subseteq \omega_1$ is stationary if it has non-empty intersection with every club in ω_1. The set of all non-stationary subsets of ω_1 form a σ-complete normal ideal.

Let T be a tree and well-order the nodes of T by a well-order $<_A$ of order type $|T|$. An order is placed on T as follows:

i) $x < y$ if $y >_T x$

ii) $x < y$ if $x \perp y$ and $z(x,y) <_A z(y,x)$.

Here $z(x,y)$ is the $>_T$-least element of $\{b : x \geq_T b\} \setminus \{b : y \geq_T b\}$. Now to every tree there is a corresponding class of linear orders. Conversely, given an order

L a class of trees can be constructed, using recursion on levels, from the non-empty convex subsets of L such that each tree T satisfies:

i) $R_0 = \{L\}$

ii) $I \in R_\alpha T \Rightarrow \exists S \subseteq R_{\alpha+1} T$ of pairwise disjoint sets with $\bigcup S = I$

iii) If $b \subseteq T$ is a chain in T and if $\bigcap b \neq \emptyset$ then $\bigcap b \in T$

Clause (ii) assures that such a construction will not be unique. For a short $\aleph_1$-tree the associated orders are of size $\aleph_1$ and are first countable – every point has a countable neighbourhood base [Tor1]. Naturally, the Dedekind completion of such an order has cardinality $2^{\aleph_0}$ and inherits the first countability property. In this way a class of continua can be connected with a short $\aleph_1$-tree. Heading in another direction, an $\aleph_1$-tree can be densely embedded into a complete Boolean algebra [Ku]. Yet another approach, mentioned for sake of completeness, is to define a notion of stationarity for $\aleph_1$-trees [Tor1] and reach some sort of classification of such trees by proving results analogous to those proved concerning stationary sets.

For an $\aleph_1$-tree, let $L(T)$, $C(T)$, and $B(T)$ respectively denote members from the class of linear orders, continua, and Boolean algebras that can be associated with T in a manner already detailed.

Definition 1. *A topological space is $\aleph_1$-Baire if the intersection of any countable collection of open dense sets is dense.*

Let $a, b \in L(T)$. Define a relation $a \equiv b$ if and only if (a, b) is countable. Note

$\equiv$ is an equivalence relation with convex equivalence classes. Let $[a]$ denote the equivalence class of $a \in L(T)$. The induced order is placed on $L(T)/\equiv$. As the set of elements below any given node in T forms an uncountable open set in $L(T)$, the size of $L(T)/\equiv$ must be $\aleph_1$.

Lemma 1. *If O is open dense in T, $\{[a] : a \in O\}$ contains an open dense set in $L(T)/\equiv$.*

Proof : Let $[a] < [b]$, note (a, b) in $L(T)$ is uncountable. Pick $c, d \in (a, b)$ with $c < d$ and $c \perp d$. Note that $\{x : x <_T d\} \subseteq (c, d)$. Now choose $e \in O \cap \{x : x <_T d\}$ and observe that $\{x : x <_T e\} \subseteq O$ and $\{[x] : x <_T e\} \subseteq ([a], [b])$ in $L(T)$. As $\{x : x <_T e\}$ is open in $L(T)$, so must $\{[x] : x <_T e\}$ be open in $L(T)/\equiv$.

Lemma 2. *If O is open dense in $L(T)/\equiv$, then $\bigcup O$ contains an open dense set in T.*

Proof : For $t \in T$ choose $a, b \in \{x : x <_T t\}$ with $a \perp b$, $a < b$, and $([a], [b]) \neq \emptyset$. Choose $([c], [d]) \subseteq ([a], [b]) \cap O$ and observe that $\{[x] : x <_T d\} \subseteq ([c], [d])$.

Theorem 1. *T is $\aleph_1$-Baire if and only if $L(T)/\equiv$ is $\aleph_1$-Baire.*

Proof : Lemma 1 and 2.

Theorem 2. *Let T be a short $\aleph_1$-tree; T satisfies the $\aleph_1$-Baire property if and only if $B(T)$ does.*

Proof : Select $f: T \to B(T)$ a dense embedding. Note that if O is open dense in T then $f(O)$ is open dense in $B(T)$. Also, if O is open dense in $B(T)$ then, using the fact that f is a dense embedding, $f^{-1}(O)$ contains an open dense set in T.

Undefined terms in the statement of Theorem 2 can be located in Chapter 5.

Theorem 3. *Let T be an ω_1-tree; T satisfies* a) *the* c.c.c or b) *separability property or* c) *the short order property iff each of the following do:*

i) $L(T)/\equiv$

ii) The dedekind completion of $L(T)/\equiv$

iii) $B(T)$.

Proof : There are eighteen statements to be proved all of which make a good exercise for the reader.

CHAPTER 2

SHOOTING CLUBS AND $\aleph_1$ – TREES

Passing a club through a stationary set has no effect on a certain class of short $\aleph_1$-trees; namely, those trees which cannot support the cardinal arithmetic neccessary to add new long branches. Presented are two results of this nature. Also, there are a number of ways to pass a club through a stationary set. Various properties of forcing notions which do the job are investigated. A short proof of the $\aleph_1$-Baire property for $T(S)$ — defined below — is introduced; this and other basic results about $T(S)$ were first proved in [BHK]. To set the stage prerequisite knowledge is discussed.

Familiarity with the technique of forcing is assumed. M denotes a countable transitive model of ZFC, P a poset in M commonly refered to as a forcing notion, and G a M-generic filter on P. M is the ground model and $M[G]$ is its extension by G. The relation $p \Vdash_P^M \psi$ reads "p forces ψ" or "$p \in G \Rightarrow \psi$ is true in $M[G]$"; of course, such a forcing language can be formalised in M.

P satisfies the κ-c.c. iff there is no pairwise incompatible subset of P of cardinality κ. The ω_1-c.c. is refered to as the c.c.c. P is κ-closed iff all chains in P of cardinality $< \kappa$ have lower bounds. P is $\aleph_1$-Baire iff the intersection of countably many open dense sets in P is dense — basic open sets are of the form $\{x : x \leq p\}$

for $p \in P$. Consequences of these properties are assumed and can be found in [Ku]. What model theory is present is basic and can be found in any standard text in logic — for example [ChK].

There are two basic methods by which to pass a club through a stationary set S. Define $p \in P(S)$ iff p is a finite collection of closed intervals in ω_1 such that:

i) $[\alpha, \beta] \in p \Rightarrow \alpha \in S$

ii) If $[\alpha, \beta]$, $[\alpha', \beta'] \in P$ then $\alpha = \alpha'$ or $[\alpha, \beta] \cap [\alpha', \beta'] = \emptyset$.

The order on P is by inclusion. Now if G is an M-generic filter over P then $C = \{\alpha :$ for some β and $p \in G,\ [\alpha, \beta] \in p\}$ is a club subset of S and P does not collapse cardinals. A more detailed account of this notion can be located in [AbSh]. There is a more natural way to shoot a club through a stationary set S — i.e. put a club inside S in the generic extension. Let $p \in T(S)$ if and only if p is a countable closed subset of S. The order on $T(S)$ is by end extension. If G is an M-generic filter over P then $\bigcup G \subset S$ is a club — the unboundedness follows from a simple density argument while closed follows from the fact that G is a branch of $T(S)$. In addition, $T(S)$ is $\aleph_1$-Baire as demonstrated in the upcoming theorem. Notice that if CH is assumed $T(S)$ will not collapse any cardinals — this follows from Theorem 1 and the fact that $T(S)$ has size $\aleph_1$.

Theorem 1. *$T(S)$ is $\aleph_1$-Baire.*

Proof : Let $\{D_i : i < \omega\}$ be a collection of open dense sets and let $p \in T(S)$.

Our wish is to show $\bigcap_{i<\omega} D_i$ contains some element below p. Let $\{M_\alpha : \alpha < \omega_1\}$ be a continuous elementary chain of countable elementary submodels of $H(\kappa)$ for some κ large enough so that all the previous sets may be elements of M_0 – see [Ku] for an account of $\langle H(\kappa), \in\rangle$. Also, stipulate that $M_\alpha \in M_\beta$ for $\alpha < \beta$ and choose a limit $\alpha \in \{\beta : M_\beta \cap \omega_1 = \beta,\ \beta < \omega_1\} \cap S$. Let $\{\alpha_i : i < \omega\}$ be cofinal in α and define $\{p_i : i < \omega\}$ satisfying:

i) $p_0 = p$

ii) $p_i \leq p_{i+1}$

iii) $p_{i+1} \setminus p_i \subset (M_{\alpha_{i+1}} \setminus M_{\alpha_i}) \cap \omega_1$

iv) $p_i \in D_i \cap M_{\alpha_i}$

The elementarity of the M_α's makes the construction proceed. Clearly $\bigcup_{i<\omega} p_i \cup \{\alpha\} \in T(S)$ and is an element of $\bigcap_{i<\omega} D_i$.

Theorem 2. *If forcing with the tree $T(S)$ puts a branch through a tree T then some level of T contains $2^{\aleph_0}$ elements.*

Proof : As before let $\{M_\alpha : \alpha < \omega_1\}$ be a continuous elementary chain of countable elementary submodels of $H(\kappa)$ for some large enough κ to embrace the definitions of the proof with $M_\alpha \in M_\beta$ for $\alpha < \beta$. Again choose $\alpha \in S$ such that $M_\alpha \cap \omega_1 = \alpha$, and let $\{\alpha_i : i < \omega\}$ be cofinal in α. Let $\mathbf{b}$ be the name for an uncountable branch in T. By induction on the levels of $\langle 2^{<\omega}, \subseteq\rangle$ define sequences $\{p_s : s \in 2^{<\omega}\}$ and $\{x_s : s \in 2^{<\omega}\}$ of elements from $T(S)$ and T respectively so that:

i) $p_0 \in M_{\alpha_0}$

ii) $p_0 \Vdash \mathbf{b}$ is a branch of size $\aleph_1$ through T, $\mathbf{b} \in M$

iii) t extends $s \Rightarrow\ p_s \leq p_t$

iv) t extends $s \Rightarrow\ x_s <_T x_t$

v) $x_{s\frown 0}$ and $x_{s\frown 1}$ are incompatible

vi) $p_s \Vdash x_s \in \mathbf{b}$

vii) r =length$(s) \Rightarrow p_s, x_s \in M_{\alpha_r}$

viii) r =length$(s) \Rightarrow \sup(M_{\alpha_{r-1}}) \leq \max(p_s)$

The only problem that may need addressing regarding the construction is v).In particular, assume the construction failed at stage n with x_s and p_s. The set $L = \{x : x < x_s,\ p \leq p_s,\ p \Vdash x \in \mathbf{b}\}$ is therefore a chain in M. So $p_s \Vdash (x < x_s$ and $x \in \mathbf{b})$ if and only if $x \in L$ — i.e. $T(S)$ has an uncountable branch.

Now for each $s \in 2^\omega$ let $p_s = \bigcup_{\beta<\omega} p_{t\restriction\beta} \cup \{\alpha\}$. Note that for $s \in 2^\omega$, $p_s \in T(S)$ and that $\{x_s : s \in 2^{<\omega}\}$ is tree isomorphic to $2^{<\omega}$. Observe that there is some ordinal β such that the constuction of the x_s's lie above level β of T. Index the branches of $\{x_s : s \in 2^{<\omega}\}$ with 2^ω and pick $x_s \in T$ extending the branch associated with $s \in 2^\omega$ such that there is $q_s \leq p_s$ with $q_s \Vdash x_s \in T$ and x_s lies at level $\beta+1$ — recognize this is possible as $\mathbf{b}$ is an uncountable branch. However, the obligation requiring $2^{\aleph_0}$ elements at level $\beta+1$ of T gives the desired conclusion.

Corollary 1. *Forcing with the tree $T(S)$ does not put a branch through a short $\aleph_1$-tree whose levels are countable.*

Proof : Apply Theorem 2.

Corollary 2. *If $\neg CH$ then forcing with $T(S)$ does not put a branch through a short $\aleph_1$-tree.*

Proof : Apply Theorem 2.

CHAPTER 3

$\aleph_1$ – BAIRE $\aleph_1$ – TREES

For a stationary set $S \subset \omega_1$ the associated tree of countable closed subsets of S ordered by end extension has the $\aleph_1$-Baire property. If a club is shot through S then the wish is characterize those short $\aleph_1$-trees which accept a new branch. The question arises as to whether the $\aleph_1$-Baire property plays a part here, or whether it can be forgotten in some way? It seems likely that short $\aleph_1$-trees with branches added by forcing with $T(S)$ are in some primitive way similar in structure to $T(S)$ but the connection is not at all transparent. Examples to clarify the role played by the $\aleph_1$-Baire property are presented, and as a result, two distinct classes of short $\aleph_1$-Baire $\aleph_1$-trees are defined.

Definition 1. *A Souslin tree is an $\aleph_1$-tree satisfying the countable chain condition.*

Lemma 1. *A Souslin tree is $\aleph_1$-Baire.*

Proof : Observe that every open dense set is of the form $\{x : c \in A, x \leq c\}$ where A is a maximal antichain. Let $O_{i<\omega}$ be a countable collection of open dense sets of the form $O_i = \{x : c \in A_i, x \leq c\}$ where A_i is a maximal antichain. The size of each A_i is countable by the c.c.c. As the collection $O_{i<\omega}$ is countable there is some $\alpha < \omega_1$ such that all elements in each A_i lie above level α of the tree. Clearly all elements below level α will be in $\bigcap_{i<\omega} O_i$ as each element below level α must be in each O_i by the maximality of C_i.

For the rest of this section fix a stationary co-stationary subset S of ω_1. Note that the existence of Souslin trees is independant of the usual axioms of set theory. This and other facts about Souslin trees can be found by consulting [J].

Theorem 1. *It is consistent that there is a short $\aleph_1$-tree that is $\aleph_1$-Baire such that for all stationary sets E of ω_1 no branches are added by forcing with $T(E)$.*

Proof : The fact that $T(E)$ is Baire ensures that no new countable branches are added. Now a Souslin tree has the $\aleph_1$-Baire property and all levels are countable; by Theorem Chapter 2, no branches of length ω_1 are added.

Assume S contains only limit points in ω_1 and that T is also stationary with no successor points and disjoint from S; also assume that $S \cup T$ has a complement that is stationary. Let A be the set of all closed bounded sets contained in $S \cup T$. Let $t \in A_E \subset A$ if for all $\alpha \leq \max(t)$, $S \cap t \cap \alpha$ and $T \cap t \cap \alpha$ are not both cofinal in α. An order $<$ is placed on A_E: $s < t$ iff t is an end extension of s. Refer to $\langle A_E, < \rangle$ as A_E.

Theorem 2. *There is a short $\aleph_1$-tree that is not $\aleph_1$-Baire such that branches are added by forcing with $T(S)$.*

Proof : Note that forcing with $T(S)$ trivially puts a branch through A_E as $T(S)$ can be thought of as a subtree of A_E. It remains to show that A_E is not $\aleph_1$-Baire. Choose a maximal antichain D_0 such that each element $t \in D_0$ satisfies $t \cap S$ is not cofinal in $\max(t)$. Pick D_1 such that each element $t \in D_1$ properly extends an element in D_0, D_1 is a maximal antichain, and $t \in D_1$ satisfies $t \cap T$ is not cofinal in $\max(t)$. In this way construct D_{2n} and D_{2n+1}. Set $O_i = \{x : d \in D_i, x \leq d\}$ and notice O_i is open dense. The claim is that $\bigcap_{i<\omega} O_i$ is empty. However, if $a \in \bigcap_{i<\omega} O_i$ then there is $d_i \in D_i$ with $d_i < d_j < a$ for $i < j < \omega$ and a end extends $\bigcup_{i<\omega} d_i$, but this means $a \notin A_E$ as both S and T are cofinal in $\bigcap_{i<\omega} d_i$.

Lemma 2. *There is a short $\aleph_1$-tree that is $\aleph_1$-Baire such that branches are added by forcing with $T(S)$.*

Proof : $T(S)$ trivially fits the requirement.

Definition 2. *A tree T is called special if there exists a map $f: T \to \omega$ such that $s < t$ implies $f(s) \neq f(t)$.*

Note that short $\aleph_1$-trees that are special exist; for example, the set of bounded subsets of the rationals ordered by end extension.

Lemma 3. *There is a short $\aleph_1$-tree that is not $\aleph_1$-Baire such that no branches are added by forcing with $T(S)$.*

Proof : Putting a long branch through a special tree would collapse $\aleph_1$.

CHAPTER 4

INTERIOR OPERATORS

Shooting clubs only adds long branches to short $\aleph_1$-trees whose levels all have size $\aleph_1$ if CH is true — see Chapter 2, Theorem 2. Consequently, for the rest of the chapter CH is assumed. The question arises as to which nodes in a tree permit long branches to pass through. An interior operator of sorts can be set up describing precisely this state of affairs. As a result, a context is provided to discuss the larger question, an eventual aim of this line of work, as to which trees accept long branches via the killing of a stationary set. Various results about the interior operator are laid down including a useful monotonicity theorem.

Let S be a stationary set and T be an $\aleph_1$-tree. Define $I_S(T) = \{x \in T : \exists p \in T(S)$ s.t. $p \Vdash x \in \mathbf{b}$ where $\mathbf{b}$ is a branch of size $\aleph_1\}$. For S and R stationary, define $S =_{NS} R$ iff $(S \setminus R) \cup (R \setminus S)$ is non-stationary.

Theorem 1 proves a homogeneity property of $T(S)$ which is to be thought of as folklore.

Theorem 1. *If $p \Vdash_{T(S)} \psi$ then $\emptyset \Vdash_{T(S)} \psi$ where ψ is a first order sentence with no names outside V.*

Proof : Let G is $T(S)$-generic and let $H = \{(s \setminus p) \cup p : s \in G\}$. To show that

H is $T(S)$-generic pick $\alpha > \max(p)$ with $\alpha \in S$ and let $G_\alpha = \{s : s \in G;\ s$ extends $\bigcup G \cap \alpha \cup \{\alpha\}\}$ and define H_α similarly. Note the trees $\{s : s \in T(S)$ extends $\bigcup G \cap \alpha \cup \{\alpha\}\}$ and $\{s : s \in T(S)$ extends $\bigcup H \cap \alpha \cup \{\alpha\}\}$ are equal. Note that G must hit every dense set in at least $\aleph_1$ places, and so G_α must hit every dense set as well since it differs from G by only a countable set. Therefore as G_α intersects any dense set in $T(S)$, H_α must do so as well. Lastly note that $M[G] = M[H]$ and apply definition of forcing.

Lemma 1. *i)* $I_S(T) \subseteq T$

ii) $I_S(I_S(T)) = I_S(T)$

iii) $I_S(T(S)) = T(S)$.

Proof : Trivial.

Theorem 2. *i)* $R \leq_{NS} S \Rightarrow I_R(T(S)) = T(S)$

ii) $R \not\leq_{NS} S \Rightarrow I_R(T(S)) = \emptyset$.

Proof : i) Trivial.

ii) It suffices to show that $R\backslash S$ remains stationary in $M[G]$. Suppose not; let $\mathbf{C}$ name a club such that $p \in T(R)$ with $p \Vdash \mathbf{C} \cap (R\backslash S) = \emptyset$. Fix $\{M_\alpha : \alpha < \omega_1\}$ as in Theorem 1, Chapter 2, and let $\alpha \in \{\beta : M_\beta \cap \omega_1 = \beta,\ \beta < \omega_1\} \cap (R\backslash S)$ be a limit. Let $\{\alpha_i : i < \omega\}$ be cofinal in α. Choose $p_i \in M_{\alpha_{i+1}}$ with $\sup(M_{\alpha_i} \cap \omega_1) \leq \sup(p_i)$ so that $p_i \Vdash x_i \in \mathbf{C} \cap R$, $\sup(M_{\alpha_i} \cap \omega_1) \leq \sup(x_i)$, $x_i < x_{i+1}$, and $p_i < p_{i+1}$. Recognize that $\bigcup_{i<\omega} p_i \cup \{\alpha\} \Vdash \alpha \in \mathbf{C} \cap (R \setminus S)$ leading to a contradiction.

Lemma 2. *Let T be a tree such that T does not embed the tree $2^{<\omega_1}$, then forcing with an ω_1-closed poset does not add any new branches in the generic extension.*

Proof : A name for an uncountable branch **b**, and the ω_1-closed property allow the construction of a subtree of T isomorphic to $2^{<\omega_1}$ much along the same vein as Theorem 2, Chapter 2. A similar result can be located in [Tor.2].

It is worthwhile to note that the above lemma goes through for posets that are strategically closed, a strictly weaker notion than ω_1-closed.

Theorem 3. *i)* $S \leq_{NS} R \Rightarrow I_R(T) \subseteq I_S(T)$.

Proof : Assume $x \in I_R(T)$ and $x \notin I_S(T)$ for sake of contradiction. Let $G \times H$ be $T(S) \times T(R)$-generic over M. Let $T_x = \{y : y <_T x\}$ and observe that T_x has no new branches added in $M[G]$. Now $T(R)^{M[G]} = T(R)^M$ as no new countable sets are added, and $I_R(T)^M \subseteq I_R(T)^{M[G]}$ as $T(R)$ is essentially closed in $M[G]$. Again, since $T(R)$ is essentially ω_1-closed in $M[G]$, and using Lemma 2, T_x has no new branches added in $M[G][H]$. However, H is also generic over M and so $x \notin I_R(T)$.

Lemma 3. *i)* $I_{\bigcup_{i\in J} R_i}(T) \subseteq \bigcap_{i\in J} I_{R_i}(T)$.

Proof : Recognize that if $\bigcup_{i\in J} R_i$ contains a club then $I_{\bigcup_{i\in J} R_i}(T) = \emptyset$; otherwise, apply Theorem 3.

CHAPTER 5

UNIVERSAL ORDERS

A class of structures C is said to have a universal element if there is a $U \in C$ such that for all $X \in C$ there is a monomorphism $F: X \hookrightarrow U$. Regarding linear orders, Shelah has shown the consistency of "ZFC+$2^{\aleph_0} = \aleph_2$+there exists a universal linear order of power $\aleph_1$". It has also been observed by Shelah that the addition of $\aleph_2$ Cohen reals to the universe will destroy all universal linear orders [Sh.1]. However, it is demonstrated in this section that certain classes of linear orders have no universal elements in an absolute sense. In particular, it is shown that there is no c.c.c or separable universal linear orders of power $2^{\aleph_0}$.

Fix a linear ordered set $\langle L, < \rangle$. Let $W(L)$ be the set of all well-ordered subsets of L. For S, $T \in W(L)$ let $S \equiv_e T$ if both S and T are cofinal in each other. Clearly, $\equiv_e$ is an equivalence relation on $W(L)$. For $A \in W(L)$ let $[A]_e$ denote its corresponding equivalence class. Now let A, $B \in W(L)/\equiv_e$ and let $A <_W B$ if for each $S \in A$ and $T \in B$ there is a $t \in T$ such that for all $s \in S$, $s < t$.

Lemma 1. *$\langle W(L)/\equiv_e, <_e \rangle$ is a linear order.*

For the remainder of the chapter assume $\langle L, < \rangle$ is a linear order with no uncountable well-ordered subsets such that $|L| = \kappa$ where $\kappa \in \{\kappa : \kappa^{\aleph_0} = \kappa\}$. Observe

$| W(L) |= \kappa^{\aleph_0} = \kappa$. To further set the stage, a proof is given that there is no order preserving embedding $F : W(L)/ \equiv_e \mapsto L$. The proof is taken directly from [HVa] and has been trivially modified to work for linear orders. Aside from a few technical calculations, most of the results of the chapter flow easily from this fact.

Theorem 1. *There is no order preserving embedding* $F\colon W(L)/ \equiv_e \to L$.

Proof : Assume $G\colon W(L) \to L$ is such an embedding. Define $\{a_i : i < \omega_1\}$ and $\{b_i : i < \omega_1\}$ by recursion as follows:

Stage 0 : $a_0 = \emptyset$, $b_0 = G(a_0)$

Stage $\beta + 1$: $a_{\beta+1} = a_\beta \cup \{b_\beta\}$, $b_{\beta+1} = G([a_{\beta+1}]_e)$

Stage β limit : $a_\beta = \bigcup_{\alpha<\beta} a_\alpha$, $b_\beta = G([a_\beta]_e)$.

A simple induction on β using the fact that G is order preserving gives that for all $a \in a_\beta$, $a <_L G([a_\beta]_e)$. This shows that for all $\beta < \omega_1$, $a_{\beta+1}$ is an end extension of a_β. So $\alpha < \omega_1$ implies that $[a_\alpha]_e \in W(L)$ and thus $G(a_\alpha) = b_\alpha < b_\beta = G(a_\beta) \in L$. However, L has no uncountable well-ordered subset.

Definition 1. *A linear order satisfies the c.c.c. condition if and only if it does not contain an uncountable collection of open disjoint sets.*

Definition 2. *A linear order is separable if and only if it possesses a countable dense subset.*

Definition 3. *A linear order is termed short if and only if it contains no uncountable well-ordered subset.*

Observation: A linear order that satisfies any combination of conditions appearing in the above definitions is neccessarily short.

Theorem 2. *The set $\{[\{a\}]_e : a \in L\}$ is dense in $W(L)/\equiv_e$.*

Proof : For $A <_W B$ where $(A, B) \neq \emptyset$ choose $a \in A$ and $b \in B$. Now pick $c \in b \setminus a$ that is bigger than every thing in a but not maximal in b. Observe that $A <_B [\{a\}]_e <_W B$.

Lemma 2. *L is c.c.c. implies $W(L)$ is c.c.c.*

Proof : Apply Theorem 2.

Lemma 3. *L is separable implies $W(L)$ is seperable.*

Proof : Apply Theorem 2.

Lemma 4. *L is short implies $W(L)$ is short.*

Proof : Apply Theorem 2.

Theorem 3. *Each of the following classes has no universal element where $\kappa^{\aleph_0} = \kappa$:*

i) c.c.c. linear orders of size κ

ii) separable linear orders of size κ

iii) c.c.c. + separable linear orders of size κ

iv) short linear orders of size κ.

Proof : Observation + Lemma 1,2,3,4,5 + Theorem 1.

Note that the following result is due to [HVa].

Theorem 4. *Given CH the class of short $\aleph_1$-trees has no universal element.*

Proof : For a given tree T the successor tree can be defined as the set of all branches of T. Observe that in the presence of CH the successor of a short $\aleph_1$-tree is also a short $\aleph_1$-tree. Now apply the argument given in Theorem 1 in the natural way.

CONCLUSION

The interior operator of Chapter 4 provides an instrument for the recovery of more detailed information on which trees accept long branches via the killing of a stationary set. One may ask if there is a stationary set S such that for all $R \leq_{NS} S$ $I_R(T)$ maintains a constant value, or is there a stationary set S such that for all $S \leq_{NS} R$, $I_R(T)$ is of constant value?

Chapter 5 has on display various classes of linear orders of power κ which have no universal element with the added proviso that $\kappa^{\aleph_0} = \kappa$. Yet nothing has been established when the proviso is removed. Also, it is known that there is no universal short $\aleph_1$-tree in the presence of CH. However, the argument turns on shallow considerations. Is it possible that there is a universal short $\aleph_1$-tree given that CH is violated?

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