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**THE COMPLEXITY OF GENERALISED COLOURINGS**

by

**Gary MacGillivray**

**B.Sc. (Honours), University of Victoria, 1985**

**M.Sc., University of Victoria, 1986**

**THESIS SUBMITTED IN PARTIAL FULFILLMENT OF  
THE REQUIREMENTS FOR THE DEGREE OF  
DOCTOR OF PHILOSOPHY**

**in the department**

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## Abstract

### The complexity of generalised colourings

by Gary MacGillivray

Let  $H$  be a fixed directed graph whose vertices are called colours. An  $H$ -colouring of a digraph  $G$  is an assignment of these colours to the vertices of  $G$  so that  $colour(g)colour(g')$  is an arc of  $H$  whenever  $gg'$  is an arc of  $G$  (that is, adjacent vertices receive adjacent colours). We study the complexity of the  $H$ -colouring problem: Given a digraph  $G$ , does there exist an  $H$ -colouring of  $G$ ?

We present some new constructions that are useful in establishing NP-completeness results for  $H$ -colouring. The majority of the thesis is devoted to using our tools, and those of other researchers, to establish complexity results when  $H$  is a member of some particular family of directed graphs.

There is, at present, no general conjecture as to exactly which  $H$ -colouring problems are NP-complete, and it appears that a complete classification may be difficult to accomplish. (There is, however, a conjecture due to Bang-Jensen and Hell, cf. below, proposing a classification for a broad, but restricted, family of digraphs.) Some order is introduced when we investigate the class of directed graphs  $H$  with the property that the presence of  $H$  as a subdigraph of a digraph  $G$  is sufficient for the  $G$ -colouring problem to be NP-complete. In this context it becomes possible to conjecture precisely which  $H$ -colouring problems are

"hereditarily hard" in the above sense. Some structural properties of hereditarily hard  $H$ -colouring problems are described, and several infinite families of problems in this class, consistent with the conjecture, are identified.

Let  $H$  be a connected digraph in which each vertex has in-degree at least one, and out-degree at least one. A conjecture of Bang-Jensen and Hell states that the  $H$ -colouring problem is NP-complete unless  $H$  is colourable by its shortest directed cycle, in which case the  $H$ -colouring problem is polynomial. We verify this conjecture for a variety of families of directed graphs including semi-complete digraphs, vertex-transitive digraphs, and others.

Finally, we describe infinite families of acyclic (resp. unicyclic) directed graphs for which the  $H$  colouring problem is polynomial, and others for which it is NP-complete.

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## 1. Introduction.

The problem of deciding whether  $k$  colours suffice to colour a given graph is one of the basic NP-complete problems [Karp, 1972]. In fact, it is believed that unless  $P=NP$ , no polynomial time algorithm can guarantee to colour an arbitrary graph with at most  $c$  times the minimum number of colours, for any constant  $c$ . This has, to date, been proved only for small values of  $c$  [Garey & Johnson, 1976 & 1979]. It has, however, been proved that any known polynomial time colouring algorithm uses  $\Omega(n(\log \log n)^2/(\log n)^2)$  colours on some  $n$ -vertex three-colourable graph [Widgerson, 1983].

Graph colouring has also been studied in a variety of other contexts. Among those of a practical nature are scheduling and storage problems [eg. Levin, 1973; Leighton, 1979], printed circuit testing [Garey et al., 1976], and combinatorial games [Nowakowski & Winkler, 1983]. Topics of theoretical interest can be found in [Albertson et. al., 1985; Häggkvist et al., 1988; etc.].  $H$ -colourings have been of interest because of their relationship to grammars and interpretations [Maurer et al., 1981]. Other generalised colourings have arisen in the study of resource allocation [Chung et al., to appear], and the four-colour-conjecture [Hell, 1974].

Let  $G$  and  $H$  be directed graphs (all graphs considered in this thesis are finite). A *homomorphism* of  $G$  to  $H$  is a function  $f:V(G) \rightarrow V(H)$  such that  $f(g)f(g')$  is an arc of  $H$  whenever  $gg'$  is an arc of  $G$ . (The existence of a homomorphism of  $G$  to  $H$  is sometimes denoted by  $G \rightarrow H$ , or when we want to emphasize that the mapping is accomplished by the function  $f$ , by  $f:G \rightarrow H$ ) The definition is similar

when  $G$  and  $H$  are undirected graphs. Since an  $n$ -colouring of a graph  $G$  is just a homomorphism  $G \rightarrow K_n$ , the term *H-colouring* of  $G$  has been employed to describe a homomorphism  $G \rightarrow H$ .

In this thesis we study the *H-colouring problem*;

**H-COL** (*H-colouring*)

INSTANCE: A directed graph  $G$ .

QUESTION: Does there exist an *H-colouring* of  $G$ ?

Each *H-colouring* problem clearly belongs to NP.

The complexity of the undirected *H-colouring* problem (i.e., the version of the problem when  $G$  and  $H$  are undirected graphs) was investigated by several authors [Maurer et al., 1981; Nešetřil, 1982; Hell & Nešetřil, 1986], and was completely determined by Hell and Nešetřil [Hell & Nešetřil, 1986], who proved that *H-COL* is NP-complete for any fixed non-bipartite graph  $H$ , and is polynomial otherwise.

Most early work on the complexity of the directed *H-colouring* problem involved polynomial time algorithms when  $H$  was a member of some fairly simple class of digraphs. For example, *H-COL* is polynomial if  $H$  is a directed path, a directed cycle, or a transitive tournament [Maurer et al., 1981]. Some NP-complete instances of the *H-colouring* problem are described in [Hell & Nešetřil, 1986; Maurer et al., 1981].

In contrast with the undirected case, no conjecture proposing a classification by complexity of directed  $H$ -colouring problems appeared among the early papers. The sequence of digraphs shown in figure 1.1, and due to E. Welzl [Welzl, 1988], may provide some insight into this situation. Observe that each digraph is obtained from its predecessor by adding a single arc, yet the complexity oscillates (assuming  $P \neq NP$ ). This is quite different from the undirected case where, once the addition of an edge creates an odd cycle, the addition of more edges never results in a bipartite graph (hence the complexity does not oscillate).

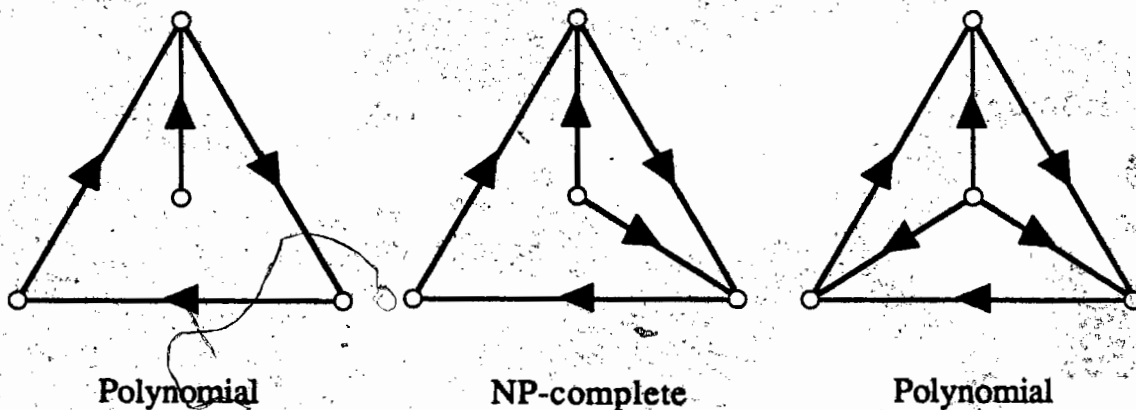


Figure 1.1. An oscillating sequence.

Before proceeding further we define a special type of homomorphism that plays an important role. If  $H$  is a subdigraph of  $H'$ , a *retraction* of  $H'$  to  $H$  is a homomorphism  $r$  of  $H'$  to  $H$  such that  $r(h) = h$  for all vertices  $h$  of  $H$ .

Other results have appeared in the literature during the preparation of this thesis [Bang-Jensen & Hell, 1988; Gutjahr, 1988; Gutjahr et al., 1988; Gutjahr,

1989]. Some of these are reviewed in the next chapter. Perhaps the most important development is the following conjecture from [Bang-Jensen & Hell, 1988].

**1.1. Conjecture.** Let  $H$  be a connected directed graph in which each vertex has in-degree at least one, and out-degree at least one. If  $H$  does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete (with respect to Turing reduction). Otherwise  $H$ -COL is polynomial.

(The last statement follows easily from Theorem 2.2.2.) This conjecture postulates a sufficient condition for NP-completeness of the  $H$ -colouring problem in a substantial portion of the set of directed graphs. The majority of this thesis is devoted to verifying Conjecture 1.1 for many large classes of digraphs.

In chapter three we describe our tools. Some of these are due to other authors, and some are new. Of particular interest is a new construction from which all of the principal constructions due to other authors can be derived as special cases.

Chapter four focusses on directed graphs  $H$  with the property that the  $G$ -colouring problem is NP-complete (sometimes with respect to Turing reduction) whenever  $H$  is a subdigraph of  $G$ . These are the "*hereditarily hard*"  $H$ -colouring problems previously mentioned. Each theorem in this chapter verifies Conjecture 1.1 for infinitely many directed graphs, and also implies NP-completeness of infinitely many  $H$ -colouring problems not covered by the conjecture.

Several large families of digraphs are classified by complexity in chapter five, including semi-complete digraphs, vertex-transitive digraphs, partitionable digraphs (a family which we introduce), and directed cycles with two chords. Classification of the vertex-transitive digraphs affirmatively answers a question due to E. Welzl (whether vertex-transitive digraphs could be classified), and classification of the last family extends a result from [Bang-Jensen & Hell, 1988]. Each major theorem in this chapter verifies Conjecture 1.1 for a family of directed graphs.

Finally, in chapter six, we direct our attention toward acyclic and unicyclic digraphs. We describe new infinite families of such digraphs for which the  $H$ -colouring problem is polynomial, and others for which it is NP-complete. In so doing we shed some light on the nature of the sequence of digraphs in figure 1.1.

## 2. Preliminaries.

This chapter presents the definitions, terminology, and preliminary results needed.

### 2.1. Definitions and Terminology.

For concepts in the theory of directed graphs we use the notation and terminology of [Bondy & Murty, 1976], subject to the additions and exceptions mentioned below. Since we assume that the reader is familiar with most basic definitions regarding directed graphs, we are briefly.

A *directed graph* (or *digraph*) is an ordered pair  $D = (V(D), E(D))$  consisting of a finite set  $V(D)$  of *vertices*; and a set  $E(D)$  of ordered pairs of (not necessarily distinct) vertices of  $D$  called *arcs*. If  $a = xy$  is an arc of  $D$ , then  $a$  is said to *join*  $x$  and  $y$ ;  $x$  is the *tail* of  $a$ ,  $y$  is its *head*. We also say that  $x$  is *adjacent to*  $y$ , and that  $y$  is *adjacent from*  $x$ . The arc  $a$  is said to be *incident* with each of the vertices  $x$  and  $y$ .

Throughout this section we let  $D$  be a directed graph.

A *loop* is an arc  $vv$  of  $D$ . If  $D$  has a loop, any directed graph is  $D$ -colourable; map all vertices to a vertex with a loop. Thus the question arises as to whether to assume our directed graphs are loopless. If  $D$  is a given digraph for which we



explore the complexity of  $D$ -COL, we assume that  $D$  has no loops because otherwise the  $D$ -colouring problem is trivially polynomial. On the other hand, if the digraph  $D$  is the result of a construction described in Chapter three, we allow loops, since in these instances the presence of a loop in  $D$  indicates that the construction has failed in a specific way, which is often a useful piece of information.

A *walk* in  $D$  is a finite sequence  $W = v_1, a_1, v_2, a_2, \dots, a_{n-1}, v_n$ , whose terms are alternately vertices and arcs, such that, for  $i = 1, 2, \dots, n-1$ , either  $a_i = v_i v_{i+1}$  or  $a_i = v_{i+1} v_i$ . We usually represent a walk by its vertex sequence alone. We sometimes treat walks as digraphs in their own right; thus we may talk about  $V(W)$ , the vertex-set of  $W$ , or  $E(W)$ , the arc-set of  $W$ , etc.. The vertex  $v_1$  is the *origin* of  $W$ ;  $v_n$  is its *terminus*. Any other vertex belonging to  $W$  is an *internal vertex* of  $W$ . We sometimes call  $W$  a  $(v_1, v_n)$ -walk. An arc of the type  $v_i v_{i+1}$  is a *forward arc* of  $W$ , while an arc of the type  $v_{i+1} v_i$  is a *backward arc* of  $W$ . The *net length* of  $W$ , denoted  $nl(W)$ , is the number of forward arcs of  $W$  minus the number of backward arcs of  $W$ .

A *path* (or *oriented path*) in  $D$  is a walk whose vertices are distinct. If such a path has origin  $u$  and terminus  $v$ , we sometimes call it a  $(u, v)$ -path. A *directed path* of length  $n$  in  $D$  (or a *directed  $n$ -path*) is a path  $P = v_0, v_1, \dots, v_n$ , such that, for  $i = 0, 1, \dots, n-1$ ,  $v_i$  is adjacent to  $v_{i+1}$ . Let  $u$  and  $v$  be vertices of  $D$ . The *distance*  $d_D(u, v)$  from  $u$  to  $v$  in  $D$  is the smallest  $k$  for which there is a directed  $(u, v)$ -path of length  $k$ . If no such directed path exists, we define  $d_D(u, v)$  to be infinite. We use  $P_n$  to denote the directed path of length  $n$ , that is, the directed graph with vertex set  $V(P_n) = \{0, 1, \dots, n\}$ , and arc set  $E(P_n) = \{(i, i+1) : i = 0, 1, \dots, n-1\}$ .

A **closed walk** in  $D$  is a walk whose origin and terminus coincide. A **cycle** (or **oriented cycle**) is a closed walk all of whose vertices, except the origin and terminus, are distinct. A **directed cycle** of length  $n$  (or **directed  $n$ -cycle**) in  $D$  is a cycle  $C = v_0, v_1, \dots, v_{n-1}, v_0$  such that, for  $i = 0, 1, \dots, n-1$ ,  $v_i$  is adjacent to  $v_{i+1}$  (where subscripts are interpreted modulo  $n$ ). Observe that a loop is a directed cycle of length one. We sometimes refer to a directed three-cycle as a directed **triangle**. We use  $C_n$  to denote the directed cycle of length  $n$ , that is, the directed graph with vertex set  $V(C_n) = \{0, 1, \dots, n-1\}$ , and arc set  $E(C_n) = \{i(i+1) : i = 0, 1, \dots, n-1\}$ , where addition is modulo  $n$ .

The **directed girth** of a directed graph is the length of its shortest directed cycle. If a digraph  $D$  has no cycles, we define the directed girth to be infinite. The **odd directed girth** of  $D$  (resp. **even directed girth** of  $D$ ) is the length of the shortest directed cycle in  $D$  with odd (resp. even) length. If  $D$  has no directed odd cycle (resp. directed even cycle), we define the odd directed girth (resp. even directed girth) to be infinite.

A **subdigraph** of  $D$  is a directed graph  $D'$  such that  $V(D) \supseteq V(D')$ , and  $E(D) \supseteq E(D')$ . We also say that  $D$  is a **superdigraph** of  $D'$ . We also say that  $D$  contains  $D'$ , and that  $D'$  is contained in  $D$ . The digraph  $D'$  is a **proper subdigraph** of  $D$  if  $D'$  is a subdigraph, but  $D'$  is not equal to  $D$ . A **spanning subdigraph** of  $D$  is a subdigraph  $D'$  of  $D$  for which  $V(D') = V(D)$ .

Let  $X$  be a subset of  $V(D)$ . The directed graph with vertex-set  $X$ , and whose arc set consists of those arcs of  $D$  with both ends in  $X$  is called the subdigraph of  $D$  induced by  $X$ , and is denoted  $D[X]$ . We use  $D - X$  to denote the induced subdigraph  $D[V(D) - X]$ . If  $X$  consists only of the vertex  $x$  we write  $D - x$  instead of  $D - \{x\}$ .

Similarly, let  $Y$  be a subset of  $E(D)$ . The directed graph with arc-set  $Y$  and whose vertex-set consists of all vertices of  $D$  incident with an arc in  $Y$  is called the subdigraph of  $D$  induced by  $Y$ , and is denoted  $D[Y]$ . We use  $D - Y$  to denote the spanning subdigraph with arc-set  $E(D) - Y$ . The directed graph obtained from  $D$  by adding a set  $Z$  of arcs is denoted by  $D + Z$ . As above, we write  $D - a$  for  $D - \{a\}$ , and  $D + a$  for  $D + \{a\}$ .

A directed graph  $D$  is *connected* if, for any two vertices  $u$  and  $v$ , there is a  $(u, v)$ -walk; otherwise  $D$  is *disconnected*. A maximal connected subdigraph of  $D$  is called a *connected component* of  $D$ . Similarly,  $D$  is *strongly connected* if, for any pair of vertices  $u$  and  $v$ , there is a directed  $(u, v)$ -path. We sometimes abbreviate strongly connected to *strong*. A maximal strong subdigraph of  $D$  is called a *strong component* of  $D$ . If  $D$  satisfies the further condition that for any vertex  $x$  of  $D$ , the directed graph  $D - x$  is strong, we say that  $D$  is *2-connected*.

A vertex that is the head of an arc with tail  $v$  is an *out-neighbour* of  $v$ ; a vertex that is the tail of an arc with head  $v$  is an *in-neighbour* of  $v$ . We use  $N^+_D(v)$  (resp.  $N^-_D(v)$ ) to denote the set of out-neighbours (resp. in-neighbours) of  $v$ . We extend this idea by defining  $N^{+k}_D(v) = \{x : \exists \text{ directed } (v, x)\text{-walk of length } k\}$ .

Similarly,  $N^k_D(v) = \{x: \exists \text{ directed } (x, v)\text{-walk of length } k\}$ . The *out-degree*  $d^+_D(v)$  of a vertex  $v$  in  $D$  is the number of arcs with tail  $v$ , that is,  $d^+_D(v) = |N^+_D(v)|$ ; the *in-degree*  $d^-_D(v)$  of  $v$  in  $D$  is the number of arcs with head  $v$ , that is,  $d^-_D(v) = |N^-_D(v)|$ . When the context is clear, we drop the subscript  $D$ ; thus we write  $d^+(v)$  instead of  $d^+_D(v)$ , and  $d^-(v)$  instead of  $d^-_D(v)$ , and so on.

A *source* of  $D$  is a vertex with in-degree zero; a *sink* is a vertex with out-degree zero. Directed graphs with no sources and no sinks play a major role in this thesis. Hence we define a *smooth digraph* to be a directed graph with no sources and no sinks.

A subset  $I$  of  $V(D)$  is an *independent set* if no two vertices in  $I$  are joined by an arc of  $D$ . We will say that  $D$  is *bipartite* if  $V(D)$  can be partitioned into two independent sets  $X$  and  $Y$ , i.e., so that every arc of  $D$  has one end in  $X$  and the other end in  $Y$ . We observe that a directed graph is bipartite if and only if it admits a homomorphism to  $C_2$ .

If  $G$  is an undirected graph, the *equivalent digraph* of  $G$  is the directed graph with vertex set  $V(G)$  and arc set  $\{xy, yx: [x, y] \in E(G)\}$ . The graph  $G$  is the *underlying graph* of its equivalent digraph. It should be clear that a graph  $G$  admits a homomorphism to a graph  $H$  if and only if the equivalent digraph of  $G$  admits a homomorphism to the equivalent digraph of  $H$ .

If both  $xy$  and  $yx$  are arcs of  $D$ , we sometimes say that  $x$  and  $y$  are joined by a *double arc* or an *undirected edge*; this situation is denoted by  $[x, y]$ . The

*undirected part of  $D$ ,  $\text{undir}(D)$* , is the subdigraph of  $D$  induced by the set of double arcs. Observe that  $\text{undir}(D)$  is the equivalent digraph of an undirected graph. We often treat  $\text{undir}(D)$  as an undirected graph; we talk about undirected paths and cycles in  $D$ , about whether  $\text{undir}(D)$  is bipartite, etc.. In our figures double arcs are drawn as undirected edges.

If  $G$  is an undirected graph, we can construct a digraph by assigning a direction or *orientation* to each edge of  $G$ . A directed graph constructed in this manner is sometimes called an *oriented graph*. A *tournament* is an oriented complete graph. A tournament is *transitive* if it contains no directed cycle. A *transitive triple* is a transitive tournament on three vertices. A *bipartite tournament* is an orientation of a complete bipartite graph.

The term *homomorphism* was defined in the introduction. At that time we also noted that homomorphisms are a generalisation of the usual graph colouring. By a *partial colouring* of  $D$  we mean a partial mapping of  $D$  to some directed graph  $H$ . Since we think of the vertices of  $H$  as colours, we can imagine that this situation represents assigning colours to some of the vertices of  $D$ . A partial colouring of  $D$  can be *extended* to an  $H$ -colouring of  $D$  if the remaining (i.e., uncoloured) vertices of  $D$  can be mapped to vertices of  $H$  so that the result is a homomorphism  $D \rightarrow H$ .

Let  $H'$  be a directed graph and let  $H$  be a subdigraph of  $H'$ . A *retraction of  $H'$  to  $H$*  was also defined in the introduction. If  $H'$  admits a retraction to  $H$ , we say that  $H$  is a *retract* of  $H'$ . A directed graph is *retract-free* (or a *core* [Hell & Nešetřil, 1986], or a *minimal graph* [Welzl, 1982]) if it does not admit a retraction

to a proper subdigraph. Every directed graph  $H$  contains a unique (up to isomorphism) subdigraph  $C$  which is retract-free, and for which there is a retraction of  $H$  to  $C$  [Welzl, 1982]. Following [Hell & Nešetřil, 1986] we call  $C$  the *core* of  $H$ . If  $H$  is a retract of  $H'$ , there are homomorphisms  $i: H \rightarrow H'$  (the *inclusion*) and  $r: H' \rightarrow H$  (a retraction); thus a given digraph is  $H'$ -colourable if and only if it is  $H$ -colourable. This allows us, when we choose, to restrict our attention to retract-free digraphs.

We now come to our final graph-theoretic definition. An *automorphism* of  $D$  is a one-to-one onto function  $f: V(D) \rightarrow V(D)$  such that  $f(x)f(y)$  is an arc of  $D$  if and only if  $xy$  is an arc of  $D$ . The set of all automorphisms of  $D$  is a group, called the *automorphism group* of  $D$ , and denoted by  $Aut(D)$ .

Let  $H$  be a retract-free digraph. We show that every homomorphism of  $H$  to itself is an onto mapping that preserves arcs, that is, an automorphism of  $H$ . Suppose not, and let  $f$  be a homomorphism of  $H$  properly into itself. Since  $H$  is finite, there exists  $n$  such that the subdigraph  $C$  which results from the composition of  $f$  with itself  $n$  times is isomorphic to the subdigraph which results from the composition of  $f$  with itself  $k$  times, for all  $k \geq n$ . By relabelling the vertices of  $C$ , we obtain a retraction of  $H$  to  $C$ , contradicting our hypothesis that  $H$  is retract-free.

Let  $H$  be a retract-free digraph. Suppose that  $H$  is a subdigraph of  $H'$ . Consider a homomorphism  $f: H' \rightarrow H$ . The restriction  $g$  of  $f$  to the copy of  $H$  in  $H'$  is a homomorphism of  $H$  to itself, i.e., an automorphism of  $H$ . It is not difficult to see that the function  $r = g^{-1} \circ f$  is a homomorphism of  $H'$  to  $H$  such that  $r(h) = h$ , for all

vertices  $h$  of  $H$ . That is,  $r$  is a retraction of  $H'$  to  $H$ . Therefore, a retract-free digraph  $H$  is a retract of a digraph  $H'$  if and only if  $H$  is both a subdigraph of  $H'$  and a homomorphic image of  $H'$ .

We now turn our attention to the concepts we need from the theory of computational complexity. We use the definitions and terminology from [Garey & Johnson, 1979] subject to one exception (Turing reduction, cf. below).

A *decision problem* is a problem which has a yes or no answer. The theory of NP-completeness is concerned only with decision problems (although the implications of the theory extend beyond this class of problems). Thus we abbreviate decision problem to *problem*. Generally speaking, a problem consists of a general question which has a yes or no answer, and a collection of parameters whose value is not specified. (In the  $H$ -colouring problem (cf. Chapter one) the parameter whose value is not specified is the directed graph  $G$ . The question is "does there exist an  $H$ -colouring of  $G$ ?") When we specify values for all of the parameters of a problem we obtain an *instance* of the problem. (We have an instance of the  $H$ -colouring problem whenever we consider a particular digraph  $G$ .)

We say that a function  $f(n)$  is  $O(g(n))$  if there are constants  $c_1$  and  $c_2$  such that  $|f(n)| \leq c_1|g(n)| + c_2$  for all  $n \geq 0$ . An algorithm is  $O(f(n))$  if the number of computational steps required by the algorithm is  $O(f(n))$ , where  $n$  is a "reasonable" measure of the problem size. (We will not attempt to define "reasonable"; instead we refer the reader to [Garey & Johnson, 1979].) If there is a polynomial  $p$  such that the number of computational steps required by the algorithm is  $O(p(n))$ , then

the algorithm is a *polynomial time algorithm*. We use  $P$  to denote the set of decision problems which are solvable in polynomial time, i.e., admit solution by a polynomial time algorithm. We sometimes say that a problem  $\Pi \in P$  is *polynomial*.

Let  $\Sigma$  and  $\Pi$  be problems. A *polynomial transformation* from  $\Sigma$  to  $\Pi$  is a function  $f$  that maps the set of instances of  $\Sigma$  to the set of instances of  $\Pi$  and satisfies two conditions: (i) a "yes" instance of  $\Sigma$  maps to a "yes" instance of  $\Pi$ , and a "no" instance of  $\Sigma$  maps to a "no" instance of  $\Pi$ , and (ii)  $f$  is computable by a polynomial time algorithm. If there exists a polynomial time transformation from  $\Sigma$  to  $\Pi$ , we say that  $\Sigma$  *polynomially transforms* to  $\Pi$  and write  $\Sigma \alpha \Pi$ . The existence of a polynomial time transformation from  $\Sigma$  to  $\Pi$  implies that if  $\Pi \in P$ , then  $\Sigma \in P$  because a composition of polynomial time algorithms is a polynomial time algorithm.

We denote by  $NP$  the set of decision problems solvable in polynomial time by a non-deterministic algorithm. It is clear that  $P$  is contained in  $NP$ . One of the fundamental questions in theoretical computer science is whether  $P = NP$ .

Let  $\Pi$  be a decision problem. We say that  $\Pi$  is *NP-complete* if  $\Pi$  belongs to  $NP$  and, for any other problem  $\Sigma \in NP$ ,  $\Sigma \alpha \Pi$ . Let  $\Psi$  be an NP-complete problem. Since every problem in  $NP$  polynomially transforms to  $\Psi$ , in order to prove that a problem  $\Pi$  is NP-complete it suffices to show  $\Psi \alpha \Pi$ . Furthermore, if some NP-complete problem belongs to  $P$ , then  $P = NP$ .



Let  $\Sigma$  and  $\Pi$  be problems. A *polynomial time Turing reduction* from  $\Sigma$  to  $\Pi$  is a function  $f$  that maps the set of instances of  $\Sigma$  to the power set of instances of  $\Pi$  and satisfies two conditions: (i) a "yes" instance of  $\Sigma$  maps to a set of instances of  $\Pi$  that contains at least one "yes" instance of  $\Pi$ , and a "no" instance of  $\Sigma$  maps to a set of "no" instances of  $\Pi$ , and (ii)  $f$  is computable by a polynomial time algorithm. If there exists a polynomial time Turing reduction from  $\Sigma$  to  $\Pi$ , we say that  $\Sigma$  *Turing reduces* to  $\Pi$  and write  $\Sigma \alpha_T \Pi$ . The existence of a polynomial time Turing reduction from  $\Sigma$  to  $\Pi$  implies that if  $\Pi \in P$ , then  $\Sigma \in P$ .

We say that a problem  $\Pi \in NP$  is  *$NP_T$ -complete* (NP-complete with respect to Turing reduction) if every problem in NP Turing reduces to it. (Note that any NP-complete problem can be considered to be  $NP_T$ -complete.) Let  $\Psi$  be an  $NP_T$ -complete problem. Since every problem in NP Turing reduces to  $\Psi$ , in order to prove that a problem  $\Pi$  is  $NP_T$ -complete it suffices to show  $\Psi \alpha_T \Pi$ . Moreover, if any  $NP_T$ -complete problem belongs to P, then  $P = NP$ .

A polynomial transformation is a one for one transformation; it takes as input one instance of a particular problem, and produces as output one instance of another problem (for which that answer to the question is the same). The requirement that only one instance of the target problem be produced is sometimes restrictive. In some of our theorems we are able to describe a one to many transformation (i.e., a Turing reduction) to the problem in question, but we are unable to describe a polynomial transformation. It should be noted, however, that all of our  $H$ -colouring problems are in NP, and all of our  $NP_T$ -complete  $H$ -colouring problems are at least as hard as any other problem in NP.

We conclude this section by describing some basic NP-complete problems that will be used in our transformations.

***k*-SATISFIABILITY** ( $k \geq 3$  fixed) (*k*-SAT) [Cook, 1971]

**INSTANCE:** A set  $U$  of boolean variables, and a collection  $C$  of (conjunctive) clauses over  $U$  such that each clause  $c \in C$  involves  $k$  variables.

**QUESTION:** Is there a satisfying truth assignment for  $C$ ?

**NOT-ALL-EQUAL *k*-SAT** ( $k \geq 3$  fixed) [Schaefer, 1978]

**INSTANCE:** A set  $U$  of variables, and a collection  $C$  of clauses over  $U$  such that each clause  $c \in C$  involves  $k$  variables.

**QUESTION:** Is there a satisfying truth assignment for  $C$  in which each clause contains at least one true literal and at least one false literal?

**Comment:** The problem remains NP-complete even if no clause contains a negated literal [Lovász, 1973]. In this case it is the problem of two-colouring a  $k$ -regular hypergraph.

**ONE-IN-*k*-SAT** ( $k \geq 3$  fixed) [Schaefer, 1978]

**INSTANCE:** A set  $U$  of variables, and a collection  $C$  of clauses over  $U$  such that each clause  $c \in C$  involves  $k$  variables.

**QUESTION:** Is there a satisfying truth assignment for  $C$  in which each clause contains exactly one true literal?

**Comment:** The problem remains NP-complete even if no clause contains a negated literal.

## 2.2. Previous work.

The launching point for this thesis is the following result of Hell and Nešetřil, which completely determined the complexity of the undirected  $H$ -colouring problem.

**2.2.1. Theorem** [Hell & Nešetřil, 1986] Let  $H$  be a fixed graph. If  $H$  contains an odd cycle then the  $H$ -colouring problem is NP-complete. Otherwise ( $H$  is bipartite) the  $H$ -colouring problem is polynomial. ■

Attention subsequently shifted to classifying directed  $H$ -colouring problems according to their complexity. The purpose of this section is to briefly survey these results.

Hell and Nešetřil [Hell & Nešetřil, 1986] remark that the indicator construction (cf. Lemma 3.8) can be used to construct NP-complete directed  $H$ -colouring problems. Moreover, the directed graph  $H$  can be made to satisfy some additional conditions; in particular, there are balanced acyclic digraphs  $H$  for which the  $H$ -colouring problem is NP-complete (a digraph is *balanced* if every oriented cycle has net length zero).

In their 1981 paper, Maurer, Sudborough and Welzl [Maurer et al., 1981] classified all three-vertex digraphs  $H$  according to the complexity of the  $H$ -colouring problem. Gutjahr has recently classified all four-vertex digraphs [Gutjahr, 1989].

Polynomial-time algorithms for several classes of directed graphs were described in [Maurer et al., 1981]: These results are summarized in the following theorem.

**2.2.2. Theorem.** [Maurer et al., 1981] The  $H$ -colouring problem is polynomial for any digraph  $H$  belonging to the following classes of directed graphs:

- (a) directed paths,  $P_n$ ,
- (b) directed cycles,  $C_n$ , and
- (c) transitive tournaments. ■

Since it suffices to consider retract-free digraphs, the  $H$ -colouring problem is polynomial if any digraph for which  $H$ -COL is polynomial is a retract of  $H$ . In particular, this is true if a directed path, directed cycle, or transitive tournament is a retract of  $H$ .

Perhaps the most significant contribution from the paper of Bang-Jensen and Hell [Bang-Jensen & Hell, 1988] is Conjecture 1.1. This paper also contained some substantial NP-completeness results concerning both sparse and dense digraphs. Conjecture 1.1 was verified for many digraphs with precisely two directed cycles, and also for bipartite tournaments (i.e., orientations of complete bipartite graphs).

By far the strongest result on the polynomial side is due to Gutjahr, Welzl, and Woeginger [Gutjahr et al., 1988]. They define an  $X$ -graph to be a directed

graph for which there is an enumeration  $v_1, v_2, \dots, v_n$  of the vertices such that if  $v_i v_j$  and  $v_k v_l$  are arcs of  $D$ , then so is  $v_{\min(i, k)} v_{\min(j, l)}$ . The main result in their paper is the following.

**2.2.3. Theorem.** [Gutjahr et al., 1988] If  $H$  is an X-graph, then the  $H$ -colouring problem is polynomial. ■

It is not difficult to see that every oriented path is an X-graph. Hence it follows from Theorem 2.2.3 that the  $H$ -colouring problem is polynomial whenever  $H$  is an oriented path. This was a long standing open problem in  $H$ -colouring.

There exist infinitely many oriented cycles which are not X-graphs. Nevertheless, Gutjahr [Gutjahr, 1989] has proved that the  $H$ -colouring problem is also polynomial whenever  $H$  is an oriented cycle.

On seeing Theorem 2.2.3, one might be inclined to conjecture that the  $H$ -colouring problem is polynomial whenever  $H$  is an oriented tree. One of the most surprising results regarding the complexity of  $H$ -colouring is this is not the case.

**2.2.4. Theorem.** [Gutjahr et al., 1988] There exists an oriented tree  $T$  for which the  $T$ -colouring problem is NP-complete. ■

The smallest such tree found to date has 288 vertices. Theorem 2.2.4 suggests that a complete classification of the complexity of directed  $H$ -colouring problems may be difficult to accomplish.

### 3. Tools.

In order to prove that a given problem  $A \in \text{NP}$  is NP-complete (resp.  $\text{NP}_T$ -complete), one must first select an NP-complete (resp.  $\text{NP}_T$ -complete) problem  $B$ , and then describe a polynomial time transformation (resp. polynomial time Turing reduction) from  $B$  to  $A$ . When  $A$  is an  $H$ -colouring problem, it is frequently the case that  $B$  is also an  $H$ -colouring problem. That is, new NP-completeness (resp.  $\text{NP}_T$ -completeness) results for  $H\text{-COL}$  can be derived from old.

The purpose of this chapter is to describe some generic transformations between  $H$ -colouring problems. The three main transformations are the indicator construction, the sub-indicator construction, and the edge sub-indicator construction [Hell & Nešetřil, 1986] (cf. Lemma 3.1.8, 3.1.9, and 3.1.10, respectively). We introduce a general construction, (cf. Lemma 3.1.11) from which all of these can be derived as special cases. In the final section of this chapter we discuss a consequence of the indicator construction.

#### 3.1. Generic Polynomial-time Reductions.

Let  $G$  and  $H$  be directed graphs. It was noted in Chapter one that if  $H$  is a retract of  $G$ , then a given digraph is  $G$ -colourable if and only if it is  $H$ -colourable. Thus  $G\text{-COL}$  and  $H\text{-COL}$  are polynomially equivalent; if one of them is polynomial (resp. NP-complete,  $\text{NP}_T$ -complete), then so is the other. We reiterate that this

allows us, when we choose, to restrict our attention to retract-free digraphs.

Our first Lemma is used implicitly throughout the entire thesis.

**3.1.1. Lemma.** [Bang-Jensen & Hell, 1988] Let  $H_1$  and  $H_2$  be disjoint directed graphs such that  $H = H_1 \cup H_2$  is retract-free. Then  $H_i\text{-COL} \alpha_T H\text{-COL}$ , for  $i=1, 2$ . ■

Hence if  $H_1\text{-COL}$  or  $H_2\text{-COL}$  is  $\text{NP}_T$ -complete then so is  $H\text{-COL}$ . On the other hand, if  $H_1\text{-COL}$  and  $H_2\text{-COL}$  are both polynomial, then there is a polynomial algorithm for  $H$ -colouring: given an input digraph  $D$ , test whether each component of  $D$  is  $H_i$ -colourable for at least one  $i$ . It therefore suffices to consider connected digraphs.

Some preparation is required before the next reduction can be described. The following lemmas concern the existence of certain directed cycles and directed walks. They are also used elsewhere in this thesis.

**3.1.2. Lemma.** [Häggkvist et al, 1987] There is a homomorphism of a directed graph  $H$  to  $C_d$  if and only if the net length of every (oriented) cycle is divisible by  $d$ . ■

Therefore a given directed graph does not admit a homomorphism to  $C_n$  just if it has a cycle of net length not divisible by  $n$ , and does not admit a homomorphism to any directed cycle of length greater than one if and only if it has

a collection  $C^1, C^2, \dots, C^k$  of cycles such that  $\gcd\{nl(C^i): i=1, 2, \dots, k\}=1$ .

**3.1.3. Lemma.** Let  $H$  be strong. There is no homomorphism of  $H$  to  $C_d$  if and only if there exists an integer  $k$  such that  $d$  does not divide  $k$ , and there is a homomorphism of  $C_k$  to  $H$ .

**Proof.**

( $\Rightarrow$ ) Suppose  $H$  does not admit a homomorphism to  $C_d$ . Let  $W$  be a closed walk in  $H$  with net length not divisible by  $d$  (the walk  $W$  exists by Lemma 3.1.2), and with the minimum number of backwards arcs among all such closed walks. If  $W$  has no backwards arcs there is nothing to prove, so we may assume that  $W$  has at least one backwards arc,  $xy$  say. Since  $H$  is strong, there is a directed  $(y, x)$ -path  $P$ . By our assumption on  $W$ , the length of the directed closed walk  $P+xy$  is a multiple of  $d$ , say  $qd$ . Let  $W' = W - xy$  (i.e., the  $(x, y)$ -section of  $W$ ). Then  $W'P$  is a closed directed walk with one fewer backwards edge than  $W$ , and

$$nl(W'P) = nl(W) + 1 + (qd-1)$$

which is not divisible by  $d$ . This contradicts the choice of  $W$ , and completes the proof of the implication.

( $\Leftarrow$ ) The image of  $C_k$  in  $H$  is a union of directed cycles. Since  $d$  does not divide  $k$ , the digraph  $H$  has a cycle of length not divisible by  $d$ . Consequently there is no homomorphism of  $H$  into  $C_d$ . ■

Lemma 3.1.3 yields a strengthening of Lemma 3.1.2 for strong digraphs.

**3.1.4. Corollary.** Let  $H$  be strong. There is a homomorphism of  $H$  into  $C_d$  if and only if the length of every directed cycle is divisible by  $d$ . ■



Therefore a given strong digraph does not admit a homomorphism to  $C_n$  just if it has a directed cycle of net length not divisible by  $n$ , and does not admit a homomorphism to any directed cycle of length greater than one if and only if it has a collection  $C^1, C^2, \dots, C^k$  of directed cycles such that  $\gcd\{nl(C^i): i=1, 2, \dots, k\}=1$ .

Let  $H$  be a smooth digraph. Then  $H$  has a directed cycle. Let  $g$  be the directed girth of  $H$ . Since no directed cycle admits a homomorphism to a larger directed cycle,  $H$  is not  $C_n$ -colourable for any  $n$  greater than  $g$ . This, together with the observation that any directed graph is  $C_1$ -colourable, allows us to talk about the largest  $d$  for which there is a homomorphism of  $H$  to  $C_d$ . (In particular, we note that a strong digraph is smooth.)

**3.1.5. Lemma.** Let  $H$  be strong, and let  $d$  be the largest integer such that there is a homomorphism  $f$  of  $H$  to  $C_d$ . For any vertex  $v$  of  $H$  there is an integer  $l_v$  (resp.  $b_v$ ) such that, for every vertex  $x$  in  $f^{-1}(f(v))$ , there is a directed  $(v, x)$ -walk of length  $l_v$  (resp. directed  $(x, v)$ -walk of length  $b_v$ ).

**Proof.**

We prove only the existence of  $l_v$ ; the existence of  $b_v$  may be established similarly. First we find an integer  $l$  such that there is a directed  $(v, y)$ -walk of length  $l$  for every vertex  $y$  in  $\{v\} \cup N^{+d}(v)$ . We then use  $l$  to define  $l_v$ .

By Corollary 3.1.4 the digraph  $H$  has a collection  $C^1, C^2, \dots, C^n$  of directed cycles such that  $\gcd\{|V(C^i)|: i=1, 2, \dots, n\}=d$ . Since  $H$  is strong, the vertex  $v$  lies on a directed cycle  $K$  of length  $kd$ , for some  $k$ . Let  $\langle d \rangle$  denote the subgroup of  $Z_{kd}$  generated by  $d$ . Then  $\langle d \rangle = \langle \{|V(C^i)|: i=1, 2, \dots, n\} \rangle$ , so there exist directed

cycles  $D^1, D^2, \dots, D^t \in \{C^1, C^2, \dots, C^n\}$  (not necessarily all distinct) such that

$$|V(D^1)| + |V(D^2)| + \dots + |V(D^t)| \equiv d \pmod{kd}.$$

For  $j=1, 2, \dots, t$ , let  $v_j$  be a vertex on  $D^j$ , let  $W^j$  be a  $(v, v_j)$ -path, and let  $X^j$  be a  $(v_j, v)$ -path. Define

$$\begin{aligned} l = & |V(D^1)| + |V(D^2)| + \dots + |V(D^t)| \\ & + |V(W^1)| + |V(W^2)| + \dots + |V(W^t)| \\ & + |V(X^1)| + |V(X^2)| + \dots + |V(X^t)|. \end{aligned}$$

It is not hard to see that  $S = W^1 D^1 X^1 W^2 D^2 X^2 \dots W^t D^t X^t$  is a directed  $(v, v)$ -walk of length  $l$ . Let  $u \in N^{+d}(v)$ . There is a directed  $(v, u)$ -walk of length  $l$ , namely  $T = W^1 X^1 W^2 X^2 \dots W^t X^t P$ , where  $P$  is the  $(v, u)$ -walk of length  $|V(D^1)| + |V(D^2)| + \dots + |V(D^t)|$  formed by traversing  $K$  repeatedly, and then using the last  $d$  arcs of  $P$  to traverse a  $(v, u)$ -path of length  $d$ .

Next we define

$$l_v = l \cdot \max\{k : \exists u \text{ such that } d(v, u) = k\}.$$

Let  $x$  be in  $f^{-1}(f(v))$ . Then  $d(v, x)$  is divisible by  $d$ . There is clearly a directed  $(v, x)$ -walk of length  $l_v$  formed by traversing  $S$   $l_v/d(v, x)$  times, traversing  $T$   $d(v, x)$  times, traversing  $K$  repeatedly, and then using the last  $d(v, x)$  arcs to traverse a directed  $(v, x)$ -path of length  $d(v, x)$ . The result follows. ■

**3.1.6. Lemma.** Let  $H$  be a strong component of a retract-free digraph  $D$ . Then  $H\text{-COL} \alpha_T D\text{-COL}$ . Furthermore, if  $H$  does not admit a homomorphism to a directed cycle of length greater than one,  $H\text{-COL} \alpha D\text{-COL}$ .

**Proof.**

Let  $d$  be the largest integer such that there is a homomorphism of  $H$  to  $C_d$ , and fix a homomorphism  $f: H \rightarrow C_d$ . Let  $G$  be a given digraph. We define a collection of digraphs,  $G_1, G_2, \dots, G_d$ , such that there is a homomorphism of  $G$  to  $H$  if and only if there is a homomorphism of some  $G_i$  to  $D$ .

There exists in  $H$  a directed path  $v_1, v_2, \dots, v_d$ . Then  $f$  assigns a different colour to each of these vertices. For  $i=1, 2, \dots, d$  let  $l_i$  and  $b_i$  be the lengths from Lemma 3.1.4 corresponding to  $v_i$ . If  $G$  admits a homomorphism to  $H$ , then there is a homomorphism of  $G$  to  $C_d$ . Since the  $C_d$ -colouring problem is polynomial, it may be assumed that a  $C_d$ -colouring  $m$  of  $G$  is known. The digraph  $G_i$  is constructed from the disjoint union of  $G$  and  $D$  by adding directed paths as follows: Let  $g$  be a vertex of  $G$ , and suppose that  $m(g)=x$ . Let  $k = i+x \pmod{d}$ . Add a directed  $(g, v_k)$ -path of length  $b_k$ , and a directed  $(v_k, g)$ -path of length  $l_k$ . The digraph  $G_i$  results from applying this construction to every vertex of  $G$ .

**CLAIM:** There is a homomorphism of  $G$  to  $H$  if and only if there is a homomorphism of some  $G_i$  to  $D$ ,

**PROOF.**

( $\Rightarrow$ ) Let  $h$  be an  $H$ -colouring of  $G$ . Then  $c = f \circ h$  is a  $C_d$ -colouring of  $G$ . Let  $g$  be a vertex of  $G$  and, without loss of generality, suppose  $m(g) = 0$ . Let  $c(g) = j$ . Consider a homomorphism of  $G_j$  to  $D$ . Since  $D$  is a retract-free, the copy of  $D$  in  $G_j$  is mapped identically onto  $D$ . Each vertex of  $G$  maps to its image in the  $H$ -colouring of  $G$  and, by Lemma 3.1.5, this partial colouring can be extended to all of the paths. Hence  $G_j \rightarrow D$ .

( $\Leftarrow$ ) Without loss of generality assume that  $G_1$  admits a homomorphism to  $D$ . Since  $D$  is retract-free, we know that  $G_1$  maps onto  $D$ . As

$H \cup G$  is contained in a strong component of  $G_1$ , it is mapped to a strong component of  $D$ . Since  $D$  is retract-free, this component is isomorphic to  $H$ . Hence there is a homomorphism of  $G$  to  $H$ .

Since the collection  $G_1, G_2, \dots, G_d$  can be constructed in polynomial time, the result follows. Furthermore, if  $H$  does not admit a homomorphism to a directed cycle of length greater than one, then  $d=1$  and the construction described above is a polynomial transformation. ■

Hence whenever we prove that  $H$ -COL is  $\text{NP}_T$ -complete for some strong digraph  $H$ , we obtain, via Lemma 3.1.6, an infinite family of  $\text{NP}_T$ -complete  $H$ -colouring problems. This is not always noted in the text.

Let  $H$  be a directed graph. The digraph  $H^\rightarrow$  (resp.  $H^\leftarrow$ ) is obtained from  $H$  by adjoining a new vertex  $x$  and adding all arcs belonging to  $\{vx: v \in V(H)\}$  (resp.  $\{xv: v \in V(H)\}$ ).

**3.1.7. Lemma.** [Gutjahr et al., 1988; Bang-Jensen et al., 1988] Both  $H^\rightarrow$ -COL  $\alpha$   $H$ -COL and  $H$ -COL  $\alpha$   $H^\rightarrow$ -COL. (Similarly, both  $H^\leftarrow$ -COL  $\alpha$   $H$ -COL and  $H$ -COL  $\alpha$   $H^\leftarrow$ -COL.)

**Proof.**

A given input digraph  $G$  admits a homomorphism to  $H$  if and only if  $G^\rightarrow$  admits a homomorphism to  $H^\rightarrow$ . Therefore  $H$ -COL  $\alpha$   $H^\rightarrow$ -COL.

On the other hand, suppose that an instance of  $H^\rightarrow$ -COL (i.e., a digraph  $D$ )

is given. Let  $D^-$  be the digraph obtained by deleting all sources of  $D$ . We claim that  $D$  admits a homomorphism to  $H^\rightarrow$  if and only if  $D^-$  admits a homomorphism to  $H$ . To extend an  $H$ -colouring of  $D^-$  to an  $H^\rightarrow$ -colouring of  $D$  map all sources of  $D$  to the vertex which was adjoined to  $H$  to get  $H^\rightarrow$  (no two such vertices are adjacent and every vertex of  $H$  is an admissible image for any of their neighbours). Conversely, in any  $H^\rightarrow$ -colouring of  $D$ , only sources can map to the vertex in  $H^\rightarrow - H$ . Hence  $D^- \rightarrow H$ . This completes the proof. ■

Thus the  $H$ -colouring problem and the  $H^\rightarrow$ -colouring problem are polynomially equivalent: if either is polynomial (or NP-complete, or NP $_T$ -complete), then so is the other. Therefore sources (resp. sinks) whose neighbourhood spans the set of all other vertices may be discarded.

Let  $I$  be a fixed digraph, and let  $u$  and  $v$  be distinct vertices of  $I$ . The *indicator construction with respect to  $(I, u, v)$*  transforms a given digraph  $H$  into the digraph  $H^*$ , defined to have the same vertex set as  $H$ , and to have as the arc set all pairs  $hh'$  for which there is a homomorphism of  $I$  to  $H$  taking  $u$  to  $h$  and  $v$  to  $h'$ . The triple  $(I, u, v)$  is called an *indicator*, and if the digraph  $H^*$  is loopless (i.e., if no homomorphism of  $I$  to  $H$  can map  $u$  and  $v$  to the same vertex), it is called a *good indicator*. If some automorphism of  $I$  maps  $u$  to  $v$  and  $v$  to  $u$ , we say that the indicator  $(I, u, v)$  is *symmetric*. (The result of the indicator construction with respect to a symmetric indicator is the equivalent digraph of an undirected graph, and can be defined to be an undirected graph [Hell & Nešetřil, 1986].) Other, more specialized, indicators are defined later.

**3.1.8. Lemma.** [Hell & Nešetřil, 1986].  $H^*$ -COL  $\alpha$  H-COL.

In applying Lemma 3.1.8 care must be taken to assure that  $H^*$  has no loops, i.e., that  $(H, u, v)$  is a good indicator. If  $H^*$  has a loop, then there is a polynomial time algorithm for  $H^*$ -colouring; map all vertices of  $G$  to a vertex with a loop.

Let  $J$  be a fixed digraph with specified vertices  $x$  and  $j_1, j_2, \dots, j_t$ . The *sub-indicator construction with respect to  $(J, x, j_1, j_2, \dots, j_t)$ , and  $h_1, h_2, \dots, h_t$*  transforms a given retract-free digraph  $H$  with specified vertices  $h_1, h_2, \dots, h_t$  to its subdigraph  $H^-$  induced by the vertex set  $V^-$  defined as follows. Let  $W$  be the digraph obtained from the disjoint union of  $J$  and  $H$  by identifying  $j_i$  and  $h_i$ ,  $i=1, 2, \dots, t$ . A vertex  $v$  of  $H$  belongs to  $V^-$  just if there is a retraction of  $W$  to  $H$  which maps  $x$  to  $v$ . The structure  $(J, x, j_1, j_2, \dots, j_t)$  is called a *sub-indicator*. The digraph  $J$  is not required to be connected. If the vertices  $j_1, j_2, \dots, j_t$  are all isolated, the outcome of the sub-indicator construction is independent of the choice of  $h_1, h_2, \dots, h_t$ . In this case we call  $(J, x, j_1, j_2, \dots, j_t)$  a *free sub-indicator* and, in order to reflect the independence of the specified vertices, refer to it as the sub-indicator construction with respect to  $(J, x, \text{free})$ .

**3.1.9. Lemma.** [Hell & Nešetřil, 1986].  $H^-$ -COL  $\alpha$  H-COL.

Similarly, let  $J$  be a fixed digraph with a specified arc  $xy$  and specified vertices  $j_1, j_2, \dots, j_t$ . The *edge\* sub-indicator construction with respect to  $(J, xy, j_1, j_2, \dots, j_t)$ , and  $h_1, h_2, \dots, h_t$*  transforms a given retract-free digraph  $H$  with specified vertices  $h_1, h_2, \dots, h_t$  into its subdigraph  $H^\wedge$  induced by the arcs of

$H$  which are images of the arc  $xy$  under retractions of  $W$  (as defined above) to  $H$ . The structure  $(J, xy, j_1, j_2, \dots, j_t)$  is called an *edge sub-indicator*. A *free edge-sub-indicator* is defined and denoted similarly to the above.

3.1.10. Lemma. [Hell & Nešetřil, 1986]  $H^{\wedge}\text{-COL} \alpha H\text{-COL}$ .

We now describe a general construction which includes the indicator construction, the sub-indicator construction, and the edge-sub-indicator construction as special cases. Let  $G$  and  $J$  be fixed digraphs with specified vertices  $g_1, g_2, \dots, g_t$ , and  $u, v, j_1, j_2, \dots, j_t$ , respectively. The *HSI construction with respect to  $(G, g_1, g_2, \dots, g_t)$  and  $(J, u, v, j_1, j_2, \dots, j_t)$*  transforms a given directed graph  $H$  into the directed graph  $H'$ , defined as follows. Let  $\text{hom}(G, H)$  denote the set of homomorphisms of  $G$  to  $H$ . The vertex set  $V(H')$  consists of  $|\text{hom}(G, H)|$  copies of  $V(H)$ . Let  $f_i$  be a homomorphism of  $G$  to  $H$ , and let  $W_i$  be the digraph constructed from  $H \cup J$  by identifying  $j_k$  and  $f_i(g_k)$ ,  $k=1, 2, \dots, t$ . There is an arc from  $x$  to  $y$  in the copy of  $V(H)$  (in  $V(H')$ ) corresponding to  $f_i$  just if there is a retraction of  $W_i$  to  $H$  which maps  $u$  to  $x$  and  $v$  to  $y$ . (Perhaps it is best to think of the construction as having three phases; a Homomorphism phase wherein  $G$  is mapped to  $H$ , a Sub-indicator phase wherein  $W_i$  is constructed and then retracted to  $H$ , and an Indicator phase which defines the arcs of  $H'$ .)

### 3.1.11. Lemma. $H'$ -COL $\alpha$ H-COL.

#### Proof.

Suppose an instance of  $H'$ -COL, i.e., a digraph  $D$ , is given. Without loss of generality,  $D$  is connected (otherwise apply the construction below to each component). Construct a digraph  ${}^1D$  from  $V(D)$ ,  $G$ , and  $|E(D)|$  copies of  $J$ , say  $J_1, J_2, \dots, J_{|E(D)|}$ , as follows. For  $i=1, 2, \dots, t$  identify  $g_i$  and all  $|E(D)|$  copies of  $j_i$ . If  $xy$  is the  $k^{\text{th}}$  arc of  $D$ , then identify the vertices  $u$  and  $v$  in the  $k^{\text{th}}$  copy of  $J$  with the vertices  $x$  and  $y$  in the copy of  $V(D)$ , respectively. Clearly the construction may be carried out in polynomial time.

We claim that  ${}^1D \rightarrow H$  if and only if  $D \rightarrow H'$ .

Suppose  $f: {}^1D \rightarrow H$ . The restriction of  $f$  to the copy of  $G$  in  ${}^1D$  is a homomorphism of  $G$  to  $H$ , say  $f_i$ . We show that the restriction of  $f$  to  $V(D)$  yields a homomorphism  $D \rightarrow H'$ . Let  $dd'$  be an arc of  $D$ . We must show that there is a retraction of  $W_i$  (as defined above) to  $H$  in which  $d$  maps to  $f(d)$  and  $d'$  maps to  $f(d')$ . This is equivalent to finding a homomorphism  $g$  of  $I$  to  $H$  such that  $g(j_k) = f(j_k)$ ,  $k = 1, 2, \dots, t$ ,  $g(d) = f(d)$  and  $g(d') = f(d')$ . The restriction of  $f$  to the copy of  $J$  corresponding to  $dd'$  in  ${}^1D$  is such a mapping, as  $u$  was identified with  $d$  and  $v$  was identified with  $d'$ . Therefore there is a homomorphism  $D \rightarrow H'$ .

Suppose  $f: D \rightarrow H'$ . Since  $D$  is connected, it maps to a connected component of  $H'$ , and hence to the subdigraph of  $H'$  induced by some copy of  $V(H)$ , say  $F$ , the one corresponding to  $f_i$ . We must construct a homomorphism  $g: {}^1D \rightarrow H$ . If  $hh'$  is the  $j^{\text{th}}$  arc of the component of  $H'$  induced by  $F$ , there is a retraction  $r_j$  of  $W_i$  to  $H$  taking  $u$  to  $h$  and  $v$  to  $h'$ .



The function  $g$  is defined as follows,

$$\begin{aligned} g(z) &= f_i(g), & g &\in V(G), \\ g(d) &= f(d), & d &\in V(D), \text{ and} \\ g(w) &= r_j(w), & w &\in V(J_j) - (V(G) \cup V(D)), \quad j=1, 2, \dots, |E(D)|. \end{aligned}$$

Then  $g: V(D) \rightarrow V(H)$ . Since  $f$ ,  $f_i$ , and  $r_j$  ( $j=1, 2, \dots, |E(D)|$ ) are homomorphisms, and each pair agrees on the intersection of their domains, the function  $g$  is a homomorphism. The result follows. ■

**Derivation of the indicator construction.** Let  $H$  be a fixed digraph, and let  $(I, u, v)$  be an indicator. Let  $H^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$ , and let  $H'$  be the result of applying the HSI construction with respect to  $(K_1, x)$  and  $(I \cup K_1, u, v, y)$  to  $H$ , where  $(\{x\}, \emptyset)$  and  $(\{y\}, \emptyset)$  are copies of  $K_1$ . Since  $u$  and  $v$  are not in the same component as  $y$ , and since there is at least one homomorphism of  $K_1$  into  $H$ , the Homomorphism and Sub-indicator parts of the construction are effectively eliminated. It is not hard to see that  $H'$  consists of  $|V(H)|$  copies of  $H^*$ , so  $H^*$  is a retract of  $H'$ .

**Derivation of the sub-indicator construction.** Suppose that  $H$  is a retract-free digraph, and let  $(J, x, j_1, j_2, \dots, j_t)$  be a sub-indicator. Let  $J'$  be the digraph constructed from two copies of  $J$  by identifying the corresponding vertices  $j_i$ ,  $i=1, 2, \dots, t$ . Let  $u, v$  be the two copies of the vertex  $x$ , and add the arc  $uv$  to  $J'$ . Let  $H^*$  be the result of applying the sub-indicator construction with respect to  $(J, x, j_1, j_2, \dots, j_t)$  and  $h_1, h_2, \dots, h_t$  to  $H$ , and let  $H'$  be the result of applying the HSI construction with respect to  $(H, h_1, h_2, \dots, h_t)$  and  $(J', u, v, j_1, j_2, \dots, j_t)$  to  $H$ . Since  $H$  is retract-free, every homomorphism of  $H$  to itself is an automorphism. It

is now not difficult to see that  $H'$  consists of  $|Aut(H)|$  disjoint copies of  $H^-$ , so  $H^-$  is a retract of  $H'$ .

**Derivation of the edge-sub-indicator construction.** Suppose that  $H$  is a retract-free digraph, and let  $(J, xy, j_1, j_2, \dots, j_l)$  be an edge sub-indicator. Let  $H^\wedge$  be the result of applying the edge sub-indicator construction with respect to  $(J, xy, j_1, j_2, \dots, j_l)$  and  $h_1, h_2, \dots, h_t$  to  $H$ , and let  $H'$  be the result of applying the HSI construction with respect to  $(H, h_1, h_2, \dots, h_t)$  and  $(J, x, y, j_1, j_2, \dots, j_l)$  to  $H$ . As above,  $H'$  consists of  $|Aut(H)|$  disjoint copies of  $H^\wedge$ , so  $H^\wedge$  is a retract of  $H'$ .

### 3.2. On the Outcome of the Indicator Construction.

In this short section we prove a useful lemma that gives information about the digraph  $H^*$  that results from the indicator construction. As a consequence we are able to show that Conjecture 1.1 is equivalent to a special case of itself.

**3.2.1. Lemma.** Let  $H$  be a connected smooth digraph. Let  $d$  be the largest positive integer such that  $H$  is  $C_d$ -colourable. Suppose  $C_d$  is not a retract of  $H$ . Then the result  $H^*$  of applying the indicator construction with respect to  $(P_d, 0, d)$  to  $H$  is a smooth digraph with exactly  $d$  connected components, none of which admits a homomorphism to a directed cycle of length greater than one. Moreover, if  $H$  is strong then so is each component of  $H^*$ .

**Proof.**

If  $d=1$ ,  $H=H^*$ . Hence assume  $d>1$ .

Since  $C_d$  is retract-free, it is a retract of  $H$  if and only if it is both a subdigraph of  $H$  and a homomorphic image of  $H$ . By hypothesis,  $C_d$  is an image of  $H$ , but it is not a retract. Hence  $H$  has no directed  $d$ -cycle. Therefore  $H^*$  is loopless.

Fix a  $C_d$ -colouring  $f$  of  $H$ . Let  $[j]$  denote the set of all vertices of  $H$  which are mapped by  $f$  to vertex  $j$  of  $C_d$ . Any two adjacent vertices of  $H^*$  receive the same colour under  $f$  because they must be joined in  $H$  by a directed path of length  $d$ . Therefore  $H^*$  has at least  $d$  connected components.

We now prove that  $H^*$  has precisely  $d$  connected components,  $H^0, H^1, \dots, H^{d-1}$ , where  $H^j$  is the subdigraph of  $H^*$  induced by  $[j]$ . Let  $u, w$  be distinct vertices in  $[j]$ . Since  $H$  is connected, there exists a  $(u, w)$ -path  $P$ . Let  $v$  be the first vertex in  $P$  which is different from  $u$ , and belongs to  $[j]$ . Let  $Q$  be the  $(u, v)$ -section of  $P$ . It suffices to show that  $H^*$  contains a  $(u, v)$ -walk.

Let us call an intermediate vertex of  $Q$  a *source of  $Q$*  (resp. *sink of  $Q$* ) if it is the tail (resp. head) of two consecutive arcs of  $Q$ . We also call  $u$  a source of  $Q$  (resp. sink of  $Q$ ) if it is the tail (resp. head) of the first arc of  $Q$ . Similarly,  $v$  is called a sink of  $Q$  (resp. source of  $Q$ ) if it is the head (resp. tail) of the last arc of  $Q$ . Let  $s_0, s_1, \dots, s_k$  be the list of sources and sinks of  $Q$  in the order they are encountered when traversing  $Q$  (thus  $u=s_0$  and  $v=s_k$ ). For  $i = 0, 1, \dots, k-1$ , the  $(s_i, s_{i+1})$ -section of  $Q$  is a directed path. Let  $l_i$  be the length of the  $(s_i, s_{i+1})$ -section of  $Q$ , and let  $l_0 = 0$ . Our choice of  $v$  implies that each of these directed paths has length less than  $d$ , and furthermore, that  $s_i \in [j]$ ,  $i = 1, 2, \dots, k-1$ . For  $i=1, 2, \dots, k-1$ , define the vertex  $t_i$  as follows. If the  $(s_{i-1}, s_i)$ -section of  $Q$  is a directed  $(s_{i-1}, s_i)$ -path of length  $l_i$ , let  $t_i$  be any vertex for which there exists a directed  $(s_i, t_i)$ -path of length  $d - l_i - l_{i+1}$ , and if the  $(s_{i-1}, s_i)$ -section of  $Q$  is a

directed  $(s_i, s_{i-1})$ -path of length  $l_i$ , let  $t_i$  be any vertex for which there exists a directed  $(t_i, s_i)$ -path of length  $d - l_i - l_{i+1}$ . The vertex  $t_i$  always exists since  $H$  is smooth, and that always  $t_i \in [j]$ .

It is not difficult to see from the definitions that, for  $i = 1, 2, \dots, k-1$ ,  $t_i$  and  $t_{i+1}$  are joined in  $H$  by a directed path of length  $d$ . Therefore they are adjacent in  $H^*$ . It is also clear that the arcs  $ut_1$  and  $t_{k-1}v$  exist (since  $u = s_0$  and  $v = s_k$ ). Thus there exists a  $(u, v)$ -walk in  $H^*$ . It follows that  $[j]$  induces a connected component of  $H^*$ . Moreover, if  $H$  is strong, then the path  $P$  can be chosen to be a directed path. Hence each component of  $H^*$  is also strong.

We now show that no component of  $H^*$  admits a homomorphism to a directed cycle of length greater than one. Assume that  $u$  is a source of  $Q$ , the argument being similar if  $u$  is a sink of  $Q$ . By our choice of  $v$  (cf. above), the path  $Q$  has net length zero or  $d$ . The vertices  $s_0, s_1, \dots, s_k$  are alternately sources and sinks of  $Q$  or sinks and sources of  $Q$ . Let  $T = u, t_1, t_2, \dots, t_{k-1}, v$  be the derived walk in  $H^*$ . By definition of  $H^*$ , the  $(u, t_i)$ -section of  $T$  has net length zero when  $i$  is even, and net length one when  $i$  is odd, for  $i = 1, 2, \dots, k-1$ .

Suppose first that  $nl(Q) = 0$ . Then among  $s_0, s_1, \dots, s_k$  there is one more source of  $Q$  than sink of  $Q$ . This implies that  $k$  must be even. Hence  $k-1$  is odd and  $nl(T) = 0$ .

Now suppose that  $nl(Q) = d$ . Then among  $s_0, s_1, \dots, s_k$  there are an equal number of sources and sinks of  $Q$ . This implies that  $k$  is odd. Hence  $k-1$  is even and  $nl(T) = 1$ .

Therefore every walk  $W$  in  $H$  whose origin and terminus belong to  $[j]$  gives rise to a walk in  $H^j$  with net length  $(1/d) nl(W)$ .

By Lemma 3.1.2,  $H$  contains a collection of closed walks,  $W_1, W_2, \dots, W_n$ ,

such that  $\gcd\{nl(W_i) : i=1, 2, \dots, n\} = d$ . By the above argument, each of these gives rise to a closed walk  $W_{ij}$  in  $H^j$  such that  $nl(W_{ij}) = (1/d) nl(W_i)$ . Therefore, for any  $j$ ,  $\gcd\{W_{ij} : i=1, 2, \dots, n\}=1$ , so  $H^j$  does not admit a homomorphism to a directed cycle of length greater than one.

Finally, since  $H$  is smooth, every vertex is the origin of a directed path of length  $d$  and the terminus of a directed path of length  $d$ . Hence each  $H^j$  is also smooth. This completes the proof. ■

**3.2.2. Corollary.** It suffices to prove Conjecture 1.1 for digraphs that admit a homomorphism to no directed cycle of length greater than one.

**Proof.**

Suppose Conjecture 1.1 is true for all connected smooth digraphs that do not admit a homomorphism to a directed cycle of length greater than one. Let  $H$  be a connected smooth digraph, and let  $d$  be the largest positive integer such that there is a homomorphism of  $H$  to  $C_d$ . Let  $H^*$  be the result of applying the indicator construction with respect to  $(P_d, 0, d)$  to  $H$ . By Lemma 3.2.1 the digraph  $H^*$  has exactly  $d$  connected components, none of which admit a homomorphism to a directed cycle of length greater than one. Let  $K$  be a connected component of the core of  $H^*$ . Then  $K$  is smooth and does not admit a homomorphism to a directed cycle of length greater than one. By hypothesis,  $K$ -COL is  $NP_T$ -complete. Therefore the  $H$ -colouring problem is  $NP_T$ -complete by Lemmas 3.1.8, 3.2.1, and 3.1.1. This completes the proof. ■

In the next chapter, we see how the above lemma extends the implications of Conjecture 1.1 to digraphs that are not smooth.

#### 4. Hereditarily Hard $H$ -colouring Problems.

Let  $C$  be an undirected odd cycle. Then the  $C$ -colouring problem is NP-complete [Maurer et al., 1981; Hell & Nešetřil, 1986]. Moreover, the theorem of Hell and Nešetřil asserts that the  $G$ -colouring problem is NP-complete for any graph  $G$  containing  $C$  (cf. Theorem 2.2.1). In a sense, we can say that the NP-completeness of  $C$ -COL is hereditary, in that it is inherited by any supergraph of  $C$ . Upon careful consideration of the proof of the above theorem, one can observe the following. The first step in the proof is equivalent to showing that it suffices to prove the result for graphs that contain a three-cycle. The remainder of the proof is equivalent to showing that the NP-completeness of  $K_3$ -COL is hereditary.

If  $P$  is not equal to  $NP$ , there are many directed graphs  $H$  for which the NP-completeness of  $H$ -COL is not hereditary in the above sense. For example, if  $H$  is the digraph constructed from  $C_4 \cup C_6$  by identifying a vertex on each directed cycle, then  $H$ -COL is NP-complete [Bang-Jensen & Hell, 1988; Gutjahr et al., 1989]. On the other hand,  $C_2$  is a retract of the digraph  $(H \cup C_2)$ , hence  $(H \cup C_2)$ -COL is polynomial.

There is, at present, no general conjecture regarding precisely which  $H$ -colouring problems are  $NP_T$ -complete. (On the other hand, we show in this chapter how Conjecture 1.1, which proposes a classification of smooth digraphs, implies a sufficient condition for NP-completeness of  $H$ -COL for many directed graphs  $H$  which are not smooth.) The fact that there are trees  $T$  for which  $T$ -COL

is NP-complete suggests that a complete classification may be difficult to accomplish. The concept of hereditarily hard  $H$ -colouring problems introduces some order into this situation. We present a conjecture as to exactly which  $H$ -colouring problems are hereditarily hard (cf. Conjecture 4.1.5'), and prove that it is equivalent to Conjecture 1.1. This provides some hope that a classification of hereditarily hard  $H$ -colouring problems may be easier to accomplish than a classification by complexity of all  $H$ -colouring problems. We identify structural properties of these "superhard" digraphs, and identify infinite families of them.

#### 4.1. The Definition and Some Properties.

Motivated by the undirected case, we say that an  $H$ -colouring problem is *hereditarily hard* if  $G$ -COL is  $\text{NP}_T$ -complete whenever  $H$  is a subdigraph of a loopless digraph  $G$ . (That is, if the  $H$ -colouring problem is so hard that the presence of  $H$  as a subdigraph of  $G$  is sufficient for  $G$ -COL to be  $\text{NP}_T$ -complete.) We call such a digraph  $H$  *superhard*.

Our first proposition describes an infinite family of superhard digraphs. Other families of superhard digraphs are described in section 4.5.

**4.1.1. Proposition.** Let  $H$  be the equivalent digraph of an undirected odd cycle. Then  $H$  is superhard.

**Proof.**

Suppose that  $H$  is a subdigraph of  $G$ . Let  $G^*$  be the undirected graph that results from applying the indicator construction with respect to  $(C_2, 0, 1)$  to  $G$ . Since  $G^*$  contains an odd cycle,  $G^*$ -COL is NP-complete. (According to our definitions,  $G^*$  is actually the equivalent digraph of an undirected graph. This is not a problem, since we have already observed (cf. section 2.1) that a graph  $F$  admits a homomorphism to the underlying simple graph corresponding to  $G^*$  if and only if the equivalent digraph of  $F$  admits a homomorphism to  $G^*$ .) The result now follows from Lemma 3.1.7. ■

Therefore the equivalent digraph of any non-bipartite undirected graph is superhard.

The set of directed graphs is partially ordered with respect to inclusion, that is,  $G <_i H$  if  $G$  is a subdigraph of  $H$ . The set of superhard digraphs is, by definition, an upper order ideal (or filter) with respect to this order.

Homomorphisms yield another partial order on the set of directed graphs, that is,  $G <_h H$  if there is a homomorphism  $H \rightarrow G$ . It follows from the next result that the set of superhard graphs is an ideal with respect to this order.

**4.1.2. Proposition.** These are equivalent:

(1) The  $G$ -colouring problem is  $NP_T$ -complete whenever  $G$  contains  $H$  (i.e.,  $H$  is superhard).

(2) The  $G$ -colouring problem is  $NP_T$ -complete whenever  $H$  admits a homomorphism to  $G$ .



**Proof.**

(1)  $\Rightarrow$  (2) Assume (1), and suppose that there is a homomorphism of  $H$  to  $G$ . Consider  $G^+ = G \cup H$ . Then  $H$  is a subdigraph of  $G^+$ , so the  $G^+$ -colouring problem is  $\text{NP}_T$ -complete. Since  $G$  is a retract of  $G^+$ , the  $G$ -colouring problem is also  $\text{NP}_T$ -complete.

(2)  $\Rightarrow$  (1) The inclusion map  $i:H \rightarrow G$  is a homomorphism. ■

We now introduce a concept which is complementary to superhardness. A *polynomial extension* of a digraph  $H$  is a loopless superdigraph  $G$  of  $H$  for which  $G$ -COL is polynomial. If  $P \neq \text{NP}$ , a directed graph  $H$  is superhard if and only if it has no polynomial extension.

Suppose that  $D$ -COL is polynomial. If there is a homomorphism of a digraph  $H$  to  $D$ , then the digraph  $D$  is a retract of  $(H \cup D)$ . Therefore the  $(H \cup D)$ -colouring problem is polynomial. Thus  $(H \cup D)$  is a polynomial extension of  $H$ . Hence any digraph that admits a homomorphism to a directed cycle of length greater than one has a polynomial extension. The complementary statement, that if  $H$  does not admit a homomorphism to a directed cycle of length greater than one then  $H$  is superhard, turns out to be equivalent to Conjecture 1.1 (cf. Theorem 4.1.6).

In the next two lemmas we explore the structure of minimal superhard digraphs.

**4.1.3. Proposition.** Let  $H$  be a digraph with connected components,  $C^1, C^2, \dots, C^n$ . Then  $H$  is superhard if and only if  $C^i$  is superhard, for some  $i \in \{1, 2, \dots, n\}$ .

**Proof.**

( $\Rightarrow$ ) If  $P=NP$ , every digraph is superhard, so there is nothing to prove.

Otherwise, assume that  $P$  is not equal to  $NP$ , and suppose  $H$  is superhard but there is no  $i \in \{1, 2, \dots, n\}$  such that  $C^i$  is superhard. Thus, for  $i=1, 2, \dots, n$ ,  $C^i$  has a polynomial extension  $X^i$ . But then  $H' = H \cup X^1 \cup X^2 \cup \dots \cup X^n$  is a polynomial extension of  $H$  (because  $H' = \cup_{1 \leq i \leq n} (C^i \cup X^i)$  and, for  $i=1, 2, \dots, n$ , the  $(C^i \cup X^i)$ -colouring problem is polynomial), a contradiction.

( $\Leftarrow$ ) Obvious. ■

**4.1.4. Proposition.** Let  $v$  be a source (sink) of  $H$ . Then  $H$  is superhard if and only if  $H-v$  is superhard.

**Proof.**

( $\Rightarrow$ ) If  $P=NP$ , there is nothing to prove. Assume that  $P \neq NP$ , and suppose that  $H-v$  is not superhard. Thus it admits a polynomial extension  $G$ . But, since  $H$  is a subdigraph of  $G \rightarrow$  and  $G \rightarrow$ -COL is polynomial by Lemma 3.1.7,  $H$  also has a polynomial extension. This contradiction proves the implication.

( $\Leftarrow$ ) Obvious. ■

Suppose  $P \neq NP$ , and consider a minimal (with respect to inclusion) superhard digraph  $H$ . By Proposition 4.1.3 the digraph  $H$  is connected and, by Proposition 4.1.4, it is smooth. Hence no acyclic or unicyclic digraph is superhard, and therefore every superhard digraph has at least two directed cycles. It can be

noted in [Maurer et al. 1981; Gutjahr 1988; Bang-Jensen and Hell 1988; Bang-Jensen et al. 1988; Gutjahr et al, 1989] that the presence of two directed cycles in a digraph  $H$  is often sufficient for  $H$ -COL to be NP-complete. Further results of this type are proved in Chapter five. Also note that all of the digraphs covered by Conjecture 1.1 have at least two directed cycles.

Every smooth digraph which we know to have a polynomial extension admits a homomorphism to a directed cycle of length greater than one. Included in this class are smooth digraphs  $H$  such that  $H$ -COL is  $NP_T$ -complete. Furthermore, we know of no smooth digraph which does not admit a homomorphism to a directed cycle of length greater than one and which has a polynomial extension. We make the following conjecture.

**4.1.5. Conjecture.** Let  $H$  be a connected smooth digraph. If  $H$  does not admit a homomorphism to a directed cycle of length greater than one, then  $H$  is superhard (the  $H$ -colouring problem is hereditarily hard). Otherwise  $H$  has a polynomial extension.

Conjecture 4.1.5 can also be formulated in terms of digraphs which may have sources or sinks. Let  $H$  be a digraph. Let  $R(H)$ , the *reduction of  $H$* , be the result of applying the sub-indicator construction with respect to  $(P_{2|V(H)|}, |V(H)|, free)$  to  $H$  (a similar use of the sub-indicator construction appears in [Bang-Jensen, 1989]). By its definition,  $R(H)$  is unique. Furthermore,  $R(H)$  is smooth. (The digraph  $R(H)$  may also be derived from  $H$  by iteratively deleting all sources and sinks, until a smooth digraph remains.)

By Proposition 4.1.4,  $H$  is superhard if and only if  $R(H)$  is superhard. (It should be clear that, by Lemma 3.1.9, if  $R(H)$ -COL is NP-complete (resp. NP $_T$ -complete), then so is  $H$ -COL. Since there are acyclic digraphs  $H$  for which  $H$ -COL is NP-complete (cf. Chapter six), and the reduction of an acyclic digraph is an empty digraph, the converse of the previous statement is false.) We have the following equivalent statement of Conjecture 4.1.5.

**4.1.5'. Conjecture.** Let  $H$  be a connected digraph. If  $R(H)$  does not admit a homomorphism to a directed cycle of length greater than one, then  $H$  is superhard (the  $H$ -colouring problem is hereditarily hard). Otherwise  $H$  has a polynomial extension.

The reduction of  $H$  can be used to expand the implications of Conjecture 1.1 to digraphs which are not smooth. Since  $R(H)$  is obtained from  $H$  via the sub-indicator construction, Lemma 3.1.9 asserts that if  $R(H)$ -COL is NP $_T$ -complete, then so is  $H$ -COL. Hence we have an extension of the NP-completeness part of Conjecture 1.1.

**1.1'. Conjecture.** Let  $H$  be a connected digraph. If the core of  $R(H)$  is not a directed cycle, then  $H$ -COL is NP-complete.

The principal difference between conjectures 4.1.5' and 1.1' is that the former proposes a complete classification of all  $H$ -colouring problems, while the latter proposes only a sufficient condition for NP-completeness of some  $H$ -colouring

problems. As was mentioned before, the conjectures on which these two conjectures are based are equivalent, which we now prove.

**4.1.6. Theorem.** Conjecture 1.1 and Conjecture 4.1.5 are equivalent.

**Proof.**

(4.1.5)  $\Rightarrow$  (1.1) Assume Conjecture 4.1.5 is true, and let  $H$  satisfy the hypotheses of Conjecture 1.1. If  $H$  does not admit a homomorphism to a directed cycle of length greater than one, there is nothing to prove. Hence assume that  $H$  admits a homomorphism to such a directed cycle. Let  $d$  be the largest positive integer such that there is a homomorphism of  $H$  to  $C_d$  (see the comment preceding Lemma 3.1.5 regarding the existence of  $d$ ). The digraph  $H$  has no directed  $d$ -cycle, otherwise  $C_d$  would be a retract of  $H$ . Let  $H^*$  be the result of applying the indicator construction with respect to  $(P_d, 0, d)$  to  $H$ . Let  $H^0$  be a connected component of  $H^*$ . By Lemma 3.2.1 the digraph  $H^0$  is smooth, and does not admit a homomorphism to a directed cycle of length greater than one. Hence it satisfies the hypotheses of Conjecture 4.1.5. Thus  $H^0$ -COL problem is  $\text{NP}_{\mathcal{T}}$ -complete, and therefore so is  $H$ -COL.

(1.1)  $\Rightarrow$  (4.1.5) Assume that Conjecture 1.1 is true, and let  $H'$  satisfy the hypotheses of Conjecture 4.1.5. Let  $G'$  be a digraph that contains  $H'$ , and let  $G$  be the core of  $G'$ . It is not hard to see that  $G$  contains a homomorphic image  $H$  of  $H'$ . Consider  $R(G)$ . Since  $H$  is smooth,  $R(G)$  contains  $H$ . Hence  $R(G)$  does not admit a retraction to a directed cycle. Moreover, since  $R(G)$  is smooth, it satisfies the hypotheses of Conjecture 1.1. Since Conjecture 1.1 is true,  $R(G)$ -COL is  $\text{NP}_{\mathcal{T}}$ -complete. Therefore the  $G$ -colouring problem is also  $\text{NP}_{\mathcal{T}}$ -complete. This completes the proof. ■

The implications of Conjecture 1.1 extend beyond the class of smooth digraphs. Taken together, Theorem 4.1.6 and Corollary 3.2.2 assert that these more general results may be obtained by proving that the  $H$ -colouring problem is  $\text{NP}_T$ -complete for each smooth digraph which does not admit a homomorphism to a directed cycle, instead of proving that the  $G$ -colouring problem is  $\text{NP}_T$ -complete whenever  $G$  contains such a digraph.

#### 4.2. An Extension of Superhardness.

Let  $H$  be a digraph that admits a homomorphism to a directed cycle of length  $n$ . Then  $H$  has a polynomial extension, namely  $H \cup C_n$ . Thus  $H$ -COL is not hereditarily hard unless  $P = \text{NP}$ . In this section we introduce a generalisation of superhardness that enables us to establish complexity theorems for superdigraphs of  $H$  that are similar to superhardness.

Our strategy is to impose enough restrictions on the superdigraphs of  $H$  to be considered so that the presence of  $H$  as a subdigraph of a digraph  $G$ , in this restricted family of digraphs, is sufficient for  $G$ -COL to be  $\text{NP}_T$ -complete. For example, let  $H$  be the digraph constructed from the equivalent digraph of  $K_3$  by subdividing every arc. Suppose  $G$  is a superdigraph of  $H$  that contains no directed two-cycle. Then  $(C_4, 0, 2)$  is a good indicator. The result  $G^*$  of applying the indicator construction with respect to  $(C_4, 0, 2)$  to  $G$  contains an undirected three-

cycle. By Proposition 4.1.1, the  $G^*$ -colouring problem is NP-complete. If we restrict our attention to digraphs with no directed two-cycle, the  $G$ -colouring problem is NP-complete for any superdigraph  $G$  of  $H$ . (More examples are given in section 4.5.)

Motivated by the above discussion, we make the following definition. Let  $\mathcal{P}$  be a set of properties. A digraph  $H$  is *superhard with respect to  $\mathcal{P}$*  if the following two conditions are satisfied: (i) the  $G$ -colouring problem is  $\text{NP}_T$ -complete whenever  $H$  is a subdigraph of a loopless digraph  $G$  which has all properties in  $\mathcal{P}$ , and (ii) at least one such  $G$  exists. (A digraph  $H$  is superhard if and only if it is superhard with respect to the empty collection of properties.) If  $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ , we sometimes abuse our notation and say that  $H$  is superhard with respect to  $P_1, P_2, \dots, P_n$ .

It follows from the definition that if  $H$  is superhard,  $H\text{-COL}$  is  $\text{NP}_T$ -complete. This is not the case if  $H$  is superhard with respect to an arbitrary collection of properties. For example, it is proved in section 5.4 that any oriented odd cycle is superhard with respect to the property " $G$  is partitionable". On the other hand, Gutjahr has recently been proved that  $C\text{-COL}$  is polynomial for any oriented cycle  $C$  [Gutjahr, 1989].

The concept of a polynomial extension can also be generalised. Let  $\mathcal{P}$  be a set of properties. A digraph  $G$  is called a *polynomial extension of  $H$  with respect to  $\mathcal{P}$*  if  $G$  has all properties in  $\mathcal{P}$ , the digraph  $H$  is a subdigraph of  $G$ , and the  $G$ -colouring problem is polynomial. It is clear that if  $P$  is not equal to  $\text{NP}$ , a digraph

is superhard with respect to  $\mathcal{P}$  if and only if it has no polynomial extension with respect to  $\mathcal{P}$ .

Most of the results in section 4.3 hold in this more general setting, although not necessarily for arbitrary property sets (some statements may not make sense with respect to property sets which forbid some of the hypotheses). More specifically, Propositions 4.1.2 and 4.1.3 hold for any property set  $\mathcal{P}$  such that  $G \cup H$  has all properties in  $\mathcal{P}$  whenever  $G$  and  $H$  both have all properties in  $\mathcal{P}$ . Proposition 4.1.4 is true for the property " $G$  has no closed directed walk of length  $k$ ", and others. In all instances, the modifications needed to the proofs are minor, and the reader should have little difficulty adding the missing details.

### 4.3. Some Families of Superhard Digraphs.

The purpose of this section is to give some examples of the digraphs discussed in sections 4.1 and 4.2. Although the focus is on superhard digraphs, we also give some examples of digraphs which are superhard with respect to the property  $L_k$ : " $G$  has no closed directed walk of length  $k$ ".

Let  $\mathcal{P}$  be a set of properties. An *sh-indicator with respect to  $\mathcal{P}$*  is an indicator  $(I, u, v)$  such that every loopless digraph  $G$  that contains a homomorphic image of  $I$  in which  $u$  and  $v$  are identified either lacks a property in  $\mathcal{P}$ , or is superhard with respect to  $\mathcal{P}$ . (That is, if  $G$  has all properties in  $\mathcal{P}$  and  $G^*$  has a



loop, then  $G$  is superhard with respect to  $\mathcal{S}$ .) An *sh-indicator* is an indicator  $(I, u, v)$  such that every loopless digraph  $G$  that contains a homomorphic image of  $I$  in which  $u$  and  $v$  are identified is superhard.

The importance of *sh*-indicators is illustrated in the following lemma.

**4.3.1. Lemma.** Let  $(I, u, v)$  be an *sh*-indicator with respect to  $\mathcal{S}$ . Let  $H^*$  be the digraph that results from applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . If  $H^*$  is superhard, then  $H$  is superhard with respect to  $\mathcal{S}$ .

**Proof.**

Let  $G$  be a digraph which has all of the properties in  $\mathcal{S}$ , and suppose  $H$  is a subdigraph of  $G$ . Let  $G^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $G$ . There are two possibilities, depending on whether  $G^*$  contains a loop. If  $G^*$  contains a loop, then  $G$  must contain a subdigraph which is a homomorphic image of  $I$  such that  $u$  and  $v$  map to the same vertex, since  $(I, u, v)$  is an *sh*-indicator. Thus  $G$  is superhard with respect to  $\mathcal{S}$ , and so  $G$ -COL is  $\text{NP}_T$ -complete. Otherwise,  $G^*$  is a loopless digraph that contains the superhard digraph  $H^*$ , so the  $G^*$ -colouring problem is  $\text{NP}_T$ -complete. Consequently  $G$ -COL is also  $\text{NP}_T$ -complete. This completes the proof. ■

Lemma 4.3.1 can be used to construct new superhard digraphs from old. For example, let  $H$  be the undirected three-cycle with  $V(H) = \{0, 1, 2\}$  and  $E(H) = \{\{0, 1\}, \{1, 2\}, \{2, 0\}\}$ . Let 3 be a new vertex, and set  $I = (H - 01) + 03$  (see figure 4.3.1). Then any homomorphic image of  $I$  in which the vertices 1 and 3 are identified is also an image of an undirected three-cycle. Hence, by Proposition

4.1.2,  $(I, 1, 3)$  is an sh-indicator. Let  $G = H$ , and let  $G'$  be the digraph constructed by replacing each arc  $xy$  of  $G$  by a copy of  $I$ , and identifying 1 with  $x$  and 3 with  $y$ . The result of applying the indicator construction with respect to  $(I, 1, 3)$  to  $G'$  is  $G$  (an undirected three-cycle). Hence  $G'$  is superhard.

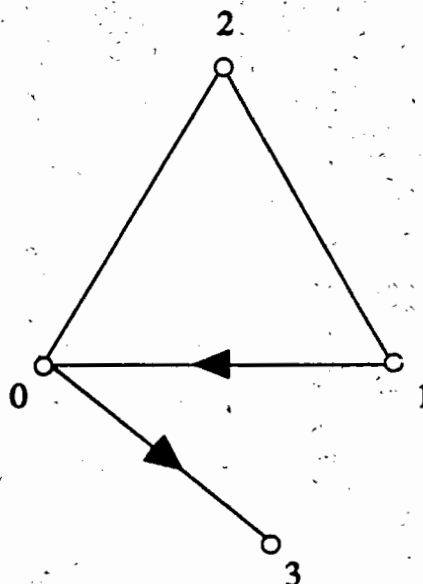


Figure 4.3.1. An example sh-indicator.

The general procedure is as follows. Suppose  $H$  is superhard, and let  $wu$  be an arc of  $H$ . Let  $v$  be a new vertex, and set  $I = (H - wu) + wv$ . Any homomorphic image of  $I$  in which  $u$  and  $v$  are identified is also an image of  $H$ . Thus, by Proposition 4.1.2,  $(I, u, v)$  is an sh-indicator. Now, let  $G$  be superhard, and let  $G'$  be the digraph obtained by replacing each arc  $xy$  of  $G$  by a copy of  $I$ , and identifying  $u$  with  $x$ , and  $v$  with  $y$ . The result of applying the indicator construction with respect to  $(I, u, v)$  to  $G'$  contains  $G$ . Hence  $G'$  is superhard.

It may also be possible to use Lemma 4.3.1 to construct digraphs which are superhard with respect to a given property set  $\mathcal{S}$ . The procedure is analogous to the above. We use Lemma 4.3.1 to construct a digraph  $H$  which is superhard with respect to the property  $L_2$ : " $G$  has no closed directed walk of length two". Since any loopless homomorphic image of  $C_4$  in which vertices 0 and 2 are identified necessarily contains a two-cycle,  $(C_4, 0, 2)$  is an sh-indicator with respect to  $L_2$ . Let  $H^*$  be the undirected 3-cycle, and let  $H$  be the digraph obtained by replacing each arc  $xy$  of  $H^*$  by a copy of  $C_4$ , identifying 0 with  $x$  and 2 with  $y$  (see figure 4.3.2). Since  $H$  has property  $L_k$ , there exists a superdigraph of  $H$  with the appropriate property. It is easy to verify that the result of applying the indicator construction with respect to  $(C_4, 0, 2)$  to  $H$  contains an undirected 3-cycle, which is superhard. Thus  $H$  is superhard with respect to  $L_2$ .

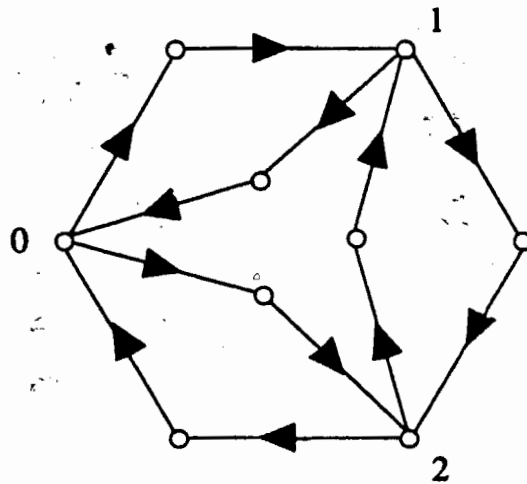


Figure 4.3.2. The digraph  $H^*$ .

The general procedure is as follows. Let  $(I, u, v)$  be a sh-indicator with respect to  $\mathcal{S}$ , and let  $H^*$  be a superhard digraph. Let  $H$  be the digraph which results from replacing every arc  $xy$  of  $H^*$  by a copy of  $(I, u, v)$ , and identifying the pairs of vertices  $u, x$  and  $v, y$ . It is not difficult to see that the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$  contains  $H^*$ . It is, however, not clear that there is a superdigraph of  $H$  with all properties in  $\mathcal{S}$ . Suppose such a digraph  $G$  exists. Let  $G^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $G$ . Since  $(I, u, v)$  is an sh-indicator,  $G^*$  is loopless. Furthermore,  $G^*$  contains  $H^*$ . Therefore  $H$  is superhard with respect to  $\mathcal{S}$ .

We now describe several infinite families of superhard digraphs. Each such directed graph gives rise to a collection of infinite families of superhard digraphs (constructed as above, via Lemma 4.3.1), and also to the infinite family of superhard digraphs that contain it.

Let  $n$  be an integer greater than or equal to three. The digraph  $W_n$ , the *wheel with  $n$  spokes*, is defined to be the digraph constructed from  $C_n \cup \{v\}$  by adding the undirected edges  $\{[v, c] : c \text{ in } V(C_n)\}$ . The digraph  $W_4$  is shown in figure 4.3.3.

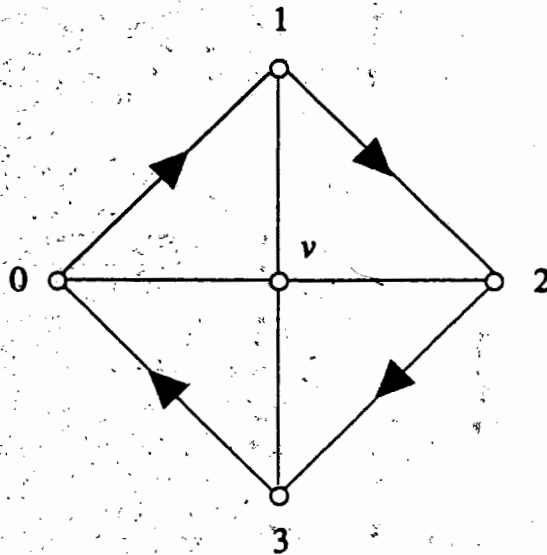


Figure 4.3.3. The digraph  $W_4$ .

**4.3.2. Theorem.** If  $n$  is not divisible by four, then  $W_n$  is superhard.

**Proof.**

Let  $(I, u, v)$  be the symmetric indicator shown in figure 4.3.4 with  $i=0$ . The digraph that results from identifying  $u$  and  $v$  is an undirected three cycle. Thus any loopless homomorphic image of  $(I, u, v)$  in which  $u$  and  $v$  are identified is also an image of an undirected three-cycle, and is therefore superhard. Since there is an automorphism of  $I$  that exchanges  $u$  and  $v$ ,  $(I, u, v)$  is a symmetric sh-indicator. The result  $W_n^*$  of applying the indicator construction with respect to  $(I, u, v)$  to  $W_n$  is the undirected graph with edge-set  $\{xy : y-x \pmod n = 2\}$ . If  $n$  is odd,  $W_n^*$  is an undirected  $n$ -cycle, and if  $n \equiv 2 \pmod 4$  it is the union of two undirected  $(n/2)$ -cycles. Since undirected odd cycles are superhard (cf. Proposition 4.1.1), the result follows from Lemma 4.3.1. ■

We believe that  $4k$ -wheels are also superhard.

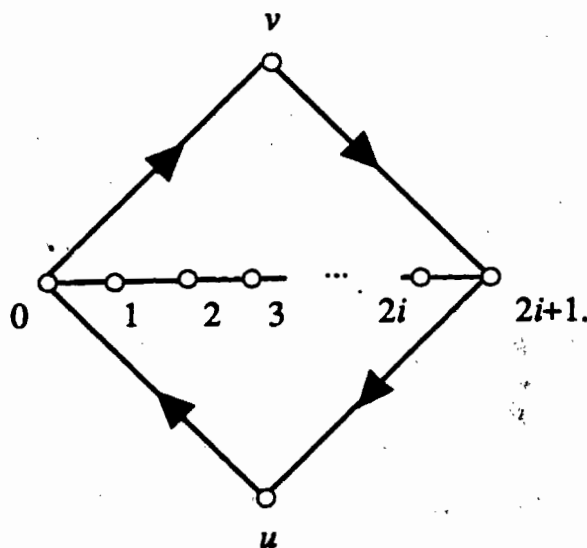


Figure 4.3.4. A useful symmetric indicator.

Let  $i$  be an integer greater than or equal to one. The digraph  $X_i$  is constructed from the equivalent digraph of an undirected cycle with vertex-set  $\{0, 1, \dots, 4i+1\}$  by adding the arcs  $0(2i), (2i)(4i), (4i)(6i), \dots, (2i+2)0$ , where computations are modulo  $4i+2$ . The digraph  $X_2$  is shown in figure 4.3.5.

**4.3.3. Theorem.** The digraph  $X_i$  is superhard.

**Proof.**

The argument is similar to Theorem 4.3.2. Let  $(I, u, v)$  be the sh-indicator shown in figure 4.3.4 (the digraph that results from identifying  $u$  and  $v$  is an undirected  $(2i+1)$ -cycle). Let  $X_i^*$  be the digraph which results from applying the indicator construction with respect to  $(I, u, v)$  to  $X_i$ . It is not hard to check that  $X_i^*$  contains the undirected  $(2i+1)$ -cycle  $0, 2i, \dots, 2i+2, 0$ . Therefore  $X_i^*$  is superhard, and the result follows from Lemma 4.3.1. ■

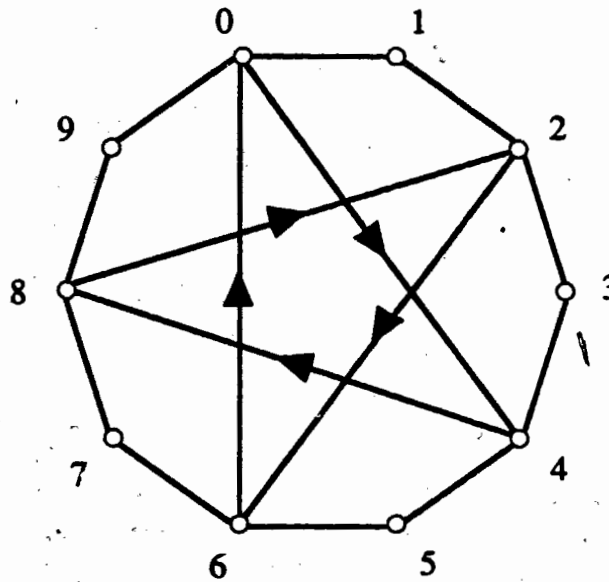


Figure 4.3.5. The digraph  $X_2$ .

Similarly, let  $T_j$  be a digraph constructed from  $C_{4j+2}$  by adding the arcs  $(0,2j)$ ,  $(2j,4j)$ , ...,  $(2j+2,0)$  and undirected odd paths between vertices 1 and  $2j+2$ , 3 and  $2j+4$ , ...,  $4j+1$  and  $2j$ . A prototype of  $T_1$  is shown in figure 4.3.6.

**4.3.4. Theorem.** Any digraph  $T_j$  is superhard.

**Proof.**

The proof is similar to the proofs of the previous two theorems. Let the longest of the undirected odd paths have length  $2i+1$ , and let  $(I, u, v)$  be the sh-indicator shown in figure 4.3.4. The result  $T_j^*$  of applying the indicator construction with respect to  $(I, u, v)$  to  $T_j$  contains the undirected odd cycle  $0, 2j, 4j, \dots, 2j+2, 0$ , which is superhard. ■

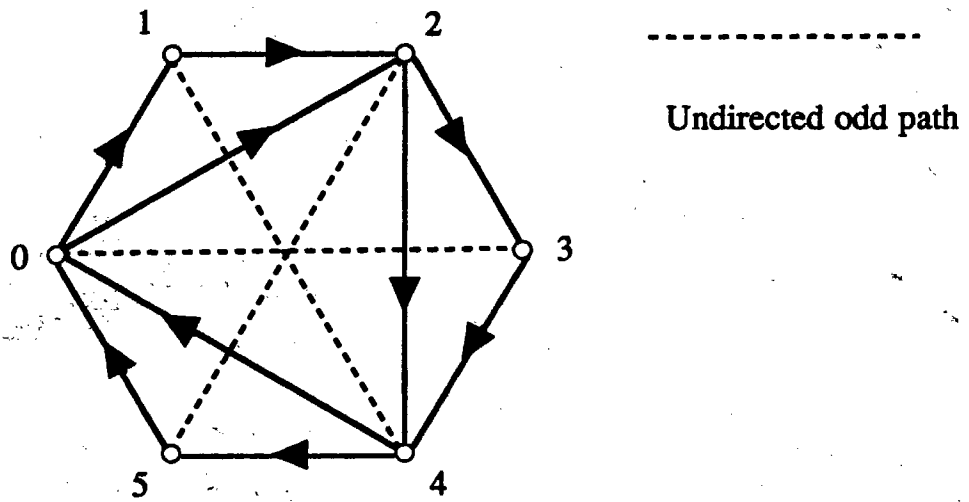


Figure 4.3.6. A prototype of  $T_1$ .

The last examples in this section concern digraphs which are superhard with respect to the property  $L_k$ : " $G$  has no closed directed walk of length  $k$ ". The set of digraphs with property  $L_2$  is precisely the set of orientations of simple graphs.  $\wedge$

Let  $G$  be a digraph. An arc  $uv$  of  $G$  is said to be *bypassed* if there is a vertex  $w$  such that the arcs  $uw$  and  $wv$  exist. Let  $i$  be an integer greater than or equal to one. Let  $B_{2i+1}$  be a digraph constructed from  $C_{2i+1}$  by adding a bypass to at least one out of every  $i$  consecutive arcs (of the  $(2i+1)$ -cycle).

**4.3.5. Theorem.** Any digraph  $B_{2i+1}$  is superhard with respect to  $L_{i+1}$ .

**Proof.**

Any homomorphic image of  $C_{2i+2}$  in which vertices 0 and  $i+1$  are identified contains a closed directed walk of length  $i+1$ . Hence  $(C_{2i+2}, 0, i+1)$  is a



symmetric sh-indicator with respect to  $L_{i+1}$ . Let  $B^*$  be the result of applying the indicator construction with respect to  $(C_{2i+2}, 0, i+1)$  to  $B_{2i+1}$ . Since, for any  $x$ , the directed  $(x, x+i+1)$ -path, along  $C_{2i+2}$  contains at least one bypassed arc, the undirected edge  $[x+i+1, x]$  is present in  $B^*$ . Thus  $B^*$  contains the undirected  $(2i+1)$ -cycle  $0, i, 2i, \dots, i+1, 0$ . Since undirected odd cycles are superhard, the result follows from Lemma 4.1.3. ■

Our final result of this section provides a method to construct digraphs that are superhard with respect to property sets other than the ones considered so far. In Lemma 4.3.1 we showed that if the result of the indicator construction (with respect to a sh-indicator) is superhard, then  $H$  is superhard with respect to a given property set. In Lemma 4.3.6 below, we show that if the result of the indicator construction is a superhard digraph, then we can find a property set  $\mathcal{P}$  such that  $H$  is superhard with respect to  $\mathcal{P}$ .

**4.3.6. Lemma.** Let  $(I, u, v)$  be an indicator. Let  $H^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . If  $H^*$  is superhard, then  $H$  is superhard with respect to the property  $P$ : " $G$  contains no homomorphic image of  $I$  in which  $u$  and  $v$  are identified" (or any other set of properties which define the same class of digraphs).

**Proof.**

Since  $H^*$  has no loops,  $H$  has property  $P$ . Furthermore,  $H$ -COL is  $\text{NP}_T$ -complete. Let  $G$  be a superdigraph of  $H$ . Let  $G^*$  be the result of applying the indicator construction to  $G$ . Then two possibilities arise; either  $G^*$  has a loop, in

which case  $G$  contains a homomorphic image of  $I$  in which  $u$  and  $v$  are identified, or  $G^*$  is superhard. That is, the  $G$ -colouring problem is  $\text{NP}_T$ -complete whenever  $H$  is a subdigraph of  $G$  and  $G$  has property  $P$ . Therefore  $H$  is superhard with respect to  $P$ . This completes the proof. ■

We conclude this section with an example of the construction from Lemma 4.3.6. Let  $I$  be the four-vertex oriented path with arc set  $ux, xy, vy$ , and let  $H$  be the digraph constructed from an undirected three cycle by replacing each arc  $xy$  with a copy of  $I$  and identifying  $u$  with  $x$ , and  $v$  with  $y$ . The result  $H^*$  of applying the indicator construction with respect to  $(I, u, v)$  to  $H$  is an undirected three-cycle, which is superhard. Thus  $H$  is superhard with respect to " $G$  contains no homomorphic image of  $I$  in which  $u$  and  $v$  are identified". This property is evidently equivalent to " $G$  has no transitive triple". Thus the  $G$ -colouring problem is NP-complete whenever the loopless directed graph  $G$  contains  $H$  and has no transitive triple.

#### 4.4. A Basis for the Superhard Digraphs.

The purpose of this section is to describe a family  $\mathcal{S}$  of directed graphs with the following property: a digraph  $D$  satisfies the conditions of Conjecture 4.1.5 if and only if some member of  $\mathcal{S}$  admits a homomorphism to  $D$ . By Theorem 4.1.2 it would suffice to prove Conjecture 4.1.5 for the digraphs in  $\mathcal{S}$ . Therefore the minimal elements of  $\mathcal{S}$  with respect to the homomorphism order can be viewed as

a basis for the set of superhard digraphs or, equivalently, as the minimal hereditarily hard  $H$ -colouring problems.

Let  $D$  be a digraph, and let  $C$  be a directed cycle in  $D$ . A vertex of  $C$  is a *vertex of attachment* if it is adjacent with some vertex in  $D-C$ . If every strong component of  $D$  is a vertex or a directed cycle, and every directed cycle has exactly one vertex of attachment, then  $D$  is called *singly attached*. The set  $\mathcal{S}$  consists of all singly attached smooth digraphs which do not admit a homomorphism to a directed cycle of length greater than one. By definition, each member of  $\mathcal{S}$  satisfies the hypotheses of Conjecture 4.1.5.

A *maximal strong component* (resp. *minimal strong component*) of a digraph  $G$  is a strong component  $C$  of  $G$  such that there exists no arc  $dc$  (resp.  $cd$ ), where  $c$  is in  $C$  and  $d$  is in  $G-C$ . Every maximal strong component of a smooth digraph  $G$  contains a directed cycle, as does every minimal strong component of  $G$ .

**4.4.1. Theorem.** Suppose  $D$  satisfies the conditions of Conjecture 4.1.5. Then there is a digraph  $H$  in  $\mathcal{S}$  which admits a homomorphism  $H \rightarrow D$ .

**Proof.**

By Lemma 3.1.2 the digraph  $D$  has a collection  $W^1, W^2, \dots, W^t$  of closed walks such that  $\gcd\{nl(W^i) : i=1, 2, \dots, t\}=1$ . Since  $D$  is connected, it has a spanning closed walk  $W^0$ . For  $i=0, 2, \dots, t$ , let  $L^i$  be an oriented  $|V(W^i)|$ -cycle such that there is a homomorphism  $f_i$  of  $L^i$  onto  $W^i$ . Then  $\gcd\{nl(L^i) : i=0, 2, \dots, t\}=1$ . Let  $M^1, M^2, \dots, M^r$  (resp.  $N^1, N^2, \dots, N^s$ ) be a collection of directed cycles, one from each maximal (resp. minimal) strong

component of  $D$ . For  $i=1, 2, \dots, r$ , let  $m_i$  be a vertex on  $M^i$  and, similarly, for  $j=1, 2, \dots, s$ , let  $n_j$  be a vertex on  $N^j$ . The digraph  $H$  is constructed from  $M^1, M^2, \dots, M^r, N^1, N^2, \dots, N^s, L^0, L^1, \dots, L^t$  by adding directed paths as follows: For  $i = 0, 1, \dots, t$ , let  $v$  be a source of  $L^i$ , or a sink of  $L^i$ . For  $k=1, 2, \dots, r$ , if there is a directed  $(f_i(v), m_k)$ -path of length  $l$  in  $D$ , then add a path of length  $l$  from  $m_k$  to  $v$  in  $H$  (all added paths are disjoint, and add  $l-2$  new vertices to  $H$ ). Similarly, for  $j=1, 2, \dots, s$ , if there is a directed  $(n_j, f_i(v))$ -path of length  $l$  in  $D$ , then add a directed path of length  $l$  from  $v$  to  $n_j$  in  $H$ . No new directed cycles are created.

Then  $H$  is in  $\mathcal{S}$  by construction. Moreover, there is a homomorphism of  $H$  onto  $D$  (every vertex of  $H$  corresponds, in a natural way, with a vertex of  $D$ , and if two vertices are adjacent in  $H$ , the vertices to which they correspond are adjacent in  $D$ ). This completes the proof. ■

The utility of Theorem 4.4.1 in settling Conjecture 4.1.5 is debatable, but the existence of the set  $\mathcal{S}$  is interesting.

## 5. Digraphs with Two Directed Cycles.

In this chapter we prove that Conjecture 1.1 is true for several large classes of digraphs. In section 5.1 we verify the conjecture for digraphs that have a spanning tournament. Tournaments were, in fact, the first large class of digraphs for which the complexity of the  $H$ -colouring problem was completely determined [Hell & MacGillivray, 1987]. The results in this section preceded Conjecture 1.1, and were the initial evidence that led to its formulation. Section 5.2 contains a classification by complexity of vertex-transitive digraphs, and of arc-transitive digraphs. We also derive necessary and sufficient conditions for a Cayley digraph to admit a homomorphism to a directed cycle. As a corollary, necessary and sufficient conditions are obtained for the core of a Cayley digraph to be a directed cycle. In section 5.3 we investigate the complexity of the  $H$ -colouring problem when  $undir(H)$  is bipartite. (By Lemma 4.1.1, if  $undir(H)$  is not bipartite, then  $H$ -COL is NP-complete.) We introduce the class of "partitionable digraphs", and completely classify them by complexity. Finally, in section 5.4, we generalise a result from [Maurer et al., 1981], and another from [Bang-Jensen & Hell, 1988].

### 5.1. Semi-complete Digraphs.

A *semi-complete digraph* is a directed graph such that for all vertices  $x$  and  $y$ , at least one of the arcs  $xy$  and  $yx$  exists. In other words, a semi-complete digraph is a digraph with a spanning tournament.

In this section we classify all semi-complete digraphs  $H$  according to the complexity of the  $H$ -colouring problem. In particular, we prove the following theorem.

**5.1.1. Theorem.** Let  $H$  be a semi-complete digraph. If  $H$  is acyclic or unicyclic, then  $H$ -COL is polynomial. Otherwise ( $H$  has at least two directed cycles)  $H$ -COL is NP-complete.

Conjecture 1.1 for semi-complete digraphs follows from Theorem 5.1.1. Although there are semi-complete digraphs with sources or sinks, Lemma 3.1.7 implies that these vertices need not be considered. Hence it suffices to prove the theorem for semi-complete digraphs that satisfy the hypotheses of the conjecture.

We first prove the first statement of Theorem 5.1.1.

**5.1.2. Lemma.** [Maurer, et al., 1981] If  $H$  is a transitive tournament (i.e., an acyclic semi-complete digraph), then  $H$ -COL is polynomial. ■

Maurer, Sudborough and Welzl proved Lemma 5.1.2 by describing a polynomial time algorithm for  $H$ -colouring. It also follows from Lemma 3.1.7, as does the remainder of the first statement of Theorem 5.1.1.

Our proofs will use some well known facts about tournaments. In particular, a tournament is strong if and only if it is hamiltonian, and if a

tournament has a directed cycle of length  $l \geq 3$ , then, it has a directed cycle of length  $k$ , for  $k = 3, 4, \dots, l$ . For more details the reader should consult [Moon, 1968; Bondy and Murty, 1976]. Since a semi-complete digraph has a spanning tournament the above conclusions are also valid for semi-complete digraphs.

**5.1.3. Lemma.** If  $H$  is a semi-complete digraph with a unique directed cycle, then  $H$ -COL is polynomial.

**Proof.**

Since  $H$  has a spanning tournament, we know that if  $H$  has directed cycle of length  $k \geq 3$  then it has directed cycles of all lengths  $l$ ,  $3 \leq l \leq k$ . Furthermore, each non-trivial strong component of  $H$  has a directed Hamilton cycle. It follows that the unique directed cycle in  $H$  has length two or three, and that all other strong components of  $H$  are trivial. That is,  $H$  may be obtained from the directed cycle by adding a sequence of sources and sinks. The result now follows from Lemma 3.1.7. ■

Taken together, Lemma 5.1.2 and 5.1.3 prove the first part of Theorem

5.1.1. The proof of the second part requires some preliminary lemmas.

**5.1.4. Lemma.** [Maurer et al, 1981] If  $H$  is a semi-complete digraph on three vertices with at least two directed cycles, then  $H$ -COL is NP-complete. ■

The next lemma follows from Gutjahr's classification by complexity of all four vertex digraphs [Gutjahr, 1988].

**5.1.5. Lemma.** If  $H$  is a semi-complete digraph on four vertices with at least two directed cycles, then  $H$ -COL is NP-complete. ■

Our next result is of interest in its own right, as it does not deal exclusively with semi-complete digraphs. Let  $D(k, l)$  denote the digraph constructed from  $C_k \cup C_l$  by adding the arcs  $\{xy: x \in V(C_k), y \in V(C_l)\}$ .

**5.1.6. Theorem.** If  $k, l \geq 2$ , then  $D(k, l)$ -COL is NP-complete. Otherwise  $D(k, l)$ -COL is polynomial.

**Proof.**

We first prove the second statement. If  $k = 1$  or  $l = 1$ , then  $D(k, l)$  has a loop; otherwise  $k = l = 0$  and  $D(k, l)$  is just a single arc. In both instances  $D(k, l)$ -COL is polynomial.

We now prove the first statement. Call  $C_k$  the *upper directed cycle* and  $C_l$  the *lower directed cycle*.

Let  $X$  and  $Y$  be the digraphs shown in figure 5.1.1 (a) and (b), respectively.

**CLAIM.** In any  $D(k, l)$ -colouring of  $X$ , exactly one of  $\{u, v\}$  is coloured on the upper directed cycle, and the other is coloured on the lower directed cycle. Moreover, any assignment of colours to  $u$  and  $v$  that satisfies the former condition can be extended to a  $D(k, l)$ -colouring of  $X$ .

**PROOF.**

Suppose to the contrary that both vertices are coloured on the upper (resp. lower) directed cycle. Then the entire outer (resp. inner) oriented cycle of  $X$  (cf. figure 5.1.1) must be coloured by the upper (resp. lower) directed

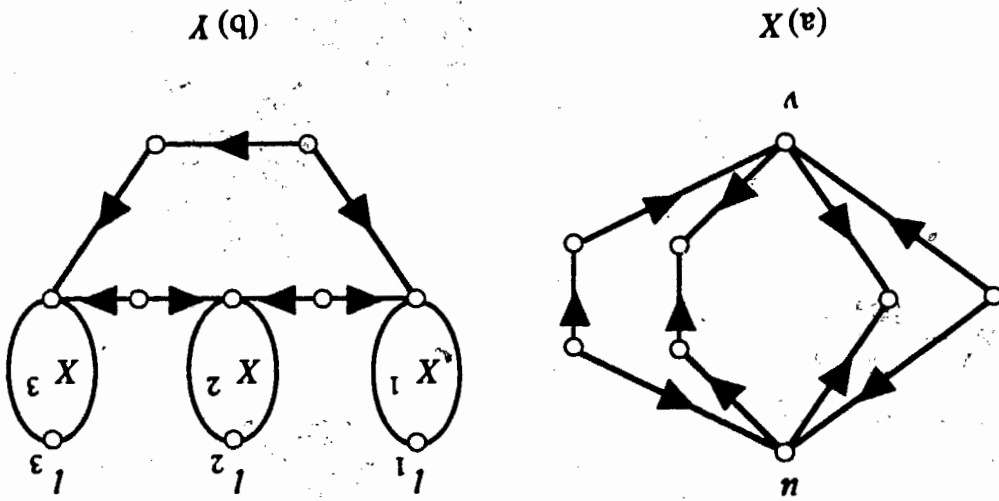


Let  $C$  be the seven-vertex oriented cycle in  $Y$  that contains the vertex  $z$ . Suppose  $l_1, l_2$ , and  $l_3$  are all coloured on the lower directed cycle. Then  $C$  must be coloured by the upper directed cycle, which is impossible since the net length of  $C$  is one. The second assertion follows easily from the preceding claim.

**PROOF.**

**CLAIM.** There is no  $D(k, l)$ -colouring of  $Y$  such that  $l_1, l_2$ , and  $l_3$  are all coloured on the lower directed cycle. On the other hand, any assignment of colours to  $l_1, l_2$ , and  $l_3$  which uses at least one colour from the upper directed cycle can be extended to a  $D(k, l)$ -colouring of  $Y$ .

Figure 5.1.1. The digraphs  $X$  and  $Y$ .



This is impossible since the net length of the outer (resp. inner) oriented cycle of  $X$  is one, and Lemma 3.1.2 implies that a cycle of net length one does not map to a directed cycle of length greater than one. This establishes (a) and (b). The last statement is easy to check.

The transformation is from 3-SAT. Accordingly, suppose we are given an instance of 3-SAT, with variables  $x_1, x_2, \dots, x_p$ , and clauses  $C^1, C^2, \dots, C^q$ .

Construct a digraph  $G$  from  $p$  copies of  $X$ , and  $q$  copies of  $Y$ , as follows. Each variable  $x_i$  ( $i=1, 2, \dots, p$ ) corresponds to a copy  $X^i$  of  $X$ , and each clause  $C^j$  ( $j=1, 2, \dots, q$ ) corresponds to a copy  $Y^j$  of  $Y$ . For each clause  $C^j = l_1 \vee l_2 \vee l_3$  ( $j=1, 2, \dots, q$ ) identify each vertex of  $Y^j$  labelled  $l_i$  with vertex  $u$  of  $X^k$  if  $l_i$  is  $x_k$ , and with vertex  $v$  of  $X^k$  if  $l_i$  is  $\neg x_k$ . Clearly, the construction may be carried out in polynomial time.

**CLAIM.** The digraph  $G$  is  $D(k, l)$ -colourable if and only if the clauses  $C^1, C^2, \dots, C^q$  are simultaneously satisfiable.

**PROOF.**

( $\Rightarrow$ ) Define a truth assignment as follows. Set  $x_i = T$  ( $i=1, 2, \dots, p$ ) just if the vertex  $u \in V(X^i)$  is coloured on the upper directed cycle (recall that the vertex  $v \in V(X^i)$  must be coloured on the other directed cycle). Consider  $Y^j$ . At least one of its three labelled vertices must be coloured on the upper directed cycle. In other words, the clause  $C^j$  contains at least one true literal. Hence the clauses are simultaneously satisfiable.

( $\Leftarrow$ ) Define a partial colouring as follows. For  $i=1, 2, \dots, p$ , if  $x_i = T$  colour vertex  $u$  (resp.  $v$ ) of  $X^i$  by vertex 0 on the upper (resp. lower) directed cycle and, if  $x_i = F$ , colour vertex  $u$  (resp.  $v$ ) of  $X^i$  by vertex 0 on the lower (resp. upper) directed cycle. Since all clauses are satisfiable, every copy  $Y^j$  of  $Y$  ( $j=1, 2, \dots, q$ ) has a labelled vertex which is coloured on the upper directed cycle. By the claims, this partial colouring can be extended to a  $D(k, l)$ -colouring of  $G$ .

The result now follows. ■

Since a semi-complete digraph has a spanning tournament, there is a natural total order on its strong components, namely  $A < B$  if and only if every vertex in  $B$  is adjacent to a vertex in  $A$ . It therefore makes sense to talk about the first strong component, or the next strong component, etc.. When we number the strong components of a semi-complete digraph as  $C^1, C^2, \dots, C^k$ , say, we always assume that  $C^1$  is the first strong component,  $C^2$  is the second strong component, and so on.

**5.1.7. Corollary.** Let  $H$  be a semi-complete digraph in which every strong component is a vertex or a directed cycle. If  $H$  has at least two directed cycles, then  $H$ -COL is NP-complete.

**Proof.**

We first show that it is enough to consider the case where  $H$  has exactly two non-trivial strong components. Suppose to the contrary that  $H$  has at least three non-trivial strong components. Consider the third non-trivial strong component, and let  $\{j_1, j_2, \dots, j_t\}$  be its vertex set. Let  $J$  be the digraph with vertex-set  $\{x, l_1, l_2, \dots, l_t\}$ , and arc-set  $\{xl_i: i=1, 2, \dots, t\}$ . Let  $H^*$  be the result of applying the sub-indicator construction with respect to  $(J, x, l_1, l_2, \dots, l_t)$  and  $j_1, j_2, \dots, j_t$  to  $H$ . Then  $H^*$  is the semi-complete digraph induced by the vertices belonging to those strong components of  $H$  up to, but not including, the third nontrivial strong component. Thus  $H^*$  has precisely two non-trivial strong components. It follows from Lemma 3.1.8 that it is sufficient to show that  $H^*$ -

*COL* is NP-complete.

We now prove the result for semi-complete digraphs  $G$  with exactly two nontrivial strong components. Let  $r$  and  $s$  be the length of the first and second directed cycle in  $G$ , respectively. Let  $G^*$  be the result of applying the sub-indicator construction with respect to  $(C_{r+s}, 0, free)$  to  $G$ . (Recall that each nontrivial strong component of  $G$  is a directed cycle.) The image of  $C_{r+s}$  in  $G$  is a union of directed cycles. That is, the trivial strong components are eliminated by the sub-indicator construction. Since  $G$  has exactly two (disjoint) directed cycles, it follows that  $G^*$  is  $D(r, s)$ . Since  $r, s \geq 2$ , the  $D(r, s)$ -colouring problem is NP-complete. Therefore *H-COL* is also NP-complete. ■

Let  $T_1, T_2, \dots, T_5$  be the tournaments shown in figure 5.1.2.

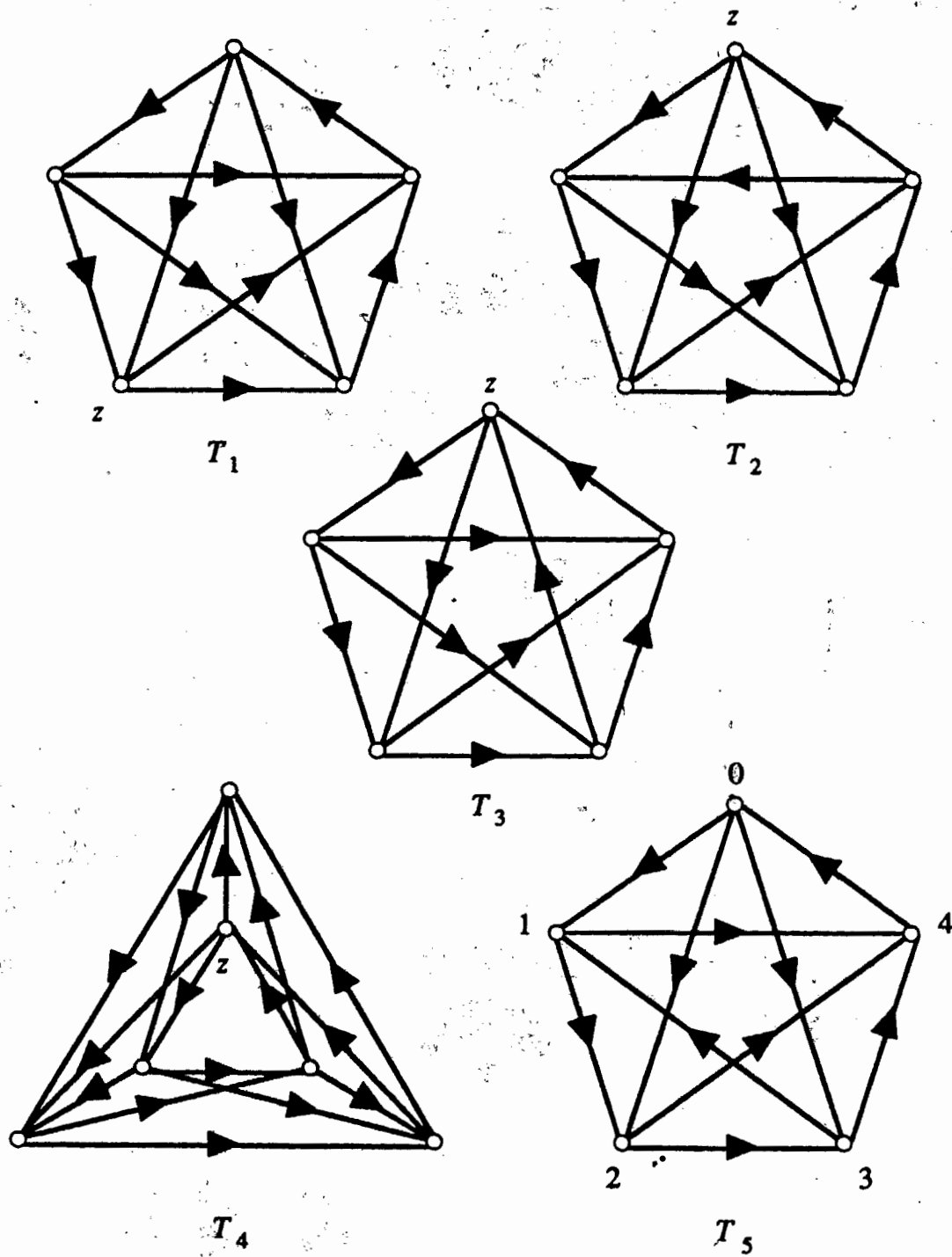


Figure 5.1.2. The tournaments  $T_1, T_2, \dots, T_5$ .

**5.1.8 Lemma.** For  $i = 1, 2, \dots, 5$ , the  $T_i$ -colouring problem is NP-complete.

**Proof.**

If  $i=1$ , the result  $T_1^*$  of applying the sub-indicator construction with respect to  $(P_6, 0, 6)$  and  $z$  to  $T_1$  is a four-vertex tournament with two directed cycles. Hence  $T_1$ -COL is NP-complete by Lemmas 3.1.8 and 5.1.5.

Similarly, if  $i=2, 3, 4$ , the result  $T_i^*$  of applying the sub-indicator construction with respect to  $(P_4, 0, 4)$  and  $z$  to  $T_i$  is a four-vertex tournament with two directed cycles. As above,  $T_i$ -COL is NP-complete by Lemmas 3.1.8 and 5.1.5.

It remains to prove that  $T_5$ -COL is NP-complete. The proof is similar to Theorem 5.1.6, except that the transformation is from 4-SAT. We will simply state the necessary claims, which can be easily verified.

**CLAIM.** In the tournament  $T_5$ ,

$$N^{+3}(i) = V(T_5), \quad i \neq 4, \text{ and}$$

$$N^{+3}(4) = V(T_5) - \{0\}.$$

Let  $X$  and  $Y$  be the digraphs shown in figure 5.1.3 (a) and (b), respectively.

**CLAIM.** In any  $T_5$ -colouring of  $X$ , one of the following holds:

(a)  $colour(u) = 4$ , and  $colour(v) \in \{1, 2, 3\}$ , or

(b)  $colour(v) = 4$ , and  $colour(u) \in \{1, 2, 3\}$ .

Moreover, the partial coloring of (a) or (b) can be extended to  $T_5$ -colourings of  $X$ .

CLAIM. There is no  $T_5$ -colouring of  $Y$  in which  $l_1, l_2, l_3$ , and  $l_4$  are all coloured by 4. On the other hand, any partial colouring of  $l_1, l_2, l_3$ , and  $l_4$  by colours in  $\{1, 2, 3, 4\}$  which uses at least one colour in  $\{1, 2, 3\}$  can be extended to a  $T_5$ -colouring of  $Y$ .

Having established these claims, the reader should have little difficulty in completing the proof. ■

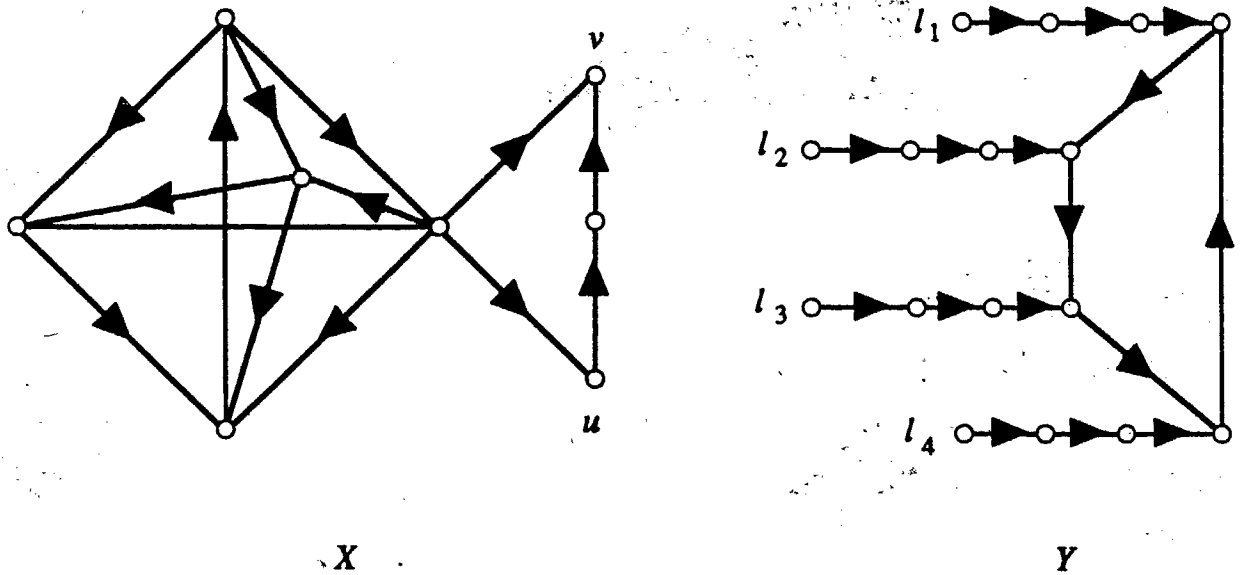


Figure 5.1.3. The digraphs  $X$  and  $Y$ .

**5.1.9. Lemma.** Let  $T$  be a strong tournament with at least five vertices. Then either

- (a) There exists a vertex  $v$  such that one of  $T[N^+(v)], T[N^{+2}(v)], T[N^-(v)], T[N^{-2}(v)]$  has at least two directed cycles, or
- (b)  $T$  is one of  $T_1, T_2, \dots, T_5$  (see figure 5.1.2).

**Proof.**

Assume (a) does not hold. We show that  $T$  is one of  $T_1, T_2, \dots, T_5$ . (It is

easy to check that (a) does not hold for  $T_1, T_2, \dots, T_5$ .)

Suppose first that  $T$  is not two-connected. Thus there exists a vertex  $x$  such that  $T-x$  is not strong. Let  $C^1, C^2, \dots, C^k$  ( $k \geq 2$ ) be the strong components of  $T-x$ . Since  $T$  is strong, some vertex of  $C^1$  (resp.  $C^k$ ) is adjacent from (resp. to)  $x$ . Every vertex of  $C^1$  (resp.  $C^k$ ) is adjacent to (resp. from) all vertices belonging to  $C^2 \cup C^3 \cup \dots \cup C^k$  (resp.  $C^1 \cup C^2 \cup \dots \cup C^{k-1}$ ). Since (a) does not hold,  $|V(C^i)| \leq 3$ ,  $i=1, 2, \dots, k$  (since any strong tournament with at least four vertices has at least two directed cycles).

We claim that  $T-x$  has at most one non-trivial strong component. Suppose  $C^r$  and  $C^s$ ,  $r < s$ , are both non-trivial strong components. If  $r > 1$ , then it is easy to check that (a) holds when  $v$  is any vertex of  $C^1$ . Thus  $r=1$ . Similarly  $s=k$ , and therefore  $|V(C^i)| = 1$ ,  $1 < i < k$ . Let  $C^1$  be the directed three-cycle  $a, b, c, a$ , and let  $C^k$  be the directed three-cycle  $d, e, f, d$ . Recall that  $x$  is adjacent to a vertex  $a$  of  $C^1$ , and is adjacent from a vertex  $d$  of  $C^k$ . Then  $N^{+2}(a) \supseteq \{x, d, e, f\}$ , so  $e$  and  $f$  also are adjacent to  $x$  (otherwise  $T[N^{+2}(a)]$  contains two directed cycles). Similarly  $x$  is adjacent to  $b$  and  $c$ . But then  $N^{+2}(d) \supseteq \{x, a, b, c, f\}$ , so  $T[N^{+2}(d)]$  has two directed cycles, a contradiction. Thus  $T-x$  has at most one non-trivial strong component.

Suppose first that  $C^1$  is the directed three-cycle  $a, b, c, a$ , and let  $V(C^k) = \{y\}$ . Then, as above,  $x$  is adjacent to  $a, b$ , and  $c$ . If  $|V(T)| = 5$ , then  $T = T_5$ . Hence assume  $|V(T)| \geq 6$ , so that  $k \geq 3$ , and let  $C^2 = \{z\}$ . Then  $N^{+2}(a) \supseteq \{c, x, y, z\}$ , which induces more than one directed cycle, leading to a contradiction. The case where  $C^k$  is a directed three cycle is similar, and also leads only to  $T = T_5$ .

Now suppose that for some  $i$ ,  $1 < i < k$ ,  $C^i$  is the directed three cycle  $a, b, c, a$ . Let  $V(C^1) = \{y\}$ , and let  $V(C^k) = \{z\}$ . Since  $T$  is strong,  $z$  is adjacent to  $x$ ,



and  $x$  is adjacent to  $y$ . Since  $N^{+2}(y) \supseteq \{a, b, c, x\}$ , and (a) does not hold, either  $x$  is adjacent to  $a, b$  and  $c$ , or  $x$  is adjacent from  $a, b$  and  $c$ . In the first case  $N^{-2}(c) \supseteq \{a, x, y, z\}$ , and in the second case  $N^{+2}(a) \supseteq \{c, x, y, z\}$ . Both of these sets of vertices induce at least two directed cycles, contrary to our assumption.

Thus  $T-x$  is a transitive tournament. Let  $V(C^i) = \{v_i\}$ ,  $i=1, 2, \dots, k$ . Since  $T$  is strong,  $x$  is adjacent to  $v_1$ , and  $v_k$  is adjacent to  $x$ . Suppose  $|V(T)|=5$ . If  $xv_2, xv_3 \in V(T)$ , or if  $v_2x, v_3x \in V(T)$ , then  $T=T_1$ . If  $xv_3, v_2x \in V(T)$ , then  $T=T_2$ . Finally, if  $xv_2, v_3x \in V(T)$ , then (a) holds, because  $N^{+2}(v_2) = \{x, v_1, v_3, v_4\}$ , which induces more than one directed cycle. Suppose now that  $|V(T)| \geq 6$ . Then  $x$  is adjacent to  $v_2$ ; otherwise  $N^{+2}(v_2) \supseteq \{v_1, v_{k-1}, v_k, x\}$ , and this set induces more than one directed cycle. Similarly  $x$  is adjacent to  $v_3$ ; otherwise  $N^{+2}(v_3) = \{v_1, v_2, v_k, x\}$ , and this vertex set induces more than one directed cycle. But now  $N^{+2}(v_1) = \{v_3, v_4, \dots, v_k, x\}$ , which induces more than one directed cycle. This completes the proof, in the case where  $T$  is not two-connected.

Now suppose that  $T$  is two-connected. If  $|V(T)| = 5$ , then  $T=T_3$  since it is the only two-connected tournament on five vertices. Hence assume  $T$  has at least six vertices. We show that if (a) does not hold, then  $T = T_4$ .

Let  $x$  be a vertex of maximum out-degree in  $T$ . We show  $N^{+2}(x) \supseteq N^-(x)$ . By our choice of  $x$ ,  $|N^+(x)| \geq 3$ . Moreover, every vertex in  $N^-(x)$  is adjacent to  $x$ . If some vertex  $y$  in  $N^-(x)$  is also adjacent to every vertex in  $N^+(x)$ , then  $|N^+(y)| > |N^+(x)|$ , contradicting the maximality of  $d^+(x)$ . Thus every vertex in  $N^-(x)$  is adjacent from a vertex in  $N^+(x)$ .

Let  $I^1, I^2, \dots, I^p$ ,  $p \geq 1$ , and  $O^1, O^2, \dots, O^q$ ,  $q \geq 1$ , be the strong components of  $T[N^-(x)]$  and  $T[N^+(x)]$ , respectively. Since (a) does not hold, each of these tournaments has at most one non-trivial strong component.

Suppose  $|V(O^1)| \geq 3$ . Then, since every vertex in  $N^-(x)$  is adjacent from a vertex in  $N^+(x)$ , and every vertex in  $N^+(x)$  is adjacent from a vertex in  $O^1$ ,  $N^{+2}(x) = V(T) - x$ . Since  $T - x$  is strong, this vertex-set induces a strong tournament with several directed cycles, which is a contradiction. Hence  $O^1$  is trivial. Let  $V(O^1) = \{o_1\}$ . Then  $N^{+2}(x) = V(T) - \{x, o_1\}$ , and thus  $T - x$  has at most one non-trivial strong component.

We claim that  $T[N^-(x)]$  is transitive. Suppose not, and let  $I_j$  be the unique strong component of  $T[N^-(x)]$ . Suppose  $j > 1$ , and let  $I_1 = \{i_1\}$ . Then  $N^{+2}(i_1) \supseteq N^+(x) \cup I_1$ , which induces a tournament with at least two directed cycles, a contradiction. Hence  $j = 1$ . If some vertex belonging to  $N^+(x) - o_1$  is adjacent to a vertex belonging to  $I_1$ , then  $N^{+2}(x)$  contains at least two directed cycles. Since every vertex belonging to  $I_1$  must be adjacent from a vertex in  $N^+(x)$ , the vertex  $o_1$  is adjacent to every vertex in  $I_1$ . But then, since  $o_1$  is adjacent to all out-neighbours of  $x$  except itself,  $d^+(o_1) > d^+(x)$ , which is a contradiction. This proves the claim. Let  $I_j = i_j$  ( $j = 1, 2, \dots, p$ ).

Next we show that  $2 \leq d^+(x) \leq 3$ . Since  $T$  is two-connected, every vertex has in-degree at least two (and out-degree at least two). Thus  $|N^-(x)| \geq 2$ . Suppose that  $d^+(x) > 3$ . Then  $N^{+2}(i_1) \supseteq N^+(x) \cup \{x\} \cup \{i_3, i_4, \dots, i_p\}$ . If some vertex belonging to  $N^+(x) - o_1$  is adjacent to a vertex belonging to  $\{i_3, i_4, \dots, i_p\}$ , then  $N^{+2}(i_1)$  contains at least two directed cycles. Since every vertex belonging to  $I_1$  must be adjacent from a vertex in  $N^+(x)$ ,  $o_1$  is adjacent to every vertex in  $\{i_3, i_4, \dots, i_p\}$ . Since we must have  $d^+(o_1) \leq d^+(x)$ , this set has size at most one, that is,  $p \leq 3$ .

We claim that  $|N^+(x)| = 3$ , and  $|N^-(x)| = 2$ . Suppose not. Let  $r$  be a vertex in  $O_q$ , and let  $s$  be a vertex in  $N^{+2}(o_1)$  that is adjacent to  $r$ . Since  $T$  is two-connected,

$r$  has out-degree at least two. Suppose  $ru \in E(T)$ . Note that  $u \in N^-(x)$ . If  $o_1$  is adjacent to any vertex belonging to  $N^-(x)$ , then  $N^{+2}(o_1) \supseteq \{r, s, u, x\}$ , which induces at least two directed cycles. Thus every vertex belonging to  $N^-(x)$  is adjacent to  $o_1$ . If  $u$  is adjacent to a vertex  $v$  in  $O_2$ , again it is easy to see that  $N^{+2}(x)$  contains at least two directed cycles. Therefore every vertex in  $O_2$  is adjacent to  $u$ . There exists a vertex  $z \in O_2$  for which there is a directed  $(z, r)$  path of length two. But then  $N^{+2}(z) \supseteq \{r, u, x, o_1\}$ , which induces at least two directed cycles. Therefore  $|N^-(x)| = 3$ . Since  $T$  has at least six vertices, we must have  $|N^-(x)| = 3$ , and  $|N^+(x)| = 2$ . The claim is now proved.

Finally we prove that  $T = T_4$ . It has been shown that  $N^+(x)$  and  $N^-(x)$  both induce transitive tournaments. Let  $O_i = \{o_i\}$ ,  $i = 1, 2, 3$ . It remains to determine the orientations of the arcs between  $N^+(x)$  and  $N^-(x)$ . Since  $T$  is two-connected,  $d^+(o_3) \geq 2$ ,  $d^-(o_1) \geq 2$ , and  $d^+(o_2) \geq 2$ . Thus  $o_3$  is adjacent to both  $i_1$  and  $i_2$ ,  $o_1$  is adjacent from at least one of  $i_1$  and  $i_2$ , and  $o_2$  is adjacent to at least one of  $i_1$  and  $i_2$ . If  $o_1$  is adjacent to one of  $i_1$  and  $i_2$ ,  $N^{+2}(o_1) \supseteq \{x, i_1, i_2, o_3\}$ , which induces at least two directed cycles. Therefore  $o_1$  is adjacent to both  $i_1$  and  $i_2$ . Similarly, if  $i_1 x \in E(T)$ , then  $N^{+2}(x)$  induces a strong tournament, and if  $i_2 x \in E(T)$ , then  $N^{+2}(i_1)$  induces a strong tournament. Therefore  $o_2$  is adjacent to both  $i_1$  and  $i_2$ , and  $T = T_4$ .

The proof of Lemma 5.1.9 is now complete. ■

Lemma 5.1.9 asserts the existence of a vertex  $v$  such that one of  $T[N^{+2}(v)]$  and  $T[N^{-2}(v)]$  has at least two directed cycles. If there is a vertex  $x$  such that  $N^+(x)$  (resp.  $N^-(x)$ ) induces at least two directed cycles,  $v$  can be chosen to be an in-neighbour (resp. out-neighbour) of  $x$ .

**5.1.10. Theorem.** If  $T$  is a tournament with at least two directed cycles, then  $T$ -COL is NP-complete.

**Proof.**

The proof is by strong induction on  $|V(T)|$ .

**BASIS.** By Lemmas 5.1.4 and 5.1.5 the statement is true if  $|V(T)| \leq 4$ .

**INDUCTION.** Assume the statement is true for all tournaments on at most  $k$  vertices, and let  $T$  be a  $(k+1)$ -vertex tournament with at least two directed cycles.

Suppose first that  $T$  is not strong. If every strong component of  $T$  is a vertex or a directed cycle, the result follows from Corollary 5.1.7. Otherwise,  $T$  has a strong component  $C$  with at least four vertices (and therefore with at least two directed cycles). Since  $C$  does not admit a homomorphism to a directed cycle, and since the  $C$ -colouring problem is NP-complete by the induction hypothesis,  $T$ -COL is NP-complete by Lemma 3.1.6.

Now suppose that  $T$  is strong. If  $T$  is one of  $T_1, T_2, \dots, T_5$ , the result follows from Lemma 5.1.8. Otherwise, let  $v$  be a vertex such that one of  $T[N^{+2}(v)]$  and  $T[N^{-2}(v)]$  has at least two directed cycles. Suppose the former case holds, the latter case being similar. Let  $T^*$  be the result of applying the sub-indicator construction with respect to  $(P_2, 2, 0)$  and  $v$  to  $T$ . Then  $T^* = T[N^{+2}(v)]$ , so the tournament  $T^*$  has at least two directed cycles. Moreover  $v \notin V(T^*)$ , so  $T^*$  has at most  $k$  vertices. By the induction hypothesis, the  $T^*$ -colouring problem is NP-complete. Hence  $T$ -COL is also NP-complete.

The result now follows by strong induction. ■

**5.1.11. Theorem.** Let  $T$  be a semi-complete digraph with at least two directed cycles and a unique two-cycle. Then  $T$ -COL is NP-complete.

**Proof.**

The proof is by strong induction on  $|V(T)|$ .

**BASIS.** By Lemmas 5.1.4 and 5.1.5 the statement is true if  $T$  has at most four vertices.

**INDUCTION.** Assume the statement is true for all such semi-complete digraphs with at most  $k$  vertices. Let  $T$  be a  $(k+1)$ -vertex semi-complete digraph with at least two directed cycles, and a unique two-cycle  $[a, b]$ . If  $T$  is not strong, then the result follows using the same argument as in the analogous case of Theorem 5.1.10. We may therefore assume that  $T$  is strong. There are two cases to consider.

**CASE 1.** There exists a vertex  $v$  which is adjacent to both  $a$  and  $b$ , or a vertex  $v'$  which is adjacent from both  $a$  and  $b$ .

We prove the result on the assumption that  $v$  exists, as the other case is similar. Let  $J_1$  be the digraph constructed from  $C_2 \cup \{x, y\}$  by adding the arcs  $y_0, y_1$ , and  $0x$ . Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(J_1, x, y)$  and  $v$  to  $T$ . Let  $V(T^-) = \{a, b\} \cup W$ . Since  $T$  is strong,  $W$  is not empty. Also,  $W = N^+(a) \cup N^+(b)$ , and  $v \notin W$  (hence  $|W| \leq k$ ). If  $T^-$  has at least two directed cycles the result follows from the induction hypothesis. Hence we may assume that  $[a, b]$  is the unique directed cycle in  $T^-$ . Therefore the tournament induced by  $W$  is transitive, and  $a$  and  $b$  are each adjacent to every vertex of  $W$ . Let  $J_2$  be the converse of  $J_1$ . Let  $w$  be in  $W$ , and let  $T^{--}$  be the result of applying the sub-indicator construction with respect to  $(J_2, x, y)$  and  $w$  to  $T$ .

Let  $V(T^-) = \{a, b\} \cup Y$ . Note that  $Y = N^-(a) \cup N^-(b)$ , and that  $w \notin Y$ . The set  $Y$  is not empty because  $v \in Y$ . Hence  $3 \leq |V(T^-)| \leq k$ . If  $T^-$  has at least two directed cycles, the result follows from the induction hypothesis. Hence we may assume that  $[a, b]$  is the unique directed cycle in  $T^-$ . It follows that  $Y$  induces a transitive tournament, and that every vertex of  $Y$  is adjacent to both  $a$  and  $b$ . Since every vertex in  $V(T) - b$  is either adjacent to  $a$ , or from  $a$ ,  
 $V(T) = \{a, b\} \cup W \cup Y$ .

Suppose  $|W| > 1$ . Since  $T$  is strong, some vertex  $w \in W$  is adjacent to a vertex  $y \in Y$ . If  $y$  is adjacent to some other vertex  $w' \in W$ , then  $N^{-2}(w')$  contains  $\{a, b, w, y\}$ . Since this set induces more than one directed cycle, we have a contradiction. Therefore every vertex of  $W$  is adjacent to  $y$ . It is easy to see that  $N^{+2}(a)$  contains  $\{a, y\} \cup W$ . Moreover,  $b \notin N^{+2}(a)$ . Let  $T^{---}$  be the result of applying the sub-indicator construction with respect to  $(P_2, 0, 1)$  and  $a$  to  $T$ . Then  $T^{---} = T[N^{+2}(a)]$ , so  $b \notin V(T^{---})$ . Thus  $T^{---}$  is a tournament with at least two directed cycles, and so the  $T^{---}$ -colouring problem is NP-complete by Theorem 5.1.10. Therefore  $T$ -COL is also NP-complete. It remains to consider the case  $|W| = 1$ . A similar argument shows that we may assume  $|Y| = 1$ . But then  $T$  has only four vertices, whence the result follows from Lemma 5.1.5. This completes the proof of case 1.

CASE 2. No vertex is adjacent to both  $a$  and  $b$ , and no vertex is adjacent from both  $a$  and  $b$ .

Let  $A = N^+(a) - b$ , and  $B = N^+(b) - a$ . Then  $V(T) = A \cup B \cup \{a, b\}$ , every vertex in  $A$  is adjacent to  $b$ , and every vertex in  $B$  is adjacent to  $a$ .

Suppose neither  $A$  nor  $B$  is empty. Assume first that  $|A| > 1$ . Let  $x$  be a

vertex of the initial strong component of  $T[A]$ . Then  $N^{+2}(x)$  contains  $\{a, b\} \cup B$ , but not  $x$ . The result  $T^-$  of applying the sub-indicator construction with respect to  $(P_2, 0, 2)$  and  $x$  to  $T$  is  $T[N^{+2}(x)]$ . The tournament  $T^-$  has at least two directed cycles. Thus  $T^-$ -COL is NP-complete. Hence  $T$ -COL is also NP-complete. Thus we may assume  $|A| = 1$ . A similar argument shows that we may also assume  $|B| = 1$ . But then  $T$  has only four vertices, whence the result follows from Lemma 5.1.5.

Now suppose that one of  $A$  and  $B$  is empty. Without loss of generality  $B$  is empty. If  $|A| = 1$  or  $2$ , the result follows from Lemma 5.1.4 or 5.1.5, respectively. Suppose that  $|A| \geq 3$ . Let  $u$  be a vertex in the initial strong component of  $T[A]$ . There exists a vertex  $v \in A$  such that there is a directed  $(u, v)$ -path of length two. Therefore  $N^{+2}(u) \supseteq \{a, b, v\}$ , which induces two directed cycles. Since  $u \notin N^{+2}(u)$ , the result follows, as before, from the sub-indicator construction and the induction hypothesis.

The result now follows by strong induction. ■

### Proof of Theorem 5.1.1.

Suppose the statement is false, and let  $T$  be a counterexample with the minimum number of vertices. That is,  $T$  is a minimum vertex semi-complete digraph with at least two directed cycles for which the  $T$ -colouring problem is not NP-complete. We derive some structural properties of  $T$ , and ultimately a contradiction.

It follows from Theorems 5.1.10 and 5.1.11 that  $T$  has at least two double arcs, and, by Lemmas 5.1.4 and 5.1.5, at least five vertices. If every strong component is a vertex or a cycle,  $T$ -COL is NP-complete by Corollary 5.1.7. Otherwise  $T$  has a strong component  $C$  with at least two directed cycles. By the minimality of  $|V(T)|$ , the  $C$ -colouring problem is NP-complete. Since  $C$  does not map to a directed cycle,  $T$ -COL is NP-complete by Lemma 3.1.6, which is a contradiction. Hence  $T$  is strong. By Lemma 4.1.1,  $\text{undir}(T)$  is bipartite.

(1) No vertex  $v$  of  $T$  has more than one directed cycle in  $T[N^+(v)]$  or  $T[N^-(v)]$ .

Suppose there exists a vertex  $v$  such that  $N^+(v)$  induces more than one directed cycle, the other case being similar. Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(P_1, 0, 1)$  and  $v$  to  $T$ . Then  $T^- = T[N^+(v)]$ . Since the semi-complete digraph  $T^-$  has at least two directed cycles, and  $v \in V(T^-)$ , the  $T^-$ -colouring problem is NP-complete. Hence  $T$ -COL is also NP-complete, which is a contradiction.



(2) Every vertex of  $T$  is incident with exactly one double arc.

First suppose there is a vertex  $v$  that is not incident with a double arc. Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(C_2, 0, free)$  to  $T$ . Then  $T^-$  is a semi-complete digraph with at least two double arcs (and hence more than one directed cycle), and fewer vertices than  $T$  (since  $v \in V(T^-)$ ). By our choice of  $T$ , the  $T^-$ -colouring problem is NP-complete. Thus  $T$ -COL is also NP-complete, which is a contradiction. Therefore every vertex is incident with at least one double arc.

Now suppose that  $b$  is a vertex of  $T$  which is incident with the double arcs  $[a, b]$  and  $[b, c]$ . Since  $undir(T)$  is bipartite,  $a$  and  $c$  are not joined by a double arc. Without loss of generality  $a$  is adjacent to  $c$ . Let  $T'$  be the semi-complete digraph  $T - \{a, b, c\}$ .

Suppose there is a vertex  $v$  of  $T'$  which is adjacent from both  $a$  and  $c$ . If  $bv \in E(T)$ , then  $N^+(v) \supseteq \{a, b, c\}$ , which induces more than one directed cycle. Otherwise,  $vb \in E(T)$ , and  $N^+(a) \supseteq \{b, c, v\}$ , which also induces more than one directed cycle.

Similarly no vertex of  $T'$  is adjacent to both  $a$  and  $c$ .

Suppose there is a vertex  $u$  of  $T'$  such that  $a$  is adjacent to  $u$  and  $u$  is adjacent to  $c$ . If  $ub \in E(T)$ , then  $N^-(c)$  contains  $\{a, b, u\}$ , contrary to (1). On the other hand, if  $uc \in E(T)$ , then  $N^+(a)$  contains  $\{b, c, u\}$ , which also induces at least two directed cycles, again a contradiction. Thus for every vertex  $t$  of  $T'$ , the vertex  $c$  is adjacent to  $t$ , and  $t$  is adjacent to  $a$ . Furthermore, by the above argument, there are no other arcs between  $V(T')$  and  $\{a, b\}$ .

Suppose there is a vertex  $x$  of  $T'$  and a double arc  $[x, b]$ . Since  $T$  has at least five vertices, there is a vertex  $y$  in  $T' - x$ . Let  $[y, z]$  be the double arc incident

with  $y$ . We know that  $z \neq c$  and  $z \neq a$ . But then  $N^+(c)$  contains the double arcs  $[b, x]$  and  $[y, z]$ , contrary to (1). Thus  $\text{undir}(T')$  is a spanning subgraph and, since  $c$  is adjacent to every vertex of  $V(T')$ ,  $T'$  consists of a single double arc  $[k, l]$ . By (1) either the vertex  $b$  is adjacent to both  $k$  and  $l$ , or is adjacent from both of them. In the former case  $N^+(b) = \{a, c, k, l\}$ , and in the latter case  $N^-(b) = \{a, c, k, l\}$ . Since this set of vertices induces more than one directed cycle, we have a contradiction.

This completes the proof of (2).

Let  $V(T) = \{a_i, b_i; i=1, 2, \dots, r\}$ , where  $[a_i, b_i]$  is a double arc  $i=1, 2, \dots, r$ . Since  $V(T) > 4$ ,  $r \geq 3$ .

**(3) The only possible configurations for the arcs between two double arcs are shown in figure 5.1.4.**

Let  $[a, b]$ , and  $[c, d]$  be double arcs. Suppose that  $a$  is adjacent to both  $c$  and  $d$ . Then, since  $N^+(a) \supseteq \{b, c, d\}$ , (1) implies that either  $b$  is adjacent to both  $c$  and  $d$  or is adjacent from both of them. Thus we have configuration (i) or (ii). Similarly, if both  $c$  and  $d$  are adjacent to  $a$ , the same two configurations arise. If no vertex is adjacent to both vertices of a double arc and no vertex is adjacent from both vertices of a double arc, the only possibility is configuration (iii).

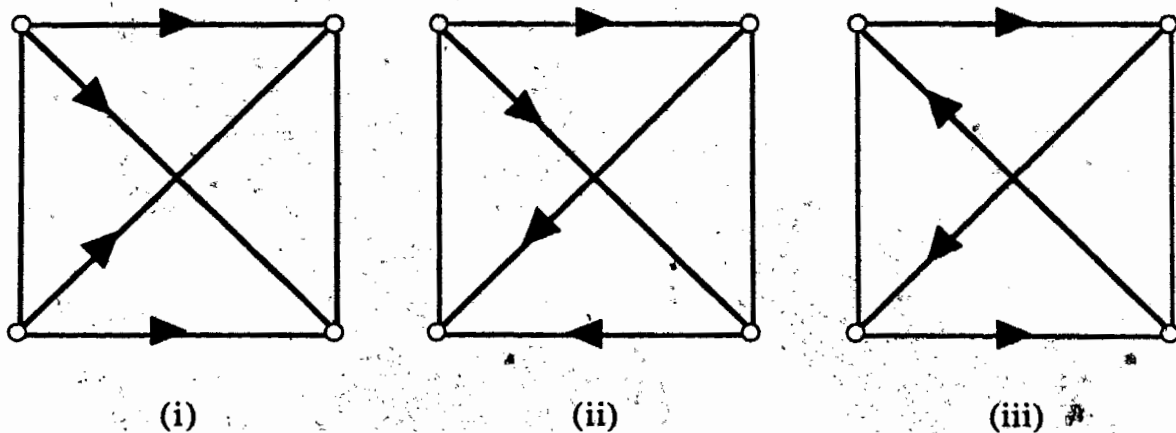


Figure 5.1.4. All possible configurations.

(4) For every vertex  $v$  of  $T$ ,  $d^+(v) = d^-(v) = r$ .

Since neither  $N^+(v)$  nor  $N^-(v)$  induces more than one double arc,  $r-1 \leq d^+(v), d^-(v) \leq r+1$ . It suffices to prove that no vertex has out-degree  $r+1$ . Suppose to the contrary that  $d^+(a_1) = r+1$ . Without loss of generality,  $N^+(a_1) = \{a_2, a_3, \dots, a_r, b_1\}$  and  $N^+(b_2) = \{b_3, b_4, \dots, b_r, b_1\}$ . Let  $J$  be the digraph constructed from  $C_2 \cup \{v\}$  by adding the arc  $v0$ . Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(J, 1, v)$  and  $a_1$  to  $T$ . Then  $V(T^-) = \{b_2, b_3, \dots, b_r, a_1, a_2\}$ , so  $T^-$  contains the directed two-cycle  $[a_2, b_2]$ . If  $T^-$  has another directed cycle, then the choice of  $T$  implies that  $T^-$ -COL is NP-complete, whence  $T$ -COL is also NP-complete. Hence  $[a_2, b_2]$  is the unique directed cycle in  $T^-$ . Therefore  $b_j$  is adjacent to  $a_2$  and  $b_2, j=3, 4, \dots, r$ . If there exists  $k, 3 \leq k \leq r$ , such that  $b_k b_1 \in E(T)$ , then  $N^+(b_k)$  contains  $\{a_1, a_2, b_1, b_2\}$ , which induces more than one directed cycle, contrary to (1). Thus  $b_1$  is adjacent to  $b_3, b_4, \dots, b_r$ . Moreover  $a_2$  is adjacent to  $b_1$ ; otherwise  $N^-(a_2)$  contains  $\{a_1, b_1, b_3\}$  which induces two directed cycles, contrary to (1). Let  $T^{--}$  be the result of applying the sub-indicator construction with respect to  $(J, 1, v)$  and  $b_3$  to  $T$ . Then  $a_1 \in V(T^{--}) \supseteq \{b_1, b_2, b_3, a_2\}$ . Since  $T^{--}$  has at least two directed cycles

and fewer vertices than  $T$ , the  $T^-$ -colouring problem is NP-complete. Therefore  $T$ -COL is also NP-complete, a contradiction. This completes the proof of (4).

**(5) Every vertex of  $T$  is adjacent to exactly one vertex of each double arc (that is, the arcs between any two double arcs form configuration (iii)).**

Suppose not. Assume that  $a_1$  is adjacent to  $a_2$  and  $b_2$ . If  $a_1$  also is adjacent to  $a_i$  or  $b_i$  for some  $i$ ,  $3 \leq i \leq r$ , we can obtain a contradiction by arguing as in (4). Thus  $a_1$  is adjacent from  $\{a_3, b_3, a_4, b_4, \dots, a_r, b_r\}$ . By (4),  $r=3$ . By applying (3) to the double arcs  $[a_1, b_1]$  and  $[a_2, b_2]$ , and then applying (4), we see that  $T$  is either  $T_6$  or  $T_7$  (cf. figure 5.1.5).

Suppose  $T=T_6$ . It is easy to check that the result  $T^-$  of applying the sub-indicator construction with respect to  $(P_2, 2, 0)$  and  $a_1$  to  $T$  is  $T-b_1$ . Since  $T^-$  has two directed cycles and fewer vertices than  $T$ , the  $T^-$ -colouring problem is NP-complete. Thus  $T$ -COL is also NP-complete, which is a contradiction.

Suppose  $T=T_7$ . Let  $J$  be the digraph constructed from  $C_2 \cup \{u, v\}$  by adding the arcs  $u0$  and  $1v$ . Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(J, v, u)$  and  $a_1$  to  $T$ . It is easy to check that  $T^- = T-a_1$  and, as above, we have a contradiction.

This completes the proof of (5).

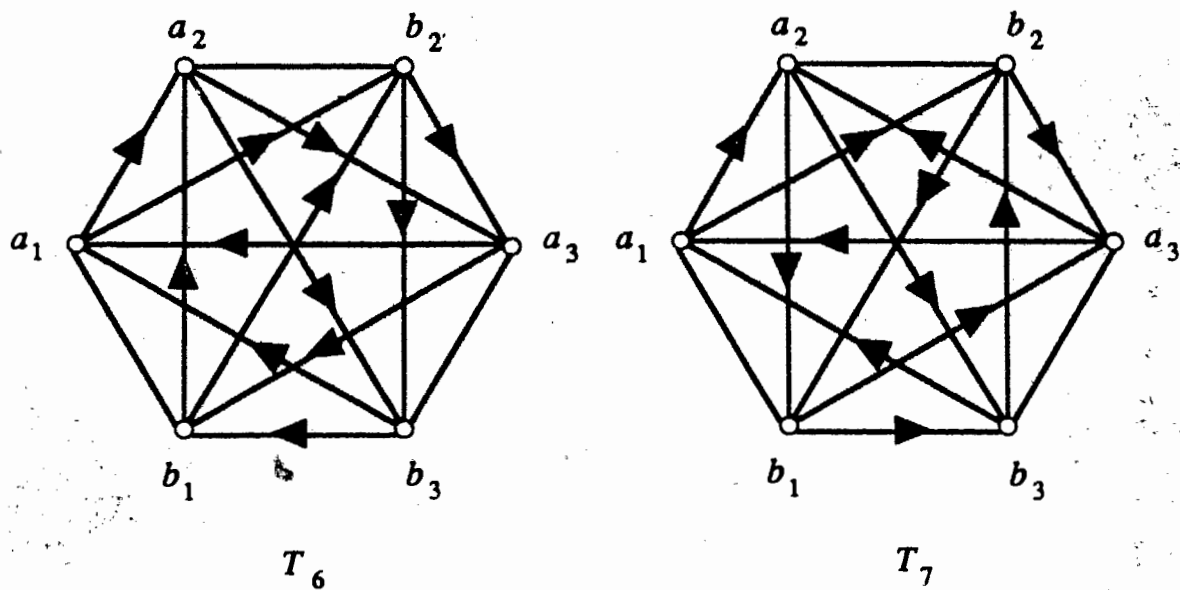


Figure 5.1.5. The tournaments  $T_6$  and  $T_7$ .

Let  $J$  be the digraph constructed from  $C_2 \cup \{u, v\}$  by adding the arcs  $u0$  and  $1v$ . Let  $T^-$  be the result of applying the sub-indicator construction with respect to  $(J, u, v)$  and  $a_1$  to  $T$ . It follows from the first paragraph of the proof of (5) that  $a_1$  is not in  $T^-$ ,  $\text{undir}(T)$  is a disjoint union of double arcs, and  $a_1$  is not adjacent to both ends of any double arc. Since, by (5), every vertex of  $T$  is adjacent to one vertex of each double arc,  $T^- = T - a_1$ . As  $r \geq 3$ , the digraph  $T^-$  has at least two directed cycles. By our choice of  $T$ , the  $T^-$ -colouring problem is NP-complete. Therefore  $T$ -COL is also NP-complete. This contradiction completes the proof of Theorem 5.1.1. ■

## 5.2. Transitive Digraphs.

In this section we completely classify the complexity of  $H$ -COL when the directed graph  $H$  is vertex-transitive or arc-transitive. For vertex-transitive digraphs, we prove that the  $H$ -colouring problem is NP-complete unless  $H$  admits a retraction to a directed cycle (cf. Theorem 5.2.4). This verifies Conjecture 1.1 for vertex-transitive directed graphs. Since there are arc-transitive digraphs with sources or sinks, their classification is slightly different (cf. Corollary 5.2.5). Note, however, that our classification implies Conjecture 1.1 for arc-transitive digraphs. In addition, we characterize, via conditions on the symbol, those Cayley digraphs that admit a homomorphism to a directed cycle (cf. Lemma 5.3.7). As a corollary, necessary and sufficient conditions are obtained for a Cayley digraph to retract to a directed cycle (cf. Corollary 5.3.8). Thus we give a structural classification of the complexity of  $H$ -colouring by Cayley digraphs.

The following three lemmas are essential to the proof of the main result of this section.

**5.2.1. Lemma.** The core of a vertex-transitive digraph is vertex-transitive.

**Proof.**

Let  $H$  be the core of  $G$ . Then there is a retraction  $r:G \rightarrow H$ . Let  $x$  and  $y$  be vertices of  $H$ , and let  $f$  be an automorphism of  $G$  such that  $f(x)=y$ . Then  $rf$  is a homomorphism of  $H$  to  $H$  and, as  $H$  is retract-free, an automorphism of  $H$ . Since  $r(f(x))=r(y)=y$ , we have that  $H$  is vertex-transitive. ■

**5.2.2. Lemma.** Let  $H$  be a directed graph and let  $(I, u, v)$  be an indicator. Let  $H^*$  be the digraph that results from applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . Then  $Aut(H)$  is a subgroup of  $Aut(H^*)$ .

**Proof.**

Since  $Aut(H)$  is a group it suffices to prove that  $Aut(H^*)$  contains  $Aut(H)$ . Let  $f$  be an automorphism of  $H$  and let  $ab$  be an arc of  $H^*$ . Then there is a homomorphism  $h: I \rightarrow H$  such that  $h(u)=a$  and  $h(v)=b$ . The function  $f \circ h$  is also a homomorphism of  $I$  to  $H$ , and  $f(h(u))=f(a)$  and  $f(h(v))=f(b)$ . Hence  $f(a)f(b)$  is also an arc of  $H^*$ . Since  $f$  is a one-to-one arc preserving map, it is an automorphism of  $H^*$ . ■

By Lemma 5.2.2, the digraph that results from applying an indicator construction to a vertex-transitive digraph is also vertex-transitive.

We now define a special type of indicator that plays a central role in the proof of Theorem 5.2.4. A *z-indicator* is an indicator  $(I, u, v)$  such that there is a vertex  $z$  that is the only neighbour of  $v$  (the vertex  $z$  may be an in-neighbour of  $v$  or an out-neighbour of  $v$ ). If  $z$  is an in-neighbour of  $v$ , we sometimes call  $(I, u, v)$  an *in-z-indicator* and, similarly, if  $z$  is an out-neighbour of  $v$ , we sometimes call  $(I, u, v)$  an *out-z-indicator*. These special indicators are important in our work on vertex-transitive digraphs because of the following lemma.

**5.2.3. Lemma.** Let  $H$  be a vertex-transitive digraph and let  $(I, u, v)$  be an in- $z$ -indicator (resp. out- $z$ -indicator). If there exists a vertex  $x$  of  $H$ , and homomorphisms  $h_1$  and  $h_2$  of  $I$  to  $H$  such that  $h_1(u)=h_2(u) = x$  and  $h_1(z) \neq h_2(z)$ , then either

(a)  $|E(H^*)| > |E(H)|$ , or

(b)  $|E(H^*)| = |E(H)|$ , and  $N^+_H(h_1(z)) = N^+_H(h_2(z))$

(resp.  $N^-_H(h_1(z)) = N^-_H(h_2(z))$ ).

**Proof.**

Since  $H$  is vertex-transitive, every vertex is a homomorphic image of the vertex  $z$ . Thus, for every vertex  $a$ , there is a vertex  $b$  such that  $N^{+*}_H(b) \supseteq N^+_H(a)$ . Hence the out-degree of a vertex does not decrease.

(Every vertex  $v$  of a vertex-transitive digraph has  $d^+(v) = d^-(v) = c$  for some constant  $c$ .) Therefore  $H^*$  has at least as many arcs as  $H$ . Suppose equality holds. Let  $r$  be the out-degree of every vertex of  $H$  and  $H^*$ . But since  $N^{+*}_H(x)$  contains both  $N^+_H(h_1(z))$  and  $N^+_H(h_2(z))$ , the vertex  $x$  has  $d^{+*}_H(x) = r$  just if these two  $r$ -sets are equal. ■

**5.2.4. Theorem.** Let  $H$  be a vertex-transitive digraph with at least one arc. Then the  $H$ -colouring problem is NP-complete unless  $H$  admits a retraction to a directed cycle. In the latter case  $H$ -COL is polynomial.

**Proof.**

We have previously noted the second statement (cf. the remark following Theorem 2.2.2). The first assertion is proved by contradiction. Let  $H$  be a counterexample with the minimum number of vertices and, within all counterexamples on  $|V(H)|$  vertices, one with the maximum number of arcs.



That is,  $H$  is a vertex-transitive digraph that does not admit a retraction to a directed cycle, and for which the  $H$ -colouring problem is not NP-complete. By Lemma 4.1.1 that  $H$  is not the equivalent digraph of a complete graph. Furthermore, the minimality of  $|V(H)|$  implies that  $H$  is retract-free. (Otherwise the core  $H'$  of  $H$  is a vertex-transitive digraph with fewer vertices than  $H$ , and which does not retract to a directed cycle. By our choice of  $H$ , the  $H'$ -colouring problem is NP-complete. Consequently  $H$ -COL is also NP-complete.) Since the result is known if  $H$  has at most four vertices [Maurer et al., 1981; Gutjahr 1988] (cf. Section 2.2), we may assume that  $H$  has at least five vertices. We may also assume without loss of generality that  $H$  is connected. Since  $H$  is not a directed cycle, each vertex has out-degree at least two. We make the following sequence of assertions about the digraph  $H$ .

(1)  $H$  does not map to a directed cycle of length greater than one.

Assume  $H$  maps to a directed cycle of length greater than one and let  $k$  be the largest positive integer such that  $H \rightarrow C_k$ . The integer  $k$  exists because  $H$  is strong, (cf. the comment preceding Lemma 3.1.5). Since the core of  $H$  is not a directed cycle,  $C_k$  is not a subdigraph of  $H$  (since  $C_k$  is retract-free, it is a retract of a given directed graph  $G$  if and only if it is both a subdigraph of  $G$  and a homomorphic image of  $G$ ). Thus  $(P_k, 0, k)$  is a good indicator. Let  $H^*$  denote the result of applying the indicator construction with respect to  $(P_k, 0, k)$  to  $H$ . By Lemma 5.2.2,  $H^*$  is vertex-transitive. Since  $H$  is strong, each colour class of the  $C_k$ -colouring induces a connected component of  $H^*$ . Thus  $H^*$  has precisely  $k$  isomorphic connected

components, and so the core of  $H^*$  is a vertex-transitive digraph with fewer vertices than  $H$ .

We claim that the core of  $H^*$  is not a directed cycle. By the choice of  $k$ , the digraph  $H$  has a collection  $C^1, C^2, \dots, C^m$  of directed cycles such that  $\gcd\{|V(C^i)|: i=1, 2, \dots, m\}=k$ . Each  $C^i$  gives rise to a directed cycle  $C^{i*}$  in  $H^*$  of length  $(1/k)|V(C^i)|$ . Hence  $\gcd\{|V(C^{i*})|: i=1, 2, \dots, m\}=1$ , and  $H^*$  does not map to a directed cycle of length greater than one. Therefore  $H^*$  does not retract to a directed cycle. Thus the  $H^*$ -colouring problem is NP-complete (by the choice of  $H$ ), and so  $H$ -COL is also NP-complete. This completes the proof of (1).

By (1),  $H$  is not an orientation of a bipartite graph.

In the remainder of this section, we omit from our proofs the observation that the digraph which results from applying the indicator construction to a vertex-transitive digraph is itself vertex-transitive.

**(2) Every vertex of  $H$  is incident with a double arc.**

Since  $H$  is vertex-transitive, it suffices to prove that  $H$  has a double arc. Suppose not. Then  $(P_2, 0, 2)$  is a good in- $z$ -indicator. Let  $H^*$  be the result of applying the indicator construction with respect to  $(P_2, 0, 2)$  to  $H$ .

We claim that  $H^*$  does not retract to a directed cycle. Since  $H$  is strong and does not map to a directed cycle of length greater than one, it has a collection  $C^1, C^2, \dots, C^m$  of directed cycles such that  $\gcd\{|V(C^i)|: i=1, 2, \dots, m\}=d$ . Each  $C^i$  gives rise to a directed cycle  $C^{i*}$  in  $H^*$ .

If  $|V(C^i)|$  is odd,  $|V(C^{i^*})|=|V(C^i)|$ , and  $|V(C^{i^*})|=(1/2)|V(C^i)|$  otherwise.

Therefore  $\gcd(|V(C^{i^*})|: i=1, 2, \dots, m)=1$ . This proves the claim.

By Lemma 5.2.3(a),  $H^*$  has at least as many arcs as  $H$ . If  $|E(H^*)| > |E(H)|$ , the  $H^*$ -colouring problem is NP-complete by the maximality of  $|E(H)|$ , and so  $H$ -COL is also NP-complete, which is again a contradiction. Suppose that equality holds. Let  $x, y, z$  be vertices of  $H$  such that  $x$  and  $y$  are both adjacent from  $z$ . By Lemma 5.2.3(b),  $N^+_H(x)=N^+_H(y)$ . Similarly, it follows from considering the indicator construction with respect to the out- $z$ -indicator  $(P_2, 2; 0)$  that  $N^-_H(x)=N^-_H(y)$ . If  $x$  and  $y$  are non-adjacent, there is a retraction  $H \rightarrow H-x$  which maps  $x$  to  $y$ , contradicting the fact that  $H$  is retract-free. If  $x$  and  $y$  are adjacent,  $N^+_H(x)=N^+_H(y)$  implies that if  $xy$  (resp.  $yx$ ) is an arc, then  $yx$  (resp.  $xy$ ) is an arc, contrary to our assumption that  $H$  has no double arc. This completes the proof of (2).

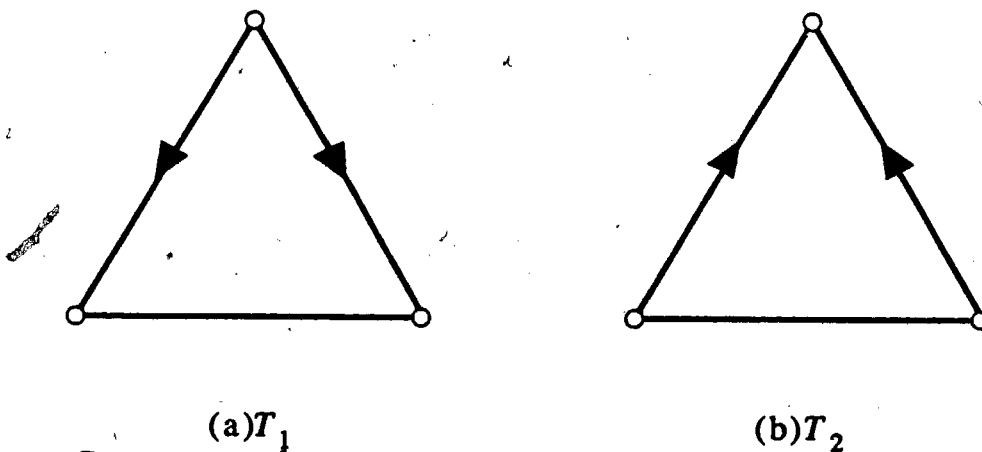


Figure 5.2.1. Subdigraphs from (3).

(3)  $H$  contains  $T_1$  and  $T_2$  (see figures 5.2.1(a) and (b), respectively).

Assume  $H$  does not contain  $T_1$ . Then the symmetric indicator  $(I, u, v)$  shown in figure 5.2.2(a) is good. Let  $H^*$  be the result of applying the

indicator construction with respect to  $(I, u, v)$  to  $H$ . Then  $H^*$  is loopless and undirected. Since  $\text{undir}(H)$  is spanning,  $H^*$  is the equivalent digraph of the underlying graph corresponding to  $H$ . Since  $H$  is not an orientation of a bipartite graph,  $H^*$  has an odd cycle, whence the  $H^*$ -colouring problem is NP-complete. Therefore  $H\text{-COL}$  is also NP-complete, which is a contradiction.

If  $H$  does not contain  $T_2$ , the argument is similar (the appropriate indicator is shown in figure 5.2.2(b)).

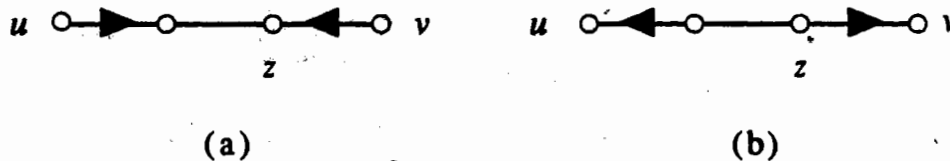


Figure 5.2.2. Indicators from (3).

(4) Every vertex of  $H$  is incident with at least two double arcs.

Suppose to the contrary that  $\text{undir}(H)$  is a disjoint union of double arcs. Then the indicator  $(I_1, u, v)$  shown in figure 5.2.3(a) is a good in- $z$ -indicator. Let  $H^*$  be the result of applying the indicator construction with respect to  $(I_1, u, v)$  to  $H$ . It is not hard to see that  $H^*$  contains a transitive triple and, therefore, does not admit a retraction to a directed cycle. By Lemma 5.2.3 the digraph  $H^*$  has at least as many arcs as does  $H$ . If  $|E(H^*)| > |E(H)|$ , then the  $H^*$ -colouring problem is NP-complete because of our choice of  $H$  and, consequently,  $H\text{-COL}$  is also NP-complete. Suppose that  $|E(H^*)| = |E(H)|$ . Let  $[x, y]$  be a double arc. Then Lemma 5.2.3 asserts

that  $N^+_H(x) = N^+_H(y)$ .

Let  $(I_2, u, v)$  be the in-z-indicator shown in figure 5.2.3(b), and let  $H^{**}$  be the result of applying the indicator construction with respect to  $(I_2, u, v)$  to  $H$ . By (2),  $E(H^{**})$  contains  $E(H)$ .

Suppose  $H^{**}$  has a loop. As  $H^{**}$  is vertex-transitive, every vertex is incident with a loop. In particular, there is a loop at  $y$ . Thus  $H$  contains a directed path of length two from  $x$  to  $y$ , say  $x, w, y$ . But, since  $N^+_H(x) = N^+_H(y)$ , if  $xw$  is an arc of  $H$ , so is  $yw$ . Therefore  $\text{undir}(H)$  is not a disjoint union of double arcs, which is a contradiction. Hence  $H^{**}$  is loopless. Since  $H$  does not map to a directed cycle of length greater than one and  $E(H^{**}) \supseteq E(H)$ , the digraph  $H^{**}$  does not retract to a directed cycle.

As above, we achieve the contradiction that the  $H$ -colouring problem is NP-complete when  $|E(H^{**})| > |E(H)|$ .

Suppose that  $E(H^{**}) = E(H)$ . Then, by Lemma 5.2.3, for any vertex  $t$  such that  $xt$  is an arc of  $H$ ,  $N^+_H(t) = N^+_H(y)$ . But  $yx$  is an arc of  $H$ , so  $tx$  must also be an arc of  $H$ . Since there exists at least one vertex  $t$  in  $N^+_H(x) \setminus \{y\}$ ,  $\text{undir}(H)$  is not a disjoint union of double arcs. This completes the proof of (4).

(5)  $H$  contains  $C_3^*$  (see figure 5.2.4).

Suppose not. Then the z-indicators  $(I_1, u, v)$  and  $(I_2, u, v)$  shown in figures 5.2.5(a) and 5.2.5(b), respectively, are good. Let  $H^*$  and  $H^{**}$  denote the result of applying the indicator construction with respect to  $(I_1, u, v)$  and  $(I_2, u, v)$ , respectively, to  $H$ . Both  $E(H^*)$  and  $E(H^{**})$  contain  $E(H)$ . If either containment is proper, we reach the contradiction that the  $H$ -colouring

problem is NP-complete by Lemma 3.1.8 and our choice of  $H$ .

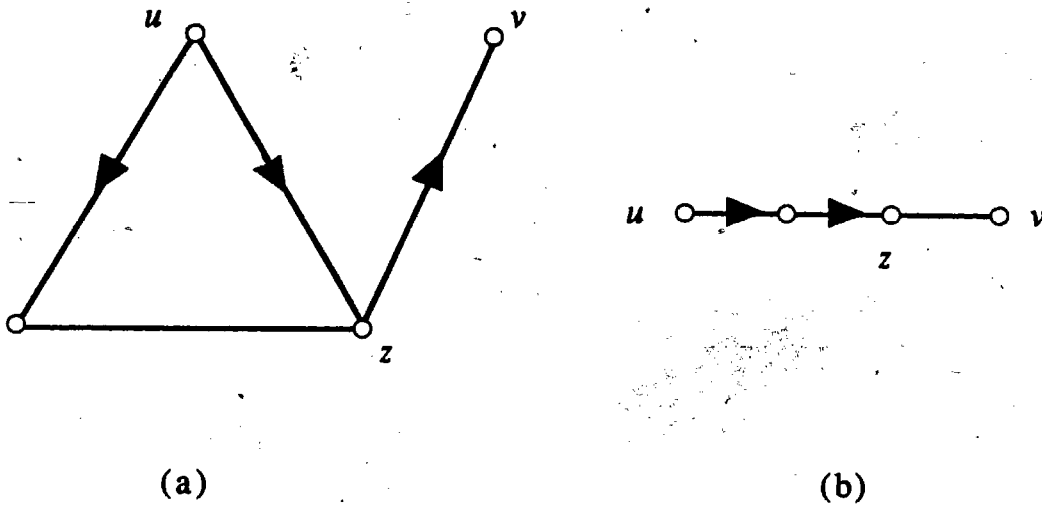


Figure 5.2.3. Indicators from (4).

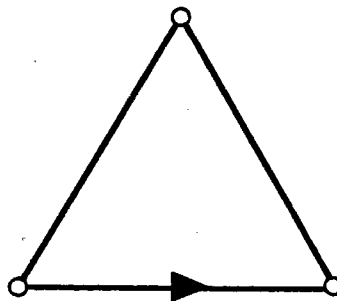


Figure 5.2.4. The digraph  $C_3^*$ .

Suppose that  $E(H)=E(H^*)=E(H^{**})$ , and let  $x, y, z$  be an undirected path of length two in  $H$ . Then, by Lemma 5.2.3,  $N^+_H(z)=N^+_H(x)$  and  $N^-_H(z)=N^-_H(x)$ . Since  $H$  does not contain  $C_3^*$ , neither  $zx$  nor  $xz$  is an arc of  $H$ . Therefore there is a retraction  $H \rightarrow H - z$  which maps  $z$  to  $x$ , contradicting

the fact that  $H$  is retract-free.



Figure 5.2.5. Indicators from (5).

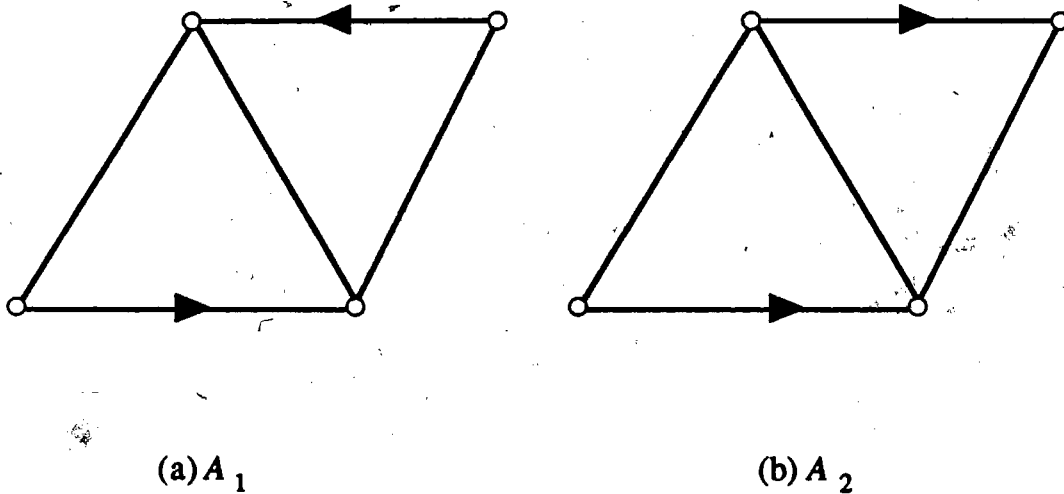


Figure 5.2.6. Subdigraphs from (6).

(6)  $H$  contains  $A_1$  or  $A_2$  (see figures 5.2.6(a) and (b), respectively).

Suppose not. Then the  $z$ -indicators  $(I_1, u, v)$  and  $(I_2, u, v)$  shown in figures 5.2.7(a) and 5.2.7(b), respectively, are good. Let  $H^*$  and  $H^{**}$  denote the result of applying the indicator construction with respect to  $(I_1, u, v)$  and  $(I_2, u, v)$ , respectively, to  $H$ . By Lemma 5.2.3, both  $E(H^*)$  and  $E(H^{**})$  contain  $E(H)$ . If either containment is proper, we have a contradiction. Suppose that

$E(H) = E(H^*) = E(H^{**})$ . Consider a homomorphic image of  $(I_1, u, v)$  in  $H$ , such that the vertex  $u$  maps to  $x$ , and  $z$  maps to  $y \neq x$  (the vertex  $y$  exists by (4)). By Lemma 5.2.3,  $N^+_H(x) = N^+_H(y)$ . Since there also exists a homomorphism of  $(I_2, u, v)$  to  $H$  such that  $u$  maps to  $x$  and  $z$  maps to  $y$ , we also have  $N^-_H(x) = N^-_H(y)$ . But then there is a retraction  $H \rightarrow H - x$  that maps  $x$  to  $y$ , which is a contradiction. This completes the proof of (6).

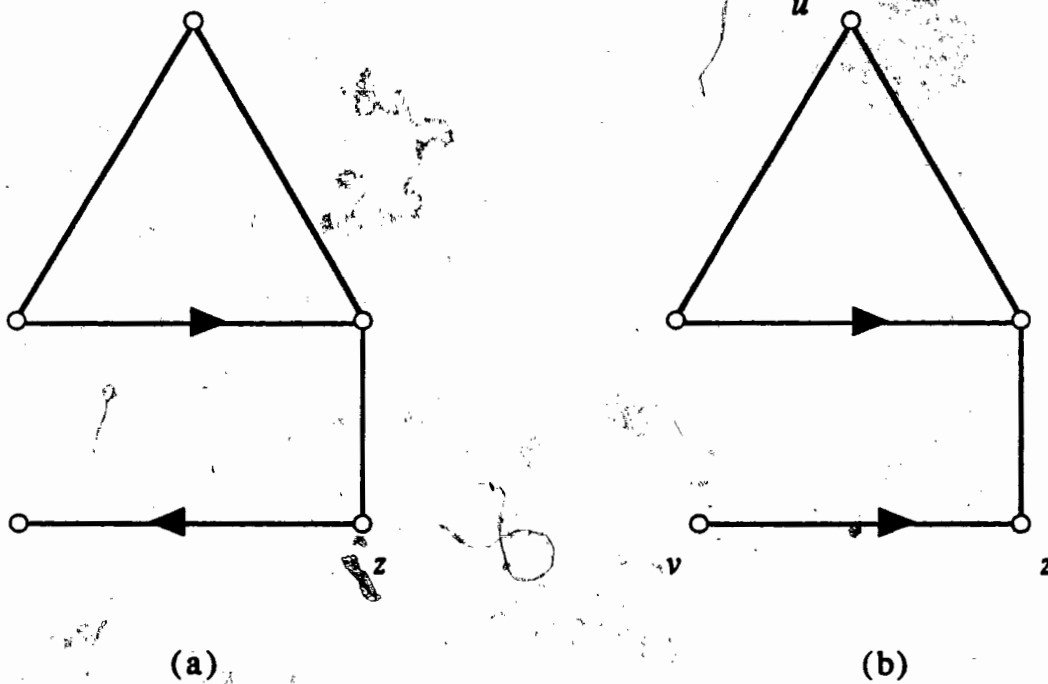


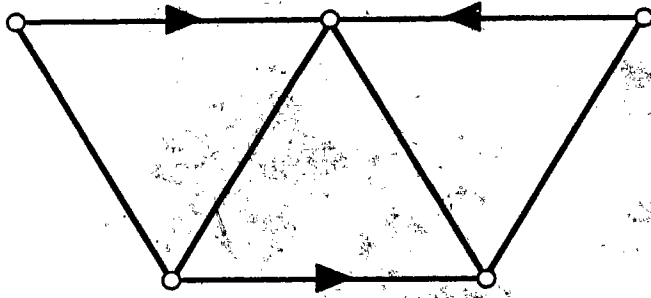
Figure 5.2.7. Indicators from (6).

(7)  $H$  contains at least one of  $X_1, X_2, X_3, X_4, X_5$  (see figures 5.2.8(a), (b), (c), (d), (e), respectively).

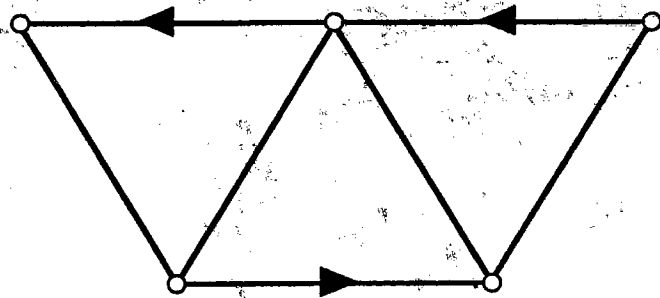
Suppose first that  $H$  contains  $A_1$ , but none of  $X_1, X_2, X_3$ . Then the indicators  $(I_1, u, v)$  and  $(I_2, u, v)$  shown in figures 5.2.9(a) and (b), respectively, are good. The remaining details are similar to those in (5), and



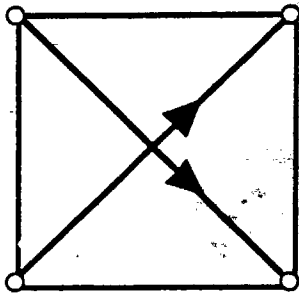
the reader should have little difficulty in completing the proof.



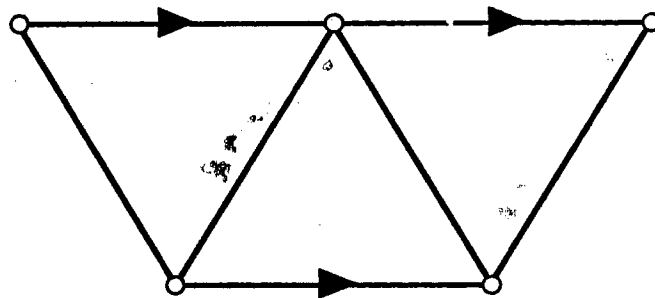
(a)  $X_1$



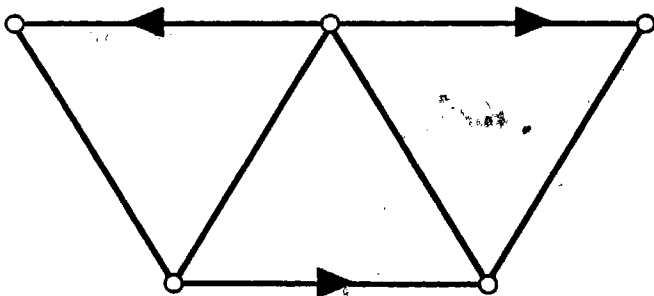
(b)  $X_2$



(c)  $X_3$



(d)  $X_4$



(e)  $X_5$

Figure 5.2.8. Subdigraphs from (7) through (11).

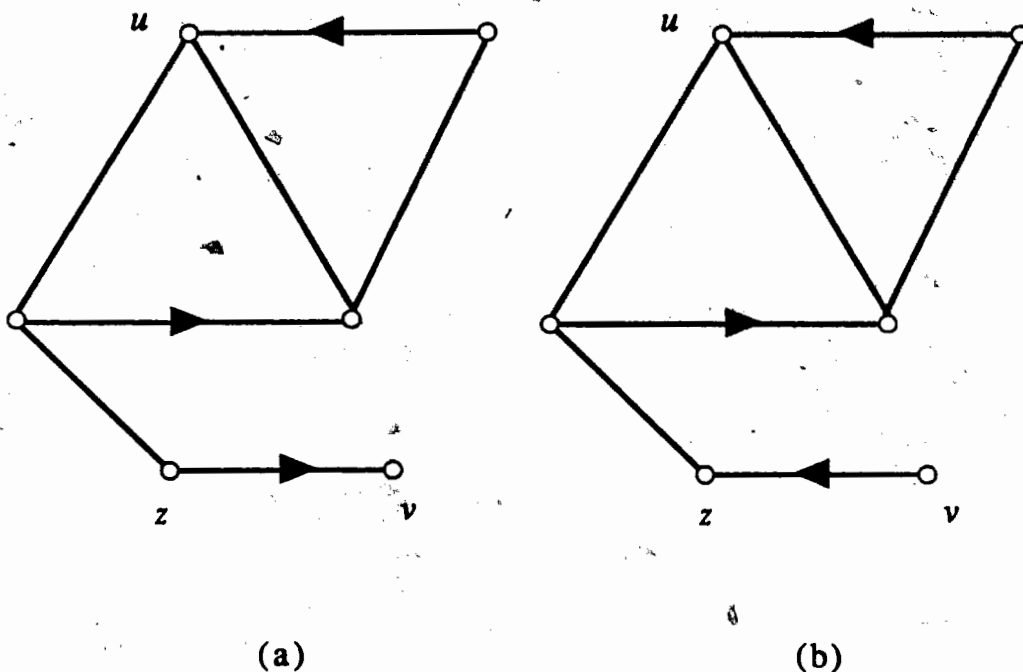


Figure 5.2.9. Indicators from (7).

Similarly, if  $H$  contains  $A_2$  but none of  $X_3, X_4, X_5$ , the indicators  $(I_3, u, v)$  and  $(I_4, u, v)$  shown in figures 5.2.9(a) and (b), respectively, are good. The details are again left for the reader.

(8)  $H$  contains neither  $X_1$  nor  $X_5$ .

We prove that if  $H$  contains  $X_1$  or  $X_5$ , then the  $H$ -colouring problem is NP-complete. Since  $X_5$  is the converse of  $X_1$ , it suffices to prove the result when  $H$  contains  $X_1$ .

Let  $x$  and  $y$  be vertices of  $H$  as shown in figure 5.2.11(a). Let  $(I_1, u, v)$  and  $(I_2, u, v)$  be the  $z$ -indicators shown in figures 5.2.11(b) and (c), respectively.

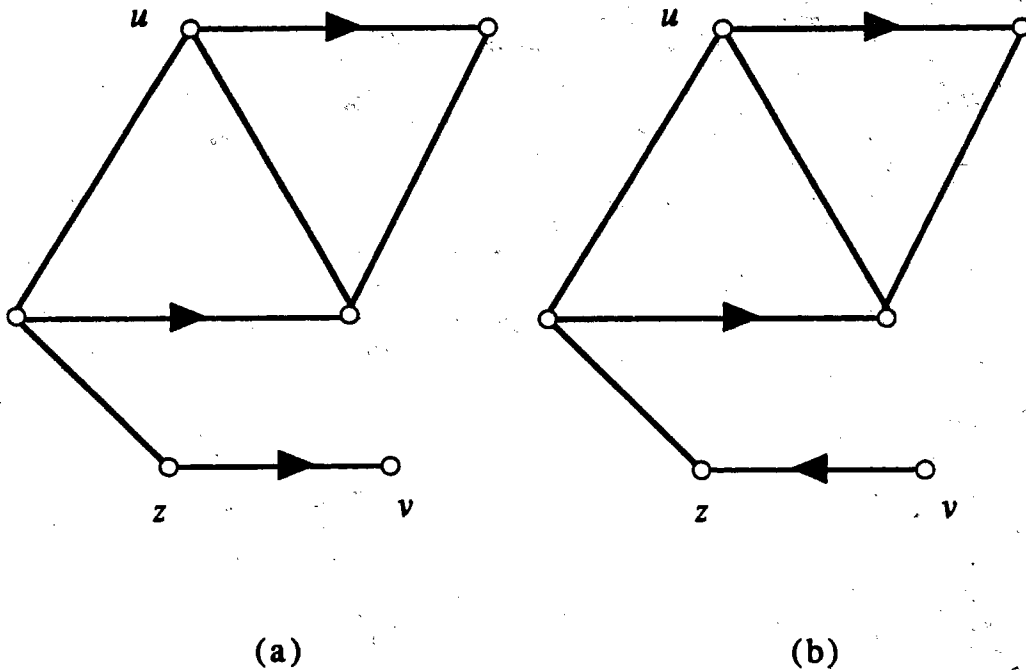


Figure 5.2.10. More indicators from (7).

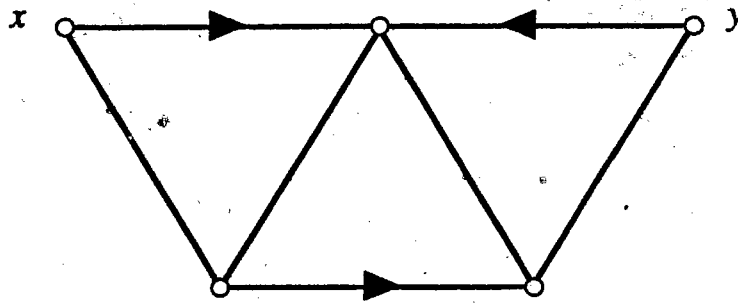
Suppose first that both indicators are good, and let  $H^*$  and  $H^{**}$  denote the result of applying the indicator construction with respect to  $(I_1, u, v)$  and  $(I_2, u, v)$ , respectively, to  $H$ . Both  $E(H^*)$  and  $E(H^{**})$  contain  $E(H)$ . If either containment is proper, the result follows from Lemma 3.1.8 and our choice of  $H$ . Hence assume  $E(H) = E(H^*) = E(H^{**})$ . Then, by Lemma 5.2.3,  $N^+_H(x) = N^+_H(y)$  and  $N^-_H(x) = N^-_H(y)$ . Therefore, either  $x$  and  $y$  are joined by a double arc, or there is a retraction  $H \rightarrow H - x$  which maps  $x$  to  $y$ . In the former case  $H$  contains an undirected five-cycle, and we are done by Lemma 4.1.1. The latter case contradicts the fact that  $H$  is retract-free.

Now suppose that one of the indicators is not good. We may assume that  $\text{undir}(H)$  is bipartite, otherwise  $H\text{-COL}$  is NP-complete by Lemma 4.1.1. Let  $C$  be a component of  $\text{undir}(H)$  and let  $(R, B)$  be a two-colouring of  $C$ . Then  $H[R]$  is a vertex-transitive graph with fewer vertices than  $H$ . If there exists a homomorphism of either  $I_1$  or  $I_2$  to  $H$  such that  $u$  and  $v$  map to the same vertex,  $H[R]$  contains a transitive triple. Therefore  $H[R]$  does not map to a directed cycle of length greater than one. By our choice of  $H$ ,  $H[R]\text{-COL}$  is NP-complete. Let  $r \in R$ . There exists an even integer  $k$  such that for every vertex  $x$  in  $R$ , there is an undirected  $(r, x)$ -walk of length  $k$ . Let  $P$  be (the equivalent digraph of) an undirected path of length  $k$ , with origin  $a$  and terminus  $b$ . Let  $H^-$  be the result of the applying the sub-indicator construction with respect to  $(P, a, b)$  and  $r$  to  $H$ . Then  $H^- = H[R]$ , and so the result follows from Lemma 3.1.9.

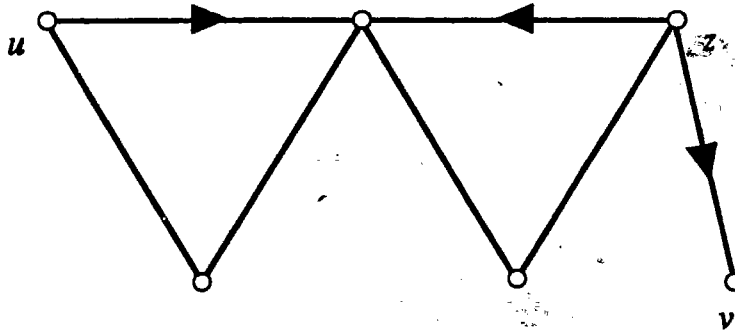
(9)  $H$  does not contain  $X_3$ .

Suppose  $H$  contains  $X_3$ . We show that  $H\text{-COL}$  is NP-complete. Let  $x$  and  $y$  be vertices of  $H$  as shown in figure 5.2.12(a). Let  $(I_1, u, v)$  be the in-z-indicator shown in figure 5.2.12(b), and let  $H^*$  be the result of applying the indicator construction with respect to  $(I_1, u, v)$  to  $H$ . Then  $E(H^*)$  contains  $E(H)$ .

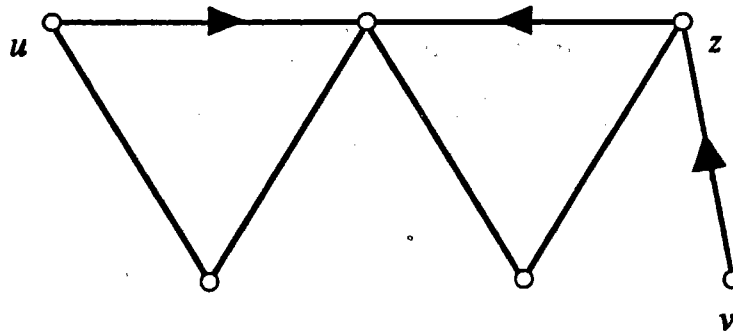
Suppose that  $E(H^*) = E(H)$ . Then, by Lemma 5.2.3,  $N^+_{H^*}(x) = N^+_H(y)$ . But  $xy$  is an arc of  $H$ , therefore  $H$  has a loop at  $y$ , which is a contradiction.



(a)



(b)



(c)

Figure 5.2.11. Configuration and indicators from (8).

Thus  $E(H^*)$  properly contains  $E(H)$ . If  $H^*$  has no loops, then the  $H$ -colouring problem is NP-complete by Lemma 3.1.8 and our choice of  $H$ . Hence we may assume that  $H^*$  has a loop. Thus  $H$  contains an undirected triangle or the graph shown in figure 5.2.13(a). In the former case  $H$ -COL is NP-complete by Lemma 4.1.1. In the latter case, let  $(I_2, u, v)$  be the

indicator shown in figure 5.2.13(b), and let  $H^{**}$  be the result of applying the indicator construction with respect to  $(I_2, u, v)$  to  $H$ . Note that  $E(H^{**})$  contains  $E(H)$ .

Suppose  $E(H^{**}) = E(H)$ . Then, as above, we see that  $H$  has a loop at  $y$ , which is a contradiction.

Thus  $E(H^{**})$  properly contains  $E(H)$ . If  $H^{**}$  has no loops, the  $H$ -colouring problem is NP-complete by Lemma 3.1.8 and our choice of  $H$ . Hence we may assume that  $H^{**}$  has a loop. Then  $H$  contains an undirected triangle, the digraph  $X_1$ , or the digraph shown in Figure 5.2.14 (a). In the first case  $H$ -COL is NP-complete. The second case contradicts (8). It remains to consider the last case. Let  $(I_3, u, v)$  be the symmetric indicator shown in figure 5.2.14(b). We may assume that  $H$  does not contain an undirected three-cycle; otherwise  $H$ -COL is NP-complete by Lemma 4.1.1. Let  $H^{***}$  be the digraph that results from applying the indicator construction with respect to  $(I_3, u, v)$  to  $H$ . It may be directly verified that  $H^{***}$  contains an undirected five-cycle. (Observe that since  $(I_3, u, v)$  is an sh-indicator, this means that the digraph  $H^{***}$  is superhard.). This completes the proof of (9).

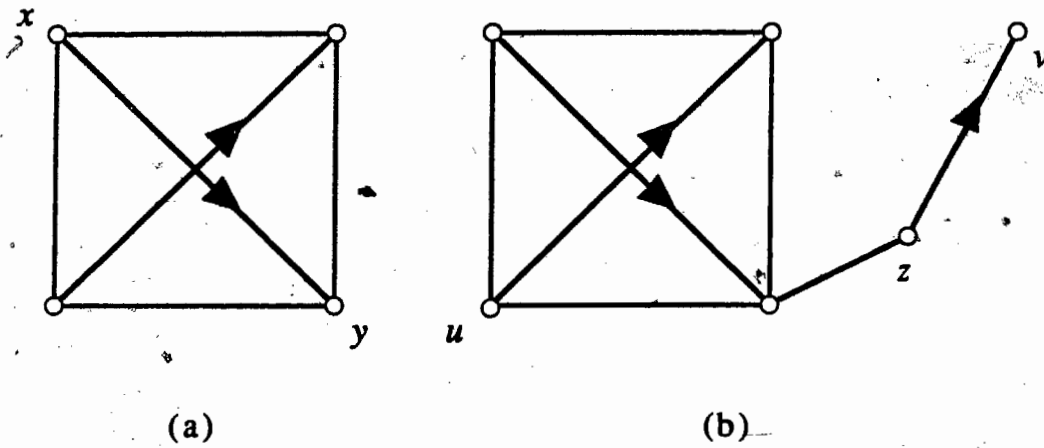


Figure 5.2.12. Configuration and indicator from (9).

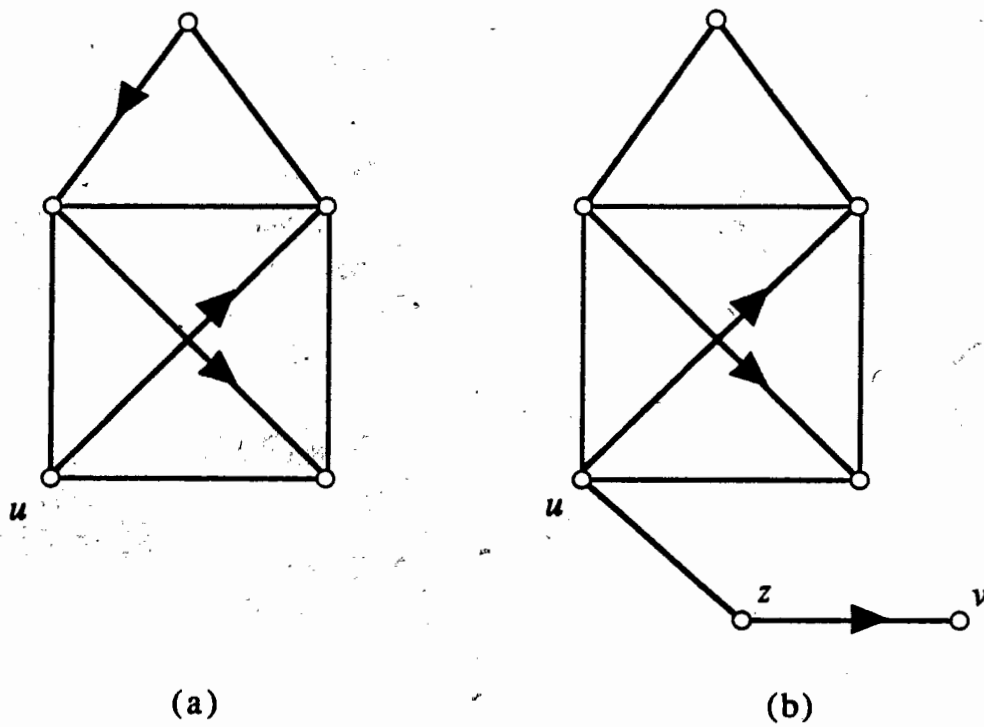
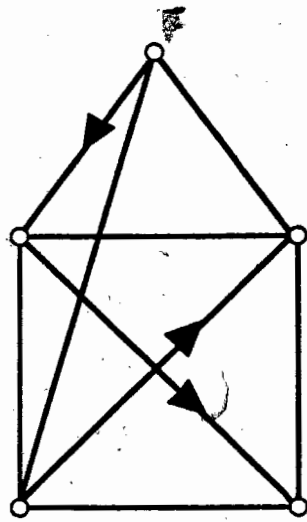
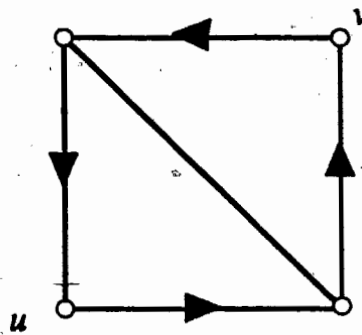


Figure 5.2.13. Subdigraph and indicator from (9).



(a)



(b)

Figure 5.2.14. Subdigraph and symmetric indicator from (9).

(10)  $H$  does not contain  $X_2$ .

Suppose to the contrary that  $H$  contains  $X_2$ . We show that  $H$ -COL is NP-complete. It may be assumed that  $H$  does not contain  $X_1$ ,  $X_3$  or  $X_5$ . Let  $x$  and  $y$  be vertices of  $H$  as shown in figure 5.2.15. Let  $(I, u, v)$  be the indicator shown in figure 5.2.9(a), and let  $H^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . Since neither  $X_1$  nor  $X_3$  is a subdigraph of  $H$ , the digraph  $H^*$  is loopless, unless  $H$  contains an undirected triangle, in which case we are done by Lemma 4.1.1. Note that  $E(H^*)$  contains  $E(H)$ . If the containment is proper, the result follows from Lemma 5.2.3 and our choice of  $H$ . Hence assume that  $E(H^*) = E(H)$ . Then by Lemma 5.2.3,  $N_H^+(x) = N_H^+(y)$ . Therefore  $yx$  is also an arc of  $H$ , and so  $H$  contains an undirected triangle. The result now follows from Lemma 4.1.1.



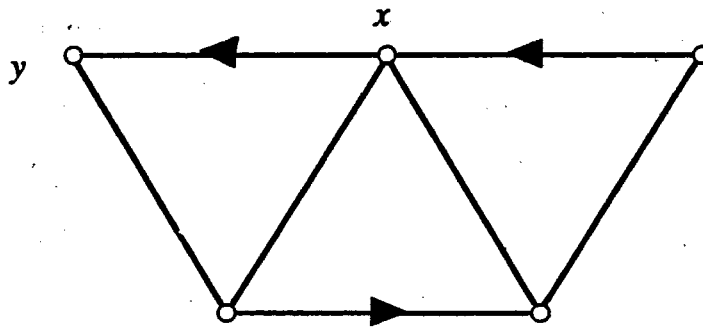


Figure 5.2.15. Configuration from (10).

(11)  $H$  does not contain  $X_4$ .

The proof is similar to (10). The indicator needed is shown in figure 5.2.9(b). The details are omitted.

Hence the digraph  $H$  cannot exist. This completes the proof of Theorem 5.2.4. ■

**5.2.5. Corollary.** Let  $H$  be an arc-transitive digraph with at least one arc. Then the  $H$ -colouring problem is NP-complete, unless  $H$  admits a retraction to  $C_n$  or  $P_1$ . In the latter case  $H$ -COL is polynomial.

**Proof.**

We have already noted the second statement (cf. the comment following Theorem 2.2.2). Let  $H$  be an arc-transitive digraph with at least one arc. We may assume without loss of generality that  $H$  has no isolated vertices. Then either  $H$  is smooth, or every vertex of  $H$  is a source or a sink. In the former case  $H$  is vertex-transitive, so the result follows from Theorem 5.2.4. In the latter case  $P_1$  is a retract. This completes the proof. ■

Since a connected vertex-transitive digraph  $H$  is strong, it admits a retraction to a directed cycle if and only if the length of every directed cycle in  $H$  is divisible by the directed girth of  $H$ . When  $H$  is a Cayley digraph on a finite group we are able to give another characterisation.

Let  $\Gamma$  be a finite group. We denote by  $\Gamma(S)$  the Cayley digraph with symbol  $S$ . That is, the digraph with vertex-set  $\Gamma$  and arc-set  $E(\Gamma(S)) = \{xy \mid yx^{-1} \in S\}$ .

It is well known that a Cayley digraph  $\Gamma(S)$  is connected just if the set  $S$  generates  $\Gamma$ . Since Cayley digraphs are vertex-transitive (because, for any  $a \in \Gamma$ , the mapping  $x \rightarrow xa$  is an automorphism), a connected Cayley digraph is strong.

**5.2.6. Lemma:** Suppose  $S$  generates  $\Gamma$  and  $H$  is a non-trivial normal subgroup of  $\Gamma$  of index  $h$ . If  $S$  is a union of cosets of  $H$ , then the Cayley digraph  $\Gamma/H(S/H)$  is a retract of  $\Gamma(S)$  ( $S/H = \{Hx \mid Hx \text{ is a subset of } S\}$ ).

**Proof.**

Let  $S = Hx_1 \cup Hx_2 \cup \dots \cup Hx_k$ , and let the collection of all cosets of  $H$  be  $Hx_1, Hx_2, \dots, Hx_h$ . We first show that  $\Gamma/H(S/H)$  is an induced subdigraph of  $\Gamma(S)$ . There is an arc from  $x_i$  to  $x_j$  in  $\Gamma(S)$  if and only if  $x_j x_i^{-1} \in Hx_m$ , for some  $m$  between 1 and  $k$ ; equivalently,  $x_j x_i$  is an arc if and only if  $Hx_j Hx_i^{-1} = Hx_m$ . Therefore  $\{x_1, x_2, \dots, x_h\}$  induces a copy  $T$  of  $\Gamma/H(S/H)$  in  $\Gamma(S)$ . It remains to show that there is a retraction  $f: \Gamma(S) \rightarrow T$ .

For each  $g \in \Gamma$ , let  $f(g)$  be the unique vertex  $x_i$  such that  $g$  is in  $Hx_i$ . Then  $f$  fixes  $V(T)$ . Let  $ab$  be an arc of  $\Gamma(S)$  and  $f(a)=x_s$  and  $f(b)=x_t$ . It is not hard to see that  $Hba^{-1}=Hx_sx_t^{-1}$ . Therefore  $x_sx_t^{-1}$  is in  $S$ , and  $x_sx_t$  is an arc of  $T$ .

Hence  $f$  is a homomorphism. This completes the proof. ■

**5.2.7. Lemma.** Let  $S$  generate  $\Gamma$ . There is a homomorphism of  $\Gamma(S)$  to  $C_h$  if and only if  $S$  is contained in a coset of a normal subgroup of  $\Gamma$  with index  $h$ .

**Proof.**

( $\Rightarrow$ ) Suppose  $f: \Gamma(S) \rightarrow C_h$ . Without loss of generality the identity  $e$  of  $\Gamma$  has  $f(e)=1$ . Let  $H=f^{-1}(1)$ . Let  $a, b$  be in  $H$ . Since  $\Gamma(S)$  is connected, there is a directed  $(e,a)$ -path of length zero modulo  $h$ , and a directed  $(e,b)$ -path of length zero modulo  $h$ . Consequently there is a directed  $(b,ab)$ -path of length zero modulo  $h$ , and a directed  $(e,ab)$ -walk of length zero modulo  $h$ . Since a  $C_h$ -colouring of a connected digraph is completely determined by the colour assigned to a single vertex, we deduce that  $f(ab)=1$ , that is,  $ab \in H$ . Hence  $H$  is a subgroup.

Let  $g \in \Gamma$  and let  $x \in H$ . There exists a directed  $(e,x)$ -path of length zero modulo  $h$ , a directed  $(e,g)$ -path of length  $r$  modulo  $h$ , and a directed  $(e,g^{-1})$ -path of length  $(-r)$  modulo  $h$  (because there is a directed  $(g^{-1},e)$ -path of length  $r$  modulo  $h$  and a closed directed walk containing both  $e$  and  $g^{-1}$  has length zero modulo  $h$ ). Therefore there is a directed  $(e, g^{-1}xg)$ -walk of length zero modulo  $h$ , that is  $f(g^{-1}xg)=1$ . Thus  $H$  is normal.

Let  $s$  be in  $S$ . The automorphism  $x \rightarrow xs$  maps each  $f^{-1}(i)$  to  $f^{-1}(i+1)$ ,  $i=1, 2, \dots, h$ , with addition modulo  $h$ . Hence each colour class of the  $C_h$ -colouring is a coset of  $H$ . Since there are  $h$  cosets and  $S$  is contained in

$f^1(2)$ , the proof of the implication is complete.

( $\Leftarrow$ ) Without loss of generality  $S=Hx$ . The result follows from Lemma 5.2.6 (the graph  $\Gamma/H(S/H)$  is connected because  $\Gamma(S)$  is strong). ■

**5.2.8. Corollary.** Suppose that  $S$  generates  $\Gamma$ . The core of  $\Gamma(S)$  is a directed cycle if and only if  $S$  is contained in a coset of a normal subgroup of index equal to the directed girth of  $\Gamma(S)$ . ■

We conclude this section by mentioning the group-theoretic analog of Lemma 5.2.6: If  $H$  is a normal subgroup of a finite group  $\Gamma$ , then  $\Gamma/H$  is cyclic if and only if there exists an  $x \in \Gamma$  such that  $\Gamma = \langle Hx \rangle$ .

### 5.3. Superdigraphs of Bipartite Graphs.

Let  $H$  be a given directed graph. By proposition 4.1.1, if  $undir(H)$  is not bipartite,  $H$ -COL is NP-complete. Thus, in order to complete the classification by complexity of all digraphs, it remains to consider those digraphs for which  $undir(H)$  is bipartite. In this section we prove some NP-completeness results concerning digraphs for which  $undir(H)$  is bipartite, and is also a spanning subdigraph. That is, digraphs  $H$  constructed from (the equivalent digraph of) an undirected bipartite graph by adding some arcs.

We begin by proving some general results in section 5.3.1. Each theorem classifies infinitely many  $H$ -colouring problems. We are, unfortunately, unable to completely classify all directed graphs such that  $undir(H)$  is spanning and connected. In section 5.3.2 we restrict our focus a little more, and introduce the family of "partitionable" digraphs. We are able to give a complete classification in this latter class.

Let  $D$  be a directed graph. It follows from the sub-indicator construction (with respect to  $(C_2, 0, free)$ ) that if  $V(undir(D))$  induces a subdigraph  $D^-$  for which  $D^-$ -COL is NP-complete, then  $D$ -COL is also NP-complete. Hence the implications of the NP-completeness results in this section extend to directed graphs for which  $undir(D)$  is not spanning. (It is, however, possible for the  $D$ -colouring problem to be NP-complete even though  $D^-$ -COL is polynomial.)

Since  $undir(D)$  is spanning, the digraph  $D$  is smooth. Thus each theorem in

this section verifies Conjecture 1.1 for infinitely many directed graphs.

### 5.3.1. Sufficient Conditions.

Let  $D'$  be a directed graph and let  $(R, B)$  be a two-colouring of  $\text{undir}(D')$ . (We often refer to these colour classes as "red" and "blue", respectively.) If  $D$  is a subdigraph of  $D'$  and all vertices of  $D$  are in the same colour class, we say that  $D$  is *monochromatic*. If it is important to distinguish which of the colour classes  $R$  and  $B$  contains  $D$ , we use the terms "red  $D$ " and "blue  $D$ ", respectively.

It is clear that  $C_2$  is a retract of any bipartite digraph that contains it. Hence we may assume throughout this section that  $H$  is not bipartite; otherwise the  $H$ -colouring problem is polynomial. We are able to prove, for non-bipartite digraphs, that the problem is NP-complete in a wide variety of circumstances.

We begin this section by considering those digraphs  $D$  for which there exists a two-colouring of  $\text{undir}(D)$  with no monochromatic  $C_3$  or no monochromatic transitive triple.

**5.3.1. Lemma.** Let  $D$  be a digraph and  $C$  be a component of  $\text{undir}(D)$ . Suppose that, with respect to the unique two-colouring  $(R, B)$  of  $C$ , the  $D[R]$ -colouring problem (resp.  $D[B]$ -colouring problem) is NP-complete. Then  $D\text{-COL}$  is also NP-complete.

**Proof.**

It suffices to prove the result when the  $D[R]$ -colouring problem is NP-complete. Let  $v \in R$ . Since  $C$  is connected, there is an (even) integer  $k$  such that for every vertex  $x \in R$  there is an undirected  $(v,x)$ -walk of length  $k$ . Let  $P$  denote the equivalent digraph of an undirected path of length  $k$ , with  $V(P) = \{0, 1, \dots, k\}$ , and  $E(P) = \{[i, i+1] : i=0, 1, \dots, k-1\}$ . The result of applying the sub-indicator construction with respect to  $(P, 0, k)$ , and  $v$  to  $D$  is the digraph  $D[R]$ . The result now follows. ■

A directed graph that plays a central role in this section is *the basic triangle*  $C_3^*$ , which is constructed from an undirected path of length two, say  $x, y, z$ , by adding the arc  $zx$ , which is called *the single arc of  $C_3^*$*  (cf. figure 5.2.4).

**5.3.2. Theorem.** Let  $D$  be a digraph that contains a basic triangle. Suppose there exists a two-colouring  $(R, B)$  of  $\text{undir}(D)$  such that  $D[R]$  is an independent set and  $D[B]$  contains no  $C_3$  (resp. no transitive triple). Then  $D\text{-COL}$  is NP-complete.

**Proof.**

Suppose  $D[B]$  has no  $C_3$  (the proof being similar when  $D[B]$  has no transitive triple). The transformation is from ONE-IN-THREE-SAT without negated variables. Suppose an instance of ONE-IN-THREE-SAT without negated variables is given, with variables  $x_1, x_2, \dots, x_n$  and clauses  $K^1, K^2, \dots, K^m$ . Let  $X$  be the digraph shown in figure 5.3.1 with  $p = 1$ . Construct a digraph  $G$  from  $\{x_1, x_2, \dots, x_n\}$  and  $m$  copies of the digraph  $X$ , say  $X_1, X_2, \dots, X_m$ , as follows. If  $K^i = x_j \vee x_k \vee x_l$ , identify the vertices  $l_1, l_2, l_3$  in  $X_i$  with  $x_j, x_k, x_l$ , respectively.

Clearly, the digraph  $G$  can be constructed in polynomial time.

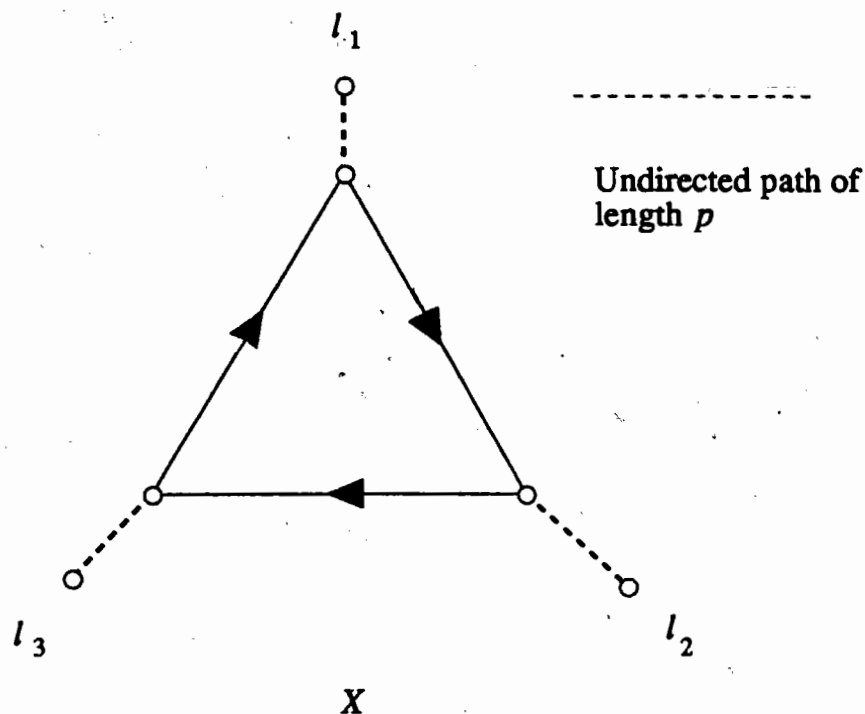


Figure 5.3.1. The digraph  $X$ .

**CLAIM.** The digraph  $G$  is  $D$ -colourable if and only if there is a truth assignment such that each clause contains exactly one true variable.

**PROOF.**

( $\Rightarrow$ ) Suppose  $G$  is  $D$ -colourable. Consider a copy  $X_i$  of  $X$ . Since  $D[R]$  is an independent set, at most one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Furthermore, since  $D[B]$  has no  $C_3$ , at least one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Therefore, exactly one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Define a truth assignment by setting  $x_j = T$  just if  $x_j$  is coloured by a vertex in  $B$ . By the above argument, each clause contains precisely one true variable.



( $\Leftarrow$ ) Suppose there is a truth assignment such that every clause contains exactly one true variable. Let  $A$  be a fixed basic triangle. Then  $A$  has necessarily the vertex set  $\{b_1, b_2, r\}$ , where  $b_1, b_2 \in B, r \in R$ , and single arc  $b_1 b_2$ . Define a partial colouring of  $G$  by assigning  $colour(x_j)=b_1$  just in case  $x_j=T$ , and  $colour(x_j)=r$  otherwise. It may be verified that, given a copy  $X_i$  of  $X$  and a partial colouring of  $l_1, l_2, l_3$  with colours from  $\{r, b_1\}$  in which exactly one of  $l_1, l_2, l_3$  is coloured  $b_1$ , the partial colouring can be extended to a  $D$ -colouring of  $X_i$  (in fact, to an  $A$ -colouring of  $X_i$ ). Hence  $G$  is  $D$ -colourable.

The claim implies that  $D$ -COL is NP-complete. ■

**5.3.3. Theorem.** Let  $D$  be a digraph such that  $undir(D)$  is a spanning subdigraph. Suppose that  $D$  contains  $C_3$ , but no two-colouring of  $undir(D)$  has a monochromatic  $C_3$ . Then the  $D$ -colouring problem is NP-complete.

**Proof.**

Let  $(R, B)$  be a two-colouring of  $undir(D)$ . There are two cases to consider.

CASE 1. Some component of  $undir(D)$  induces a subdigraph which contains three-cycles  $r_1, r_2, b, r_1$  and  $b_1, b_2, r, b_1$ , where  $r_1, r_2 \in R$ , and  $b_1, b_2 \in B$ .

The transformation is from NOT-ALL-EQUAL-THREE-SAT without negated variables. Suppose an instance of NOT-ALL-EQUAL-THREE-SAT without negated variables is given, with variables  $x_1, x_2, \dots, x_n$  and clauses  $K^1, K^2, \dots, K^m$ . Let  $Y$  be the digraph shown in figure 5.3.1, where  $p$  is chosen to be an odd integer such that there is an undirected walk of length  $p$  from  $r$  to each of  $b, b_1, b_2$ , and from  $b$  to each of  $r, r_1, r_2$ . Construct a digraph  $G$  from

$\{x_1, x_2, \dots, x_n\}$  and  $m$  copies of  $Y$ , say  $Y_1, Y_2, \dots, Y_m$ , as follows. If  $K^i = x_j \vee x_k \vee x_l$ , then identify vertices  $l_1, l_2, l_3$  in  $Y_i$  with  $x_j, x_k, x_l$ , respectively. Clearly the digraph  $G$  can be constructed in polynomial time.

**CLAIM.** The digraph  $G$  is  $D$ -colourable if and only if there is a satisfying truth assignment in which each clause contains at least one true variable and at least one false variable.

**PROOF.**

( $\Rightarrow$ ) Suppose that  $G$  is  $D$ -colourable. Consider  $Y_i$ . Since  $D$  has no monochromatic  $C_3$ , not all of the vertices  $l_1, l_2, l_3$  can be coloured by members of  $R$ . Similarly, these vertices can not all be coloured by members of  $B$ . Thus two of  $l_1, l_2, l_3$  are coloured by members of  $R$  and the remaining one is coloured by a vertex in  $B$ , or vice-versa. Define a truth assignment by setting  $x_j = T$  just if vertex  $x_j$  is coloured by a member of  $R$ . By the above argument, every clause contains a true variable and a false variable.

( $\Leftarrow$ ) Suppose there is a truth assignment such that every clause contains at least one true variable, and at least one false variable. Define a partial colouring of  $G$  by assigning  $colour(x_j) = b$  just in case  $x_j = T$ , and  $colour(x_j) = r$  otherwise. It may be directly verified that, given a copy  $Y_i$  of  $Y$  and a partial colouring of  $l_1, l_2, l_3$  with colours from  $\{r, b\}$  in which at least one, but not all of  $l_1, l_2, l_3$ , is coloured  $b$ , the partial colouring can be extended to a  $D$ -colouring of  $Y_i$ . Hence  $G$  is  $D$ -colourable.

This completes the proof of case 1.

CASE 2. The subdigraph induced by every component of  $\text{undir}(D)$  contains only three-cycles with two vertices from  $R$  and one vertex from  $B$ , or vice-versa.

By switching colours in some components of  $\text{undir}(D)$ , it may be assumed that every three-cycle has two vertices in  $R$  and one vertex in  $B$ . Let  $A = r_1, r_2, b$  be a fixed copy of  $C_3$  in the subdigraph induced by some component of  $\text{undir}(D)$ , where  $r_1, r_2 \in R$  and  $b \in B$ . Let  $Y$  be the digraph shown in figure 5.3.1, and choose  $p$  such that there is an undirected walk of length  $p$  from  $b$  to  $r_1$  and to  $r_2$ . The transformation is once again from ONE-IN-THREE-SAT without negated variables. Suppose an instance of ONE-IN-THREE-SAT without negated variables is given, with variables  $x_1, x_2, \dots, x_n$  and clauses  $K^1, K^2, \dots, K^m$ . Construct a digraph  $G$  from  $\{x_1, x_2, \dots, x_n\}$  and  $m$  copies of  $Y$ , say  $Y_1, Y_2, \dots, Y_m$ , as follows. If  $K^i = x_j \vee x_k \vee x_l$ , then identify vertices  $l_1, l_2, l_3$  in  $Y_i$  with  $x_j, x_k, x_l$  respectively. Clearly, the digraph  $G$  can be constructed in polynomial time.

CLAIM. The digraph  $G$  is  $D$ -colourable if and only if there is a truth assignment such that each clause contains exactly one true variable.

PROOF.

( $\Rightarrow$ ) Suppose  $G$  is  $D$ -colourable. Consider a copy  $Y_i$  of  $Y$ . Since  $D[R]$  has no three-cycle, at least one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Since  $D[B]$  has no three-cycle, at least one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Moreover, since there is no three-cycle which has two vertices in  $B$ , exactly one of  $l_1, l_2, l_3$  is coloured by a vertex in  $B$ . Define a truth assignment by setting  $x_j = T$  just if  $x_j$  is coloured by a vertex in  $B$ . By the above argument, each clause contains precisely one true variable.

( $\Leftarrow$ ) Suppose there is a truth assignment such that every clause contains exactly one variable with the value T. Define a partial colouring of  $G$  by assigning  $colour(x_j)=r_1$  just in case  $x_j=T$ , and  $colour(x_j)=b$  otherwise. It may be directly verified that, given a copy  $Y_i$  of  $Y$ , and a partial colouring of  $l_1, l_2, l_3$  with colours from  $\{r_1, b\}$  in which exactly one of  $l_1, l_2, l_3$  is coloured  $r_1$ , the partial colouring can be extended to a  $D$ -colouring of  $Y_i$ . Hence  $G$  is  $D$ -colourable.

This completes the proof of case 2. ■

The next result generalises Theorem 5.3.2.

**5.3.4. Theorem.** Let  $D$  be a digraph such that  $undir(D)$  is a spanning subdigraph. Suppose that  $D$  contains a basic triangle. If  $D$  does not contain a monochromatic transitive triple, the  $D$ -colouring problem is NP-complete.

**Proof.**

The proof is similar to the proof of Theorem 5.3.3. The component which corresponds to each clause is shown in figure 5.3.2. The details are left to the reader. ■

We now turn our attention to superdigraphs of undirected bipartite graphs that contain the digraph  $B_3$ , the directed triangle with all arcs bypassed, defined in section 4.3. (Recall that an arc  $xy$  is said to be *bypassed* if there is a vertex  $z$  such that the arcs  $xz$  and  $zy$  both exist. The vertex  $z$  is called a *bypass vertex* for  $xy$ .) We have already proved that  $B_3$  is superhard with respect to the property " $G$  has no directed two-cycle" (cf. Lemma 4.3.5). We show that  $B_3$  is also

superhard with respect to the property " $undir(G)$  is spanning and connected".

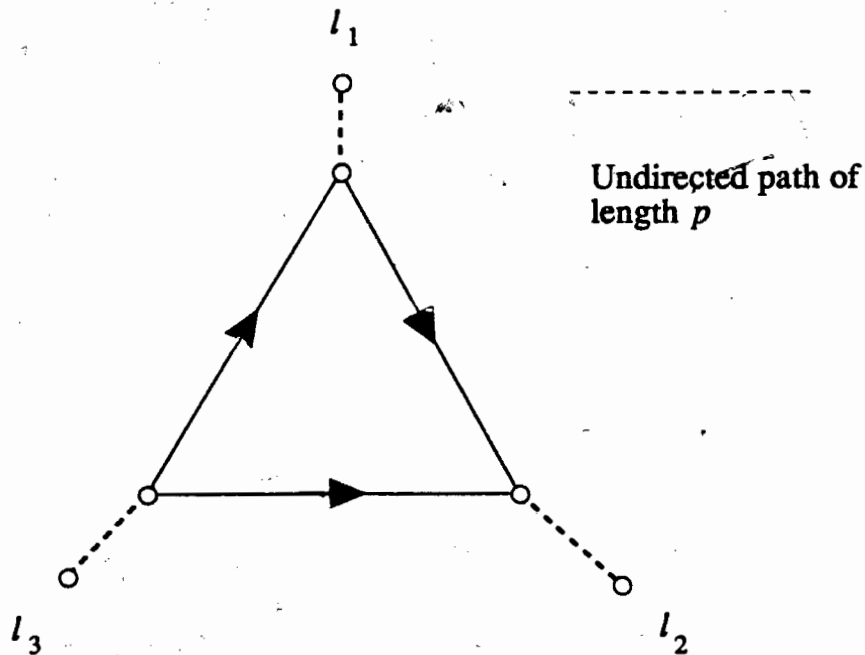


Figure 5.3.2: Clause component for Theorem 5.3.4.

**5.3.5. Lemma.** Let  $D$  be a digraph such that  $undir(D)$  is a connected spanning subdigraph. Suppose there exists a two-colouring  $(R, B)$  of  $undir(D)$  such that there is a monochromatic  $C_3$  in which all arcs are bypassed. If the bypass vertices  $x_1, x_2, x_3$  are all in the same colour class, then  $D-COL$  is NP-complete.

**Proof.**

Without loss of generality assume the  $C_3$  in the statement of the theorem is red. If  $x_1, x_2, x_3 \in R$ , then the  $D[R]$ -colouring problem is NP-complete by

Lemmas 5.3.1 and 4.3.5. Consequently the  $D$ -colouring problem is also NP-complete. Suppose that  $x_1, x_2, x_3$  are in  $B$ . Then the  $D$ -colouring problem is NP-complete by Lemma 4.3.4 (each  $x_i$  is joined to every vertex on the three-cycle by an undirected odd path because  $\text{undir}(G)$  is spanning and connected). ■

**5.3.6. Theorem.** Let  $D$  be a digraph such that  $\text{undir}(D)$  is spanning and connected. Suppose that  $(R, B)$  is a two-colouring of  $\text{undir}(D)$  such that there is a monochromatic  $C_3$  in which all arcs are bypassed. Then  $D$ -COL is NP-complete.

**Proof.**

There are three cases to consider.

CASE 1. There is a monochromatic  $C_3$  in which all arcs are bypassed, and the bypass vertices are all in the same colour class.

Then  $D$ -COL is NP-complete by Lemma 5.3.5.

CASE 2. There are monochromatic three-cycles  $C, C'$  with bypass vertices  $a, b, c$ , and  $a', b', c'$ , respectively, where  $a, b, c' \in R$ , and  $a', b', c \in B$ .

It may be assumed that case 1 does not hold simultaneously. The transformation is from NOT-ALL-EQUAL-THREE-SAT without negated variables. Let  $Z$  be the digraph shown in figure 5.3.3, where  $p$  is chosen so that any two vertices in the same colour class are joined by an undirected walk of length  $p$  (the (even) length  $p$  exists because  $\text{undir}(D)$  is spanning and connected). Let an instance of NOT-ALL-EQUAL-THREE-SAT without negated variables be given, with variables  $x_1, x_2, \dots, x_n$  and clauses  $K^1, K^2, \dots, K^m$ . Construct a digraph  $G$  from  $\{x_1, x_2, \dots, x_n\}$  and  $m$  copies of  $Z$ , say

$Z_1, Z_2, \dots, Z_m$ , as follows. Suppose  $K_i = x_j \vee x_k \vee x_l$ . Then identify the vertices  $l_1, l_2, l_3$  in  $Z_i$  with  $x_j, x_k, x_l$ , respectively. Clearly, the digraph  $G$  can be constructed in polynomial time.

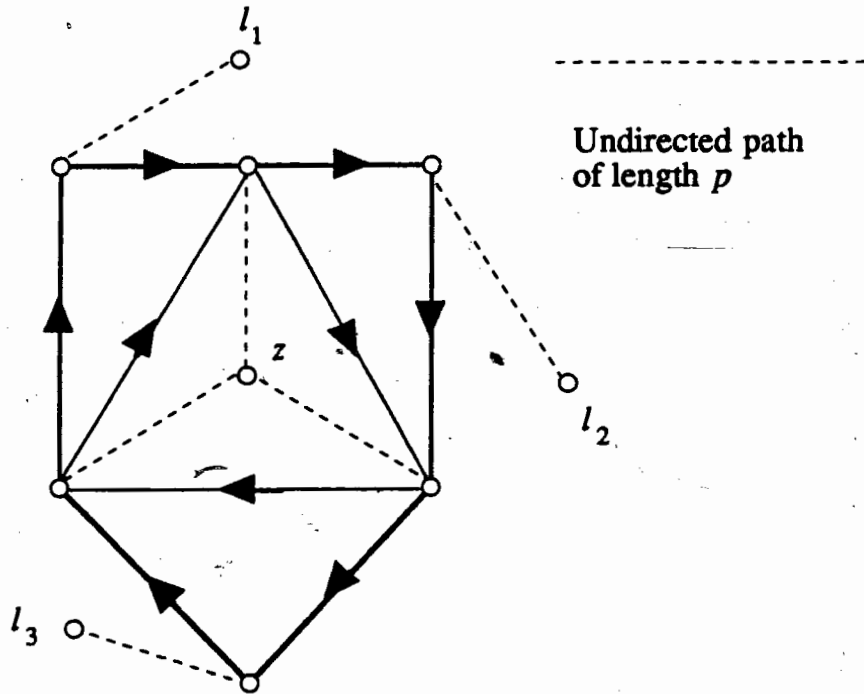


Figure 5.3.3. The digraph  $Z$ .

**CLAIM.** The digraph  $G$  is  $D$ -colourable if and only if there is a satisfying truth assignment in which each clause has at least one true variable, and at least one false variable.

**PROOF.**

Consider a  $D$ -colouring of  $G$ . It is not hard to see that the three-cycle in each copy of  $Z$  must be monochromatic (it must map into the same colour class as  $z$ ). The remaining details are similar to the proof of the analogous claim

in Theorem 5.3.3, and are left to the reader.

This completes the proof of case 2.

CASE 3. Every monochromatic  $C_3$  with bypasses on all arcs has red two bypass vertices and one blue bypass vertex, or vice-versa.

The transformation is from ONE-IN-THREE-SAT without negated variables. The construction of the digraph  $G$  is identical to case 2. The remaining details are similar to those in the analogous case of Theorem 5.3.3, and are left to the reader.

All three cases have been considered, and the result follows. ■

**5.3.7. Theorem.** Let  $D$  be a digraph such that  $undir(D)$  is a connected spanning subdigraph. If  $D$  contains  $B_3$ , then the  $D$ -colouring problem is NP-complete. (That is, the digraph  $B_3$  is superhard with respect to the property " $undir(D)$  is spanning and connected".)

**Proof.**

By Theorem 5.3.6, it suffices to consider the case when no copy of  $B_3$  in  $D$  contains a monochromatic  $C_3$ . The proof is identical to the proof of Theorem 5.3.3, except that bypasses must be added to all arcs of the three cycle in the digraph in figure 5.3.1. The details may be easily supplied by the reader. ■



Let  $n > 1$ , and let  $F_n$ , the  $n$ -fan, be the digraph with vertex set  $\{0, 1, \dots, n\}$ , and arc set  $\{01, 10, 02, 20, \dots, 0n, n0\} \cup \{12, 23, \dots, (n-1)n\}$ . The following corollary is useful in the next section.

**5.3.8. Corollary.** For any  $n > 3$ , the digraph  $F_n$  is superhard with respect to " $undir(D)$  is spanning and connected".

**Proof.**

If  $n > 3$ , then  $F_n$  contains  $B_3$ . ■

### 5.3.2. Partitionable Digraphs.

In this subsection we introduce a sub-class of superdigraphs of bipartite graphs which we call "partitionable". We give a complete classification by complexity of the digraphs in this class.

Throughout this section, a *single arc* of a digraph  $D$  is an arc  $xy \in E(D)$  such that  $yx \notin E(D)$ .

We say that a digraph  $D$  is *partitionable* if the following two conditions are satisfied:

- (i)  $undir(D)$  is a spanning subdigraph of  $D$ , and
- (ii) There is a two-colouring of  $undir(D)$ , such that every single arc is monochromatic.

Another way to express condition (ii) is the following: There exists a two-colouring of  $\text{undir}(D)$  such that all undirected edges are between the colour classes and all single arcs are within the colour classes. Partitionable digraphs are precisely the digraphs obtained from (the equivalent digraph of) an undirected bipartite graph by adding monochromatic arcs.

Let  $D$  be a partitionable digraph. We have previously noted that  $D$  may be assumed to be non-bipartite, otherwise  $D\text{-COL}$  is polynomial (cf. page 112). In the remainder of this section we show that the  $D$ -colouring problem is NP-complete for all non-bipartite partitionable digraphs  $D$ . We prove the following theorem.

**5.3.9. Theorem.** Let  $D$  be a partitionable digraph. If  $D$  is bipartite, then  $D\text{-COL}$  is polynomial. Otherwise ( $D$  contains an oriented odd cycle),  $D\text{-COL}$  is NP-complete.

The proof requires some preliminary lemmas. Recall the digraph  $T_1$  defined in section 5.2 (cf. figure 5.2.1(a)).

**5.3.10. Lemma.** Let  $D$  be a digraph for which  $\text{undir}(D)$  is a spanning subdigraph, and suppose  $T_1$  is not a subdigraph of  $D$ . If  $D$  is not bipartite, then the  $D$ -colouring problem is NP-complete.

**Proof.**

This is assertion (3) in the proof of Theorem 5.2.1. ■

**5.3.11. Corollary.** If  $D$  is a non-bipartite partitionable digraph that does not contain a basic triangle, then the  $D$ -colouring problem is NP-complete. ■

Let  $(J_1, x, z)$  and  $(J_2, x, z)$  be the sub-indicators shown in figure 5.3.4. These are important in the proof of Theorem 5.3.9.

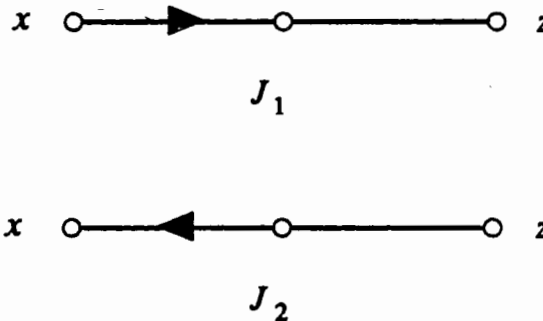


Figure 5.3.4. Important indicators.

**5.3.12. Lemma.** Let  $D$  be a partitionable digraph, and let  $(R, B)$  be a two-colouring of  $\text{undir}(D)$ . Let  $r \in R$ , and let  $i \in \{1, 2\}$ . Let  $D^-$  be the result of applying the sub-indicator construction with respect to  $(J_i, x, z)$ , and  $r$ , to  $D$ . If  $r'$  is a red vertex that belongs to  $V(D^-) - r$ , then there is an undirected path of length two in  $D$  joining  $r$  and  $r'$ .

**Proof.**

This follows immediately from the definition of a partitionable digraph. ■

There are obviously other versions of Lemma 5.3.12 corresponding to the possible cases that arise when the sub-indicator is  $(J_i, z, x)$ , or the construction takes place with respect to a vertex in  $B$ . For brevity, these are not stated, but they are used.

### **Proof of Theorem 5.3.9.**

It remains to show that if  $D$  is a partitionable digraph that contains a basic triangle, then the  $D$ -colouring problem is NP-complete. Suppose that  $D$  is a counterexample with the minimum number of vertices. That is,  $D$  is a partitionable digraph that contains a basic triangle, with the minimum number of vertices such that the  $D$ -colouring problem is not NP-complete.

We would like to use the sub-indicator construction to deduce various structural properties of  $D$ . There is, unfortunately, a difficulty: Let  $D^-$  be the result of applying a sub-indicator construction to  $D$ . The digraph  $\text{undir}(D^-)$  may not be spanning, consequently  $D^-$  may not be partitionable. This is not a severe handicap, as the following fact (\*) turns out to be sufficient.

(\*) Suppose that the result  $D^-$  of applying a sub-indicator construction to  $D$  contains a basic triangle. Then  $D = D^-$ .

### **Proof of (\*).**

Let  $D^{--}$  be the result of applying the sub-indicator construction with respect to  $(C_2, 0, \text{free})$  to  $D^-$ . Then  $D^{--}$  is a partitionable digraph (it is an induced subdigraph of  $D$ , and, by the construction,  $\text{undir}(D^{--})$  is a spanning subdigraph) that contains a basic triangle. Suppose  $D^{--}$  is a proper subdigraph of  $D$ . Then our choice of  $D$  implies that the  $D^{--}$ -colouring problem is NP-complete. Hence the

$D$ -colouring problem is also NP-complete, a contradiction. Therefore  $D = D^-$ .

The theorem follows from the sequence of claims below in which we use (\*) to derive some structural properties of  $D$  and, ultimately, a contradiction.

**(1) The digraph  $\text{undir}(D)$  is connected.**

Let  $C$  be a fixed basic triangle. Without loss of generality, assume that the single arc  $pq$  of  $C$  is red. Let  $b$  be the third vertex of  $C$ . Let  $D^-$  be the result of applying the sub-indicator construction with respect to  $(J_1, z, x)$ , and  $r$  to  $D$ . Then  $D^-$  contains a basic triangle and by (\*),  $D = D^-$ . Hence, by Lemma 5.3.12, there is in  $D$  an undirected path of length two from  $r$  to any other red vertex. Therefore there is also an undirected path of length one or three from  $r$  to any blue vertex. This completes the proof of (1).

The next two claims concern forbidden subdigraphs of  $D$ . Recall the digraph  $F_n$  defined in section 5.2 and the digraph  $W_3$  defined in section 4.3.

**(2) The digraph  $D$  contains neither  $F_4$  nor  $W_3$ .**

The digraph  $W_3$  is superhard, and the  $F_4$  is superhard with respect to " $\text{undir}(D)$  is spanning and connected" (cf Theorem 4.3.2, and Corollary 5.3.8, respectively).

**(3) The digraph  $D$  does not contain  $F_3$ .**

Suppose  $D$  contains a copy of  $F_3$  labelled as shown in figure 5.3.5. Since  $D$  contains neither  $F_4$  nor  $W_3$ , there is no vertex  $p$  such that  $pr, [p, q] \in E(D)$ . Let

$D^-$  be the result of applying the indicator construction with respect to  $(J_2, x, z)$ , and  $q$  to  $D$ . It is not hard to check that  $q, s, t \in V(D^-)$ , but  $r \notin V(D^-)$ . Hence  $D^-$  contains a basic triangle and has fewer vertices than  $D$ . By our choice of  $D$ , the  $D^-$ -colouring problem is NP-complete. Therefore the  $D$ -colouring problem is also NP-complete, which is a contradiction.

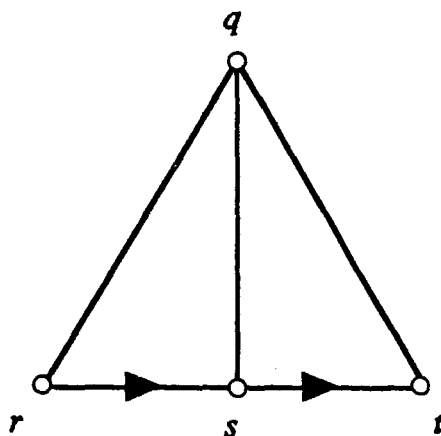


Figure 5.3.5. Configuration for (3).

(4) If  $ab$  is a single arc in a basic triangle of  $D$ , then there is no  $d$  (resp.  $x$ ) in  $V(D)$  such that  $bd$  (resp.  $xa$ ) is also a single arc.

We prove only that the vertex  $d$  can not exist. Let  $c$  be the third vertex of the basic triangle. Suppose to the contrary that  $d$  exists. Let  $D^-$  be the result of applying the sub-indicator construction with respect to  $(J_2, z, x)$ , and  $b$  to  $D$ . Since  $D^-$  contains the basic triangle induced by  $\{a, b, c\}$ , we have by (\*) that  $D = D^-$ . Since  $D$  is partitionable, this implies that there exists a vertex  $e$  and double arcs  $[b, e]$  and  $[d, e]$ . By (3),  $e \neq c$ . Let  $D^{--}$  be the result of applying the sub-indicator construction with respect to  $(J_2, x, z)$  and  $c$  to  $D^-$ . Then

$b, d, e \in V(D^-)$ , but since  $D^-$  does not contain  $F_3$ ,  $a \notin V(D^-)$ . Hence  $D^-$  contains a basic triangle (induced by  $\{b, d, e\}$ ), and has fewer vertices than  $D$ . By our choice of  $D$ , the  $D^-$ -colouring problem is NP-complete. Therefore  $D$ -COL is also NP-complete. This contradiction completes the proof of (4).

We define a *red basic triangle* ( resp. *blue basic triangle*) to be a basic triangle in which the single arc is red (resp. blue).

(5) Let  $r \in R$ . Then there is a red basic triangle  $C$ , with  $V(C)=\{r_1, r_2, b\}$ , and  $r_1 r_2 \in R$ , such that  $[r, b]$  is a double arc.

The statement is true if  $r$  is in a red basic triangle. Suppose that  $r$  is not in a red basic triangle. Let  $C'$  be a red basic triangle and let  $V(C')=\{r', r'', b'\}$ , with single arc  $r'r''$ . Let  $D^-$  be the result of applying the sub-indicator construction with respect to  $(J_1, z, x)$  and  $r'$  to  $D$ . Then  $D^-$  contains a basic triangle, so by (\*),  $D=D^-$  and hence  $r$  and  $r'$  are joined by an undirected path of length two, say  $r', b, r$ . If  $b$  and  $r''$  are adjacent the proof is complete. Suppose that  $b$  and  $r''$  are not adjacent. Since  $b$  is a vertex of  $D^-$ , the definition of  $V(D^-)$  implies that there exists a vertex  $r'''$ , an arc  $r'r'''$ , and a double arc  $[r''', b]$ . The 3-set  $\{r', r''', b\}$  induces the desired basic triangle.

(6) Let  $b \in B$ . Then there is a blue basic triangle  $C$ , with  $V(C)=\{b_1, b_2, r\}$ , and  $b_1 b_2 \in B$  such that  $[b, r]$  is a double arc.

The proof is similar to (5).

**(7) If  $D$  has a red basic triangle (resp. blue basic triangle) then  $D[R]$  (resp.  $D[B]$ ) has no directed path of length two.**

We prove the result on the assumption that there is a red basic triangle (the proof of the other case being similar). Suppose to the contrary that  $D$  has a red basic triangle and  $D[R]$  has a directed path of length two, say  $r, r', r''$ . By (5) there exists a red basic triangle  $C$ , with vertex set  $\{u, v, b\}$ ,  $b \in B$ , single arc  $uv$ , and such that there is a double arc  $[r, b]$ . Let  $D^-$  and  $D^{\sim}$  denote the result of applying the sub-indicator construction with respect to  $(J_1, z, x)$  and  $r_1$ , and  $(J_2, z, x)$  and  $r_2$  to  $D$ , respectively. Since both  $D^-$  and  $D^{\sim}$  contain the basic triangle  $C$ , (\*) implies that  $D = D^- = D^{\sim}$ . By Lemma 5.2.12 there is an undirected path of length two between  $u$  and  $r'$ , and also between  $v$  and  $r'$ . Let  $D^{\sim\sim}$  be the result of applying the sub-indicator construction with respect to  $(J_2, z, x)$  and  $r'$  to  $D$ . Then  $D^{\sim\sim}$  contains the basic triangle  $C$ , so (\*) implies that  $D^{\sim\sim} = D$ . Hence there is also an undirected path of length two joining  $r''$  and  $r''$ , say  $r', b', r''$ . Thus  $\{r', r'', b\}$  induces a basic triangle. But  $rr' \in E(D)$ , contrary to (4). This completes the proof of (7).

**(8) Neither  $R$  nor  $B$  is an independent set.**

If  $R$  is an independent set, then  $D$  satisfies the hypotheses of Theorem 5.3.2; it contains a basic triangle and, since  $D[B]$  has no directed path of length two, it does not contain a transitive triple. Hence the  $D$ -colouring problem is NP-complete, which is a contradiction.



**(9) Suppose  $D$  contains a red (resp. blue) basic triangle. Then every vertex in  $B$  (resp.  $R$ ) is in a red (resp. blue) basic triangle.**

Suppose that  $\{r_1, r_2, b'\}$ ,  $b' \in B$ , induces a red basic triangle. Let  $b \in B$ . Let  $D^-$ , and  $D^{---}$  be the result of applying the sub-indicator construction with respect to  $(J_1, z, x)$  and  $r_1$ , and  $(J_2, z, x)$  and  $r_2$  to  $D$ , respectively. Since both  $V(D^-)$  and  $V(D^{---})$  contain the basic triangle induced by  $\{r_1, r_2, b'\}$ , (\*) implies that  $D = D^- = D^{---}$ . Thus there exists a vertex  $r'$  such that  $r_1 r'$  is an arc and  $[r', b]$  is a double arc. Moreover there is an undirected path of length two joining  $r_1$  and  $r'$ , and another such path joining  $r_2$  and  $r'$ . Let  $D^{----}$  be the result of applying the sub-indicator construction with respect to  $(J_2, z, x)$  and  $r'$  to  $D$ . Since  $r_1, r_2, b \in V(D^{----})$ , the digraph  $D^{----}$  contains a basic triangle, hence (\*) implies that  $D^{----} = D$ . Therefore there exists a vertex  $r''$  such that  $r'' r'$  is an arc and  $r''$  is adjacent to  $b$  via a double arc. But now  $\{r', r'', b\}$  induces a red basic triangle. This completes the proof of (9).

**(10) The digraph  $D$  contains a red basic triangle and a blue basic triangle.**

Without loss of generality, suppose there is a red basic triangle. Let  $bb'$  be an arc of  $D[B]$  (the existence of such an arc is guaranteed by (8)). By (9), every vertex in  $B$  is in a red basic triangle. Suppose  $\{r_1, r_2, b\}$  and  $\{r', r'', b'\}$  each induce a basic triangle, with single arcs  $r_1 r_2$  and  $r' r''$ , respectively. Let  $D^-$  be the result of applying the sub-indicator construction with respect to  $(J_1, z, x)$  and  $r_1$  to  $D$ . Since  $r_1, r_2, b \in V(D^-)$ , (\*) asserts that  $D = D^-$ . Consequently there is an undirected path of length two joining  $r_1$  and  $r'$ , and another such path between  $r_1$  and  $r''$ . Let  $D^{---}$  be the result of applying the sub-indicator construction with respect to  $(J_2, x, z)$  and  $r_1$  to  $D (=D^-)$ . Then  $r', r'', b \in V(D^{---})$ , so (\*) implies that

$D=D^{\sim}$ . Since  $b \in V(D^{\sim})$ , there is a vertex  $b_1$  such that  $b_1b$  is an arc, and  $[b_1, r_1]$  is a double arc. The 3-set  $\{b_1, b, r_1\}$  induces a blue basic triangle.

By (10),  $D$  has no monochromatic directed path of length two. Hence  $D$  satisfies the hypotheses of Theorem 5.3.3, leading to the contradiction that the  $D$ -colouring problem is NP-complete. This completes the proof of Theorem 5.3.10. ■

As a corollary we derive an NP-completeness result for a class of digraphs that are not partitionable, and that do not necessarily have the property that  $\text{undir}(D)$  is spanning. For all  $n > 2$ , the digraph  $W_n$  (cf Lemma 4.3.2) belongs to this class. Let  $0 \leq a_1 < a_2 < \dots < a_t < n$  be integers and  $PW(a_1, a_2, \dots, a_t)$  be the digraph constructed from  $C_n \cup \{v\}$  by adding the double arcs  $\{[v, a_i] : i=1, 2, \dots, t\}$ . We call  $PW(a_1, a_2, \dots, a_t)$  a *partial wheel*.

**5.1.13. Corollary.** If  $PW(a_1, a_2, \dots, a_t)$  is bipartite, then  $PW(a_1, a_2, \dots, a_t)$ -COL is polynomial. Otherwise  $PW(a_1, a_2, \dots, a_t)$ -COL is NP-complete.

**Proof.**

We have previously noted the first statement (cf. the second paragraph of Section 5.3.1). Suppose  $PW(a_1, a_2, \dots, a_t)$  is not bipartite. Then there exists  $i$  such that  $a_{i+1}-a_i$  is odd. Let  $k = \min \{a_{j+1}-a_j : 1 \leq j \leq t \text{ and } a_{j+1}-a_j \text{ is odd}\}$ . Let  $(I, u, v)$  be the indicator constructed by adjoining two-cycles to the endvertices  $u$  and  $v$  of a directed path of length  $k$ . Let  $H^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . If  $n=2k$ , the digraph  $H^*$  is an undirected triangle. Otherwise, the core of  $H^*$  is a partitionable digraph that contains a basic

triangle. In both cases  $H^*$ -COL is NP-complete. Therefore  $H$ -COL is also NP-complete. ■□

In this chapter we have described a variety of sufficient conditions for NP-completeness of the  $D$ -colouring problem when  $D$  is a superdigraph of an undirected bipartite graph. Based on our results, we make the following conjecture.

**5.1.14. Conjecture.** Let  $D$  be a superdigraph of an undirected bipartite graph, and suppose that  $undir(D)$  is a spanning subdigraph. If  $D$  is not bipartite, then  $D$ -COL is NP-complete. Otherwise  $D$ -COL is polynomial.

We note that this is a special case of Conjecture 1.1. Even this restricted conjecture seems difficult. As a next step, one might try to show that it is true in the presence of the additional condition " $undir(D)$  is connected". By using the indicator construction in a manner similar to [Hell & Nešetřil, 1986], it can be shown that it would suffice to prove the latter strengthened conjecture for digraphs that contain a basic triangle. Other results in this thesis can be used to deduce a variety of structural properties of a hypothetical counterexample to Conjecture 5.3.14.

## 5.4. More on the Effect of Two Directed Cycles.

In this section we generalise a result of Maurer, Sudborough and Welzl [Maurer et al, 1981], and also a result of Bang-Jensen and Hell [Bang-Jensen & Hell, 1988; Gutjahr et al., 1989]. Our results add to the list of sparse digraphs  $H$  with two directed cycles for which the  $H$ -colouring problem is NP-complete.

We begin by extending some work of Maurer, Sudborough and Welzl. Let  $C_{n,k}$  be a digraph obtained from a directed  $n$ -cycle by replacing  $k$  arcs with double arcs. It has been proved [Maurer et al, 1981] that if  $n$  is odd, then  $C_{n,1}$ -COL is NP-complete. When  $n$  is even, the core of  $C_{n,k}$  is a directed two-cycle. Thus  $C_{n,k}$ -COL is polynomial. A complete classification of the complexity of  $C_{n,k}$ -COL is given below.

**5.4.1. Theorem.** If  $n$  is even or  $k=0$ , then  $C_{n,k}$ -COL is polynomial. Otherwise ( $n$  is odd and  $k > 0$ )  $C_{n,k}$ -COL is NP-complete.

**Proof.**

It remains to prove that if  $n$  is odd and  $2 \leq k \leq n$ , then  $C_{n,k}$ -COL is NP-complete. Let  $C^*$  be the digraph that results from applying the indicator construction with respect to  $(P_{n-2}, 0, n-1)$  to  $C_{n,k}$ . Since the directed odd girth of  $C_{n,k}$  is  $n$ , the digraph  $C^*$  is loopless. Furthermore, each double arc of  $C_{n,k}$  is also a double arc of  $C^*$ .

We claim that the vertices incident with double arcs induce a semi-complete digraph. Suppose  $[u,v]$  and  $[x,y]$  are distinct double arcs. The arcs of

$C_{n,k}$  belonging to the directed  $n$ -cycle give rise to a directed  $(u,x)$ -path and a directed  $(x,u)$ -path. Moreover, exactly one of these paths has odd length. Since both  $u$  and  $x$  are incident with double arcs, this implies that there is either a directed  $(u,x)$ -walk of length  $n-2$  or a directed  $(x,u)$ -walk of length  $n-2$ . Hence one of  $ux$  and  $xu$  is an arc of  $C^*$ . This proves the claim.

Let  $c$  be any vertex of  $C^*$  and  $C^{*-}$  be the digraph which results from applying the sub-indicator construction with respect to  $(C_2, 0, free)$  to  $C^*$ . Then  $C^{*-}$  is a semi-complete digraph with at least two directed cycles, and therefore, by Theorem 5.1.1,  $C^{*-}$ -COL is NP-complete. Thus  $C_{n,k}$ -COL is also NP-complete. This completes the proof. ■

We now generalise the following result.

**5.4.2. Theorem.** [Bang-Jensen & Hell, 1988; Gutjahr et al., 1989] Let  $H$  be a digraph of the form  $D_1$  or  $D_2$  (see figure 5.4.1). If  $H$  does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete. Otherwise,  $H$ -COL is polynomial. ■

Theorem 5.4.2 states, as a special case, that if  $D$  is a digraph constructed from a directed cycle by adding a chord, then  $D$ -COL is NP-complete unless  $D$  admits a retraction to a directed cycle. That is, Conjecture 1.1 is true for directed cycles with one chord.

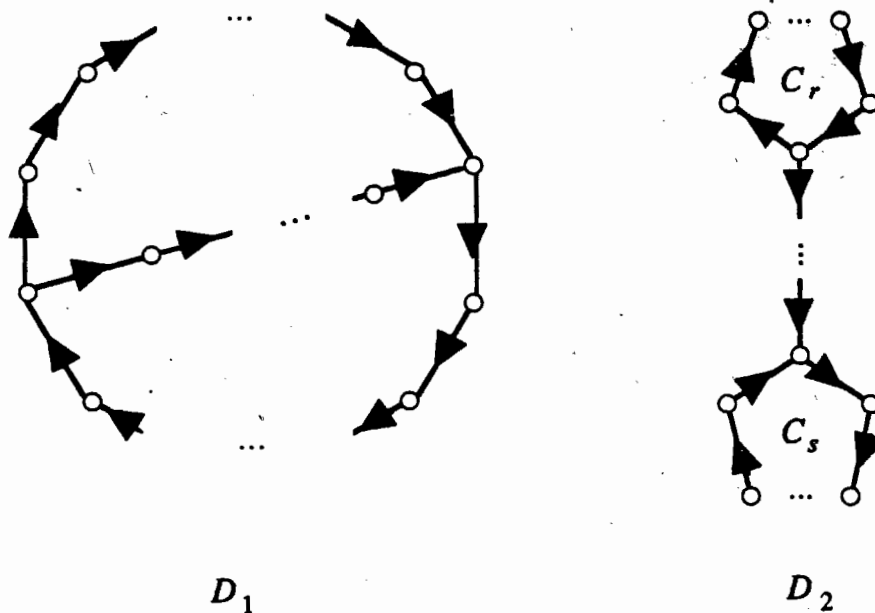


Figure 5.4.1. Digraphs with two directed cycles.

Let  $H$  be a directed graph constructed from a directed  $n$ -cycle by adding two chords. Then, depending on the relative orientation of the chords,  $H$  is of one of four types; an example of each type is shown in figure 5.4.2. We now prove that the  $H$ -colouring problem is NP-complete unless  $H$  retracts to a directed cycle. That is, Conjecture 1.1 is true for directed cycles with two chords.

**5.4.3. Theorem.** Let  $H$  be a directed graph that is constructed from a directed cycle by adding two chords. If  $H$  does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete. Otherwise,  $H$ -COL is polynomial.

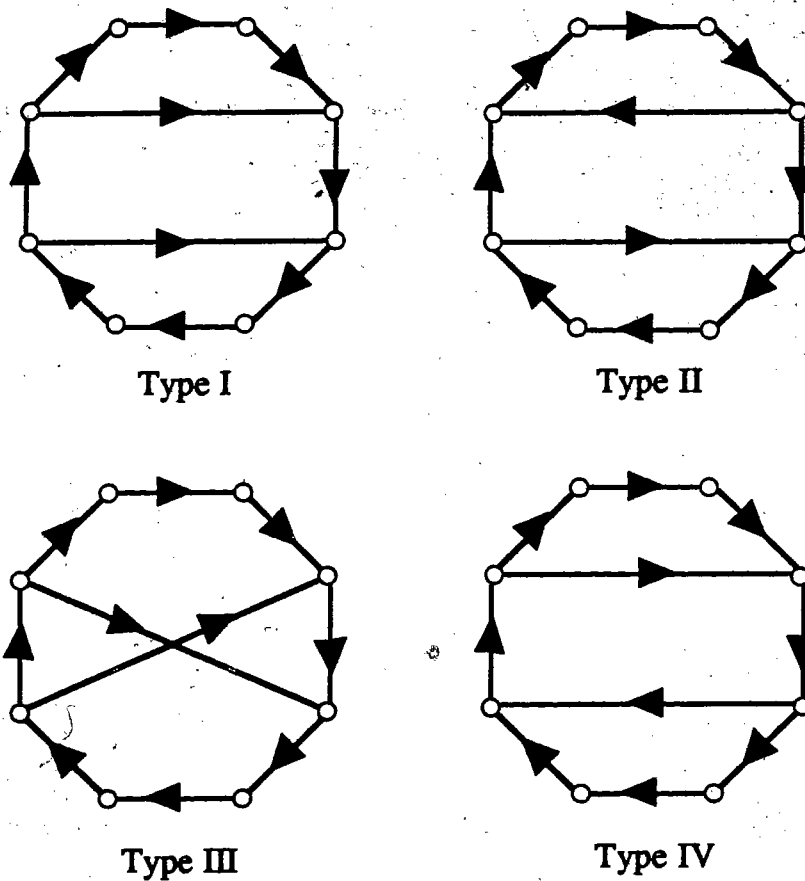


Figure 5.4.2. The four possible orientations of the chords.

We have previously noted the second statement. The proof of the first statement is divided into four lemmas, depending on the type of  $H$ .

**5.4.4. Lemma.** If  $H$  is of type I and does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete.

**Proof.**

Let  $H$  be of type I. Then  $H$  has exactly three directed cycles, say of lengths  $n$ ,  $a$ , and  $b$ , respectively. Without loss of generality assume  $n > a > b$ . Suppose that  $H$  does not admit a retraction to a directed cycle. Then  $b$  does not divide both  $a$

and  $n$ . There are two cases to consider.

CASE 1.  $b$  does not divide  $a$ .

Let  $H^*$  be the result of applying the sub-indicator construction with respect to  $(C_a, 0, \text{free})$  to  $H$ . Then  $H^*$  is the subdigraph of  $H$  induced by the vertex set of the directed  $a$ -cycle. Since  $b$  does not divide  $a$ , the digraph  $H^*$  does not admit a retraction to a directed cycle. Hence  $H^*$ -COL is NP-complete by Theorem 5.4.2, and therefore  $H$ -COL is also NP-complete.

CASE 2.  $b$  divides  $a$ .

Since the directed  $b$ -cycle is not a retract of  $H$ ,  $b$  does not divide  $n$ . Let  $H^-$  be the result of applying the edge sub-indicator construction with respect to  $(C_{n+b}, 0, \text{free})$  to  $H$ . It is clear that every arc, except the chord  $e$  that forms the directed  $a$ -cycle, belongs to a closed directed walk of length  $n+b$ . If  $e$  also belongs to such a closed walk, then there are integers  $\alpha$  and  $\beta$  such that  $n+b = \alpha a + \beta b = \gamma b$ . Therefore  $b$  divides  $n$ , which is a contradiction. Hence  $H^- = H - e$ , and so  $H^-$ -COL is NP-complete by Theorem 5.4.2. Therefore  $H$ -COL is also NP-complete.

All cases have been considered. ■

**5.4.5. Lemma.** If  $H$  is of type II and does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete.



**Proof.**

Let  $H$  be of type II. The digraph  $H$  can have three or four directed cycles. When  $H$  has four directed cycles, the chords have both endpoints in common. In this case,  $H$  is also of type IV. We defer consideration of this case to Lemma 5.4.7. Hence assume  $H$  has exactly three directed cycles, say of length  $n$ ,  $a$ , and  $b$ . Without loss of generality  $n > a \geq b$ . Suppose that  $H$  does not admit a retraction to a directed cycle. Then  $b$  does not divide both  $a$  and  $n$ . Let the directed  $n$ -cycle be  $0, 1, \dots, n-1, 0$ . There are four cases to consider.

**CASE 1.  $b$  does not divide  $a$ .**

Let  $uv$  (resp.  $xy$ ) be the chord that belongs to the directed  $a$ -cycle (resp. directed  $b$ -cycle). Without loss of generality assume  $u \neq y$ ; otherwise consider the converse of  $H$ . Let  $J$  be the directed graph constructed by identifying the terminal vertex of a directed path of length  $n-a-1$  with a vertex on a directed  $a$ -cycle. Let  $0$  be the label of the initial vertex of the directed path. Let  $H^*$  be the result of applying the sub-indicator construction with respect to  $(J, 0, \text{free})$  to  $H$ . It is not hard to see that the core of  $H^*$  is of the form  $D_2$ . Since  $b$  does not divide  $a$ ,  $H^*$ -COL is NP-complete by Theorem 5.4.2. Hence  $H$ -COL is also NP-complete.

**CASE 2.  $b$  divides  $a$ , and  $b < a$ .**

Since the directed  $b$ -cycle is not a retract,  $b$  does not divide  $n$ . Let  $e$  be the chord that belongs to the directed  $a$ -cycle. Every arc except  $e$  belongs to a closed directed walk of length  $n + b$ . If the arc  $e$  also belongs to such a closed directed walk, then either  $n+a \leq n+b$ , or  $a$  divides  $n+b$ . The former case is impossible since  $a > b$ , and the latter case is also impossible since, if  $b$  divides  $a$ ,

then  $b$  divides  $n$ . Let  $H^-$  be the result of applying the edge sub-indicator construction with respect to  $(C_{n+b}, 0, \text{free})$  to  $H$ . Then  $H^- = H - e$ , where  $e$  is the chord of the  $n$ -cycle that forms the  $a$ -cycle. Since  $H$  is of the form  $D_1$ , and does not admit a retraction to a directed cycle,  $H^-$ -COL is NP-complete by Theorem 5.4.2. Therefore  $H$ -COL is also NP-complete.

CASE 3.  $a=b$ , and there exists a vertex  $x$  on an  $a$ -cycle such that the vertex  $x+a$  is also on an  $a$ -cycle.

Since  $C_a$  is not a retract, the digraph  $H$  is retract-free. Relabel the vertices so that  $x$  is labelled 0,  $x+1$  is labelled 1, and so on. (That is, subtract  $x$  from the label of each vertex, where computations are modulo  $n$ .) Let  $k \geq 0$ . If  $u$  is a vertex on a directed  $a$ -cycle then the set of vertices reachable from  $u$  by a directed walk of length  $ka$  is  $\{0, a, 2a, \dots, ka\}$ . Let  $m$  be the order of the element  $a$  in  $Z_n$ . Note that  $m > 2$  (if  $m=2$  then  $2a=n$ , whence  $C_a$  is a retract).

We show that NOT-ALL-EQUAL  $m$ -SAT without negated variables polynomially transforms to H-COL. Suppose an instance of NOT-ALL-EQUAL  $m$ -SAT without negated variables is given, with variables  $x_1, x_2, \dots, x_p$ , and clauses  $K^1, K^2, \dots, K^q$ . Construct a digraph  $G$  from  $H$ ,  $\{x_1, x_2, \dots, x_p\}$ , and  $q$  copies of  $C_n$ , say  $C^1, C^2, \dots, C^q$ , by adding directed paths as follows. Vertex 0 in  $H$  is joined to each vertex  $x_j$  ( $j = 1, 2, \dots, p$ ) by a directed path of length  $a$ . Vertex 0 in  $H$  is also joined to vertex 0 on each  $C^l$  ( $l = 1, 2, \dots, q$ ) by a directed path of length  $ma$ . If the  $r^{\text{th}}$  variable in clause  $K^s$  is  $x_r$ , then join  $x_r$  to vertex  $ra$  of  $C^s$  by a directed path of length  $(m-2)a$ . Clearly the digraph  $G$  is constructible in polynomial time.

CLAIM.  $G$  is  $H$ -colourable if and only if there is a satisfying truth assignment in which each clause contains at least one true variable and at least one false variable.

PROOF.

( $\Rightarrow$ ) Consider a homomorphism of  $G$  to  $H$ . Since  $H$  is retract-free, the copy of  $H$  in  $G$  must map onto  $H$ . We may therefore assume that every vertex of  $H$  maps to itself. Thus each vertex  $x_i$  ( $i=1, 2, \dots, n$ ) maps to 0 or  $a$ . Moreover, each  $C^j$  maps onto the directed  $n$ -cycle in  $H$  (because  $a$  does not divide  $n$ ), and vertices  $\{0, a, 2a, \dots, (m-1)a\}$  of  $C^j$  map, in cyclic order, to the corresponding set of vertices of  $H$ . Define a truth assignment by setting  $x_i = T$  just if  $x_i$  maps to 0.

Consider  $C^s$ . Recall that there is no directed  $(0, -a)$ -walk of length  $(m-2)a$ . Let  $v$  be the vertex of  $C^s$  that maps to  $-a$ , and let  $x_r$  be the vertex joined to  $v$  by a copy of  $P_{(m-2)a}$ . Then  $x_r$  must map to  $a$ . Hence  $K^s$  contains a false variable. Similarly  $K^s$  contains a true variable.

( $\Leftarrow$ ) Suppose such a truth assignment exists. Define an  $H$ -colouring of  $G$  as follows. Every vertex of the copy of  $H$  is coloured by itself. For  $i = 1, 2, \dots, t$ , if  $x_i = T$ , then colour  $x_i$  by 0, otherwise colour  $x_i$  by  $a$ . This partial colouring extends to all of the directed paths joining the  $x_i$ 's to the copy of  $H$ . Consider  $K^s$ . There exists  $t$  such that the  $t^{\text{th}}$  variable  $l_t$  in  $K^s$  is true, and the  $(t+1)^{\text{th}}$  variable  $l_{t+1}$  is false. Colour vertex  $la$  of  $C^s$  by  $(-2a)$  and vertex  $(l+1)a$  of  $C^s$  by  $(-a)$ . This completely determines the colouring of  $C^s$ . Furthermore, this partial colouring can be extended to all of the directed  $(m-2)a$ -paths joining  $C^s$  to  $x^i$  ( $x^i \in K^s$ ), and to the directed  $m$ -path joining  $H$  to  $C^s$ . Therefore  $G$  is  $H$ -colourable.

This completes the proof of case 3.

CASE 4.  $a=b$  and for every vertex  $x$  on a directed  $a$ -cycle the vertex  $x+a$  is not on a directed  $a$ -cycle.

Without loss of generality, the vertex 0 is on a directed  $a$ -cycle. Since  $C_a$  is not a retract, the digraph  $H$  is retract-free. Let  $m$  be the order of the element  $a$  in  $Z_n$ . Note that  $m > 2$  (if  $m=2$  then  $2a=n$ , whence  $C_a$  is a retract).

CLAIM. Each directed  $a$ -cycle contains at least two elements of  $\langle a \rangle$ .

PROOF.

Each directed  $a$ -cycle contains the same number of elements of  $\langle a \rangle$ . If this number is one, then  $a$  divides  $n$  and  $C_a$  is a retract, which is a contradiction.

Let  $I$  be the directed graph constructed from a directed path of length  $(m-1)a$  as follows. Identify vertex 0 (on the path) with a vertex on a directed  $a$ -cycle, and identify  $(m-1)a$  with a vertex on a second directed  $a$ -cycle. For  $i=1, 2, \dots, a$ , add a directed path of length  $n-1$  from  $i$  to  $i-1$ . Let  $H^*$  be the result of applying the indicator construction with respect to  $(I, 0, 1)$  to  $H$ . There is no  $H$ -colouring of  $I$  such that the vertex  $a$  is coloured by a vertex on a directed  $a$ -cycle (otherwise  $colour(0)$  and  $colour(0)+a = colour(a)$  are vertices of  $H$  that are both on a directed  $a$ -cycle, which is a contradiction). Let  $A$  be the set of vertices of  $H$  which are on directed  $a$ -cycles. Let  $x \in A$  and consider an  $H$ -colouring of  $I$  such that  $colour(0)=x$ . Since vertex  $a$  of  $I$  does not map to a vertex on a directed  $a$ -cycle, the possible images of vertex  $(m-1)a$  of  $I$  are those vertices which also lie on a directed  $a$ -cycle, and are reachable from vertex  $x+a$  of  $H$  by a directed walk of length  $(m-2)a$ . Thus  $colour(ma) \in \{x+2a, x+3a, \dots, x+(m-1)a\} \cap A$ , so

$colour((m-2)a) \neq colour(0)$ . Thus  $H^*$  is loopless. Moreover the vertex  $(m-1)a$  can be coloured by any vertex in the set  $(\langle a \rangle + x) - \{x\}$ . The claim now implies that  $H^*$  contains an undirected  $K_4$ . Thus  $H^*$ -COL is NP-complete, and so  $H$ -COL is also NP-complete.

All cases have been considered. ■

**5.4.6. Lemma.** If  $H$  is of type III and does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete.

**Proof.**

Let  $H$  be of type III. Then  $H$  has three directed cycles, say of lengths  $n$ ,  $a$ , and  $b$ . Without loss of generality assume  $n > a \geq b$ . Suppose that the core of  $H$  is not a directed cycle. We may further assume that the chords have no common vertex, since this occurrence is covered under Lemmas 5.4.4 and 5.4.7. There are three cases to consider.

**CASE 1.**  $b$  does not divide  $a$ .

The argument is similar to case 1 of Lemma 5.4.5, and uses the same sub-indicator.

**CASE 2.**  $b$  divides  $a$ , and  $b < a$ .

Since the directed  $b$ -cycle is not a retract,  $b$  does not divide  $n$ . The remaining details are identical to those of case 2 of Lemma 5.4.5.

CASE 3.  $a=b$ .

Then there is a vertex  $x$  on a directed  $a$ -cycle such that  $x+a$  is also on a directed  $a$ -cycle. The remaining details are identical to those of case 3 of Lemma 5.4.5.

The result now follows. ■

**5.4.7. Lemma.** If  $H$  is of type IV and does not admit a retraction to a directed cycle, then  $H$ -COL is NP-complete.

**Proof.**

The digraph  $H$  has four directed cycles, say of lengths  $n$ ,  $a$ ,  $b$ , and  $c$ . Without loss of generality assume  $n > a \geq b > c$ . Note that  $n=a+b-c$ . Suppose that the core of  $H$  is not a directed cycle. There are four cases to consider.

CASE 1.  $c$  divides  $b$ .

Then the subdigraph of  $H$  induced by the vertex set of the directed  $a$ -cycle is a retract. Since  $C_c$  is not a retract of  $H$ ,  $c$  does not divide  $a$ . Consequently the core of  $H$  is of the form  $D_1$ , and  $H$ -COL is NP-complete by Theorem 5.4.2.

CASE 2.  $c$  does not divide  $b$ , and  $b < a$ .

Let  $H^-$  be the result of applying the sub-indicator construction with respect to  $(C_b, 0, \text{free})$  to  $H$ . Then  $H^-$  consists of a directed  $b$ -cycle plus a chord that belongs to the directed  $c$ -cycle. That is,  $H^-$  is of the form  $D_1$ . Since  $c$  does not divide  $b$ ,  $H^-$ -COL is NP-complete by Theorem 5.4.2, and therefore  $H$ -COL is also

NP-complete.

CASE 3.  $a=b$  and  $c$  does not divide  $n$ .

Since  $C_c$  is not a retract,  $c$  does not divide  $a$ . Let  $m$  be the order of the element  $a$  in  $Z_n$ . If  $m=2$ , then  $2a=n$ , and hence  $c=0$ , which is a contradiction. Therefore  $m>2$ . Let  $Q_r$  ( $r \geq a-1$ ) denote the  $(r+1)$ -vertex digraph constructed from  $P_r$  by adding the arcs  $\{(i, i-a+1) : i = a-1, a, \dots, r\}$ . Since any  $a-1$  consecutive arcs along the directed  $r$ -path must belong to an image of a directed  $a$ -cycle and  $c$  does not divide  $a$ , no image of  $Q_r$  in  $H$  contains a directed  $c$ -cycle. This effectively eliminates the use of the directed  $c$ -cycle. The transformation is from NOT-ALL-EQUAL  $m$ -SAT without negated variables, and is identical to case 3 of Lemma 5.4.5, except that wherever  $P_r$  appears in the construction,  $Q_r$  should be used.

CASE 4.  $a=b$ , and  $c$  divides  $n$ .

Since  $C_c$  is not a retract,  $c$  does not divide  $a$ . Let  $J$  be the directed graph constructed by identifying the initial vertex of a directed path of length  $a-2$  with a vertex on a directed  $a$ -cycle. Let  $x$  be the terminal vertex of the directed path. Let the vertices of  $H$  be numbered cyclically such that vertex 0 is the terminal vertex of one of the chords. Let  $H^-$  be the result of applying the sub-indicator construction with respect to  $(J, x, \text{free})$  to  $H$ . It may be directly verified that  $H^- = H - \{2a-1\}$ . Consequently the core of  $H^-$  is of the form  $D_1$  and, since  $c$  does not divide  $a$ ,  $H^-$ -COL is NP-complete. Therefore  $H$ -COL is also NP-complete.

All cases have been considered. ■

Theorem 5.4.3 generalises Theorem 5.4.2 for digraphs of the form  $D_1$  (cf. figure 5.4.1). Our next result generalises the same theorem for digraphs of the form  $D_2$ . Let  $Q = q_0, q_1, \dots, q_q$  be an oriented path, and let  $r$  and  $s$  be integers. Let  $H$  be the digraph constructed from  $C_r \cup C_s \cup Q$  as follows. Let  $V(C_r) = \{r_0, r_1, \dots, r_{r-1}\}$ , and  $E(C_r) = \{r_i r_{i+1} : i=0, 1, \dots, r-1\}$ . Similarly, let  $V(C_s) = \{s_0, s_1, \dots, s_{s-1}\}$  and  $E(C_s) = \{s_i s_{i+1} : i=0, 1, \dots, s-1\}$ . Identify the vertices  $q_0$  and  $q_q$  with  $r_0$  and  $s_0$ , respectively. That is,  $H$  is of the form  $D_2$  except that the directed path has been replaced by an oriented path.

**5.4.8. Theorem.** If  $r$  divides  $s$  or  $s$  divides  $r$ , then  $H$ -COL is polynomial.

Otherwise ( $r$  does not divide  $s$  and  $s$  does not divide  $r$ )  $H$ -COL is NP-complete.

**Proof.**

If  $r$  divides  $s$ , then the directed  $r$ -cycle is a retract, and if  $s$  divides  $r$ , then the directed  $s$ -cycle is a retract. In either case, the  $H$ -colouring problem is polynomial.

Suppose  $r$  does not divide  $s$  and  $s$  does not divide  $r$ . By Theorem 5.4.2 we may assume that  $Q$  is not a directed path. Let  $k = nl(Q)$ , and let  $a$  and  $b$  be any integers such that  $a, b > |V(Q)|$ ,  $a \equiv 1 - k \pmod{rs}$ , and  $b \equiv 0 \pmod{rs}$ . Let  $I$  be the oriented path constructed from  $P_a \cup P_b \cup Q$  by identifying the terminal vertex of  $P_a$  with  $q_0$ , and the initial vertex of  $P_b$  with  $q_q$ . Then  $nl(I) \equiv 1 \pmod{rs}$ . Let  $u$  be the initial vertex of  $P_a$  and let  $v$  be the terminal vertex of  $P_b$ . Let  $H^*$  be the result of applying the indicator construction with respect to  $(I, u, v)$  to  $H$ . We make the following assertions about the digraph  $H^*$ .



(1) Every internal vertex of  $Q$  is either isolated, a source of  $H^*$ , or a sink of  $H^*$ .

Let  $q_i$  be an internal vertex of  $Q$ . If there is no directed path between  $q_i$  and either  $q_0$  or  $q_q$ , then the vertex  $q_i$  is isolated in  $H^*$ , since there is no homomorphism of  $P_a$  to  $Q$  that maps  $u$  to  $q_i$ . Suppose there is a directed path from  $q_i$  to  $q_q$ . Since  $Q$  is not a directed path, there is no directed walk of length  $b$  that ends at  $q_i$ . Similarly, if there is a directed path from  $q_i$  to  $q_0$ , there is no directed walk of length  $b$  that ends at  $q_i$ . Hence  $q_i$  is a source of  $H^*$ . The existence of a directed path from  $q_0$  or  $q_q$  to  $q_i$  similarly implies that  $q_i$  is a sink of  $H^*$ .

(2)  $H^*$  is loopless.

By (1) no internal vertex of  $Q$  is incident with a loop. Suppose  $r_i$  is incident with a loop. Then there is a homomorphism of  $I$  to  $H$  which takes both  $u$  and  $v$  to  $r_i$ . Let  $W$  be the walk in  $H$  determined by the image of  $I$ . Since  $I$  and  $Q$  have the same number of sources (and sinks), no vertex of  $W$  is on the directed  $s$ -cycle. That is,  $W$  is contained in the subdigraph  $(C_r \cup Q) - q_q$ . Since the net length of each  $(q_0, q_0)$ -section  $W$  is zero, it follows that  $nl(W) \equiv 0 \pmod{r}$ , which is a contradiction (recall that  $nl(I) \equiv 1 \pmod{rs}$ ). Similarly, no vertex of  $C_s$  is incident with a loop.

(3)  $H^*$  contains both  $C_r$  and  $C_s$ .

This is clear since  $nl(I) \equiv 1 \pmod{rs}$ .

**(4) Neither  $C_r$  nor  $C_s$  has a chord.**

Consider a homomorphism of  $I$  into  $H$  that takes  $u$  to  $r_i$  and  $v$  to  $r_j$ . By (2)  $i \neq j$ . Arguing as in (2), the image of  $I$  is contained in the subdigraph  $(C_r \cup Q) - q_q$ . Since the net length of each  $(q_0, q_0)$ -section of the image of  $I$  is zero, the net length of the walk defined by image of  $I$  is congruent to  $(j - i)$  modulo  $r$ . Therefore  $j = i + 1 \pmod{r}$ , and  $C_r$  has no chord. Similarly  $C_s$  has no chord.

**(5) The arc  $s_{s-k+1}r_0$  exists.**

We describe the necessary homomorphism of  $I$  to  $H$ . Map  $u$  to  $s_{s-k+1}$ . Since  $a \equiv 1-k \pmod{rs}$  the first vertex in the copy of  $Q$  in  $I$  maps to  $s_0 = q_0$ . Now map each vertex of the copy of  $Q$  in  $I$  to the corresponding vertex of  $Q$ , and map the copy of  $P_b$  in  $I$  to  $C_r$ . Since  $b \equiv 0 \pmod{rs}$ , the vertex  $v$  maps to  $r_0$ .

**(6) The arc  $r_{r-k+1}s_0$  exists if and only if  $Q$  is self-converse.**

( $\Rightarrow$ ) If the arc exists, then the copy of  $Q$  in  $I$  must map onto the copy of  $Q^{-1}$  in  $H$ .

( $\Leftarrow$ ) The argument is similar to (5).

If  $Q$  is self-converse, then  $nl(Q) = 0$ . Hence the arcs from (5) and (6) are  $s_{s-1}r_0$  and  $r_{r-1}s_0$ , respectively.

**(7) There are no other arcs between  $C_r$  and  $C_s$ .**

Consider a homomorphism of  $I$  to  $H$  in which  $u$  maps to a vertex on one of the directed cycles and  $v$  maps to a vertex on the other directed cycle. It is not hard to see that the copy of  $Q$  in  $I$  must map onto  $Q$ . Since homomorphisms to

directed cycles are completely determined by the image of a single vertex, this forces  $u$  to map to  $s_{s-1}$  and  $v$  to map to  $r_0$ , or  $u$  to map to  $r_{r-1}$  and  $v$  to map to  $s_0$ , (depending on the orientation of the supposed arc).

Thus the structure of  $H^*$  is completely determined. Let  $H^{*-}$  be the result of applying the sub-indicator construction with respect to  $(P_2, 1, free)$  to  $H^*$ . Then  $H^{*-}$  is the subdigraph of  $H^*$  induced by  $V(H^*) - \{q_1, q_2, \dots, q_{q-1}\}$ . If  $Q$  is not self-converse,  $H^{*-}$  consists of a directed  $r$ -cycle and a directed  $s$ -cycle joined by an arc. Since  $r$  does not divide  $s$  and  $s$  does not divide  $r$ ,  $H^{*-}$ -COL is NP-complete by Theorem 5.4.2. On the other hand, if  $Q$  is self-converse,  $H^{*-}$  consists of a directed cycle with two chords and is of type II. The lengths of the cycles are  $r+s$ ,  $r$ , and  $s$ . Since  $r$  does not divide  $s$  and  $s$  does not divide  $r$ ,  $H^{*-}$ -COL is NP-complete by Theorem 5.4.3. Therefore  $H$ -COL is also NP-complete. This completes the proof. ■

## 6. Acyclic and Unicyclic Digraphs.

In the previous two chapters we have concentrated on complexity results for smooth digraphs. As we have seen, there are several natural conjectures about which of these  $H$ -colouring problems are  $\text{NP}_T$ -complete. There are, however, many digraphs with sources or sinks, and at present there are very few complexity results for digraphs in this class (cf. Section 2.2).

This chapter is devoted to extending the collection of complexity results for  $H$ -colouring by acyclic or unicyclic digraphs. We identify infinite families in each of these classes of digraphs for which  $H\text{-COL}$  is NP-complete, and others for which it is polynomial. In so doing we shed some light on the sequence of digraphs in Chapter one, where the complexity of the  $H$ -colouring problem was oscillatory (cf. figure 1.1). We also describe an acyclic digraph  $D$  with six vertices and seven arcs for which  $D\text{-COL}$  is NP-complete. This is the smallest such digraph discovered so far.

Let  $n \geq 3$ ,  $t \geq 1$ , and let  $a_1, a_2, \dots, a_t$  be integers such that

$$0 \leq a_1 < a_2 < \dots < a_t < n-1.$$

The digraph  $U(n; a_1, a_2, \dots, a_t)$  is constructed from  $C_n \cup \{v\}$  by adding the arcs  $\{va_i; i=1, 2, \dots, t\}$ . When the context is clear, for brevity we denote this digraph by  $U$ . Note that  $U$  is unicyclic. If the integers  $d_i = a_{i+1} - a_i$ , are all equal ( $i=1, 2, \dots, t$ , and subscripts modulo  $t$ ), we say that  $U$  is symmetric, and we use  $d$  to denote the common value of  $a_{i+1} - a_i$ .

Suppose  $D \rightarrow H$ . We denote by  $[i]_H$  the set of all vertices of  $D$  that map to vertex  $i$  of  $H$ . If  $H = C_n$ , this notation is abbreviated to  $[i]_n$ ; in this case we also use  $X_i$  (resp.  $Y_i$ ) to denote the set of sources (resp. sinks) in  $[i]$ . When the context is clear, the subscript is omitted.

**6.1. Lemma.** Let  $U$  be symmetric and let  $G$  be a digraph. Then a connected digraph  $G \rightarrow U$  if and only if  $G \rightarrow C_d$  and there exists  $i$  such that  $G-X_i \rightarrow C_n$ .

**Proof.**

( $\Rightarrow$ ) Clearly  $G \rightarrow U \rightarrow C_d$ . Suppose  $v \in [i]_n$  then, since only sources of  $G$  can map to  $v$ ,  $G-X_i \rightarrow U-\{v\} = C_n$ .

( $\Leftarrow$ ) Consider a fixed  $C_d$ -colouring  $c$  of  $G$ . Without loss of generality assume  $i = 0$ . Any vertex that is adjacent from a vertex in  $X_0$  belongs to  $[1]_d$ .

Furthermore,  $d$  divides the net length of any path between two vertices in  $[1]_d$ .

Hence, in any  $C_n$ -colouring of  $G-X_0$ , the colours assigned to these vertices can be assumed to differ by a multiple of  $d$ . If  $X_0$  is empty, then  $G \rightarrow C_n \rightarrow U$ . Otherwise,

consider a vertex  $x \in X_0$ , and let  $y$  be an out-neighbour of  $x$ . Choose a  $C_n$ -colouring of  $G-X_0$  such that  $y \in [a_1]_n$ . Every element of  $N^+(x)$  is coloured by a vertex in  $\{a_1, a_2, \dots, a_{nd}\}$ . If we give all vertices in  $X_0$  colour  $v$ , the result is a  $U$ -colouring of  $G$ . ■

**6.2. Corollary.** If  $U$  is symmetric, then  $U$ -COL is polynomial.

**Proof.**

The algorithm is implied in the above proof. It is described more formally in figure 6.1. Each step may clearly be carried out in polynomial time. ■

1. Find a  $C_d$ -colouring of  $G$ .
2. For  $i=0, 1, \dots, d-1$  do
  - Find  $X_i$ .
  - Find  $G-X_i$ .
  - If  $G-X_i$  is  $C_n$ -colourable then answer YES and stop.
3. Answer NO.

Figure 6.1. Polynomial algorithm for  $U$ -colouring.

Our next lemma classifies many non-symmetric  $U(n; a_1, a_2, \dots, a_d)$ -colouring problems. Certain special oriented paths play a central role in its proof. Let  $Z$  be the four vertex oriented path shown in figure 6.2 (a). More generally, let  $Z_r$  be the oriented path formed by juxtaposing  $r$  copies of  $Z$  and identifying the corresponding endpoints. The digraph  $Z_3$  is shown in figure 6.2 (b).



(a)  $Z = Z_1$



(b)  $Z = Z_3$

Figure 6.2. Sample "zigzag" paths.

**6.3. Lemma.** Let  $U = U(n; 0, 1, a_3, a_4, \dots, a_t)$ . Suppose there is a unique  $i$  such that  $d_i = r = \max\{d_j; j=1, 2, \dots, t\}$ . Then

(a) a partial colouring of  $Z_{2^{r-1}}$  in which  $\text{colour}(s)=a_i$  and  $\text{colour}(t)\neq a_{i+1}-1, v$  can be extended to a  $U$ -colouring of  $Z_{2^{r-1}}$ ,

(b) there is no  $U$ -colouring of  $Z_{2^{r-1}}$  in which  $\text{colour}(s)=a_i$  and  $\text{colour}(t)=a_{i+1}-1$  or  $v$ ; and

(c) a partial colouring of  $Z_{2^{r-1}}$  in which  $\text{colour}(s)\neq a_i$  and  $\text{colour}(t)\neq v$  can be extended to a  $U$ -colouring of  $Z_{2^{r-1}}$ .

**Proof.**

Direct verification. ■

**6.4. Lemma.** If  $\max\{d_k; k=1, 2, \dots, t\}$  is unique, then  $U(n; 0, 1, a_3, a_4, \dots, a_t)$ -COL is NP-complete.

**Proof.**

Suppose the unique maximum value is  $d_i$ . The transformation is from  $n$ -SAT. Suppose an instance of  $n$ -SAT is given, with variables  $x_1, x_2, \dots, x_p$  and clauses  $K^1, K^2, \dots, K^q$ . Construct a digraph  $G$  as follows. Each variable (resp. each clause) corresponds to a directed  $n$ -cycle, say  $X^1, X^2, \dots, X^p$  (resp.

$C^1, C^2, \dots, C^q$ ). We make connections between these directed cycles by adding copies of  $Z_{2^{r-1}}$ , as described below. (When we "join" a vertex in  $X^w$  to a vertex in  $C^k$  by a copy of  $Z_{2^{r-1}}$ , we identify the vertex  $s$  of  $Z_{2^{r-1}}$  with the vertex belonging to  $X^w$ , and we identify the vertex  $t$  of  $Z_{2^{r-1}}$  with the vertex belonging to  $C^k$ .)

Suppose that  $K^j = l_1 \vee l_2 \vee \dots \vee l_n$ . If  $l_m$  is  $x_a$  then join the  $m^{\text{th}}$  vertex on  $C^j$  to all vertices on  $X^a$  which have opposite parity from  $a_i$ . Otherwise the  $m^{\text{th}}$  vertex on  $C^j$

is joined to all vertices on  $X^a$  with the same parity as  $a_i$ .

CLAIM. The digraph  $G$  is  $U$ -colourable if and only if the clauses  $K^1, K^2, \dots, K^q$  have a satisfying truth assignment.

PROOF.

( $\Rightarrow$ ) Suppose there is a homomorphism  $f: G \rightarrow U$ . Each directed  $n$ -cycle of  $G$  must map onto the directed  $n$ -cycle of  $U$ . Define a truth assignment by setting  $x_a = T$  just if the parity of the label of the unique vertex of  $X^a$  in  $[a_i]_H$  is different from the parity of  $a_i$ . The colour  $z$  assigned to the  $m^{\text{th}}$  vertex  $y$  of  $C^j$  must be compatible with the colours given to all of the vertices joined to it by copies of  $Z_{2r-1}$ . That is, there must be a homomorphism of  $Z_{2r-1}$  to  $H$  taking  $s$  to  $\text{colour}(y)$  and  $t$  to  $z$ . If a vertex joined to  $y$  is coloured  $a_i$  then  $\text{colour}(y) \neq a_{i+1}-1$ . That is, if the corresponding literal is false, then the colour of the  $m^{\text{th}}$  vertex of  $C^j$  is not  $a_i$ . Since a colouring exists, not all literals in any clause can be false.

( $\Leftarrow$ ) Suppose there exists a satisfying truth assignment. Define a colouring as follows. If  $x_a = T$ , then colour  $X^a$  so that the unique vertex coloured  $a_i$  has different parity from  $a_i$ . Since each clause contains a true literal, each directed cycle  $C^j$  has a vertex which is not joined to  $a_i$  by a copy of  $Z_{2r-1}$ . By Lemma 6.3, this partial colouring can be extended to a  $U$ -colouring of  $G$ .

Since the digraph  $G$  can be constructed in polynomial time, the result follows. ■

We are now in a position to give a classification of all digraphs  $U(n; a_1, a_2)$  and  $U(n; a_1, a_2, a_3)$ . (It is clear that if  $t=0$  or  $1$ , then  $C_n$  is a retract, implying that



$U$ -COL is polynomial.)

**6.5. Corollary.** Let  $2 \leq t \leq 3$ . If  $U(n; a_1, a_2, \dots, a_t)$  is symmetric, then  $U(n; a_1, a_2, \dots, a_t)$ -COL is polynomial. Otherwise  $U(n; a_1, a_2, \dots, a_t)$ -COL is NP-complete.

**Proof.**

The first statement follows immediately from Theorem 6.3. The case  $t=2$  follows from Lemma 6.4 because if  $U(n; a_1, a_2)$  is not symmetric, then  $\max\{d_1, d_2\}$  is uniquely achieved.

Consider  $U(n; a_1, a_2, a_3)$ . If  $\max\{d_1, d_2, d_3\}$  is uniquely achieved, the result follows from Lemma 6.4. Otherwise, we may assume without loss of generality that  $r = d_1 = d_2 > d_3$ . Let  $U^*$  be the result of applying the indicator construction with respect to  $(P_r, 0, r)$  to  $U(n; a_1, a_2, a_3)$ . We show that  $U^*$ -COL is NP-complete.

Suppose first that  $\gcd\{r, n\} = 1$ . Then  $U^* = U(n; 1, 2, p)$ . If  $p-2 \neq 1-p \pmod{n}$ , then  $\max\{1, p-2, 1-p\}$  is unique, whence the result follows from Lemma 6.4. If  $p-2 = 1-p \pmod{n}$ , we must have  $p-2 = (n+1)/2$ . Let  $U^{**}$  be the result of applying the indicator construction with respect to  $(P_{(n+1)/2}, 0, (n+1)/2)$  to  $U^*$ . Since  $\gcd\{(n+1)/2, n\} = 1$  and  $2-p = p-1 = (n+1)/2$ , the digraph  $U^{**}$  is isomorphic to  $U(n; 1, 2, 3)$ . Therefore the  $U^{**}$ -colouring problem is NP-complete, so  $U^*$ -COL is also NP-complete.

Now suppose that  $\gcd\{r, n\} = k$ . Let  $n' = n/k$ . Since  $U$  is not symmetric  $n' > 3$ . It is easy to see that the directed cycle in  $U^*$  has length  $n'$ . If the vertices  $a_1, a_2, a_3$  all belong to the same connected component of  $U^*$ , the result follows via an argument similar to the above. Otherwise, the core of  $U^*$  is isomorphic to

$U(n'; 1, 2)$ , whence  $U^*$ -COL is NP-complete by the argument in the first paragraph.

We have shown in all cases that  $U^*$ -COL is NP-complete. Therefore  $U$ -COL is also NP-complete. ■

**Conjecture.** If  $U$  is asymmetric, then  $U$ -COL is NP-complete.

We now turn our attention to acyclic digraphs. In analogy with the unicyclic digraphs discussed above, let  $n \geq 3$ ,  $t \geq 1$ , and let  $a_1, a_2, \dots, a_t$  be integers such that

$$0 \leq a_1 < a_2 < \dots < a_t < n.$$

The digraph  $A(n; a_1, a_2, \dots, a_t)$  is constructed from  $P_n \cup \{v\}$  by adding the arcs  $\{va_i; i=1, 2, \dots, t\}$ . When the context is clear, for brevity we denote this digraph by  $A$ . Note that  $A$  is acyclic. If the integers  $d_i = a_{i+1} - a_i$  are all equal ( $i=1, 2, \dots, t$ ), we say that  $A$  is symmetric, and we denote by  $d$  to denote the common value of  $a_{i+1} - a_i$  (where  $a_1 - a_t$  is calculated modulo  $n$ ). Each digraph  $A$  may be obtained from a symmetric unicyclic digraph of the type described above by splitting vertex 0 into two independent vertices, say  $0'$  and  $0''$ , adding the arcs  $n0'$  and  $0''1$  and, if  $v0$  was an arc of  $U$ , also the arcs  $v0'$  and  $v0''$ .

We first describe an infinite class of polynomial  $A$ -colouring problems.

**6.6. Lemma.** Let  $A$  be symmetric. Then a digraph  $G \rightarrow A$  if and only if  $G \rightarrow U(n; 0, d, 2d, \dots, (n-1)d)$  and there exists  $i$  such that  $i \equiv a_1 \pmod{d}$  and  $[i]_U$  contains only sources and sinks.

**Proof.**

( $\Rightarrow$ ) This is clear, since  $A \rightarrow U$ , and only sources of  $G$  (resp. sinks of  $G$ ) can map to vertex 0 (resp.  $n$ ) of  $A$ .

( $\Leftarrow$ ) Suppose  $G \rightarrow U$  and there exist  $i$  such that  $i \equiv a_0 \pmod{d}$  and  $[i]$  contains only sources and sinks. We can define an  $A$ -colouring of  $G$  by mapping the sources in  $[i]$  to vertex 0 of  $A$ , the sinks in  $[i]$  to vertex  $n+1$  of  $A$ ,  $[v]$  to vertex  $v$  of  $A$ , and for  $j = i+1, i+2, \dots, j-1$ , mapping  $[j]$  to vertex  $j-i$  of  $A$ . A moment's reflection should convince the reader that this is an  $A$ -colouring of  $G$ . ■

**6.7. Corollary.** If  $\hat{A}$  is symmetric, then  $A$ -COL is polynomial. ■

We now describe an infinite family of NP-complete  $A$ -colouring problems. The smallest of these digraphs  $A$  has six vertices and eight arcs, and is the smallest acyclic digraph for which the  $H$ -colouring problem is known to be NP-complete. Gutjahr has proved that there is no such digraph on four vertices [Gutjahr, 1988].

**6.8. Theorem.** Let  $A$  be the digraph constructed from  $P_4 \cup \{v\}$  by adding the arcs  $\{v0, v1, v3, v4\}$ . Then  $A$ -COL is NP-complete.

**Proof.**

The transformation is from ONE-IN-THREE SAT without negated variables. Accordingly, let an instance of ONE-IN-THREE SAT without negated variables be given, with variables  $x_1, x_2, \dots, x_p$  and clauses  $C^1, C^2, \dots, C^q$ .

Construct a digraph  $G$  from  $\{x_1, x_2, \dots, x_p\}$  and  $q$  copies of the digraph  $Y$  shown in figure 6.3 as follows. If  $C^i = l_1 \vee l_2 \vee l_3$  ( $1 \leq i \leq q$ ) then identify the vertices  $y_1, y_2,$

and  $y_3$  in the  $i^{\text{th}}$  copy of  $Y$  with  $l_1$ ,  $l_2$ , and  $l_3$ , respectively. Clearly the digraph  $G$  can be constructed in polynomial time.

**CLAIM.** The digraph  $G$  is  $A$ -colourable if and only if there exists a truth assignment in which each clause has exactly one true variable.

**PROOF.**

( $\Rightarrow$ ) In any  $A$ -colouring of  $Y$ , the vertices  $z_1, z_2, z_3$  are coloured by a vertex on  $P_4$  and one of these vertices is coloured by 2. Thus exactly one of  $y_1, y_2, y_3$  is coloured by 1. Define a truth assignment by setting  $x_i = T$  just if  $colour(x_i) = 1$ . By the above argument, each clause contains exactly one true variable.

( $\Leftarrow$ ) Define a partial colouring of  $G$  by setting  $colour(x_i) = 2$  if  $x_i = T$ , and  $colour(x_i) = v$  otherwise. This partial colouring can be extended to an  $A$ -colouring of each copy of  $Y$ , and hence to an  $A$ -colouring of  $G$ .

The result now follows. ■

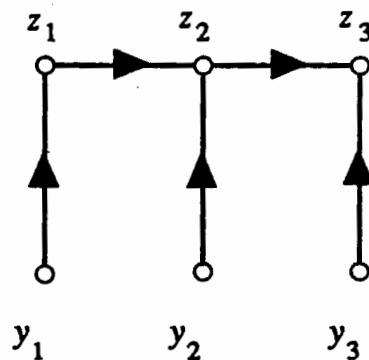


Figure 6.3. The digraph  $Y$ .

Let  $A^+$  be any digraph constructed from  $P_n \cup \{v\}$  by adding some arcs from  $v$  to the directed  $n$ -path and such that  $A^+$  contains  $H$ .

**6.9. Corollary.** The  $A^+$ -colouring problem is NP-complete.

**Proof.**

The proof is almost identical to Theorem 6.8, only the digraph  $Y$  is different. We only sketch the argument, the remaining details may be easily supplied by the reader. Suppose that there is a copy of  $A$  induced by vertices  $\{v, k, k+1, \dots, k+4\}$ . Let  $r = k$  if  $x_1$  is an arc of  $A^+$  and  $k \neq 1$  and  $(k-1)$  otherwise. Let  $s = n - k - 4$ . The digraph  $Y^+$  is constructed from  $P_r$ ,  $P_s$ , and  $Y$  by identifying vertex  $r$  of  $P_r$  with  $z_1$ , and vertex  $0$  of  $P_s$  with  $z_3$ . In any  $A^+$ -colouring of  $Y^+$ , vertices  $z_1$ ,  $z_2$ , and  $z_3$  are coloured by  $\{k, k+1, \dots, k+4\}$ , and exactly one of them is coloured  $k+2$ . Conversely, any partial colouring of  $y_1, y_2, y_3$  by  $k+2$ , and  $v$  such that exactly one  $y_j$  ( $1 \leq j \leq 3$ ) is coloured  $v$  can be extended to an  $A^+$ -colouring of  $Y^+$ .

■

Gutjahr has recently proved that  $A(4; 0, 2, 3)$ -COL is polynomial [Gutjahr, 1989]. This provides an example of a polynomial asymmetric  $A$ -colouring problem and another mystery to solve.

In this chapter we have contributed to the study of the  $H$ -colouring problem when the digraph  $H$  is acyclic or unicyclic. There is some suggestion in our results that the automorphism group may play a role in any classification.

## 7. References.

- Albertson, M., P. Catlin and L. Gibbons, 1985. Homomorphisms of 3-chromatic graphs II. *Congressus Num.* 47, 19-28.
- Bang-Jensen, J., 1989. On the complexity of generalized colourings of directed graphs. Internal Report No. 3, Odense Universitet (Institut for Matematik og Datalogi).
- Bang-Jensen, J. and P. Hell, 1988. The effect of two cycles on the complexity of colourings by directed graphs. *Discrete Applied Math.* (to appear).
- Bang-Jensen, J., P. Hell and G. MacGillivray, 1988. The complexity of colourings by semicomplete digraphs. *SIAM J. Discrete Math.* 1, 281-298.
- Bloom, G. and S. Burr, 1987. On unavoidable digraphs in orientations of graphs. *J. Graph Theory* 11, 453-462.
- Bondy, J. A. and U. S. R. Murty, 1976. *Graph Theory with Applications.* Elsevier Press, New York.
- Brouwer, A. E. and H. J. Veldman, 1987. Contractability and NP-completeness. *J. Graph Theory* 11, 71-79.
- Chung, F.R.K., R. Graham and M. Saks, 1989. A dynamic location problem for graphs. *Combinatorica* (to appear).
- Cook, S., 1971. On the complexity of theorem-proving procedures. *Proc. 3<sup>rd</sup> ACM Symposium on Theory of Computing*, ACM New York, 151-158.
- Fellner, W.D., 1982. On minimal graphs. *Theoretical Comp. Sci.* 17, 103-110.

- Fortune, S., J. Hopcroft and J. Wyllie, 1980. The directed subgraph homeomorphism problem. *Theoretical Comp. Sci.* 10, 111-121.
- Garey, M. R. and D. S. Johnson, 1979. *Computers and Intractability*. W. H. Freedman, San Francisco.
- Garey, M.R. and D.S. Johnson, 1976. The complexity of near-optimal graph colouring. *J.A.C.M.* 23, 43-49.
- Garey, M.R., D.S. Johnson and H.C. So, 1976. An application of graph colouring to printed circuit testing. *IEEE Trans. on Circuits and Systems*, CAS 23, 591-598.
- Gutjahr, W., 1988. *Farbung durch gerichtete Graphen*. Diplomarbeit, Institute for Information Processing, IIG, Technical University Graz.
- Gutjahr, W., 1989. Ph.D. Thesis. Technical University Graz, (in preparation).
- Gutjahr, W., E. Welzl and G. Woeginger, 1989. Polynomial graph colourings. Technical Report B-88-06, Freie Universitat Berlin.
- Gutjahr, W., E. Welzel and G. Woeginger, 1989. Personal communication.
- Häggkvist, R., P. Hell, D. J. Miller and V. Neuman Lara 1988. On multiplicative graphs and the product conjecture. *Combinatorica* 8, 63-74.
- Häggkvist, R. and P. Hell, 1989. Universality of A-mote graphs. Preprint.
- Hedrlin, Z. and A. Pultr, 1965. Symmetric relations (undirected graphs) with given semigroups. *Montash. Math* 69, 318-322.
- Hell, P., 1974. Absolute planar retracts and the four-colour conjecture. *J.C.T. B* 17, 5-10.

Hell, P. and G. MacGillivray, 1987. The complexity of tournament colourings of digraphs. Technical Report LCCR-12, Laboratory for Computer Communications and Research, Simon Fraser University.

Hell, P. and J. Nešetřil, 1986. On the complexity of H-colouring. J.C.T. Ser B (to appear).

Johnson, D.S., 1974. Worst case behaviour of graph colouring algorithms. Proc. 5<sup>th</sup> Southeast conference on Combinatorics, Graph Theory and Computing, Utilitas Math., 513-527.

Johnson, D.S., 1982. The NP-completeness column - An ongoing guide. J. Algorithms 3, 89-99.

Karp, R., 1972. Reducibility among combinatorial problems. in R.E. Miller and J.W. Thatcher (eds) Complexity of Computer Computations, Plenum Press, New York, 85-103.

Komarek, P., 1984. Some new good characterisations of directed graphs. Cas. pro Pest. Mat. 51, 348-354.

Levin, L.A., 1973. Universal sequential search problems. Problems of Information Transmission 9, 265-266.



Lovász, L., 1973. Coverings and colorings of hypergraphs. Proc. 4<sup>th</sup> Southeast Conference on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Publishing, Winnipeg, 3-12.

Leighton, F.T., 1979. A graph colouring algorithm for large scheduling problems. J. Res. Nat. B. Standards 84, 489-496.

Maurer, H. A., J. H. Sudborough and E. Welzl, 1981. On the complexity of the general colouring problem. Information and Control, 51, 123-145.

Maurer, H.A., A. Salomaa and D. Wood, 1981. Colourings and interpretations: a connection between graphs and grammar forms. Discrete Appl. Math. 3, 119-135.

Moon, J.W., 1966. On subtournaments of a tournament. Canadian Math Bull., 9, 297-301.

Moon, J.W., 1969. Topics on Tournaments. Holt, Reinhart, Winston, New York.

Nešetřil, J., 1982. Representations of graphs by means of products and their complexity. MFCS, 94-102.

Nešetřil, J. and A. Pultr, 1978. On classes of relations determined by subobjects and factorobjects. Discrete Math. 22, 287-300.

Schaefer, T.J., 1978. The complexity of satisfiability problems. Proc. 10<sup>th</sup> ACM Symp. on Theory of Computing, A.C.M., New York, 216-226.

Welzl, E., 1982. Colour families are dense. Theoretical Comp. Sci. 17, 29-41.

Welzl, E., 1987. Personal communication.

Wigderson, A., 1983. Improving the performance guarantee for approximate graph colouring. J.A.C.M. 30, 729-735.