WREATH PRODUCTS AND VARIETIES OF INVERSE SEMIGROUPS

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WREATH PRODUCTS AND VARIETIES OF INVERSE SEMIGROUPS

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ABSTRACT

Given two varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, define Wr($\mathscr{U}$, $\mathscr{V}$) to be the variety of inverse semigroups generated by wreath products of semigroups in $\mathscr{U}$ with semigroups in $\mathscr{V}$. The principal result of this work is a description of the fully invariant congruence on the free inverse semigroup corresponding to Wr($\mathscr{U}$, $\mathscr{V}$) in terms of the fully invariant congruences corresponding to $\mathscr{U}$ and $\mathscr{V}$, where $\mathscr{U}$ and $\mathscr{V}$ are varieties of inverse semigroups. This description makes use of a graphical representation of inverse semigroups with presentations, due to Stephen, which is the inverse semigroup theoretic analogue to the Cayley graphs of group theory. We further show that Wr, considered as a binary operator on the lattice $\mathscr{L}(\mathscr{I})$ of varieties of inverse semigroups, is associative. Thus, the lattice of varieties of inverse semigroups is a semigroup ($\mathscr{L}(\mathscr{I})$, Wr) under the operation Wr.

Using these results we investigate properties possessed by varieties of the form Wr($\mathscr{U}$, $\mathscr{V}$). We show that when $\mathscr{U}$ is a group variety, Wr($\mathscr{U}$, $\mathscr{V}$) is the more familiar Mal'cev product variety $\mathscr{U} \circ \mathscr{V}$. The principal result also provides us with a solution to the word problem for the relatively free objects in Wr($\mathscr{U}$, $\mathscr{V}$) given solutions to the word problem for the relatively free objects in the varieties $\mathscr{U}$ and $\mathscr{V}$. We show that when the varieties $\mathscr{U}$ and $\mathscr{V}$ have E-unitary covers over the group varieties $\mathscr{W}$ and $\mathscr{X}$, respectively, then Wr($\mathscr{U}$, $\mathscr{V}$) has E-unitary covers over the group variety Wr($\mathscr{W}$, $\mathscr{X}$). Further properties of varieties of this form are presented as well as a discussion of the basic properties of the semigroup ($\mathscr{L}(\mathscr{I})$,Wr). We conclude this work by showing that several special intervals in $\mathscr{L}(\mathscr{I})$ corresponding to ν-classes and whose maximum member is of the form Wr($\mathscr{U}$, $\mathscr{B}^1$) are infinite, where $\mathscr{U}$ is a variety of abelian groups and $\mathscr{B}^1$ is the variety of inverse semigroups generated by the five element Brandt semigroup with an identity adjoined.

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Introduction

An inverse semigroup S is a set with an associative binary operation, usually referred to as multiplication, and a unary operation of inversion which satisfies the property that every element of S has a unique inverse in the sense of von Neumann. V.V. Wagner in 1952 was the first to study inverse semigroups, though he called them 'generalized groups' and defined them as regular semigroups in which the idempotents commute. In 1953 Liber proved that the two definitions are in fact equivalent. Preston later (and independently) rediscovered this class of semigroups and called them 'inverse semigroups', the name most widely used today.

Every inverse semigroup is isomorphic to a semigroup of partial one-to-one transformations on a nonempty set. This is the substance of the Wagner Representation Theorem which is the inverse semigroup theoretic analogue to the Cayley Representation Theorem of group theory. Indeed, the Wagner representation of a group is the Cayley representation of that group. Thus, just as we often find it convenient to think of groups as permutations, we often think of inverse semigroups as semigroups of partial one-to-one transformations.

Given two inverse semigroups, a new inverse semigroup can be obtained by forming their wreath product. By the wreath product S wr (T,I) of S and T, where T is a semigroup of partial one-to-one transformations on the set I, we mean the set of pairs (ψ,β) where $\beta \in T$, ψ is a mapping from I into S and the domains of ψ and β are equal, with products defined by

$$(\psi_1,\beta_1)(\psi_2,\beta_2) = (\psi_1{}^{\beta_1}\psi_2,\beta_1\beta_2) \qquad ((\psi_1,\beta_1),(\psi_2,\beta_2) \in \text{S wr (T,I)})$$

where, for all i in the domain of $\beta_1\beta_2$, $i\psi_1{}^{\beta_1}\psi_2 = (i\psi_1)(i\beta_1\psi_2)$. Note that, given any two inverse semigroups S and T, we can always form the wreath product of S and T by taking the Wagner representation of T. Wreath products are of fundamental importance in the study of inverse semigroups and play a central role in our investigations.

Inverse semigroups are determined by associativity and the laws $xx^{-1}x = x$, $(x^{-1})^{-1} = x$ and $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$. Thus, the class of inverse semigroups (considered as algebras of type $\langle 2,1\rangle$) forms a variety and we may approach the study of inverse semigroups from the perspective of their lattice of varieties. This approach not only suggests a possible framework from which we may tackle classification problems, but it has proved itself essential in the study of the structure of inverse semigroups by the identities they satisfy.

The focus of our investigations is a binary operator Wr on the lattice of varieties of inverse semigroups. Given two varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, we define $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ to be the variety generated by wreath products of members of $\mathscr{U}$ with members of $\mathscr{V}$ which are represented as partial one-to-one transformations on some nonempty set. The principal result of this thesis is a characterization of the fully invariant congruence on the free inverse semigroup corresponding to $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ in terms of the fully invariant congruences corresponding to $\mathscr{U}$ and $\mathscr{V}$, where $\mathscr{U}$ and $\mathscr{V}$ are varieties of inverse semigroups. Our motivation for studying this class of varieties is essentially twofold. First of all, every completely semisimple inverse semigroup is a subdirect product of inverse subsemigroups of wreath products of the form G wr $\mathscr{I}(I)$, where G is a group and $\mathscr{I}(I)$ is the inverse semigroup of all partial one-to-one transformations on the nonempty set I. Thus, every variety whose free objects are completely semisimple (and they are many) is generated by inverse subsemigroups of wreath products. Secondly, the relation ν defined on the lattice of varieties of inverse semigroups by $\mathscr{U} \,\nu\, \mathscr{V}$ if and only if $\mathscr{U} \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G}$ and $\mathscr{U} \vee \mathscr{G} = \mathscr{V} \vee \mathscr{G}$, where $\mathscr{G}$ is the variety of all groups, is a congruence [**K1**] and, by a result due to Reilly [**Re1**], if $\mathscr{V}$ is a combinatorial variety, the ν-class of $\mathscr{V} \vee \mathscr{U}$, for some variety of groups $\mathscr{U}$, is the interval $[\mathscr{U} \vee \mathscr{V}, \mathscr{U} \circ \mathscr{V}]$, where $\mathscr{U} \circ \mathscr{V}$ is the Mal'cev product of $\mathscr{U}$ and $\mathscr{V}$. It turns out that whenever $\mathscr{U}$ is a variety of groups, $\mathrm{Wr}(\mathscr{U},\mathscr{V}) = \mathscr{U} \circ \mathscr{V}$, and so a description of the fully invariant congruence corresponding to $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ is of some interest.

There is a connection between these two motivating factors and this connection forms the basis for our principal result, which is generalized beyond the specific classes of varieties mentioned above. The first factor is closely related to representations by right translations, which we must 'decode' in order to determine the laws of the varieties mentioned in the second factor. This 'decoding' is made possible by yet another representation of inverse semigroups, this time as directed inverse word graphs over some label set X. This representation, due to Stephen [S], is called the Schützenberger representation and is the inverse semigroup analogue to the Cayley graphs of group theory. Unlike the group case, in which there is one underlying graph representing a group with respect to some presentation, an inverse semigroup (with presentation) has one underlying graph for each $\mathscr{D}$-class. When considering whether the variety $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ satisfies the identity $u = v$, where u and v are words over some alphabet X, we look at, first of all, whether the variety $\mathscr{V}$ satisfies $u = v$ and if so, whether $\mathscr{U}$ satisfies an identity determined by the paths labelled by u and v in the Schützenberger representation of u and v relative to the presentation $(X;\rho(\mathscr{V}))$, where $\rho(\mathscr{V})$ is the fully invariant congruence on the free inverse semigroup corresponding to $\mathscr{V}$.

It turns out that Wr is an associative operator on the lattice of varieties of inverse semigroups which, when restricted to the lattice of varieties of groups, is the well-known product operator. While the lattice of varieties of groups under Wr is freely generated by its indecomposable members, the same cannot be said for the lattice of varieties of inverse semigroups. We can however, use our description of the fully invariant congruence corresponding to $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ to discover some interesting results concerning familiar classes of varieties, including Mal'cev products of the form $\mathscr{U} \circ \mathscr{V}$ where $\mathscr{U}$ is a variety of groups and varieties whose free objects are E-unitary and their subvarieties.

The following is a brief outline of each chapter of this thesis.

Chapter 2 is devoted to preliminary material required in the sequel including fundamental results and definitions of inverse semigroup theory as well as the basics on the Wagner representation, the Translational Hull of an inverse semigroup, Varieties of inverse semigroups and Schützenberger graphs.

Since wreath products figure prominently in our investigations, Chapter 3 is concerned with the basic results we will require in subsequent chapters on this subject. The first section of this chapter deals with the definition of wreath product. Section 2 deals with showing that completely semisimple inverse semigroups are isomorphic to subdirect products of inverse subsemigroups of semigroups of the form G wr (T,I) where G is a group and T is an antigroup. The final section of chapter 3 presents some basic facts concerning wreath products of inverse semigroups.

In Chapter 4 we present our Main Theorem which characterizes the fully invariant congruence on the free inverse semigroup corresponding to Wr($\mathscr{U}$,$\mathscr{V}$) in terms of the fully invariant congruences corresponding to $\mathscr{U}$ and $\mathscr{V}$, for varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups. The first section introduces the notion of the doubly labelled Schützenberger graph and from it we define, for any given word w and variety $\mathscr{V}$, an associated word dependent upon $\mathscr{V}$ called the derived word of w with respect to $\mathscr{V}$. The derived word of w with respect to $\mathscr{V}$ is an 'encoding' of the path labelled by w in the Schützenberger graph of w with respect to $\mathscr{V}$. We use this encoding in our Main Theorem, which is the subject of Section 2. Section 3 is concerned with basic properties of varieties of the form Wr($\mathscr{U}$,$\mathscr{V}$), including the result that, when $\mathscr{U}$ is a group variety, Wr($\mathscr{U}$,$\mathscr{V}$) = $\mathscr{U} \circ \mathscr{V}$, the Mal'cev product of $\mathscr{U}$ and $\mathscr{V}$. The last section of this chapter deals with the associativity of the Wr operator.

Consequences of the Main Theorem are presented in Chapter 5. In section 1 we show that the Wr($\mathscr{U}$,$\mathscr{V}$)-free semigroups have solvable word problem if both the $\mathscr{U}$-free and the $\mathscr{V}$-free semigroups have solvable word problem and also that Wr($\mathscr{U}$,$\mathscr{V}$) is locally finite if and only if both $\mathscr{U}$ and $\mathscr{V}$ are locally finite. Section 2 contains results concerning E-unitary covers which utilize a graphical description, due to Meakin and Margolis, of varieties of the form $\mathscr{U}^{max} = [w = w^2 : w = w^2$ is a law in $\mathscr{U}]$, where $\mathscr{U}$ is a variety of groups. The third section is devoted to results concerning varieties of the form Wr($\mathscr{S}$,$\mathscr{V}$). It turns out that Wr($\mathscr{S}$,$\mathscr{V}$) is the largest variety satisfying those identities $w = w^2$ that are satisfied by $\mathscr{V}$. This chapter concludes with some basic results concerning the semigroup of varieties of inverse semigroups under the operation of Wr.

In the final chapter we look at the intervals $[\mathscr{U} \vee \mathscr{B}^1, \mathscr{U} \circ \mathscr{B}^1]$ where $\mathscr{U}$ is a variety of abelian groups and $\mathscr{B}^1$ is the variety generated by a special six-element inverse semigroup (the five-element Brandt semigroup with an identity adjoined). For each of these intervals, we construct an infinite chain of varieties using only a minimal knowledge of the relatively free object on a countably infinite set in the variety $\mathscr{B}^1$.

CHAPTER TWO

Preliminaries

The fundamental definitions and results of inverse semigroup theory which are required in the sequel are presented in this chapter. The principle source used is *Inverse Semigroups* by Mario Petrich **[P]**. For the fundamentals of semigroup theory, the reader is referred to Clifford and Preston **[CP]**. The material on Schützenberger graphs comes from Stephen **[S]**. For basic universal algebraic results concerning varieties, we refer the reader to either Burris and Sankapanavar **[BS]** or Grätzer **[G]**. It is assumed that the reader is familiar with the notion of a lattice and the basic definitions and results concerning lattices. A standard text on this subject is Birkhoff's *Lattice Theory* **[Bi1]**. Most of the results of sections 2.3 through 2.7 can be found in **[P]**. We will cite the reference to **[P]** when the result is stated and provide the original source in the final paragraphs of these sections.

2.1 Semigroups

A *semigroup* is a pair $(S,\cdot)$ where S is a set and $\cdot$ is an associative binary operation, usually referred to as multiplication. Unless there is the possibility of ambiguity, we denote the semigroup $(S,\cdot)$ by S and denote products in S by juxtaposition. A familiar example of a semigroup is the set of functions on a nonempty set X under the operation of composition.

A semigroup may possess special elements which are distinguished by certain characteristics. Let S be a semigroup.

An element $s \in S$ is an *identity* if $sx = xs = x$, for all $x \in S$. If S possesses an identity then it is unique and is denoted by 1 or 1_S if we wish to emphasize that it is the identity of S. A semigroup which has an identity is called a *monoid*. Given an arbitrary semigroup S, we define S^1 to be S if S is a monoid or $(S \cup \{1\}, \cdot)$ with $1 \cdot x = x \cdot 1 = x$, for all $x \in S$, if S is not a monoid. It is easy to see that S^1 is a monoid.

An element $s \in S$ is a *zero* if $sx = xs = s$, for all $x \in S$. If S possesses a zero then it is unique and is denoted by 0 or 0_S if we wish to emphasize that it is the zero of S. The semigroup S^0 is defined to be S, if S possesses a zero, or $(S \cup \{0\}, \cdot)$ with $0 \cdot x = x \cdot 0 = 0$, for all $x \in S$, otherwise. If T is a subset of S, but not a semigroup with 0, and T satisfies the property that, for any $a,b,c \in T$, ab and (ab)c are elements of T if and only if bc and a(bc) are elements of T, then we define T^0 to be the set $T \cup \{0\}$ with multiplication $\cdot$ given by $t_1 \cdot t_2 = t_1 t_2$ if $t_1 t_2 \in T$, $t_1 \cdot t_2 = 0$, otherwise and $0 \cdot t_1 = t_1 \cdot 0 = 0$, for all $t_1, t_2 \in T$. One easily verifies that T^0 is a semigroup.

An element $e \in S$ is an *idempotent* if $e = e^2$. The set of idempotents of S is denoted E_S. The relation $\leq$ on E_S defined by $e \leq f$ if and only if $e = ef = fe$, for all $e,f \in E_S$, is a partial order and is called the *natural partial order* of E_S. If S has no zero, an idempotent $e \in E_S$ is *primitive* if it is minimal in the natural partial order of E_S. If S has a zero, $e \in E_S$ is *primitive* if it is minimal in $E_S \setminus \{0\}$.

2.2 Inverse semigroups

Let S be a semigroup. An element $s \in S$ is *regular* if there exists an $x \in S$ such that $s = sxs$. The semigroup S is said to be *regular* if all its elements are regular. The element x is an *inverse* of s if $s = sxs$ and $x = xsx$.

A regular semigroup whose idempotents commute is an *inverse semigroup*. An equivalent definition of inverse semigroup is a semigroup in which each element has a unique inverse **[P;II.1.2]**. The former definition is due to Wagner **[Wa1]** who was the first to study inverse semigroups, though he called them 'generalized groups'. The latter definition is due to Liber who, in [L], showed that the two definitions are equivalent. An inverse semigroup which is also a monoid is called an *inverse monoid.* For any element s in an inverse semigroup S, we denote the unique inverse of s by s^{-1}.

The set of partial one-to-one transformations on a nonempty set X under the operation of composition is an important example of inverse semigroups. This semigroup is called the *symmetric inverse semigroup on* X and is denoted by $\mathscr{I}(X)$. It is easy to verify that if S is an inverse semigroup then both S^1 and S^0 are inverse semigroups. Moreover, if T is a subset of S such that $t \in T$ implies that $t^{-1}, tt^{-1} \in T$, and T satisfies the property mentioned in the definition of T^0, then T^0 is an inverse semigroup.

2.3 Fundamentals

Throughout this section S is an inverse semigroup.

Inverse semigroups are partially ordered algebras. Define the relation $\leq$ on S by

$$s \leq t \iff s = et \text{ for some } e \in E_S \qquad (s,t \in S).$$

It is a simple task to verify that $\leq$ is a partial order on S. The relation $\leq$ is called the *natural partial order* on S. The following are equivalent characterizations of $\leq$ (See **[P;II.1.6]**):

$$\begin{aligned} s \leq t &\iff s = te \text{ for some } e \in E_S \\ &\iff s = ss^{-1}t \\ &\iff s = ts^{-1}s \qquad (s,t \in S). \end{aligned}$$

Observe that the natural partial order on S restricted to E_S coincides with the natural partial order on E_S defined in the previous section.

Let S be an inverse semigroup. A subset T of S is an *inverse subsemigroup* of S if T is closed under the operations of S; that is, for all $t_1,t_2 \in T$, $t_1t_2 \in T$ and $t_1^{-1} \in T$. It is not true in general that a subsemigroup of an inverse semigroup is an inverse semigroup. An example which illustrates this is $T = \{ (1 \to 2), \emptyset \}$, where $(1 \to 2)$ is the member of $\mathscr{I}(\{1,2\})$ with domain $\{1\}$ which maps 1 to 2. T is a subsemigroup, but not an inverse subsemigroup, of $\mathscr{I}(\{1,2\})$. If S is a monoid and T is a subsemigroup of S such that

$1_S \in T$, then T is an *inverse submonoid* of S. If K is a subset of S then the *inverse subsemigroup of* S *generated by* K is the intersection of all subsemigroups of S containing K. We say that the inverse subsemigroup T of S is *full* if $E_S \subseteq T$, and we say that T is *closed* if, for all $x \in T$, $y \in S$, $x \leq y$ implies that $y \in T$. The *closure of* T in S, denoted $T\omega$, is the set $\{ s \in S : s \geq t \text{ for some } t \in T \}$. If $T\omega = T$ then we say that T is *closed.*

A nonempty subset I of S is a *right ideal* if

$$IS = \{ts : s \in S, t \in I\} \subseteq I.$$

A nonempty subset I of S is a *left ideal* if

$$SI = \{st : s \in S, t \in I\} \subseteq I.$$

A subset I of S is a (two-sided) *ideal* of S if it is both a right ideal and a left ideal. Equivalently, I is an ideal of S if

$$SIS = \{s_1ts_2 : s_1,s_2 \in S, t \in I\}.$$

For any element $s \in S$, the *principal right ideal generated by* s is the intersection of all right ideals containing s and is denoted by R(s). The *principal left ideal generated by* s and the *principal ideal generated by* s are defined similarly and are denoted by L(s) and J(s), respectively. It is not difficult to show that $R(s) = sS$, $L(s) = Ss$ and $J(s) = SsS$.

An inverse semigroup is *simple* if it has no proper ideals. If S has a zero, then S is *0-simple* if $S^2 \neq 0$ and S has no proper nonzero ideals. A simple inverse semigroup possessing a primitive idempotent is called a *completely simple* inverse semigroup and likewise, a 0-simple inverse semigroup possessing a primitive idempotent is called a *completely 0-simple* inverse semigroup. The intersection, if nonempty, of all ideals of S is called the *kernel* of S. Note that the kernel of S, if it exists, is a simple semigroup.

The relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ on S, called *Green's relations*, are of fundamental importance and are defined as follows. For all $s,t \in S$,

$$s \,\mathcal{R}\, t \iff R(s) = R(t);$$

$$s \,\mathcal{L}\, t \iff L(s) = L(t);$$

$s \mathscr{J} t \iff J(s) = J(t)$;

$s \mathscr{H} t \iff s \mathscr{R} t$ and $s \mathscr{L} t$;

$s \mathscr{D} t \iff$ there exists an $x \in S$ such that $s \mathscr{R} x$ and $x \mathscr{L} t$.

Clearly, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$ and $\mathscr{H}$ are equivalence relations. Furthermore, it can be shown that $\mathscr{D}$ is an equivalence relation which can equivalently be defined by $s \mathscr{D} t$ if and only if there exists an $x \in S$ such that $s \mathscr{L} x$ and $x \mathscr{R} t$.

For any $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{J}, \mathscr{H}, \mathscr{D}\}$, define the *$\mathscr{K}$-class* of $s \in S$ by $K_s = \{x \in S : s \mathscr{K} x\}$. For $\mathscr{K} \in \{\mathscr{R}, \mathscr{L}, \mathscr{J}\}$ there is a partial order on the $\mathscr{K}$-classes of S given by $K_s \leq K_t$ if and only if $K(s) \subseteq K(t)$.

The following is a list of basic results concerning Green's relations in inverse semigroups.

Lemma 2.3.1. Let S be an inverse semigroup.

a) Every $\mathscr{R}$-class and every $\mathscr{L}$-class of S contains exactly one idempotent **[P;II.1.2]**;

b) If e is an idempotent of S then H_e is a maximal subgroup of S and conversely, if G is a maximal subgroup of S then G is an $\mathscr{H}$-class of S **[P;I.7.5,I.7.6]**;

c) For any $s,t \in S$,

$s \mathscr{R} t \iff ss^{-1} = tt^{-1}$,

$s \mathscr{L} t \iff s^{-1}s = t^{-1}t$,

$s \mathscr{D} t \iff$ there exists an $x \in S$ such that $ss^{-1} = xx^{-1}$ and $x^{-1}x = t^{-1}t$;

d) For any $e,f \in E_S$, $J(e) \subseteq J(f)$ if and only if $e = aa^{-1}$ and $a^{-1}a \leq f$, for some $a \in S$.

(both (c) and (d) are from **[P;II.1.7]**)

In fact, property a) is an equivalent definition of an inverse semigroup.

A homomorphism from S into a semigroup T is a function ϕ such that, for all $s,t \in S$, $(s\phi)(t\phi) = st\phi$. Any homomorphic image of an inverse semigroup is necessarily an inverse semigroup [**P; II.1.10**]. Furthermore, by the definition of inverse semigroup, homomorphisms preserve inverses. That is, if ϕ is a homomorphism of S into T and $s \in S$, then $s^{-1}\phi = (s\phi)^{-1}$.

A congruence ρ on S is an equivalence relation satisfying the property that, for all $s,t,x \in S$, $s \rho t$ implies that $xs \rho xt$ and $sx \rho tx$. If ρ is a congruence on S then S / ρ is an inverse semigroup with multiplication given by $(s\rho)(t\rho) = st\rho$. S / ρ is called the *quotient semigroup* induced by ρ. We denote by ω and ε the universal relation on S and the identical relation on S, respectively. The set of all congruences on an inverse semigroup S forms a complete lattice under inclusion with greatest element ω and least element ε.

There is a strong connection between congruences and homomorphisms. Given a homomorphism $\phi : S \rightarrow T$, there is an associated congruence ϕ^* on S defined by $s \phi^* t$ if and only if $s\phi = t\phi$, for all $s,t \in S$. Conversely, given a congruence ρ on S, there is an associated homomorphism $\rho^{\#} : S \rightarrow S / \rho$ given by $s\rho^{\#} = s\rho$, for all $s \in S$.

Because congruences (and hence homomorphisms) play such an important role in our investigations, we present here some basic facts concerning congruences and list some special types.

Any congruence on an inverse semigroup S is uniquely determined by the union of its classes which contain idempotents and by its restriction to E_S. Let ρ be a congruence on S. Define the *trace* and *kernel* of ρ by

$$\text{tr } \rho = \rho \cap (E_S \times E_S)$$

$$\ker \rho = \{s \in S : s \rho e \text{ for some } e \in E_S\},$$

respectively. ρ is the unique congruence on S with trace equal to tr ρ and kernel equal to ker ρ [**P;III.1.5**]. If we think of tr as a mapping from the lattice of congruences on S into the lattice of congruences on E_S, then tr is a complete lattice homomorphism [**P;III.2.5**].

Likewise, ker, considered as a mapping from the lattice of congruences on S into the lattice of kernels (of congruences) of S, is a complete $\cap$ - homomorphism **[P;III.4.8]**.

For any congruences ρ and τ on S such that $\rho \subseteq \tau$, define the relation τ / ρ on S / ρ by $(x\rho)\ (\tau / \rho)\ (y\rho)$ if and only if $x \tau y$. Then τ / ρ is a confruence on S / ρ and $(S / \rho) / (\tau / \rho) \cong S / \tau$, **[P;I.4.15]**.

Let I be an ideal of S. Then the relation ρ_I on S defined by

$$s \rho_I t \iff s,t \in I \text{ or } s = t \qquad (s,t \in S)$$

is a congruence and is called the *Rees congruence* on S relative to I. The quotient semigroup S / ρ_I induced by ρ_I is called the *Rees quotient* semigroup (See **[P;I.5.3]**).

A congruence ρ on S is *idempotent separating* if, for any $e,f \in E_S$, $e \rho f$ implies that $e = f$. Thus, ρ is idempotent separating if and only if $\text{tr}\, \rho = \varepsilon$, the identical relation. Equivalently, ρ is idempotent separating if and only if $\rho \subseteq \mathscr{H}$ **[P;III.3.2]**. We denote by μ_S the greatest idempotent separating congruence on S. That μ_S exists is guaranteed by the fact that it is characterized by being the greatest congruence on S contained in $\mathscr{H}$. A further characterization is given by

$$s \mu_S t \iff s^{-1}es = t^{-1}et \text{ for all } e \in E_S \qquad (s,t \in S).$$

A congruence ρ on S is *idempotent pure* if E_S is the union of ρ-classes. That is, ρ is idempotent pure if for all $s \in S$, $e \in E_S$, $s \rho e$ implies that $s \in E_S$. Thus ρ is idempotent pure if and only if $\ker \rho = E_S$. A useful characterization is ρ is idempotent pure if and only if $\rho \cap \mathscr{R} = \varepsilon$, **[P;III.4.2]**.

A congruence ρ on S is a *group congruence* if S / ρ is a group. The least group congruence on S, denoted σ_S, is given by

$$s \sigma_S t \iff se = te \text{ for some } e \in E_S$$

$$\iff s \geq x,\ t \geq x \text{ for some } x \in S \qquad (s,t \in S) \qquad \textbf{[P;III.5.2]}.$$

The last concept which we introduce in this section is that of direct product. If $\{S_i\}_{i \in I}$ is a family of inverse semigroups, their *direct product* is the inverse semigroup with underlying set the Cartesian product $\prod_{i \in I} S_i$ and coordinatewise multiplication. If, for all $i \in I$, $S_i = S$, then we write S^I, and call this direct product the *direct power* of S by I.

An inverse semigroup S is a *subdirect product* of an indexed family $\{S_i\}_{i \in I}$ of inverse semigroups if

i) S is an inverse subsemigroup of $\prod_{i \in I} S_i$;

ii) $(S)\pi_i = S_i$ for each $i \in I$ where π_i is the i^{th} projection map.

An embedding $\alpha : S \to \prod_{i \in I} S_i$ is a *subdirect embedding* if $(S)\alpha$ is a subdirect product of the S_i.

Green's relations are named for J.A. Green who introduced them in 1951 **[GrJ]**. The natural partial order on inverse semigroups was introduced by Wagner in **[Wa1]**. He was also the first to show that a congruence on an inverse semigroup is completely determined by its classes containing idempotents **[Wa2]**. The kernel-trace approach to the study of congruences on an inverse semigroup is due to Scheiblich **[Sc]**. This approach differs from the traditional 'kernel normal system' approach which we do not use here. That tr is a homomorphism was proved by Reilly-Scheiblich **[RS]** and D.G. Green showed that ker is a $\cap$-homomorphism in **[GrD]**. Munn **[Mu2]** showed that idempotent separating congruences are contained in $\mathcal{H}$ and Howie **[Ho]** proved the existence of μ, the greatest idempotent separating congruence. The characterizations of σ are due to Munn **[Mu1]** and Wagner **[Wa2]**.

2.4 Special Classes

There are several important classes of inverse semigroups which we find necessary to distinguish. The following is a list of those classes which figure prominently in our investigations.

2.4.1. Groups. It is immediate from the definition of an inverse semigroup that all groups are inverse semigroups. Furthermore, the class of completely simple inverse semigroups coincides with the class of groups. We denote the class of all groups by $\mathcal{G}$.

2.4.2. Semilattices and Clifford semigroups. A *semilattice* is an inverse semigroup in which every element is an idempotent. Such a semigroup is called a semilattice because under the natural partial order it forms a meet semilattice. Moreover, any meet semilattice Y is a semilattice under the operation given by $e \cdot f = e \wedge f$, for all $e,f \in Y$. Note that, for any inverse semigroup S, E_S is a semilattice.

A *Clifford semigroup* is an inverse semigroup which is a semilattice of groups. That is, the inverse semigroup S is a Clifford semigroup if there is a congruence ρ on S such that S / ρ is a semilattice and each of the ρ-classes is a group.

2.4.3. Brandt semigroups. A completely 0-simple inverse semigroup is a *Brandt semigroup*.

Let G be a group and I a nonempty set. Let $B(G,I) = I \times G \times I \cup \{0\}$, where $0 \notin I \times G \times I$, with multiplication $(i, g, j) \cdot (j, h, k) = (i, gh, k)$ and all other products equal to 0. It is a simple task to verify that with this multiplication B(G,I) is an inverse semigroup. In fact, an inverse semigroup S is a Brandt semigroup if and only if S is isomorphic to B(G,I) for some group G and nonempty set I [**P;II.3.5**]. The 'smallest' Brandt semigroup which is not a semilattice of groups is isomorphic to B(G,I) for $G = \{1\}$ and $|I| = 2$, and is denoted by B_2. We sometimes refer to B_2 as the five-element Brandt semigroup.

An inverse semigroup which is a subdirect product of Brandt semigroups and/or groups is called a *strict inverse semigroup*. A property that characterizes strict inverse semigroups is *$\mathcal{D}$-majorization* : For any $e,f,g \in E_S$, $e \geq f$, $e \geq g$, $f \mathcal{D} g$ imply $f = g$

[P;II.4.5]. Note that, in particular, if e and f are two comparable idempotents belonging to the same $\mathscr{D}$-class then e = f.

2.4.4. Completely semisimple inverse semigroups. Let S be an inverse semigroup. For every $a \in S$, define $I(a) = J(a) \setminus J_a = \{ s \in J(a) : J(s) \neq J(a) \}$. Whenever $I(a) \neq \varnothing$, I(a) is an ideal of S. The Rees quotient semigroup J(a) / I(a), where $J(a) / \varnothing = J(a)$, is called a *principal factor* of S. A semigroup in which every principal factor is completely simple or completely 0-simple is a *completely semisimple* semigroup. Thus, an inverse semigroup is completely semisimple if and only if all of its principal factors are Brandt semigroups or groups. Indeed, at most one principal factor of a completely semisimple inverse semigroup can be a group and that is the kernel, if it exists. Note that in a completely semisimple inverse semigroup $\mathscr{D} = \mathscr{J}$.

2.4.5. Combinatorial inverse semigroups. An inverse semigroup is *combinatorial* if the Green's relation $\mathscr{H}$ is the identical relation. That is, an inverse semigroup is combinatorial if its maximal subgroups are trivial.

2.4.6. Cryptic inverse semigroups. An inverse semigroup is *cryptic* if the Green's relation $\mathscr{H}$ is a congruence.

2.4.7. Antigroups. An inverse semigroup S is an *antigroup* if ε is the only congruence on S contained in $\mathscr{H}$. Equivalently, S is an antigroup if and only if $\mu_S = \varepsilon$. Note that all combinatorial inverse semigroups are antigroups and a cryptic inverse semigroup is an antigroup if and only if it is combinatorial. We denote the class of all antigroups by $\mathscr{A}$.

2.4.8. E-untary inverse semigroups. An inverse semigroup S is E-unitary if and only if, for all $a \in S$, $e \in E_S$, $a \geq e$ implies that $a \in E_S$. Equivalently, S is E-unitary if and only if σ_S, the least group congruence on S, is idempotent pure.

Semilattices of groups were introduced by Clifford in **[Cl]**. In the same paper, Clifford also showed that Brandt semigroups are isomorphic to B(G,I) for some group G and some nonempty set I, though he was considering Brandt groupoids (first studied by H. Brandt in 1927) with a zero adjoined with all undefined products set to zero. Munn **[Mu2]**, was the first to recognize that Brandt semigroups (that is, Brandt groupoids with a zero adjoined and all undefined products set to zero) were precisely the inverse completely 0-simple semigroups. What we call antigroups was introduced by Wagner, though another terminology, 'fundamental inverse semigroup', was coined by Munn. E-unitary inverse semigroups were first studied by Saitô **[Sa]** who called them proper and later by, among others, McAlister **[McA1]**, who called them reduced inverse semigroups. A great deal of research has concerned itself with E-unitary inverse semigroups; we mention only the work done by McAlister on the P-representation of E-unitary inverse semigroups **[McA1,McA2]**.

2.5 The Wagner representation

A fundamental result in group theory is the Cayley representation theorem which states that every group is isomorphic to a permutation group. The analagous result in the theory of inverse semigroups is the Wagner representation of an inverse semigroup by partial one-one transformations of a set. Every inverse semigroup is isomorphic to an inverse semigroup of one-one partial transformations on a nonempty set.

For any $\beta \in \mathscr{I}(X)$, the symmetric inverse semigroup on X, we denote by **d**β and **r**β the domain of β and the range of β, respectively.

Theorem 2.5.1 [P;IV.1.6]. Let S be an inverse semigroup. For each $s \in S$, let $\beta_s \in \mathscr{I}(S)$ be defined by

$$x\beta_s = xs \qquad [\, x \in \mathbf{d}\beta_s = Ss^{-1} \,].$$

Then the mapping

$$\beta : S \rightarrow \mathscr{I}(S) \text{ defined by } s\beta = \beta_s$$

is an embedding of S into $\mathscr{I}(S)$.

The Wagner representation of S restricted to a given $\mathscr{R}$-class provides another representation of S, though in general it is not faithful (that is, not one-to-one). For a given $\mathscr{R}$-class R of S, we call the following representation the *Wagner representation of S restricted to R*.

Theorem 2.5.2. Let S be an inverse semigroup and let R be a fixed $\mathscr{R}$-class of S. For each $s \in S$, let $\alpha_s \in \mathscr{I}(R)$ be defined by

$$x\alpha_s = xs \qquad [\, x \in \mathbf{d}\alpha_s = \{\, y \in R : ys \in R \,\} \,]$$

Then the mapping

$$\alpha : S \rightarrow \mathscr{I}(R) \text{ defined by } s\alpha = \alpha_s$$

is a homomorphism.

Proof: Let $s \in S$ and suppose that for some $x,y \in \mathbf{d}\alpha_s = \{\, y \in R : ys \in R \,\}$, $x\alpha_s = y\alpha_s$. Then $xs = ys$ and x,y and $xs = ys$ are $\mathscr{R}$-related. But then $x = xx^{-1}x = (xs)(xs)^{-1}x = xss^{-1}x^{-1}x = xss^{-1} = yss^{-1} = yss^{-1}y^{-1}y = yy^{-1}y = y$. Therefore, α_s is indeed an element of $\mathscr{I}(R)$.

Let $s, t \in S$. In order to show that α is a homomorphism we must show that $\alpha_s\alpha_t = \alpha_{st}$. We first compare their domains.

$$\mathbf{d}\alpha_s = \{ y \in R : ys \in R \}$$

$$\mathbf{d}\alpha_t = \{ y \in R : yt \in R \}$$

$$\mathbf{d}\alpha_{st} = \{ y \in R : yst \in R \}.$$

Therefore, $\mathbf{d}\alpha_s\alpha_t = \{ y \in R : ys \in R \text{ and } yst \in R \}$ and this is a subset of $\mathbf{d}\alpha_{st}$. On the other hand, if $y \in \mathbf{d}\alpha_{st}$, then y and yst are $\mathscr{R}$-related and so there is some $z \in S$ such that ystz = y. But then y and ys are $\mathscr{R}$-related and so $y \in \mathbf{d}\alpha_s\alpha_t$. Thus, $\alpha_s\alpha_t$ and α_{st} have identical domains. Since (xs)t = x(st) for all x in their common domain, $\alpha_s\alpha_t = \alpha_{st}$. As a result, α is a homomorphism. •

The Wagner representation is due to Wagner [**Wa1**] and was discovered independently by Preston [**Pr**].

2.6 The translational hull of an inverse semigroup

Though it plays a minor role in our investigations, the translational hull of an inverse semigroup has strong connections with the Wagner representation and the Schützenberger representations (discussed below in §2.8), both of which figure prominently in the sequel.

Let S be an inverse semigroup. A transformation ρ on S is a *right translation* of S if, for all $x,y \in S$, $(xy)\rho = x(y\rho)$. Likewise, a transformation λ is a *left translation* if $\lambda(xy) = (\lambda x)y$, for all $x,y \in S$. If, in addition, the left translation λ and the right translation ρ satisfy $x(\lambda y) = (x\rho)y$, for all $x,y \in S$, then the two are *linked* and the pair (λ,ρ) is a *bitranslation*. The set of all bitranslations on S under the operation of componentwise composition is an inverse semigroup and is called the *translational hull of* S [**P;V.1.4**]. We denote this semigroup by $\Omega(S)$. We note that either of the projection maps on $\Omega(S)$ is a monomorphism [**P;V.1.2**].

For any $s \in S$, the functions λ_s and ρ_s defined by $\lambda_s x = sx$ and $x\rho_s = xs$, for all $x \in S$, are left and right translations, respectively. In fact, (λ_s,ρ_s) is a bitranslation and so is a member of $\Omega(S)$. The mapping

$$\pi : s \rightarrow (\lambda_s,\rho_s) \qquad\qquad (s \in S),$$

is a monomorphism of S into $\Omega(S)$ and is called the *canonical homomorphism of* S *into* $\Omega(S)$.

It turns out that $\Omega(S)$ is isomorphic to the idealizer of the Wagner representation of S in $\mathcal{I}(S)$ **[P;V.1.3]**. That is, $\Omega(S)$ is isomorphic to the largest inverse subsemigroup of $\mathcal{I}(S)$ containing the Wagner representation of S as an ideal. Perhaps more to the point, $\Omega(S)$ is isomorphic to the idealizer of the Wagner representation of S in the inverse semigroup of all one-to-one partial right translations on S (a partial one-to-one right translation on S is a right translation whose domain is a left ideal of S; see [**P;V.2**]). It is this fact which makes plain the connection between the translational hull of S and both the Wagner representation and the Schützenberger representations of S.

If S is an ideal of the inverse semigroup V then V is an *ideal extension* of S (by the Rees quotient semigroup V / S). The translational hull is particularly useful when considering ideal extensions of inverse semigroups S for which we know $\Omega(S)$.

Let V be an ideal extension of S. For each $v \in V$, define

$$\lambda^v s = vs \qquad \text{and} \qquad s\rho^v = sv \qquad\qquad (s \in S).$$

Then the mapping

$$\tau(V{:}S) : V \rightarrow \Omega(S)$$

defined by

$$v\tau(V{:}S) = (\lambda^v,\rho^v) \qquad\qquad (v \in V)$$

is a homomorphism of V into $\Omega(S)$ which extends π. Moreover, $\tau(V{:}S)$ is the unique extension of π to a homomorphism of V into $\Omega(S)$ **[P;I.9.2]**. $\tau(V{:}S)$ is called the *canonical homomorphism of* V *into* $\Omega(S)$.

Theorem 2.6.1. Let S be a completely semisimple inverse semigroup and let D be a $\mathcal{D}$-class of S which is not the kernel of S. Let $I = \{ x \in S : J_x \not\geq D \}$. Then S / I is an ideal extension of the Brandt semigroup D^0 and the image of S / I in $\Omega(D^0)$ under the canonical homomorphism is isomorphic to the Wagner representation of S restricted to any $\mathcal{R}$-class belonging to D.

Proof: First of all, identify S / I with $(S \setminus I)^0$.

Let R be an $\mathcal{R}$-class of S contained in D. Let α be the Wagner representation of S restricted to R and denote $s\alpha$ by α_s, for all $s \in S$. Let T be the projection of $\tau(S / I : D^0)$ onto its second coordinate. The elements of T are right translations of the form ρ^v, for $v \in S / I$. We first prove the following statement:

Let $\phi : S \to S / I$ be the natural homomorphism of S onto the Rees quotient semigroup S / I. Let $s,t \in S$. Then $\alpha_s = \alpha_t$ if and only if $\rho^{s\phi} = \rho^{t\phi}$.

First of all, observe that if a and as both belong to the same $\mathcal{D}$-class in a completely semisimple inverse semigroup, then a $\mathcal{R}$ as. This is because $aa^{-1} \geq ass^{-1}a^{-1}$ and $D_a{}^0$ is a Brandt semigroup (and hence satisfies $\mathcal{D}$-majorization) and so $aa^{-1} = ass^{-1}a^{-1}$. Secondly, observe that if $s \in S \setminus I$ then $\{x \in R : xs \in R\} \neq \varnothing$. To see this, note that $D \subseteq J(s)$ and so, by Lemma 2.3.1 (d), there is an $a \in D$ such that $a^{-1}a \leq ss^{-1}$. Thus, $a^{-1}a = a^{-1}ass^{-1}$ and $as \neq 0$. But if a and as both belong to the Brandt semigroup D^0, then a $\mathcal{R}$ as. Let $y \in R$ be such that y $\mathcal{L}$ a. Then y $\mathcal{R}$ ys and , as a result the set $\{x \in R : xs \in R\}$ is nonempty.

If $\rho^{s\phi} = \rho^{t\phi}$ then for all $x \in D^0$, $xs\phi = xt\phi$. If $s \notin I$, then $\{x \in R : xs \in R\} \neq \varnothing$ and for all $x \in \{x \in R : xs \in R\}$, $xs\phi \neq 0$. Thus, $s\phi = 0$ if and only if $t\phi = 0$. If $s \notin I$ then we must have $xs = xt$ for all $x \in \{x \in R : xs \in R\}$ since ϕ is one-to-one on $S \setminus I$. Likewise, we must have that $xs = xt$ for all $x \in \{x \in R : xt \in R\}$. Therefore,

$\{x \in R : xs \in R\} = \{x \in R : xt \in R\}$ and $\alpha_s = \alpha_t$. If $s \in I$ then $s\phi = 0$ and so, for all $x \in D^0$, $0 = xs\phi = xt\phi$. Consequently,
$\{x \in R : xs \in R\} = \{x \in R : xt \in R\} = \varnothing$, and $\alpha_s = \alpha_t$.

Conversely, if $\alpha_s = \alpha_t$ then for all
$x \in \{x \in R : xs \in R\} = \{x \in R : xt \in R\}$, $xs = xt$. If $s \in I$ then $\{x \in R : xs \in R\} = \{x \in R : xt \in R\} = \varnothing$ and so $t \in I$. Then $s\phi = t\phi = 0$ and $\rho^{s\phi} = \rho^{t\phi}$. So suppose that $s \notin I$ and hence that $t \notin I$. Let $x \in D^0$ and let $y \in R$ be such that $y \mathscr{L} x$. Now $ys \in D$ if and only if $ys \in R$ and so $ys = yt$. Since $y \mathscr{L} x$, $x = xy^{-1}y$ and so $xs = xy^{-1}ys = xy^{-1}yt = xt$. Therefore, $\rho^{s\phi} = \rho^{t\phi}$ and the claim is proved.

By what we have just done, it follows that the mapping

$$\Theta : \alpha_s \to \rho^{s\phi} \qquad\qquad (s \in S)$$

is a well-defined bijection from the restricted Wagner representation onto the image of the projection onto the second coordinate of $\Omega(S / I)$. Since ϕ is a homomorphism, so is Θ. The projection map of $\Omega(S / I)$ onto its second coordinate is an isomorphism and so the desired result is obtained. •

The connection between the Wagner representation restricted to an $\mathscr{R}$-class R and the translational hull of a semigroup related to the $\mathscr{D}$-class containing R was made in a more general setting by Petrich (See **[Pe1]** or **[Pe2]**).

Ponizovskiĭ **[Po]** first proved that the translational hull of an inverse semigroup is an inverse semigroup. The relationship between the translational hull of an inverse semigroup S and the semigroup of all one-to-one partial right translations on S was established by McAlister by way of Schein's work on permissable subsets.

2.7 Varieties

A nonempty class of algebras $\mathscr{C}$ of the same type is a *variety* if it is closed under subalgebras, homomorphic images and direct products. By a theorem due to Birkhoff, an equivalent definition of variety is an equationally defined class of algebras of the same type. That is, if $\mathscr{F}$ is a nonempty family of equations over a language $\mathscr{L}$, then the class $\mathscr{V}$ of all algebras of type $\mathscr{L}$ satisfying each identity in $\mathscr{F}$ is a variety.

If $\mathscr{U}$ is a variety contained in the variety $\mathscr{V}$ then $\mathscr{U}$ is a *subvariety* of $\mathscr{V}$. It is apparent from the definition of variety that the intersection of a nonempty family of varieties contained in the variety $\mathscr{V}$ is also a variety contained in $\mathscr{V}$. Consequently, the collection of subvarieties of a variety $\mathscr{V}$ forms a complete lattice under inclusion, which we denote by $\mathscr{L}(\mathscr{V})$.

Given a class $\mathscr{C}$ of algebras, each member of which belongs to the variety $\mathscr{V}$, the *variety generated by* $\mathscr{C}$ is the intersection of all varieties contained in $\mathscr{V}$ which contain $\mathscr{C}$. We write $\langle \mathscr{C} \rangle$ to denote this variety. If $\mathscr{C}$ consists of the single algebra S, we write $\langle S \rangle$ instead of $\langle \mathscr{C} \rangle$. If $\mathscr{U}$ is a subvariety of $\mathscr{V}$ defined by the equations Σ then we write $\mathscr{U} = [\Sigma]$. If Σ is a finite set of equations $\{u_1 = v_1, \ldots, u_n = v_n\}$ we will often write $\mathscr{U} = [u_1 = v_1, \ldots, u_n = v_n]$ instead of $[\Sigma]$. We sometimes refer to the equations Σ which define the variety $\mathscr{V}$ as *laws*.

A refinement of our first definition of variety is the so-called HSP Theorem. If $\mathscr{C}$ is a class of algebras belonging to the variety $\mathscr{V}$, the variety $\langle \mathscr{C} \rangle$ consists of homomorphic images of subalgebras of direct products of algebras in $\mathscr{C}$.

If $\mathscr{V}$ is a variety and X is a nonempty set then $\mathscr{V}$ possesses a free algebra $F\mathscr{V}(X)$ on X which has the universal mapping property. In fact, up to isomorphism, this free algebra is the unique algebra in $\mathscr{V}$ with the universal mapping property freely generated by a set of generators of size $|X|$. Thus, $F\mathscr{V}(X)$ may be defined as the unique algebra F in $\mathscr{V}$, up to isomorphism, which satisfies: Let $\iota : X \to F$ map X injectively onto a set of generators of F. Then for any $S \in \mathscr{V}$ and any mapping $\phi : X \rightarrow S$, there is a unique homomorphism $\phi^* : F \rightarrow S$ which extends ϕ. That is, there is a unique homomorphism $\phi^* : F \rightarrow S$ such that, for all $x \in X$, $x\phi = x\phi^*$.

The class of all semigroups forms a variety as does the class of all monoids (considered as algebras with a binary operation and a nullary operation (constant)). The free semigroup on the set X consists of all nonempty finite sequences of elements of X, called *words*, over X, called an *alphabet*, given the multiplication of concatenation (or juxtaposition). We denote the free semigroup on X by X^+. The free monoid on X, denoted X^*, consists of all words over X including the empty word, which serves as the identity of X^*.

An inverse semigroup S is *subdirectly irreducible* if for every subdirect embedding $\alpha : S \rightarrow \prod_{i\in I} S_i$ there is an $i \in I$ such that $\alpha\pi_i$ is an isomorphism.
The following is an equivalent definition of subdirectly irreducible and can be found in any Universal Algebra text.

Theorem 2.7.1 [BS;II.8.4]. An inverse semigroup S is subdirectly irreducible if and only if S is trivial or there is a minimum congruence in $\mathscr{C}(S)\backslash\{\varepsilon\}$ where $\mathscr{C}(S)$ is the lattice of congruences on S and ε is the equality relation.

The following useful theorem is due to Birkhoff.

Theorem 2.7.2 [BS;II.9.7]. Every variety $\mathscr{V}$ of inverse semigroups is completely determined by its subdirectly irreducible members.

Inverse semigroups, considered as algebras with a binary operation and a unary operation, is determined by associativity and the equations $x = xx^{-1}x$, $(x^{-1})^{-1} = x$ and

$x^{-1}xy^{-1}y = y^{-1}yx^{-1}x$. Consequently, the class of all inverse semigroups forms a variety and we may consider the lattice of varieties of inverse semigroups.

Let X^{-1} denote a set disjoint from X and in one-to-one correspondence with X via $x \leftrightarrow x^{-1}$. This correspondence can be extended to a unary operation on $(X \cup X^{-1})^+$ by defining $(x^{-1})^{-1} = x$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $x \in X$, $a,b \in (X \cup X^{-1})^+$. Throughout $(X \cup X^{-1})^+$ will denote the free semigroup on $X \cup X^{-1}$ with involution $^{-1}$. The *Wagner congruence* is the least congruence ρ on $(X \cup X^{-1})^+$ such that $(a,aa^{-1}a) \in \rho$ and $(aa^{-1}bb^{-1},bb^{-1}aa^{-1}) \in \rho$, for all $a,b \in (X \cup X^{-1})^+$. If ρ is the Wagner congruence, then $(X \cup X^{-1})^+/\rho$ is the free inverse semigroup on X **[P;VIII.1.1]**. For any word w over $X \cup X^{-1}$ we will write w for $w\rho$ and refer to elements of the free inverse semigroup on X as words over $X \cup X^{-1}$. For any word $w \in (X \cup X^{-1})^+$, we define the *content of w* by $c(w) = \{ x \in X : x \text{ or } x^{-1} \text{ occurs in } w \}$.

A congruence ρ on an inverse semigroup S is *fully invariant* if it is invariant under all endomorphisms of S. That is, if $u \,\rho\, w$ and ϕ is an endomorphism of S, then $(u\phi)\,\rho\,(w\phi)$. The set of all fully invariant congruences on S, denoted $\mathcal{FI}(S)$, is a complete sublattice of the lattice of congruences on S. Let X be a countably infinite set and consider the free inverse semigroup $F\mathcal{I}(X)$. For any variety $\mathcal{V}$ of inverse semigroups, the relation $\rho(\mathcal{V})$ defined on $F\mathcal{I}(X)$ by $u\,\rho(\mathcal{V})\,w$ if and only if $u = w$ is a law in $\mathcal{V}$ is a fully invariant congruence on $F\mathcal{I}(X)$. Conversely, given a fully invariant congruence ρ on $F\mathcal{I}(X)$, let $\mathcal{V}(\rho)$ be the variety of inverse semigroups determined by the set of identities $u = w$, where $u\,\rho\,w$. Then the mappings $\rho : \mathcal{V} \to \rho(\mathcal{V})$ and $\mathcal{V} : \rho \to \mathcal{V}(\rho)$ are mutually inverse order antiisomorphisms of $\mathcal{L}(\mathcal{I})$ and $\mathcal{FI}(F\mathcal{I}(X))$ **[P;I.11.11]**. We sometimes refer to $\rho(\mathcal{V})$ as the fully invariant congruence corresponding to $\mathcal{V}$. We will often find it necessary to consider fully invariant congruences on $F\mathcal{I}(Y)$, for some set Y, and $F\mathcal{I}(X)$ at the same time. Under these conditions, we will write $\rho_Y(\mathcal{V})$ to mean the fully invariant congruence on $F\mathcal{I}(Y)$ corresponding to $\mathcal{V}$, and simply $\rho(\mathcal{V})$ for the fully invariant congruence on $F\mathscr{I}(X)$ corresponding to $\mathscr{V}$. Throughout, X is assumed to be a fixed countably infinite set, unless otherwise stated.

The variety $\mathscr{V}$ of inverse semigroups is said to be *combinatorial* if all the members are combinatorial. Equivalently, $\mathscr{V}$ is a combinatorial variety if and only if $\mathscr{V} \cap \mathscr{G} = \mathscr{T}$, the trivial variety (defined by the law $x = y$) if and only if $\mathscr{V} \subseteq [x^n = x^{n+1}]$, for some $n \in \omega$ **[P;XII.1.10]**. Likewise, the variety $\mathscr{V}$ is completely semisimple or cryptic if every member of $\mathscr{V}$ is completely semisimple or cryptic, respectively.

Let S and T be inverse semigroups and let G be a group. T is an *E-unitary cover of S over G* if T is E-unitary, there exists an idempotent separating homomorphism of T onto S and $T / \sigma_T \cong G$. If $\mathscr{U}$ is a variety of groups then the inverse semigroup variety $\mathscr{V}$ *has E-unitary covers over* $\mathscr{U}$ if, for every $S \in \mathscr{V}$, there is a group $G \in \mathscr{U}$ for which there is an E-unitary cover of S over G. A variety $\mathscr{V}$ of inverse semigroups *has E-unitary covers* if, for every $S \in \mathscr{V}$, there is an E-unitary cover of S in $\mathscr{V}$.

Theorem 2.7.3 [PR;3.3,5.4]. Let $\mathscr{V}$ be a variety of inverse semigroups. Then the following statements are equivalent:

i) $\mathscr{V}$ has E-unitary covers;

ii) the $\mathscr{V}$-free objects in $\mathscr{V}$ are E-unitary;

iii) the $\mathscr{V}$-free object on a countably infinite set is E-unitary;

iv) $\mathscr{V}$ has E-unitary covers over $\mathscr{V} \cap \mathscr{G}$.

Theorem 2.7.4 [PR;5.7]. Let $\mathscr{V}$ be a variety of inverse semigroups and $\mathscr{U}$ a variety of groups. Then $\mathscr{V}$ has E-unitary covers over $\mathscr{U}$ if and only if $\mathscr{V} \subseteq [\, u^2 = u : u^2 = u \text{ is a law in } \mathscr{U} \,]$.

We will use the following notation. If $\mathscr{U}$ is a variety of inverse semigroups, we denote by $\mathscr{U}^{max}$ the variety of inverse semigroups [$u^2 = u : u^2 = u$ is a law in $\mathscr{U}$] and by $\mathscr{U}_M{}^{max}$ the variety of inverse monoids [$u^2 = u : u^2 = u$ is a law in $\mathscr{U}$].

Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. The *Mal'cev product* of $\mathscr{U}$ and $\mathscr{V}$, denoted by $\mathscr{U} \circ \mathscr{V}$, is the collection of those inverse semigroups S for which there exists a congruence ρ on S with the property that $e\rho \in \mathscr{U}$ for all $e \in E_S$ and $S/\rho \in \mathscr{V}$; we say that ρ *witnesses* that $S \in \mathscr{U} \circ \mathscr{V}$.

In general, $\mathscr{U} \circ \mathscr{V}$ is not a variety. For example, if $\mathscr{V}$ is any nontrivial group variety and $\mathscr{U} = \mathscr{S}$, the variety of semilattices, then the five element Brandt semigroup B_2 is a member of $\langle \mathscr{U} \circ \mathscr{V} \rangle$ but B_2 is not a member of $\mathscr{U} \circ \mathscr{V}$. To see that $B_2 \notin \mathscr{U} \circ \mathscr{V}$, observe that any congruence ρ for which B_2/ρ is a group must be the universal relation and hence any idempotent ρ-class is just B_2 which is not a semilattice. On the other hand, since B_2 has an E-unitary cover over any nontrivial group variety ([**PR**] or [**P;XII.9.8**]), $B_2 \in \langle \mathscr{U} \circ \mathscr{V} \rangle$ ([**PR**] or [**P;XII.9.11**]).

However, when $\mathscr{U}$ is a variety of groups, $\mathscr{U} \circ \mathscr{V}$ is a variety [See [**P**; **XII 8.3**] or [**Ba**]]. Note that, if $\mathscr{V}$ and $\mathscr{W}$ are varieties such that $\mathscr{V} \subseteq \mathscr{W}$ then, for any variety $\mathscr{U}$, $\mathscr{U} \circ \mathscr{V} \subseteq \mathscr{U} \circ \mathscr{W}$ and $\mathscr{V} \circ \mathscr{U} \subseteq \mathscr{W} \circ \mathscr{U}$.

Lemma 2.7.5. Let $\mathscr{U}$ be a variety of groups and let $\mathscr{V}$ be a variety of inverse semigroups. Then $S \in \mathscr{U} \circ \mathscr{V}$ implies that $S/\mu_S \in \mathscr{V}$. Moreover,
$\operatorname{tr} \rho(\mathscr{V}) = \operatorname{tr} \rho(\mathscr{U} \circ \mathscr{V})$.

Proof: If ρ witnesses that $S \in \mathscr{U} \circ \mathscr{V}$, then ρ is idempotent separating and so $\rho \subseteq \mu_S$. Now, S/μ_S is isomorphic to $(S/\rho)/(\mu_S/\rho)$ and $S/\rho \in \mathscr{V}$ so we may conclude that
$S/\mu_S \in \mathscr{V}$.

If A is an antigroup belonging to $\mathscr{U} \circ \mathscr{V}$, then $A/\mu_S \cong A \in \mathscr{V}$. Thus,

$(\mathcal{U} \circ \mathcal{V}) \cap \mathcal{A} \subseteq \mathcal{V} \cap \mathcal{A}$. Since $\mathcal{V} \subseteq \mathcal{U} \circ \mathcal{V}$, we have $\mathcal{V} \cap \mathcal{A} \subseteq (\mathcal{U} \circ \mathcal{V}) \cap \mathcal{A}$. Therefore, $(\mathcal{U} \circ \mathcal{V}) \cap \mathcal{A} = \mathcal{V} \cap \mathcal{A}$. It follows from **[P; XII.2]** that $\mathcal{V} \vee \mathcal{G} = (\mathcal{U} \circ \mathcal{V}) \vee \mathcal{G}$, and hence, $\operatorname{tr} \rho(\mathcal{V}) = \operatorname{tr} \rho(\mathcal{U} \circ \mathcal{V})$. •

Mal'cev products play an important role in the study of varieties of inverse semigroups. For example, if $\mathcal{U}$ is a group variety and $\mathcal{V}$ is a combinatorial variety, then $\mathcal{U} \circ \mathcal{V}$ is the maximum variety in the ν-class of $\mathcal{U} \vee \mathcal{V}$, where ν is the congruence on $\mathcal{L}(\mathcal{I})$ defined by $\mathcal{V}_1 \,\nu\, \mathcal{V}_2$ if and only if $\mathcal{V}_1 \cap \mathcal{G} = \mathcal{V}_2 \cap \mathcal{G}$ and $\mathcal{V}_1 \vee \mathcal{G} = \mathcal{V}_2 \vee \mathcal{G}$, for all $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{L}(\mathcal{I})$, (See, for e.g., **[P; XII.2, XII.3]**). For strict inverse varieties it turns out that the ν-classes are trivial (and in fact $\mathcal{L}(\mathcal{S}\mathcal{I})$ is isomorphic to three copies of $\mathcal{L}(\mathcal{G})$, the so-called 'first three layers' of $\mathcal{L}(\mathcal{I})$) **[P;XII.4.16]**, but this is by no means true throughout $\mathcal{L}(\mathcal{I})$ as we shall see in Chapter Six. For further information on Mal'cev products we refer the reader to [P] or **[R1]**.

Before we proceed, we provide a list of notation introduced in this section as well as the notation we will use for certain special varieties and classes of inverse semigroups.

Varieties and classes:

$\mathcal{I}$ — the variety of all inverse semigroups

$\mathcal{T}$ — $[\, x = y \,]$ the trivial variety

$\mathcal{G}$ — $[\, xx^{-1} = yy^{-1} \,]$ the variety of all groups

$\mathcal{S}$ — $[\, x = x^2 \,]$ the variety of semilattices

$\mathcal{S}\mathcal{U}$ — $[\, xx^{-1} = x^{-1}x,\ u_\alpha v_\alpha^{-1} = (u_\alpha v_\alpha^{-1})^2 \,]_{\alpha \in A}$ the variety of semilattices of groups in $\mathcal{U}$ (Clifford semigroups over $\mathcal{U}$) where $\mathcal{U} = [u_\alpha = v_\alpha]_{\alpha \in A}$

$\mathcal{SG}$ — $[xx^{-1} = x^{-1}x]$ the variety of Clifford semigroups or the variety of semilattices of groups

$\mathcal{B}$ — $\langle B_2 \rangle = [xyx^{-1} = (xyx^{-1})^2]$ the variety generated by the five-element Brandt semigroup

$\mathcal{SI}$ — $[(xyx^{-1})(xyx^{-1})^{-1} = (xyx^{-1})^{-1}(xyx^{-1})]$ the variety of strict inverse semigroups

$\mathcal{B}^1$ — $\langle B_2^1 \rangle$ the variety generated by the five-element Brandt semigroup with an identity adjoined

$\mathcal{A}_n$ — the variety of abelian groups of exponent n

$\mathcal{AG}$ — the variety of abelian groups

$\mathcal{C}_n$ — $[x^n = x^{n+1}]$ for every natural number n

$\mathcal{U}^{max}$ — $[u^2 = u : u^2 = u$ is a law in $\mathcal{U}]$

$\mathcal{U} \circ \mathcal{V}$ — the Mal'cev product of the varieties $\mathcal{U}$ and $\mathcal{V}$ (not necessarily a variety)

$\mathcal{A}$ — the class of all antigroups (not a variety)

Further Notation

$\mathcal{L}(\mathcal{V})$ — the lattice of all subvarieties of $\mathcal{V}$

$\langle \mathcal{C} \rangle$ — the variety of inverse semigroups generated by the nonempty class $\mathcal{C}$ of inverse semigroups; when $\mathcal{C} = \{S\}$, we write $\langle S \rangle$ instead of $\langle \mathcal{C} \rangle$

$[\Sigma]$ — the variety of inverse semigroups satisfying u = w for all equations u = w in Σ

$w \in E$ — the equation $w = w^2$

$F\mathcal{V}(X)$ — the $\mathcal{V}$-free inverse semigroup on X

$c(w)$ — for a word w over $X \cup X^{-1}$, the content of w

$\rho(\mathscr{V})$ — for a variety $\mathscr{V}$ of inverse semigroups, the fully invariant congruence on $F\mathscr{I}(X)$ corresponding to $\mathscr{V}$

Many of the results we have mentioned here are of a fundamental nature and can be found in virtually any text on Universal Algebra ([**Gr**] or [**BS**], for example); we do, however, mention Birkhoff's important paper [**Bi2**] of 1935 in which he proved his famous theorem that $\mathscr{V}$ is a variety if and only if $\mathscr{V}$ is an equational class. The Wagner congruence is, of course, due to Wagner [**Wa3**]. Completely semisimple varieties were studied by Reilly [**Re2**]. The congruence ν was introduced by Kleiman who is responsible for the result cited on the first three layers of $\mathscr{L}(\mathscr{I})$ [**K1**]. Reilly [**Re2**] also studied the congruence ν and showed that $\mathscr{L}(\mathscr{I})$ is not a modular lattice. For results concerning the Mal'cev product of inverse semigroup varieties we refer the reader to Reilly [**Re1**], and for results concerning E-unitary covers we refer the reader to [**PR**].

2.8 Presentations and Schützenberger graphs

A *presentation* of an inverse semigroup is a pair $P = (X;R)$ where R is a binary relation on $F\mathscr{I}(X)$. If $P = (X;R)$, the inverse semigroup presented by P is $F\mathscr{I}(X) / \theta$ where θ is the congruence on $F\mathscr{I}(X)$ generated by R. Equivalently, we may consider $P = (X;R)$, where R is a binary relation on $(X \cup X^{-1})^+$. Then the inverse semigroup presented by P is $(X \cup X^{-1})^+ / \tau$, where τ is the congruence on $(X \cup X^{-1})^+$ generated by $R \cup \rho$. We will consider only those presentations for which R (and hence θ) is $\rho(\mathscr{V})$ for some variety $\mathscr{V}$ of inverse semigroups.

The definitions and results of this section can be found in Stephen [S] to which we refer the reader for additional information concerning Schützenberger graphs.

A *labelled digraph* Γ over a nonempty set X consists of a set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, where $E(\Gamma) \subseteq V \times X \times V$. An edge $(v_1,x,v_2) \in E(\Gamma)$ is *labelled* by x and *directed* from v_1 to v_2. We call v_1 the *initial* or *start* vertex and v_2 the *terminal* or *end* vertex of the edge (v_1,x,v_2). A *path* p is a sequence of edges such that the end vertex of an edge in the sequence is the start vertex of the next edge in the sequence.

Γ is *strongly connected* if, given any two vertices $v_1, v_2 \in V(\Gamma)$, there is a path p from v_1 to v_2. We will often call a path from v_1 to v_2 a v_1-v_2 *walk*. An *inverse word graph* Γ over $X \cup X^{-1}$ is a strongly connected labelled digraph over $X \cup X^{-1}$ satisfying the condition: $(v_1,x,v_2) \in E(\Gamma)$ implies $(v_2,x^{-1},v_1) \in E(\Gamma)$, for all $x \in X \cup X^{-1}$. An inverse word graph Γ is *deterministic* if all edges directed away from a vertex are labelled by different letters, and *injective* if all edges directed toward a vertex are labelled by different letters. Thus, a deterministic inverse word graph over $X \cup X^{-1}$ is necessarily injective.

If Γ and Γ' are inverse word graphs over $X \cup X^{-1}$, a *V-homomorphism* $\phi: \Gamma \to \Gamma'$ is a map on the vertices of Γ which preserves incidence, orientation and labelling. More precisely, ϕ is a pair of functions $\phi_V: V(\Gamma) \to V(\Gamma')$ and $\phi_E: E(\Gamma) \to E(\Gamma')$ such that $(v_1,x,v_2)\phi_E = (v_1\phi_V,x,v_2\phi_V)$. ϕ is a *V-monomorphism* if it is one–one on the vertices of Γ; a *V-epimorphism* if it is surjective on both the set of edges and the set of vertices of Γ; a *V-isomorphism* if it is both a V-monomorphism and a V-epimorphism. An *inverse birooted word graph* is a triple (s, Γ, e) where Γ is an inverse word graph and s and e are distinguished vertices called, respectively, the *start* and *end vertices*.

Let $P = (X;R)$ be a fixed presentation of the inverse semigroup S with τ the corresponding congruence on $F\mathscr{I}(X)$. Let $w \in S$ and R_w the $\mathscr{R}$-class of w in S. The *Schützenberger graph of* R_w *with respect to P* is the labelled digraph $\Gamma(w)$, where

$V(\Gamma(w)) = R_w$

$E(\Gamma(w)) = \{ (v_1,x,v_2) : v_1,v_2 \in R_w, x \in X \cup X^{-1}$ and $v_1(x\tau) = v_2 \}$.

Dually, we define the *Schützenberger graph of* L_w *with respect to P* to be the labelled digraph $\Delta(w)$ with

$V(\Delta(w)) = L_w$

$E(\Delta(w)) = \{ (v_1,x,v_2) : v_1,v_2 \in L_w, x \in X \cup X^{-1}$ and $(x\tau)v_1 = v_2 \}$.

Lemma 2.8.1 [S; 3.1]. Let $v \in S$, $\Gamma(v)$ be the Schützenberger graph of R_v with respect to P, $v_1,v_2,v \in R_v$, $e = vv^{-1}$ and $w \in (X \cup X)^+$.

a) $\Gamma(v)$ is a deterministic inverse word graph;

b) $v_1(w\tau) = v_2$ if and only if w labels a v_1-v_2 walk;

c) $(w\tau) \geq v$ if and only if w labels an e-v walk;

The lemma above can be dualized for $\Delta(v)$ for any $\mathscr{L}$-class L_v of S. We remark that if S is a group, then for any $w \in S$, $\Gamma(w)$ is the Cayley graph of S (See [S; 3.7]). For a discussion of Cayley graphs, we refer the reader to [W].

The following lemma characterizes Green's relations on S in terms of the Schützenberger graphs of S.

Lemma 2.8.2 [S; 3.4]. Let v_1, $v_2 \in S$ and let $e = v_1v_1^{-1}$ anf $f = v_2v_2^{-1}$. Then

a) $v_1 \mathscr{D} v_2$ if and only if there exists a V-isomorphism $\phi : \Gamma(v_1) \rightarrow \Gamma(v_2)$;

b) $v_1 \mathscr{R} v_2$ if and only if there exists a V-isomorphism $\phi : \Gamma(v_1) \rightarrow \Gamma(v_2)$ such that $e\phi = f$.

c) $v_1 \mathscr{L} v_2$ if and only if there exists a V-isomorphism $\phi : \Gamma(v_1) \rightarrow \Gamma(v_2)$ such that $v_1\phi = v_2$.

d) $v_1 \mathscr{H} v_2$ if and only if there exist V-isomorphisms $\phi,\psi : \Gamma(v_1) \to \Gamma(v_2)$ such that $e\phi = f$ and $v_1\psi = v_2$.

e) $v_1 = v_2$ if and only if there exists a V-isomorphism $\phi : \Gamma(v_1) \to \Gamma(v_2)$ such that $e\phi = f$ and $v_1\phi = v_2$.

For any $v \in S$, the *Schützenberger representation of* v (*with respect to P*) is the birooted inverse word graph $(vv^{-1},\Gamma(v),v)$. We will also use $\Gamma(v)$ to denote the birooted graph and specify the roots whenever required. We are considering presentations in which the relation R is always a fully invariant congruence on $F\mathscr{I}(X)$ corresponding to some variety $\mathscr{V}$. Thus, for any word $w \in (X \cup X^{-1})^+$ and congruence $\rho(\mathscr{V})$, we will write $\Gamma(w)$ (or $\Gamma_{\mathscr{V}}(w)$ if we wish to emphasize the variety being considered) to denote $(ww^{-1}\rho(\mathscr{V}), \Gamma(w\rho(\mathscr{V})), w\rho(\mathscr{V}))$ with respect to $P = (X; \rho(\mathscr{V}))$, and call $\Gamma_{\mathscr{V}}(w)$ the *Schützenberger representation of w with respect to* $\mathscr{V}$. We remark that the Schützenberger representation of the free inverse semigroup is the representation of $F\mathscr{I}(X)$ by birooted inverse word trees, which is due to Munn **[Mu4]** (See Stephen [S] for the connection between Schützenberger graphs of the free inverse semigroup and Munn trees). For further properties of Schützenberger graphs, we refer the reader to Stephen [S].

The following result will be used throughout, but is presented here so that we may look at what are probably the simplest examples of Schützenberger graphs relative to some variety.

Proposition 2.8.3. If $w \in (X \cup X^{-1})^+$, then $\Gamma_{\mathscr{S}}(w)$ is just a single vertex with $2|c(w)|$ loops. For each $x \in c(w)$ there is precisely one loop labelled x and one loop labelled x^{-1}.

Proof: For any $u,v \in (X \cup X^{-1})^+$, $u\,\rho(\mathscr{S})\,v$ if and only if $c(u) = c(v)$. Furthermore, $u\rho(\mathscr{S})\ \mathscr{R}\ ua\rho(\mathscr{S})$, for some $a \in (X \cup X^{-1})^+$, if and only if a or a^{-1} is an element of c(u). From these two facts and the definition of Schützenberger graph, one easily obtains the desired result. •

Examples. 1)

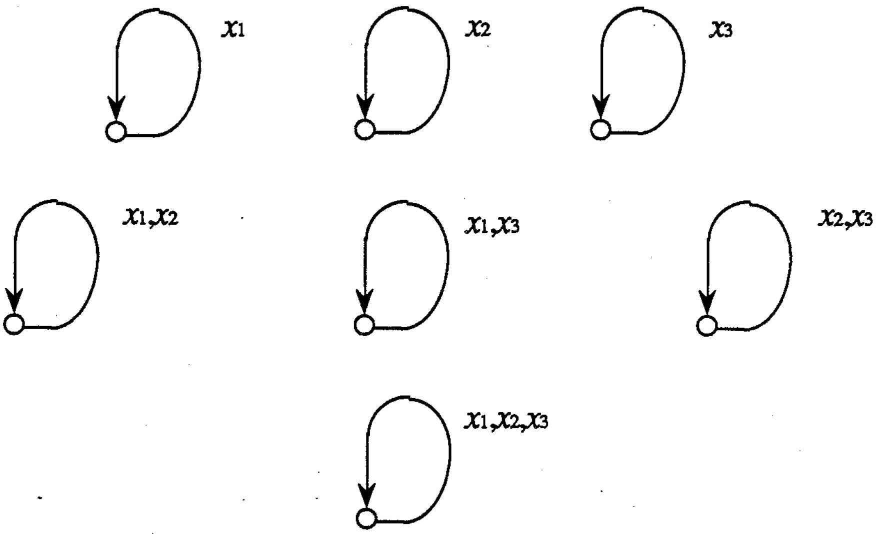

Figure 2.1. Schützenberger graphs in the free semilattice on three generators.

The graphs above in Figure 2.1 form the collection of Schützenberger graphs (up to V-isomorphism) of the free semilattice on three generators (see Proposition 2.8.3). We follow the standard practice of providing edges labelled by $x \in X$ but not edges labelled by elements of X^{-1} as these edges are implicitly determined by those edges labelled by elements of X. Also, we follow the convention of drawing a single edge with more than one label if there are several edges between two given vertices.

2) Figure 2.2 is the Schützenberger graph of the word $w = x_1x_2x_1^{-1}x_2^{-1}$ with respect to the variety $\mathcal{B}^1$. The proof of this can be found in Theorem 6.1.7.

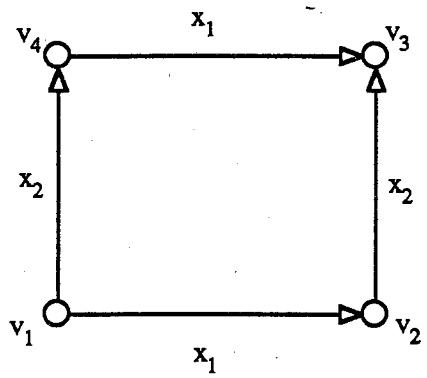

Figure 2.2. The Schützenberger graph $\Gamma_{\mathcal{S}}(w)$.

This example will be used again in the sequel to illustrate concepts related to Schützenberger graphs.

CHAPTER THREE
Wreath Products

In this chapter we present the definition of the wreath product of two inverse semigroups. Our definition is a slightly more general definition than that of Houghton [**H**] and a generalization to arbitrary inverse semigroups of Petrich's definition of the (right) wreath product of a group and an inverse semigroup [**P**]. The restriction to groups of our definition is dual to the definition of standard (unrestricted) wreath product found in Neumann [**N**], as Neumann writes her operators on the left and we write our operators on the right. The only material of this chapter required for the sequel can be found in section 1. The material in section 2 serves as motivation for the work in subsequent chapters, particularly chapter 6. The third section contains some structural results concerning wreath products and, while these results are of independent interest, they are not required for the remaining chapters.

3.1 Definition of wreath product

Let S and T be inverse semigroups and suppose that T is an inverse subsemigroup of $\mathscr{I}(I)$, the symmetric inverse semigroup on I. Let ${}^{I}S$ denote the set of functions (written on the right) from subsets of I into S. For any $\psi \in {}^{I}S$, denote the domain of ψ by $\mathbf{d}\psi$. Define a multiplication on ${}^{I}S$ by

$$i\,(\psi \cdot \psi') = (i\psi) \cdot (i\psi') \qquad [i \in \mathbf{d}\psi \cap \mathbf{d}\psi'].$$

For any $\beta \in \mathscr{I}(I)$ and $\psi \in {}^{I}S$, we define a mapping ${}^{\beta}\psi$ by

$$i\,({}^{\beta}\psi) = (i\beta)\psi \qquad [i \in \mathbf{d}\beta,\ i\beta \in \mathbf{d}\psi\,].$$

The *(right) wreath product of* S *and* T is the set

$$S \text{ wr } T = \{\ (\psi,\beta) \in {}^{I}S \times T : \mathbf{d}\psi = \mathbf{d}\beta\ \}$$

with multiplication given by

$$(\psi,\beta)\cdot(\psi',\beta') = (\psi^{\beta}\psi',\beta\beta').$$

If T is an inverse subsemigroup of $\mathscr{I}(I)$, we will sometimes write (T,I) for T if we wish to emphasize the set I on which T acts. We will write (T,T) to denote the Wagner representation of T by partial right translations.

Our definition of wreath product follows that of Houghton [H]. In [H] the wreath product W(S,T) of inverse semigroups S and T is, in our notation, S wr (T,T) where T is given the Wagner representation by partial right translations. Our notation follows Petrich [P;V.4].

Proposition 3.1.1. Let S and (T,I) be inverse semigroups.

a) S wr (T,I) is a semigroup;

b) S wr (T,I) is regular;

c) If $(\psi,\beta) \in$ S wr (T,I) then (ψ,β) is an idempotent if and only if β is the identity map on $\mathbf{d}\beta$ and for all $i \in \mathbf{d}\beta$, $i\psi \in E_S$.

d) S wr (T,I) is an inverse semigroup. If $(\psi,\beta) \in$ S wr (T,I) then the inverse of (ψ,β) is the pair (ψ^{-1},β^{-1}) where β^{-1} is the inverse of β in (T,I) and for all $i \in \mathbf{d}\beta^{-1}$, $i\psi^{-1} = [i\beta^{-1}\psi]^{-1}$.

Proof: a) Let $(\psi,\beta), (\psi',\beta') \in$ S wr (T,I).

Then

$$\begin{aligned} i \in \mathbf{d}\psi^{\beta}\psi' &\Leftrightarrow i \in \mathbf{d}\psi \text{ and } i \in \mathbf{d}^{\beta}\psi' \\ &\Leftrightarrow i \in \mathbf{d}\psi = \mathbf{d}\beta \text{ and } i\beta \in \mathbf{d}\psi' = \mathbf{d}\beta' \\ &\Leftrightarrow i \in \mathbf{d}\beta\beta'. \end{aligned}$$

Therefore, $(\psi,\beta)(\psi',\beta') = (\psi^{\beta}\psi',\beta\beta') \in$ S wr (T,I) and S wr (T,I) is closed under the operation defined above.

Next, let (ψ_1,β_1), (ψ_2,β_2) and $(\psi_3,\beta_3) \in$ S wr (T,I).

Then

$$[(\psi_1,\beta_1)(\psi_2,\beta_2)](\psi_3,\beta_3) = (\psi_1{}^{\beta_1}\psi_2,\beta_1\beta_2)(\psi_3,\beta_3)$$
$$= ((\psi_1{}^{\beta_1}\psi_2)^{\beta_1\beta_2}\psi_3,(\beta_1\beta_2)\beta_3)$$

and

$$(\psi_1,\beta_1)[(\psi_2,\beta_2)(\psi_3,\beta_3)] = (\psi_1,\beta_1)(\psi_2{}^{\beta_2}\psi_3,\beta_2\beta_3) = (\psi_1{}^{\beta_1}(\psi_2{}^{\beta_2}\psi_3),\beta_1(\beta_2\beta_3)).$$

Since $(\beta_1\beta_2)\beta_3 = \beta_1(\beta_2\beta_3)$ and S wr (T,I) is closed under the operation, we need only check that the first components agree on $\mathbf{d}\beta_1\beta_2\beta_3$. Let $i \in \mathbf{d}\beta_1\beta_2\beta_3$. Then

$$i(\psi_1{}^{\beta_1}\psi_2)^{\beta_1\beta_2}\psi_3 = (i\psi_1{}^{\beta_1}\psi_2)(i\beta_1\beta_2\psi_3)$$
$$= [(i\psi_1)(i\beta_1\psi_2)](i\beta_1\beta_2\psi_3)$$
$$= (i\psi_1)[(i\beta_1\psi_2)(i\beta_1\beta_2\psi_3)] \quad \text{(associativity of S)}$$
$$= (i\psi_1)[i\beta_1(\psi_2{}^{\beta_2}\psi_3)]$$
$$= i[\psi_1{}^{\beta_1}(\psi_2{}^{\beta_2}\psi_3)].$$

It follows that the operation is associative and so S wr (T,I) is a semigroup.

b) Let $(\psi,\beta) \in$ S wr (T,I). Define (ψ',β') by setting $\beta' = \beta^{-1}$, $\mathbf{d}\psi' = \mathbf{d}\beta'$ and $j\psi' = [j\beta^{-1}\psi]^{-1}$ for all $j \in \mathbf{d}\beta'$. It is immediate that $(\psi',\beta') \in$ S wr (T,I). We have $(\psi,\beta)(\psi',\beta')(\psi,\beta) = (\psi^{\beta}\psi'^{\beta\beta'}\psi,\beta\beta'\beta)$. Since $\beta' = \beta^{-1}$, $\beta\beta'\beta = \beta$ and $\beta\beta'$ is the identity map on $\mathbf{d}\beta$. Therefore, for all $i \in \mathbf{d}\beta = \mathbf{d}\beta\beta'\beta = \mathbf{d}\psi^{\beta}\psi'^{\beta\beta'}\psi$,

$$i\,\psi^{\beta}\psi'^{\beta\beta'}\psi = (i\psi)(i\beta\psi')(i\psi)$$
$$= (i\psi)(i\beta\beta^{-1}\psi)^{-1}(i\psi)$$
$$= (i\psi)(i\psi)^{-1}(i\psi)$$
$$= i\psi.$$

It now follows that $(\psi,\beta)(\psi',\beta')(\psi,\beta) = (\psi,\beta)$ and so S wr (T,I) is regular.

c) Let $(\psi,\beta) \in$ S wr (T,I). Then (ψ,β) is an idempotent in S wr (T,I) means that $(\psi^{\beta}\psi,\beta\beta) = (\psi,\beta)$. But $\beta\beta = \beta$ and $\psi^{\beta}\psi = \psi$ if and only if β is the identity map on its domain and for all $i \in \mathbf{d}\beta = \mathbf{d}\psi$, $i\psi \in E(S)$.

d) If (ψ,β) and (ψ',β') are idempotents in S wr (T,I), then

$$
\begin{aligned}
(\psi,\beta)(\psi',\beta') &= (\psi^{\beta}\psi',\beta\beta') \\
&= (\psi\psi',\beta\beta') \\
&= (\psi'\psi,\beta'\beta) \\
&= (\psi'^{\beta'}\psi,\beta'\beta) \\
&= (\psi',\beta')(\psi,\beta).
\end{aligned}
$$

Therefore, the idempotents of S wr (T,I) commute which, combined with the fact that S wr (T,I) is regular, implies that S wr (T,I) is an inverse semigroup.

If $(\psi,\beta) \in$ S wr (T,I) then define $(\psi,\beta)^{-1}$ to be the pair (ψ^{-1},β^{-1}) where $\psi^{-1} \in {}^{I}S$ and $\beta^{-1} \in T$ are defined by

$$\mathbf{d}\beta^{-1} = \mathbf{d}\psi^{-1} = \{\, i\beta : i \in \mathbf{d}\beta \,\},$$

β^{-1} is the inverse of β in T and

$$i\psi^{-1} = (\, i\beta^{-1}\psi)^{-1} \qquad (\, i \in \mathbf{d}\beta^{-1} \,).$$

We have seen in the proof of the regularity of S wr (T,I) that $(\psi,\beta)(\psi,\beta)^{-1}(\psi,\beta) = (\psi,\beta)$. We also have that

$$
\begin{aligned}
(\psi,\beta)^{-1}(\psi,\beta)(\psi,\beta)^{-1} &= (\psi^{-1}\,{}^{\beta^{-1}}\psi\,{}^{\beta^{-1}\beta}\psi^{-1},\beta^{-1}\beta\beta^{-1}) \\
&= (\psi^{-1}\,{}^{\beta^{-1}}\psi\psi^{-1},\beta^{-1}).
\end{aligned}
$$

For any $i \in \mathbf{d}\beta^{-1}$,

$$
\begin{aligned}
i\psi^{-1}\,{}^{\beta^{-1}}\psi\psi^{-1} &= (i\psi^{-1})(i\beta^{-1}\psi)(i\psi^{-1}) \\
&= (i\beta^{-1}\psi)^{-1}(i\beta^{-1}\psi)(i\beta^{-1}\psi)^{-1} \\
&= (i\beta^{-1}\psi)^{-1} \\
&= (i\psi^{-1}).
\end{aligned}
$$

Therefore, (ψ^{-1},β^{-1}) is the inverse of (ψ,β) in S wr (T,I). Note that we may equivalently define ψ^{-1} by

$$j\beta\psi^{-1} = (\, j\psi\,)^{-1} \qquad (\, j \in \mathbf{d}\beta \,). \qquad \bullet$$

Remark. For any (ψ,β) belonging to S wr (T,I), we have written $(\psi,\beta)^{-1}$ as (ψ^{-1},β^{-1}) even though the definition of ψ^{-1} depends on β. This is not to suggest that if (ψ,β') is another member of S wr (T,I), then the first coordinate of $(\psi,\beta')^{-1}$ is the same as the first coordinate of $(\psi,\beta)^{-1}$. We use ψ^{-1} to avoid notational difficulties and simply note that when ψ^{-1} is used, the member of (T,I) to which it is paired will be understood.

In [N], Neumann defines the (unrestricted) wreath product A Wr B of the groups A and B as follows.

$$A \text{ Wr } B = B \times A^B$$

with products defined by

$$(b,\phi)(c,\psi) = (bc,\phi^c\psi)$$

where, for all $y \in B$, $\phi^c(y) = \phi(yc^{-1})$.

Let A^d be the group defined as A with multiplication $*$ given by, for all $g,h \in A$, $g * h = h\cdot g$, with this last product as in A. Then A Wr B is antiisomorphic to A^d wr (B,B) using the definition in section 3.1 with B given its Wagner (Cayley) representation:

Define $\Theta : A \text{ Wr } B \to A^d \text{ wr } (B,B)$ by setting $(b,\varphi)\Theta = (\varphi',\alpha_{b^{-1}})$ where φ' is defined by $y\varphi' = \varphi(y)$ for all $y \in B$ and $\alpha_{b^{-1}}$ is the permutation corresponding to b^{-1} in the Wagner representation of B. Then for all $(b,\varphi),(c,\psi) \in A \text{ Wr } B$,

$$(c,\psi)\Theta(b,\varphi)\Theta = (\psi',\alpha_{c^{-1}})(\varphi',\alpha_{b^{-1}}) = (\psi'\,{}^{\alpha_{c^{-1}}}\varphi',\alpha_{c^{-1}b^{-1}})$$

while

$$(bc,\varphi^c\psi)\Theta = ((\varphi^c\psi)',\alpha_{c^{-1}b^{-1}}).$$

For all $y \in B$,

$$y(\varphi^c\psi)' = \varphi^c\psi(y) = \varphi(yc^{-1}) \cdot \psi(y)$$

with this product in A and

$$y\psi'\,{}^{\alpha_{c^{-1}}}\varphi' = (y\psi')*(yc^{-1}\varphi') = (\psi(y))*(\varphi(yc^{-1})) = \varphi(yc^{-1}) \cdot \psi(y).$$

Thus, $(c,\psi)\Theta(b,\varphi)\Theta = (bc,\varphi^c\psi)\Theta$. Θ is easily seen to be a bijection and so Θ is an antiisomorphism.

As a consequence of these remarks, the results concerning wreath products of groups and product varieties of groups found in [N] are valid in the context presented here.

We conclude this section with a remark concerning wreath products of semigroups. In the study of finite semigroups and automata theory wreath products play a significant role (see, for example, [E]). In general, however, the definition of wreath product for semigroups does not ensure that the wreath product of two inverse semigroups will be an inverse semigroup, as Houghton points out in [H]. In fact, the wealth of research on wreath products and pseudovarieties of semigroups did not serve as motivation for our investigations, though some of the ideas presented here have their analogues in finite semigroup theory.

3.2 Subdirectly irreducible inverse semigroups in completely semisimple varieties

The principal factors of a completely semisimple inverse semigroup S are Brandt semigroups and groups. In fact, at most one principal factor of S can be a group and this is the case only if S possesses a minimum ideal which is a group. If D is a $\mathcal{J}$-class of S, but not the minimum $\mathcal{J}$-class of S, then the Rees quotient semigroup corresponding to the ideal of S consisting of those elements x for which $\mathcal{J}_x \not\geq D$ is an ideal extension of the Brandt semigroup D^0. The canonical homomorphism of this ideal extension of D^0 into the translational hull $\Omega(D^0)$ of D^0 is one-to-one on D. Consequently, it can be shown that S can be subdirectly embedded into a product of inverse subsemigroups of $\Omega(D^0_\alpha)$, where the D_α are the non-minimum $\mathcal{J}$-classes of S, and possibly a group. For any non-minimum $\mathcal{J}$-class D_α of S, the translational hull of D^0_α is a wreath product of a group G and $\mathcal{I}(I)$, where G and I depend on D^0_α. Thus, wreath products play an important role in the study of completely semisimple inverse semigroups. In fact, as we will discover in subsequent chapters, wreath products of inverse semigroups in general prove to be useful tools in studying varieties of inverse semigroups.

The following two theorems make clear the connection between wreath products and completely semisimple inverse semigroups and are of fundamental importance.

Theorem 3.2.1 [P;V.4.6]. For any Brandt semigroup $S = B(G,I)$, we have $\Omega(S) \cong G \text{ wr } \mathscr{I}(I)$.

In light of Theorem 3.2.1, wreath products of the form $G \text{ wr } \mathscr{I}(I)$ are related to ideal extensions of Brandt semigroups. The following result is a general description of ideal extensions of Brandt semigroups which we will find useful. For a semigroup S with zero, we denote S with its zero removed by S*.

Theorem 3.2.2[P;V.4.7]. Let $S = B(G,I)$ be a Brandt semigroup and Q be an inverse semigroup with zero disjoint from S. Let $\varphi : Q^* \to G \text{ wr } \mathscr{I}(I)$, denoted by $\varphi : q \to (\psi_q, \beta_q)$, be a partial homomorphism such that $|\mathbf{d}\beta_q\beta_r| \le 1$ if $qr = 0$ in Q. On $V = S \cup Q^*$ define a multiplication $*$ by : for $q,r \in Q^*$, $(i,g,j) \in S$,

$$(i, g, j) * q = (i, g(j\psi_q), j\beta_q) \qquad \text{if } j \in \mathbf{d}\beta_q,$$

$$q * (i, g, j) = (i\beta_q^{-1}, (i\beta_q^{-1}\psi_q)g, j) \qquad \text{if } i \in \mathbf{r}\beta_q,$$

and if $qr = 0$ in Q,

$$q * r = (k\beta_q^{-1}, (k\beta_q^{-1}\psi_q)(k\psi_r), k\beta_r) \qquad \text{if } \{k\} = \mathbf{r}\beta_q \cap \mathbf{d}\beta_r,$$

$a * b = ab$ if $a,b \in S$, or $a,b \in Q^*$ and $ab \neq 0$, and all other products equal to zero. Then V is an ideal extension of S by Q. Conversely, every ideal extension of S by Q can be so constructed.

The first result of this section states that every completely semisimple inverse semigroup is isomorphic to a subdirect product of ideal extenxions of Brandt semigroups. This is nothing new. We refer the reader to **[Pe1]** and **[Pe2]**. We use the term *kernel of S* in this section to mean the intersection of all nonzero ideals of S. That is, the kernel of S is the minimum nonzero ideal of S, if it exists.

Lemma 3.2.3. Let S be a completely semisimple inverse semigroup. Then S is isomorphic to a subdirect product of ideal extensions of Brandt semigroups and possibly a group. Each of these ideal extensions of Brandt semigroups is an inverse subsemigroup of G wr $\mathscr{I}(I)$ where G and I are determined by kernel Brandt semigroup.

Proof: Let $\{D_\alpha : \alpha \in A\}$ be the collection of $\mathscr{D}$-classes (or equivalently, $\mathscr{J}$-classes) of S. For each $\alpha \in A$, let $I_\alpha = \{x \in S : J_x \ngeq D_\alpha\}$. Then I_α is an ideal of S and the Rees quotient S / I_α is an ideal extension of D_α^0 or $I_\alpha = \varnothing$. Observe that if $I_\alpha = \varnothing$, then D_α is the kernel of S and so must be a group. As S is completely semisimple, D_α^0 is a Brandt semigroup for each α in A. Suppose that $D_\alpha^0 \cong B(G_\alpha, K_\alpha)$. Let $\tau_\alpha : S / I_\alpha \to G_\alpha$ wr $\mathscr{I}(K_\alpha)$ be the canonical homomorphism of S / I_α into the translational hull of D_α^0. If S possesses a kernel group D_α, then τ_α is understood to be the canonical mapping of S into $\Omega(D_\alpha) = D_\alpha$. Recall that for each α, τ_α is one-to-one on D_α. Let ϕ_α be the natural homomorphism of S onto S / I_α, for each α in A. Define

$$\Phi : S \to \prod_{\alpha \in A} \tau_\alpha(S / I_\alpha) \subseteq \prod_{\alpha \in A} (G_\alpha \text{ wr } \mathscr{I}(K_\alpha))$$

by $(s\Phi)\pi_\alpha = s\phi_\alpha\tau_\alpha$, where if D_α is the kernel of S, G_α wr $\mathscr{I}(K_\alpha)$ is understood to be G_α. Φ is clearly a homomorphism. Let $x,y \in S$ and suppose that $x\Phi = y\Phi$. If $J_x \neq J_y$ then either $J_x \ngeq J_y$ or $J_y \ngeq J_x$. If $J_x \ngeq J_y$ and $y \in D_\alpha$ then $x\phi_\alpha\tau_\alpha \neq y\phi_\alpha\tau_\alpha$ and so

$x\Phi \neq y\Phi$. Likewise, if $J_y \not\geq J_x$ then $x\Phi \neq y\Phi$. If $J_x = J_y$ and $x,y \in D_\alpha$, then $x\phi_\alpha\tau_\alpha = y\phi_\alpha\tau_\alpha$ implies that $x = y$ as $\phi_\alpha\tau_\alpha$ is one-to-one on D_α. It follows that Φ is an embedding which is rather obviously subdirect. •

We will call an inverse subsemigroup S of G wr $\mathscr{I}$(I) *k-full* if it contains all $(\psi,\beta) \in$ G wr $\mathscr{I}$(I) such that $|\mathbf{d}\beta| = |\mathbf{d}\psi| \leq 1$. That is, S is a k-full subsemigroup of G wr $\mathscr{I}$(I) if S contains the Brandt semigroup of which G wr $\mathscr{I}$(I) is an extension.

Lemma 3.2.4. Let S be an ideal extension of B(G,I) and let φ be a congruence on G. Define a binary relation φ^* on S by

$$x\, \varphi^*\, y \Leftrightarrow \text{i) } x,y \in B(G,I),\ x = (i,g,j),\ y = (i,h,j) \text{ and } g\, \varphi\, h, \text{ or}$$
$$\text{ii) } x = y.$$

Then φ^* is a congruence on S. Moreover, if θ^* is a congruence on S and θ is its restriction to some group $\mathscr{H}$-class of B(G,I), then i) $\varphi \subseteq \theta$ implies that $\varphi^* \subseteq \theta^*$; and ii) $\theta^* \subseteq \varphi^*$ implies that $\theta \subseteq \varphi$.

Proof: It is easy to see that φ^* is an equivalence relation. Suppose that $x\,\varphi^*\,y$ and let $z \in S$. If $x = y$ then $zx = zy$ and $xz = yz$. If $x = (i,g,j)$ and $y = (i,h,j)$ with $g\,\varphi\,h$ then a) if $z = (i',g',j')$ then $zx\,\varphi^*\,zy$ and $xz\,\varphi^*\,yz$ because φ is a congruence; b) using Theorem 3.2.2, $xz = (i,g(j\psi_z),j\beta_z)$ and $yz = (i,h(j\psi_z),j\beta_z)$ where $(\psi_z,\beta_z) \in$ G wr $\mathscr{I}$(I). Since φ is a congruence, $xz\,\varphi^*\,yz$. Likewise, Theorem 3.2.2 also implies that $zx\,\varphi^*\,zy$. Thus, φ^* is a congruence.

Let θ^* be a congruence on S and suppose that θ is the restriction of θ^* to the group $\mathscr{H}$-class $H = \{(i,g,i) : g \in G\}$. If $x, y \in S$ and $x\,\varphi^*\,y$ then either $x = y$, in which case $x\,\theta^*\,y$, or $x = (j,g',k)$, $y = (j,h',k)$ and $g'\,\varphi\,h'$. But if $g'\,\varphi\,h'$ then $(i,g',i)\;\theta\;(i,h',i)$ and so for any $j,k \in I$, $(j,g',k) = (j,1_G,i)(i,g',i)(i,1_G,k)\;\theta^*\;(j,1_G,i)(i,h',i)(i,1_G,k) = (j,h',k)$. Therefore, $\varphi^* \subseteq \theta^*$. Now suppose that $\theta^* \subseteq \varphi^*$. From the definition of φ^* we have that

$\varphi^*|_H = \varphi$. That is, (i,g,i) φ^* (i,h,i) if and only if g φ h. Thus, $\theta^* \subseteq \varphi^*$ implies that $\theta \subseteq \varphi$. •

Theorem 3.2.5. Let S be a completely semisimple inverse semigroup. Then S is subdirectly irreducible if and only if S is a subdirectly irreducible group or S is a k-full inverse subsemigroup of G wr $\mathscr{I}$(I) for some set I and some subdirectly irreducible group G.

Proof: Let S be a subdirectly irreducible completely semisimple inverse semigroup. By Lemma 3.2.3 above, S is isomorphic to a k-full inverse subsemigroup of $\Omega(D^0)$ for some $\mathscr{D}$-class D of S or S is a group. If S is a group then it is a subdirectly irreducible group, so assume that S is isomorphic to a k-full inverse subsemigroup of $\Omega(D^0)$ where $D^0 \cong B(G,I)$, since S is completely semisimple. By Theorem 3.2.1, we need only show that G is subdirectly irreducible. Let θ be the minimum non-equality congruence on S (where we think of S as a k-full inverse subsemigroup of $\Omega(B(G,I))$). Then θ is contained in the Rees congruence relative to D^0. If (x,y) generates θ and x is not $\mathscr{H}$-related to y then it is not difficult to show that θ must be the Rees congruence relative to D^0. [If $x = (i_1,g_1,j_1)$ and $y = (i_2,g_2,j_2)$ then for any $(i_3,g_3,j_3) \in B(G,I)$,

$(i_3,g_3,j_3) = (i_3,g_3g_1^{-1},i_1)(i_1,g_1,j_1)(j_1,1_G,j_3)$ and

$(i_3,g_3g_1^{-1},i_1)(i_1,g_1,j_1)(j_1,1_G,j_3)\ \theta\ (i_3,g_3g_1^{-1},i_1)(i_2,g_2,j_2)(j_1,1_G,j_3) \neq 0$ if and only if $i_1 = i_2$ and $j_1 = j_2$ if and only if x $\mathscr{H}$ y. Therefore, every $(i_3,g_3,j_3) \in B(G,I)$ is θ-related to 0 and so θ is the Rees congruence relative to D^0.] By Lemma 3.2.4, G must be simple and hence subdirectly irreducible. So suppose that (x,y) generates θ and x $\mathscr{H}$ y. Let φ be any non-identity congruence on G. Then $\theta \subseteq \varphi^*$ and so by Lemma 3.2.4 (ii), the restriction of θ to any group $\mathscr{H}$-class, $\theta^* \subseteq \varphi$. Thus, G has a minimum non-identity congruence and so must be subdirectly irreducible.

Conversely, suppose that S is a k-full inverse subsemigroup of G wr $\mathscr{I}(I)$ where G is subdirectly irreducible. We identify the minimum non-zero ideal of S with B(G,I). Let φ be the minimum non-identity congruence on G. We claim that φ^* is the minimum non-identity congruence on S. Let θ be the non-identity congruence on S generated by the pair (x,y). Since S is k-full and $x \neq y$, there is a $z \in B(G,I)$ such that $z \leq x$, $z \nleq y$ (or $z \leq y$, $z \nleq x$). Then $z = zz^{-1}x \;\theta\; zz^{-1}y \neq z$ and z and $zz^{-1}y$ are $\mathscr{D}$-related. If z and $zz^{-1}y$ are not $\mathscr{H}$-related then it is not difficult to show that θ contains the Rees congruence relative to the ideal B(G,I) which in turn contains φ^*. If $z \mathscr{H} zz^{-1}y$ then suppose that $z = (i,g,j)$ and $zz^{-1}y = (i,h,j)$ where $g \neq h$. Then $(i,g,i) = (i,g,j)(j,1_G,i) \;\theta\; (i,h,j)(j,1_G,i) = (i,h,i)$ and so θ restricted to the group $\mathscr{H}$-class $H = \{(i,g,i) : g \in G\}$ is not the equality. Therefore, φ is contained in θ restricted to H and so by Lemma 3.2.4, $\varphi^* \subseteq \theta$. It now follows that S is subdirectly irreducible. •

The subdirectly irreducible completely semisimple inverse semigroups are not only inverse subsemigroups of wreath products of the form G wr $\mathscr{I}(I)$ for some subdirectly irreducible group G, but in fact inverse subsemigroups of wreath products of the form G wr (T,I) where G is a subdirectly irreducible group and (T,I) is a k-full antigroup.

Lemma 3.2.6. Let S be a k-full inverse subsemigroup of G wr $\mathscr{I}(I)$, for some group G and some nonempty set I, and let π denote the natural homomorphism of S into $\mathscr{I}(I)$ given by $(\psi,\beta)\pi = \beta$ for all $(\psi,\beta) \in S$. Then $S\pi$ is an antigroup.

Proof: Let μ denote the greatest idempotent separating congruence on $S\pi$ and suppose that $\beta_1 \mu \beta_2$ for some $\beta_1, \beta_2 \in S\pi$. Since $\mu \subseteq \mathscr{H}$, $\beta_1 \mathscr{H} \beta_2$ and, as a consequence $\beta_1\beta_1^{-1} = \beta_2\beta_2^{-1}$, whence $\mathbf{d}\beta_1 = \mathbf{d}\beta_2$.

Let $i \in \mathbf{d}\beta_1 = \mathbf{d}\beta_2$. Since S is a k-full inverse subsemigroup of G wr $\mathscr{I}(I)$, the element β of $\mathscr{I}(I)$ defined by $\mathbf{d}\beta = \{i\}$ and $i\beta = i$, is an idempotent of $S\pi$. By the definition of μ, $\beta_1^{-1}\beta\beta_1 = \beta_2^{-1}\beta\beta_2$. Now $i\beta_1 \in \mathbf{d}\beta_1^{-1}\beta\beta_1$ and so $(i\beta_1)\beta_1^{-1}\beta\beta_1 = (i\beta_1)\beta_2^{-1}\beta\beta_2$. But $(i\beta_1)\beta_1^{-1}\beta\beta_1 = i\beta\beta_1 = i\beta_1$ and, in order for $(i\beta_1)\beta_2^{-1}\beta\beta_2$ to be defined, we must have that $(i\beta_1)\beta_2^{-1} = i$ and so $(i\beta_1)\beta_2^{-1}\beta\beta_2 = i\beta\beta_2 = i\beta_2$. Thus, $i\beta_1 = i\beta_2$ and, since our choice of i was arbitrary, it follows that $\beta_1 = \beta_2$. Consequently, $S\pi$ is an antigroup. •

Theorem 3.2.7. Let $\mathscr{V}$ be a completely semisimple variety of inverse semigroups. Then $\mathscr{V}$ is generated by those members of $\mathscr{V}$ which are subdirectly irreducible groups and inverse subsemigroups of wreath products of subdirectly irreducible groups and k-full antigroups.

Proof: $\mathscr{V}$ is completely determined by its subdirectly irreducible members. By Theorem 3.2.5, these are subdirectly irreducible groups and k-full inverse subsemigroups of wreath products of a subdirectly irreducible group and $\mathscr{I}(I)$ for some I. A k-full inverse subsemigroup of a wreath product of a subdirectly irreducible group and $\mathscr{I}(I)$ is an inverse subsemigroup of a wreath product of a subdirectly irreducible group and a k-full antigroup, by Lemma 3.2.6. •

3.3 Isomorphic wreath products and connections with varieties

This section contains some structural results concerning wreath products of inverse semigroups and some connections with varieties.

Lemma 3.3.1. Let T and A be inverse semigroups. Then T wr (A,A) can be embedded in $\mathscr{I}(T \times A)$.

Proof: Define $\Theta : T \text{ wr } (A,A) \to \mathscr{I}(T \times A)$ by $(\psi,\beta)\Theta = f_{(\psi,\beta)}$ where

$$\mathbf{d}f_{(\psi,\beta)} = \{ (t,a) : a \in \mathbf{d}\beta \text{ and } t \in T(a\psi)^{-1} \}$$

and

$$(t,a)f_{(\psi,\beta)} = (t(a\psi),a\beta).$$

We first show that $f_{(\psi,\beta)} \in \mathscr{I}(T \times A)$. Suppose that for some $(t_1,a_1),(t_2,a_2) \in \mathbf{d}f_{(\psi,\beta)}$ we have that $(t_1,a_1)f_{(\psi,\beta)} = (t_2,a_2)f_{(\psi,\beta)}$. Then $(t_1(a_1\psi),a_1\beta) = (t_2(a_2\psi),a_2\beta)$ and so $t_1(a_1\psi) = t_2(a_2\psi)$ and $a_1\beta = a_2\beta$. Since β is one–to–one, it follows that $a_1 = a_2 = a$, say. As a consequence, we have that both t_1 and t_2 belong to $T(a\psi)^{-1}$ and that $t_1(a\psi) = t_2(a\psi)$. Therefore, $t_1 = t_1(a\psi)(a\psi)^{-1} = t_2(a\psi)(a\psi)^{-1} = t_2$. Thus, $(t_1,a_1) = (t_2,a_2)$ and so $f_{(\psi,\beta)}$ is one–to–one and $f_{(\psi,\beta)} \in \mathscr{I}(T \times A)$.

Let $(\psi_1,\beta_1),(\psi_2,\beta_2) \in T \text{ wr } (A,A)$. Let f_1 denote $(\psi_1,\beta_1)\Theta$, f_2 denote $(\psi_2,\beta_2)\Theta$ and f_3 denote $(\psi_1{}^{\beta_1}\psi_2,\beta_1\beta_2)\Theta$. In order to show that Θ is a homomorphism we must show that $f_1f_2 = f_3$. Our first step is to show that $\mathbf{d}f_1f_2 = \mathbf{d}f_3$. From the definition of Θ we have that

$$\mathbf{d}f_1 = \{(t,a) : a \in \mathbf{d}\beta_1 \text{ and } t \in T(a\psi_1)^{-1}\},$$

$$\mathbf{d}f_2 = \{(t,a) : a \in \mathbf{d}\beta_2 \text{ and } t \in T(a\psi_2)^{-1}\},$$

$$\mathbf{d}f_3 = \{(t,a) : a \in \mathbf{d}\beta_1\beta_2 \text{ and } t \in T(a(\psi_1{}^{\beta_1}\psi_2))^{-1}\}.$$

It follows that

$$\begin{aligned}\mathbf{d}f_1f_2 &= \{(t,a) : a \in \mathbf{d}\beta_1, t \in T(a\psi_1)^{-1} \text{ and } (t(a\psi_1),a\beta_1) \in \mathbf{d}f_2\}\\ &= \{(t,a) : a \in \mathbf{d}\beta_1, a\beta_1 \in \mathbf{d}\beta_2, t \in T(a\psi_1)^{-1} \text{ and}\\ &\qquad t(a\psi_1) \in T(a\beta_1\psi_2)^{-1}\}\\ &= \{(t,a) : a \in \mathbf{d}\beta_1\beta_2, t \in T(a\psi_1)^{-1} \text{ and } t(a\psi_1) \in T(a\beta_1\psi_2)^{-1}\}.\end{aligned}$$

If $(t,a) \in \mathbf{d}f_3$ then $t \in T(a(\psi_1{}^{\beta_1}\psi_2))^{-1} = T((a\psi_1)(a\beta_1\psi_2))^{-1} = T(a\beta_1\psi_2)^{-1}(a\psi_1)^{-1}$ and so $t \in T(a\psi_1)^{-1}$ and $t(a\psi_1) \in T(a\beta_1\psi_2)^{-1}(a\psi_1)^{-1}(a\psi_1) = T(a\beta_1\psi_2)^{-1}(a\psi_1)^{-1}(a\psi_1)(a\beta_1\psi_2)(a\beta_1\psi_2)^{-1} \subseteq T(a\beta_1\psi_2)^{-1}$. Moreover, $a \in \mathbf{d}\beta_1\beta_2$ and so $(t,a) \in \mathbf{d}f_1f_2$. On the other hand, if $(t,a) \in \mathbf{d}f_1f_2$ then $t \in T(a\psi_1)^{-1}$ and $t(a\psi_1) \in T(a\beta_1\psi_2)^{-1}$ and so

$$t = t(a\psi_1)(a\psi_1)^{-1} \in T(a\beta_1\psi_2)^{-1}(a\psi_1)^{-1} = T(a(\psi_1{}^{\beta_1}\psi_2))^{-1}.$$

Also $a \in \mathbf{d}\beta_1\beta_2$ and so $(t,a) \in \mathbf{d}f_3$. Therefore, $\mathbf{d}f_1f_2 = \mathbf{d}f_3$.

Let $(t,a) \in \mathbf{d}f_1f_2 = \mathbf{d}f_3$. Then

$$\begin{aligned}(t,a)f_1f_2 &= (t(a\psi_1),a\beta_1)f_2 \\ &= (t(a\psi_1)(a\beta_1\psi_2),\ a\beta_1\beta_2) \\ &= (t(a(\psi_1{}^{\beta_1}\psi_2)),a\beta_1\beta_2) \\ &= (t,a)f_3.\end{aligned}$$

It now follows that Θ is a homomorphism.

Finally, we show that Θ is one–to–one. Suppose that $(\psi_1,\beta_1)\Theta = (\psi_2,\beta_2)\Theta = f$. Then $\mathbf{d}f = \{(t,a) : a \in \mathbf{d}\beta_1 \text{ and } t \in T(a\psi_1)^{-1}\} = \{(t,a) : a \in \mathbf{d}\beta_2 \text{ and } t \in T(a\psi_2)^{-1}\}$ and $(t,a)f = (t(a\psi_1),a\beta_1) = (t(a\psi_2),a\beta_2)$. Let $a \in \mathbf{d}\beta_1 = \mathbf{d}\psi_1$. Then $((a\psi_1)^{-1},a) \in \mathbf{d}f$ and so $a \in \mathbf{d}\beta_2$ whence $\mathbf{d}\beta_1 \subseteq \mathbf{d}\beta_2$. Symmetrically, we obtain that $\mathbf{d}\beta_2 \subseteq \mathbf{d}\beta_1$ and so $\mathbf{d}\beta_1 = \mathbf{d}\beta_2$. For any $a \in \mathbf{d}\beta_1 = \mathbf{d}\beta_2$,

$$((a\psi_1)^{-1},a)f = ((a\psi_1)^{-1}(a\psi_1),a\beta_1) = ((a\psi_1)^{-1}(a\psi_2),a\beta_2).$$

Thus, $a\beta_1 = a\beta_2$ and so $\beta_1 = \beta_2$. Furthermore,
$(a\psi_1)^{-1}(a\psi_1) = (a\psi_1)^{-1}(a\psi_2)$ and we can likewise obtain
$(a\psi_2)^{-1}(a\psi_2) = (a\psi_2)^{-1}(a\psi_1)$ by considering $((a\psi_2)^{-1},a)f$. We thus have that

$$\begin{aligned}a\psi_1 &= (a\psi_1)(a\psi_1)^{-1}(a\psi_1) \\ &= (a\psi_1)(a\psi_1)^{-1}(a\psi_2) \\ &= (a\psi_1)(a\psi_1)^{-1}(a\psi_2)(a\psi_2)^{-1}(a\psi_2) \\ &= (a\psi_1)(a\psi_1)^{-1}(a\psi_2)(a\psi_2)^{-1}(a\psi_1) \\ &= (a\psi_2)(a\psi_2)^{-1}(a\psi_1)(a\psi_1)^{-1}(a\psi_1) \\ &= (a\psi_2)(a\psi_2)^{-1}(a\psi_1) \\ &= (a\psi_2)(a\psi_2)^{-1}(a\psi_2) \\ &= a\psi_2.\end{aligned}$$

Therefore, $\psi_1 = \psi_2$ and as a consequence, Θ is a monomorphism.

Thus, Θ is an embedding of T wr (A,A) into $\mathscr{I}(T \times A)$. •

We will call the representation of T wr (A,A) described in Lemma 3.3.1 the *cartesian representation of* T wr (A,A) and write (T wr A, T $\times$ A) to denote this representation.

Lemma 3.3.2. Let S,T and A be inverse semigroups. Then

$$[S \text{ wr } (T,T)] \text{ wr } (A,A) \cong S \text{ wr } (T \text{ wr } A, T \times A).$$

Proof: Let $(\Psi,B) \in [S \text{ wr } (T,T)] \text{ wr } (A,A)$. Set $a\Psi = (\psi_a,\beta_a)$, for all $a \in \mathbf{d}\Psi = \mathbf{d}B$. Let $\Gamma \in \mathscr{I}(T \times A)$ be defined by setting $\mathbf{d}\Gamma = \{(t,a) : a \in \mathbf{d}B \text{ and } t \in \mathbf{d}\beta_a\}$ and defining $(t,a)\Gamma = (t\beta_a, aB)$. Define Φ, a partial map from $T \times A$ to S by setting $\mathbf{d}\Phi = \mathbf{d}\Gamma$ and defining $(t,a)\Phi = t\psi_a$. Now Γ corresponds to the pair (ψ',B) in the cartesian representation of T wr A, where for all $a \in \mathbf{d}B$, $a\psi'$ is the element of T which maps to β_a in the Wagner representation of T. Thus, the pair $(\Phi,\Gamma) \in S \text{ wr } (T \text{ wr } A, T \times A)$.

Define $\Theta : [S \text{ wr } (T,T)] \text{ wr } (A,A) \to S \text{ wr } (T \text{ wr } A, T \times A)$ by mapping (Ψ,B) to (Φ,Γ), as above.

We first show that Θ is a homomorphism.

Let $(\Psi_1,B_1), (\Psi_2,B_2) \in [S \text{ wr } (T,T)] \text{ wr } (A,A)$ and set $(\Psi_1,B_1)\Theta = (\Phi_1,\Gamma_1)$, $(\Psi_2,B_2)\Theta = (\Phi_2,\Gamma_2)$ and $(\Psi_1{}^{B_1}\Psi_2, B_1B_2)\Theta = (\Phi_3,\Gamma_3)$. We must show that $(\Phi_1{}^{\Gamma_1}\Phi_2, \Gamma_1\Gamma_2) = (\Phi_3,\Gamma_3)$. For all $a \in \mathbf{d}B_i$ set $a\Psi_i = (\psi_{i_a},\beta_{i_a})$, $i = 1,2$, and set $a(\Psi_1{}^{B_1}\Psi_2) = (\psi_a,\beta_a)$.

$$\mathbf{d}\Gamma_1 = \{(t,a) : a \in \mathbf{d}B_1 \text{ and } t \in \mathbf{d}\beta_{1_a}\},$$

$$\mathbf{d}\Gamma_2 = \{(t,a) : a \in \mathbf{d}B_2 \text{ and } t \in \mathbf{d}\beta_{2_a}\},$$

$$\mathbf{d}\Gamma_3 = \{(t,a) : a \in \mathbf{d}B_1B_2 \text{ and } t \in \mathbf{d}\beta_a\}.$$

Now

$$\mathbf{d}\Gamma_1\Gamma_2 = \{(t,a) : a \in \mathbf{d}B_1,\ aB_1 \in \mathbf{d}B_2,\ t \in \mathbf{d}\beta_{1_a} \text{ and } t\beta_{1_a} \in \mathbf{d}\beta_{2_c}\},$$ where

$c = aB_1$, while for all $a \in \mathbf{d}B_1B_2$,

$$\begin{aligned} a(\Psi_1{}^{B_1}\Psi_2) &= (a\Psi_1)(aB_1\Psi_2) \\ &= (\psi 1_a, \beta 1_a)(\psi 2_c, \beta 2_c) \\ &= (\psi 1_a{}^{\beta 1_a}\psi 2_c, \beta 1_a\beta 2_c) \end{aligned}$$

and so, as a consequence,

$$\mathbf{d}\Gamma_3 = \{(t,a) : a \in \mathbf{d}B_1B_2 \text{ and } t \in \mathbf{d}\beta 1_a\beta 2_c\} = \mathbf{d}\Gamma_1\Gamma_2.$$

Also, for any $(t,a) \in \mathbf{d}\Gamma_1\Gamma_2 = \mathbf{d}\Gamma_3$, $(t,a)\Gamma_3 = (t\beta 1_a\beta 2_c, aB_1B_2)$, while $(t,a)\Gamma_1\Gamma_2 = (t\beta 1_a, aB_1)\Gamma_2 = (t\beta 1_a\beta 2_c, aB_1B_2)$ and so $\Gamma_1\Gamma_2 = \Gamma_3$. Moreover, for any $(t,a) \in \mathbf{d}\Gamma_1\Gamma_2 = \mathbf{d}\Gamma_3$,

$$\begin{aligned} (t,a)\Phi_1{}^{\Gamma_1}\Phi_2 &= (t,a)\Phi_1\,(t,a)\Gamma_1\Phi_2 \\ &= (t\psi 1_a)(t\beta 1_a, c)\Phi_2 \\ &= (t\psi 1_a)(t\beta 1_a\psi 2_c) \end{aligned}$$

and

$$\begin{aligned} (t,a)\Phi_3 &= t\psi_a \\ &= t(\psi 1_a{}^{\beta 1_a}\psi 2_c) \\ &= (t\psi 1_a)(t\beta 1_a\psi 2_c). \end{aligned}$$

Therefore, $\Phi_3 = \Phi_1{}^{\Gamma_1}\Phi_2$ which combined with $\Gamma_3 = \Gamma_1\Gamma_2$ implies that Θ is a homomorphism.

Let $(\Psi_1,B_1), (\Psi_2,B_2) \in [S \text{ wr } (T,T)] \text{ wr } (A,A)$ and suppose that $(\Psi_1,B_1)\Theta = (\Phi,\Gamma) = (\Psi_2,B_2)\Theta$. For all $a \in \mathbf{d}B_1$ set $a\Psi_1 = (\psi 1_a, \beta 1_a)$ and for all $a \in \mathbf{d}B_2$ set $a\Psi_2 = (\psi 2_a, \beta 2_a)$. By the definition of Θ, we have

$$\begin{aligned} \mathbf{d}\Gamma &= \{(t,a) : a \in \mathbf{d}B_1 \text{ and } t \in \mathbf{d}\beta 1_a\} \\ &= \{(t,a) : a \in \mathbf{d}B_2 \text{ and } t \in \mathbf{d}\beta 2_a\}, \end{aligned}$$

and for all $(t,a) \in \mathbf{d}\Gamma$

$$\begin{aligned} (t\beta 1_a, aB_1) &= (t\beta 2_a, aB_2), \\ t\psi 1_a &= t\psi 2_a. \end{aligned}$$

Since T is given the Wagner representation in S wr (T,T), for all $a \in dB_1$, $d\beta 1_a \neq \varnothing$ and for all $a \in dB_2$, $d\beta 2_a \neq \varnothing$. Thus, given $a \in dB_1$ there is a $t \in d\beta 1_a$ so that $(t,a) \in d\Gamma$ and so $aB_1 = aB_2$. Therefore, $dB_1 \subseteq dB_2$ and B_1 and B_2 agree on the domain of B_1. Symmetrically we obtain that $dB_2 \subseteq dB_1$ and B_1 and B_2 agree on the domain of B_2, and so, as a consequence, $B_1 = B_2$. Moreover, we have that $d\Psi_1 = d\Psi_2$, and so in order to show that Θ is a monomorphism, it remains to show that for all $a \in dB_1 = dB_2$, $(\psi 1_a, \beta 1_a) = (\psi 2_a, \beta 2_a)$. From the definition of Γ we have that $t \in d\beta 1_a$ if and only if $(t,a) \in d\Gamma$ if and only if $t \in d\beta 2_a$ and so $d\beta 1_a = d\beta 2_a$. Furthermore, for any $t \in d\beta 1_a = d\beta 2_a$ by the definition of Θ, $t\beta 1_a = t\beta 2_a$ and so $\beta 1_a = \beta 2_a$. Also, $d\beta 1_a = d\beta 2_a$ implies that $d\psi 1_a = d\psi 2_a$ and again by the definition of Θ, $\psi 1_a = \psi 2_a$. It follows that $a\Psi_1 = a\Psi_2$. Therefore, $(\Psi_1, B_1) = (\Psi_2, B_2)$ and Θ is a monomorphism.

Finally, we show that Θ is surjective. Let $(\Phi, f) \in$ S wr (T wr A, T $\times$ A) and suppose that $f = f_{(\psi,\beta)}$ for some $(\psi,\beta) \in$ T wr A. Consider the pair (Ψ,β) where, for all $a \in d\beta$, $a\Psi = (\psi_a, \beta_a)$ and $d\beta_a = \{t \in T : (t,a) \in df\}$ and $t\beta_a = t(a\psi)$, $t\psi_a = (t,a)\Phi$. Now β_a is the representation of $(a\psi)$ in (T,T) and so $(\psi_a, \beta_a) \in$ S wr (T,T). Also, $\beta \in (A,A)$ and so $(\Psi,\beta) \in$ [S wr (T,T)] wr (A,A). We claim that $(\Psi,\beta)\Theta = (\Phi, f)$. Set $(\Psi,\beta)\Theta = (\Psi\Theta, \beta\Theta)$. Then

$$\begin{aligned} d\beta\Theta &= \{(t,a) : a \in d\beta \text{ and } t \in d\beta_a\} \\ &= \{(t,a) : a \in d\beta \text{ and } (t,a) \in df\} \\ &= \{(t,a) : (t,a) \in df\} \\ &= df, \end{aligned}$$

and, for all $(t,a) \in df = d\beta\Theta$,

$$\begin{aligned} (t,a)\beta\Theta &= (t\beta_a, a\beta) \\ &= (t(a\psi), a\beta) \\ &= (t,a)f. \end{aligned}$$

Also, for all $(t,a) \in \mathbf{d}\Phi = \mathbf{d}f = \mathbf{d}\beta\Theta = \mathbf{d}(\Psi\Theta)$, $(t,a)\Psi\Theta = t\psi_a = (t,a)\Phi$. It follows that $(\Psi,\beta)\Theta = (\Phi,f)$ and so Θ is surjective. Therefore, Θ is an isomorphism. •

The following proposition is a collection of simple properties of wreath products which suggest a connection between A wr B and the variety it generates.

Proposition 3.3.3. Let A and B be inverse semigroups and let $\{A_i\}_{i\in I}$ be a collection of inverse semigroups.

a) If S is an inverse subsemigroup of A then S wr B is an inverse subsemigroup of A wr B.

b) If $\alpha: A \to S$ is an epimorphism then there exists an epimorphism μ: A wr B $\to$ S wr B.

c) $\prod_{i\in I} A_i$ wr B can be embedded in $\prod_{i\in I}(A_i$ wr B).

Proof: a) If $(\psi,\beta) \in$ S wr B then $\mathbf{d}\beta = \mathbf{d}\psi$ and for all $i \in \mathbf{d}\beta$, $i\psi \in S \subseteq A$. Therefore, $(\psi,\beta) \in$ A wr B. Since S wr B is an inverse semigroup, it is an inverse subsemigroup of A wr B.

b) Define μ: A wr B $\to$ S wr B by $(\psi,\beta)\mu = (\psi^*,\beta)$ where ψ^* is defined by setting $\mathbf{d}\psi^* = \mathbf{d}\beta$ and for all $i \in \mathbf{d}\psi^*$, defining $i\psi^* = (i\psi)\alpha$. It is clear that $(\psi^*,\beta) \in$ S wr B. It follows from the definition of the multiplication in wreath products that μ is a homomorphism provided that for any $(\psi_1,\beta_1),(\psi_2,\beta_2) \in$ A wr B, we have $\psi_1{}^{*\beta_1}\psi_2{}^* = (\psi_1{}^{\beta_1}\psi_2)^*$. From the definition of the multiplication we have that $\mathbf{d}\psi_1{}^{*\beta_1}\psi_2{}^* = \mathbf{d}(\psi_1{}^{\beta_1}\psi_2)^* = \mathbf{d}\beta_1\beta_2$. Let $i \in \mathbf{d}\beta_1\beta_2$. Then

$$
\begin{aligned}
i(\psi_1{}^{\beta_1}\psi_2)^* &= [i(\psi_1{}^{\beta_1}\psi_2)]\alpha \\
&= [(i\psi_1)(i\beta_1\psi_2)]\alpha \\
&= (i\psi_1)\alpha\,(i\beta_1\psi_2)\alpha
\end{aligned}
$$

$= (i\psi_1{}^*)((i\beta_1)\psi_2{}^*)$

$= i(\psi_1{}^{*\beta_1}\psi_2{}^*).$

Therefore, μ is a homomorphism.

Let $(\psi,\beta) \in S$ wr B. Define $(\psi',\beta) \in A$ wr B by, for all $i \in \mathbf{d}\beta$, $i\psi' \in (i\psi)\alpha^{-1}$. Then $(\psi',\beta)\mu = ((\psi')^*,\beta)$ and for all $i \in \mathbf{d}\beta$, $i(\psi')^* = (i\psi')\alpha = i\psi$. Thus, μ is an epimorphism.

c) Define Φ: $(\prod_{i\in I}A_i)$ wr B $\rightarrow \prod_{i\in I}(A_i$ wr B$)$ by

$$(\psi,\beta)\Phi = (\psi_i,\beta)_{i\in I}$$

where if $i \in \mathbf{d}\beta$ and $i\psi = (a_j)_{j\in I}$, then $i\psi_j = a_j$.

Suppose that $(\psi_1,\beta_1),(\psi_2,\beta_2) \in (\prod_{i\in I}A_i)$ wr B. In order to show that Φ is a homomorphism, we must show that for all $j \in I$ and for all $i \in \mathbf{d}\beta_1\beta_2$, $i((\psi_1)_j{}^{\beta_1}(\psi_2)_j) = i(\psi_1{}^{\beta_1}\psi_2)_j$. Suppose that $i\psi_1 = (a_j)_{j\in I}$ and $(i\beta_1)\psi_2 = (b_j)_{j\in I}$. Then $i(\psi_1)_j = a_j$ and $(i\beta_1)(\psi_2)_j = b_j$ and so $i((\psi_1)_j{}^{\beta_1}(\psi_2)_j) = i(\psi_1)_j(i\beta_1)(\psi_2)_j = a_jb_j$. On the other hand, $i(\psi_1{}^{\beta_1}\psi_2)_j = j^{th}$ coordinate of $i\psi_1{}^{\beta_1}\psi_2 = j^{th}$ coordinate of $(i\psi_1)(i\beta_1\psi_2)$. But this is just the j^{th} coordinate of $(a_j)_{j\in I}\cdot(b_j)_{j\in I}$ which is simply a_jb_j. Therefore, Φ is indeed a homomorphism.

Suppose now that $(\psi_1,\beta_1)\Phi = (\psi_2,\beta_2)\Phi$. From the definition of Φ we obtain that $\beta_1 = \beta_2$ and for all $j \in I$, $(\psi_1)_j = (\psi_2)_j$. For all $i \in \mathbf{d}\beta_1 = \mathbf{d}\beta_2$, $i(\psi_1)_j$ is the j^{th} coordinate of $i\psi_1$ and $i(\psi_2)_j$ is the j^{th} coordinate of $i\psi_2$. Therefore, $i\psi_1$ and $i\psi_2$ agree in each of their coordinates and so $i\psi_1 = i\psi_2$. This is true for all $i \in \mathbf{d}\beta_1 = \mathbf{d}\beta_2$ and so it follows that Φ is a monomorphism. Thus, $(\prod_{i\in I}A_i)$ wr B can be embedded in $\prod_{i\in I}(A_i$ wr B$)$. •

Corollary 3.3.4. Let $\mathscr{V}$ be a variety of inverse semigroups and suppose that A generates $\mathscr{V}$. Then for any $S \in \mathscr{V}$, S wr B $\in \langle$ A wr B $\rangle$.

Proof: If $S \in \mathscr{V}$ then S is a homomorphic image of an inverse subsemigroup T of a direct power A^I of A, for some index set I. By Proposition 3.3.8 (c), A^I wr B can be embedded in $(A \text{ wr } B)^I$ and as a consequence, $A^I \text{ wr } B \in \langle A \text{ wr } B \rangle$. By Proposition 3.3.8 (a), T wr B is an inverse subsemigroup of A^I wr B, since T is an inverse subsemigroup of A^I. Thus, $T \text{ wr } B \in \langle A \text{ wr } B \rangle$. S is a homomorphic image of T and so, by Proposition 3.3.8 (b), there is an epimorphism of T wr B onto S wr B. Therefore, $S \text{ wr } B \in \langle A \text{ wr } B \rangle$. •

CHAPTER FOUR
The Principal Result

Given two varieties $\mathcal{U}$ and $\mathcal{V}$ of inverse semigroups, denote by Wr($\mathcal{U}$,$\mathcal{V}$) the variety generated by wreath products of semigroups in $\mathcal{U}$ with semigroups in $\mathcal{V}$. The principal result of this chapter is a description of the fully invariant congruence on $F\mathcal{I}(X)$ corresponding to Wr($\mathcal{U}$, $\mathcal{V}$) in terms of $\rho(\mathcal{U})$ and $\rho(\mathcal{V})$ for any pair of varieties $\mathcal{U}$ and $\mathcal{V}$ of inverse semigroups. Our description makes use of the Schützenberger graphs of the $\mathcal{V}$-free inverse semigroup given by the presentation $P = (X; \rho(\mathcal{V}))$. For any words w and v over X, Wr($\mathcal{U}$,$\mathcal{V}$) satisfies the equation w = v if and only if $\mathcal{V}$ satisfies w = v and $\mathcal{U}$ satisfies an equation dependent upon the paths in the Schützenberger representation of w (and hence v) relative to $\mathcal{V}$ labelled by w and v. Given two varieties $\mathcal{U}$ and $\mathcal{V}$, we can thus describe a more 'complicated' variety both in terms of its generators and the equations it satisfies if we know the equations satisfied by $\mathcal{U}$ and $\mathcal{V}$.

The first section of this chapter deals with associating the path labelled by w in the Schützenberger representation of the word w relative to the variety $\mathcal{V}$ with a word over some alphabet Y. This enables us to prove the main result of this chapter which is concerned with describing the fully invariant congruence corresponding to Wr($\mathcal{U}$,$\mathcal{V}$) in terms of the fully invariant congruences corresponding to $\mathcal{U}$ and $\mathcal{V}$. The third section concerns itself with basic properties of the Wr operator, including the result that when $\mathcal{U}$ is a group variety then Wr($\mathcal{U}$,$\mathcal{V}$) is the more familiar Mal'cev product variety $\mathcal{U} \circ \mathcal{V}$. Finally, it is shown in the fourth section that Wr is an associative operator and so $\mathcal{L}(\mathcal{I})$ is a semigroup under the operation of Wr.

4.1 Doubly Labelled Schützenberger Graphs

For any word w over X we require an 'encoding' of the path labelled by w in the Schützenberger representation of w with respect to $\mathscr{V}$ as a word over some alphabet Y. In order to do this we extend our definition of Schützenberger graph to what we call the *doubly labelled Schützenberger graph*.

Definition 4.1.1. Let $\mathscr{V}$ be a variety of inverse semigroups and ρ the fully invariant congruence on $F\mathscr{I}(X)$ corresponding to $\mathscr{V}$. Let $w \in (X \cup X^{-1})^+$ and let $\Gamma_{\mathscr{V}}(w)$ be the Schützenberger graph of w in the $\mathscr{V}$-free inverse semigroup on X. Let Y be a countably infinite set and Y^{-1} a set disjoint from Y and in one-to-one correspondence with Y via $y \leftrightarrow y^{-1}$. Assume that $X \cup X^{-1}$ and $Y \cup Y^{-1}$ are disjoint. From $\Gamma_{\mathscr{V}}(w)$ we obtain the *doubly labelled Schützenberger graph* $\overline{\Gamma_{\mathscr{V}}(w)}$ *of w relative to* $\mathscr{V}$, as follows:

$$\overline{\Gamma_{\mathscr{V}}(w)} = (\Gamma_{\mathscr{V}}(w), \lambda_w)$$

where

$$\lambda_w: E(\Gamma_{\mathscr{V}}(w)) \rightarrow Y \cup Y^{-1}$$

satisfies

(i) $(v_1,x,v_2) \in E(\Gamma_{\mathscr{V}}(w))$ and $x \in X$ implies that $\lambda_w(v_1,x,v_2) \in Y$;

(ii) $\lambda_w(v_2,x^{-1},v_1) = [\lambda_w(v_1,x,v_2)]^{-1}$;

(iii) $\lambda_w(v_1,x,v_2) = \lambda_w(v_3,z,v_4)$ implies that $v_1 = v_3$, $v_2 = v_4$, and $x = z$.

We call x the *primary label* and $\lambda_w(v_1,x,v_2)$ the *secondary label* of the edge (v_1,x,v_2). Condition (iii) says that no two distinct edges have the same secondary label, condition (ii) says that inverse edges have inverse secondary labels and condition (i) is just convenient. Thus, the doubly labelled Schützenberger graph of w is just $\Gamma_{\mathscr{V}}(w)$ with a secondary label attached to each edge such that inverse edges have inverse secondary labels and no two distinct edges have the same secondary label.

We define the *derived word* $d_{\mathcal{V}}(w)$ of w relative to $\mathcal{V}$ as follows:

Let v_1 and v_2 be the start and end vertices respectively, of the Schützenberger graph $\Gamma_{\mathcal{V}}(w)$ of w relative to $\mathcal{V}$. Then w labels a v_1-v_2 walk in $\overline{\Gamma_{\mathcal{V}}(w)}$ by primary labels, by Lemma 2.8.1 (b) and the definition of the Schützenberger representation of w with respect to $\mathcal{V}$. Let $e_1,\ldots,e_n$ be the edge sequence corresponding to this walk. Define $d_{\mathcal{V}}(w)=\lambda_w(e_1)\lambda_w(e_2)\ldots\lambda_w(e_n) \in (Y\cup Y^{-1})^+$. That is, $d_{\mathcal{V}}(w)$ is just the word obtained by taking the secondary labels from each edge in our v_1-v_2 walk.

Note that if $w = a_1\ldots a_k$, $a_i \in X\cup X^{-1}$ for $i = 1,\ldots,k$, and $d_{\mathcal{V}}(w) = b_1\ldots b_m$, $b_i \in Y\cup Y^{-1}$, then $m = k$ and if e is the edge corresponding to a_i in the start-end path labelled by w in $\Gamma_{\mathcal{V}}(w)$ then $b_i = \lambda_w(e)$ is the secondary label of e in $\overline{\Gamma_{\mathcal{V}}(w)}$. Note also that w is an instance of its derived word $d_{\mathcal{V}}(w)$ relative to $\mathcal{V}$. That is, w can be obtained from $d_{\mathcal{V}}(w)$ by a substitution of variables.

Example.

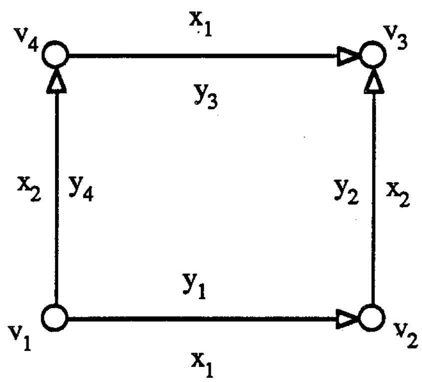

Figure 4.1. The doubly labelled Schützenberger graph $\overline{\Gamma_{\mathcal{B}^1}(w)}$.

Figure 4.1 is the doubly labelled Schützenberger graph of the word $w = x_1x_2x_1^{-1}x_2^{-1}$ relative to the variety $\mathcal{B}^1$, the variety of inverse semigroups generated by the five-element

Brandt semigroup with an identity adjoined. Both the start vertex and the end vertex are v_1. Reading directly from the graph, we have that the derived word of w with respect to $\mathcal{B}^1$ is $d_{\mathcal{B}^1}(w) = y_1y_2y_3^{-1}y_4^{-1}$.

Proposition 4.1.2 Let $\mathcal{V}$ be a variety of inverse semigroups and let $w \in X \cup X^{-1}$. Suppose that $w\ \rho(\mathcal{V})\ w^2$. Then $d_{\mathcal{V}}(w^2) = [d_{\mathcal{V}}(w)]^2$.

Proof: Let the two roots in the Schützenberger representation of w with respect to $\mathcal{V}$ be s and e. Since $w\ \rho(\mathcal{V})\ w^2$ we have that $w\ \rho(\mathcal{V})\ ww^{-1}$ and so, as a consequence, s = e. By Lemma 2.8.1 (c), w and w^2 both label s-s walks in $\overline{\Gamma_{\mathcal{V}}(w)}$ by primary labels. By Lemma 2.8.1 (a), $\Gamma_{\mathcal{V}}(w)$ is deterministic which implies that the s-s walk labelled by w^2 is just the s-s walk labelled by w taken twice. Thus, $d_{\mathcal{V}}(w^2) = [d_{\mathcal{V}}(w)]^2$, as required. •

Proposition 4.1.3. Let $\mathcal{V}$ be a variety of inverse semigroups and let v and w be words over $X \cup X^{-1}$. Then $v\ \rho\ w$ if and only if $d_{\mathcal{V}}(v)\ \rho\ d_{\mathcal{V}}(w)$, where ρ is the Wagner congruence.

Proof: The Wagner congruence ρ is generated by the relation

$$\varphi = \{ (aa^{-1}a,a) : a \in (X \cup X^{-1})^+ \} \cup \{ (aa^{-1}bb^{-1},bb^{-1}aa^{-1}) : a,b \in (X \cup X^{-1})^+ \}$$

[P;VIII.1.1].

Now, $w\ \rho\ v$ if and only if $w = v$ or

$$\begin{aligned} w &= x_1c_1y_1 \\ x_1d_1y_1 &= x_2c_2y_2 \\ x_2d_2y_2 &= x_3c_3y_3 \\ &\vdots \end{aligned}$$

$x_k d_k y_k = v,$

for some words x_i, y_i, c_i, d_i such that $c_i \varphi d_i$ or $d_i \varphi c_i$, for $i = 1,\ldots,k$.

If $w = v$, then $d_{\mathscr{V}}(w) = d_{\mathscr{V}}(v)$ and so $d_{\mathscr{V}}(w) \rho d_{\mathscr{V}}(v)$. Otherwise, we proceed by induction on k.

If $w = x_1c_1y_1$, $x_1d_1y_1 = v$ and $c_1 \varphi d_1$ then $w \rho v$ implies that $w \rho(\mathscr{V}) v$ and so both w and v label s-e walks in $\Gamma_{\mathscr{V}}(w)$. Because $\Gamma_{\mathscr{V}}(w)$ is deterministic, $d_{\mathscr{V}}(w) = xcy$ and $d_{\mathscr{V}}(v) = xdy$, where $x,y,c,d \in (Y \cup Y^{-1})^+$ and c,d depend upon the paths labelled by c_1 and d_1 in the Schützenberger graph of w with respect to $\mathscr{V}$. If $c_1 = a$ and $d_1 = aa^{-1}a$ then $d = cc^{-1}c$ since $\Gamma_{\mathscr{V}}(x_1c_1y_1)$ is deterministic. That is, the path labelled by d_1 must be the path labelled by c_1 followed by the path labelled by c_1 in reverse followed by the path labelled by c_1. Likewise, if $d_1 = a$ and $c_1 = aa^{-1}a$ then $c = dd^{-1}d$. Thus, in this case, $d_{\mathscr{V}}(w) = xcy \rho xdy = d_{\mathscr{V}}(v)$. If $c_1 = aa^{-1}bb^{-1}$ and $d_1 = bb^{-1}aa^{-1}$ then $c = u_1u_1^{-1}u_2u_2^{-1}$ and $d = u_2u_2^{-1}u_1u_1^{-1}$, again because $\Gamma_{\mathscr{V}}(x_1c_1y_1)$ is deterministic and the paths labelled by aa^{-1} and bb^{-1} both start and end at the same vertex. Thus, $d_{\mathscr{V}}(w) = xu_1u_1^{-1}u_2u_2^{-1}y \rho xu_2u_2^{-1}u_1u_1^{-1}y = d_{\mathscr{V}}(v)$. In either case, we have that $d_{\mathscr{V}}(w) \rho d_{\mathscr{V}}(v)$.

If $k > 1$, then $d_{\mathscr{V}}(w) \rho d_{\mathscr{V}}(x_kc_ky_k)$ and $d_{\mathscr{V}}(x_kc_ky_k) \rho d_{\mathscr{V}}(v)$, by the induction hypothesis, and so $d_{\mathscr{V}}(w) \rho d_{\mathscr{V}}(v)$.

Conversely, w and v are instances of $d_{\mathscr{V}}(w)$ and $d_{\mathscr{V}}(v)$, respectively, whence $d_{\mathscr{V}}(w) \rho d_{\mathscr{V}}(v)$ implies that $w \rho v$. ●

In the following lemma and throughout this thesis we use the following shorthand notation. For any words v and w over some alphabet Z and any variety $\mathscr{V}$ of inverse semigroups, we write $w \leq_{\mathscr{V}} v$ to mean $w\rho(\mathscr{V}) \leq v\rho(\mathscr{V})$ in the natural partial order on the $\mathscr{V}$-free object over the set Z.

Lemma 4.1.4. Let $w = a_1 \ldots a_k$ and $v = d_1 \ldots d_m$ with $w \ \rho(\mathscr{V})\ v$. Set $d_{\mathscr{V}}(w) = b_1 \ldots b_k$ and $d_{\mathscr{V}}(v) = c_1 \ldots c_m$, where we construct both $d_{\mathscr{V}}(w)$ and $d_{\mathscr{V}}(v)$ from the same doubly labelled Schützenberger graph $\overline{\Gamma_{\mathscr{V}}(w)}$. Then

a) $b_i = c_j \quad \Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_{i-1} d_j \ldots d_m$ and $a_i = d_j$

b) $b_i = c_j^{-1} \Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_i d_j \ldots d_m$ and $a_i = d_j^{-1}$

$\Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_{i-1} d_{j+1} \ldots d_m$ and $a_i = d_j^{-1}$.

Proof: The proofs of a) and b) are similar. We provide a proof of a). Let s and e be the start and end vertices, respectively, corresponding to w and v in $\overline{\Gamma_{\mathscr{V}}(w)}$ (and so also in $\Gamma_{\mathscr{V}}(w)$). If $b_i = c_j$ then a_i and d_j are primary labels for the same edge in $\overline{\Gamma_{\mathscr{V}}(w)}$ and so $a_i = d_j$. Moreover, $a_1 \ldots a_{i-1} d_j \ldots d_m$ must label an s-e walk by primary labels in $\overline{\Gamma_{\mathscr{V}}(w)}$ and so, by Lemma 2.8.1 (c), $a_1 \ldots a_{i-1} d_j \ldots d_m \geq_{\mathscr{V}} w$. Conversely, if $a_i = d_j$ and $a_1 \ldots a_{i-1} d_j \ldots d_m \geq_{\mathscr{V}} w$ then, by Lemma 2.8.1 (c), $a_1 \ldots a_{i-1} d_j \ldots d_m$ must label an s-e walk by primary labels $\overline{\Gamma_{\mathscr{V}}(w)}$. Since both w and v label s-e walks by primary labels in $\overline{\Gamma_{\mathscr{V}}(w)}$ and since $\Gamma_{\mathscr{V}}(w)$ is deterministic by Lemma 2.8.1 (a), we must have that a_i and d_j are primary labels for the same edge. It follows that $b_i = c_j$. •

Remark. If we take v = w in Lemma 4.1.4 we obtain

a) $b_i = b_j \quad \Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_{i-1} a_j \ldots a_k$ and $a_i = a_j$.

b) $b_i = b_j^{-1} \quad \Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_i a_j \ldots a_k$, and $a_i = a_j^{-1}$

$\Leftrightarrow w \leq_{\mathscr{V}} a_1 \ldots a_{i-1} a_{j+1} \ldots a_k$, and $a_i = a_j^{-1}$.

4.2 The Main Theorem

Definition 4.2.1. For every pair $\mathscr{U}$ and $\mathscr{V}$ of varieties of inverse semigroups, let $Wr(\mathscr{U},\mathscr{V}) = \langle\, S \text{ wr } (T,I) : S \in \mathscr{U} \text{ and } T \in \mathscr{V} \rangle$.

Varieties of the form $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ will be the focus of our investigations throughout this chapter and chapter five. Our first task is to describe the fully invariant congruence on the free inverse semigroup corresponding to $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ in terms of the fully invariant congruences corresponding to the varieties $\mathscr{U}$ and $\mathscr{V}$. Observe that, for any varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, $\mathscr{U}, \mathscr{V} \subseteq \mathrm{Wr}(\mathscr{U}, \mathscr{V})$. This fact will be used throughout this text without explicit reference.

Definition 4.2.2. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. Define a relation $\Phi(\mathscr{U},\mathscr{V})$ on $F\mathscr{I}(X)$ as follows:

$$u\ \Phi(\mathscr{U},\mathscr{V})\ w \Leftrightarrow u\ \rho(\mathscr{V})\ w \text{ and } d_{\mathscr{V}}(u)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(w)\,,$$

where $d_{\mathscr{V}}(u)$ and $d_{\mathscr{V}}(w)$ are both obtained from the same doubly labelled Schützenberger graph $\overline{\Gamma_{\mathscr{V}}(w)}$.

Observe that $\Phi(\mathscr{U},\mathscr{V})$ is an equivalence relation. We will see in Theorem 4.2.3 that it is not only an equivalence relation, but a fully invariant congruence on $F\mathscr{I}(X)$. Note that, if we think of $\Phi(\mathscr{U},\mathscr{V})$ as a relation on $(X \cup X^{-1})^+$, then by Proposition 4.1.3, the Wagner congruence $\rho \subseteq \Phi(\mathscr{U},\mathscr{V})$ and so, as a relation on $F\mathscr{I}(X)$, $\Phi(\mathscr{U},\mathscr{V})$ is well-defined.

Example. If we let $\mathscr{V}$ be the variety $\mathscr{S}$ of semilattices and $\mathscr{U}$ be any variety of inverse semigroups then $d_{\mathscr{V}}(u)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(w)$ if and only if $u\ \rho(\mathscr{U})\ w$ because $d_{\mathscr{S}}(u)$ is just a relabelling of u, for any word u over X (see Proposition 2.8.3 and the example which accompanies it). Thus, u $\Phi(\mathscr{U},\mathscr{S})$ w if and only if u $\rho(\mathscr{V})$ w and u $\rho(\mathscr{U})$ w. That is, Φ is just $\rho(\mathscr{S} \vee \mathscr{U})$ in this case.

Example. If $\mathscr{V}$ is any variety of inverse semigroups then u $\Phi(\mathscr{S},\mathscr{V})$ w if and only if u $\rho(\mathscr{V})$ w and $d_{\mathscr{V}}(u)$ ρ $d_{\mathscr{V}}(w)$, where ρ is the Wagner congruence. By Proposition 4.1.3, $d_{\mathscr{V}}(u)$ ρ $d_{\mathscr{V}}(w)$ if and only if u ρ w. Thus, u $\Phi(\mathscr{S},\mathscr{V})$ w if and only if u $\rho(\mathscr{S} \vee \mathscr{V})$ w. That is, $\Phi(\mathscr{S},\mathscr{V})$ is just ρ.

The following is the principal result of this work. It connects the variety $Wr(\mathscr{U},\mathscr{V})$ to the relation $\Phi(\mathscr{U},\mathscr{V})$.

Theorem 4.2.3. Let $Wr(\mathscr{U},\mathscr{V}) = \langle T \text{ wr } (F,I) : T \in \mathscr{U}, F \in \mathscr{V} \rangle$. Then

$$\Phi(\mathscr{U},\mathscr{V}) = \rho(Wr(\mathscr{U},\mathscr{V})).$$

Proof: For ease of notation, set $\Phi(\mathscr{U},\mathscr{V}) = \Phi$ and $\rho(Wr(\mathscr{U},\mathscr{V})) = \rho$.

We first show that $\Phi \subseteq \rho$. Suppose that u Φ w. We will show that u = w is a law in $Wr(\mathscr{U},\mathscr{V})$. It is sufficient to show that every S = T wr (F,I) which is in the generating set of $Wr(\mathscr{U},\mathscr{V})$ satisfies u = w.

Let

$u = u(x_1,\ldots,x_n) = a_1\ldots a_k$ and $w = w(x_1,\ldots,x_n) = d_1\ldots d_m$

where

$c(u) \cup c(w) = \{x_1,\ldots,x_n\}$ and $a_i, d_j \in X \cup X^{-1}$ for $i = 1,\ldots,k$ and $j = 1,\ldots,m$.

Let S = T wr (F,I) where $T \in \mathscr{U}$ and $F \in \mathscr{V}$. Let $(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n) \in S$ and suppose that

$u[(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)] = (\psi,\beta)$,

$w[(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)] = (\psi',\beta')$.

Let $\pi : S \to F$ be given by $\pi : (\varphi, \alpha) \to \alpha$, for all $(\varphi, \alpha) \in S$.

Then π is an epimorphism of S onto F which we shall call the *natural homomorphism of* S *onto* F. Since $F \in \mathscr{V}$, it follows from the hypothesis that $F = S\pi$ satisfies $u = w$.

Therefore,

$$\begin{aligned} \beta &= [u((\psi_1,\beta_1),\ldots,(\psi_n,\beta_n))]\pi \\ &= u[(\psi_1,\beta_1)\pi,\ldots,(\psi_n,\beta_n)\pi] \\ &= w[(\psi_1,\beta_1)\pi,\ldots,(\psi_n,\beta_n)\pi] \\ &= [w((\psi_1,\beta_1),\ldots,(\psi_n,\beta_n))]\pi \\ &= \beta'. \end{aligned}$$

Thus, $\beta = \beta'$ and $\mathbf{d}\psi = \mathbf{d}\beta = \mathbf{d}\beta' = \mathbf{d}\psi'$, so that all we need to show is that $i\psi = i\psi'$ for all $i \in \mathbf{d}\psi$.

We will write

$$\psi_{a_i} = \begin{cases} \psi_j & \text{if } a_i = x_j \\ \psi_j^{-1} & \text{if } a_i = x_j^{-1} \end{cases}$$

$$\beta_{a_i} = \begin{cases} \beta_j & \text{if } a_i = x_j \\ \beta_j^{-1} & \text{if } a_i = x_j^{-1} \end{cases}$$

We will also write $\beta_{a_1\ldots a_i}$ for $\beta_{a_1}\beta_{a_2}\cdots\beta_{a_i}$. Observe that with this notation $\psi_{a_i^{-1}} = \psi_{a_i}^{-1}$ and $\beta_{a_i^{-1}} = \beta_{a_i}^{-1}$.

Let $d_{\mathscr{V}}(u) = b_1\ldots b_k$ and $d_{\mathscr{V}}(w) = c_1\ldots c_m$. Let $i \in \mathbf{d}\psi = \mathbf{d}\beta$. We first prove the following statements.

1) $b_p = b_j \;\Rightarrow\; (i\beta_{a_1\ldots a_{p-1}})\psi_{a_p} \;=\; (i\beta_{a_1\ldots a_{j-1}})\psi_{a_j}$;

2) $b_p = b_j^{-1} \;\Rightarrow\; (i\beta_{a_1\ldots a_{p-1}})\psi_{a_p} \;=\; [(i\beta_{a_1\ldots a_{j-1}})\psi_{a_j}]^{-1}$;

3) $b_p = c_j \;\Rightarrow\; (i\beta_{a_1\ldots a_{p-1}})\psi_{a_p} \;=\; (i\beta_{d_1\ldots d_{j-1}})\psi_{d_j}$;

4) $b_p = c_j^{-1} \Rightarrow (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p} = [(i\beta_{d_1 \ldots d_{j-1}})\psi_{d_j}]^{-1}$;

5) $c_p = c_j \Rightarrow (i\beta_{d_1 \ldots d_{p-1}})\psi_{d_p} = (i\beta_{d_1 \ldots d_{j-1}})\psi_{d_j}$;

6) $c_p = c_j^{-1} \Rightarrow (i\beta_{d_1 \ldots d_{p-1}})\psi_{d_p} = [(i\beta_{d_1 \ldots d_{j-1}})\psi_{d_j}]^{-1}$.

1) Suppose that $b_p = b_j$ for $p < j \leq k$. Then $a_p = a_j$ and $u' = a_1 \ldots a_{p-1}a_j \ldots a_k \geq_{\mathscr{V}} u$, by Lemma 4.1.4. Again $S\pi \in \mathscr{V}$ and so $\beta'' = \beta_{a_1 \ldots a_{p-1}a_j \ldots a_k} \geq \beta_{a_1 \ldots a_k} = \beta$. This means that $\mathbf{d}\beta \subseteq \mathbf{d}\beta''$ and β and β'' agree when both are defined, and so $i\beta = i\beta''$. But then, $(i\beta_{a_1 \ldots a_{j-1}})\beta_{a_j \ldots a_k} = i\beta = i\beta'' = (i\beta_{a_1 \ldots a_{p-1}})\beta_{a_j \ldots a_k}$. Since $\beta_{a_j \ldots a_k}$ is one-to-one, we have $i\beta_{a_1 \ldots a_{j-1}} = i\beta_{a_1 \ldots a_{p-1}}$. This, combined with $a_p = a_j$, gives $(i\beta_{a_1 \ldots a_{j-1}})\psi_{a_j} = (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p}$.

2) Suppose that $b_p = b_j^{-1}$ with $p < j \leq k$. Then $a_p = a_j^{-1}$ and $u' = a_1 \ldots a_p a_j \ldots a_k \geq_{\mathscr{V}} u$, by Lemma 4.1.4. Again $S\pi \in \mathscr{V}$ and so $\beta'' = \beta_{a_1 \ldots a_p a_j \ldots a_k} \geq \beta_{a_1 \ldots a_k} = \beta$. This means that $\mathbf{d}\beta \subseteq \mathbf{d}\beta''$ and β and β'' agree when both are defined. As in 1) we obtain $i\beta_{a_1 \ldots a_p} = i\beta_{a_1 \ldots a_{j-1}}$. Then

$$\begin{aligned}
[(i\beta_{a_1 \ldots a_{j-1}})\psi_{a_j}]^{-1} &= [(i\beta_{a_1 \ldots a_p})\psi_{a_p{-1}}]^{-1} \\
&= [(i\beta_{a_1 \cdots a_p})\psi_{a_p}{}^{-1}]^{-1} \\
&= [[(i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p}]^{-1}]^{-1} \text{ (definition of } \psi_{a_p}{}^{-1}) \\
&= (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p}.
\end{aligned}$$

3) Suppose that $b_p = c_j$. By Lemma 4.1.4, $a_p = d_j$ and $u' = a_1 \ldots a_{p-1}d_j \ldots d_m \geq_{\mathscr{V}} u$. Since $S\pi \in \mathscr{V}$, we have that $\beta'' = \beta_{a_1 \ldots a_{p-1}d_j \ldots d_m} \geq \beta_{a_1 \ldots a_k} = \beta$. This means that $\mathbf{d}\beta \subseteq \mathbf{d}\beta''$ and both β and β'' agree on $\mathbf{d}\beta$. In particular, $i\beta'' = i\beta = i\beta'$. But then, $(i\beta_{a_1 \ldots a_{p-1}})\beta_{d_j \ldots d_m} = i\beta'' = i\beta' = (i\beta_{d_1 \ldots d_{j-1}})\beta_{d_j \ldots d_m}$. Since $\beta_{d_j \ldots d_m}$ is one-to-one, $i\beta_{a_1 \ldots a_{p-1}} = i\beta_{d_1 \ldots d_{j-1}}$. Combining this with the fact that $a_p = d_j$ gives $(i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p} = (i\beta_{d_1 \ldots d_{j-1}})\psi_{d_j}$.

4) Suppose that $b_p = c_j^{-1}$. By Lemma 4.1.4, $a_p = d_j^{-1}$ and $u' = a_1 \ldots a_p d_j \ldots d_m \geq_{\mathscr{V}} u$. Again $S\pi \in \mathscr{V}$ implies that $\beta'' = \beta_{a_1 \ldots a_p d_j \ldots d_m} \geq \beta_{a_1 \ldots a_k} = \beta$. This means that $\mathbf{d}\beta \subseteq \mathbf{d}\beta''$ and both β and β'' agree on $\mathbf{d}\beta$. In particular, $i\beta'' = i\beta = i\beta'$. But then, $(i\beta_{a_1 \ldots a_p})\beta_{d_j \ldots d_m} = i\beta'' = i\beta' = (i\beta_{d_1 \ldots d_{j-1}})\beta_{d_j \ldots d_m}$. Since $\beta_{d_j \ldots d_m}$ is one-to-one, $i\beta_{a_1 \ldots a_p} = i\beta_{d_1 \ldots d_{j-1}}$.

$$\begin{aligned}
[(i\beta_{d_1 \ldots d_{j-1}})\psi_{d_j}]^{-1} &= [(i\beta_{a_1 \ldots a_p})\psi_{a_p^{-1}}]^{-1} \\
&= [(i\beta_{a_1 \ldots a_p})\psi_{a_p}{}^{-1}]^{-1} \\
&= [[(i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p}]^{-1}]^{-1} \text{ (definition of } \psi_{a_p}{}^{-1}) \\
&= (i\beta_{a_1 \ldots a_{p-1}})\psi_{a_p}.
\end{aligned}$$

5) and 6) The proofs use Lemma 4.1.4 and are similar to the proofs of 1) and 2).

Multiplying $u[(\psi_1,\beta_1),\ldots(\psi_n,\beta_n)]$ from left to right we obtain

$$i\psi = (i\psi_{a_1})(i\beta_{a_1}\psi_{a_2})(i\beta_{a_1a_2}\psi_{a_3})\cdots(i\beta_{a_1 \ldots a_{k-1}}\psi_{a_k}).$$

Likewise, we obtain

$$i\psi' = (i\psi_{d_1})(i\beta_{d_1}\psi_{d_2})(i\beta_{d_1d_2}\psi_{d_3})\cdots(i\beta_{d_1 \ldots d_{m-1}}\psi_{d_m}).$$

By 1)–6) above, the expressions on the right-hand side are instances of $d_{\mathscr{V}}(u)$ and $d_{\mathscr{V}}(w)$ by the same substitution of variables. Since $T \in \mathscr{U}$, T satisfies $d_{\mathscr{V}}(u) = d_{\mathscr{V}}(w)$ and so, as a consequence, $i\psi = i\psi'$. It now follows that $(\psi,\beta) = (\psi',\beta')$ and hence that T wr (F,I) satisfies $u = w$. Therefore, the generating semigroups of $Wr(\mathscr{U},\mathscr{V})$ satisfy $u = w$ and so $Wr(\mathscr{U},\mathscr{V})$ also satisfies $u = w$, whence $\Phi \subseteq \rho$.

Before we prove that $\rho \subseteq \Phi$, we require a construction and a preliminary lemma.

Construction 4.2.4. Let $w, u \in (X \cup X)^+$ be such that $w\ \rho(\mathscr{V})\ u$ and let $\overline{\Gamma_{\mathscr{V}}(w)}$ be their doubly labelled Schützenberger graph relative to $\mathscr{V}$. Let s and e be the start and end vertices, respectively, corresponding to w (and u) in $\overline{\Gamma_{\mathscr{V}}(w)}$ and let V denote the set of vertices of $\overline{\Gamma_{\mathscr{V}}(w)}$. Suppose that $c(w) \cup c(u) = \{x_1, \ldots, x_m\}$ and $c(d_{\mathscr{V}}(w)) \cup c(d_{\mathscr{V}}(u)) = \{y_1, \ldots, y_n\}$, where $x_1, \ldots x_m \in X$ and $y_1, \ldots y_n \in Y$. Here X is the set of primary labels and Y is the set of secondary labels in $\overline{\Gamma_{\mathscr{V}}(w)}$. Let T be any inverse semigroup and let $t_1, \ldots, t_n \in T$. We use $\{ x_1, \ldots, x_m \}$, $\{ y_1, \ldots, y_n \}$ and $t_1, \ldots, t_n$ to construct an inverse semigroup S as follows.

For $i = 1, \ldots, m$ let $s_i = (\psi_i, \beta_i)$ where $\psi_i \in T^V$, $\beta_i \in \mathscr{I}(V)$ are defined by:

$\mathbf{d}\beta_i = \mathbf{d}\psi_i = \{v \in V : (v, x_i, v') \in E(\overline{\Gamma_{\mathscr{V}}(w)}) \text{ for some } v' \in V\}$

and for $v \in \mathbf{d}\beta_i = \mathbf{d}\psi_i$,

$$v\beta_i = v' \qquad \text{where } (v, x_i, v') \in E(\overline{\Gamma_{\mathscr{V}}(w)}),$$

$$v\psi_i = \begin{cases} t_k & \text{if } \lambda_w(v, x_i, v') = y_k, \\ t & \text{if } \lambda_w(v, x_i, v') \notin \{y_1, \ldots, y_n\}. \end{cases}$$

Here, t is some fixed element of T.

Then $s_i \in T \text{ wr } \mathscr{I}(V)$, for $i = 1, \ldots, m$. Let S be the inverse subsemigroup of $T \text{ wr } \mathscr{I}(V)$ generated by $\{s_1, \ldots, s_m\}$. Note that S depends on T, $t_1, \ldots, t_n$, $\{ x_1, \ldots, x_m \}$, $\{ y_1, \ldots, y_n \}$ and $\overline{\Gamma_{\mathscr{V}}(w)}$.

Observe that if u is a word in $\{ x_1,\ldots,x_m,x_1^{-1},\ldots,x_m^{-1} \}^+$ and $(\psi,\beta) = s$ is the element of S obtained from u by substituting s_j for x_j, $j = 1,\ldots,m$, then for all $v \in \mathbf{d}\beta$, u labels a v-v' walk by primary labels in $\overline{\Gamma_{\mathscr{V}}(w)}$ if and only if $v\beta = v'$.

Lemma 4.2.5. Let $\mathscr{V}$ be a variety of inverse semigroups and suppose that T is an inverse semigroup. Let $u, w \in (X \cup X)^+$ be such that $u\,\rho(\mathscr{V})\,w$ and set $F = F\mathscr{V}(X)$. Let S be as constructed in 4.2.4 using any $t_1,\ldots,t_n \in T$ and $\overline{\Gamma_{\mathscr{V}}(w)}$. Let (F,F) be the Wagner representation of F by partial right translations. Then $S \in \langle T \text{ wr } (F,F) \rangle$. If T is a member of the variety $\mathscr{U}$ then $S \in Wr(\mathscr{U},\mathscr{V})$.

Proof: Let R_w be the $\mathscr{R}$-class of $w\rho(\mathscr{V})$ in $F\mathscr{V}(X)$.

Define

$$\theta : T \text{ wr } (F,F) \to T \text{ wr } \mathscr{I}(R_w) \text{ by}$$

$$\theta : (\psi,\beta) \to (\psi\theta,\beta\theta) \text{ where}$$

$\mathbf{d}\psi\theta = \mathbf{d}\beta\theta = \{ u \in \mathbf{d}\beta : u \in R_w, u\beta \in R_w \}$ and for all $u \in \mathbf{d}\beta\theta$, $u\beta\theta = u\beta$, $u\psi\theta = u\psi$.

We first show that θ is a homomorphism. Observe that θ as defined maps T wr (F,F) into T wr $\mathscr{I}(R_w)$. Let $(\psi_1, \beta_1), (\psi_2, \beta_2) \in T$ wr (F,F). Now F is given the Wagner representation by partial right translations of itself, so there exist $v_1, v_2 \in F$ such that $\mathbf{d}\beta_1 = Fv_1^{-1}$, $\mathbf{d}\beta_2 = Fv_2^{-1}$ and for all v in the domain of β_1, $v\beta_1 = vv_1$ and for all v in the domain of β_2, $v\beta_2 = vv_2$.

Since $(\psi_1,\beta_1)\theta\,(\psi_2,\beta_2)\theta = (\psi_1\theta\, {}^{\beta_1\theta}\psi_2\theta,\ \beta_1\theta\beta_2\theta)$ and $(\psi_1,\beta_1)(\psi_2,\beta_2)\theta = ((\psi_1{}^{\beta_1}\psi_2)\theta,\ \beta_1\beta_2\theta)$, we must show that $\beta_1\beta_2\theta = \beta_1\theta\beta_2\theta$ and $\psi_1\theta\, {}^{\beta_1\theta}\psi_2\theta = (\psi_1{}^{\beta_1}\psi_2)\theta$.

The domain of $\beta_1\theta\beta_2\theta$ is the set $\{ u : u \in R_w, u\beta_1 \in R_w \text{ and } u\beta_1\beta_2 \in R_w \}$, while the domain of $\beta_1\beta_2\theta$ is the set $\{ u : u \in R_w \text{ and } u\beta_1\beta_2 \in R_w \}$. But if $u \in R_w$ and $u\beta_1\beta_2 \in R_w$ then $u \in R_w$ and $uv_1v_2 \in R_w$ and so $uv_1 \in R_w$. As a consequence, $\mathbf{d}\beta_1\beta_2\theta = \mathbf{d}\beta_1\theta\beta_2\theta$ and so, for all $v \in \mathbf{d}\beta_1\beta_2\theta$, $v\beta_1\beta_2\theta = v\beta_1\beta_2 = v\beta_1\theta\beta_2\theta$.

Since $\mathbf{d}\beta_1\beta_2\theta = \mathbf{d}\beta_1\theta\beta_2\theta$ we have $\mathbf{d}\psi_1\theta\,{}^{\beta_1\theta}\psi_2\theta = \mathbf{d}(\psi_1{}^{\beta_1}\psi_2)\theta$. For any $v \in \mathbf{d}\beta_1\beta_2\theta$, $v(\psi_1{}^{\beta_1}\psi_2)\theta = v(\psi_1{}^{\beta_1}\psi_2) = (v\psi_1)(v\beta_1\psi_2)$, while $v\psi_1\theta\,{}^{\beta_1\theta}\psi_2\theta = (v\psi_1\theta)(v\beta_1\theta\psi_2\theta) = (v\psi_1)(v\beta_1\psi_2)$ since $v \in \mathbf{d}\beta_1\theta\beta_2\theta$ implies that $v\beta_1\theta = v\beta_1$ and $(v\beta_1)\psi_2\theta = v\beta_1\psi_2$. Therefore, $\psi_1\theta\,{}^{\beta_1\theta}\psi_2\theta = (\psi_1{}^{\beta_1}\psi_2)\theta$. It now follows that θ is a homomorphism.

We now claim that S is an inverse subsemigroup of the image of θ in T wr $\mathscr{I}(R_w)$. It is enough to show that each generator of S is in the image of θ. Let $s_i = (\psi_i,\beta_i)$ be a generator of S. Then,

$$\begin{aligned} \mathbf{d}\psi_i = \mathbf{d}\beta_i &= \{\, v \in R_w : (v, x_i, v') \in E(\overline{\Gamma_{\mathscr{V}}(w)}) \text{ for some } v' \in R_w \,\} \\ &= \{\, v \in R_w : v\, x_i\rho(\mathscr{V}) \in R_w \,\} \\ &\subseteq Fx_i^{-1}\rho(\mathscr{V}), \end{aligned}$$

where the last containment follows from the more general fact that if a and ax are $\mathscr{R}$–related elements of the same inverse semigroup, then $a = axx^{-1}$.

We choose $(\psi,\beta) \in$ T wr (F,F) as follows. Let β be the representation in the Wagner representation of F of $x_i\rho(\mathscr{V})$ so that $\mathbf{d}\beta = Fx_i^{-1}\rho(\mathscr{V})$. Let ψ be any mapping from $\mathbf{d}\beta$ into T such that, for any $v \in R_w \cap \mathbf{d}\beta$ such that $v\beta \in R_w$, $v\psi = v\psi_i$. Such a ψ exists since $\mathbf{d}\psi_i \subseteq \mathbf{d}\beta$. We clearly have that $(\psi,\beta) \in$ T wr (F,F).

Consider now $(\psi,\beta)\theta = (\psi\theta,\beta\theta)$, where

$$\begin{aligned} \mathbf{d}\psi\theta = \mathbf{d}\beta\theta &= \{\, v \in \mathbf{d}\beta : v \in R_w \text{ and } v\beta \in R_w \,\} \\ &= \{\, v \in \mathbf{d}\beta : v \in R_w \text{ and } vx_i\rho(\mathscr{V}) \in R_w \,\} \\ &= \{\, v \in Fx_i^{-1}\rho(\mathscr{V}) : v \in R_w \text{ and } vx_i\rho(\mathscr{V}) \in R_w \,\} \\ &= \{\, v \in R_w : v = vx_ix_i^{-1}\rho(\mathscr{V}) \text{ and } vx_i\rho(\mathscr{V}) \in R_w \,\} \\ &= \{\, v \in R_w : vx_i\rho(\mathscr{V}) \in R_w \,\} \\ &= \mathbf{d}\beta_i = \mathbf{d}\psi_i. \end{aligned}$$

Moreover, for any $v \in \mathbf{d}\beta\theta$, $v\beta\theta = v\beta = vx_i\rho(\mathscr{V}) = v\beta_i$ and $v\psi\theta = v\psi = v\psi_i$ by our choice of (ψ,β).

Thus, $(\psi,\beta)\theta = (\psi_i,\beta_i)$ and so S is an inverse subsemigroup of the image of θ in T wr $\mathscr{I}(R_w)$. It follows that $S \in \langle T \text{ wr } (F,F) \rangle$. If T is a member of the variety $\mathscr{U}$ then T wr (F,F) $\in$ Wr($\mathscr{U}$,$\mathscr{V}$) and so we also have that S $\in$ Wr($\mathscr{U}$,$\mathscr{V}$). •

We now show that $\rho \subseteq \Phi$. Suppose that u = w is a law in Wr($\mathscr{U}$,$\mathscr{V}$). Since $\mathscr{V} \subseteq$ Wr($\mathscr{U}$,$\mathscr{V}$), $\mathscr{V}$ satisfies u = w. Therefore, to prove the theorem, we need only show that $d_{\mathscr{V}}(u)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(w)$.

Suppose that $c(u) \cup c(w) = \{x_1,\ldots,x_m\}$ and that $c(d_{\mathscr{V}}(u)) \cup c(d_{\mathscr{V}}(w)) = \{y_1,\ldots,y_n\}$. Let T be any generator of $\mathscr{U}$ (for e.g., we may take $T = F\mathscr{U}(X)$). It is sufficient to show that T satisfies $d_{\mathscr{V}}(u) = d_{\mathscr{V}}(w)$.

Let $t_1,\ldots,t_n \in T$, and let S be the inverse semigroup which is constructed as in 4.2.4 using $t_1,\ldots,t_n$ and $\overline{\Gamma_{\mathscr{V}}(w)}$. By Lemma 4.2.5, S $\in$ Wr($\mathscr{U}$,$\mathscr{V}$) and so S satisfies u = w. Therefore, with $s_i = (\psi_i,\beta_i)$, $u(s_1,\ldots,s_m) = w(s_1,\ldots,s_m) = (\psi,\beta)$, say. Let v be the start vertex of u and w in $\overline{\Gamma_{\mathscr{V}}(w)}$ (u and w have the same start vertex in $\overline{\Gamma_{\mathscr{V}}(w)}$ since $u\ \rho(\mathscr{V})\ w$). Set $u = d_1\ldots d_k$, where $d_i \in X \cup X^{-1}$.

As before, we write

$$\psi_{d_i} = \begin{cases} \psi_j & \text{if } d_i = x_j \\ \psi_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases}$$

$$\beta_{d_i} = \begin{cases} \beta_j & \text{if } d_i = x_j \\ \beta_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases}$$

Again, we write $\beta_{d_1\ldots d_i}$ for $\beta_{d_1}\beta_{d_2}\ldots\beta_{d_i}$.

Now u labels a v-vβ walk in $\overline{\Gamma_{\mathscr{V}}(w)}$ (see the observation made after the construction) and the edge sequence corresponding to this walk is $(v,d_1,v\beta_{d_1})$, $(v\beta_{d_1},d_2,v\beta_{d_1d_2})$, $(v\beta_{d_1d_2},d_3,v\beta_{d_1d_2d_3}),\ldots,(v\beta_{d_1\ldots d_{k-1}},d_k,v\beta_{d_1\ldots d_k})$. By definition, for any

$i < k$, $v\beta d_1 \ldots d_{i-1}\psi d_i = t_q$ if and only if the edge between $v\beta d_1 \ldots d_{i-1}$ and $v\beta d_1 \ldots d_i$ with primary label d_i has secondary label y_q. Thus,

$$\begin{aligned} v\psi &= (v\psi d_1)(v\beta d_1\psi d_2)(v\beta d_1 d_2\psi d_3)\ldots(v\beta d_1\ldots d_{k-1}\psi d_k) \\ &= d_{\mathscr{V}}(u)[t_1,\ldots,t_n]. \end{aligned}$$

Similarly, we obtain $v\psi = d_{\mathscr{V}}(w)[t_1,\ldots,t_n]$ and so we conclude that $d_{\mathscr{V}}(u)[t_1,\ldots,t_n] = d_{\mathscr{V}}(w)[t_1,\ldots,t_n]$. Since the t_i were chosen arbitrarily, we have T (and hence $\mathscr{U}$) satisfies $d_{\mathscr{V}}(u) = d_{\mathscr{V}}(w)$. Therefore, $d_{\mathscr{V}}(u)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(w)$ and $u\ \Phi\ w$. •

Theorem 4.2.6. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. Let $(F\mathscr{V}(X),F\mathscr{V}(X))$ be the Wagner representation of $F\mathscr{V}(X)$ by partial right translations. If T generates $\mathscr{U}$ then T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ generates $Wr(\mathscr{U},\mathscr{V})$. In particular, $F\mathscr{U}(X)$ wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ generates $Wr(\mathscr{U},\mathscr{V})$.

Proof: Clearly $\langle$ T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ $\rangle \subseteq Wr(\mathscr{U},\mathscr{V})$. Thus, to prove the corollary we need only show that if T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ satisfies the equation $u = w$ then $\mathscr{V}$ satisfies $u = w$ and $\mathscr{U}$ satisfies $d_{\mathscr{V}}(u) = d_{\mathscr{V}}(w)$.

Since $F\mathscr{V}(X) \in \langle$ T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ $\rangle$ we have that $\mathscr{V} \subseteq \langle$ T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ $\rangle$ and so $\mathscr{V}$ satisfies $u = w$. Since T generates $\mathscr{U}$, it is sufficient to show that T satisfies $d_{\mathscr{V}}(u) = d_{\mathscr{V}}(w)$. We may now use the proof of $\rho \subseteq \Phi$ in Theorem 4.2.3 to demonstrate this, noting that any semigroup S, as constructed in 4.2.4, which is used in this proof is a member of the variety $\langle$ T wr $(F\mathscr{V}(X),F\mathscr{V}(X))$ $\rangle$ by Lemma 4.2.5. •

Remark. We cannot in general replace $F\mathscr{V}(X)$ in Theorem 4.2.6 by an arbitrary inverse semigroup which generates $\mathscr{V}$. An example well known to group theorists illustrates this (cf [**N;22.23**]):

Let $\mathscr{A}_2$ be the variety of abelian groups of exponent 2 and let C_2 be the cyclic group of order 2. $F\mathscr{A}_2(X)$ wr C_2 is nilpotent of class 2 and so satisfies the identity $[[x,y],z] = [[x,y],z]^2$. On the other hand $Wr(\mathscr{A}_2,\mathscr{A}_2)$ does not satisfy this identity. One can demonstrate this directly by showing that $F\mathscr{A}_2$ wr ${C_2}^2$ does not satisfy $[[x,y],z] = [[x,y],z]^2$ or, by appealing to Theorem 4.2.3, one can simply note that $\mathscr{A}_2$ does not satisfy the identity $y_1y_2y_3^{-1}y_4^{-1}y_5y_6y_7y_8^{-1}y_9^{-1}y_5^{-1} \in E$; that is, $d_{\mathscr{A}_2}([[x,y],z])$ is not a law in $\mathscr{A}_2$.

Example. The following diagram (Figure 4.2) is the Schützenberger graph of $w = x_1x_2x_1^{-1}x_2^{-1}$ relative to the variety $\mathscr{B}^1$, where $v_1 = s = e$, the start and end vertices corresponding to w. Here $d_{\mathscr{V}}(w) = y_1y_2y_3^{-1}y_4^{-1}$.

From this we can conclude that, for any nontrivial group variety $\mathscr{U}$, $Wr(\mathscr{U},\mathscr{B}^1) = \mathscr{U} \circ \mathscr{B}^1$ does not satisfy the equation

$$x_1x_2x_1^{-1}x_2^{-1} = (x_1x_2x_1^{-1}x_2^{-1})^2,$$

since no group variety other than the trivial variety satisfies the equation $y_1y_2y_3^{-1}y_4^{-1} = (y_1y_2y_3^{-1}y_4^{-1})^2$. Moreover, $Wr(\mathscr{U},\mathscr{V})$ does not satisfy $x_1x_2x_1^{-1}x_2^{-1} \in E$ whenever $\mathscr{V} \supseteq \mathscr{B}^1$ and $\mathscr{U} \notin \{\mathscr{T},\mathscr{S}\}$. This is a consequence of the fact that only $\mathscr{T}$ and $\mathscr{S}$ satisfy the equation $y_1y_2y_3^{-1}y_4^{-1} \in E$ and Proposition 4.3.1, the first result of the next section.

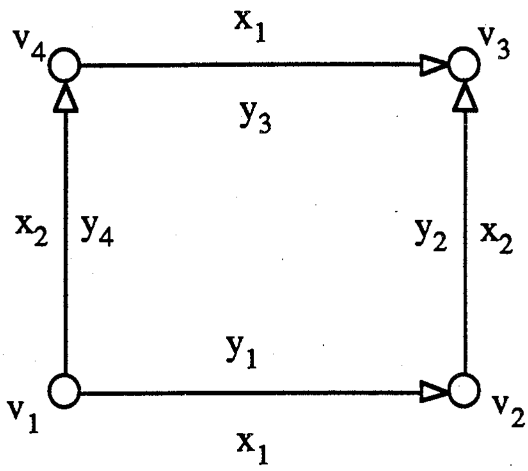

Figure 4.2. The doubly labelled Schützenberger graph $\overline{\Gamma_{\mathscr{B}}1(w)}$.

4.3 Basic Properties of Wr(_,_)

This section is devoted to several consequences of the main theorem discussed in the previous section. We first present some properties of $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ and then show that when $\mathscr{U}$ is a group variety, $\mathrm{Wr}(\mathscr{U},\mathscr{V})$ is the more familiar variety $\mathscr{U} \circ \mathscr{V}$, the Mal'cev product of $\mathscr{U}$ and $\mathscr{V}$.

Proposition 4.3.1. Let $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$ and $\mathscr{X}$ be varieties of inverse semigroups. If $\mathscr{U} \subseteq \mathscr{W}$ and $\mathscr{V} \subseteq \mathscr{X}$, then $\mathrm{Wr}(\mathscr{U},\mathscr{V}) \subseteq \mathrm{Wr}(\mathscr{W},\mathscr{X})$.

Proof: This is immediate from the definition of Wr(_,_). •

Proposition 4.3.2. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. Then $\{ S \text{ wr } (T,I) : S \in \mathscr{U}, T \in \mathscr{V} \} \subseteq \mathscr{U} \circ \mathscr{V}$ and hence, $\mathrm{Wr}(\mathscr{U},\mathscr{V}) \subseteq \langle \mathscr{U} \circ \mathscr{V} \rangle$.

Proof: Let S and (T,I) be inverse semigroups with $S \in \mathscr{U}$ and $(T,I) \in \mathscr{V}$. Let π be the natural map of S wr (T,I) onto T and let ρ be the congruence induced by π. Let $e \in E_S$.

Then $e = (\psi, \beta)$ where for all $i \in \mathbf{d}\psi = \mathbf{d}\beta$, $i\beta = i$ and $i\psi \in E_S$. Therefore, $e\rho = \{ (\psi', \beta) : \mathbf{d}\psi' = \mathbf{d}\beta \}$. Since β is the identity map on its domain we have, for all ψ', ψ'' with $\mathbf{d}\psi' = \mathbf{d}\psi'' = \mathbf{d}\beta$, that $(\psi', \beta)(\psi'', \beta) = (\psi'\psi'', \beta)$. Therefore, the map $\phi : e\rho \to S^{\mathbf{d}\beta}$ defined by $(\psi', \beta)\phi = \psi'$ is a homomorphism. It is clearly one–to-one and so the fact that $S^{\mathbf{d}\beta} \in \mathscr{U}$ implies that $e\rho \in \mathscr{U}$. Since $S \text{ wr } (T,I) / \rho \cong T \in \mathscr{V}$, we have that $S \text{ wr } (T,I) \in \mathscr{U} \circ \mathscr{V}$. Since $\mathrm{Wr}(\mathscr{U}, \mathscr{V})$ is generated by $\{ S \text{ wr } (T,I) : S \in \mathscr{U}, T \in \mathscr{V} \}$, it follows that $\mathrm{Wr}(\mathscr{U}, \mathscr{V}) \subseteq \langle \mathscr{U} \circ \mathscr{V} \rangle$. •

When $\mathscr{U}$ is a group variety it turns out that $\mathscr{U} \circ \mathscr{V}$ is a variety and $\mathrm{Wr}(\mathscr{U}, \mathscr{V}) = \mathscr{U} \circ \mathscr{V}$. In order to show this, we require a special case of a result due to Houghton.

Lemma 4.3.3 [H]. Let S be an inverse semigroup and let ρ be an idempotent separating homomorphism of S onto T. Then there is a monomorphism of S into $(\ker \rho) \text{ wr } (T,T)$ where T is given the Wagner representation by partial right translations.

Theorem 4.3.4. Let $\mathscr{U}$ be a variety of groups and $\mathscr{V}$ a variety of inverse semigroups. Then $\mathrm{Wr}(\mathscr{U}, \mathscr{V}) = \mathscr{U} \circ \mathscr{V}$.

Proof: Note that, in the setting of the theorem, $\mathscr{U} \circ \mathscr{V}$ is a variety.

Let $S \in \mathscr{U} \circ \mathscr{V}$ and let ρ be the congruence which witnesses this. Then ρ is idempotent separating and so, by Lemma 4.3.3, S can be embedded in $(\ker \rho) \text{ wr } S/\rho$, where S/ρ is given its Wagner representation. Since $\ker \rho$ is a semilattice of groups belonging to $\mathscr{U}$, $(\ker \rho) \text{ wr } S/\rho \in \mathrm{Wr}(\mathscr{SU}, \mathscr{V})$ and hence $S \in \mathrm{Wr}(\mathscr{SU}, \mathscr{V})$. Therefore, $\mathscr{U} \circ \mathscr{V} \subseteq \mathrm{Wr}(\mathscr{SU}, \mathscr{V})$.

Now, $\mathrm{Wr}(\mathscr{U}, \mathscr{V}) \subseteq \mathscr{U} \circ \mathscr{V}$, by Proposition 4.3.2 and $\mathscr{U} \circ \mathscr{V} \subseteq \mathrm{Wr}(\mathscr{SU}, \mathscr{V})$. By Theorem 4.2.3, $\ker \rho(\mathrm{Wr}(\mathscr{SU}, \mathscr{V})) = \ker \rho(\mathrm{Wr}(\mathscr{U}, \mathscr{V}))$ since $d_{\mathscr{V}}(w) \ \rho(\mathscr{U}) \ d_{\mathscr{V}}(w^2)$ if and only if $d_{\mathscr{V}}(w)\ \rho(\mathscr{SU})\ d_{\mathscr{V}}(w^2)$. Therefore, $\ker \rho(\mathscr{U} \circ \mathscr{V}) = \ker \rho(Wr(\mathscr{U},\mathscr{V}))$. But, $\mathscr{V} \subseteq Wr(\mathscr{U},\mathscr{V}) \subseteq \mathscr{U} \circ \mathscr{V}$ and $\mathrm{tr}\ \rho(\mathscr{V}) = \mathrm{tr}\ \rho(\mathscr{U} \circ \mathscr{V})$ by Lemma 2.7.5, so that $\mathrm{tr}\ \rho(Wr(\mathscr{U},\mathscr{V})) = \mathrm{tr}\ \rho(\mathscr{U} \circ \mathscr{V})$. It now follows that $\rho(Wr(\mathscr{U},\mathscr{V})) = \rho(\mathscr{U} \circ \mathscr{V})$ and hence that $Wr(\mathscr{U},\mathscr{V}) = \mathscr{U} \circ \mathscr{V}$. •

It is not true in general that $Wr(\mathscr{U},\mathscr{V}) = \langle \mathscr{U} \circ \mathscr{V} \rangle$ for varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, as the following example illustrates.

Example. Let $\mathscr{U} = \mathscr{B}$, the variety of inverse semigroups generated by the five element Brandt semigroup B_2 and let $\mathscr{V} = \mathscr{S}$, the variety of semilattices. $\mathscr{B}$ is defined by the identity $xyx^{-1} = (xyx^{-1})^2$ (See **[P;XII.4.8]**,**[K1]** or **[R2]**). Let $w = xyx^{-1}$. Now $w\ \rho(\mathscr{S})\ w^2$ and $d_{\mathscr{S}}(w)$ is just a relabelling of w since the Schützenberger graph of w relative to $\mathscr{S}$ has no two edges with the same (primary) label. (In fact, $\Gamma_{\mathscr{S}}(w)$ is just a single vertex with four loops labelled by x,y,x^{-1},y^{-1} — see Proposition 2.8.3.) Therefore, $d_{\mathscr{S}}(w)\ \rho(\mathscr{B})\ d_{\mathscr{S}}(w^2)$. By Theorem 4.2.3, $Wr(\mathscr{B},\mathscr{S})$ satisfies the equation $w = w^2$, and so $Wr(\mathscr{B},\mathscr{S}) \subseteq \mathscr{B}$. Clearly $\mathscr{B} \subseteq Wr(\mathscr{B},\mathscr{S})$, and so $\mathscr{B} = Wr(\mathscr{B},\mathscr{S})$. But $\mathscr{B}^1$, the variety of inverse semigroups generated by the five element Brandt semigroup with an identity adjoined, denoted by B_2^1, is contained in $\langle \mathscr{B} \circ \mathscr{S} \rangle$ since B_2^1 is a semilattice of B_2 and $\{1\}$. Since $\mathscr{B} \neq \mathscr{B}^1$, we have $Wr(\mathscr{B},\mathscr{S}) \neq \langle \mathscr{B} \circ \mathscr{S} \rangle$.

Proposition 4.3.5. Let $\mathscr{U},\mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. Then $Wr(\mathscr{U} \vee \mathscr{V}, \mathscr{W}) = Wr(\mathscr{U}, \mathscr{W}) \vee Wr(\mathscr{V}, \mathscr{W})$.

Proof: Set $\rho = \rho(Wr(\mathscr{U} \vee \mathscr{V}, \mathscr{W}))$. Then, for any $u,v \in (X \cup X^{-1})^+$ we have

$$u\ \rho\ v \Leftrightarrow u\ \rho(\mathscr{W})\ v \text{ and } d_{\mathscr{W}}(u)\ \rho(\mathscr{U} \vee \mathscr{V})\ d_{\mathscr{W}}(v) \quad \text{(Theorem 4.2.3)}$$

$$\Leftrightarrow u\ \rho(\mathscr{W})\ v \text{ and } d_{\mathscr{W}}(u)\ \rho(\mathscr{U}) \cap \rho(\mathscr{V})\ d_{\mathscr{W}}(v)$$

$\Leftrightarrow$ u $\rho(\mathscr{W})$ v and $d_{\mathscr{W}}(u)\, \rho(\mathscr{U})\, d_{\mathscr{W}}(v)$ and $d_{\mathscr{W}}(u)\, \rho(\mathscr{V})\, d_{\mathscr{W}}(v)$

$\Leftrightarrow$ u $\rho(\mathrm{Wr}(\mathscr{U}, \mathscr{W}))$ v and u $\rho(\mathrm{Wr}(\mathscr{V}, \mathscr{W}))$ v (Theorem 4.2.3)

$\Leftrightarrow$ u $\rho(\mathrm{Wr}(\mathscr{U}, \mathscr{W})) \cap \rho(\mathrm{Wr}(\mathscr{V}, \mathscr{W}))$ v

$\Leftrightarrow$ u $\rho(\mathrm{Wr}(\mathscr{U}, \mathscr{W}) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{W}))$ v. •

We do not know whether $\mathrm{Wr}(\mathscr{U}, \mathscr{W}) \wedge \mathrm{Wr}(\mathscr{V}, \mathscr{W}) = \mathrm{Wr}(\mathscr{U} \wedge \mathscr{V}, \mathscr{W})$ for arbitrary varieties $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$; but the equality does hold when $\mathscr{U}$ and $\mathscr{V}$ are varieties of groups and $\mathscr{W}$ is a combinatorial variety. The proof of this requires further results which are presented in the following chapter and so we leave this proposition until section 5.3. Another case in which this equality holds is when $\mathscr{U}$,$\mathscr{V}$ and $\mathscr{W}$ are varieties of groups **[N;21.23]**.

Remark. Let $\mathscr{U}$,$\mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. It is not true in general that $\mathrm{Wr}(\mathscr{U}, \mathscr{V} \wedge \mathscr{W}) = \mathrm{Wr}(\mathscr{U}, \mathscr{V}) \wedge \mathrm{Wr}(\mathscr{U}, \mathscr{W})$, nor is it true in general that $\mathrm{Wr}(\mathscr{U}, \mathscr{V} \vee \mathscr{W}) = \mathrm{Wr}(\mathscr{U}, \mathscr{V}) \vee \mathrm{Wr}(\mathscr{U}, \mathscr{W})$.

Consider $\mathrm{Wr}(\mathscr{S}, \mathscr{G} \wedge \mathscr{B})$, where $\mathscr{S}$ is the variety of semilattices, $\mathscr{G}$ is the variety of groups and $\mathscr{B}$ is the variety of inverse semigroups generated by the five element Brandt semigroup.

$$\mathrm{Wr}(\mathscr{S}, \mathscr{G} \wedge \mathscr{B}) = \mathrm{Wr}(\mathscr{S}, \mathscr{T}) = \mathscr{S},$$

while

$$\mathrm{Wr}(\mathscr{S}, \mathscr{G}) \wedge \mathrm{Wr}(\mathscr{S}, \mathscr{B}) = \mathscr{G}^{\max} \wedge \mathscr{B}^{\max} = \mathscr{I} \wedge \mathscr{B} = \mathscr{B} \neq \mathscr{S}.$$

(We will prove in Theorem 5.3.3 that, for any variety of inverse semigroups $\mathscr{V}$, $\mathrm{Wr}(\mathscr{S}, \mathscr{V}) = \mathscr{V}^{\max}$.)

Now consider $\mathrm{Wr}(\mathscr{S}, \mathscr{A}_2 \vee \mathscr{A}_3)$ where $\mathscr{A}_2$ and $\mathscr{A}_3$ are , respectively, the variety of abelian groups of exponent 2 and the variety of abelian groups of exponent 3. It is clear that $\mathrm{Wr}(\mathscr{S}, \mathscr{A}_2) \vee \mathrm{Wr}(\mathscr{S}, \mathscr{A}_3) \subseteq \mathrm{Wr}(\mathscr{S}, \mathscr{A}_2 \vee \mathscr{A}_3) = \mathrm{Wr}(\mathscr{S}, \mathscr{A}_6)$, where $\mathscr{A}_6$ is the variety of abelian groups of exponent 6. The containment is proper however, as both $Wr(\mathscr{S}, \mathscr{A}_2)$ and $Wr(\mathscr{S}, \mathscr{A}_3)$ satisfy the equation $x^3 = x^9$, while $Wr(\mathscr{S}, \mathscr{A}_6)$ does not satisfy $x^3 = x^9$. This can be seen by considering the Cayley graphs of Z_2, Z_3 and Z_6 and using Theorem 4.2.3.

Lemma 4.3.6. Let T be a full, closed inverse subsemigroup of S. Then σ_S saturates T and T/σ_T is isomorphic to the subgroup of S/σ_S consisting of those σ_S–classes contained in T.

Proof: Let $t_1, t_2 \in T$. By the definition of σ_S, $t_1 \; \sigma_S \; t_2$ if and only if for some idempotent e in S, $t_1e = t_2e$. Since T is full, it follows from the definition of σ_T that $t_1 \; \sigma_S \; t_2$ if and only if $t_1 \; \sigma_T \; t_2$. Thus, σ_S restricted to T is σ_T. If $t \; \sigma_S \; s$ for some $t \in T$ and $s \in S$ then for some idempotent e in S, we have $te = se \leq s$. Since T is full, $te \in T$ and so, since T is closed, $s \in T$. Thus, σ_S saturates T and these σ_S–classes form a group isomorphic to T/σ_T. ●

Theorem 4.3.7. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. Then

$$Wr(\mathscr{U},\mathscr{V}) \cap \mathscr{G} = Wr(\mathscr{U} \cap \mathscr{G}, \mathscr{V} \cap \mathscr{G}) = (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G}) = \langle \mathscr{U} \circ \mathscr{V} \rangle \cap \mathscr{G}.$$

Proof: Let $S \in \mathscr{U} \circ \mathscr{V}$ and let ρ be the congruence on S which witnesses this. We consider S/σ_S, the maximal group homomorphic image of S. Now $\sigma_S \subseteq \rho \vee \sigma_S$ and so $\rho \vee \sigma_S / \sigma_S$ is a congruence on S/σ_S. Moreover, $(S/\sigma_S)/(\rho \vee \sigma_S / \sigma_S) \cong S / \rho \vee \sigma_S$ and $S / \rho \vee \sigma_S \cong (S/\rho) / (\rho \vee \sigma_S / \rho) \in \mathscr{V}$, since $S/\rho \in \mathscr{V}$. Therefore, $(S/\sigma_S)/(\rho \vee \sigma_S / \sigma_S) \in \mathscr{V} \cap \mathscr{G}$. The single idempotent $(\rho \vee \sigma_S / \sigma_S)$–class is just $\ker (\rho \vee \sigma_S / \sigma_S) = \{ x\sigma_S : x \in (\ker \rho)\omega \}$ since $e\sigma_S \; (\rho \vee \sigma_S / \sigma_S) \; x\sigma_S$ if and only if $e \; (\rho \vee \sigma_S) \; x$; that is, $x\sigma_S \in \ker (\rho \vee \sigma_S / \sigma_S)$ if and only if $x \in \ker(\rho \vee \sigma_S) = (\ker \rho)\omega$ [P;III.5.5]. Now $(\ker \rho)\omega$ is a closed, full inverse subsemigroup of S and so it follows from Lemma 4.3.6 that $\ker(\rho \vee \sigma_S/\sigma_S) \cong (\ker \rho)\omega / \sigma_{(\ker \rho)\omega}$.

We claim that $(\ker \rho)\omega / \sigma_{(\ker \rho)\omega} \cong \ker \rho / \sigma_{(\ker \rho)}$. To see this observe that if $s \in (\ker \rho)\omega$ then there is a $t \in \ker \rho$ such that $t \leq s$. But this means that $t = se$ for some idempotent e in $(\ker \rho)\omega$ and hence in $\ker \rho$. Thus, $te = see = se$ and so $t \; \sigma_{(\ker \rho)\omega} \; s$. Moreover, for any $t_1, t_2 \in \ker \rho$, $t_1 \; \sigma_{(\ker \rho)\omega} \; t_2$ if and only if $t_1 \; \sigma_{\ker \rho} \; t_2$ since $\ker \rho$ is full. It follows from these remarks that

$$(\ker \rho)\omega / \sigma_{(\ker \rho)\omega} \cong \ker \rho / \sigma_{(\ker \rho)} \in (\mathscr{U} \circ \mathscr{S}) \cap \mathscr{G} = \mathscr{U} \cap \mathscr{G}.$$

It now follows that $\ker(\rho \vee \sigma_S/\sigma_S) \in \mathscr{U} \cap \mathscr{G}$. Thus, the congruence $\rho \vee \sigma_S/\sigma_S$ on S/σ_S witnesses that $S/\sigma_S \in (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G})$.

Let $G \in \mathrm{Wr}(\mathscr{U},\mathscr{V}) \cap \mathscr{G}$. Then $G \in \langle \mathscr{U} \circ \mathscr{V} \rangle \cap \mathscr{G}$. Since $\mathscr{U} \circ \mathscr{V}$ is closed under the formation of direct products and subsemigroups **[P;XII.8.2]**, $G = S/\rho$ for some $S \in \mathscr{U} \circ \mathscr{V}$. We have just shown that $S/\sigma_S \in (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G})$ so we must have that $G \in (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G})$ since G is a homomorphic image of S/σ_S. Therefore,

$$\mathrm{Wr}(\mathscr{U},\mathscr{V}) \cap \mathscr{G} \subseteq \langle \mathscr{U} \circ \mathscr{V} \rangle \cap \mathscr{G} \subseteq (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G}) = \mathrm{Wr}(\mathscr{U} \cap \mathscr{G}, \mathscr{V} \cap \mathscr{G})$$

by Theorem 4.3.4. It follows immediately from Proposition 4.3.1 that

$$\mathrm{Wr}(\mathscr{U} \cap \mathscr{G}, \mathscr{V} \cap \mathscr{G}) \subseteq \mathrm{Wr}(\mathscr{U},\mathscr{V}) \cap \mathscr{G}$$

and so

$$\mathrm{Wr}(\mathscr{U},\mathscr{V}) \cap \mathscr{G} = \mathrm{Wr}(\mathscr{U} \cap \mathscr{G}, \mathscr{V} \cap \mathscr{G}) = (\mathscr{U} \cap \mathscr{G}) \circ (\mathscr{V} \cap \mathscr{G}) = \langle \mathscr{U} \circ \mathscr{V} \rangle \cap \mathscr{G}. \qquad \bullet$$

4.4 The Associativity of Wr

The binary operator Wr on the lattice of varieties of inverse semigroups is, in fact, an associative operator and so $(\mathscr{L}(\mathscr{I}), \mathrm{Wr})$ is a semigroup. The proof of this makes use of Theorem 4.2.2 — the description of the fully invariant congruence on the free inverse semigroup corresponding to $\mathrm{Wr}(\mathscr{U}, \mathscr{V})$, for any pair of varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, and Lemma 4.1.4 — the description of the derived word relative to the variety $\mathscr{V}$.

We say that the two equations $u_1 = u_2$ and $v_1 = v_2$ are *equivalent* if each is a consequence of the other. Another way of saying this is $u_1 = u_2$ and $v_1 = v_2$ are equivalent if and only if, for any variety $\mathscr{U}$ of inverse semigroups, $\mathscr{U}$ satisfies the equation $u_1 = u_2$ if and only if $\mathscr{U}$ satisfies the equation $v_1 = v_2$.

Lemma 4.4.1. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups and let $v,w \in (X \cup X^{-1})^+$ be such that $w\,\rho(\mathscr{U})\,v$ and $d_{\mathscr{U}}(w)\,\rho(\mathscr{V})\,d_{\mathscr{U}}(v)$ (or, equivalently, $w\,\rho(Wr(\mathscr{V},\mathscr{U})\,v$). Set

$w = a_1 \ldots a_n$, $v = b_1 \ldots b_m$, where $a_i, b_j \in X \cup X^{-1}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.

$d_{\mathscr{U}}(w) = c_1 \ldots c_n$, $d_{\mathscr{U}}(v) = d_1 \ldots d_m$, where both words are constructed from the same doubly labelled Schützenberger graph $\overline{\Gamma_{\mathscr{U}}(w)}$.

$d_{\mathscr{V}}(d_{\mathscr{U}}(w)) = e_1 \ldots e_n$, $d_{\mathscr{V}}(d_{\mathscr{U}}(v)) = f_1 \ldots f_m$, where both words are constructed from the same doubly labelled Schützenberger graph $\overline{\Gamma_{\mathscr{V}}(d_{\mathscr{U}}(w))}$.

$d_{Wr(\mathscr{V},\mathscr{U})}(w) = g_1 \ldots g_n$, $d_{Wr(\mathscr{V},\mathscr{U})}(v) = h_1 \ldots h_m$, where both words are constructed from the same doubly labelled Schützenberger graph $\overline{\Gamma_{Wr(\mathscr{V},\mathscr{U})}(w)}$.

Then the equations $d_{\mathscr{V}}(d_{\mathscr{U}}(w)) = d_{\mathscr{V}}(d_{\mathscr{U}}(v))$ and $d_{Wr(\mathscr{V},\mathscr{U})}(w) = d_{Wr(\mathscr{V},\mathscr{U})}(v)$ are equivalent.

Proof: We prove the stronger statement that each of the two equations is a one-to-one relabelling of the other. That is, we prove the following statements.

For all i,j,

1) $g_i = h_j \quad \Leftrightarrow e_i = f_j$;

2) $g_i = h_j^{-1} \Leftrightarrow e_i = f_j^{-1}$;

3) $g_i = g_j \quad \Leftrightarrow e_i = e_j$;

4) $g_i = g_j^{-1} \Leftrightarrow e_i = e_j^{-1}$;

5) $h_i = h_j \quad \Leftrightarrow f_i = f_j$;

6) $h_i = h_j^{-1} \Leftrightarrow f_i = f_j^{-1}$.

1) First of all, observe that

$$g_i = h_j \quad \Leftrightarrow \quad w \leq_{Wr(\mathscr{V},\mathscr{U})} a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

(Lemma 4.1.4 since $w\ \rho(Wr(\mathscr{V}, \mathscr{U}))\ v$)

$$\Leftrightarrow \quad w\ \rho(Wr(\mathscr{V}, \mathscr{U}))\ ww^{-1}a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

$$\Leftrightarrow \quad w\ \rho(\mathscr{U})\ ww^{-1}a_1 \ldots a_{i-1} b_j \ldots b_m,$$

$$d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(ww^{-1}a_1 \ldots a_{i-1} b_j \ldots b_m) \text{ and } a_i = b_j$$

(Theorem 4.2.2)

Next, observe that

$$e_i = f_j \quad \Leftrightarrow \quad d_{\mathscr{U}}(w) \leq_{\mathscr{V}} c_1 \ldots c_{i-1} d_j \ldots d_m \text{ and } c_i = d_j$$

(Lemma 4.1.4 since $d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(v)$)

$$\Leftrightarrow \quad (c_1 \ldots c_n)\ \rho(\mathscr{V})\ (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m \text{ and } c_i = d_j$$

$$\Leftrightarrow \quad (c_1 \ldots c_n)\ \rho(\mathscr{V})\ (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m$$

$$w \leq_{\mathscr{U}} a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j$$

(Lemma 4.1.4 since $w\ \rho(\mathscr{U})\ v$).

$$\Leftrightarrow \quad (c_1 \ldots c_n)\ \rho(\mathscr{V})\ (c_1 \ldots c_n)(c_1 \ldots c_n)^{-1} c_1 \ldots c_{i-1} d_j \ldots d_m$$

$$w\ \rho(\mathscr{U})\ ww^{-1}a_1 \ldots a_{i-1} b_j \ldots b_m \text{ and } a_i = b_j.$$

By the hypothesis $w\ \rho(\mathscr{U})\ v$ and so if $w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ we must have that $d_{\mathscr{U}}(ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m) = (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_{i-1}d_j\ldots d_m$. This is because each of w, v and $ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ label s-e paths in $\overline{\Gamma_{\mathscr{U}}(w)}$ by primary labels. It is clear that $ww^{-1}a_1\ldots a_{i-1}$ corresponds to the path labelled $(c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_{i-1}$ by secondary labels, since $\overline{\Gamma_{\mathscr{U}}(w)}$ is deterministic. While there may be many paths in $\overline{\Gamma_{\mathscr{U}}(w)}$ labelled $b_j\ldots b_m$, the part of $ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ labelled $b_j\ldots b_m$ ends at vertex e. Since the s-e path labelled v ends at e, both the $b_j\ldots b_m$ of v and the $b_j\ldots b_m$ of $ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ must correspond to the same edges, since $\overline{\Gamma_{\mathscr{U}}(w)}$ is deterministic. It follows that $d_{\mathscr{U}}(ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m) = (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_{i-1}d_j\ldots d_m$. From these remarks we have that

$w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m,\ d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m)$ and $a_i = b_j$

$$\Leftrightarrow$$

$(c_1\ldots c_n)\ \rho(\mathscr{V})\ (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_{i-1}d_j\ldots d_m,\ w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ and $a_i = b_j$.

From this it follows that $g_i = h_j$ if and only if $e_i = f_j$.

2) We proceed in a similar manner:

$g_i = h_j^{-1} \Leftrightarrow\ w \leq_{Wr(\mathscr{V},\mathscr{U})} a_1\ldots a_ib_j\ldots b_m$ and $a_i = b_j^{-1}$

(Lemma 4.1.4 since $w\ \rho(Wr(\mathscr{V}, \mathscr{U})\ v$)

$\Leftrightarrow\ w\ \rho(Wr(\mathscr{V}, \mathscr{U})\ ww^{-1}a_1\ldots a_ib_j\ldots b_m$ and $a_i = b_j^{-1}$

$\Leftrightarrow\ w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_ib_j\ldots b_m,$

$d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(ww^{-1}a_1\ldots a_ib_j\ldots b_m)$ and $a_i = b_j^{-1}$

(Theorem 4.2.2)

Also,

$e_i = f_j^{-1} \Leftrightarrow\ d_{\mathscr{U}}(w) \leq_{\mathscr{V}} c_1\ldots c_id_j\ldots d_m$ and $c_i = d_j^{-1}$

(Lemma 4.1.4 since $d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(v)$)

$\Leftrightarrow \quad (c_1\ldots c_n)\ \rho(\mathscr{V})\ (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_id_j\ldots d_m$ and $c_i = d_j^{-1}$

$\Leftrightarrow \quad (c_1\ldots c_n)\ \rho(\mathscr{V})\ (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_id_j\ldots d_m$

$w \leq_{\mathscr{U}} a_1\ldots a_ib_j\ldots b_m$ and $a_i = b_j^{-1}$

(Lemma 4.1.4 since $w\ \rho(\mathscr{U})\ v$).

$\Leftrightarrow \quad (c_1\ldots c_n)\ \rho(\mathscr{V})\ (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_id_j\ldots d_m$

$w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_ib_j\ldots b_m$ and $a_i = b_j^{-1}$.

We have by the hypothesis $w\ \rho(\mathscr{U})\ v$ and so if $w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m$ we must have, as in 1), that $d_{\mathscr{U}}(ww^{-1}a_1\ldots a_ib_j\ldots b_m) = (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_id_j\ldots d_m$. It follows that

$w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_ib_j\ldots b_m,\ d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(ww^{-1}a_1\ldots a_ib_j\ldots b_m)$ and $a_i = b_j^{-1}$

$\Leftrightarrow$

$(c_1\ldots c_n)\ \rho(\mathscr{V})\ (c_1\ldots c_n)(c_1\ldots c_n)^{-1}c_1\ldots c_id_j\ldots d_m,\ w\ \rho(\mathscr{U})\ ww^{-1}a_1\ldots a_ib_j\ldots b_m$

and $a_i = b_j^{-1}$.

Consequently, $g_i = h_j^{-1}$ if and only if $e_i = f_j^{-1}$.

The proofs of 3), 4), 5) and 6) are similar, noting the remark immediately following Lemma 4.1.4. •

Theorem 4.4.2. The operator Wr is associative.

Proof: Let $\mathscr{W}$, $\mathscr{V}$ and $\mathscr{U}$ be varieties of inverse semigroups. For any $w, v \in F\mathscr{I}(X)$,

$w\ \rho(\mathrm{Wr}(\mathscr{W},\mathrm{Wr}(\mathscr{V},\mathscr{U})))\ v \Leftrightarrow w\ \rho(\mathrm{Wr}(\mathscr{V},\mathscr{U}))\ v$ and

$d_{\mathrm{Wr}(\mathscr{V},\mathscr{U})}(w)\ \rho(\mathscr{W})\ d_{\mathrm{Wr}(\mathscr{V},\mathscr{U})}(v)$

(Theorem 4.2.3)

$\Leftrightarrow w\ \rho(\mathscr{U})\ v,\ d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(v)$ and

$d_{\mathrm{Wr}(\mathscr{V},\mathscr{U})}(w)\ \rho(\mathscr{W})\ d_{\mathrm{Wr}(\mathscr{V},\mathscr{U})}(v)$

(Theorem 4.2.3)

$\Leftrightarrow$ w $\rho(\mathscr{U})$ v, $d_{\mathscr{U}}(w)\ \rho(\mathscr{V})\ d_{\mathscr{U}}(v)$ and

$d_{\mathscr{V}}(d_{\mathscr{U}}(w))\ \rho(\mathscr{W})\ d_{\mathscr{V}}(d_{\mathscr{U}}(v))$

(Lemma 4.4.1)

$\Leftrightarrow$ w $\rho(\mathscr{U})$ v and

$d_{\mathscr{U}}(w)\ \rho(\mathrm{Wr}(\mathscr{W},\mathscr{V}))\ d_{\mathscr{U}}(v)$

(Theorem 4.2.3)

$\Leftrightarrow$ w $\rho(\mathrm{Wr}(\mathrm{Wr}(\mathscr{W},\mathscr{V}),\mathscr{U}))$ v.

Therefore, $\rho(\mathrm{Wr}(\mathscr{W},\mathrm{Wr}(\mathscr{V},\mathscr{U}))) = \rho(\mathrm{Wr}(\mathrm{Wr}(\mathscr{W},\mathscr{V}),\mathscr{U}))$ and as a consequence, $\mathrm{Wr}(\mathscr{W},\mathrm{Wr}(\mathscr{V},\mathscr{U})) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{W},\mathscr{V}),\mathscr{U})$. Thus, the operator Wr is associative. •

Theorem 4.4.3. $\mathscr{L}(\mathscr{I})$ is a monoid with zero under the operation Wr.

Proof: That ($\mathscr{L}(\mathscr{I})$, Wr) is a semigroup is a consequence of Theorem 4.4.2. Since $\mathrm{Wr}(\mathscr{U},\mathscr{T}) = \mathrm{Wr}(\mathscr{T},\mathscr{U}) = \mathscr{U}$, for any variety $\mathscr{U}$, the variety $\mathscr{T}$ is an identity for ($\mathscr{L}(\mathscr{I})$,Wr) and so ($\mathscr{L}(\mathscr{I})$,Wr) is a monoid. For any variety $\mathscr{U}$ of inverse semigroups, $\mathrm{Wr}(\mathscr{I},\mathscr{U}) = \mathrm{Wr}(\mathscr{U},\mathscr{I}) = \mathscr{I}$ and so $\mathscr{I}$ is a zero of the monoid ($\mathscr{L}(\mathscr{I})$,Wr). •

There are several natural questions which arise as a result of Theorem 4.4.3. For example: Which varieties are idempotents? Which varieties, if any, can be expressed as a product of two non-trivial varieties? Which familiar classes of varieties of inverse semigroups form a subsemigroup of ($\mathscr{L}(\mathscr{I})$,Wr)? What is the structure of the semigroup ($\mathscr{L}(\mathscr{I})$,Wr)? Is it free? Many of these problems do not have obvious solutions. In Chapter 5, once we have equipped ourselves with some facts, we discuss some of these questions.

We conclude this section with some results concerning generators of varieties of the form $\mathrm{Wr}(\mathscr{U},\mathscr{V})$.

Theorem 4.4.4. $F(Wr(\mathscr{U},\mathscr{V}))(X)$ can be embedded in $F(\mathscr{U})$ wr $(F(\mathscr{V}), F(\mathscr{V}))$.

Proof: Set $\rho = \rho(Wr(\mathscr{U},\mathscr{V}))$. Let $X = \{ x_i : i \in \omega \}$ and
$Y = \cup \{ x_{i_u} : u \in F(\mathscr{V})x_i^{-1}\rho(\mathscr{V}) \}$, where the union is over all $i \in \omega$.

Define

$$\Theta : F(Wr(\mathscr{U},\mathscr{V}))(X) \to F(\mathscr{U})(Y) \text{ wr } (F(\mathscr{V})(X),F(\mathscr{V})(X))$$

as follows:

For each $i \in \omega$, set

$$(x_i\rho)\Theta = (\psi_i,\beta_i)$$

Here β_i corresponds to $x_i\rho(\mathscr{V})$; that is,

$$d\beta_i = F(\mathscr{V})(x_i^{-1}\rho(\mathscr{V})),$$

$$u\beta_i = ux_i\rho(\mathscr{V}) \qquad (u \in d\beta_i),$$

$$u\psi_i = x_{i_u}\rho(\mathscr{U}) \qquad (u \in d\beta_i).$$

It is immediate that Θ maps $\{ x_i\rho : i \in \omega \}$ into $F(\mathscr{U})(Y)$ wr $(F(\mathscr{V})(X),F(\mathscr{V})(X))$. Since $F(\mathscr{U})(Y)$ wr $(F(\mathscr{V})(X),F(\mathscr{V})(X))$ is a member of $Wr(\mathscr{U},\mathscr{V})$ and $F(Wr(\mathscr{U},\mathscr{V}))(X)$ is $Wr(\mathscr{U},\mathscr{V})$–free, we let Θ be the unique extension of Θ thus far defined, to $F(Wr(\mathscr{U},\mathscr{V}))$.

Let $w = a_1 \ldots a_n$ and $v = b_1 \ldots b_m$ where $a_i,b_j \in X \cup X^{-1}$, $i = 1,\ldots,n$ and $j = 1,\ldots,m$. Suppose that $w\rho\Theta = v\rho\Theta$. As before we write

$$\psi_{d_i} = \begin{cases} \psi_j & \text{if } d_i = x_j \\ \psi_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases}$$

$$\beta_{d_i} = \begin{cases} \beta_j & \text{if } d_i = x_j \\ \beta_j^{-1} & \text{if } d_i = x_j^{-1} \end{cases}$$

Again, we write $\beta_{d_1 \ldots d_i}$ for $\beta_{d_1}\beta_{d_2} \ldots \beta_{d_i}$.

Now

$$w\rho\Theta = (\psi_{a_1},\beta_{a_1})\ldots(\psi_{a_n},\beta_{a_n})$$
$$= (\psi_{a_1}{}^{\beta_{a_1}}\psi_{a_2}{}^{\beta_{a_1a_2}}\psi_{a_3}\ldots{}^{\beta_{a_1\ldots a_{n-1}}}\psi_{a_n}, \beta_{a_1\ldots a_n})$$

and

$$v\rho\Theta = (\psi_{b_1},\beta_{b_1})\ldots(\psi_{b_m},\beta_{b_m})$$
$$= (\psi_{b_1}{}^{\beta_{b_1}}\psi_{b_2}{}^{\beta_{b_1b_2}}\psi_{b_3}\ldots{}^{\beta_{b_1\ldots b_{m-1}}}\psi_{b_m}, \beta_{b_1\ldots b_m}).$$

First of all, $\beta_{a_1\ldots a_n} = \beta_{b_1\ldots b_m}$ and so $w\ \rho(\mathscr{V})\ v$. Secondly, observe that $ww^{-1}\rho(\mathscr{V}) = vv^{-1}\rho(\mathscr{V}) \in \mathbf{d}\beta_{a_1\ldots a_n} = \mathbf{d}\beta_{b_1\ldots b_m}$. Thus,

$$[ww^{-1}\rho(\mathscr{V})]\psi_{a_1}{}^{\beta_{a_1}}\psi_{a_2}{}^{\beta_{a_1a_2}}\psi_{a_3}\ldots{}^{\beta_{a_1\ldots a_{n-1}}}\psi_{a_n}$$
$$= [ww^{-1}\rho(\mathscr{V})\psi_{a_1}][ww^{-1}\rho(\mathscr{V})\beta_{a_1}\psi_{a_2}]\ldots[ww^{-1}\rho(\mathscr{V})\beta_{a_1\ldots a_{n-1}}\psi_{a_n}]$$
$$= y_1\ldots y_n\ \rho(\mathscr{U}),\text{ where } y_i = ww^{-1}\rho(\mathscr{V})\beta_{a_1\ldots a_{i-1}}\psi_{a_i} \in Y \cup Y^{-1}$$
$$= [ww^{-1}\rho(\mathscr{V})]\psi_{b_1}{}^{\beta_{b_1}}\psi_{b_2}{}^{\beta_{b_1b_2}}\psi_{b_3}\ldots{}^{\beta_{b_1\ldots b_{m-1}}}\psi_{b_m}$$
$$= [ww^{-1}\rho(\mathscr{V})\psi_{b_1}][ww^{-1}\rho(\mathscr{V})\beta_{b_1}\psi_{b_2}]\ldots[ww^{-1}\rho(\mathscr{V})\beta_{b_1\ldots b_{m-1}}\psi_{b_m}]$$
$$= z_1\ldots z_m\ \rho(\mathscr{U}),\text{ where } z_i = ww^{-1}\rho(\mathscr{V})\beta_{b_1\ldots b_{i-1}}\psi_{b_i} \in Y \cup Y^{-1}.$$

Now observe that

$$y_i = z_j \iff ww^{-1}\rho(\mathscr{V})\beta_{a_1\ldots a_{i-1}}\psi_{a_i} = ww^{-1}\rho(\mathscr{V})\beta_{b_1\ldots b_{j-1}}\psi_{b_j}$$
$$\iff ww^{-1}a_1\ldots a_{i-1}\rho(\mathscr{V})\psi_{a_i} = ww^{-1}b_1\ldots b_{j-1}\rho(\mathscr{V})\psi_{b_j}$$
$$\iff a_i = b_j \text{ and } ww^{-1}a_1\ldots a_{i-1}\rho(\mathscr{V})\ ww^{-1}b_1\ldots b_{j-1}$$
$$\iff a_i = b_j \text{ and } ww^{-1}a_1\ldots a_{i-1}b_j\ldots b_m\ \rho(\mathscr{V})ww^{-1}v\ \rho(\mathscr{V})\ w$$
$$\iff a_i = b_j \text{ and } a_1\ldots a_{i-1}b_j\ldots b_m \geq_{\mathscr{V}} w$$

and

$$y_i = z_j^{-1} \iff ww^{-1}\rho(\mathscr{V})\beta_{a_1\ldots a_{i-1}}\psi_{a_i} = [ww^{-1}\rho(\mathscr{V})\beta_{b_1\ldots b_{j-1}}\psi_{b_j}]^{-1}$$
$$\iff ww^{-1}a_1\ldots a_{i-1}\rho(\mathscr{V})\psi_{a_i} = ww^{-1}b_1\ldots b_{j-1}\rho(\mathscr{V})\beta_{b_j}\psi_{b_j}{}^{-1}$$
$$\iff ww^{-1}a_1\ldots a_{i-1}\rho(\mathscr{V})\psi_{a_i} = ww^{-1}b_1\ldots b_j\rho(\mathscr{V})\psi_{b_j^{-1}}$$
$$\iff a_i = b_j^{-1} \text{ and } ww^{-1}a_1\ldots a_{i-1}\rho(\mathscr{V})\ ww^{-1}b_1\ldots b_j$$
$$\iff a_i = b_j^{-1} \text{ and } ww^{-1}a_1\ldots a_{i-1}b_{j+1}\ldots b_m\rho(\mathscr{V})\ ww^{-1}v\ \rho(\mathscr{V})\ w$$
$$\iff a_i = b_j^{-1} \text{ and } a_1\ldots a_{i-1}b_{j+1}\ldots b_m \geq_{\mathscr{V}} w$$

We similarly obtain

$$y_i = y_j \iff a_i = a_j \text{ and } a_1 \ldots a_{i-1} a_j \ldots a_n \geq_{\mathscr{V}} w$$

$$y_i = y_j^{-1} \iff a_i = a_j^{-1} \text{ and } a_1 \ldots a_{i-1} a_{j+1} \ldots a_n \geq_{\mathscr{V}} w$$

$$z_i = z_j \iff b_i = b_j \text{ and } b_1 \ldots b_{i-1} b_j \ldots b_m \geq_{\mathscr{V}} w$$

$$z_i = z_j^{-1} \iff b_i = b_j^{-1} \text{ and } b_1 \ldots b_{i-1} b_{j+1} \ldots b_m \geq_{\mathscr{V}} w$$

Since $y_1 \ldots y_n \, \rho(\mathscr{U}) \, z_1 \ldots z_m$ we have by the above and Lemma 4.1.4 that $d_{\mathscr{V}}(w) \, \rho(\mathscr{U}) \, d_{\mathscr{V}}(v)$. It now follows from Theorem 4.2.2 that $w \, \rho \, v$ and hence that Θ is a monomorphism. Thus, Θ is an embedding of $F(Wr(\mathscr{U},\mathscr{V}))(X)$ into $F(\mathscr{U})$ wr $(F(\mathscr{V}), F(\mathscr{V}))$. •

Corollary 4.4.5. For any pair of varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups, the variety $Wr(\mathscr{U},\mathscr{V})$ is generated by $F(\mathscr{U})$ wr $(F(\mathscr{V}), F(\mathscr{V}))$.

Proof: By the definition of Wr, $F(\mathscr{U})$ wr $(F(\mathscr{V}), F(\mathscr{V})) \in Wr(\mathscr{U},\mathscr{V})$ and so $\langle F(\mathscr{U}) \text{ wr } (F(\mathscr{V}), F(\mathscr{V})) \rangle \subseteq Wr(\mathscr{U},\mathscr{V})$. On the other hand, by Theorem 4.4.4, $F(Wr(\mathscr{U},\mathscr{V}))(X) \in \langle F(\mathscr{U}) \text{ wr } (F(\mathscr{V}), F(\mathscr{V})) \rangle$, and so $Wr(\mathscr{U},\mathscr{V}) \subseteq \langle F(\mathscr{U}) \text{ wr } (F(\mathscr{V}), F(\mathscr{V})) \rangle$. Thus, $\langle F(\mathscr{U}) \text{ wr } (F(\mathscr{V}), F(\mathscr{V})) \rangle = Wr(\mathscr{U},\mathscr{V})$. •

Lemma 4.4.6. Let A and S be inverse semigroups and suppose that T is an inverse subsemigroup of S. Then

1) If T' is isomorphic to T then A wr (T',T') is isomorphic to A wr (T,T);

2) A wr $(T,T) \in \langle A \text{ wr } (S,S) \rangle$.

Proof: 1) Let Φ be the isomorphism from T to T'. Let $(\psi,\beta) \in$ A wr (T,T) with $\mathbf{d}\beta = Tt^{-1}$ (and so β corresponds to $t \in T$). Define $(\psi',\beta') \in$ A wr (T',T') by setting $\mathbf{d}\beta' = T'(t^{-1})\Phi$ and defining $u\beta' = (u\Phi^{-1}\beta)\Phi = u(t\Phi)$ (and so β' corresponds to $t\Phi$) and $u\psi' = u\Phi^{-1}\psi$ (for all $u \in \mathbf{d}\beta'$). The map which sends (ψ,β) to (ψ',β') is the desired isomorphism: Let $(\psi,\beta),(\varphi,\alpha) \in A \text{ wr } (T,T)$. To see that this map is a homomorphism, it is enough to show that $\beta'\alpha' = (\beta\alpha)'$ and that $\psi'^{\beta'}\varphi' = (\psi^{\beta}\varphi)'$. It is clear that $\beta'\alpha' = (\beta\alpha)'$, so let $u \in \mathbf{d}\beta'\alpha' = \mathbf{d}(\beta\alpha)'$. $u(\psi^{\beta}\varphi)' = (u\Phi^{-1})\psi^{\beta}\varphi = (u\Phi^{-1}\psi)(u\Phi^{-1}\beta\varphi)$, while $u(\psi'^{\beta'}\varphi') = (u\psi')(u\beta'\varphi') = (u\Phi^{-1}\psi)(u\Phi^{-1}\beta\Phi\Phi^{-1}\varphi) = (u\Phi^{-1}\psi)(u\Phi^{-1}\beta\varphi)$. Thus, the map is a homomorphism. Since Φ is an isomorphism, it is not difficult to varify that this map is indeed a bijection and hence, an isomorphism.

2) If $(\psi,\beta) \in A \text{ wr } (T,T)$ then β corresponds to some $t \in T \subseteq S$ and $\mathbf{d}\beta = Tt^{-1}$. Let β' be the element of (S,S) corresponding to t. Then $\mathbf{d}\beta' = St^{-1} \supseteq Tt^{-1}$ and it follows that there exists a ψ' such that $(\psi',\beta') \in A \text{ wr } (S,S)$ and ψ' restricted to Tt^{-1} is ψ. Given any identity satisfied by A wr (S,S) to see that A wr (T,T) satisfies this identity, observe that for any substitution of variables from A wr (T,T), say $(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)$, the identity holds by substituting $(\psi_1',\beta_1'),\ldots,(\psi_n',\beta_n')$ and so it must hold when substituting $(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)$. •

Theorem 4.4.7. Let $\mathscr{U},\mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. Then

$$\begin{aligned}\langle F\mathscr{U}(X) \text{ wr } F(\mathrm{Wr}(\mathscr{V},\mathscr{W}))\rangle &= \langle F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X), F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X))\rangle\\ &= \langle F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X), F\mathscr{V}(X) \times F\mathscr{W}(X))\rangle\\ &= \langle [F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X),F\mathscr{V}(X))] \text{ wr } (F\mathscr{W}(X),F\mathscr{W}(X))\rangle\\ &= \langle F(\mathrm{Wr}(\mathscr{U},\mathscr{V}))(X) \text{ wr } (F\mathscr{W}(X),F\mathscr{W}(X))\rangle\end{aligned}$$

Proof:

$$\begin{aligned}\mathrm{Wr}(\mathscr{U},\mathrm{Wr}(\mathscr{V},\mathscr{W})) &= \langle F\mathscr{U}(X) \text{ wr } F(\mathrm{Wr}(\mathscr{V},\mathscr{W}))\rangle\\ &\subseteq \langle F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X), F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X))\rangle\\ &\qquad\qquad (\text{Theorem 4.4.4 and Lemma 4.4.6})\\ &\subseteq \mathrm{Wr}(\mathscr{U},\mathrm{Wr}(\mathscr{V},\mathscr{W})) \quad (\text{by the definition of Wr})\end{aligned}$$

and,

$$\begin{aligned}
\mathrm{Wr}(\,\mathrm{Wr}(\mathscr{U},\mathscr{V}),\mathscr{W}) &= \langle\, [F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X),F\mathscr{V}(X))] \text{ wr } (F\mathscr{W}(X),F\mathscr{W}(X))\,\rangle \\
&= \langle\, F\mathscr{U}(X) \text{ wr } (F\mathscr{V}(X) \text{ wr } F\mathscr{W}(X),\, F\mathscr{V}(X) \times F\mathscr{W}(X))\,\rangle \\
&\qquad\qquad (\text{ Theorem 3.3.3 }) \\
&\subseteq \mathrm{Wr}(\mathscr{U},\mathrm{Wr}(\mathscr{V},\mathscr{W})).
\end{aligned}$$

Therefore, by the associativity of Wr, all of these varieties are the same. •

CHAPTER FIVE

Consequences

Armed with the main result of the previous chapter, we set about proving various properties of $\mathrm{Wr}(\mathcal{U},\mathcal{V})$ for a given pair of varieties $\mathcal{U}$ and $\mathcal{V}$. We first discuss general properties preserved under the Wr operation. Included among these are that the $\mathrm{Wr}(\mathcal{U},\mathcal{V})$-free semigroups have solvable word problem (or, $\mathrm{Wr}(\mathcal{U},\mathcal{V})$ has decidable equational theory) whenever the $\mathcal{V}$-free semigroups and the $\mathcal{U}$-free semigroups have solvable word problem. Also, if $\mathcal{U}$ and $\mathcal{V}$ are locally finite then so is $\mathrm{Wr}(\mathcal{U},\mathcal{V})$. In the second section we discuss properties preserved under Wr which are more inverse semigroup related. Included here are results concerning E-unitary covers. The penultimate section is devoted to showing that $\mathrm{Wr}(\mathcal{S},\mathcal{V})$ is in fact the largest variety of inverse semigroups satisfying the same idempotent laws as $\mathcal{V}$. In the final section we look at some basic properties of the semigroup $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$.

5.1 Further properties of Wr

Theorem 5.1.1. Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of inverse semigroups. If $F\mathcal{U}(X)$ and $F\mathcal{V}(X)$ have solvable word problems then so does $F\mathrm{Wr}(\mathcal{U},\mathcal{V})(X)$.

Proof: By Theorem 4.2.3, we need only show that we can determine whether or not $d_{\mathcal{V}}(w)\ \rho(\mathcal{U})\ d_{\mathcal{V}}(u)$ whenever $w\ \rho(\mathcal{V})\ u$.

Suppose that $w = a_1\ldots a_m$ and $u = d_1\ldots d_k$ where $c(w) \cup c(u) = \{x_1,\ldots,x_n\}$ and $a_i,d_j \in X \cup X^{-1}$. We construct words $v_1 = b_1\ldots b_m$ and $v_2 = c_1\ldots c_k$ over $X \cup X^{-1}$ satisfying: for $i < j \leq m$,

$$b_i = b_j \quad \Leftrightarrow \quad a_i = a_j \text{ and } a_1\ldots a_{i-1}a_j\ldots a_m \geq_{\mathcal{V}} a_1\ldots a_m,$$

$$b_i = b_j^{-1} \quad \Leftrightarrow \quad a_i = a_j^{-1} \text{ and } a_1\ldots a_i a_j\ldots a_m \geq_{\mathcal{V}} a_1\ldots a_m,$$

for $i < j \leq k$,

$$c_i = c_j \quad \Leftrightarrow \quad d_i = d_j \text{ and } d_1 \ldots d_{i-1} d_j \ldots d_k \geq_{\mathscr{V}} d_1 \ldots d_k,$$

$$c_i = c_j^{-1} \quad \Leftrightarrow \quad d_i = d_j^{-1} \text{ and } d_1 \ldots d_i d_j \ldots d_k \geq_{\mathscr{V}} d_1 \ldots d_k,$$

and for any i,j,

$$b_i = c_j \quad \Leftrightarrow \quad a_i = d_j \text{ and } a_1 \ldots a_{i-1} d_j \ldots d_k \geq_{\mathscr{V}} a_1 \ldots a_k,$$

$$b_i = c_j^{-1} \quad \Leftrightarrow \quad a_i = d_j^{-1} \text{ and } a_1 \ldots a_i d_j \ldots d_k \geq_{\mathscr{V}} a_1 \ldots a_k.$$

It is clear from Lemma 4.1.4 that v_1 and v_2 are one-to-one relabellings of $d_{\mathscr{V}}(w)$ and $d_{\mathscr{V}}(u)$ via the correspondence

$$\lambda_w(e_{a_i}) \leftrightarrow b_i$$

$$\lambda_w(e_{d_j}) \leftrightarrow c_j$$

where e_{a_i} (e_{d_j}) is the edge in $\overline{\Gamma_{\mathscr{V}}(w)}$ corresponding to a_i (d_j) in the path in $\Gamma_{\mathscr{V}}(w)$ labelled by w (u). It follows that $v_1\ \rho(\mathscr{U})\ v_2$ if and only if $d_{\mathscr{V}}(w)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(u)$. Since $F\mathscr{U}(X)$ has solvable word problem, we can determine whether or not $v_1\ \rho(\mathscr{U})\ v_2$ and hence, whether or not $d_{\mathscr{V}}(w)\ \rho_Y(\mathscr{U})\ d_{\mathscr{V}}(u)$. Therefore, if $F\mathscr{V}(X)$ and $F\mathscr{U}(X)$ have solvable word problems then so does $FWr(\mathscr{U},\mathscr{V})(X)$.

We have used the fact that, if $\mathscr{V}$ has solvable word problem, then the natural partial order $\leq_{\mathscr{V}}$ is solvable, since $w \leq_{\mathscr{V}} u$ if and only if $w\ \rho(\mathscr{V})\ ww^{-1}u$. •

Corollary 5.1.2. If $\mathscr{U}$ is a group variety and $\mathscr{V}$ is any variety of inverse semigroups, then $F\mathscr{U}\circ\mathscr{V}(X)$ has solvable word problem if both $F\mathscr{U}(X)$ and $F\mathscr{V}(X)$ have solvable word problems.

Proof: This follows immediately from 4.3.4 and 5.1.1. •

A variety $\mathscr{V}$ is said to be *locally finite* if every finitely generated member of $\mathscr{V}$ is finite. Equivalently, $\mathscr{V}$ is locally finite if and only if every $\mathscr{V}$-free inverse semigroup on a finite set of generators is finite.

Theorem 5.1.3. $Wr(\mathscr{U},\mathscr{V})$ is locally finite if and only if both $\mathscr{U}$ and $\mathscr{V}$ are locally finite.

Proof: Since both $\mathscr{U}$ and $\mathscr{V}$ are contained in $Wr(\mathscr{U},\mathscr{V})$, if $Wr(\mathscr{U},\mathscr{V})$ is locally finite then so are $\mathscr{U}$ and $\mathscr{V}$.

Suppose that $\mathscr{U}$ and $\mathscr{V}$ are locally finite but $Wr(\mathscr{U},\mathscr{V})$ is not. Then for some $n \in \omega$, the $Wr(\mathscr{U},\mathscr{V})$–free inverse semigroup on n generators is not finite. Let X_n be a subset of X of cardinality n. It follows that there exists an infinite set of words $\{ w_i : i \in \omega \}$ over $X_n \cup X_n^{-1}$ such that, for all $i,j \in \omega$, w_i is not $\rho(Wr(\mathscr{U},\mathscr{V}))$–equivalent to w_j. Since $\mathscr{V}$ is locally finite, we have as a consequence of Theorem 4.2.3 that there exists an infinite subset $\{ w_{i_j} : j \in \omega \}$ of $\{ w_i : i \in \omega \}$ such that, for all $j,k \in \omega$, $w_{i_j} \, \rho(\mathscr{V}) \, w_{i_k}$ but $d(w_{i_j})$ is not $\rho(\mathscr{U})$–equivalent to $d(w_{i_k})$.

Let V be the set of vertices and E the set of edges incident to any of the paths from s to e in $\Gamma_{\mathscr{V}}(w_{i_j})$ labelled by the w_{i_k}, $k \in \omega$, where s and e are the start and end vertices, respectively, of $\Gamma_{\mathscr{V}}(w_{i_j})$.

Let $v \in V$. Then for some w_{i_k}, v is incident to the path from s to e in $\Gamma_{\mathscr{V}}(w_{i_j})$ labelled by w_{i_k}. Thus, there is an initial segment w' of w_{i_k} such that w' labels an s–v walk in $\Gamma_{\mathscr{V}}(w_{i_j})$. It follows from the definition of Schützenberger graphs that $w_{i_k}w_{i_k}^{-1}w' \, \rho(\mathscr{V}) \, v$. Since $c(w_{i_k}w_{i_k}^{-1}w') \subseteq X_n$ and since $F\mathscr{V}(X_n)$ is finite by our hypothesis, there can only be finitely many members of V.

Let $(v_1,z,v_2) \in E$. Then $v_1,v_2 \in V$ and $z \in \{ x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1} \}$. Since V is finite, we have $| E | \leq | V |^2 \cdot 2n$ and so E is finite.

Since E is finite, it follows from the definition of the derived word that there exists an $m \in \omega$ such that $c(d(w_{i_j})) \subseteq \{ y_1,\ldots,y_m \} = Y_m$ for all $j \in \omega$. Therefore, $\{ d(w_{i_j}) \, \rho(\mathscr{U}) : j \in \omega \}$ is contained in the $\mathscr{U}$–free inverse semigroup on Y_m. Since $\mathscr{U}$ is locally finite, the $\mathscr{U}$–free inverse semigroup on Y_m is finite and as a consequence,

$\{ d(w_{i_j})\ \rho(\mathscr{U}) : j \in \omega \}$ must be finite, contradicting the assertion above that, for all $j,k \in \omega$, $d(w_{i_j})$ is not $\rho(\mathscr{U})$–equivalent to $d(w_{i_k})$.

Therefore, if $\mathscr{U}$ and $\mathscr{V}$ are locally finite then $Wr(\mathscr{U},\mathscr{V})$ is locally finite. •

In **[K1]**, Kleiman proved that $\mathscr{L}(\mathscr{S}\mathscr{I})$, the lattice of varieties of strict inverse semigroups, is isomorphic to three copies of $\mathscr{L}(\mathscr{G})$, the lattice of group varieties and that every strict inverse variety $\mathscr{V}$ is equal to $(\mathscr{V} \cap \mathscr{G}) \vee (\mathscr{V} \cap \mathscr{B})$. As a consequence, the ν-class of a strict inverse variety is trivial. Thus, for any group variety $\mathscr{U}$, $\mathscr{U} \circ \mathscr{S} = \mathscr{U} \vee \mathscr{S} = Wr(\mathscr{U},\mathscr{S})$ and $\mathscr{U} \circ \mathscr{B} = \mathscr{U} \vee \mathscr{B} = Wr(\mathscr{U},\mathscr{S})$. This is also an immediate consequence of Theorem 4.2.3 , Proposition 2.8.3 and the following result.

Theorem 5.1.4 [Re3]. The collection of Schützenberger graphs corresponding to the variety $\mathscr{B}$ is the collection of all finite birooted inverse word graphs in which each label (from $X \cup X^{-1}$) occurs at most once.

Reilly in fact showed that the $\mathscr{B}$-free semigroup on countably infinite X can be represented faithfully by birooted labelled digraphs which, as it turns out, are the Schützenberger graphs of $F\mathscr{B}(X)$. Stephen showed directly that the Schützenberger graphs of $F\mathscr{B}(X)$ are the ones mentioned above. We remark that, in the following theorem, we do not need to know what the Schützenberger representation relative to $\mathscr{B}$ of a given word is, but simply that in its Schützenberger representation relative to $\mathscr{B}$ each label occurs at most once.

Theorem 5.1.5. If $\mathscr{V} \in \{ \mathscr{T},\mathscr{S},\mathscr{B} \}$ then $Wr(\mathscr{U},\mathscr{V}) = \mathscr{U} \vee \mathscr{V}$ for any variety $\mathscr{U}$ of inverse semigroups.

Proof: It follows from the definition of the derived word, Theorem 5.1.4 and Proposition 2.8.3 that for $\mathcal{V} \in \{ \mathcal{T}, \mathcal{S}, \mathcal{B} \}$, $d_{\mathcal{V}}(w)$ is a relabelling of w in $Y \cup Y^{-1}$ for any word $w \in (X \cup X^{-1})^{+}$. It follows from Theorem 4.2.3 that $Wr(\mathcal{U},\mathcal{V}) = \mathcal{U} \vee \mathcal{V}$. •

5.2 E-unitary covers

The results of this section are concerned with conditions under which varieties of the form $Wr(\mathcal{V},\mathcal{W})$ have E-unitary covers and E-unitary covers over some group variety. If we are to use Theorem 4.2.3 in this effort, we require some information about the Schützenberger graphs of $F\mathcal{U}^{max}(X)$, for $\mathcal{U}$ a variety of groups. This information is provided by a graphical representation of $F\mathcal{U}_M{}^{max}(X)$ due to Meakin and Margolis [MM] which we present forthwith.

Let $\mathcal{U}$ be a variety of groups. Then $P = (X ; \rho(\mathcal{U}))$ is a presentation of $F\mathcal{U}(X)$ for which $\{ x\rho(\mathcal{U}) : x \in X \}$ freely generates $F\mathcal{U}(X)$. Let $\Gamma(X ; \rho(\mathcal{U}))$ denote the Cayley graph of $F\mathcal{U}(X)$ relative to P.

Let $M(X ; \rho(\mathcal{U})) = \{ (\Gamma,g) : g \in F\mathcal{U}(X)$ and Γ is a finite connected subgraph of $\Gamma(X ; \rho(\mathcal{U}))$ containing 1 and g as vertices $\}$, where 1 is the identity of $F\mathcal{U}(X)$. For each finite subgraph Γ' of $\Gamma(X ; \rho(\mathcal{U}))$ and each $g \in F\mathcal{U}(X)$, let $g\Gamma'$ be the subgraph of $\Gamma(X; \rho(\mathcal{U}))$ obtained by acting on Γ' on the left. The set of vertices of $g\Gamma'$ is $\{ gh: h$ is a vertex of $\Gamma' \}$ and the edges of $g\Gamma'$ are of the form (gh, x, ghx) whenever (h, x, hx) is an edge in Γ'. Observe that $g\Gamma'$ is V-isomorphic to Γ'.

On $M(X ; \rho(\mathcal{U}))$ define a multiplication by setting

$$(\Gamma, g).(\Gamma', g') = (\Gamma \cup g\Gamma' , gg')$$

where $\Gamma \cup g\Gamma'$ is the graph whose vertices and edges are the union of the vertices and edges of Γ and $g\Gamma'$.

Theorem 5.2.1 [MM] . $M(X; \rho(\mathscr{V}))$ is an E-unitary inverse monoid generated by the graphs $(\Gamma_x, x\rho(\mathscr{V}))$ for $x \in X$, where Γ_x is the graph with vertex set $\{1, x\rho(\mathscr{V})\}$ and (directed) edge set $\{(1, x, x\rho(\mathscr{V})), (x\rho(\mathscr{V}), x^{-1}, 1)\}$. Furthermore, $M(X; \rho(\mathscr{V}))$ is (isomorphic to) the relatively free X-generated inverse monoid in the variety $\mathscr{V}_M{}^{max}$. $M(X; \rho(\mathscr{V}))$ satisfies the following properties :

i) $(\Gamma, g)\ \mathscr{R}\ (\Gamma', g')$ if and only if $\Gamma = \Gamma'$;

ii) $(\Gamma, g)\ \mathscr{L}\ (\Gamma', g')$ if and only if $g^{-1}\Gamma = (g')^{-1}\Gamma'$;

iii) $(\Gamma, g)\ \mathscr{D}\ (\Gamma', g')$ if and only if Γ is V-isomorphic to Γ';

iv) (Γ, g) is an idempotent if and only if $g = 1$.

We are interested in varieties of inverse semigroups (as opposed to monoids) and so to make use of Theorem 5.2.1, we require the following result.

Lemma 5.2.2 . Let $\mathscr{V}$ be a variety of groups. Then $F\mathscr{V}^{max}(X)$ is isomorphic (as inverse semigroups) to $F\mathscr{V}_M{}^{max}(X) \setminus \{1\}$, where 1 is the identity of $F\mathscr{V}_M{}^{max}(X)$.

Proof: We first must show that $S = F\mathscr{V}_M{}^{max}(X) \setminus \{1\}$ is an inverse semigroup. If S is not an inverse semigroup then there is a $w \in (X \cup X^{-1})^+$ such that $\mathscr{V}_M{}^{max}$ satisfies the equation $w = 1$. Let $a \in X \cup X^{-1}$ be the initial letter in w; that is, $w = aw'$ for some $w' \in (X \cup X^{-1})^*$. Then , since aa^{-1} is a left identity for w, $\mathscr{V}_M{}^{max}$ must satisfy the equation $aa^{-1} = 1$. But then for any $v \in \mathscr{V}_M{}^{max}$ we must have $vv^{-1} = 1$ and $v^{-1}v = 1$ (by substituting v for a in the first case and v^{-1} for a in the latter case). Thus, $F\mathscr{V}_M{}^{max}(X)$ has a single $\mathscr{H}$-class and so must be a group. Since this is not the case, $F\mathscr{V}_M{}^{max}(X) \setminus \{1\}$ must be an inverse semigroup.

If $T \in \mathscr{U}^{max}$, then the monoid $T^1 \in \mathscr{U}_M{}^{max}$. This follows from the observation that the set of identities $\{ u = u^2 : u = u^2$ is a law in $\mathscr{U} \}$ is closed under deletion. That is, if $A \subseteq c(u)$ and u_A is the word obtained from u by deleting all occurrences of x and x^{-1} in u for all $x \in A$, then $u_A = u_A{}^2$ is a law in $\mathscr{U}$.

Finally, we show that S is free in $\mathscr{U}^{max}$. Let T be an inverse semigroup in $\mathscr{U}^{max}$ and let $f : X \rightarrow T$ be a function. Then $T^1 \in \mathscr{U}_M{}^{max}$ and f can be extended uniquely to a homomorphism ψ of $F\mathscr{U}_M{}^{max}(X)$ into T^1. Since S and T are inverse semigroups, ψ maps S into T and hence maps only the identity of $F\mathscr{U}_M{}^{max}(X)$ onto the identity of T. Thus, $\chi = \psi|_S$ is a homomorphism of S into T which extends f. If θ is another homomorphism of S into T extending f, then the uniqueness of ψ implies that $\theta = \chi$. Therefore, χ is the unique homomorphism of S into T extending f. It follows that $S \cong F\mathscr{U}^{max}(X)$. •

Remark. The argument above can be extended in the obvious way to obtain the following result. If $\mathscr{V}_M$ is a variety of inverse monoids which is not a variety of groups, then

1) for all $w \in (X \cup X^{-1})^+$, $\mathscr{V}_M$ does not satisfy the equation $w = 1$;

2) if Σ is a basis of identities for $\mathscr{V}_M$ such that no equation in Σ contains an occurrence of 1, then $F\mathscr{V}_M(X) \setminus \{1\}$ is isomorphic to the relatively free object on X in the variety of inverse semigroups defined by Σ.

It is not difficult to verify that the identity of $M(X ; \rho(\mathscr{U}))$ is $(\Gamma_1,1)$, where Γ_1 is the graph consisting of the single vertex $\{1\}$ and no edges. Thus, $M(X ; \rho(\mathscr{U})) \setminus \{(\Gamma_1,1)\}$ is isomorphic to $F\mathscr{U}^{max}(X)$ via the map χ which takes $(\Gamma_x, x\rho(\mathscr{U}))$ to $x\rho(\mathscr{U}^{max})$ for all $x \in X$.

Lemma 5.2.3. Let $w \in (X \cup X^{-1})^+$ and suppose that

$(\Gamma, g) \in M(X ; \rho(\mathscr{U})) \setminus \{(\Gamma_1,1)\}$ is such that $(\Gamma, g)\chi = w\rho(\mathscr{U}^{max})$. Then the Schützenberger graph $\Gamma_{\mathscr{U}^{max}}(w)$ of w relative to $\mathscr{U}^{max}$ is V-isomorphic to Γ.

Proof: By Theorem 5.2.1, $(\Gamma, g) \mathscr{R} (\Gamma', g')$ if and only if $\Gamma = \Gamma'$ and so there is a one-to-one correspondence between the vertices of Γ and the members of the $\mathscr{R}$-class of (Γ, g). Let the vertex set of Γ, $V(\Gamma) = \{g_1,\ldots,g_n\}$. Then the function $\phi_V : V(\Gamma) \to V(\Gamma_{\mathscr{U}^{max}}(w))$ defined by $g_i \phi_V = (\Gamma, g_i)\chi$, $i = 1,\ldots,n$, is a one-to-one map of the vertices of Γ onto the vertices of $\Gamma_{\mathscr{U}^{max}}(w)$. We then define $\phi_E : E(\Gamma) \to E(\Gamma_{\mathscr{U}^{max}}(w))$ by $(g_i, x, g_j)\phi_E = ((\Gamma,g_i)\chi, x, (\Gamma,g_j)\chi)$, for all edges (g_i, x, g_j) in Γ. Now,

$$\begin{aligned}
(g_i, x, g_j) \in E(\Gamma) \quad &\Rightarrow \quad g_j = g_i\, x\rho(\mathscr{U})\\
&\Rightarrow \quad (\Gamma, g_i)(\Gamma_x, x) = (\Gamma \cup g_i\Gamma_x, g_i\, x\rho(\mathscr{U})) = (\Gamma, g_j)\\
&\Rightarrow \quad (\Gamma, g_i)\chi\, (\Gamma_x, x)\chi = (\Gamma, g_j)\chi\\
&\Rightarrow \quad (\Gamma, g_i)\chi\, x\rho(\mathscr{U}^{max}) = (\Gamma, g_j)\chi\\
&\Rightarrow \quad ((\Gamma, g_i)\chi, x, (\Gamma, g_j)\chi) \in E(\Gamma_{\mathscr{U}^{max}}(w)).
\end{aligned}$$

Thus, ϕ_E maps edges of Γ to edges of $\Gamma_{\mathscr{U}^{max}}(w)$. Clearly ϕ_E is one-to-one. To see that ϕ_E is onto, let (v_1, x, v_2) be an edge in $\Gamma_{\mathscr{U}^{max}}(w)$. Then $v_1 = (\Gamma, g_i)\chi$ and $v_2 = (\Gamma, g_j)\chi$, say. By the definition of Schützenberger graph, $(\Gamma, g_i)\chi\, x\rho(\mathscr{U}^{max}) = (\Gamma, g_j)\chi$ which implies that $(\Gamma, g_i)(\Gamma_x, x) = (\Gamma, g_j)$ since χ is an isomorphism and maps (Γ_x, x) onto $x\rho(\mathscr{U}^{max})$. But then, $(\Gamma \cup g_i\Gamma_x, g_ix) = (\Gamma, g_j)$ and so $\Gamma \cup g_i\Gamma_x = \Gamma$ and $g_i\, x\rho(\mathscr{U}) = g_j$, whence (g_i, x, g_j) is an edge in Γ. Since $(g_i, x, g_j)\phi_E = ((\Gamma, g_i)\chi, x, (\Gamma, g_j)\chi) = (v_1, x, v_2)$, we have that ϕ_E is surjective. Therefore, $\phi = (\phi_V, \phi_E)$ is a V-isomorphism of Γ onto $\Gamma_{\mathscr{U}^{max}}(w)$. Finally, $g\phi_E = w\rho(\mathscr{U}^{max})$ and $1\phi_E = (\Gamma,1)\chi = ww^{-1}\rho(\mathscr{U}^{max})$, since χ is an isomorphism and so must map the idempotent of the $\mathscr{R}$-class of (Γ,g) onto the idempotent of the $\mathscr{R}$-class of $(\Gamma,g)\chi = w\rho(\mathscr{U}^{max})$. Thus, ϕ is a V-isomorphism which maps roots to roots, as required. ∎

If (Γ, g) and (Γ', g') are members of $M(X; \rho(\mathcal{U}))$ then both Γ and Γ' are V–embeddable in $\Gamma \cup g\Gamma'$, since $g\Gamma'$ is V-isomorphic to Γ'. Lemma 5.2.3 then says that if u and w are words in X, then $\Gamma_{\mathcal{U}max}(u)$ and $\Gamma_{\mathcal{U}max}(w)$ are both V-embeddable in $\Gamma_{\mathcal{U}max}(uw)$. Indeed, we have the following:

Lemma 5.2.4. Let $\mathcal{V}$ be a variety of inverse semigroups which has E-unitary covers over its group part and let $\mathcal{U}$ be a variety of groups. Let $u,w \in (X \cup X^{-1})^+$. If

$$d_{\mathcal{U}max}(uw)\ \rho(\mathcal{V})\ d_{\mathcal{U}max}(uw)^2 \text{ and } d_{\mathcal{U}max}(u)\ \rho(\mathcal{V})\ d_{\mathcal{U}max}(u)^2$$

then

$$d_{\mathcal{U}max}(w)\ \rho(\mathcal{V})\ d_{\mathcal{U}max}(w)^2.$$

Proof: Let χ be the isomorphism which maps $M(X; \rho(\mathcal{U})) \setminus \{ (\Gamma_1, 1) \}$ onto $F\mathcal{U}^{max}(X)$. Let $u\rho(\mathcal{U}^{max}) = (\Gamma,g)\chi$ and $w\rho(\mathcal{U}^{max}) = (\Gamma',g')\chi$. Then $uw\rho(\mathcal{U}^{max}) = (\Gamma \cup g\Gamma', gg')\chi$. Now both Γ and $g\Gamma'$ are V-embeddable in $\Gamma \cup g\Gamma'$ and so by Lemma 5.2.3, $\Gamma_{\mathcal{U}max}(u)$ and $\Gamma_{\mathcal{U}max}(w)$ are V-embeddable in $\Gamma_{\mathcal{U}max}(uw)$. Let Y_1, Y_2 and Y_3 be the secondary label sets of $\overline{\Gamma_{\mathcal{U}max}(u)}$, $\overline{\Gamma_{\mathcal{U}max}(w)}$ and $\overline{\Gamma_{\mathcal{U}max}(uw)}$, respectively. We may assume that Y_1 and Y_2 are disjoint.

Define $f: Y_1 \cup Y_2 \to Y_3$ as follows:

If $y \in Y_1$ then y labels an edge e_{y_1} in $\Gamma_{\mathcal{U}max}(u)$. Since $\Gamma_{\mathcal{U}max}(u)$ is V–isomorphic to Γ, e_{y_1} corresponds to an edge e_{y_2} in Γ which in turn corresponds to an edge e_{y_3} in $\Gamma \cup g\Gamma'$ via the obvious embedding. Since $\Gamma \cup g\Gamma'$ is V–isomorphic to $\Gamma_{\mathcal{U}max}(uw)$, e_{y_3} corresponds to an edge e_{y_4} in $\Gamma_{\mathcal{U}max}(uw)$. Define yf to be the secondary label $\lambda_{uw}(e_{y_4})$ in $\overline{\Gamma_{\mathcal{U}max}(uw)}$.

If $y \in Y_2$ then y labels an edge e_{y_1} in $\Gamma_{\mathcal{U}max}(w)$. Since $\Gamma_{\mathcal{U}max}(w)$ is V–isomorphic to Γ', e_{y_1} correspond to an edge e_{y_2} in Γ' which in turn corresponds to an edge e_{y_3} in $g\Gamma'$ via the obvious V-isomorphism. Now, e_{y_3} corresponds to an edge e_{y_4} in

$\Gamma \cup g\Gamma'$ via the obvious embedding and since $\Gamma \cup g\Gamma'$ is V–isomorphic to $\Gamma_{\mathcal{U}^{max}}(uw)$, e_{y_4} corresponds to an edge e_{y_5} in $\Gamma_{\mathcal{U}^{max}}(uw)$. Define yf to be the secondary label $\lambda_{uw}(e_{y_5})$ in $\overline{\Gamma_{\mathcal{U}^{max}}(uw)}$.

It follows that f is one–to–one on Y_1 and one–to–one on Y_2 (but not necessarily one–to–one on $Y_1 \cup Y_2$). Furthermore, f maps $Y_1 \cup Y_2$ onto Y_3 which is a consequence of Lemma 5.2.3 and the fact that the edge set of $\Gamma \cup g\Gamma'$ is the union of the edge sets of Γ and $g\Gamma'$. Also, f extends uniquely to a homomorphism (also denoted f) which maps $(Y_1 \cup Y_2)^+$ onto $Y_3{}^+$. It follows from our definition of f that $d_{\mathcal{U}^{max}}(uw) = (d_{\mathcal{U}^{max}}(u)f)(d_{\mathcal{U}^{max}}(w)f)$. By the hypothesis, $d_{\mathcal{U}^{max}}(uw)\ \rho(\mathcal{V})\ d_{\mathcal{U}^{max}}(uw)^2$ and $d_{\mathcal{U}^{max}}(u)\ \rho(\mathcal{V})\ d_{\mathcal{U}^{max}}(u)^2$ and so, $d_{\mathcal{U}^{max}}(u)f\ \rho(\mathcal{V})\ (d_{\mathcal{U}^{max}}(u)f)^2$, since f is one-to-one on Y_1. Thus, $d_{\mathcal{U}^{max}}(uw) \leq_{\mathcal{V}} d_{\mathcal{U}^{max}}(w)f$. By Theorem 2.7.3, $F\mathcal{V}(Y_3)$ is E-unitary and so, as a consequence, $d_{\mathcal{U}^{max}}(w)f\ \rho(\mathcal{V})\ (d_{\mathcal{U}^{max}}(w)f)^2$. But f is one-to-one on Y_2, so $d_{\mathcal{U}^{max}}(w)\ \rho(\mathcal{V})\ d_{\mathcal{U}^{max}}(w)^2$. •

Corollary 5.2.5. Let $\mathcal{V}$ be a variety of inverse semigroups which has E-unitary covers (over $\mathcal{V} \cap \mathcal{G}$) and let $\mathcal{U}$ be a variety of groups. Then $FWr(\mathcal{V},\mathcal{U}^{max})(X)$ is E-unitary.

Proof: Set $\rho(Wr(\mathcal{V},\mathcal{U}^{max}) = \rho$ and write d(_) for $d_{\mathcal{U}^{max}}$(_).

Let $e,w \in (X \cup X^{-1})^+$ be such that $e\ \rho\ ew$ where $e\ \rho\ e^2$. By Theorem 4.2.3 we have $e\ \rho(\mathcal{U}^{max})\ ew$, $e\ \rho(\mathcal{U}^{max})\ e^2$, $d(e)\ \rho(\mathcal{V})\ d(ew)$ and $d(e)\ \rho(\mathcal{V})\ d(e^2) = d(e)^2$ with this last equality holding by Proposition 4.1.2 since $e\ \rho(\mathcal{U}^{max})\ e^2$. Now $F\mathcal{U}^{max}(X)$ is E-unitary by Theorems 2.7.3 and 2.7.4, so $w\ \rho(\mathcal{U}^{max})\ w^2$. But $d(e)\ \rho(\mathcal{V})\ d(e)^2$ and $d(ew)\ \rho(\mathcal{V})\ d(e)$ which implies $d(ew)\ \rho(\mathcal{V})\ d(ew)^2$. Thus, by Lemma 5.2.4, $d(w)\ \rho(\mathcal{V})\ d(w)^2 = d(w^2)$ where again the last equality holds by Proposition 4.1.2. Theorem 4.2.3 now gives $w\ \rho\ w^2$. Therefore, $FWr(\mathcal{V},\mathcal{U}^{max})(X)$ is E-unitary. •

Corollary 5.2.6. Let $\mathscr{V}$ be a variety of inverse semigroups which has E-unitary covers and let $\mathscr{U}$ be a variety of groups. Then $Wr(\mathscr{V},\mathscr{U}^{max})$ has E-unitary covers (over $Wr(\mathscr{V},\mathscr{U}^{max}) \cap \mathscr{G}$).

Proof: By Corollary 5.2.5 and Theorem 2.7.3. •

Theorem 5.2.7. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of groups and let $\mathscr{X}$ and $\mathscr{W}$ be varieties of inverse semigroups such that $\mathscr{X}$ has E-unitary covers over $\mathscr{U}$ and $\mathscr{W}$ has E-unitary covers over $\mathscr{V}$. Then $Wr(\mathscr{X},\mathscr{W})$ has E-unitary covers over $Wr(\mathscr{U},\mathscr{V}) = \mathscr{U} \circ \mathscr{V}$.

Proof: We know that $Wr(\mathscr{X},\mathscr{W}) \subseteq Wr(\mathscr{U}^{max},\mathscr{V}^{max})$ by the hypothesis and Proposition 4.3.1. The theorem will follow from Corollary 5.2.6 and Theorem 2.7.4 if we can show that $Wr(\mathscr{U}^{max},\mathscr{V}^{max}) \cap \mathscr{G} = Wr(\mathscr{U},\mathscr{V})$. This follows immediately from Theorem 4.3.7, however, we include the following argument as it deals with this specific case and provides us with a better 'intuitive feel' for why this result should be true.

Set $U = \{ w \in (X \cup X^{-1})^+ : w \text{ is a law in } \mathscr{U} \}$ and

$V = \{ w \in (X \cup X^{-1})^+ : w \text{ is a law in } \mathscr{V} \}$.

Let $U(V) = \{ u(v_1,\ldots,v_n) : u = u(x_1,\ldots,x_n) \in U \text{ and } v_1,\ldots,v_n \in V \}$.

Our first claim is that $Wr(\mathscr{U}^{max},\mathscr{V}^{max}) \subseteq [\, w = w^2 : w \in U(V) \,]$. It is sufficient to show that $S \text{ wr } (T,I)$ satisfies $w = w^2$ for all $S \in \mathscr{U}^{max}$, $(T,I) \in \mathscr{V}^{max}$ and $w \in U(V)$. Let $w \in U(V)$, say $w = u(v_1,\ldots,v_n)$, where $v_i = v_i[x_1,\ldots,x_{m(i)}]$, for $i = 1,\ldots,n$, and $u = u[x_1,\ldots,x_n]$. Suppose that for an arbitrary substitution of variables, v_i takes the value (ψ_i,β_i) in $S \text{ wr } (T,I)$, for $i = 1,\ldots,n$. Since $T \in \mathscr{V}^{max}$ and $\mathscr{V}^{max}$ satisfies the identities $v_i = v_i^2$, for $i = 1,\ldots,n$, each β_i is an idempotent in (T,I). That is, each β_i is the identity map on its domain. We wish to show that $u[(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)]$ is an idempotent in $S \text{ wr } (T,I)$. Let $u = a_1 \ldots a_k$. Using the same notation as before, we wish to show that

$(\psi_{a_1}\beta_{a_1}\psi_{a_2}\ldots\beta_{a_1\ldots a_{k-1}}\psi_{a_k},\beta_{a_1}\ldots_{a_k})$ is an idempotent in S wr (T,I). Since each of the β_is is an idempotent, we have that $\beta_{a_1}\ldots_{a_k}$ is an idempotent. Moreover, for all $i \in \mathbf{d}\beta_{a_1}\ldots_{a_k}$,

$$i(\psi_{a_1}\beta_{a_1}\psi_{a_2}\ldots\beta_{a_1\ldots a_{k-1}}\psi_{a_k}) = (i\psi_{a_1})(i\beta_{a_1}\psi_{a_2})\ldots(i\beta_{a_1}\ldots_{a_{k-1}}\psi_{a_k}) = (i\psi_{a_1})(i\psi_{a_2})\ldots(i\psi_{a_k})$$

since each β_i is the identity map on its domain. Since $S \in \mathcal{U}^{max}$, S satisfies the equation $u = u^2$. Thus, $(i\psi_{a_1})(i\psi_{a_2})\ldots(i\psi_{a_k})$ and hence, $i(\psi_{a_1}\beta_{a_1}\psi_{a_2}\ldots\beta_{a_1\ldots a_{k-1}}\psi_{a_k})$ is an idempotent of S. It follows from Proposition 3.1.1 (c) that $u[(\psi_1,\beta_1),\ldots,(\psi_n,\beta_n)]$ is an idempotent of S wr (T,I). Therefore, S wr (T,I) satisfies the equation $w = w^2$. From this we obtain that $Wr(\mathcal{U}^{max},\mathcal{V}^{max}) \subseteq [\ w = w^2 : w \in U(V)\ ]$.

Our second claim is that $[\ w = w^2 : w \in U(V)\ ] \cap \mathcal{G} = \mathcal{U} \circ \mathcal{V}$. Observe that $U\rho(\mathcal{G}) = \{\ u\rho(\mathcal{G}) : u \in U\ \}$ and $V\rho(\mathcal{G}) = \{\ v\rho(\mathcal{G}) : v \in V\ \}$ are the fully invariant subgroups of $F\mathcal{G}(X)$ corresponding to $\mathcal{U}$ and $\mathcal{V}$, respectively. It follows from Neumann [**N;21.12**] that $\{\ w = w^2 : w \in U(V)\ \} \cup \{\ xx^{-1} = yy^{-1}\}$ forms a basis of identities for $\mathcal{U} \circ \mathcal{V}$.

We may now conclude that

$$Wr(\mathcal{U}^{max},\mathcal{V}^{max}) \cap \mathcal{G} \subseteq [\ w = w^2 : w \in U(V)\ ] \cap \mathcal{G} = \mathcal{U} \circ \mathcal{V}.$$

By Theorem 4.3.4 and Proposition 4.3.1,

$$\mathcal{U} \circ \mathcal{V} = Wr(\mathcal{U},\mathcal{V}) \subseteq Wr(\mathcal{U}^{max},\mathcal{V}^{max}) \cap \mathcal{G}.$$

Thus, $Wr(\mathcal{U}^{max},\mathcal{V}^{max}) \cap \mathcal{G} = Wr(\mathcal{U},\mathcal{V})$ and so Theorem 5.2.7 is proved. •

Corollary 5.2.8. Let $\mathcal{V}$ be a variety of inverse semigroups.

1) If $\mathcal{V}$ has E-unitary covers then, for any group variety $\mathcal{U}$,
$\mathcal{U} \circ \mathcal{V} = \mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}$.

2) If $\mathcal{V}$ has E-unitary covers over the group variety $\mathcal{U}$ then, for any group variety $\mathcal{W}$,
$(\mathcal{W} \circ \mathcal{U}) \vee \mathcal{V} = (\mathcal{W} \circ \mathcal{U}) \vee (\mathcal{W} \circ \mathcal{V})$.

Proof: 1) By Theorem 5.2.7, $\mathcal{U} \circ \mathcal{V}$ has E-unitary covers over $\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})$. Therefore, $\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \subseteq \mathcal{U} \circ \mathcal{V} \subseteq [\ \mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})\ ]^{max}$. But $\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \subseteq \mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}$

and $\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V} \subseteq \mathcal{U} \circ \mathcal{V} \subseteq [\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})]^{max}$. Thus, $\ker \rho(\mathcal{U} \circ \mathcal{V}) = \ker \rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V})$. On the other hand, $tr(\mathcal{V}) = tr(\mathcal{U} \circ \mathcal{V})$ by Lemma 2.7.5 and $tr\ \rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}) = tr\ [\rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})) \cap \rho(\mathcal{V})] = tr\ \rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})) \cap tr\ \rho(\mathcal{V})$ and $tr\ \rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G})) \cap tr\ \rho(\mathcal{V}) = tr\ \rho(\mathcal{V})$. Thus, $tr(\mathcal{U} \circ \mathcal{V}) = tr\ \rho(\mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V})$. We thus obtain $\mathcal{U} \circ \mathcal{V} = \mathcal{U} \circ (\mathcal{V} \cap \mathcal{G}) \vee \mathcal{V}$.

2) By Theorem 5.2.7, $\mathcal{W} \circ \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{U})^{max}$ and by the hypothesis $\mathcal{V} \subseteq \mathcal{U}^{max}$ and hence, $\mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{U})^{max}$. Thus, $\mathcal{W} \circ \mathcal{U} \subseteq (\mathcal{W} \circ \mathcal{U}) \vee \mathcal{V} \subseteq (\mathcal{W} \circ \mathcal{U})^{max}$ and $\mathcal{W} \circ \mathcal{U} \subseteq (\mathcal{W} \circ \mathcal{U}) \vee (\mathcal{W} \circ \mathcal{V}) \subseteq (\mathcal{W} \circ \mathcal{U})^{max}$. Therefore, $\ker \rho((\mathcal{W} \circ \mathcal{U}) \vee \mathcal{V}) = \ker \rho((\mathcal{W} \circ \mathcal{U}) \vee (\mathcal{W} \circ \mathcal{V}))$. On the other hand, $tr\ \rho[(\mathcal{W} \circ \mathcal{U}) \vee \mathcal{V}] = tr\ [\rho(\mathcal{W} \circ \mathcal{U}) \cap \rho(\mathcal{V})] = tr\ \rho(\mathcal{W} \circ \mathcal{U}) \cap tr\ \rho(\mathcal{V}) = tr\ \rho(\mathcal{V})$ and $tr\ \rho[(\mathcal{W} \circ \mathcal{U}) \vee (\mathcal{W} \circ \mathcal{V})] = tr[\rho(\mathcal{W} \circ \mathcal{U}) \cap \rho(\mathcal{W} \circ \mathcal{V})] = tr\ \rho(\mathcal{W} \circ \mathcal{U}) \cap tr\ \rho(\mathcal{W} \circ \mathcal{V}) = tr\ \rho(\mathcal{W} \circ \mathcal{U}) \cap tr\ \rho(\mathcal{V})$ (by Lemma 2.7.5) $= tr\ \rho(\mathcal{V})$. It now follows that $(\mathcal{W} \circ \mathcal{U}) \vee \mathcal{V} = (\mathcal{W} \circ \mathcal{U}) \vee (\mathcal{W} \circ \mathcal{V})$. ●

5.3 Wr($\mathcal{S}$, $\mathcal{V}$)

The principal result of this section is the following. For any variety $\mathcal{V}$ of inverse semigroups, Wr($\mathcal{S}$,$\mathcal{V}$) is the largest variety of inverse semigroups which satisfies the equations $w = w^2$ whenever $\mathcal{V}$ satisfies $w = w^2$. Throughout this section we will use the following convention. If $w \in (X \cup X^{-1})^+$ and $\mathcal{V}$ is a variety of inverse semigroups, we will write $w_{\mathcal{V}}$ to denote $w\rho(\mathcal{V})$.

Theorem 5.3.1. Let $\mathcal{U} \subseteq \mathcal{V}$ be varieties of inverse semigroups and let ρ be the congruence on $F\mathcal{V}(X)$ such that $F\mathcal{V}(X) / \rho \cong F\mathcal{U}(X)$. Then ρ is idempotent pure if and only if for every $w \in (X \cup X^{-1})^+$, $\Gamma_{\mathcal{V}}(w)$ is V-embeddable in $\Gamma_{\mathcal{U}}(w)$.

Proof: Suppose that ρ is idempotent pure and let $w \in (X \cup X^{-1})^+$. Define a map ϕ on $\mathscr{R}_{w_\mathscr{V}}$, the set of vertices of $\Gamma_\mathscr{V}(w)$, by setting $v\phi = v\rho$. Green's relation $\mathscr{R}$ is preserved under homomorphism so ϕ maps $\mathscr{R}_{w_\mathscr{V}}$ into $\mathscr{R}_{w_\mathscr{V}\rho}$, which is the vertex set of $\Gamma_\mathscr{U}(w)$ since, for any $v \in (X \cup X^{-1})^+$, $v_\mathscr{V}\rho = v_\mathscr{U}$. If (v_1,x,v_2) is an edge in $\Gamma_\mathscr{V}(w)$ then $v_1x_\mathscr{V} = v_2$ and so $(v_1\rho)(x_\mathscr{V}\rho) = (v_2\rho)$. But this means that $(v_1\phi)x_\mathscr{U} = v_2\phi$ from which it follows that $(v_1\phi,x,v_2\phi)$ is an edge in $\Gamma_\mathscr{U}(w)$. Therefore, ϕ is a V-homomorphism.

Suppose that $v_1\phi = v_2\phi$ for some $v_1,v_2 \in \mathscr{R}_{w_\mathscr{V}}$. Then $v_1 \ \rho \ v_2$ and so $v_1 = v_2$ since $\rho \cap \mathscr{R} = \varepsilon$ whenever ρ is idempotent pure. Thus, ϕ is a V-embedding of $\Gamma_\mathscr{V}(w)$ into $\Gamma_\mathscr{U}(w)$.

Conversely, suppose that $\Gamma_\mathscr{V}(w)$ is V-embeddable in $\Gamma_\mathscr{U}(w)$ for every $w \in (X \cup X^{-1})^+$. Let $e,a \in (X \cup X^{-1})^+$ be such that $e_\mathscr{V} = e_\mathscr{V}^2$ and $e_\mathscr{V} \ \rho \ a_\mathscr{V}$. Then $a_\mathscr{U} = aa^{-1}{}_\mathscr{U}$. If ϕ is the V-embedding of $\Gamma_\mathscr{V}(a)$ into $\Gamma_\mathscr{U}(a)$ then $a_\mathscr{V}\phi = a_\mathscr{U}$ and $aa^{-1}{}_\mathscr{V}\phi = aa^{-1}{}_\mathscr{U}$, since ϕ maps roots to roots. Since ϕ is one-to-one on the vertices of $\Gamma_\mathscr{V}(a)$ and $a_\mathscr{U} = aa^{-1}{}_\mathscr{U}$, we must have that $a_\mathscr{V} = aa^{-1}{}_\mathscr{V}$ and so ρ is idempotent pure. •

Lemma 5.3.2. Let $\mathscr{U} \subseteq \mathscr{V}$ be varieties of inverse semigroups and suppose that $\mathscr{U}^{max} = \mathscr{V}^{max}$. If ρ is the congruence on $F\mathscr{V}(X)$ such that $F\mathscr{V}(X) / \rho \cong F\mathscr{U}(X)$ then ρ is idempotent pure.

Proof: Let $w,a \in (X \cup X^{-1})^+$ be such that $w_\mathscr{V} = w_\mathscr{V}^2$ and $w_\mathscr{V} \ \rho \ a_\mathscr{V}$.
Then $a_\mathscr{V} \ \rho \ a_\mathscr{V}^2$; that is, $a_\mathscr{U} = a_\mathscr{U}^2$. But then $a_{\mathscr{U}^{max}} = a_{\mathscr{U}^{max}}^2$ and so, since $\mathscr{V} \subseteq \mathscr{V}^{max} = \mathscr{U}^{max}$, we have that $a_\mathscr{V} = a_\mathscr{V}^2$ and as a consequence, ρ is idempotent pure. •

Theorem 5.3.3. Let $\mathscr{V}$ be a variety of inverse semigroups. Then $Wr(\mathscr{S},\mathscr{V}) = \mathscr{V}^{max}$.

Proof: First of all, observe that $Wr(\mathscr{S},\mathscr{V}) \subseteq \mathscr{V}^{max}$ because, for any $w \in (X \cup X^{-1})^+$, $w \ \rho(Wr(\mathscr{S},\mathscr{V})) \ w^2$ if and only if $w \ \rho(\mathscr{V}) \ w^2$ and $d_\mathscr{V}(w) \ \rho(\mathscr{S}) \ d_\mathscr{V}(w^2) = d_\mathscr{V}(w)^2$, by

Theorem 4.2.3 with the last equality by Proposition 4.1.2, and so $w\ \rho(Wr(\mathcal{S},\mathcal{V}))\ w^2$ if and only if $w\ \rho(\mathcal{V})\ w^2$.

Let ρ_1,ρ_2 be the congruences on $F\mathcal{V}^{max}(X)$ such that

$$F\mathcal{V}^{max}(X) / \rho_1 \cong FWr(\mathcal{S},\mathcal{V})(X)$$

$$F\mathcal{V}^{max}(X) / \rho_2 \cong F\mathcal{V}(X)$$

and let ρ_3 be the congruence on $FWr(\mathcal{S},\mathcal{V})(X)$ such that

$$FWr(\mathcal{S},\mathcal{V})(X) / \rho_3 \cong F\mathcal{V}(X).$$

From the preceding lemma we obtain that ρ_1,ρ_2 and ρ_3 are idempotent pure and so, by the theorem above, for all $w \in (X \cup X^{-1})^+$, $\Gamma_{\mathcal{V}max}(w)$ is V-embeddable in $\Gamma_{Wr(\mathcal{S},\mathcal{V})}(w)$ which in turn is V-embeddable in $\Gamma_{\mathcal{V}}(w)$. Let $w,u \in (X \cup X^{-1})^+$ be such that $w_{\mathcal{V}max} = w_{\mathcal{V}max}{}^2$, $u_{\mathcal{V}max} = u_{\mathcal{V}max}{}^2$ and $w_{\mathcal{V}max}\ \rho_1\ u_{\mathcal{V}max}$. By Theorem 4.2.3, $w_{\mathcal{V}} = u_{\mathcal{V}}$ and $c(d_{\mathcal{V}}(w)) = c(d_{\mathcal{V}}(u))$. It follows that $\Gamma_{\mathcal{V}}(w) = \Gamma_{\mathcal{V}}(u)$ and both u and w label $ww^{-1}{}_{\mathcal{V}}$ - $w_{\mathcal{V}}$ paths in $\Gamma_{\mathcal{V}}(w)$. Moreover, the $ww^{-1}{}_{\mathcal{V}}$ - $w_{\mathcal{V}}$ path labelled by u in $\Gamma_{\mathcal{V}}(w)$ uses only the edges in the $ww^{-1}{}_{\mathcal{V}}$ - $w_{\mathcal{V}}$ path in $\Gamma_{\mathcal{V}}(w)$ labelled by w. Thus, u labels a $ww^{-1}{}_{\mathcal{V}}$ - $w_{\mathcal{V}}$ path in the subgraph of $\Gamma_{\mathcal{V}}(w)$ consisting of the $ww^{-1}{}_{\mathcal{V}}$ - $w_{\mathcal{V}}$ path labelled by w. Since $\Gamma_{\mathcal{V}max}(w)$ is V-embeddable in $\Gamma_{\mathcal{V}}(w)$, this subgraph is V-embeddable in $\Gamma_{\mathcal{V}max}(w)$ and so u labels a $ww^{-1}{}_{\mathcal{V}max}$ - $w_{\mathcal{V}max}$ path in $\Gamma_{\mathcal{V}max}(w)$. By Lemma 2.8.1 (c), we have that $u_{\mathcal{V}max} \geq w_{\mathcal{V}max}$. In a similar fashion we may demonstrate that $w_{\mathcal{V}max} \geq u_{\mathcal{V}max}$ and so obtain that $w_{\mathcal{V}max} = u_{\mathcal{V}max}$. As a consequence, we have that ρ_1 is an idempotent separating congruence. But the only idempotent pure and idempotent separating congruence on any inverse semigroup is the identical relation ε. Thus, $\rho_1 = \varepsilon$ and $F\mathcal{V}^{max}(X) \cong FWr(\mathcal{S},\mathcal{V})(X)$. Therefore, $Wr(\mathcal{S},\mathcal{V}) = \langle\ FWr(\mathcal{S},\mathcal{V})(X)\ \rangle = \langle\ F\mathcal{V}^{max}(X)\ \rangle = \mathcal{V}^{max}$. ■

We now present some immediate consequences of the preceding Theorem in light of some of the principle results obtained thus far.

Corollary 5.3.4. Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups.

a) $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V})$;

b) If $\mathscr{U} = \mathscr{U}^{\max}$ then $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{U},\mathscr{V})$;

c) $\mathrm{Wr}(\mathscr{U},\mathscr{V}^{\max}) = \mathrm{Wr}(\mathscr{U} \vee \mathscr{S}, \mathscr{V}) = \mathrm{Wr}(\mathscr{U},\mathscr{V}) \vee \mathscr{V}^{\max}$;

d) If $\mathscr{U}$ is not a variety of groups then $\mathrm{Wr}(\mathscr{U},\mathscr{V}^{\max}) = \mathrm{Wr}(\mathscr{U}, \mathscr{V})$;

e) $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}^{\max})$;

Proof: a) $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{S}, \mathrm{Wr}(\mathscr{U},\mathscr{V}))$ by Theorem 5.3.3. Since Wr is associative, we have $\mathrm{Wr}(\mathscr{S}, \mathrm{Wr}(\mathscr{U},\mathscr{V})) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{S},\mathscr{U}),\mathscr{V}) = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V})$, again by Theorem 5.3.3.

b) If $\mathscr{U} = \mathscr{U}^{\max}$ then $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V})$, by part a) and $\mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}) = \mathrm{Wr}(\mathscr{U},\mathscr{V})$ by our assumption.

c) $\mathrm{Wr}(\mathscr{U},\mathscr{V}^{\max}) = \mathrm{Wr}(\mathscr{U}, \mathrm{Wr}(\mathscr{S},\mathscr{V}))$ by Theorem 5.3.3. By the associativity of Wr we have that $\mathrm{Wr}(\mathscr{U}, \mathrm{Wr}(\mathscr{S},\mathscr{V})) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{U},\mathscr{S}),\mathscr{V})$ and $\mathrm{Wr}(\mathrm{Wr}(\mathscr{U},\mathscr{S}),\mathscr{V}) = \mathrm{Wr}(\mathscr{U} \vee \mathscr{S}, \mathscr{V})$ by Theorem 5.1.5. By Proposition 4.3.5, $\mathrm{Wr}(\mathscr{U} \vee \mathscr{S}, \mathscr{V}) = \mathrm{Wr}(\mathscr{U},\mathscr{V}) \vee \mathrm{Wr}(\mathscr{S}, \mathscr{V}) = \mathrm{Wr}(\mathscr{U},\mathscr{V}) \vee \mathscr{V}^{\max}$.

d) If $\mathscr{U}$ is not a variety of groups then $\mathscr{U} \vee \mathscr{S} = \mathscr{U}$. By part c) above, $\mathrm{Wr}(\mathscr{U},\mathscr{V}^{\max}) = \mathrm{Wr}(\mathscr{U},\mathscr{V})$.

e) For any variety $\mathscr{U}$ of inverse semigroups, $\mathscr{U}^{\max}$ is not a variety of groups. By part d) above, $\mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}^{\max}) = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V})$ and so, by part a), $[\mathrm{Wr}(\mathscr{U},\mathscr{V})]^{\max} = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}^{\max})$. •

If we let $\mathscr{U}$ and $\mathscr{V}$ be varieties of groups in Corollary 5.3.4 (e), then we have that $(\mathscr{U} \circ \mathscr{V})^{\max} = \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}^{\max})$. Thus, if the variety $\mathscr{X}$ has E-unitary covers over $\mathscr{U}$ and the variety $\mathscr{W}$ has E-unitary covers over $\mathscr{V}$ then $\mathscr{X} \subseteq \mathscr{U}^{\max}$ and $\mathscr{W} \subseteq \mathscr{V}^{\max}$ and so $\mathrm{Wr}(\mathscr{X},\mathscr{W}) \subseteq \mathrm{Wr}(\mathscr{U}^{\max},\mathscr{V}^{\max}) = (\mathscr{U} \circ \mathscr{V})^{\max}$. Consequently, $\mathrm{Wr}(\mathscr{X},\mathscr{W})$

has E-unitary covers over $\mathscr{U} \circ \mathscr{V} = \mathrm{Wr}(\mathscr{U},\mathscr{V})$. As a result, Theorem 5.2.7 is just a special case of the results of this section.

Corollary 5.3.5. Let $\mathscr{W}$ and $\mathscr{X}$ be non-trivial varieties of inverse semigroups and suppose that $\mathscr{W}$ is a strict inverse semigroup with group part $\mathscr{U}$ and combinatorial part $\mathscr{V}$.

a) If $\mathscr{X}$ is not a group variety then $\mathrm{Wr}(\mathscr{X}, \mathscr{W}) = \mathrm{Wr}(\mathscr{X}, \mathscr{U})$ unless $\mathscr{U} = \mathscr{T}$ in which case $\mathrm{Wr}(\mathscr{X}, \mathscr{W}) = \mathscr{X} \vee \mathscr{V}$. If $\mathscr{X}$ is a group variety then $\mathrm{Wr}(\mathscr{X}, \mathscr{W}) = \mathrm{Wr}(\mathscr{X}, \mathscr{U}) \vee \mathscr{V}$.

b) If $\mathscr{W}$ is not a group variety then $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U},\mathscr{X}) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{X})$. If $\mathscr{W}$ is a group variety then $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U},\mathscr{X})$.

c) If $\mathscr{X}$ is a strict inverse variety with group part $\mathscr{U}^*$ and combinatorial part $\mathscr{V}^*$ then

i) $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U},\mathscr{U}^*) \vee \mathscr{V}^*$ if $\mathscr{W}$ is a group variety;

ii) $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{W}, \mathscr{U}^*)$ if $\mathscr{W}$ is not a group variety and $\mathscr{U}^* \neq \mathscr{T}$;

iii) $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathscr{W} \vee \mathscr{X}$ if $\mathscr{W}$ is not a group variety and $\mathscr{U}^* = \mathscr{T}$.

Proof: It follows from [**K1**] (See [**P;XII.2 and XII.3**]) that if $\mathscr{W}$ is a strict inverse variety with group part $\mathscr{U}$ and combinatorial part $\mathscr{V}$ then $\mathscr{W} = \mathscr{U} \vee \mathscr{V}$.

a) $\mathrm{Wr}(\mathscr{X}, \mathscr{W}) = \mathrm{Wr}(\mathscr{X}, \mathscr{U} \vee \mathscr{V}) = \mathrm{Wr}(\mathscr{X}, \mathrm{Wr}(\mathscr{U},\mathscr{V}))$ by Theorem 5.1.5 and $\mathrm{Wr}(\mathscr{X}, \mathrm{Wr}(\mathscr{U},\mathscr{V})) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{X},\mathscr{U}),\mathscr{V})$ by the associativity of Wr. Also from Theorem 5.1.5 we have that $\mathrm{Wr}(\mathrm{Wr}(\mathscr{X},\mathscr{U}),\mathscr{V}) = \mathrm{Wr}(\mathscr{X},\mathscr{U}) \vee \mathscr{V}$. If $\mathscr{X}$ is not a group variety then $\mathscr{S} \subseteq \mathscr{X}$ and so $\mathscr{V} \subseteq \mathrm{Wr}(\mathscr{S},\mathscr{U})$ whenever $\mathscr{U}$ is not trivial since $\mathscr{V}$ has E-unitary covers over every nontrivial group variety and so is contained in $\mathscr{U}^{\max}$ by Theorem 2.7.4. But $\mathrm{Wr}(\mathscr{S},\mathscr{U}) \subseteq \mathrm{Wr}(\mathscr{X},\mathscr{U})$ and as a consequence, $\mathrm{Wr}(\mathscr{X},\mathscr{U}) \vee \mathscr{V} = \mathrm{Wr}(\mathscr{X},\mathscr{U})$. If $\mathscr{U}$ is trivial then $\mathrm{Wr}(\mathscr{X},\mathscr{U}) \vee \mathscr{V} = \mathscr{X} \vee \mathscr{V}$.

b) $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U} \vee \mathscr{V}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{X}) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{X})$ by Theorem 5.1.5 and Proposition 4.3.5. If $\mathscr{W}$ is a group variety then $\mathscr{V} = \mathscr{T}$ and so $\mathrm{Wr}(\mathscr{V}, \mathscr{X}) = \mathscr{X}$. Therefore, $\mathrm{Wr}(\mathscr{U}, \mathscr{X}) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{X}) \vee \mathscr{X} = \mathrm{Wr}(\mathscr{U}, \mathscr{X})$.

c) If $\mathscr{X} = \mathscr{U}^* \vee \mathscr{V}^*$ and $\mathscr{W}$ is a group variety then by part b),

$\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{U}^* \vee \mathscr{V}^*) = \mathrm{Wr}(\mathscr{U}, \mathrm{Wr}(\mathscr{U}^*, \mathscr{V}^*))$ by Theorem 5.1.5 and by the associatitvity of Wr, $\mathrm{Wr}(\mathscr{U}, \mathrm{Wr}(\mathscr{U}^*, \mathscr{V}^*)) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{U}, \mathscr{U}^*), \mathscr{V}^*)$. But $\mathrm{Wr}(\mathrm{Wr}(\mathscr{U}, \mathscr{U}^*), \mathscr{V}^*) = \mathrm{Wr}(\mathscr{U}, \mathscr{U}^*) \vee \mathscr{V}^*$. On the other hand, if $\mathscr{X} = \mathscr{U}^* \vee \mathscr{V}^*$ and $\mathscr{W}$ is not a group variety then by part b),

$\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{U}^* \vee \mathscr{V}^*) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{U}^* \vee \mathscr{V}^*)$. Using Theorem 5.1.5 and the associativity of Wr, we obtain $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{U}^*) \vee \mathscr{V}^* \vee \mathrm{Wr}(\mathscr{V}, \mathscr{U}^*) \vee \mathscr{V}^*$. But if $\mathscr{W}$ is not a group variety then $\mathscr{V} \neq \mathscr{T}$ and so, as in part a), if $\mathscr{U}^*$ is not trivial, we have that $\mathscr{V}^* \subsetneq \mathrm{Wr}(\mathscr{V}, \mathscr{U}^*)$ and so

$\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathrm{Wr}(\mathscr{U}, \mathscr{U}^*) \vee \mathrm{Wr}(\mathscr{V}, \mathscr{U}^*) = \mathrm{Wr}(\mathscr{U} \vee \mathscr{V}, \mathscr{U}^*) = \mathrm{Wr}(\mathscr{W}, \mathscr{U}^*)$. If $\mathscr{U}^*$ is trivial then $\mathrm{Wr}(\mathscr{W}, \mathscr{X}) = \mathscr{U} \vee \mathscr{V}^* \vee \mathscr{V} = \mathscr{W} \vee \mathscr{X}$. •

Proposition 5.3.6. Let $\mathscr{U}, \mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. If $\mathscr{W}$ is not a variety of groups then $\mathscr{U}^{max} = \mathscr{V}^{max}$ implies that $\mathrm{Wr}(\mathscr{W}, \mathscr{U}) = \mathrm{Wr}(\mathscr{W}, \mathscr{V})$.

Proof: If $\mathscr{U}^{max} = \mathscr{V}^{max}$ then $\mathrm{Wr}(\mathscr{W}, \mathscr{U}^{max}) = \mathrm{Wr}(\mathscr{W}, \mathscr{V}^{max})$ and so, by Theorem 5.3.3 and the associativity of Wr, $\mathrm{Wr}(\mathrm{Wr}(\mathscr{W}, \mathscr{S}), \mathscr{U}) = \mathrm{Wr}(\mathrm{Wr}(\mathscr{W}, \mathscr{S}), \mathscr{V})$. By Theorem 5.1.5, $\mathrm{Wr}(\mathscr{W}, \mathscr{S}) = \mathscr{W} \vee \mathscr{S} = \mathscr{W}$ since $\mathscr{W}$ is not a variety of groups. Therefore, $\mathrm{Wr}(\mathscr{W}, \mathscr{U}) = \mathrm{Wr}(\mathscr{W}, \mathscr{V})$. •

Proposition 5.3.7. Let $\mathscr{U}, \mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. If $\mathscr{U}^{max} = \mathscr{V}^{max}$ then $\mathrm{Wr}(\mathscr{U}, \mathscr{W})^{max} = \mathrm{Wr}(\mathscr{V}, \mathscr{W})^{max}$, but the converse is not true.

Proof: By Corollary 5.3.4, $\mathrm{Wr}(\mathscr{U}, \mathscr{W})^{max} = \mathrm{Wr}(\mathscr{U}^{max}, \mathscr{W})$ and $\mathrm{Wr}(\mathscr{U}^{max}, \mathscr{W}) = \mathrm{Wr}(\mathscr{V}^{max}, \mathscr{W})$ by the hypothesis. Again by Corollary 5.3.4, $\mathrm{Wr}(\mathscr{V}^{max}, \mathscr{W}) = \mathrm{Wr}(\mathscr{V}, \mathscr{W})^{max}$, and so $\mathrm{Wr}(\mathscr{U}, \mathscr{W})^{max} = \mathrm{Wr}(\mathscr{V}, \mathscr{W})^{max}$. As for the converse, consider the wreath-closed variety $\mathscr{C}_2 = [\, x^2 = x^3 \,]$. Now $\mathscr{B}$ and $\mathscr{B}^1$ are both contained in $\mathscr{C}_2$, so $\mathrm{Wr}(\mathscr{B}, \mathscr{C}_2) = \mathscr{C}_2 = \mathrm{Wr}(\mathscr{B}^1, \mathscr{C}_2)$ but $\mathscr{B}^{max} = \mathscr{B} \neq (\mathscr{B}^1)^{max}$. •

Theorem 5.3.3 deals with varieties which satisfy the same 'kernel identities'; that is, identities of the form $w = w^2$. The following results deal with 'trace identities' and are the companion results to Theorem 5.3.3.

Theorem 5.3.8. Let $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ be varieties of inverse semigroups. If $\operatorname{tr} \rho(\mathscr{V}) = \operatorname{tr} \rho(\mathscr{U})$ then $\operatorname{tr} \rho(\mathrm{Wr}(\mathscr{U},\mathscr{W})) = \operatorname{tr} \rho(\mathrm{Wr}(\mathscr{V},\mathscr{W}))$.

Proof: Let v and w be idempotents in $F\mathscr{I}(X)$ and suppose that $v \; \rho(\mathrm{Wr}(\mathscr{U},\mathscr{W})) \; w$. Then, by Theorem 4.2.3, $v \; \rho(\mathscr{W}) \; w$ and $d_{\mathscr{W}}(v) \; \rho(\mathscr{U}) \; d_{\mathscr{W}}(w)$. By Lemma 4.1.3, both $d_{\mathscr{W}}(v)$ and $d_{\mathscr{W}}(w)$ are idempotents of $F\mathscr{I}(Y)$. Consequently, $v \; \rho(\mathscr{W}) \; w$ and $d_{\mathscr{W}}(v) \; \rho(\mathscr{V}) \; d_{\mathscr{W}}(w)$, and so $v \; \rho(\mathrm{Wr}(\mathscr{V},\mathscr{W})) \; w$. Similarly, $v \; \rho(\mathrm{Wr}(\mathscr{V},\mathscr{W})) \; w$ implies that $v \; \rho(\mathrm{Wr}(\mathscr{U},\mathscr{W})) \; w$, and the result follows. •

Corollary 5.3.9. For any varieties $\mathscr{U}$ and $\mathscr{V}$ of inverse semigroups,

$$\operatorname{tr} \rho(\mathrm{Wr}(\mathscr{U},\mathscr{V})) = \operatorname{tr} \rho(\mathrm{Wr}(\mathscr{U} \vee \mathscr{G}, \mathscr{V})),$$

and

$$\mathrm{Wr}(\mathscr{U},\mathscr{V}) \vee \mathscr{G} = \mathrm{Wr}(\mathscr{U} \vee \mathscr{G}, \mathscr{V}).$$

Proof: By Theorem 5.3.8, since $\operatorname{tr} \rho(\mathscr{U}) = \operatorname{tr} \rho(\mathscr{U} \vee \mathscr{G})$ for any variety $\mathscr{U}$ of inverse semigroups **[P;XII.2.2]**. Also by **[P;XII.2.2]**,

$$\mathrm{Wr}(\mathscr{U},\mathscr{V}) \vee \mathscr{G} = \mathrm{Wr}(\mathscr{U} \vee \mathscr{G}, \mathscr{V}) \vee \mathscr{G} = \mathrm{Wr}(\mathscr{U} \vee \mathscr{G}, \mathscr{V}).$$ •

It is just a conjecture that $\mathrm{Wr}(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V}) = \mathrm{Wr}(\mathscr{U}_1, \mathscr{V}) \wedge \mathrm{Wr}(\mathscr{U}_2, \mathscr{V})$, for varieties $\mathscr{U}_1, \mathscr{U}_2$ and $\mathscr{V}$, but we do have the following special case, as promised at the end of section 4.3.

Proposition 5.3.10. Let $\mathscr{U}_1$ and $\mathscr{U}_2$ be varieties of groups and let $\mathscr{V}$ be a combinatorial variety of inverse semigroups. Then

$$Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V}) = Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V}).$$

Consequently, the mapping

$$\chi_{\mathscr{V}}: \mathscr{L}(\mathscr{G}) \to \mathscr{L}(\mathscr{I}) \text{ defined by } \mathscr{U} \to Wr(\mathscr{U}, \mathscr{V}) \quad (\mathscr{U} \in \mathscr{L}(\mathscr{G}))$$

is a lattice homomorphism. Moreover, $\chi_{\mathscr{V}}$ is one-to-one and so is an embedding of $\mathscr{L}(\mathscr{G})$ into $\mathscr{L}(\mathscr{I})$.

Proof: First of all, $Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V}) \subseteq Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V})$, by Proposition 4.3.1. Now,

$$Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V}) \cap \mathscr{G} = (Wr(\mathscr{U}_1, \mathscr{V}) \cap \mathscr{G}) \cap (Wr(\mathscr{U}_2, \mathscr{V}) \cap \mathscr{G}),$$

and by Theorem 4.3.7, this expression is $\mathscr{U}_1 \wedge \mathscr{U}_2$. Therefore, both $Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V})$ and $Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V})$ have the same group parts. By Corollary 5.3.9,

$$Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V}) \vee \mathscr{G} = Wr(\mathscr{G}, \mathscr{V})$$

and

$$\begin{aligned}(Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V})) \vee \mathscr{G} &= (Wr(\mathscr{U}_1, \mathscr{V}) \vee \mathscr{G}) \wedge (Wr(\mathscr{U}_2, \mathscr{V}) \vee \mathscr{G}) \\ &\qquad \text{([P;XII.2.8])} \\ &= Wr(\mathscr{G}, \mathscr{V}).\end{aligned}$$

It follows that both $Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V})$ and $Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V})$ belong to the same ν-class and so $Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V}) \subseteq Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V})$, since $Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V})$ is the maximum member of its ν-class. Therefore,

$$Wr(\mathscr{U}_1 \wedge \mathscr{U}_2, \mathscr{V}) = Wr(\mathscr{U}_1, \mathscr{V}) \wedge Wr(\mathscr{U}_2, \mathscr{V}).$$

By Proposition 4.3.5 and what we have just done, the map $\chi_{\mathscr{V}}$ is a homomorphism. To see that it is one-to-one, observe that $Wr(\mathscr{U}_1, \mathscr{V}) = Wr(\mathscr{U}_2, \mathscr{V})$ implies that $\mathscr{U}_1 = Wr(\mathscr{U}_1, \mathscr{V}) \cap \mathscr{G} = Wr(\mathscr{U}_2, \mathscr{V}) \cap \mathscr{G} = \mathscr{U}_2$, by Theorem 4.3.7, since $\mathscr{V}$ is combinatorial. •

5.4 Some facts about the semigroup $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$

Before we address some of the questions concerning the monoid $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$ alluded to at the end of the previous chapter, we introduce some terminology and notation.

Following the standard nomenclature of group theory, we call a variety of inverse semigroups *indecomposable* if it cannot be written as the product of two non-trivial factors. An obvious example is the variety $\mathscr{A}_p$, the variety of abelian groups of exponent p, for some prime p. If $\mathscr{A}_p = \mathrm{Wr}(\mathscr{U},\mathscr{V})$ then both $\mathscr{U}$ and $\mathscr{V}$ are subvarieties of $\mathscr{A}_p$, and hence each must be either $\mathscr{A}_p$ or $\mathscr{T}$. Since $\mathrm{Wr}(\mathscr{A}_p,\mathscr{A}_p) \neq \mathscr{A}_p$, it follows that $\mathscr{A}_p$ is indecomposable. A less obvious class of indecomposable varieties is the class of nilpotent varieties of groups [**N;24.34**].

We define a variety $\mathscr{V}$ of inverse semigroups to be *wreath-closed* if for every pair of varieties $\mathscr{U},\mathscr{W} \subseteq \mathscr{V}$, $\mathrm{Wr}(\mathscr{U},\mathscr{W}) \subseteq \mathscr{V}$. The most obvious example of a wreath-closed variety is $\mathscr{I}$, the variety of all inverse semigroups.

Proposition 5.4.1. Let $\mathscr{V}$ be a variety of inverse semigroups. Then $\mathscr{V}$ is an idempotent in $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$ if and only if $\mathscr{V}$ is wreath-closed.

Proof: If $\mathscr{V}$ is an idempotent then $\mathrm{Wr}(\mathscr{V},\mathscr{V}) = \mathscr{V}$. If $\mathscr{U},\mathscr{W} \subseteq \mathscr{V}$ then $\mathrm{Wr}(\mathscr{U},\mathscr{W}) \subseteq \mathrm{Wr}(\mathscr{V},\mathscr{V}) = \mathscr{V}$ and so $\mathscr{V}$ is wreath-closed. On the other hand, if $\mathscr{V}$ is wreath-closed then, in particular, $\mathrm{Wr}(\mathscr{V},\mathscr{V}) \subseteq \mathscr{V}$. Since $\mathscr{V} \subseteq \mathrm{Wr}(\mathscr{V},\mathscr{V})$, we have that $\mathrm{Wr}(\mathscr{V},\mathscr{V}) = \mathscr{V}$, and $\mathscr{V}$ is an idempotent. •

Exactly which varieties are wreath-closed is not obvious, though we can narrow down the class of candidates significantly. In the process we discover a familiar class of varieties which forms a subsemigroup of $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$.

Proposition 5.4.2. If $\mathscr{U}$ and $\mathscr{V}$ are combinatorial varieties of inverse semigroups, then $Wr(\mathscr{U},\mathscr{V})$ is combinatorial. The only non-combinatorial varieties which are wreath-closed are $\mathscr{G}$ and $\mathscr{I}$. Included among the combinatorial wreath-closed varieties are $\mathscr{T}$, $\mathscr{B}$ and $\mathscr{C}_n$ for all $n \in \omega$ (we remind the reader that $\mathscr{C}_0 = \mathscr{T}$ and $\mathscr{C}_1 = \mathscr{S}$).

Proof: Let $n \in \omega$. It is not too difficult to see that the Schützenberger graph of x^n relative to the variety $\mathscr{C}_n$ is just a single vetex with loops labelled x and x^{-1}. It follows that $d_{\mathscr{C}_n}(x^n)$ is y^n and $d_{\mathscr{C}_n}(x^{n+1})$ is y^{n+1} for some $y \in Y \cup Y^{-1}$. It then follows from Theorem 4.2.3 that $Wr(\mathscr{C}_n,\mathscr{C}_n)$ satisfies the equation $x^n = x^{n+1}$ and hence that $Wr(\mathscr{C}_n,\mathscr{C}_n) \subseteq \mathscr{C}_n$. As a result, not only have we shown that $\mathscr{C}_n$ is wreath closed for all $n \in \omega$, but, since every combinatorial variety is contained in some $\mathscr{C}_m$ for some $m \in \omega$, we have that if $\mathscr{U}$ and $\mathscr{V}$ are combinatorial varieties then so is $Wr(\mathscr{U},\mathscr{V})$.

Since $Wr(\mathscr{B},\mathscr{B}) = \mathscr{B} \vee \mathscr{B} = \mathscr{B}$, $\mathscr{B}$ is a wreath-closed variety. Since $Wr(\mathscr{U},\mathscr{V})$ is a group variety if and only if $\mathscr{U}$ and $\mathscr{V}$ are both group varieties, $\mathscr{G}$ is a wreath-closed variety. Clearly both $\mathscr{T}$ and $\mathscr{I}$ are wreath-closed varieties. Suppose that $\mathscr{V}$ is some arbitrary wreath-closed variety. If $\mathscr{V}$ is a group variety then $\mathscr{V}$ is wreath-closed if and only if $\mathscr{V} = \mathscr{G}$ or $\mathscr{T}$ [N;23.32]. Let $\mathscr{U} = \mathscr{V} \cap \mathscr{G}$ be the group part of $\mathscr{V}$. Since $Wr(\mathscr{U},\mathscr{U}) \subseteq Wr(\mathscr{V}, \mathscr{V}) \cap \mathscr{G} = \mathscr{V} \cap \mathscr{G} = \mathscr{U}$, we must have that $\mathscr{U}$ is a wreath-closed variety and so must be either $\mathscr{G}$ or $\mathscr{T}$. Since $Wr(\mathscr{S},\mathscr{G}) = \mathscr{G}^{max} = \mathscr{I}$, the only wreath-closed varieties containing $\mathscr{G}$ are $\mathscr{I}$ and $\mathscr{G}$ itself. It now follows that all wreath-closed varieties which do not belong to $\{\mathscr{G},\mathscr{I}\}$ are combinatorial varieties. •

Corollary 5.4.3. The class of combinatorial varieties of inverse semigroups forms a subsemigroup of $(\mathscr{L}(\mathscr{I}),Wr)$.

Proof: By Proposition 5.4.2, the class of combinatorial varieties forms a subsemigroup of $(\mathscr{L}(\mathscr{I}),Wr)$. •

The definition of wreath-closed suggests the following connection between wreath-closed varieties and certain subsemigroups of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$.

Proposition 5.4.4. If $\mathcal{V}$ is a wreath-closed variety then the interval $[\mathcal{T}, \mathcal{V}]$ is a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$. Moreover, $\mathcal{V}$ is a zero of this subsemigroup.

Proof: By the definition of wreath-closed, $[\mathcal{T}, \mathcal{V}]$ is closed under the operation Wr. That $\mathcal{V}$ is a zero for $[\mathcal{T}, \mathcal{V}]$ follows from the fact that, for any $\mathcal{U} \in [\mathcal{T}, \mathcal{V}]$, $\mathcal{V} \subseteq \mathrm{Wr}(\mathcal{V},\mathcal{U}) \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \mathrm{Wr}(\mathcal{U},\mathcal{V}) \subseteq \mathcal{V}$. •

Corollary 5.4.5. The lattice of varieties of groups forms a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$ with identity $\mathcal{T}$ and zero $\mathcal{G}$. For each $n \in \omega$, the interval $[\mathcal{T}, \mathcal{C}_n]$ is a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$ with identity $\mathcal{T}$ and zero $\mathcal{C}_n$. Also $\{\mathcal{T}, \mathcal{S}, \mathcal{B}\}$ is a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$ which is also a three-element chain (semilattice).

Proof: This follows from Propositions 5.4.2 and 5.4.4. That $\{\mathcal{T}, \mathcal{S}, \mathcal{B}\}$ is a semilattice follows from Proposition 5.4.1 and the fact that each variety is wreath-closed. From Proposition 5.4.4 we obtain that $\mathcal{B}$ is a zero for $\mathcal{S}$ which in turn is a zero for $\mathcal{T}$ and hence, $\{\mathcal{T}, \mathcal{S}, \mathcal{B}\}$ is a chain. •

Not all subsemigroups of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$ have a direct connection with wreath closed varieties, as the following illustrates.

Theorem 5.4.6. Let $\mathcal{F}_1$ be a subsemigroup of $(\mathcal{L}(\mathcal{G}),\mathrm{Wr})$ and let $\mathcal{F}_2$ be the family of varieties of inverse semigroups which have E-unitary covers over some variety in $\mathcal{F}_1$. Then $\mathcal{F}_2$ is a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$.

Proof: Let $\mathcal{U}, \mathcal{V} \in \mathcal{F}_2$ and suppose that $\mathcal{U}$ has E-unitary covers over $\mathcal{W} \in \mathcal{F}_1$ and $\mathcal{V}$ has E-unitary covers over $\mathcal{X} \in \mathcal{F}_1$. By Theorem 5.2.7, $\mathrm{Wr}(\mathcal{U},\mathcal{V})$ has E-unitary covers over $\mathrm{Wr}(\mathcal{W}, \mathcal{X})$. Since $\mathcal{F}_1$ is a subsemigroup of $\mathcal{L}(\mathcal{G})$, $\mathrm{Wr}(\mathcal{W}, \mathcal{X}) \in \mathcal{F}_1$ and as a consequence, $\mathrm{Wr}(\mathcal{U},\mathcal{V}) \in \mathcal{F}_2$. •

Proposition 5.4.7. Let $\mathcal{V}$ be a variety of inverse semigroups. The interval $[\mathcal{V}, \mathcal{I}]$ is a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$. If $\mathcal{V} = \mathcal{B}$ or $\mathcal{S}$ then $\mathcal{V}$ is a right identity of the semigroup $[\mathcal{V}, \mathcal{I}]$. Consequently, the only indecomposable varieties in $\mathcal{L}(\mathcal{I})$ are the indecomposable group varieties.

Proof: If $\mathcal{U}$ and $\mathcal{W}$ are varieties in the interval $[\mathcal{V}, \mathcal{I}]$ then $\mathcal{V} \subseteq \mathcal{U} \subseteq \mathrm{Wr}(\mathcal{U},\mathcal{W})$ and so $[\mathcal{V}, \mathcal{I}]$ is closed under the operation Wr. By Theorem 5.1.5, if $\mathcal{V}$ is either $\mathcal{B}$ or $\mathcal{S}$ then, for any $\mathcal{U} \in [\mathcal{V}, \mathcal{I}]$, $\mathrm{Wr}(\mathcal{U},\mathcal{V}) = \mathcal{U} \vee \mathcal{V} = \mathcal{U}$. As a result, any variety $\mathcal{V}$ which contains $\mathcal{S}$ cannot be indecomposable since $\mathrm{Wr}(\mathcal{V},\mathcal{S}) = \mathcal{V}$. That is, the only indecomposable varieties are the indecomposable group varieties. •

Some familiar classes of varieties of inverse semigroups do not form a subsemigroup of $(\mathcal{L}(\mathcal{I}),\mathrm{Wr})$.

Proposition 5.4.8. $\mathrm{Wr}(\mathcal{U},\mathcal{V})$ need not be completely semisimple if both $\mathcal{U}$ and $\mathcal{V}$ are completely semisimple. $\mathrm{Wr}(\mathcal{U},\mathcal{V})$ need not be cryptic if both $\mathcal{U}$ and $\mathcal{V}$ are cryptic.

Proof: Consider $\mathrm{Wr}(\mathcal{S}, \mathcal{G})$. Both of $\mathcal{S}$ and $\mathcal{G}$ are completely semisimple cryptic varieties but $\mathrm{Wr}(\mathcal{S}, \mathcal{G}) = \mathcal{I}$ which is neither completely semisimple nor cryptic. •

As far as Green's relations are concerned, we have the following. By a $\mathcal{J}$-trivial semigroup we mean a semigroup S in which $s \mathcal{J} t$ implies that $s = t$, for all $s,t \in S$.

Theorem 5.4.9. $(\mathcal{L}(\mathcal{S}),\mathrm{Wr})$ is a $\mathcal{J}$-trivial semigroup.

Proof: If the variety $\mathcal{U}$ is in the principal ideal generated by the variety $\mathcal{V}$ then $\mathcal{V} \subseteq \mathcal{U}$ by the definition of the operator Wr. Thus, if $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{J}$-related in $(\mathcal{L}(\mathcal{S}),\mathrm{Wr})$ then $\mathcal{U} = \mathcal{V}$. Therefore, $(\mathcal{L}(\mathcal{S}),\mathrm{Wr})$ is $\mathcal{J}$-trivial. •

A well-known result from the study of varieties of groups is that the semigroup of group varieties other than $\mathcal{G}$ is freely generated by the indecomposable varieties [**N;23.4**]. That is, every variety of groups can be uniquely factored as a product of non-trivial indecomposable varieties. This is not true for $\mathcal{L}(\mathcal{S})$, nor is it true for any of the intervals $[\mathcal{T}, \mathcal{C}_n]$, $n \in \omega$.

Proposition 5.4.10. $(\mathcal{L}(\mathcal{S}),\mathrm{Wr})$ is not freely generated by its indecomposable members. None of the subsemigroups $[\mathcal{T}, \mathcal{C}_n]$, $n \geq 2$, is freely generated by its indecomposable members.

Proof: Consider the variety $\mathcal{B}^1$. $\mathcal{S} \subseteq \mathcal{B}^1 \neq (\mathcal{B}^1)^{max}$ and so, by Proposition 5.3.6, $\mathrm{Wr}(\mathcal{B}^1,\mathcal{B}^1) = \mathrm{Wr}(\mathcal{B}^1,(\mathcal{B}^1)^{max})$ and so none of the semigroups mentioned in the statement of the theorem possess the property of unique factorization. As a result, none of the semigroups mentioned in the theorem are freely generated by their indecomposable members. •

Theorem 5.4.11. $(\mathcal{L}(\mathcal{G}),\mathrm{Wr})$ is a homomorphic image (as well as a subsemigroup) of $(\mathcal{L}(\mathcal{S}),\mathrm{Wr})$.

Proof: Define the mapping $\Theta : \mathcal{L}(\mathcal{S}) \rightarrow \mathcal{L}(\mathcal{G})$ by $\mathcal{V}\Theta = \mathcal{V} \cap \mathcal{G}$. Since $\mathrm{Wr}(\mathcal{U},\mathcal{V}) \cap \mathcal{G} = \mathrm{Wr}(\mathcal{U} \cap \mathcal{G},\mathcal{V} \cap \mathcal{G})$, for all $\mathcal{U},\mathcal{V} \in \mathcal{L}(\mathcal{S})$, it follows that Θ is a homomorphism. Since $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{S})$, Θ is surjective. •

Since $\mathscr{L}(\mathscr{G})$ is freely generated by its indecomposable members, the free semigroup on the indecomposable varieties of groups is a homomorphic image of $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$.

While the relation ν on the lattice of varieties is a congruence, the relation ν on the semigroup $(\mathscr{L}(\mathscr{I}),\mathrm{Wr})$ is only a right congruence.

Theorem 5.4.12. The relation ν on $\mathscr{L}(\mathscr{I})$ is a right (semigroup) congruence but not a (semigroup) congruence.

Proof: Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups and suppose that $\mathscr{U}\ \nu\ \mathscr{V}$. Then, for any variety $\mathscr{W}$,

$$\begin{aligned}\mathrm{Wr}(\mathscr{U},\mathscr{W})\cap\mathscr{G} &= \mathrm{Wr}(\mathscr{U}\cap\mathscr{G},\mathscr{W}\cap\mathscr{G}) && \text{(Theorem 4.3.7)}\\ &= \mathrm{Wr}(\mathscr{V}\cap\mathscr{G},\mathscr{W}\cap\mathscr{G}) && (\text{since } \mathscr{U}\ \nu\ \mathscr{V})\\ &= \mathrm{Wr}(\mathscr{V},\mathscr{W})\cap\mathscr{G} && \text{(Theorem 4.3.7)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Wr}(\mathscr{U},\mathscr{W})\vee\mathscr{G} &= \mathrm{Wr}(\mathscr{U}\vee\mathscr{G},\mathscr{W}) && \text{(Corollary 5.3.9)}\\ &= \mathrm{Wr}(\mathscr{V}\vee\mathscr{G},\mathscr{W}) && (\text{since } \mathscr{U}\ \nu\ \mathscr{V})\\ &= \mathrm{Wr}(\mathscr{V},\mathscr{W})\vee\mathscr{G} && \text{(Corollary 5.3.9).}\end{aligned}$$

Therefore, $\mathrm{Wr}(\mathscr{U},\mathscr{W})\ \nu\ \mathrm{Wr}(\mathscr{V},\mathscr{W})$ and so ν is a right (semigroup) congruence.

To see that ν is not a semigroup congruence, consider the following varieties. Since $\mathscr{B}^1$ is combinatorial, the ν-class of $\mathscr{A}_2\vee\mathscr{B}^1$, where $\mathscr{A}_2$ is the variety of abelian groups of exponent two, is the interval $[\,\mathscr{A}_2\vee\mathscr{B}^1, \mathscr{A}_2\circ\mathscr{B}^1\,]$. Thus, $\mathscr{A}_2\vee\mathscr{B}^1$ and $\mathscr{A}_2\circ\mathscr{B}^1$ are ν-related. We claim that $\mathrm{Wr}(\mathscr{S},\mathscr{A}_2\vee\mathscr{B}^1)$ is not ν-related to the variety $\mathrm{Wr}(\mathscr{S},\mathscr{A}_2\circ\mathscr{B}^1)$. First of all, by Theorem 5.3.3,

$$\mathrm{Wr}(\mathscr{S},\mathscr{A}_2\vee\mathscr{B}^1)=(\mathscr{A}_2\vee\mathscr{B}^1)^{\max}=\mathscr{A}_2^{\max}$$

and, by Theorem 5.3.3, the associativity of Wr and Theorem 4.3.4,

$Wr(\mathscr{S},\mathscr{A}_2 \circ \mathscr{B}^1) = Wr(\mathscr{S},Wr(\mathscr{A}_2,\mathscr{B}^1) = Wr(Wr(\mathscr{S},\mathscr{A}_2),\mathscr{B}^1) = Wr(\mathscr{A}_2^{max},\mathscr{B}^1)$.

Let w be the word $x_1x_2x_1^{-1}x_2^{-1}$. Now $\mathscr{A}_2^{max}$ satisfies the identity $ww^{-1} = w^{-1}w$. This can easily be seen by considering the Cayley graph of the $\mathscr{A}_2$-free group and using Theorem 4.2.3. $\mathscr{B}^1$ also satisfies this identity as it is contained in $\mathscr{A}_2^{max}$. The Schützenberger graph of ww^{-1} (and $w^{-1}w$) is the one given in Figures 2.2, 4.1 and 4.2. From the Schützenberger graph we read $d_{\mathscr{V}}(ww^{-1}) = y_1y_2y_3^{-1}y_4^{-1}y_4y_3y_2^{-1}y_1^{-1}$ and $d_{\mathscr{V}}(w^{-1}w) = y_4y_3y_2^{-1}y_1^{-1}y_1y_2y_3^{-1}y_4^{-1}$. While it is true that $\mathscr{A}_2$ satisfies the identity $d_{\mathscr{V}}(ww^{-1}) = d_{\mathscr{V}}(w^{-1}w)$, $\mathscr{A}_2^{max}$ does not. This is because in the Cayley graph of the $\mathscr{A}_2$-free group on $\{y_1,y_2,y_3,y_4\}$ (which is a 4-cube), the paths corresponding to $d_{\mathscr{V}}(ww^{-1})$ and $d_{\mathscr{V}}(w^{-1}w)$ do not use precisely the same set of edges. It follows that $Wr(\mathscr{A}_2^{max},\mathscr{B}^1)$ does not satisfy the identity $ww^{-1} = w^{-1}w$. Therefore, the fully invariant congruences corresponding to $Wr(\mathscr{S}, \mathscr{A}_2 \vee \mathscr{B}^1)$ and $Wr(\mathscr{S}, \mathscr{A}_2 \circ \mathscr{B}^1)$ do not have the same trace and, as a consequence, these two varieties cannot be ν-related. It follows that ν is not a semigroup congruence on $(\mathscr{L}(\mathscr{I}),Wr)$. •

CHAPTER SIX

An Infinite Chain of Varieties

As was pointed out in the previous chapter, Kleiman [**K1**] showed that $\mathcal{L}(\mathcal{S}\mathcal{I})$ is isomorphic to three copies of $\mathcal{L}(\mathcal{G})$ and that each of the intervals $[\mathcal{S}, \mathcal{S} \vee \mathcal{G}]$ and $[\mathcal{B}, \mathcal{B} \vee \mathcal{G}]$ is isomorphic to $\mathcal{L}(\mathcal{G})$ (and so, as a consequence, $\mathcal{L}(\mathcal{S}\mathcal{I})$ is a modular lattice). $\mathcal{L}(\mathcal{S}\mathcal{I})$ is sometimes referred to colloquially as the first three layers of the lattice $\mathcal{L}(\mathcal{I})$. The 'fourth' layer, $[\mathcal{B}^1, \mathcal{B}^1 \vee \mathcal{G}]$, is not nearly as nice. While it is a modular lattice (the collection of congruences on an inverse semigroup which have the same trace forms a complete modular sublattice of the lattice of congruences on that semigroup), the ν-classes of its members are not all trivial and, as a result, $\mathcal{L}(\mathcal{B}^1 \vee \mathcal{G})$ is not modular, and hence $\mathcal{L}(\mathcal{I})$ is not modular ([**Re2**] provides one example). In this chapter we show that the ν-class of $\mathcal{B}^1 \vee \mathcal{A}$, for any abelian group variety $\mathcal{A}$, contains an infinite chain of varieties and so is far from being trivial. The technique used is interesting in that we are only required to know the Schützenberger graphs of a given collection of words with respect to $\mathcal{B}^1$ (and not the entire $\mathcal{B}^1$-free object on countably infinite X) in order to construct inverse semigroups which are then shown to generate distinct varieties. We remark that the variety $\mathcal{B}^1$ has proved to be rather enigmatic. Even though it is generated by a small (6-element) inverse semigroup and $\mathcal{L}(\mathcal{B}^1)$ is just a 4-element chain, its members are not easily characterized and, as Kleiman proved in [K2], it is not defined by a finite set of identities.

6.1 The variety $\mathcal{B}^1$

In this section we construct inverse semigroups which belong to the variety $\mathcal{B}^1$ which, in subsequent sections, will be used to construct inverse semigroups in $Wr(\mathcal{U},\mathcal{B}^1)$, where $\mathcal{U}$ is a variety of abelian groups of exponent n, for some $n \in \omega$. These semigroups will be used to define an infinite collection of varieties in the interval $[\mathcal{U} \vee \mathcal{B}^1, Wr(\mathcal{U},\mathcal{B}^1)]$. Throughout the remainder of this chapter ρ will denote the fully invariant congruence on $F\mathcal{I}(X)$ corresponding to $\mathcal{B}^1$.

Before we proceed, we require some notation. For any word $w \in X \cup X^{-1}$, denote by w_A the word obtained from w by deleting all occurrences of variables not in A. For example, $(x_1x_2x_1^{-1}x_3x_2x_1)_{\{x_1\}}$ is the word $x_1x_1^{-1}x_1$.

Lemma 6.1.1. Let w and v be words over $X \cup X^{-1}$. Then $w \rho v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \, \rho(\mathcal{B}) \, v_A$.

Proof: $w \rho v$ if and only if B_2^1 satisfies the equation $w = v$. Since B_2^1 possesses an identity, B_2^1 satisfies the equation $w = v$ if and only if B_2 satisfies $w_A = v_A$ for all $A \subseteq c(w_A) = c(v_A)$. This is equivalent to $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \, \rho(\mathcal{B}) \, v_A$. •

Corollary 6.1.2. Let w and v be words over $X \cup X^{-1}$. Then $w \rho v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \rho v_A$.

Proof: If $w \rho v$ then by Lemma 6.1.1, $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \, \rho(\mathcal{B}) \, v_A$. But then for any $A \subseteq c(w) = c(v)$, for all $B \subseteq A$, $B \neq \emptyset$, $w_B \, \rho(\mathcal{B}) \, v_B$ and so by Lemma 6.1.1, $w_A \rho v_A$. On the other hand, if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \rho v_A$, then for all $A \subseteq c(w)$, $A \neq \emptyset$, $w_A \, \rho(\mathcal{B}) \, v_A$. As a consequence of Lemma 6.1.1, $w \rho v$. •

Lemma 6.1.3. If $S \in \mathscr{B}^1$ then $S^1 \in \mathscr{B}^1$.

Proof: Suppose that $\mathscr{B}^1$ satisfies the equation $w = v$, where $c(w) = c(v) = \{x_1,\ldots,x_n\}$. Let $s_1,\ldots,s_n$ be arbitrarily chosen elements of S^1 with repetitions allowed. Suppose that $s_{i_1},\ldots,s_{i_k}$ are each the identity of S^1. Then S^1 satisfies $w[s_1,\ldots,s_n] = v[s_1,\ldots,s_n]$ if S satisfies $w_A[s_1,\ldots,s_n] = v_A[s_1,\ldots,s_n]$ where $A = \{x_1,\ldots,x_n\}\backslash\{x_{i_1},\ldots,x_{i_k}\}$. Since $S \in \mathscr{B}^1$, S does satisfy $w_A[s_1,\ldots,s_n] = v_A[s_1,\ldots,s_n]$ by Corollary 6.1.2 and so, as a result, $w[s_1,\ldots,s_n] = v[s_1,\ldots,s_n]$ is true in S^1. Since the s_i were chosen arbitrarily, S^1 satisfies the equation $w = v$. Therefore, $S^1 \in \mathscr{B}^1$. •

We require some further notation for this section. Let $w \in (X \cup X^{-1})^+$. We write $w \equiv v$ to mean w and v are identical words, letter for letter, over a common alphabet (in this case $X \cup X^{-1}$). We say the word v is a *cyclic shift* of w if $w \equiv u_1u_2$ and $v \equiv u_2u_1$ for words u_1,u_2 over the alphabet of w. For each $n \in \omega$, we denote by τ_n the equation $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1} \in E$. Observe that if w is the word $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ then any cyclic shift of w can be written $y_1y_2\ldots y_ny_1^{-1}y_2^{-1}\ldots y_n^{-1}$.

The remainder of 6.1 is devoted to a construction of a family of inverse semigroups $\{S(\tau_n): n \in \omega\}$ each of which belongs to the variety $\mathscr{B}^1$. For each $n \in \omega$, $S(\tau_n)$ is obtained from the $\mathscr{B}^1$-free inverse semigroup by first identifying the ideal consisting of those elements whose $\mathscr{R}$-class does not lie above the $\mathscr{R}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}\rho$ (which results in an ideal extension of the $\mathscr{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}\rho$, a Brandt semigroup) and then mapping this semigroup into the translational hull of the principal factor corresponding to the $\mathscr{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}\rho$. In order to do this we require some knowledge of the $\mathscr{D}$-class of $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}\rho$.

Lemma 6.1.4. Let $w = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ and suppose that $v = y_1y_2\ldots y_ny_1^{-1}y_2^{-1}\ldots y_n^{-1}$ is a cyclic shift of w. Let $a \in X \cup X^{-1}$.

a) $v\rho$ is an idempotent;

b) $(va\rho) \mathscr{R} (v\rho)$ if and only if $a = y_1$ or $a = y_n$.

Proof: a) $\mathscr{B}^1$ has E-unitary covers over the variety $\mathscr{A}_2$ of abelian groups of exponent two and so is contained in $\mathscr{A}_2^{max}$. Since $\mathscr{A}_2$ satisfies the equation $v = v^2$, $\mathscr{A}_2^{max}$ and hence $\mathscr{B}^1$ satisfies $v = v^2$. Thus, $v\rho$ is an idempotent.

b) Since $v\rho$ is an idempotent, if $a = y_1$ or $a = y_n$ then $(va\rho) \mathscr{R} (v\rho)$. On the other hand, suppose that $(va\rho) \mathscr{R} (v\rho)$. Then $vaa^{-1}v^{-1} \rho vv^{-1}$ and so $c(va) = c(v)$. Thus, $a \in c(v)$. But $(va\rho) \mathscr{R} (v\rho)$ also implies that $vaa^{-1} \rho v$. If $a = y_i^{-1}$ for some i, then $(vaa^{-1})_{\{y_i\}} = y_iy_i^{-1}y_i^{-1}y_i \rho_{\mathscr{B}} y_i^2$, while $v_{\{y_i\}} = y_iy_i^{-1} \not\rho_{\mathscr{B}} y_i^2$ and so, by Lemma 6.1.2, $vaa^{-1} \not\rho v$. Therefore, $a = y_i$ for some i. If $1 < i < n$ then $(vaa^{-1})_{\{y_1,y_i,y_n\}} = y_1y_iy_ny_1^{-1}y_i^{-1}y_n^{-1}y_iy_i^{-1}$ and $v_{\{y_1,y_i,y_n\}} = y_1y_iy_ny_1^{-1}y_i^{-1}y_n^{-1}$. If a is any non-idempotent element of B_2, then substituting a for y_1 and y_n and substituting a^{-1} for y_i, yields that $(vaa^{-1})_{\{y_1,y_i,y_n\}} \not\rho_{\mathscr{B}} v_{\{y_1,y_i,y_n\}}$. As a consequence, y_i must be either y_1 or y_n. •

Lemma 6.1.5. Let $w = x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$ and suppose that u is an initial segment of w with $w \equiv uu'$. Let $a \in X \cup X^{-1}$. Then $wu\rho \mathscr{R} wua\rho$ if and only if a is the initial letter of u′ or a^{-1} is the terminal letter of u, unless u is the empty word, in which case a^{-1} is the terminal letter of u′.

Proof: First suppose that $wu\rho \mathscr{R} wua\rho$. $wu\rho = uu'u\rho \mathscr{L} u'u\rho$ since u′u is a cyclic shift of w and any cyclic shift of w is an idempotent modulo ρ. Therefore, $wu\rho \mathscr{R} wua\rho$ if and only if $u'u\rho \mathscr{R} u'ua\rho$. (This follows from the more general result that $t \mathscr{L} s$ implies that $t \mathscr{R} ta$ if and only if $s \mathscr{R} sa$) Since u′u is a cyclic shift of w, we have by Lemma 6.1.4 that a is either the initial letter of u′ or a^{-1} is the terminal letter of u. For the converse, first note that if a is the initial letter of u′ then ua is an initial segment of w and so, since $w\rho$ is an idempotent, $wu\rho \mathcal{R} wua\rho$. If a^{-1} is the terminal letter of u then letting $u \equiv u^*a^{-1}$ we obtain that $wua \equiv wu^*a^{-1}a \equiv u^*a^{-1}u'u^*a^{-1}a$. Since $a^{-1}u'u^*$ is a cyclic shift of w, $a^{-1}u'u^*\rho$ is an idempotent by Lemma 6.1.4 (a) and as a result, $wua \equiv wu^*a^{-1}a \equiv u^*a^{-1}u'u^*a^{-1}a \;\rho\; u^*a^{-1}aa^{-1}u'u^* \;\rho\; u^*a^{-1}u'u^* \equiv uu'u^* \equiv wu^*$. It is now immediate that $wu\rho \mathcal{R} wu^*\rho = wua\rho$. Note that if u is the empty word then the statement becomes $w\rho \mathcal{R} wa\rho$ if and only if a is the initial letter of w or a^{-1} is the terminal letter of w (which is the terminal letter of u′, in this case), by Lemma 6.1.4. •

Lemma 6.1.6. Let $w = x_1x_2...x_nx_1^{-1}x_2^{-1}...x_n^{-1}$. For any word v over $X \cup X^{-1}$, $w\rho \mathcal{R} v\rho$ if and only if $v \;\rho\; wu$ for some initial segment u of w.

Proof: Suppose that $w\rho \mathcal{R} v\rho$, say $wa_1...a_k \;\rho\; v$, where $a_1,...,a_k \in X \cup X^{-1}$. We prove by induction on k that $wa_1...a_k \;\rho\; v$ implies that $wa_1...a_k \;\rho\; wu$ for some initial segment u of w. If k = 1 then $wa_1\rho \mathcal{R} w\rho$ implies by Lemma 6.1.4 that $a_1 = x_1$ or x_n. If $a = x_1$ then a_1 is an initial segment of w already. If $a_1 = x_n$ then $wa_1 \;\rho\; wwx_n$.
Now $wwx_n \equiv x_1...x_nx_1^{-1}...x_{n-1}^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_{n-1}^{-1}]x_n^{-1}x_n \;\rho$
$x_1...x_nx_1^{-1}...x_{n-1}^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_{n-1}^{-1}]$ since $[x_n^{-1}x_1...x_nx_1^{-1}...x_{n-1}^{-1}]$ is a cyclic shift of w and so $[x_n^{-1}x_1...x_nx_1^{-1}...x_{n-1}^{-1}]\rho$ is an idempotent.
But $x_1...x_nx_1^{-1}...x_{n-1}^{-1}[x_n^{-1}x_1...x_nx_1^{-1}...x_{n-1}^{-1}] \equiv wx_1...x_nx_1^{-1}...x_{n-1}^{-1}$ and so as a consequence, $v \;\rho\; wx_1...x_nx_1^{-1}...x_{n-1}^{-1}$. Now suppose that k > 1. $wa_1...a_k\rho \mathcal{R} w\rho$ implies that $w\rho \mathcal{R} wa_1...a_{k-1}\rho$ and so, by the induction hypothesis, $wa_1...a_{k-1} \;\rho\; wu$ for some initial segment u of $w \equiv uu'$. By Lemma 6.1.5, $wu\rho \mathcal{R} wua_k\rho$ implies that a_k is the initial letter of u′ or a_k^{-1} is the terminal letter of u. If a is the initial letter of u′ then $v \;\rho\; wa_1...a_k \;\rho\; wua_k$ and ua_k is an initial segment of w. If a_k^{-1} is the terminal letter of u then setting $u \equiv b_1...b_m$ we obtain that $v \;\rho\; wa_1...a_k \;\rho\; wua_k$ and $wua_k \equiv wb_1...b_mb_m^{-1}$ $\equiv b_1...b_{m-1}[b_mu' b_1...b_{m-1}]b_mb_m^{-1} \;\rho\; b_1...b_{m-1}[b_mu' b_1...b_{m-1}]$ since $[b_mu' b_1...b_{m-1}]$ is a cyclic shift of w and so must be an idempotent modulo ρ. But

$b_1 \ldots b_{m-1}[b_m u' b_1 \ldots b_{m-1}] \equiv wb_1 \ldots b_{m-1}$ and so $v\ \rho\ wb_1 \ldots b_{m-1}$ and $b_1 \ldots b_{m-1}$ is an initial segment of w. Since $w\rho$ is an idempotent, the converse is immediate. •

Schützenberger graphs provide a concise, visual representation of a $\mathscr{R}$-class. Because of this, in the following theorem we describe the $\mathscr{R}$-classes of the words $\{x_1x_2 \ldots x_nx_1^{-1}x_2^{-1} \ldots x_n^{-1} : n \in \omega, n > 1\}$ relative to the variety $\mathscr{B}^1$ in this way.

Theorem 6.1.7. Let $w = x_1x_2 \ldots x_nx_1^{-1}x_2^{-1} \ldots x_n^{-1}$. The following graph is V-isomorphic to the Schützenberger graph of w relative to $\mathscr{B}^1$, where v_1 is both the start and end vertex.

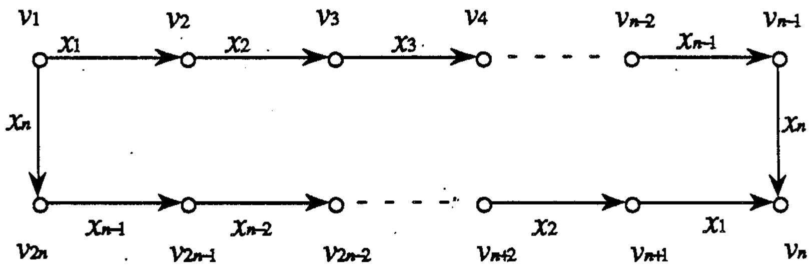

Figure 6.1. The Schützenberger graph of $w = x_1x_2 \ldots x_nx_1^{-1}x_2^{-1} \ldots x_n^{-1}$ with respect to $\mathscr{B}^1$.

Proof: By Lemma 6.1.6 there are at most 2n vertices in the Schützenberger graph Γ of w relative to $\mathscr{B}^1$ as there are 2n initial segments of w not identical to w. It is a simple exercise to verify, using Lemma 6.1.1, that if u and u′ are two proper initial segments of w (that is, u nor u′ is identical to w) then $wu\ \rho\ wu'$ implies that $u \equiv u'$. By Lemma 6.1.5, $(wu_1\rho, x, wu_2\rho)$ is an edge of Γ if and only if x^{-1} is the terminal letter of u_1 or x is the initial letter of u_1', where $u_1u_1' \equiv w$. If x is the initial letter of u_1', then wu_2 and wu_1x are ρ-equivalent with both u_1x and u_2 initial segments of w. Thus, $u_1x \equiv u_2$. If x^{-1} is the terminal letter of u_1 then $u_1 \equiv u_1{}^*x^{-1}$ and $wu_1{}^*x^{-1}x \ \rho \ wu_2$. Since $wu_1{}^*\rho \ \mathscr{R} \ wu_1 \equiv wu_1{}^*x^{-1}\rho$, we have that $wu_1{}^* \ \rho \ wu_1{}^*x^{-1}x \ \rho \ wu_2$. Since both $u_1{}^*$ and u_2 are initial segments of w, $wu_1{}^* \equiv wu_2$ and so $wu_2x^{-1} \equiv wu_1$. Finally, if u_1 is the empty word and x^{-1} is the terminal letter of w then x^{-1} is the terminal letter of $ww \equiv ww^*x^{-1} \ \rho \ w$ and hence, $ww^*x^{-1}x \ \rho \ wu_2$. But, $ww^*x^{-1}x \ \rho \ ww^*$ and both w^* and u_2 are initial segments of w, so $wu_2 \equiv ww^*$, whence $wu_2x^{-1} \equiv ww$.

It follows from these remarks that Γ is V-isomorphic to the graph described above via the map which sends $wu\rho$ to $v_{|u|+1}$, for all proper initial segments u of w. •

Definition 6.1.8. Let F be the $\mathscr{B}^1$-free inverse semigroup on $X = \{x_i : i \in \omega\}$. Let w_n be the word $x_1 \ldots x_n x_1{}^{-1} \ldots x_n{}^{-1}$ for each $n \in \omega$. Denote the ideal $\{v \in F : J_v \ngeq J_{w_n\rho}\}$ of F by $I(\tau_n)$ and let $J(\tau_n) = F / I(\tau_n)$. Now $J(\tau_n)$ is an ideal extension of $J^0_{w_n\rho}$ which is isomorphic to $B(\{1\},2n)$. Let $S(\tau_n)$ be the image of $J(\tau_n)$ under the canonical homomorphism into the translational hull $\Omega(J^0_{w_n\rho})$ of $J^0_{w_n\rho}$.

Lemma 6.1.9. $S(\tau_n) \in \mathscr{B}^1$ and $S(\tau_n)^1 \in \mathscr{B}^1$, for all $n \in \omega$, $n \geq 2$.

Proof: $S(\tau_n)$ is a homomorphic image of the $\mathscr{B}^1$-free inverse semigroup on X and so is an element of $\mathscr{B}^1$. $S(\tau_n)^1 \in \mathscr{B}^1$ by Lemma 6.1.3. •

In the following section we will use the $S(\tau_n)$ to construct a family of inverse semigroups which belong to $Wr(\mathscr{A}_m, \mathscr{B}^1)$ but not to $\mathscr{A}_m \vee \mathscr{B}^1$, for $m \in \omega$. Before we do so, we describe the $S(\tau_n)$.

Lemma 6.1.10. $S(\tau_n)$ is isomorphic to the Wagner representation of the $\mathscr{B}^1$-free inverse semigroup on X restricted to $R_{w_n\rho}$.

Proof: By Theorem 2.6.1, since the $\mathscr{B}^1$-free inverse semigroup is completely semisimple. •

An added advantage to using the Schützenberger graph description in Theorem 6.1.7 is that we can read directly from the graph the image of any word of $J(\tau_n)$ under the canonical homomorphism into $\Omega(J^0_{w_n\rho}) \cong \mathscr{I}(R_{w_n\rho})$. $S(\tau_n)$ is generated by the image of the x_i under the canonical homomorphism and, for each $i = 1,\ldots,n$, the domain of the image of x_i is the set of vertices v for which there is an edge labelled by x_i starting at v and v is mapped to the terminal vertex of that edge. It is straightforward to verify that $S(\tau_n)$ is (isomorphic to) the inverse subsemigroup of $\mathscr{I}(R_{w_n\rho})$ generated by $\{\alpha_i : i = 1,\ldots,n\}$ where for each i,

$$d\alpha_i = \{w_n x_1 \ldots x_{i-1}\rho,\ w_n x_1 \ldots x_n x_1^{-1} \ldots x_i^{-1}\rho\}$$

and

$$\begin{aligned} w_n x_1 \ldots x_{i-1}\rho\alpha_i &= w_n x_1 \ldots x_i\rho \\ w_n x_1 \ldots x_n x_1^{-1} \ldots x_i^{-1}\rho\alpha_i &= w_n x_1 \ldots x_n x_1^{-1} \ldots x_i^{-1} x_i \\ &\rho\ w_n x_1 \ldots x_n x_1^{-1} \ldots x_{i-1}^{-1}. \end{aligned}$$

6.2 Inverse semigroups in $Wr(\mathscr{A}_m,\mathscr{B}^1)$

The semigroups constructed in section 6.1 can be used to construct semigroups in $Wr(\mathscr{A}_m,\mathscr{B}^1)$ for $m \in \omega$. By Lemma 6.1.10, $S(\tau_n)$ can be represented as an inverse subsemigroup of $\mathscr{I}(R_{w_n})$ for all $n \in \omega$. Thus, for any group G belonging to $\mathscr{A}_m$, $m \in \omega$, $G \text{ wr } (S(\tau_n),R_{w_n}) \in Wr(\mathscr{A}_m,\mathscr{B}^1)$. The semigroups we construct in this section are inverse subsemigroups of semigroups of this form and so belong to $Wr(\mathscr{A}_m,\mathscr{B}^1)$.

For each $n \in \omega$, $n \geq 2$, let C_n denote the cyclic group of order n.

Definition 6.2.1. Let $m,n \in \omega$, $m,n \geq 2$. Let 1 denote the identity of C_m and let g be a generator of C_m. Let $A_{m,n} \subseteq C_m$ wr $(S(\tau_n),R_{w_n})$ be defined as follows:

Let $\{\alpha_i : i = 1,\ldots,n\}$ be the generators of $S(\tau_n)$ as described at the end of the previous section. For $i = 1,\ldots,n\text{-}1$, define the map ϕ_i from R_{w_n} into C_m by setting $\mathbf{d}\phi_i = \mathbf{d}\alpha_i = \{w_n x_1 \ldots x_{i-1}\rho,\ w_n x_1 \ldots x_n x_1^{-1} \ldots x_i^{-1}\rho\}$ and defining $(w_n x_1 \ldots x_{i-1}\rho)\phi_i = 1$, $(w_n x_1 \ldots x_n x_1^{-1} \ldots x_i^{-1}\rho)\phi_i = 1$. Define the map ϕ_n from R_{w_n} into C_m by setting $\mathbf{d}\phi_n = \mathbf{d}\alpha_n = \{w_n x_1 \ldots x_{n-1}\rho,\ w_n\rho\}$ and defining $(w_n x_1 \ldots x_{n-1}\rho)\phi_n = 1$, $(w_n\rho)\phi_n = g$. Then $(\phi_i,\alpha_i) \in C_m$ wr $(S(\tau_n),R_{w_n})$ for $i = 1,\ldots,n$.

Let $A_{m,n} = \{(\psi,\beta) \in C_m \text{ wr } (S(\tau_n),R_{w_n}) : |\mathbf{d}\psi| = |\mathbf{d}\beta| \leq 1\}$

$\cup\ \{(\phi_i,\alpha_i) : i = 1,\ldots,n\}$.

Define $T_{m,n}$ to be the inverse subsemigroup of C_m wr $(S(\tau_n),R_{w_n})$ generated by $A_{m,n}$. Observe that $T_{m,n}$ is an ideal extension of a Brandt semigroup over the group C_m. It is not difficult to see that $T_{m,n}$ is in fact the following:

$$\{(\psi,\beta) \in C_m \text{ wr } (S(\tau_n),R_{w_n}) : |\mathbf{d}\psi| = |\mathbf{d}\beta| \leq 1\} \ \cup$$
$$\{(\phi_i,\alpha_i),(\phi_i,\alpha_i)^{-1},(\phi_i,\alpha_i)(\phi_i,\alpha_i)^{-1},(\phi_i,\alpha_i)^{-1}(\phi_i,\alpha_i) : i = 1,\ldots,n\}.$$

Lemma 6.2.2. For each $m,n \in \omega$, $m,n \geq 2$,

a) $T_{m,n} \in \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$ but $T_{m,n} \notin \mathscr{B}^1$;

b) $T^1_{m,n} \in \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$ but $T^1_{m,n} \notin \mathscr{B}^1$;

c) $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n} \rangle \subseteq \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$;

d) $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T^1_{m,n} \rangle \subseteq \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$.

Proof: $T^1_{m,n}$ is an inverse subsemigroup of C_m wr $(S(\tau_n)^1,R_{w_n})$ and $S(\tau_n)^1 \in \mathscr{B}^1$ by Lemma 6.1.9. Thus, $T^1_{m,n} \in \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$ by the definition of the Wr operator. As a consequence, $T_{m,n} \in \mathrm{Wr}(\mathscr{A}_m,\mathscr{B}^1)$ since $T_{m,n}$ is an inverse subsemigroup of $T^1_{m,n}$. On the other hand, $T_{m,n}$ is an ideal extension of a Brandt semigroup over C_m and so contains a subgroup isomorphic to C_m. Thus, $T_{m,n} \notin \mathscr{B}^1$ since $\mathscr{B}^1$ is a combinatorial variety. Since

$T_{m,n}$ is an inverse subsemigroup of $T^1_{m,n}$, we also have that $T^1_{m,n} \notin \mathscr{B}^1$. This proves both a) and b).

Both $T^1_{m,n}$ and $T_{m,n}$ contain subgroups isomorphic to C_m and so $\mathscr{A}_m \subseteq \langle T^1_{m,n} \rangle$ and $\mathscr{A}_m \subseteq \langle T_{m,n} \rangle$ since $\mathscr{A}_m$ is generated by C_m. The natural homomorphism onto the second coordinate maps $T_{m,n}$ onto an inverse semigroup isomorphic to $S(\tau_n) \in \mathscr{B}^1$, and maps $T^1_{m,n}$ onto an inverse semigroup isomorphic to $S(\tau_n)^1 \in \mathscr{B}^1$. Since both $S(\tau_n)$ and $S(\tau_n)^1$ contain copies of B^1_2, it follows that $\mathscr{B}^1 \subseteq \langle T^1_{m,n} \rangle$ and $\mathscr{B}^1 \subseteq \langle T_{m,n} \rangle$. Consequently, we have that $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n} \rangle$ and $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T^1_{m,n} \rangle$. It is immediate from parts a) and b) that $\langle T_{m,n} \rangle \subseteq \mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ and $\langle T^1_{m,n} \rangle \subseteq \mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$. This completes the proofs of c) and d). •

Lemma 6.2.3. Let $m,n \in \omega$, $m,n \geq 2$. Neither $T_{m,n}$ nor $T^1_{m,n}$ satisfies the equation τ_n.

Proof: Substitute (ϕ_i,α_i) for x_i, $i = 1,\ldots,n$. •

In the following lemma we use the term *kernel* to mean the minimum nonzero ideal of an inverse semigroup, if it exists.

Lemma 6.2.4. Let $m,n \in \omega$, $m,n \geq 2$. $T_{m,n}$ satisfies the equation τ_k for $k < n$.

Proof: Towards a contradiction, suppose that $T_{m,n}$ does not satisfy τ_k for some $k < n$. Assume that k is the least such integer and let $(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k) \in T_{m,n}$ be such that $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}[(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)] = (\psi,\beta)$ is not an idempotent in $T_{m,n}$.

We first make a few observations.

i) $|d\beta| = 1$: If $|d\beta| = 0$ then we immediately have that (ψ,β) is an idempotent. If $|d\beta| = 2$ then the (ψ_i,β_i) all belong to the same $\mathscr{D}$-class, namely, the $\mathscr{D}$-class D of (ψ,β). [This is because $T_{m,n}$ is completely semisimple and so $\mathscr{D} = \mathscr{J}$. Thus, the $\mathscr{D}$-class of

(ψ,β) is contained in the $\mathscr{D}$-class of (ψ_i,β_i) for all i. But if $|\mathbf{d}\beta| = 2$, then the $\mathscr{D}$-class of (ψ,β) is a maximal $\mathscr{D}$-class in $T_{m,n}$ and so (ψ,β) is $\mathscr{D}$-related to (ψ_i,β_i) for all i.] But D^0 is a Brandt semigroup and as such satisfies τ_k. Since $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}[(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)] = (\psi,\beta)$ in D^0 and $(\psi,\beta) \neq 0$, we conclude that, in this case, (ψ,β) is an idempotent. The only remaining possibility is that $|\mathbf{d}\beta| = 1$.

ii) If $\mathbf{d}\beta = \{v\}$ then $v\beta = v$. We know that β is an idempotent of $(S(\tau_n),R_{w_n})$ since the natural homomorphism of $T_{m,n}$ onto its second coordinate has image $S(\tau_n)$ which, by Lemma 6.1.9, is a member of $\mathscr{B}^1$ and $\mathscr{B}^1$ satisfies the equation τ_k. Thus, $v\beta = v$.

iii) If (ψ,β) is not an idempotent then for any cyclic shift $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}$ of $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}$ we have that $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}[(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)]$ is not an idempotent. To see this note that if $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}$ is a cyclic shift of $x_1 \ldots x_k x_1^{-1} \ldots x_k^{-1}$ then $y_1 \ldots y_n y_1^{-1} \ldots y_n^{-1}[(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)] = (\psi',\beta')$ can be expressed as $(\varphi_1,\gamma_1)(\varphi_2,\gamma_2)$ where $(\psi,\beta) = (\varphi_2,\gamma_2)(\varphi_1,\gamma_1)$. If $\{v\} = \mathbf{d}\beta$ then $v\gamma_2 \in \mathbf{d}\beta'$ because $v\gamma_2\gamma_1\gamma_2 = v\gamma_2$ since $v\gamma_2\gamma_1 = v\beta = v$. Then $v\gamma_2\psi' = (v\gamma_2\varphi_1)(v\gamma_2\gamma_1\varphi_2) = (v\gamma_2\varphi_1)(v\varphi_2) = (v\varphi_2)(v\gamma_2\varphi_1)$ since C_m is abelian. But $(v\varphi_2)(v\gamma_2\varphi_1) = v\psi$ which is not an idempotent and so, as a result, (ψ',β') is not an idempotent.

iv) For some $i \in \{1,\ldots,k\}$, $(\psi_i,\beta_i) = (\phi_n,\alpha_n)$ or $(\phi_n,\alpha_n)^{-1}$. By ii), if $\mathbf{d}\beta = \{v\}$ then $v\beta = v$. Therefore, if (ψ,β) is not an idempotent then $v\psi$ is not the identity of C_m. The only elements of $T_{m,n}$ which can contribute non-identity elements to $v\psi$ are those (ψ,β) for which $|\mathbf{d}\beta| = 1$, (ϕ_n,α_n) and $(\phi_n^{-1},\alpha_n^{-1})$. Now $v\psi = (v\psi_1)(v\beta_1\psi_2) \ldots (v\beta_1 \ldots \beta_{k-1}\psi_k)(v\beta_1 \ldots \beta_k\psi_1^{-1})(v\beta_1 \ldots \beta_k\beta_1^{-1}\psi_2^{-1}) \ldots (v\beta_1 \ldots \beta_k\beta_1^{-1} \ldots \beta_{k-1}^{-1}\psi_k^{-1})$. If (ψ_i,β_i) is such that $|\mathbf{d}\beta_i| = 1$, then in this factorization of $v\psi$, ψ_i contributes $v\beta_1 \ldots \beta_{i-1}\psi_i = g$, say, and $v\beta_1 \ldots \beta_k\beta_1^{-1} \ldots \beta_{i-1}^{-1}\psi_i^{-1} = g^{-1}$, since g^{-1} is the only element of $r\psi_i^{-1}$. Thus, the contributions to this factorization of $v\psi$ by ψ_i cancel and so, if (ψ,β) is not an idempotent, one of the (ψ_i,β_i) must be (ϕ_n,α_n) or $(\phi_n,\alpha_n)^{-1}$.

v) None of the (ψ_i,β_i) is an idempotent. This follows from the general observation that if $e = e^2$ and aebec is not an idempotent then $aebec = aea^{-1}(abc)c^{-1}ec$ and so abc cannot be an idempotent. Thus, (ψ_i,β_i) an idempotent contradicts the minimality of k.

As a consequence of the aforementioned observations, the following assumptions concerning the (ψ_i,β_i) can be made. First of all, by iii) and iv) we may assume that $(\psi_1,\beta_1) = (\phi_n,\alpha_n)$. Secondly, assume that the k-tuple $\langle(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)\rangle$ contains a maximal number of elements from the kernel of $T_{m,n}$ among the collection of k-tuples from $T_{m,n}$ whose first element is (ϕ_n,α_n) and which witness that $T_{m,n}$ does not satisfy τ_k.

There are two stages to the remainder of the proof. The first stage is showing that exactly one of the (ψ_i,β_i) is a member of the kernel of $T_{m,n}$. We do this in four parts.

1) For any $i \in \{1,\ldots,k\}$, both (ψ_i,β_i) and (ψ_{i+1},β_{i+1}) do not belong to the kernel of $T_{m,n}$.

Suppose that both (ψ_i,β_i) and (ψ_{i+1},β_{i+1}) belong to the kernel of $T_{m,n}$. If $\mathbf{d}\beta_i = \{v_i\}$ and $\mathbf{d}\beta_{i+1} = \{v_{i+1}\}$ then $v_i\beta_i = v_{i+1}$ since $\beta_i\beta_{i+1} \neq 0$ and $v_{i+1}\beta_{i+1} = v_i$ since $\beta_i^{-1}\beta_{i+1}^{-1}\neq 0$. It follows that

$$v_i\beta_i\beta_{i+1} = v_i \quad \text{and} \quad v_{i+1}\beta_{i+1}\beta_i = v_{i+1}$$

and

$$\begin{aligned}(v_{i+1}\psi_i^{-1})(v_{i+1}\beta_i^{-1}\psi_{i+1}^{-1}) &= (v_i\beta_i\psi_i^{-1})(v_i\psi_{i+1}^{-1})\\ &= (v_i\psi_i)^{-1}(v_i\beta_{i+1}^{-1}\psi_{i+1})^{-1}\\ &= (v_i\psi_i)^{-1}(v_{i+1}\psi_{i+1})^{-1}\\ &= (v_{i+1}\psi_{i+1})^{-1}(v_i\psi_i)^{-1} \quad \text{(since } C_m \text{ is abelian)}\\ &= [(v_i\psi_i)(v_{i+1}\psi_{i+1})]^{-1}\end{aligned}$$

As a consequence of this we have that

$$x_1\ldots x_{i-1}x_{i+2}\ldots x_kx_1^{-1}\ldots x_{i-1}^{-1}x_{i+2}^{-1}\ldots x_k^{-1}[(\psi_1,\beta_1),\ldots,(\psi_{i-1},\beta_{i-1}),(\psi_{i+2},\beta_{i+2}),\ldots,(\psi_k,\beta_k)]$$

is equal to (ψ,β), which is not an idempotent by assumption. Thus, $T_{m,n}$ does not satisfy the equation τ_{k-2}, contrary to our choice of k. Note that under these conditions, $k \geq 3$, by observation iv). In the case $k = 3$, the conclusion is that $T_{m,n}$ does not satisfy τ_1 which is absurd since all inverse semigroups satisfy the equation $xx^{-1} \in E$.

2) If (ψ_i,β_i) is an element of the kernel then

i) if $\mathbf{d}\beta_i = \{wx_1...x_j\rho\}$, then $wx_1...x_j\rho\beta_i = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$;

ii) if $\mathbf{d}\beta_i = \{wx_1...x_nx_1^{-1}...x_j^{-1}\rho\}$, then $wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i = wx_1...x_j\rho$;

i) We have assumed that $(\psi_1,\beta_1) = (\phi_n,\beta_n)$ and so $i \neq 1$. Let $\mathbf{d}\beta_{i-1} = \{v_1,v_2\}$ (By (1) $|\mathbf{d}\beta_{i-1}| = 2$), and suppose that $v_1\beta_{i-1} = u_1$ and $v_2\beta_{i-1} = u_2$. Now, $\beta_{i-1}\beta_i \neq 0$ so one of u_1 and u_2 must be $wx_1...x_j\rho$, say $u_1 = wx_1...x_j\rho$. Also, $\beta_{i-1}^{-1}\beta_i^{-1} \neq 0$ so one of v_1 and v_2 must be $wx_1...x_j\rho\beta_i$. If $v_1 = wx_1...x_j\rho\beta_i$ then (ψ_{i-1},β_{i-1}) can be replaced by $(\hat{\psi}, \hat{\beta})$ where $\mathbf{d}\,\hat{\beta} = \{v_1\}$ and $v_1\hat{\beta} = u_1$ and $v_1\hat{\psi} = v_1\psi_{i-1}$. This new substitution witnesses that $T_{m,n}$ does not satisfy τ_k which contradicts 1), above (that is, this new substitution yields $T_{m,n}$ does not satisfy τ_{k-2} following the argument in (1), above). Thus, $v_2 = wx_1...x_j\rho\beta_i$. By observation (v), β_{i-1} is α_p or α_p^{-1} for some $p \in \{1,...,n\}$. If $\beta_{i-1} = \alpha_p$ then $v_1\beta_{i-1} = wx_1...x_j\rho$ implies that $v_1x_p\rho = wx_1...x_j\rho$ and hence that either $p = j$ or $j = n$, $p = 1$ and $v_1\ \rho\ wx_1...x_{j-1}$ or $v_1\ \rho\ wx_1...x_nx_1^{-1}$. Thus, $wx_1...x_j\rho\beta_i = v_2 = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$, by the definition of α_p or $wx_1...x_n\rho\beta_i = v_2 = w\rho$, which is what we want to prove..

If $\beta_{i-1} = \alpha_p^{-1}$ then $v_1\beta_{i-1} = wx_1...x_j\rho$ implies that $v_1x_p^{-1}\rho = wx_1...x_j\rho$ and hence that $v_1\ \rho\ wx_1...x_p$ and $p = j + 1$. Note that in this case $j \neq n$ since if u is an initial segment of w, then $wux_p^{-1}\ \rho\ wx_1...x_n$ is impossible by Lemma 6.1.5. Therefore, $wx_1...x_j\rho\beta_i = v_2 = wx_1...x_nx_1^{-1}...x_{p-1}^{-1}\rho\ wx_1...x_nx_1^{-1}...x_j^{-1}$, by the definition of α_p^{-1}.

ii) As in (i) we can assume that $\mathbf{d}\beta_{i-1} = \{v_1, wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i\}$ and that $v_1\beta_{i-1} = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$. Again, by observation (v), we may assume that $\beta_{i-1} = \alpha_p$ or α_p^{-1}. If $\beta_{i-1} = \alpha_p$ then $v_1x_p\rho = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$ and hence $p = j + 1$ and $v_1\ \rho\ wx_1...x_nx_1^{-1}...x_{j+1}^{-1}$. Note that if $j = n$, $wx_1...x_nx_1^{-1}...x_j^{-1}\rho\ w$ and so for any initial segment u of w, $wux_p \rho w$ is impossible, by Lemma 6.1.5. Therefore, by the definition of α_p, $wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i = wx_1...x_j\rho$.

If $\beta_{i-1} = \alpha_p^{-1}$ then $v_1x_p^{-1}\rho = wx_1...x_nx_1^{-1}...x_j^{-1}\rho$ and so $p = j$ and $v_1 \rho wx_1...x_nx_1^{-1}...x_{j-1}^{-1}$ or $j = n$, $p = 1$, $v_1 \rho wx_1$. By the definition of α_p^{-1}, $wx_1...x_nx_1^{-1}...x_j^{-1}\rho\beta_i = wx_1...x_j\rho$ and if $j = n$, $p = 1$, $w\rho\beta_i = v_2 = wx_1...x_n\rho$.

3) At most one of the (ψ_i,β_i) belongs to the kernel of $T_{m,n}$. Suppose that (ψ_j,β_j) and (ψ_{j+p},β_{j+p}) are two members of the kernel of $T_{m,n}$ and they are the first two such elements appearing in the sequence $\{(\psi_1,\beta_1),...,(\psi_k,\beta_k)\}$. Let $\mathbf{d}\beta_j = \{v_1\}$, $\mathbf{d}\beta_{j+p} = \{u_1\}$, $v_1\beta_j = v_2$ and $v_1\psi_j = g_1$, and $u_1\beta_{j+p} = u_2$ and $u_1\psi_{j+p} = g_2$. The claim is that if (ψ,β) is not an idempotent then neither is the following:

$x_1...x_{j-1}x_{j+1}^{-1}...x_{j+p-1}^{-1}x_{j+p+1}...x_kx_1^{-1}...x_{j-1}^{-1}x_{j+1}...x_{j+p-1}x_{j+p+1}^{-1}...x_k^{-1}$ when (ψ_i,β_i) is substituted for x_i for all x_i appearing in the expression. If the claim is correct then $T_{m,n}$ does not satisfy τ_{k-2}, contrary to our assumptions. Since (ψ_j,β_j) and (ψ_{j+p},β_{j+p}) do not contribute to $v\psi$ (where $v = \mathbf{d}\beta$) it is sufficient to show that the above expression in the second coordinate is identical to β. Now, with $\mathbf{d}\beta = \{v\}$

$v\beta_1...\beta_{j-1} = v_1$;

$v_1 \in \mathbf{d}x_{j+1}^{-1}...x_{j+p-1}^{-1}[(\psi_{j+1},\beta_{j+1}),...,(\psi_{j+p-1},\beta_{j+p-1})]$ and

$v_1\beta_{j+1}^{-1}...\beta_{j+p-1}^{-1} = u_2$;

$u_2 \in \mathbf{d}x_{j+p+1}...x_kx_1^{-1}...x_{j-1}^{-1}[(\psi_{j+p+1},\beta_{j+p+1}),...,(\psi_k,\beta_k),(\psi_1,\beta_1),...,(\psi_{j-1},\beta_{j-1})]$

$u_2\beta_{j+p+1}...\beta_k\beta_1^{-1}...\beta_{j-1}^{-1} = v_2$;

$v_2 \in \mathbf{d}x_{j+1}...x_{j+p-1}[(\psi_{j+1},\beta_{j+1}),...,(\psi_{j+p-1},\beta_{j+p-1})]$ and

$v_2\beta_{j+1}...\beta_{j+p-1} = u_1$;

$u_1 \in \mathbf{d}x_{j+p+1}^{-1}...x_k^{-1}[(\psi_{j+p+1},\beta_{j+p+1}),...,(\psi_k,\beta_k)]$ and

$u_1\beta_{j+p+1}^{-1}...\beta_k^{-1} = v\beta = v$.

It now follows that at most one of the (ψ_i,β_i) belongs to the kernel of $T_{m,n}$.

4) Exactly one of the (ψ_i,β_i) is a member of the kernel of $T_{m,n}$. First of all, observe that if none of the (ψ_i,β_i) belong to the kernel then each (ψ_i,β_i) is (ϕ_p,α_p) or $(\phi_p,\alpha_p)^{-1}$ for some p. By the definition of the α_p, if $v\beta_1\ldots\beta_k \in \mathbf{d}\beta_1^{-1}$ then $v\beta_1\ldots\beta_k\beta_1^{-1} = v$. This is because if $v = wu\rho$ for some initial segment u of w then $v\beta_1\ldots\beta_k = wu'\rho$ for some initial segment u′ of w and the difference between the lengths of u and u′ is not greater than k and hence strictly less than n. It follows that $v\beta_1\ldots\beta_k$ must be $v\beta_1$. By the same reasoning we can conclude that, for all $1 \leq i \leq k$, $v\beta_1\ldots\beta_k\beta_1^{-1}\ldots\beta_i^{-1} = v\beta_1\ldots\beta_{i-1}$. Since $\mathbf{d}\beta = \{v\}$, we can replace each (ψ_i,β_i) with an element of the kernel and conclude that if (ψ,β) is not an idempotent then neither is the result of this new substitution. But this cannot be since the kernel of $T_{m,n}$ is a Brandt semigroup over an abelian group and so satisfies the equation τ_k. Therefore, exactly one of the (ψ_i,β_i) belongs to the kernel of $T_{m,n}$. This completes the first stage of the proof.

Let (ψ_j,β_j) be the only member of $\{(\psi_1,\beta_1),\ldots,(\psi_k,\beta_k)\}$ which belongs to the kernel of $T_{m,n}$. Let $\mathbf{d}\beta_j = \{v_1\}$, $v_1\beta_j = v_2$ and $v_1\psi_j = g_1$. We consider the following two cases: i) $v_1\ \rho\ wx_1\ldots x_p$; and ii) $v_1\ \rho\ wx_1\ldots x_nx_1^{-1}\ldots x_p^{-1}$.

i) If $v_1\ \rho\ wx_1\ldots x_p$ then $v_2 = wx_1\ldots x_nx_1^{-1}\ldots x_p^{-1}\rho$ by the first stage, part 2). Since $(\psi_1,\beta_1) = (\phi_n,\alpha_n)$ and $k < n$, by the constraints on the (ψ_i,β_i) discussed thus far, for some $1 < q < j$, $(\psi_q,\beta_q) = (\phi_n,\alpha_n)^{-1}$. Assume q is the least such integer. Because $k < n$ and each of the (ψ_i,β_i) is either (ϕ_h,α_h) or $(\phi_h,\alpha_h)^{-1}$, for some h, for $1 < i \leq q$, as a consequence of the definitions of the (ϕ_h,α_h), we have that $v\beta_1\ldots\beta_q = v$ and $(v\psi_1)(v\beta_1\psi_2)\ldots(v\beta_1\ldots\beta_{q-1}\psi_q) = 1$. In a likewise manner we obtain that

$$(v\beta_1\ldots\beta_k)\beta_1^{-1}\ldots\beta_q^{-1} = v\beta_1\ldots\beta_k$$

and

$$[(v\beta_1\ldots\beta_k)\psi_1^{-1}][(v\beta_1\ldots\beta_k)\beta_1^{-1}\psi_2^{-1}]\ldots[(v\beta_1\ldots\beta_k)\beta_1^{-1}\ldots\beta_{q-1}^{-1}\psi_q^{-1}] = 1.$$

As a result, $x_{q+1}\ldots x_k x_{q+1}^{-1}\ldots x_k^{-1}[(\psi_{q+1},\beta_{q+1}),\ldots,(\psi_k,\beta_k)]$ is not an idempotent if (ψ,β) is not an idempotent, contrary to our choice of k.

ii) If $v_1 \; \rho \; wx_1\ldots x_n x_1^{-1}\ldots x_p^{-1}$ then $v_2 \; \rho \; wx_1\ldots x_p$. Using a similar argument to that used in (i) above, we can assume that (ψ_1,β_1) is the only (ψ_i,β_i) equal to (ϕ_n,α_n) for $i < j$. Moreover, the same argument can be used to show that at most one of the (ψ_i,β_i) is equal to (ϕ_n,α_n) for $j < i \leq k$. In this case, by the constraints on the (ψ_i,β_i) and the definitions of the (ϕ_i,α_i) and their inverses, (ψ_k,β_k) is equal to (ϕ_n,α_n). Thus, the only (ψ_i,β_i) equal to (ϕ_n,α_n) are (ψ_1,β_1) and (ψ_k,β_k). But for any inverse semigroup, $axaa^{-1}ya^{-1}$ is not an idempotent implies that xy is not an idempotent. It would then follow that $T_{m,n}$ does not satisfy the equation τ_{k-2}, a contradiction.

The proof is complete if we can show that, for $n > 2$, $T_{m,n}$ satisfies τ_2. This is not difficult to verify directly: Suppose that $(\psi,\beta) \in T_{m,n}$ is such that $(\phi_n,\alpha_n)(\psi,\beta)(\phi_n,\alpha_n)^{-1}(\psi,\beta)^{-1}$ is not an idempotent. Since $\mathscr{B}^1$ does satisfy τ_2, we have that $\alpha_n\beta\alpha_n^{-1}\beta^{-1}$ is an idempotent. Thus, for all $v \in \mathbf{d}\alpha_n\beta\alpha_n^{-1}\beta^{-1} \subseteq \mathbf{d}\alpha_n$, $v\alpha_n\beta\alpha_n^{-1}\beta^{-1} = v$. Therefore, both v and $v\alpha_n$ (which are not equal) are in the domain of β. For either v in the domain of α_n, there is no pair (ψ,β) in $T_{m,n}$ such that $\mathbf{d}\beta = \{v,v\alpha_n\}$. It follows that $T_{m,n}$ must satisfy τ_2. •

Lemma 6.2.5. Let $m,n \in \omega$, $m,n \geq 2$. $T^1_{m,n}$ satisfies the equation τ_k for $k < n$, but $T^1_{m,n}$ does not satisfy the equation τ_k for $k \geq n$.

Proof: This is an immediate consequence of Lemma 6.2.4. •

Remark. The only property of the varieties $\mathscr{A}_m$ that we used in the construction of the $T_{m,n}$'s was that they each satisfied the equations τ_n, $n \in \omega$. This is also true of the variety $\mathscr{AG}$, the variety of abelian groups. Thus, in a similar way, we can construct a family of inverse semigroups $\{ T^1_n \}$ such that , for each n, T^1_n satisfies the equations τ_k, for $k < n$, but T_n^1 does not satisfy the equations τ_k, for $k \geq n$. Moreover, for each $n \in \omega$, $\mathscr{AG} \vee \mathscr{B}^1 \subseteq \langle T_n{}^1 \rangle \subseteq \mathscr{AG} \circ \mathscr{B}^1$.

6.3 A class of varieties in the interval $[\mathscr{A}_m, \mathscr{B}^1]$

The inverse semigroups defined in the previous section can be used to define an infinite collection of varieties in the interval $[\mathscr{A}_m, \mathscr{B}^1]$. Once it is established that the interval $[\mathscr{A}_m, \mathscr{B}^1]$ is infinite, it can then be shown that other intervals which coincide with v-classes are infinite.

Notation 6.3.1. Let $m \in \omega$. For each $n \in \omega$, define the variety $\mathscr{V}_{m,n}$ to be the variety of inverse semigroups generated by $\{ T^1_{m,k} : k \geq n \}$.

Proposition 6.3.2. Let $m,n \in \omega$, with $m,n > 1$.

a) $\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$;

b) $\mathscr{V}_{m,n}$ does not satisfy τ_j for $j \geq n$;

c) $\mathscr{V}_{m,n} \supset \mathscr{V}_{m,n+1}$.

Proof: a) By Lemma 6.2.5, $T^1_{m,k}$ satisfies τ_j for $j < k$. Therefore, each generator of $\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$, and hence $\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$.

b) By Lemma 6.2.3, $T^1_{m,n}$ does not satisfy τ_n. Since $T^1_{m,n}$ is a generator of $\mathscr{V}_{m,n}$, the equation τ_n is not satisfied by $\mathscr{V}_{m,n}$.

c) $\{ T^1_{m,k} : k \geq n \} \supset \{ T^1_{m,k} : k \geq n + 1 \}$ and so

$\mathscr{V}_{m,n} = \langle T^1_{m,k} : k \geq n \rangle \supset \langle T^1_{m,k} : k \geq n + 1 \rangle = \mathscr{V}_{m,n+1}$. •

As a consequence of Proposition 6.3.2, the collection of varieties of inverse semigroups $\{ \mathscr{V}_{m,n} : n > 1 \}$ forms an infinite chain in the lattice of varieties of inverse semigroups. Furthermore, by Lemma 6.2.2, $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \mathscr{V}_{m,n} \subseteq \mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$. Since

$Wr(\mathcal{A}_m, \mathcal{B}^1) = \mathcal{A}_m \circ \mathcal{B}^1$, by Theorem 4.3.4 , and the ν-class of $\mathcal{A}_m \vee \mathcal{B}^1$ is the interval $[\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m \circ \mathcal{B}^1]$, we have the following result.

Theorem 6.3.3. The ν-class of the variety $\mathcal{A}_m \vee \mathcal{B}^1$ possesses an infinite descending chain of varieties.

Using Theorem 6.3.3, we can show that other intervals in $\mathcal{L}(\mathcal{I})$ are infinite.

Lemma 6.3.4. Let $\mathcal{V} \in [\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_n \circ \mathcal{B}^1]$, where $\mathcal{A}_m$ is the variety of abelian groups of exponent m, and let $\mathcal{U} \in [\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m^{max}]$. Then

$$\ker \rho(\mathcal{U} \vee \mathcal{V}) = \ker \rho(\mathcal{V}) \quad \text{and} \quad \operatorname{tr} \rho(\mathcal{U} \vee \mathcal{V}) = \operatorname{tr} \rho(\mathcal{U}).$$

Proof: $\mathcal{A}_m \subseteq \mathcal{V}$ and so $\mathcal{A}_m^{max} \subseteq \mathcal{V}^{max}$. Therefore,

$$\mathcal{V} \subseteq \mathcal{U} \vee \mathcal{V} \subseteq \mathcal{A}_m^{max} \vee \mathcal{V} \subseteq \mathcal{V}^{max} \vee \mathcal{V} = \mathcal{V}^{max}.$$

Since $\ker \rho(\mathcal{V}) = \ker \rho(\mathcal{V}^{max})$, it follows that $\ker \rho(\mathcal{U} \vee \mathcal{V}) = \ker \rho(\mathcal{V})$.

Also,

$$\mathcal{U} \subseteq \mathcal{U} \vee \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{V} \vee \mathcal{G} = \mathcal{U} \vee (\mathcal{A}_m \vee \mathcal{B}^1) \vee \mathcal{G} = \mathcal{U} \vee \mathcal{G}.$$

Since $\operatorname{tr} \rho(\mathcal{U}) = \operatorname{tr} \rho(\mathcal{U} \vee \mathcal{G})$, we have that $\operatorname{tr} \rho(\mathcal{U} \vee \mathcal{V}) = \operatorname{tr} \rho(\mathcal{U})$. •

Theorem 6.3.5. Let $\mathcal{U} \in [\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m^{max}]$. Then the interval $[\mathcal{U}, (\mathcal{A}_m \circ \mathcal{B}^1) \vee \mathcal{U}]$ contains an infinite descending chain.

Proof: The function $\theta : [\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m \circ \mathcal{B}^1] \to [\mathcal{U}, (\mathcal{A}_m \circ \mathcal{B}^1) \vee \mathcal{U}]$ defined by $\mathcal{V}\theta = \mathcal{V} \vee \mathcal{U}$ is one-to-one on $[\mathcal{A}_m \vee \mathcal{B}^1, \mathcal{A}_m \circ \mathcal{B}^1]$ by Lemma 6.3.4 and the fact that all varieties $\mathcal{V}$ in this interval are such that $\operatorname{tr} \rho(\mathcal{V}) = \operatorname{tr} \rho(\mathcal{A}_m \vee \mathcal{B}^1)$. Clearly θ is order-preserving, and the result follows from Theorem 6.3.3. •

Corollary 6.3.6. Let $\mathscr{U}$ be a combinatorial variety contained in $\mathscr{A}_m{}^{max}$ and containing $\mathscr{B}^1$. Then the $\vee$-class of $\mathscr{U} \vee \mathscr{A}_m$, that is, $[\mathscr{U} \vee \mathscr{A}_m, \mathscr{A}_m \circ \mathscr{U}]$, contains an infinite descending chain.

Proof: By Theorem 6.3.5 since $\mathscr{U} \vee \mathscr{A}_m \subseteq [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m{}^{max}]$. •

Remark. The results of this section are true for the variety $\mathscr{A}\mathscr{G}$ as well. That is, defining the variety $\mathscr{V}_n$ to be the variety of inverse semigroups generated by $\{ T_n^1 : k \geq n\}$, the analogous results to Proposition 6.3.2 hold and replacing $\mathscr{A}_m$ by $\mathscr{A}\mathscr{G}$ in the remaining results of this section yields valid statements.

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