MAL'CEV PRODUCTS AND RELATED TOPICS ON THE LATTICES OF VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

by

SHUHUA ZHANG

B.Sc. Sichuan Normal University 1982

M.Sc. Lanzhou University 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

December 1991

ISBN 0-315-78264-1

APPROVAL

Name: Shuhua Zhang

Degree: Ph. D. (Mathematics)

Title of Thesis: Mal'cev Products and Related Topics on The Lattice of Varieties of Completely Regular Semigroups

Examining Committee:

Chairman : Dr. A. H. Lachlan

Dr. N. R. Reilly
Senior Supervisor

Dr. T. C. Brown

Dr. A. R. Freedman

*Dr. A. Mekler was absent due to illness

Dr. A. Mekler

Dr. P. R. Jones
External Examiner
Professor
Department of Mathematics
Marquette University
Milwaukee, Wisconsin

Date Approved: December 5, 1991

Title of Thesis/Project/Extended Essay

Mal'cev Products and Related Topics on the Lattice of Varieties of Completely Regular Semigroups

Author: ______________
(signature)

Shuhua Zhang
(name)

Dec. 5, 1991
(date)

ABSTRACT

Completely regular semigroups are semigroups which are unions of groups. They form a variety $\mathcal{CR}$ of unary semigroups, determined by identities $xx^{-1}x = x$, $(x^{-1})^{-1} = x$ and $xx^{-1} = x^{-1}x$. The lattice of subvarieties of $\mathcal{CR}$ will be denoted by $\mathcal{L}(\mathcal{CR})$.

Given $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, their *Mal'cev product* $\mathcal{U} \circ \mathcal{V}$ consists of those completely regular semigroups S which possess a congruence ρ such that $S/\rho \in \mathcal{V}$ and $e\rho \in \mathcal{U}$ ($e^2 = e$). In general $\mathcal{U} \circ \mathcal{V}$ need not again be a variety. We define $< \mathcal{U} \circ \mathcal{V} >$ to be the variety of completely regular semigroups generated by $\mathcal{U} \circ \mathcal{V}$. This thesis is devoted to a detailed study of the varieties of this form.

Chapter 1 provides an introduction. Chapter 2 contains all the preliminary material needed in this thesis. The first section of Chapter 3 studies joins of congruences on Rees matrix semigroups. This enables us to extend a result of Jones [J5] by showing that $\mathcal{U} \circ \mathcal{V}$ is again a variety if $\mathcal{U} \in \mathcal{L}(\mathcal{C})$ and $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, where $\mathcal{C}$ denotes the variety of all central completely simple semigroups. We also introduce the concept of CR-relational morphism in this chapter. This makes it possible to describe the varieties of the form $< \mathcal{U} \circ \mathcal{V} >$. This description plays an important role in subsequent chapters.

Chapter 4 is devoted to study the varieties of the form $< \mathcal{U} \circ \mathcal{G} >$, where $\mathcal{G}$ denotes the variety of all groups. We first study the least full and self-conjugate subsemigroup $C^*(S)$ of a completely regular semigroup S. This enables us to introduce the operator C^*, and characterize $< \mathcal{U} \circ \mathcal{G} >$. The operator C^* is considered in detail. As a consequence, we extend a result of Petrich and Reilly [PR7] by showing that the well known operator C is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$, where $\mathcal{OG}$ denotes the variety of all orthogroups. By restricting C^* to completely simple semigroup varieties, we show that the order of C^* is infinite and the Mal'cev product is not associative on

$\mathcal{L}(\mathcal{CS})$. The semigroup generated by the operators C^* and C is determined here. We also describe $\mathcal{V}^{(C^*)^i}$, $\mathcal{V} \in [\mathcal{RB}, \mathcal{CS}]$ and $i \geq 0$, in terms of $\mathcal{E}$-invariant normal subgroups of the free group over a countably infinite set.

Chapter 5 is devoted to study the varieties of the forms $< \mathcal{V} \circ \mathcal{Y} >$ with $\mathcal{Y} \in \{ \mathcal{LZ}, \mathcal{RZ}, \mathcal{RB} \}$. We first provide descriptions for these varieties. Operators on $\mathcal{L}(\mathcal{CR})$ related to these varieties and various relationships between these operators are studied in detail throughout the rest of this chapter.

As consequences of results obtained in the previous chapters, we describe the varieties of the forms $< \mathcal{V} \circ \mathcal{Y} >$ with $\mathcal{Y} \in \{ \mathcal{LG}, \mathcal{RG}, \mathcal{ReG}, \mathcal{SG}, \mathcal{LNB}, \mathcal{RNB}, \mathcal{NB} \}$ in the final chapter.

ACKNOWLEDGEMENTS

I wish to thank my Senior Supervisor, Dr. Norman R. Reilly, for his great help throughout my Ph. D. program and his encouragement, guidance and patience during the preparation of this thesis.

Thanks also to my wife Yan for her unflagging support over the years. Financial support from the Department of Mathematics and Statistics is much appreciated.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

Completely regular semigroups are semigroups which are unions of groups. This conception unifies many important classes of semigroups, such as idempotent semigroups, groups and completely simple semigroups, etc. Basic information about such semigroups can be found in Howie [**How1**]. The fundamental structure theorem for completely regular semigroups, due to Clifford, states that a semigroup S is completely regular if and only if S is a semilattice of completely simple semigroups. This theorem is the cornerstone of the whole theory. From their inception to the present day, completely regular semigroups have attracted a wide attention among researchers in semigroups, particularly in recent years.

A *variety* is a class consists of similar algebras which is closed for taking homomorphic images, direct products and subalgebras. By a famous theorem, due to Birkhoff, an equivalent definition of variety is an equationally defined class of similar algebras. It is rather easy to see that the class $\mathcal{CR}$ of all completely regular semigroups is closed with respect to homomorphic images and direct products. However, the additive group of integers is completely regular but has the infinite cyclic semigroup of positive integers, which is not completely regular, as a subsemigroup. Thus $\mathcal{CR}$ do not constitute a variety of semigroups. Fortunately, the class $\mathcal{CR}$ of all completely regular semigroups, considered as algebras with the binary operation of multiplication and the unary operation of inversion within each subgroup, is a variety determined by the identities : $x(yz) = (xy)z$, $x = xx^{-1}x$, $(x^{-1})^{-1} = x$ and $xx^{-1} = x^{-1}x$. Thus, the study of completely regular semigroups may be approached from the perspective of their lattice of varieties. The lattice $\mathcal{L}(\mathcal{CR})$ of all subvarieties of $\mathcal{CR}$ has been the subject of intensive study in recent years. Many authors have investigated varieties of completely regular semigroups. Various approaches to these investigations showed that $\mathcal{L}(\mathcal{CR})$, even though complex, can be successfully studied both locally and globally. For an extensive bibliography up to 1989 see N.R. Reilly **[Rei3]**. The present thesis is a contribution to this subject.

Given subvarieties $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{CR}$, their *Mal'cev product* $\mathcal{U} \circ \mathcal{V}$ consists of those completely regular semigroups S which possess a congruence ρ such that $S/_{\rho} \in \mathcal{V}$ and each class $e\rho \in \mathcal{U}$ ($e^2 = e$). In general $\mathcal{U} \circ \mathcal{V}$ is a quasivariety, and need not again be a variety. But it has been showed that in many important cases a variety is obtained. Mal'cev products are of fundamental importance in the study of the lattice $\mathcal{L}(\mathcal{CR})$ of all subvarieties of $\mathcal{CR}$ and play a central role in our investigations.

Given two varieties $\mathcal{U}$ and $\mathcal{V}$ of completely regular semigroups, we define $<\mathcal{U} \circ \mathcal{V}>$ to be the variety of completely regular semigroups generated by the Mal'cev product $\mathcal{U} \circ \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$. This thesis is devoted to a detailed study of the varieties of this form.

The following is a brief outline of each chapter of this thesis.

Chapter 2 presents the preliminary material required in the sequel. In particular, the following topics are included : semigroups, completely simple semigroups, completely regular semigroups, congruences and homomorphisms, ideals, Green's relations, varieties, and free objects.

The first section of Chapter 3 studies joins of congruences on Rees matrix semigroups. Section 2 introduces the most important definition of this thesis, the Mal'cev product, and also extends a result of Jones [J5] by showing that if $\mathcal{U} \in \mathcal{L}(\mathcal{G})$, $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, then $\mathcal{U} \circ \mathcal{V}$ is again a variety (C.A. Vachuska also obtained this result in

[V]), where $\mathscr{C}$ denotes the variety of all central completely simple semigroups. Section 3 presents alternative descriptions of the operators T_l, T_r, T, K, T_l^* and T_r^* in terms of Mal'cev products and identities. Section 4 introduces the concept of CR-relational morphism, and establishes a number of properties of CR-relational morphisms. The final section of this chapter presents a useful description of the varieties of the form $\langle \mathcal{V}\circ\mathcal{Y} \rangle$ in terms of CR-relational morphisms. This description will be used in subsequent chapters.

Chapter 4 is devoted to study the varieties of the form $\langle \mathcal{V}\circ\mathcal{G} \rangle$, where $\mathcal{G}$ denotes the variety of all groups. The first section studies the least full and self-conjugate subsemigroup $C^*(S)$ of a completely regular semigroup S. Section 2 presents the description of the least group, semilattice of groups and orthogroup congruences on S in terms of $C^*(S)$. In Section 3 we show that $\mathcal{V}^{C^*} = \{ S \in \mathcal{CR} \mid C^*(S) \in \mathcal{V} \} = \langle \mathcal{V}\circ\mathcal{G} \rangle$ for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, by using the description obtained in Chapter 3. Section 4 deals with showing that the operator C^* commutes with the operators K, T_l, T_r, T, T_l^* and T_r^*. By restricting our attention to $\mathcal{L}(\mathcal{OG})$, Section 5 extends a result of Petrich and Reilly [PR7] by showing that the operator C is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$, where $\mathcal{OG}$ denotes the variety of all orthogroups. Though the action of C^* on $\mathcal{L}(\mathcal{CS})$ is complicated, a number of interesting and important results about C^* on $\mathcal{L}(\mathcal{CS})$ are obtained in Section 6, where $\mathcal{CS}$ denotes the variety of all completely simple semigroups. In particular, we show that the order of the operator C^* is infinite and the Mal'cev product is not associative on $\mathcal{L}(\mathcal{CS})$. Further, for $\mathcal{V} \in [\mathcal{RG}, \mathcal{CS}]$ and $n \geq 1$, we describe $\mathcal{V}^{(C^*)^n}$ in terms of $\mathcal{E}$-invariant normal subgroups of the free group over a countably infinite set. The final section of this chapter contains some remarks on the operator C^*.

Chapter 5 is devoted to study the varieties of the forms $\langle \mathcal{V}\circ\mathcal{Y} \rangle$ with $\mathcal{Y} \in \{ \mathcal{LZ}, \mathcal{RZ}, \mathcal{RB} \}$. Section 1 provides simple identities for these varieties in terms of those for

$\mathcal{U}$. Operators on $\mathcal{L}(\mathcal{CR})$ related to these varieties and various relationships between these operators are studied in details throughout the rest of this chapter.

As consequences of results obtained in the previous chapters, we describe the varieties of the forms $\langle \mathcal{U} \circ \mathcal{V} \rangle$ with $\mathcal{V} \in \{ \mathcal{LG}, \mathcal{RG}, \mathcal{RBG}, \mathcal{SG}, \mathcal{LNB}, \mathcal{RNB}, \mathcal{NB} \}$ in the final chapter.

Chapter 2

Preliminaries

The fundamental definitions and results of completely regular semigroup theory which required in the sequel are presented in this chapter. For the fundamentals of semigroup theory, the reader is referred to Clifford and Preston **[CP]**, Howie **[How1]** and Petrich **[Pe1]**. For background on varieties of algebras the reader is referred to Burris and Sankapanavar **[BS]**, Grätzer **[Gr2]**, and McKenzie, McNulty and Taylor **[MMT]**. For background on varieties of semigroups, the reader is referred to Evans **[Ev]**. The reader is assumed familiar with the fundamentals of lattice theory (see Grätzer **[Gr1]** for the appropriate background). Most of the results in Section § 2.6 can be found in Pastijn and Petrich **[PP2]**.

§ 2.1 Semigroups

A *semigroup* S is a pair (S, •) where S is a nonempty set and • is an associative binary operation, usually referred to as multiplication. Unless there is the possibility of ambiguity, we denote the semigroup (S, •) by S and denote products in S by juxtaposition.

Throughout the rest of this section S is a semigroup.

Certain elements of a semigroup have special properties relative to the multiplication and play an important role in the study of the subject.

An element e of S is a *left* (*right*) *identity* of S if ex = x (xe = x), for all x ∈ S; a *two-sided identity* (or simply an *identity*) of S if it is both a left and a right identity of

S. If S possesses an identity then it is unique and is denoted by 1 or 1_S if we wish to emphasize that it is the identity of S. We define S^1 to be S if S has an identity or $S \cup \{1\}$ with $1x = x1 = x$, for all $x \in S$, if S does not have an identity. A semigroup which has an identity is called a *monoid.*

An element $z \in S$ is a *zero* if $zx = xz = z$, for all $x \in S$. If S possesses a zero then it is unique and is denoted by 0 or 0_S if we wish to emphasize that it is the zero of S. The semigroup S^0 is defined to be S, if S possesses a zero, or $S \cup \{0\}$ with $0x = x0 = 0$, for all $x \in S$, otherwise.

An element $s \in S$ is *regular* if there exists an $x \in S$ such that $s = sxs$. S is called *regular* if every element of S is regular.

Let $s \in S$. An element x of S is an *inverse* of S if $s = sxs$ and $x = xsx$. For any $s \in S$ denote by V(s) the set of inverses of s in S. S is called an *inverse semigroup* if $|V(s)| = 1$ for every $s \in S$.

Let $s, t \in S$. Then s and t are said to *commute* with each other if $st = ts$. S is called *commutative* if all of its elements commute with each other. An element of S which commutes with every element of S is called a *central element* of S. The set of all central elements of S is either empty or a subsemigroup of S, and in the latter case is called the *centre* of S.

An element $e \in S$ is an *idempotent* if $e = e^2$. The set of idempotents of S is denoted by E(S). The relation $\leq$ on E(S) defined by $e \leq f$ if and only if $e = ef = fe$, for all $e, f \in E(S)$, is a partial order and is called the *natural partial order* of E(S). An element e of S without zero is *primitive* if it is minimal relative to the partial order on E(S); i.e., $f^2 = f = ef = fe$ implies $f = e$. An *idempotent semigroup*, or simply a *band*, is a semigroup in which all elements are idempotent. A commutative band is a *semilattice*. We denote by $\mathcal{S}$ and $\mathcal{B}$, respectively, the classes of all semilattices and all bands.

S is *left cancellative* if for any a, b, x $\in$ S, xa = xb implies a = b; *right cancellative* if ax = bx implies a = b; *cancellative* if it is both left and right cancellative; *weakly cancellative* if ax = bx and xa = xb imply a = b. S is *left reductive* if for any a, b $\in$ S, xa = xb for all x $\in$ S implies a = b; *right reductive* if ax = bx for all x $\in$ S implies a = b; *reductive* if it is both left and right reductive; *weakly reductive* xa = xb and ax = bx for all x $\in$ S imply a = b.

A nonempty subset T of S is a *subsemigroup* of S if it is closed under the operation of S; i.e., if a, b $\in$ T then ab $\in$ T. If A is an arbitrary nonempty subset of S, then the set

$$\{ s_1 \dots s_m \mid s_i \in A \text{ and m is arbitrary} \}$$

is *the subsemigroup of* S *generated* by A, denoted by [A]. If S = [A] we shall say that A is a *set of generators* for S or a *generating set* of S.

A nonempty subset T of S is a *left ideal* of S if s $\in$ S, t $\in$ T imply st $\in$ T; T is a *right ideal* if s $\in$ S, t $\in$ T imply ts $\in$ T; T is a *two-sided ideal* (or simply an *ideal*) if it is both a left and right ideal. An ideal of S different from S is a *proper ideal*. The intersection of all left ideals of S containing a nonempty subset T of S is the *left ideal generated* by T. A left ideal generated by a one-element set { a } is the *principal left ideal generated* by a, and will be denoted by L(a). The corresponding definitions are valid for right ideals with notation R(a), and two-sided ideals with notation J(a). If a $\in$ S, then $L(a) = S^1a$, $R(a) = aS^1$ and $J(a) = S^1aS^1$.

Let S and T be semigroups. A mapping $\varphi : S \longrightarrow T$ is a *homomorphism* of S into T if for all a, b $\in$ S, we have $(a\varphi)(b\varphi) = (ab)\varphi$. If φ is one-to-one, then φ is an *isomorphism* or *embedding* of S into T, and S is said to be *embeddable* in T. If there is a homomorphism of S into T, T is a homomorphic image of S; further, S and T are *isomorphic* if there is an isomorphism of S onto T; if so, we write $S \cong T$. A homomorphism of S into itself is an *endomorphism*; a one-to-one endomorphism of S onto itself is an *automorphism.*

If $\{ S_\alpha \}_{\alpha \in I}$ is a family of semigroups, their *direct product* is the semigroup defined on the Cartesian product $\Pi_{\alpha \in I} S_\alpha$ with coordinatewise multiplication. The notation for the direct product is $\Pi_{\alpha \in I} S_\alpha$ except when I is finite, say $I = \{ 1, 2, ..., n \}$, in which case we write $S_1 \times S_2 \times ... \times S_n$. Any semigroup isomorphic to a direct product of semigroups S_α is itself a direct product of S_α, $\alpha \in I$.

Let $\{ S_\alpha \}_{\alpha \in I}$ be a family of semigroups, let $S = \Pi_{\alpha \in I} S_\alpha$ and π_α denote the projection homomorphism $\pi_\alpha : S \longrightarrow S_\alpha$. Any semigroup S' isomorphic to a subsemigroup T of S such that $T\pi_\alpha = S_\alpha$ for all $\alpha \in I$ is a *subdirect product* of semigroups S_α, $\alpha \in I$. A semigroup S is *subdirectly irreducible* if it has the property : whenever $S \subseteq \Pi_{\alpha \in I} S_\alpha$ is a subdirect product, then one of the projection homomorphisms π_α is one-to-one.

Note that if S is a subdirect product of semigroups $\{ S_\alpha \}_{\alpha \in I}$, then each S_α is a homomorphic image of S.

The following result is of universal-algebraic character, and is useful.

Theorem 2.1.1 [Pe1]. Every semigroup is a subdirect product of subdirectly irreducible semigroups.

The above theorem remains valid if in it we substitute " the class of all semigroups " by any class of semigroups closed under taking homomorphic images. For example, it follows that every (idempotent, commutative, or both) semigroup is a subdirect product of subdirectly irreducible (idempotent, commutative, or both) semigroups.

§ 2.2 Equivalences and congruences

A *binary relation* ρ on a set S is a subset of the Cartesian product $S \times S$. We will write $a \rho b$ and say that a and b are ρ *-related* if $(a, b) \in \rho$ and ρ call simply a *relation.*

If ρ and σ are relations on S, their *composition* $\rho \circ \sigma$ is defined as follows: $(a, b) \in \rho \circ \sigma$ if and only if there exists $c \in S$ such that $(a, c) \in \rho$ and $(c, b) \in \sigma$. The binary operation is associative.

A relation ρ on S is

reflexive if $a \rho a$,

symmetric if $a \rho b$ implies $b \rho a$,

transitive if $a \rho b$ and $b \rho c$ imply $a \rho c$

for all $a, b, c \in S$.

A reflexive, symmetric, transitive relation ρ is an *equivalence relation*; its classes are ρ -classes and the ρ -class containing an element a will be denoted by $a\rho$. The relation ρ on S for which $a \rho b$ if and only if $a = b$ is the *equality relation* on S and will be denoted by ε_S; the relation ρ on S for which $a \rho b$ for all $a, b \in S$ is the *universal relation* on S and will be denoted by ω_S. Both ε_S and ω_S are equivalence relations. We denote by $\Sigma(S)$ the set of equivalence relations on S.

Also, for any $\lambda, \rho \in \Sigma(S)$, we will write $[\lambda, \rho] = \{ \sigma \in \Sigma(S) \mid \lambda \subseteq \sigma \subseteq \rho \}$.

An equivalence relation ρ on a semigroup S is a *left congruence* if for all a, b, c $\in S$, $a \rho b$ implies $ca \rho cb$, a *right congruence* if $a \rho b$ implies $ac \rho bc$; ρ is a *congruence* if it is both a left and a right congruence. We denote by $\Lambda(S)$ the set of congruences on S.

Let ρ, σ be relations on a set [semigroup] S. The *equivalence relation [congruence] generated by* ρ is the least equivalence relation [congruence] on S containing ρ; it is denoted by ρ^*. The join $\rho \vee \sigma$ of ρ and σ is the equivalence relation [congruence] generated by $\rho \cup \sigma$.

Both $\Sigma(S)$ and $\Lambda(S)$ are closed under intersections. If S is a semigroup, then $\Lambda(S)$ is a sublattice of the lattice $\Sigma(S)$. The reader is referred to Howie **[How1]** for more information about those two lattices.

Lemma 2.2.1 [How1]. If ρ is an equivalence on a semigroup S, then

$$\rho^0 = \{ (a, b) \in S \times S \mid xay \, \rho \, xby \text{ for all } x, y \in S^1 \}$$

is the largest congruence on S contained in ρ.

Lemma 2.2.2 [How1]. Let ρ_0 be a reflexive symmetric relation on a semigroup S. Let ρ consist of all pairs (a, b) such that there exist $s_i, t_i, p_i, q_i \in S^1$ with $p_i \, \rho_0 \, q_i$ (i = 1, ... , n) and

$$\begin{aligned} a &= s_1 p_1 t_1 \\ s_1 q_1 t_1 &= s_2 p_2 t_2 \\ s_2 q_2 t_2 &= s_3 p_3 t_3 \\ &\vdots \\ s_n p_n t_n &= b. \end{aligned}$$

Then ρ is the congruence on S generated by ρ_0.

Lemma 2.2.3 [How1]. Let ρ, σ be equivalences on a set S [congruences on a semigroup S]. If $a, b \in S$, then $a \, \rho \vee \sigma \, b$ if and only if for some n there exist elements $x_1, x_2, \ldots, x_{2n-1}$ in S such that

$$a \, \rho \, x_1 \, , x_1 \, \sigma \, x_2 \, , \; x_2 \, \rho \, x_3 \, , \; \ldots \; , x_{2n-1} \, \sigma \, b \, .$$

Let ρ be a congruence on a semigroup S. Then the set S/ρ of all ρ-classes with the multiplication $(a\rho)(b\rho) = (ab)\rho$ is the *quotient semigroup* relative to the congruence ρ.

Lemma 2.2.4 [Howl]. For any congruences ρ and σ on a semigroup S such that $\rho \subsetneq \sigma$, define a relation σ/ρ on S/ρ by

$$(a\rho)\ (\sigma/\rho)\ (b\rho) \iff a\ \sigma\ b.$$

Then σ/ρ is a congruence on S/ρ and $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

There is a strong connection between congruences and homomorphisms. Given a homomorphism ϕ of a semigroup S into a semigroup T, there is an associated congruence $\phi^{\#}$ on S defined by a $\phi^{\#}$ b if and only if $a\phi = b\phi$, for all a, b $\in$ S. Conversely, given a congruence ρ on a semigroup S, there is an associated homomorphism $\rho^{\#} : S \longrightarrow S/\rho$ given by $s\rho^{\#} = s\rho$, for all $s \in S$ **[Howl]**.

Let I be an ideal of a semigroup S. Then the relation ρ_I on S defined by

$$a\ \rho_I\ b \iff a, b \in I \text{ or } a = b \quad (a, b \in S)$$

is a congruence and is called the *Rees congruence* on S relative to I. The quotient semigroup S/ρ_I induced by ρ_I is called the *Rees quotient semigroup* relative to I and is denoted by S/I **[Pe1]**.

Lemma 2.2.5 (Lallement's Lemma). Let ρ be a congruence on a regular semigroup S and $a \in S$ be such that $a\rho \in E(S/\rho)$. Then $a\rho = e\rho$ for some $e \in E(S)$.

The close correspondence between congruences and homomorphisms enables us to obtain the following alternative version of Lallement's Lemma:

Lemma 2.2.6 [Howl]. If $\phi : S \longrightarrow T$ is a homomorphism from a regular semigroup S onto a semigroup T. Then $S\phi$ is regular. If $f \in E(T)$ then there exists $e \in E(S)$ such that $e\phi = f$.

If ρ is a congruence on a semigroup S and T is a subsemigroup of S, $\rho|_T$ will denote the restriction $\rho \cap (T \times T)$ of ρ to T.

The following two concepts will be used extensively.

If $\mathscr{C}$ is any class of semigroups, S is a semigroup and ρ is a congruence on S, then ρ is a $\mathscr{C}$*-congruence* if $S/\rho \in \mathscr{C}$. If $\mathscr{C}$ is the class of all semilattices, $\mathscr{C}$-congruences are called *semilattice congruences*; one defines analogously *band congruences, left zero congruences*, etc.

For example, a congruence ρ on a semigroup S is a semilattice congruence if and only if for all $x, y \in S$, $xy \,\rho\, yx$, $x^2 \,\rho\, x$. Similar expressions hold for other congruences.

Let $\mathscr{C}$ be a class of semigroups. A semigroup S is a *semilattice of semigroups belonging* to $\mathscr{C}$ if there exists a semilattice congruence on S all of whose classes belong to $\mathscr{C}$. The concepts: a *band of* or a *left zero semigroups of semigroups belong to* $\mathscr{C}$, are defined analogously.

The following result will prove useful.

Lemma 2.2.7 [Pe3]. Let $\{\rho_\alpha\}_{\alpha \in I}$ be a family of congruences on a semigroup S such that $\cap_{\alpha \in I} \rho_\alpha = \varepsilon_S$. Then S is a subdirect product of semigroups S/ρ_α, $\alpha \in I$.

§ 2.3 Green's relations

Green's relations are named for J.A. Green who introduced them in 1951 [Gre]. These relations have played a fundamental role in the development of semigroup theory.

In any semigroup S, the relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{H}$ and $\mathscr{D}$ defined on S by

$$a \,\mathscr{R}\, b \iff aS^1 = bS^1,$$

$$a \,\mathscr{L}\, b \iff S^1a = S^1b,$$

$$a \,\mathscr{J}\, b \iff S^1aS^1 = S^1bS^1,$$

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R} \text{ and } \mathscr{D} = \mathscr{L} \circ \mathscr{R},$$

are *Green's relations* (or *equivalences*) on S. Note that $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence; further, $\mathcal{R}$ and $\mathcal{L}$ commute, and that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is an equivalence relation **[CP]**.

For any $\mathcal{K} \in \{ \mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D} \}$, define the $\mathcal{K}$ - class of $s \in S$ by $K_s = \{ x \in S \mid s \mathcal{K} x \}$.

The next lemma is known as *Green's Lemma.*

Lemma 2.3.1 [Gre]. Let a and b be $\mathcal{R}$ -related elements of a semigroup S. By hypothesis there exist s, s' $\in S^1$ such that as = b and bs' = a. Then the mappings

$$\sigma : x \longrightarrow xs \quad (x \in L_a),$$

$$\sigma' : y \longrightarrow ys' \quad (y \in L_b),$$

are mutually inverse, $\mathcal{R}$ -related preserving, one-to-one mappings L_a onto L_b, and of L_b onto L_a, respectively.

The next result is known as *Green's Theorem.*

Lemma 2.3.2 [Gre]. If a, b and ab all belong to the same $\mathcal{H}$-class H of a semigroup S, then H is a subgroup of S. In particular, any $\mathcal{H}$-class containing an idempotent is a subgroup of S.

Lemma 2.3.3 [PC]. Every idempotent e in a semigroup S is a left identity of R_e, a right identity of L_e, and the identity of H_e.

Lemma 2.3.4 [How1]. Let e, f be idempotents in a semigroup S. Then

(i) $e \mathcal{L} f$ if and only if $ef = e, fe = f$;

(ii) $e \mathcal{R} f$ if and only if $ef = f, fe = e$;

(iii) $e \mathscr{D} f$ if and only if there exists $a \in S$ and $a' \in V(a)$ such that $e = aa'$, $f = a'a$; further, if $e \mathscr{D} f$, then $H_e \cong H_f$.

Lemma 2.3.5 [How1]. In a regular semigroup each $\mathscr{R}$-class and each $\mathscr{L}$-class contains at least one idempotent.

Lemma 2.3.6 [H1]. Let S be a regular subsemigroup of a semigroup T. Then Green's relations $\mathscr{L}, \mathscr{R}, \mathscr{H}$ on S are the restrictions of those on T.

§ 2.4 Completely simple semigroups

Let S be a semigroup. Then S is *simple* if $\mathscr{J} = S \times S$; and S is *completely simple* if it is simple and contains a primitive idempotent.

The next two results give some useful characterizations of completely simple semigroups.

Lemma 2.4.1 [Pe1]. The following conditions on a semigroup S are equivalent.

(i) S is completely simple.

(ii) S is regular and all its idempotents are primitive.

(iii) S is regular and weakly cancellative.

Lemma 2.4.2 [Pe1]. Let S be a completely simple semigroup and let $e, f \in E(S)$. Then the following statements hold.

(i) $\mathscr{H}$ is a congruence on S, and $H_a = aSa = G_a$ — the maximal subgroup of S containing a.

(ii) For any $a, b \in S$, $ab \in G_e$ implies $aSb \subseteq G_e$.

(iii) $ef = e$ implies $fe = f$.

(iv) ef = f implies fe = e.

We denote by $\mathcal{CS}$ the class of all completely simple semigroups.

For any 4-tuple (G; I, Λ; P) where G is a group, I and Λ are nonempty sets and $P : (\lambda, i) \longrightarrow p_{\lambda i}$ is a function from $\Lambda \times I$ to G, let $\mathcal{M}(G; I, \Lambda; P) = G \times I \times \Lambda$ together with multiplication

$$(g; i, \lambda)(h; j, \mu) = (gp_{\lambda j}h; i, \mu).$$

It is a straightforward exercise to show that $\mathcal{M}(G; I, \Lambda; P)$ is a completely simple semigroup. This construction is due to Rees [Re] and $\mathcal{M}(G; I, \Lambda; P)$ is therefore called the *Rees $I \times \Lambda$ matrix semigroup over the group G with the sandwich matrix P*. For convenience, we sometimes write [λ, i] for $p_{\lambda i}$. We will usually call such a semigroup a ***Rees matrix semigroup***. However, Rees matrix semigroups are much more than examples of completely simple semigroups.

Theorem 2.4.3 [Pe1]. Let S be a completely simple semigroup; fix $g \in E(S)$, and let $G = G_g$,

$$I = \{ e \in E(S) \mid eg = e \}, \quad \Lambda = \{ f \in E(S) \mid gf = f \},$$

$P = (p_{fe})$ where $p_{fe} = fe$. Then the mapping χ defined by

$$\chi : a \longrightarrow (gag; e, f) \qquad (a \in S)$$

where $ag \in G_e$, $ga \in G_f$, is an isomorphism of S onto $T = \mathcal{M}(G; I, \Lambda; P)$.

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup. The sandwich matrix P is *normalized* if there exists $1 \in I \cap \Lambda$ such that $p_{\lambda 1} = e = p_{1i}$, for all $\lambda \in \Lambda$ and $i \in I$, where e denote the identity of G. A point which will be of importance in § 3.1, is that the sandwich matrix P defined in Theorem 2.4.3 is normalized.

We can sum up the structure theorem of completely simple semigroups in the following form.

Theorem 2.4.4 [Pe1]. The following conditions on a semigroup S are equivalent.

(i) S is completely simple.

(ii) S is isomorphic to a Rees matrix semigroup with normalized sandwich matrix.

(iii) S is isomorphic to a Rees matrix semigroup.

Lemma 2.4.5 [Pe1]. Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup and $a = (g; i, \lambda) \in S$. Then

(i) $L_a = \{ (g'; j, \lambda) \mid g' \in G, j \in I \}$.

(ii) $R_a = \{ (g'; i, \mu) \mid g' \in G, \mu \in \Lambda \}$.

(iii) $H_a = \{ (g'; i, \lambda) \mid g' \in G \}$.

The following useful result can be derived easily from Howie **[How2]**.

Lemma 2.4.6 [How2]. Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup with normalized P. Then $[E(S)] = \mathcal{M}(< P >; I, \Lambda; P)$ where $< P >$ is the subgroup of G generated by the entries of P.

We now introduce some rather special yet important classes of completely simple semigroups.

A semigroup S is a *rectangular band* if aba = a, for all a, b ∈ S. A semigroup S is a *rectangular group* if it is isomorphic to the direct product of a rectangular band and a group. The class of all rectangular group [rectangular band] will be denoted by $\mathcal{RG}$ [$\mathcal{RB}$].

Lemma 2.4.7 [How1]. The following conditions on a semigroup S are equivalent.

(i) $S \in \mathcal{RG}$.

(ii) S is regular and $E(S) \in \mathcal{RB}$.

(iii) $S \in \mathscr{CS}$ and E(S) is a subsemigroup of S.

The class of all groups will be denoted by $\mathscr{G}$. Of course any group is a rectangular group so that $\mathscr{G} \subseteq \mathscr{R}e\mathscr{G}$.

A semigroup S is a *left* [*right*] *zero* if ab = a [ab = b], for all a, b $\in$ S. The class of all left [right] zero semigroups will be denoted by $\mathscr{LZ}$ [$\mathscr{RZ}$]. A semigroup S is a *left* [*right*] *group* if it is a direct product of a left [right] zero semigroup and a group. The class of all left [right] groups will be denoted by $\mathscr{LG}$ [$\mathscr{RG}$].

Lemma 2.4.8 [How1]. The following conditions on a semigroup S are equivalent.

(i) S is a left [right] group.

(ii) S is completely simple and E(S) is a left [right] zero semigroup.

(iii) S is regular and $\mathscr{L} = S \times S$ [$\mathscr{R} = S \times S$].

It follows from Lemma 2.4.8 that $\mathscr{G} = \mathscr{LG} \cap \mathscr{RG}$.

§ 2.5 Completely regular semigroups

An element a of a semigroup S is *completely regular* if a = axa and ax = xa for some x $\in$ S; S is *completely regular* if all its elements are completely regular.

Lemma 2.5.1 [Pe1]. The following conditions on an element a of a semigroup S are equivalent.

(i) a is completely regular.

(ii) a has an inverse with which it commutes.

(iii) H_a is a subgroup of S.

Lemma 2.5.2 [Pe1]. The following conditions on a semigroup S are equivalent.

(i) S is completely regular.

(ii) For every $a \in S$, $a \in aSa^2$.

(iii) S is a union of (disjoint) groups.

(iv) Every $\mathcal{H}$-class of S is a group.

Let $\mathcal{CR}$ denote the class of all completely regular semigroups and for any $a \in S \in \mathcal{CR}$, let a^{-1} denote the inverse of a in the (group) $\mathcal{H}$-class H_a and let a^0 denote the element $aa^{-1} = a^{-1}a$, the identity of the group H_a.

Let S be the disjoint union of the semigroups S_α ($\alpha \in Y$), where Y is a semilattice and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Then S is said to be a *semilattice of the semigroups* S_α, $\alpha \in Y$, and we write $S = \cup_{\alpha \in Y} S_\alpha$, and refer to the semigroups S_α as the components of S. The importance of this concept in the theory of completely regular semigroups was revealed by the following theorem.

Theorem 2.5.3 [CP]. Let $S \in \mathcal{CR}$. Then $\mathcal{D} = \mathcal{J}$ is a congruence, each $\mathcal{J}$ -class is a completely simple semigroup and $S/\mathcal{J}$ is a semilattice. Thus S is a semilattice of its $\mathcal{J}$ -classes.

For use in later chapters, we gather the following basic properties of completely regular semigroups.

Lemma 2.5.4. Let $S \in \mathcal{CR}$ and let T be a subsemigroup of S. Then $T \in \mathcal{CR}$ if and only if T is closed under inverses; that is, $a^{-1} \in T$ for any $a \in T$.

Lemma 2.5.5. Let $S, T \in \mathcal{CR}$, then $S \times T \in \mathcal{CR}$ and $(s, t)^{-1} = (s^{-1} , t^{-1})$ for any $(s, t) \in S \times T$.

Lemma 2.5.6. Let $S, T \in \mathscr{CR}$ and $\varphi : S \longrightarrow T$ be a surjective homomorphism of S onto T. Then

(i) $a^{-1}\varphi = (a\varphi)^{-1}$ for any $a \in S$.

(ii) For any $t \in T$, there exists $s \in S$ such that $s\varphi = t$ and $s^{-1}\varphi = t^{-1}$.

By the above facts, $\mathscr{CR}$ is closed with respect to products and homomorphic images. However, $\mathscr{CR}$ is not closed under subsemigroups. Thus $\mathscr{CR}$ is not a variety of semigroups. However, $\mathscr{CR}$ may be regarded as a class of algebras with the operations of (binary) multiplication and (unary) inversion. As such $\mathscr{CR}$ forms a variety defined by the identities

$$x(yz) = (xy)z,\ x = xx^{-1}x,\ x^{-1}x = xx^{-1},\ (x^{-1})^{-1} = x.$$

With the earlier notation, we shall write $x^0 = x^{-1}x = xx^{-1}$.

One observation that is sometimes helpful is the following.

Lemma 2.5.7 [PR3]. The variety $\mathscr{CR}$ satisfies the identity

$$(xy)^{-1} = (xy)^0y^{-1}(yx)^0x^{-1}(xy)^0 .$$

For any $S \in \mathscr{CR}$, let C(S) denote the subsemigroup of S generated by the idempotents of S, i.e., C(S) = [E(S)].

Lemma 2.5.8 [Fi]. For any $S \in \mathscr{CR}$, $C(S) \in \mathscr{CR}$.

As a particular case of [H2, Theorem 2], we have the following useful observation.

Lemma 2.5.9. For $S = \cup_{\alpha\in Y} S_\alpha \in \mathscr{CR}$, we have $C(S) = \cup_{\alpha\in Y} C(S_\alpha)$.

The rest of this section is devoted to several important classes of completely regular semigroups.

Lemma 2.5.10 [Pe1]. The following conditions on a semigroup S are equivalent.

(i) S is a band of groups.

(ii) S is completely regular and $\mathscr{H}$ is a congruence.

(iii) S is regular and $a^2bS = abS$, $Sab^2 = Sab$ for all $a, b \in S$.

We denote by $\mathscr{BG}$ the class of all *bands of groups*.

A completely regular semigroup S is an *orthogroup* if E(S) forms a subsemigroup. We denote by $\mathscr{OG}$ the class of all orthogroups.

Lemma 2.5.11 [Pe1]. The following conditions on a semigroup S are equivalent.

(i) $S \in \mathscr{OG}$.

(ii) Every $\mathscr{D}$-class is a rectangular group.

Let $\mathscr{SG}$ denote the class of all *semilattices of groups*. Then we have the following result.

Lemma 2.5.12 [How1]. The following conditions on a semigroup S are equivalent.

(i) $S \in \mathscr{SG}$.

(ii) S is regular and its idempotents lie in its centre.

(iii) S is isomorphic to a subdirect product of a group and a semilattice.

§ 2.6 Congruences on completely regular semigroups

Throughout this section, let S denote a completely regular semigroup.

Let ρ be a congruence on S. Then the *kernel* of ρ is

$$\ker \rho = \{\, a \in S \mid a \,\rho\, a^0 \,\}$$
$$= \cup_{e \in E(S)} e\rho$$

and the *trace* of ρ is

$$\operatorname{tr} \rho = \rho \mid_{E(S)} .$$

The key observation about the kernel and trace of a congruence is that in combination they completely determine the congruence.

Lemma 2.6.1 [PP2]. Let ρ be a congruence on S. Then, for any elements a, b ∈ S,

$$a \,\rho\, b \iff a^0 \operatorname{tr} \rho\ b^0 \text{ and } ab^{-1} \in \ker \rho .$$

This leads to natural questions concerning the nature of those subsets of S which are kernels of congruences and those equivalence relations on E(S) which are the traces of congruences.

A subset K of S is said to be a *normal subset* of S if it satisfies the following conditions:

(i) $E(S) \subseteq K$,

(ii) $k \in K \Rightarrow k^{-1} \in K$,

(iii) $xy \in K \Rightarrow yx \in K$ $(x, y \in S)$,

(iv) $x, x^0 y \in K \Rightarrow xy \in K$ $(x, y \in S)$.

For any subset K of S, we denote by π_K the largest congruence on S for which K is a union of π_K-classes. Then

$$\pi_K = \{\, (a, b) \in S \times S \mid xay \in K \text{ if and only if } xby \in K, \text{ for all } x, y \in S^1 \,\}.$$

Theorem 2.6.2 [PP2]. Let K be a subset of S. Then the following statements are equivalent.

(i) K is a normal subset of S.

(ii) K is the kernel of some congruence on S.

(iii) K is the kernel of π_K.

When (i) — (iii) hold, $\{ (k, k^0) \mid k \in K \}^*$ is the smallest congruence and π_K is the largest congruence on S with the kernel K.

Let τ be an equivalence relation on E(S). Then τ is a *normal equivalence* if it satisfies the following condition:

$$e \,\tau\, f \iff (xey)^0 \,\tau\, (xfy)^0 \quad (x, y \in S^1).$$

Theorem 2.6.3 [PP2]. Let τ be an equivalence relation on E(S). Then the following conditions are equivalent.

(i) τ is a normal equivalence.

(ii) τ is the trace of some congruence on S.

(iii) $\tau = \operatorname{tr} \tau^*$.

When (i) — (iii) hold, then τ^* is the smallest congruence and $(\mathcal{H} \circ \tau \circ \mathcal{H})^0$ is the largest congruence on S with trace τ.

We refer the reader to either Pastijn and Petrich **[PP2]** or Reilly **[Rei3]** for results concerning when a normal subset and a normal equivalence can be combined to the kernel and trace of a single congruence.

Let the kernel relation K and the trace relation T be defined on $\Lambda(S)$ as follows:

$$\lambda \,K\, \rho \iff \ker \lambda = \ker \rho \quad (\lambda, \rho \in \Lambda(S)),$$

$$\lambda \,T\, \rho \iff \operatorname{tr} \lambda = \operatorname{tr} \rho \quad (\lambda, \rho \in \Lambda(S)).$$

Clearly, K and T are both equivalence relations, and $K \cap T = \varepsilon$, the identical relation on $\Lambda(S)$. We have the following interesting observations.

Lemma 2.6.4 [PP2]. Let $\lambda, \rho \in \Lambda(S)$. Then

(i) $\lambda \, K \, \rho \Leftrightarrow \lambda \cap \mathscr{H} = \rho \cap \mathscr{H}$.

(ii) $\lambda \, T \, \rho \Leftrightarrow \lambda \vee \mathscr{H} = \rho \vee \mathscr{H}$.

Let $\mathscr{K}(S)$ denote the set of all normal subsets of S ordered by set theoretic inclusion. Then $\mathscr{K}(S)$ is a complete lattice with respect to the operations

$$K_1 \wedge K_2 = K_1 \cap K_2 \text{ and } K_1 \vee K_2 = \cap\{ K \in \mathscr{K}(S) \mid K_1 \cup K_2 \subseteq K \}.$$

Theorem 2.6.5 [PP2]. The mapping

$$\ker : \rho \longrightarrow \ker \rho \qquad (\rho \in \Lambda(S)).$$

is a complete $\cap$-homomorphism of $\Lambda(S)$ onto $\mathscr{K}(S)$ which induces the relation K on $\Lambda(S)$. For all $\rho \in \Lambda(S)$ the K -class of ρ is an interval $[\rho_K , \rho^K]$, where $\rho_K = (\rho \cap \mathscr{H})^*$ and $\rho^K = \pi_{\ker \rho}$.

Unfortunately, K is not always a congruence **[Rei3]**.

Let $\mathscr{T}(S)$ denote the set of all normal equivalence relations on E(S). Then $\mathscr{T}(S)$ is complete lattice with respect to the operations

$$\sigma \wedge \tau = \sigma \cap \tau \text{ and } \sigma \vee \tau = \cap\{ \rho \in \mathscr{T}(S) \mid \sigma \cup \tau \subseteq \rho \}.$$

Theorem 2.6.6 [PP2]. The mapping

$$\mathrm{tr} : \rho \longrightarrow \mathrm{tr}\, \rho \qquad (\rho \in \Lambda(S)).$$

is a complete homomorphism of $\Lambda(S)$ onto $\mathscr{T}(S)$ inducing the relation T on $\Lambda(S)$. Moreover, for each $\rho \in \Lambda(S)$, the T -class of ρ is an interval $[\rho_T , \rho^T]$, where $\rho_T = (\mathrm{tr}\, \rho)^*$ and $\rho^T = (\rho \vee \mathscr{H})^0$.

In contrast to the fact that K need not be a congruence on $\Lambda(S)$, we have that T is a complete congruence on $\Lambda(S)$, by Theorem 2.6.6.

Two additional relations on $\Lambda(S)$ associated with the other Green's relations $\mathscr{L}$ and $\mathscr{R}$ are defined as follows:

$$\lambda\, T_l\, \rho \;\Leftrightarrow\; \lambda/(\lambda\cap\rho) \subseteq \mathscr{L} \text{ and } \rho/(\rho\cap\lambda) \subseteq \mathscr{L} \quad (\lambda, \rho \in \Lambda(S)),$$

$$\lambda\, T_r\, \rho \;\Leftrightarrow\; \lambda/(\lambda\cap\rho) \subseteq \mathscr{R} \text{ and } \rho/(\rho\cap\lambda) \subseteq \mathscr{R} \quad (\lambda, \rho \in \Lambda(S)).$$

We refer to T_l as the *left trace relation* and to T_r as the *right trace relation* on $\Lambda(S)$.

For any congruence $\rho \in \Lambda(S)$, the *left trace* and *right trace* of ρ are defined to be

$$\text{ltr}\, \rho = (\rho \vee \mathscr{L})^0 \quad \text{and} \quad \text{rtr}\, \rho = (\rho \vee \mathscr{R})^0.$$

Then an equivalent characterization of the relations T_l and T_r is given by the following: for $\lambda, \rho \in \Lambda(S)$,

$$\lambda\, T_l\, \rho \;\Leftrightarrow\; \text{ltr}\, \lambda = \text{ltr}\, \rho \quad \text{and} \quad \lambda\, T_r\, \rho \;\Leftrightarrow\; \text{rtr}\, \lambda = \text{rtr}\, \rho.$$

The parallelism between the relations T, T_l and T_r is brought out strongly in the next result.

Theorem 2.6.7 [PP2]. The mappings

$$\rho \longrightarrow \rho \vee \mathscr{H}, \quad \rho \longrightarrow \rho \vee \mathscr{L}, \quad \rho \longrightarrow \rho \vee \mathscr{R}$$

are complete homomorphisms of the lattice $\Lambda(S)$ into the lattice $\Sigma(S)$ inducing the relations T, T_l and T_r, respectively. Consequently, the relations T, T_l and T_r are complete congruences on $\Lambda(S)$.

Since T_l and T_r are complete congruences, it follows that all the T_l-classes and T_r-classes are intervals. For any $\rho \in \Lambda(S)$, we define ρ_{T_l}, ρ_{T_r}, ρ^{T_l} and ρ^{T_r} by setting

$$\rho T_l = [\, \rho_{T_l}, \rho^{T_l} \,] \quad \text{and} \quad \rho T_r = [\, \rho_{T_r}, \rho^{T_r} \,].$$

In order to give more explicit descriptions of the endpoints of T_l- and T_r-classes, it is convenient to introduce the following relations. Define

$$e \leq_l f \;\Leftrightarrow\; e = ef \qquad (e, f \in E(S))$$

and define the relation $\leq_r$ dually.

Lemma 2.6.8 [PP2]. Let $\rho \in \Lambda(S)$. Then

(i) $\rho_{T_l} = (\rho \cap \leq_r)^*$ and $\rho^{T_l} = (\rho \vee \mathscr{L})^0$.

(ii) $\rho_{T_r} = (\rho \cap \leq_l)^*$ and $\rho^{T_r} = (\rho \vee \mathscr{R})^0$.

The next result sets out some important basic connections between the relations K, T, T_l and T_r.

Lemma 2.6.9 [PP2]. Let $\rho \in \Lambda(S)$. Then

(i) $\rho_K \vee \rho_T = \rho = \rho^K \cap \rho^T$.

(ii) $\rho_{T_l} \vee \rho_{T_r} = \rho_T$ and $\rho^{T_l} \cap \rho^{T_r} = \rho^T$.

(iii) $T_l \cap T_r = T$.

This leads to the following diagram from [PP2].

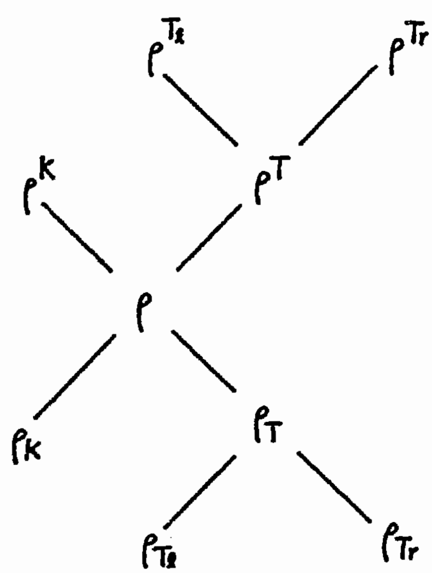

Figure 2.1.

As we will see later, the most important applications of these concepts are in the lattice of varieties of completely regular semigroups.

In the remainder of this section we briefly discuss several congruences on S, which will be needed in the sequel.

Let $\rho \in \Lambda(S)$. Then ρ is *idempotent pure* if $\ker \rho = E(S)$; ρ is *idempotent separating* if $\mathrm{tr}\, \rho = \varepsilon$ or, equivalently, $\rho \subseteq \mathcal{H}$.

The following observation is straightforward.

Lemma 2.6.10. Let $\rho \in \Lambda(S)$. Then ρ is idempotent pure if and only if $\rho \cap \mathcal{H} = \varepsilon_S$.

Let $\mu = \mu_S$ be the largest idempotent separating congruence on S, and let $\tau = \tau_S$ be the largest idempotent pure congruence on S.

Lemma 2.6.11 [H2 and J5]. Let $S \in \mathcal{CR}$. Then

$$\mu = \mathcal{H}^0$$
$$= \{ (a, b) \in S \times S \mid a^0 = b^0 \text{ and } a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S), e \le a^0 \}$$
$$= \{ (a, b) \in S \times S \mid a^{-1}(a^0ea^0)^0a = b^{-1}(b^0eb^0)^0b \text{ for all } e \in E(S) \}.$$

It is not hard to see that

$$\tau = \{ (a, b) \in S \times S \mid xay \in E(S) \text{ if and only if } xby \in E(S), \text{ for all } x, y \in S^1 \}.$$

Whilst this description is not very useful in practice, the following simple description of $\tau \cap \mathcal{D}$ will be needed in the sequel.

Lemma 2.6.12 [J5]. Let $S \in \mathcal{CR}$. Then

$$\tau \cap \mathcal{D} = \{ (a, b) \in \mathcal{D} \mid (xay)(xby)^{-1} \in E(S) \text{ for all } x, y \in S \}.$$

Lemma 2.6.13. Let $S \in \mathcal{CR}$. Then

(i) $\mathscr{L}^0 = \{ (a, b) \in S \times S \mid (xa)^0 = (xaxb)^0$ and $(xb)^0 = (xbxa)^0$ for all $x \in S^1 \}$

$= \{ (a, b) \in S \times S \mid xa = xa(xb)^0$ and $xb = xb(xa)^0$ for all $x \in S^1 \}$.

(ii) $\mathscr{R}^0 = \{ (a, b) \in S \times S \mid (ax)^0 = (bxax)^0$ and $(bx)^0 = (axbx)^0$ for all $x \in S^1 \}$

$= \{ (a, b) \in S \times S \mid ax = (bx)^0 ax$ and $bx = (ax)^0 bx$ for all $x \in S^1 \}$.

Proof. (i) Since $S \in \mathscr{CR}$, then for any $a, b \in S$, we have

$$a \mathscr{L} b \iff a \mathscr{H} ab \text{ and } b \mathscr{H} ba$$
$$\iff a = ab^0 \text{ and } b = ba^0 .$$

Note that $\mathscr{L}$ is a right congruence, then for any $a, b \in S$, we have

$$a \mathscr{L}^0 b \iff xa \mathscr{L} xb \text{ for all } x \in S^1$$
$$\iff xa \mathscr{H} xaxb \text{ and } xb \mathscr{H} xbxa$$
$$\iff (xa)^0 = (xaxb)^0 \text{ and } (xb)^0 = (xbxa)^0 \text{ for all } x \in S^1$$
$$\iff xa = xa(xb)^0 \text{ and } xb = xb(xa)^0 \text{ for all } x \in S^1.$$

Hence, (i) is obtained.

(ii) This is the dual of (i). #

§ 2.7 Varieties of completely regular semigroups

We begin with some familiar but necessary background on varieties of algebras. The reader is referred to Burris and Sankappanavar **[BS]**, Grätzer **[Gr2]**, and McKenzie, McNulty and Taylor **[MMT]** for details.

By an ***algebra***, we will mean a nonempty set together with one or more operations. Familiar examples are groups, lattices, semigroups, etc. By a *variety* or *equational class* of algebras we shall mean a class of algebras of the same type defined by a set of identities.

For any class $\mathscr{C}$ of algebras, let

$P\mathscr{C}$ = the class of all direct products of members in $\mathscr{C}$

$H\mathscr{C}$ = the class of all homomorphic images of members in $\mathscr{C}$

$S\mathscr{C}$ = the class of all subalgebras of members in $\mathscr{C}$.

The next result is known as *Birkhoff's Theorem.*

Theorem 2.7.1 [Gr2]. Let $\mathscr{C}$ be a class of algebras of the same type. Then $\mathscr{C}$ is a variety if and only if $\mathscr{C} = HSP\mathscr{C}$.

One useful consequence of Birkhoff's Theorem is a characterization of the variety generated by a class of algebras of the same type.

Lemma 2.7.2 [Gr2]. For any class $\mathscr{C}$ of algebras of the same type, the smallest variety containing $\mathscr{C}$ is $HSP\mathscr{C}$.

We call $HSP\mathscr{C}$ the *variety generated by* $\mathscr{C}$ and denoted by $< \mathscr{C} >$. Thus $< \mathscr{C} > = HSP\mathscr{C}$. If $\mathscr{C}$ consists of the single algebra S, we write $< S >$ instead of $< \mathscr{C} >$, and call this *the variety generated by* S.

If $\mathscr{U}$ is a variety defined by the identities Σ then we write $\mathscr{U} = [\ \Sigma\ ]$. If Σ is a finite set of identities $\{ u_1 = v_1 , \dots , u_n = v_n \}$ we will often write $\mathscr{U} = [u_1 = v_1 , \dots , u_n = v_n]$ instead of $[\ \Sigma\ ]$. We sometimes refer to the identities which define the variety $\mathscr{U}$ as *laws*.

If $\mathscr{U}$ is a variety contained in the variety $\mathscr{V}$ then $\mathscr{U}$ is a subvariety of $\mathscr{V}$. For any variety $\mathscr{W}$, it is easily verified that the subvarieties of $\mathscr{W}$ constitute a complete lattice with respect to the operations

$$\mathscr{U} \wedge \mathscr{V} = \mathscr{U} \cap \mathscr{V}, \ \mathscr{U} \vee \mathscr{V} = \cap\{ \mathscr{S} : \mathscr{U} \subseteq \mathscr{S} \text{ and } \mathscr{V} \subseteq \mathscr{S} \}.$$

We shall denote this lattice of subvarieties of $\mathscr{W}$ by $\mathscr{L}(\mathscr{W})$.

Let $\mathscr{V}$ be a variety of algebras, X be a nonempty set, F = F(X) be an algebra in $\mathscr{V}$ generated by X and $\theta : X \longrightarrow F$ be the embedding of X into F. If, for all algebras A in $\mathscr{V}$ and all mappings $\alpha : X \longrightarrow A$, there exists a unique homomorphism $\beta : F \longrightarrow A$ which "extends" α, that is, such that $x\theta\beta = x\alpha$, for all $x \in X$, then F is said to have the *universal mapping property for* $\mathscr{V}$ over X or to be a *free object* in $\mathscr{V}$ over X. In fact, up to isomorphism, such free object in $\mathscr{V}$ over X is unique. This leads us to refer to the *free object* in $\mathscr{V}$ over X, or the *relatively free object* in $\mathscr{V}$ over X, and to be denoted by $F_{\mathscr{V}}(X)$. Then for any $S \in \mathscr{V}$ and any mapping $\phi : X \longrightarrow S$, there is a unique homomorphism $\phi^*: F_{\mathscr{V}}(X) \longrightarrow S$ which extends ϕ.

P is an *operator* if for every class K of algebras, K P is also a class of algebras.

As we mentioned in § 2.5, completely regular semigroups, considered as algebras with the binary operation of multiplication and the unary operation of inversion within each subgroup, is a variety determined by the identities

(1) $$x(yz) = (xy)z, \ x = xx^{-1}x, \ x^{-1}x = xx^{-1}, \ (x^{-1})^{-1} = x.$$

Consequently we may consider the lattice of varieties of completely regular semigroups.

Let X be a nonempty set. The *free semigroup* on X consists of all nonempty finite sequences of elements of X, called *words*, over X, called an *alphabet*, given the multiplication of concatenation. We denote the free semigroup on X by X^+. The *free monoid* over X, denoted by X^*, consists of all words over X including the empty word, which serves as the identity of X^*.

The description in [C] and [Rei2] of the free completely regular semigroup $F_{\mathscr{CR}}(X)$, that is, the free object in the variety $\mathscr{CR}$ over X is via a description of the free unary semigroup U over X; that is, the free object over X in the variety of all semigroups with a unary operation. Let $Y = X \cup \{ (\ ,\ )^{-1} \}$, where " (" and " $)^{-1}$ " are two distinct elements not in X. By Clifford [C], U is the smallest subsemigroup of the free semigroup Y^+ on Y such that $X \subseteq U$ and $(w)^{-1} \in U$ for all $w \in U$. As a notational convenience we write w^{-1} instead of $(w)^{-1}$ in U. Let ζ be the congruence on U generated by the pairs $(w, ww^{-1}w)$, $(ww^{-1} , w^{-1}w)$ and $((w^{-1})^{-1} , w)$ for all $w \in U$. Then $F_{\mathscr{CR}}(X) = U/\zeta$ ([C], [Rei2]). Every element of $F_{\mathscr{CR}}(X)$ can be written in the form w with $w \in U$. We henceforth assume that $w = v$ in $F_{\mathscr{CR}}(X)$ if and only if the identity $w = v$ is a consequence of the identities $x = xx^{-1}x$, $x^{-1}x = xx^{-1}$, $(x^{-1})^{-1} = x$. Thus if $\mathscr{V}$ is a subvariety of $\mathscr{CR}$, then we shall write the identities that hold in $\mathscr{V}$ in the form $w = v$ where $w, v \in U$.

A congruence ρ on a completely regular semigroup S is *fully invariant* if it is invariant under all endomorphisms ϕ of S, that is, if $a \rho b$ implies $(a\phi) \rho (b\phi)$ for all endomorphisms ϕ of S. The set of all fully invariant congruences on S, denoted by FCON(S), is a complete sublattice of the lattice of congruences on S.

Fundamental to the discussion of varieties is the standard correspondence between varieties and fully invariant congruences.

Theorem 2.7.3 ([Gr2], [Rei2]). Let X be a nonempty set. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, let

$$\rho_{\mathscr{V}} = \cap\{ \gamma \in \Lambda(F_{\mathscr{CR}}(X)) \mid F_{\mathscr{CR}}(X)/\gamma \in \mathscr{V} \}.$$

Then $\rho_{\mathscr{V}}$ is a fully invariant congruence on $F_{\mathscr{CR}}(X)$ and, identifying x with $x\rho_{\mathscr{V}}$, $F_{\mathscr{CR}}(X)/\rho_{\mathscr{V}}$ is the free object in $\mathscr{V}$ over X.

Let X be infinite and for any fully invariant congruence ρ on $F_{\mathscr{CR}}(X)$, let $\mathscr{V}_\rho$ denote the subvariety of $\mathscr{CR}$ defined by the family of identities

$$u = v \quad \text{for all } (u, v) \in \rho.$$

Then the mappings

$$\mathscr{V} \longrightarrow \rho_{\mathscr{V}} \quad \text{and} \quad \rho \longrightarrow \mathscr{V}_\rho$$

are mutually inverse order anti-isomorphisms between the lattices $\mathcal{L}(\mathcal{CR})$ and FCON($F_{\mathcal{CR}}(X)$).

We sometimes refer to $\rho_{\mathcal{V}}$ and $\mathcal{V}_\rho$ as the fully invariant congruence corresponding to $\mathcal{V}$ and the variety corresponding to the fully invariant congruence ρ, respectively. Throughout the rest of this thesis, X is assumed to be a fixed countably infinite set, unless otherwise stated; and Γ the lattice of fully invariant congruences on $F_{\mathcal{CR}}(X)$.

In Theorem 2.6.7, we saw that the relations T, T_l and T_r are complete congruences on $\Lambda(S)$ for any $S \in \mathcal{CR}$. A notable absentee from this list was K. We now have:

Theorem 2.7.4 ([P], [Po2]). K is a complete congruence on Γ.

Thus K, T, T_l and T_r are all complete congruences on Γ. Under the anti-isomorphism in Theorem 2.7.3, these carry to complete congruences on $\mathcal{L}(\mathcal{CR})$:

$$\mathcal{U}\,K\,\mathcal{V} \Leftrightarrow \rho_{\mathcal{U}}\,K\,\rho_{\mathcal{V}}, \quad \mathcal{U}\,T\,\mathcal{V} \Leftrightarrow \rho_{\mathcal{U}}\,T\,\rho_{\mathcal{V}}$$

$$\mathcal{U}\,T_l\,\mathcal{V} \Leftrightarrow \rho_{\mathcal{U}}\,T_l\,\rho_{\mathcal{V}}, \quad \mathcal{U}\,T_l\,\mathcal{V} \Leftrightarrow \rho_{\mathcal{U}}\,T_l\,\rho_{\mathcal{V}}.$$

The classes of any complete congruences are intervals and so it is convenient to denote the intervals for these congruences as follows:

$$\mathcal{U}K = [\mathcal{U}_K, \mathcal{U}^K], \quad \mathcal{U}T = [\mathcal{U}_T, \mathcal{U}^T]$$

$$\mathcal{U}T_l = [\mathcal{U}_{T_l}, \mathcal{U}^{T_l}], \quad \mathcal{U}T_r = [\mathcal{U}_{T_r}, \mathcal{U}^{T_r}]$$

Theorem 2.7.5 ([P], [Po1], [Po2]). The mappings

$$\mathcal{U} \longrightarrow \mathcal{U}^K, \ \mathcal{U} \longrightarrow \mathcal{U}_{T_l}, \ \mathcal{U} \longrightarrow \mathcal{U}_{T_r} \quad (\mathcal{U} \in \mathcal{L}(\mathcal{CR}))$$

are complete endomorphisms of $\mathcal{L}(\mathcal{CR})$ inducing the congruences K, T_l and T_r. The mappings

$$\mathcal{U} \longrightarrow \mathcal{U}^T, \ \mathcal{U} \longrightarrow \mathcal{U}^{T_l}, \ \mathcal{U} \longrightarrow \mathcal{U}^{T_r} \quad (\mathcal{U} \in \mathcal{L}(\mathcal{CR}))$$

are complete $\cap$-endomorphisms of $\mathscr{L}(\mathscr{CR})$ but are not $\vee$-homomorphisms.

Surprisingly, the mapping

$$\mathscr{V} \longrightarrow \mathscr{V}_T \quad (\mathscr{V} \in \mathscr{L}(\mathscr{CR}))$$

is not an endomorphism of $\mathscr{L}(\mathscr{CR})$ (see Petrich and Reilly [**PR8,** Proposition 7.6]). It will be important to point out that K, T, T_l and T_r are all idempotent operators on $\mathscr{L}(\mathscr{CR})$. Also, more discussion about these operators will take place later in this thesis. For more information, the reader is also referred to Pastijn **[P]**, Polák (**[Po1]**, **[Po2]**, **[Po3]**), Petrich and Reilly (**[PR6]**, **[PR8]**).

One question about $\mathscr{L}(\mathscr{CR})$ that remained unanswered for a considerable time was whether or not it is a modular lattice (a lattice L is *modular* if $\alpha \leq \gamma \Rightarrow \alpha\vee(\beta\wedge\gamma) = (\alpha\vee\beta)\wedge\gamma$, $\alpha, \beta, \gamma \in L$). Rasin **[R2]** showed that $\mathscr{L}(\mathscr{CS})$ is modular. Hall and Jones **[HJ]** showed that $\mathscr{L}(\mathscr{BG})$ is modular. The question was finally answered with the aid of Polák's Theorem by Pastijn:

Theorem 2.7.6 [P]. $\mathscr{L}(\mathscr{CR})$ is modular.

§ 2.9 Special symbols

For convenience we provide a list of notation introduced in this chapter as well as the notation we will use in the rest of this thesis.

The following special subvarieties of $\mathscr{CR}$ will be involved in this thesis:

$\mathscr{T}$ — the variety of one element semigroups = [x = y].

$\mathscr{S}$ — the variety of semilattices = [$x^2 = x$, xy = yx].

$\mathscr{LZ}$ — the variety of left zero semigroups = [xy = x].

$\mathscr{RZ}$ — the variety of right zero semigroups = [xy = y].

$\mathcal{RB}$ — the variety of rectangular bands = $[\ x^2 = x,\ xyx = x\ ]$.

$\mathcal{LNB}$ — the variety of left normal bands = $[\ x^2 = x,\ xyz = xzy\ ]$.

$\mathcal{RNB}$ — the variety of right normal bands = $[\ x^2 = x,\ xyz = yxz\ ]$.

$\mathcal{NB}$ — the variety of normal bands = $[\ x^2 = x,\ axya = ayxa\ ]$.

$\mathcal{LRB}$ — the variety of left regular bands = $[\ x^2 = x,\ xyx = xy\ ]$.

$\mathcal{RRB}$ — the variety of right regular bands = $[\ x^2 = x,\ xyx = yx\ ]$.

$\mathcal{ReB}$ — the variety of regular bands = $[\ x^2 = x,\ axya = axaya\ ]$.

$\mathcal{B}$ — the variety of bands = $[\ x^2 = x\ ]$.

$\mathcal{G}$ — the variety of groups = $[\ x^0 = y^0\ ]$.

$\mathcal{AG}$ — the variety of abelian groups = $[\ x^0 = y^0,\ xy = yx\ ]$.

$\mathcal{A}_n$ — the variety of abelian groups of exponent n = $[\ x^0 = y^0,\ xy = yx,\ x^n = x^0\ ]$.

$\mathcal{LG}$ — the variety of left groups = $[\ x^0y^0 = x^0\ ]$.

$\mathcal{RG}$ — the variety of right groups = $[\ x^0y^0 = y^0\ ]$.

$\mathcal{ReG}$ — the variety of rectangular groups = $[\ x^0y^0x^0 = x^0\ ]$.

$\mathcal{CS}$ — the variety of completely simple semigroups = $[\ (xyz)^0 = (xz)^0\ ]$.

$\mathcal{A}$ — the variety of completely simple semigroups over abelian groups

$$= [\ (xyz)^0 = (xz)^0,\ x^0yx = xyx^0\ ].$$

$\mathcal{C}$ — the variety of central completely simple semigroups

$$= [\ (xyz)^0 = (xz)^0,\ x^0y^0x = xy^0x^0\ ].$$

$\mathcal{D}$ — the variety $\{\ S \in \mathcal{CS} \mid [\ E(S)\ ] \in \mathcal{A}\ \}$

$$= [\ (xyz)^0 = (xz)^0,\ xy^0x^0z^0x = xz^0x^0y^0x\ ].$$

$\mathcal{SG}$ — the variety of semilattices of groups = $[\ x^0y^0 = y^0x^0\ ]$.

$\mathcal{OG}$ — the variety of orthogroups = $[\ x^0y^0 = (x^0y^0)^0\ ]$.

$\mathcal{BG}$ — the variety of bands of groups = $[\ (x^0y^0)^0 = (xy)^0\ ]$.

$\mathcal{ROG}$ — the variety of regular orthogroups =$[\ x^0y^0 = (x^0y^0)^0,\ ax^0y^0a = ax^0a^0y^0a\ ]$.

$\mathcal{NBG}$ — the variety of normal bands of groups = $[\ (axya)^0 = (\ ayxa)^0\ ]$.

$\mathcal{U}\mathcal{B}\mathcal{G}$ — the variety of those bands of groups such that $S/_{\mathcal{H}} \in \mathcal{U} \in \mathcal{L}(\mathcal{B})$.

$\mathcal{CR}$ — the variety of completely regular semigroups.

Moreover,

$\mathcal{L}(\mathcal{U})$ — the lattice of subvarieties of $\mathcal{U}$.

$<\mathcal{C}>$ — the variety of completely regular semigroups generated by the nonempty class of completely regular semigroups $\mathcal{C}$.

$\mathcal{U} \circ \mathcal{V}$ — the Mal'cev product of the varieties $\mathcal{U}$ and $\mathcal{V}$ (not necessarily a variety).

$[\, u_\alpha = v_\alpha : \alpha \in I \,]$ — the variety of all completely regular semigroups satisfying the identities $u_\alpha = v_\alpha$ ($\alpha \in I$).

[A, B] — the interval of a lattice with minimum A and maximum B.

X — a fixed countably infinite set.

U — the free unary semigroup over X.

c(u) — the set of variables from X appearing in $u \in U$.

$F_{\mathcal{U}}(X)$ — the free object in the variety $\mathcal{U}$ over X.

Γ — the lattice of fully invariant congruences on $F_{\mathcal{CR}}(X)$.

$\rho_{\mathcal{V}}$ — the fully invariant congruence on $F_{\mathcal{CR}}(X)$ corresponding to the variety $\mathcal{V}$

$\mathcal{V}_\rho$ — the variety corresponding to the fully invariant congruence ρ on $F_{\mathcal{CR}}(X)$.

Further notation

$\Sigma(S)$ — the lattice of equivalences on the set or semigroup S.

$\Lambda(S)$ — the lattice of congruences on the semigroup S.

FCON(S) — the lattice of fully invariant congruences on the semigroup S.

E(S) — the set of idempotents of the semigroup S.

$C(S)$ — the subsemigroup of S generated by E(S).

$C^*(S)$ — the least full and self-conjugate subsemigroup of S.

$V(a)$ — the set of inverses of a in S.

$x^0 = x^{-1}x = xx^{-1}$, for any $x \in S \in \mathscr{CR}$.

$CR(S, T)$ — the set of CR-relational morphisms from S into T ($S, T \in \mathscr{CR}$).

End S — the semigroup of all endomorphisms of the semigroup S.

μ — the largest idempotent separating congruence on S.

τ — the largest idempotent pure congruence on S.

$\sigma_{\mathscr{V}}$ — the least congruence on S such that $S/\sigma_{\mathscr{V}} \in \mathscr{V} \in \mathscr{L}(\mathscr{CR})$.

ε — the identity relation.

$\operatorname{tr} \rho$ — the trace of the congruence ρ.

$\ker \rho$ — the kernel of the congruence ρ.

$\mathscr{L}, \mathscr{R}, \mathscr{H}, \mathscr{J}$ and $\mathscr{D}$ — Green's relations.

ρ^0 — the largest congruence contained in the equivalence relation ρ.

ρ^* — the congruence generated by the relation ρ.

$\rho|_T$ — the restriction of the relation ρ to T.

$\bar{\theta}$ — the equivalence relation on A induced by the mapping θ of the set A.

Y^+ — the free semigroup on the nonempty set Y.

Y^* — the free monoid on the set Y.

$h(w)$ — the first variable which appears in $w \in Y^+$($w \in U$).

$t(w)$ — the last variable which appears in $w \in Y^+$($w \in U$).

$\mathcal{T}_Y$ — the semigroup of all transformations on the set Y.

Chapter 3

Mal'cev Products and CR -relational Morphisms

Given two varieties $\mathcal{U}$ and $\mathcal{V}$ of completely regular semigroups, denote by $\mathcal{U}\circ\mathcal{V}$ the class of all completely regular semigroups S on which there is a congruence ρ such that the idempotent ρ -classes are in $\mathcal{U}$ and $S/_{\rho} \in \mathcal{V}$. The class $\mathcal{U}\circ\mathcal{V}$ is said to be the Mal'cev product of $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{CR}$. Our definition is a specialization of Mal'cev's original definition [M]. In general $\mathcal{U}\circ\mathcal{V}$ need not again be a variety. We define $< \mathcal{U}\circ\mathcal{V} >$ to be the variety of completely regular semigroups generated by $\mathcal{U}\circ\mathcal{V}$.

The first section of this chapter studies joins of congruences on Rees matrix semigroups. This enables us to extend a result of Jones [J5] by showing that $\mathcal{U}\circ\mathcal{V}$ is again a variety if $\mathcal{U}\in\mathcal{L}(\mathcal{C})$ and $\mathcal{V}\in\mathcal{L}(\mathcal{CR})$. We introduce the concept of CR-relational morphism in the fourth section. This makes it possible to describe the varieties of the form $< \mathcal{U}\circ\mathcal{V} >$. This description will plays an important role in subsequent chapters.

§ 3.1 Congruences on Rees matrix semigroups

Congruences on Rees matrix semigroups have been described completely in terms of the admissible triples. The details of this treatment can be found in either Howie [How1] or Lallement [L1]. Here we present a special form of such description discovered by Tamura [T].

Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup with normalized sandwich matrix P. Let N be a normal subgroup of G. Define P_N and π_N on I and Λ respectively, as follows:

$$i\, P_N\, j \;\Leftrightarrow\; p_{\lambda i}\, p_{\lambda j}^{-1} \in N \quad \text{for every } \lambda \in \Lambda;$$

$$\lambda\, \pi_N\, \mu \;\Leftrightarrow\; p_{\lambda i}\, p_{\mu i}^{-1} \in N \quad \text{for every } i \in I.$$

Then P_N and π_N are equivalence relations on I and Λ respectively. Let $P \subseteq P_N$ and $\pi \subseteq \pi_N$ be equivalence relations on I and Λ respectively, and define the relation $(N; P, \pi)$ on S by:

$$(g_1 ; i_1, \lambda_1)\, (N; P, \pi)\, (g_2 ; i_2, \lambda_2) \;\Leftrightarrow\; Ng_1 = Ng_2,\ i_1\, P\, i_2 \text{ and } \lambda_1\, \pi\, \lambda_2.$$

We then have

Theorem 3.1.1 [T]. The relation $(N; P, \pi)$ is a congruence on S. Conversely given a congruence θ on S there exists $N \triangleleft G$, $P \subseteq P_N$ and $\pi \subseteq \pi_N$ such that $\theta = (N; P, \pi)$.

The following useful result can be derived easily from Howie [How1].

Lemma 3.1.2 [How1]. Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup with normalized P. If $\alpha = (N_\alpha; P_\alpha , \pi_\alpha)$ and $\rho = (N_\rho; P_\rho , \pi_\rho)$ are congruences on S. Then

(i) $\alpha \subseteq \rho \;\Leftrightarrow\; P_\alpha \subseteq P_\rho , \pi_\alpha \subseteq \pi_\rho$ and $N_\alpha \subseteq N_\rho$.

(ii) $\alpha \cap \rho = (N_\alpha \cap N_\rho; P_\alpha \cap P_\rho, \pi_\alpha \cap \pi_\rho)$.

(iii) $\alpha \vee \rho = (N_\alpha N_\rho; P_\alpha \vee P_\rho, \pi_\alpha \vee \pi_\rho)$.

Lemma 3.1.3. Let $\theta = (N; P, \pi)$ be a congruence on $S = \mathcal{M}(G; I, \Lambda; P)$ with normalized P. Then

(i) θ is idempotent pure $\Leftrightarrow$ $N = \{ e \}$, where e denotes the identity of G.

(ii) θ is idempotent separating $\Leftrightarrow$ $P = \varepsilon_I, \pi = \varepsilon_\Lambda$.

Proof. (i) *Necessity.* Assume that $N \neq \{ e \}$, then there exist $g_1, g_2 \in N$ such that $g_1 \neq g_2$ and $Ng_1 = Ng_2$. Let $i \in I$ and $\lambda \in \Lambda$, then $(g_1; i, \lambda)$ and $(g_2; i, \lambda)$ are two distinct elements of S such that $(g_1; i, \lambda) \, \theta \cap \mathscr{H} \, (g_2; i, \lambda)$, and so $\theta \cap \mathscr{H} \neq \varepsilon_S$, a contradiction, by Lemma 2.6.10.

Sufficiency. It suffices to show that $\theta \cap \mathscr{H} = \varepsilon_S$. Let $(g_1 ; i_1, \lambda_1) \, \theta \cap \mathscr{H} \, (g_2 ; i_2, \lambda_2)$. By Lemma 2.4.5, we have $i_1 = i_2$ and $\lambda_1 = \lambda_2$. Since $(g_1 ; i_1, \lambda_1) \, \theta \, (g_2 ; i_2, \lambda_2)$, then $Ng_1 = Ng_2$, so that $g_1 = g_2$. Hence $\theta \cap \mathscr{H} = \varepsilon_S$.

(ii) *Necessity.* Assume that $P \neq \varepsilon_I$, then there exist $i_1, i_2 \in I$ such that $i_1 \neq i_2$ and $i_1 \, P \, i_2$. Let $\lambda \in \Lambda$, $e = (p_{\lambda i_1}^{-1}; i_1, \lambda)$ and $f = (p_{\lambda i_2}^{-1}; i_2, \lambda)$. Then e and f are two distinct idempotents of S. Since $i_1 \, P \, i_2$, then $p_{\lambda i_1} p_{\lambda i_2}^{-1} \in N$, so that $p_{\lambda i_2}^{-1} p_{\lambda i_1} = p_{\lambda i_1}^{-1}(p_{\lambda i_1} p_{\lambda i_2}^{-1}) p_{\lambda i_1} \in N$, since $N \lhd G$. Thus $Np_{\lambda i_1}^{-1} = Np_{\lambda i_2}^{-1}$, and so $e \, \theta \, f$, a contradiction. Hence $P = \varepsilon_I$. In a similar way, one can show that $\pi = \varepsilon_\Lambda$.

Sufficiency. Note that $E(S) = \{ (p_{\lambda i}^{-1}; i, \lambda) \mid i \in I, \lambda \in \Lambda \}$, the assertion follows easily. #

The following straightforward corollary will be used throughout this section without explicit reference.

Corollary 3.1.4. Let $\theta = (N; P, \pi)$ be a congruence on $S = \mathscr{M}(G; I, \Lambda; P)$ with normalized P and $e = (p_{\lambda i}^{-1}; i, \lambda)$ and $f = (p_{\mu j}^{-1}; j, \mu) \in E(S)$. Then $e \, \theta \, f$ if and only if $i \, P \, j$ and $\lambda \, \pi \, \mu$.

Lemma 3.1.5. Let S be a completely simple semigroup and let $\alpha, \rho \in \Lambda(S)$, with ρ idempotent pure. Then for any $e \in E(S)$, we have $H_e \cap e(\alpha \vee \rho) = H_e \cap e\alpha$.

Proof. By Theorem 2.4.3, we may assume that $S = \mathscr{M}(G; I, \Lambda; P)$ with normalized P. By Theorem 3.1.1 and Lemma 3.1.3 (i), we may assume that $\alpha = (N_\alpha; P_\alpha, \pi_\alpha)$ and $\rho = (\{e\}; P_\rho, \pi_\rho)$. It follows from Lemma 3.1.2 (iii) that

$$\alpha \vee \rho = (N_\alpha; P_\alpha \vee P_\rho, \pi_\alpha \vee \pi_\rho).$$

For any $e \in E(S)$, we have $e = (p_{\lambda i}^{-1}; i, \lambda)$ for some $i \in I$ and $\lambda \in \Lambda$. By Theorem 3.1.1,

$$H_e \cap e(\alpha \vee \rho) = \{ (g; i, \lambda) \mid g \in N_\alpha p_{\lambda i}^{-1} \}$$
$$= H_e \cap e\alpha,$$

as required. #

Definition 3.1.6. Let $S \in \mathscr{CR}$ and $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$. A congruence ρ on S is *over* $\mathscr{V}$ if $e\rho \in \mathscr{V}$ for each $e \in E(S)$.

The next result can be derived easily from Jones [J5]. We sketch the proof for completeness.

Lemma 3.1.7 [J5]. Let $S \in \mathscr{CR}$ and let $\alpha, \rho \in \Lambda(S)$, with ρ idempotent pure. Then for any $\mathscr{V} \in \{ \mathscr{LG}, \mathscr{RG}, \mathscr{G} \}$, α is over $\mathscr{V}$ implies $(\alpha \vee \rho)/_\rho$ is over $\mathscr{V}$.

Proof. (i) $\mathscr{V} = \mathscr{LG}$. To show that $(\alpha \vee \rho)/_\rho$ is over $\mathscr{LG}$, it suffices to show that $(\alpha \vee \rho)/_\rho \subseteq \mathscr{L}$, by Lemma 2.4.8. So let $a, b \in S$, with $(a\rho, b\rho) \in (\alpha \vee \rho)/_\rho$, that is, $a (\alpha \vee \rho) b$. Then there exist $a_i, b_i \in S$ ($i = 0, 1, \ldots, n$) such that

$$a = a_0 \,\alpha\, b_0 \,\rho\, a_1 \alpha\, b_1 \ldots a_n \,\alpha\, b_n = b, \qquad \text{by Lemma 2.2.3.}$$

Since $\alpha \subseteq \mathscr{L}$, then $a_i \mathscr{L} b_i$, and so $(a_i\rho) \mathscr{L} (b_i\rho)$ in $S/_\rho$. Thus

$$a\rho = (a_0\rho) \mathscr{L} (b_0\rho) = (a_1\rho) \mathscr{L} (b_1\rho) = \ldots = (a_n\rho) \mathscr{L} (b_n\rho) = b\rho,$$

and whence $(a\rho) \mathscr{L} (b\rho)$ in $S/_\rho$, as required.

(ii) $\mathscr{V} = \mathscr{RG}$. This is the dual of (i).

(iii) $\mathscr{V} = \mathscr{G}$. Since $\mathscr{G} = \mathscr{LG} \cap \mathscr{RG}$, the assertion follows easily. #

Before proceeding, we need some preparation.

Notation 3.1.8. Let $\mathscr{A}$ denote the variety of all completely simple semigroups with abelian groups.

We here provide some simple characterizations of the variety $\mathscr{A}$.

Lemma 3.1.9 ([PR5], [R2]). The following conditions on a completely simple semigroup S are equivalent.

(i) $S \in \mathscr{A}$.

(ii) S satisfies the identity $x^0yx = xyx^0$.

(iii) S satisfies the identity $x^2yx = xyx^2$.

Definition 3.1.10. A completely simple semigroup S is *central* if the product of any two idempotents of S lies in the centre of the containing maximal subgroup. We denote by $\mathscr{C}$ the variety of all central completely simple semigroups.

Lemma 3.1.11 [PR5]. The following conditions on a completely simple semigroup S are equivalent.

(i) S is central.

(ii) In every (respectively, some) Rees matrix representation $\mathcal{M}(G; I, \Lambda; P)$ of S with normalized P, all entries of P lie in the centre of G.

(iii) S satisfies the identity $x^0y^0x = xy^0x^0$.

(iv) $S \in \mathscr{A} \vee \mathscr{G}$.

The lattice of all subvarieties of $\mathscr{C}$ has been described completely by Petrich and Reilly [PR4] in the following way.

Notation 3.1.12. For $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$, let I($\mathcal{V}$) denote the class of all idempotent generated members of $\mathcal{V}$ and let < I($\mathcal{V}$) > denote the variety of completely simple semigroups generated by I($\mathcal{V}$).

Notation 3.1.13. Let $\mathcal{AG}$ denote the variety of all abelian groups. For each $k \geq 1$, let $\mathcal{A}_k$ denote the variety of abelian groups of exponent k.

Lemma 3.1.14 [N]. (i) $\mathcal{AG} = [\, x^0 = y^0,\ xy = yx \,]$.

(ii) $\mathcal{A}_k = [\, x^0 = y^0,\ xy = yx,\ x^k = x^0 \,]$, $k \geq 1$.

(iii) Every subvariety of $\mathcal{AG}$ is either $\mathcal{AG}$ or $\mathcal{A}_k$ for some $k \geq 1$.

Theorem 3.1.15 [PR4]. The mapping

$$\zeta :\ \mathcal{V} \longrightarrow (\mathcal{V} \cap \mathcal{RB},\ < I(\mathcal{V}) > \cap\, \mathcal{AG},\ \mathcal{V} \cap \mathcal{G}) \qquad (\ \mathcal{V} \in \mathcal{L}(\mathcal{C})\ )$$

is an isomorphism of $\mathcal{L}(\mathcal{C})$ onto the subdirect product

$$\{\ (\mathcal{W}, \mathcal{X}, \mathcal{Y}\ ) \in \mathcal{L}(\mathcal{RB}) \times \mathcal{L}(\mathcal{AG}) \times \mathcal{L}(\mathcal{G}) \ \mid\ \mathcal{X} \subseteq \mathcal{Y},\ \mathcal{W} \neq \mathcal{RB}\ \Rightarrow\ \mathcal{X} = \mathcal{T}\ \}.$$

Moreover, if $\mathcal{V} \in \mathcal{L}(\mathcal{C})$ and $\mathcal{V}\zeta = (\mathcal{W}, \mathcal{X}, \mathcal{Y}\ )$, then

$$\mathcal{V} = \{\ S \in \mathcal{C} \ \mid\ S/_{\mathcal{H}} \in \mathcal{W},\ \text{subgroups of } [\ E(S)\ ] \text{ lie in } \mathcal{X},\ \text{subgroups of } S \text{ lie in } \mathcal{Y}\ \}.$$

In the next proposition, the case $\mathcal{V} = \mathcal{OBG}$ has been obtained by Jones [J5]. Here we provide an alternative proof of this fact.

Proposition 3.1.16. Let S be a completely simple semigroup and let $\alpha, \rho \in \Lambda(S)$, with ρ idempotent pure. Then for $\mathcal{V} \in \{\ \mathcal{OBG}, \mathcal{C}\ \}$, α is over $\mathcal{V}$ implies that $\alpha \vee \rho$ is over $\mathcal{V}$.

Proof. By Theorem 2.4.3, we may assume that $S = \mathcal{M}(G; I, \Lambda; P)$ with normalized P. By Theorem 3.1.1 and Lemma 3.1.3(i), we may assume that $\alpha = (\, N_\alpha; P_\alpha, \pi_\alpha \,)$ and $\rho = (\, \{e\}; P_\rho, \pi_\rho \,)$. Thus, $\alpha \vee \rho = (\, N_\alpha; P_\alpha \vee P_\rho, \pi_\alpha \vee \pi_\rho \,)$ by Lemma 3.1.2 (iii). Let e and f be two arbitrarily elements of E(S) with $e\ (\ \alpha \vee \rho\ )\ f$. Then there exist $e_i,\ f_i \in E(S)$ ($i = 0, 1, \dots, n$) such that

$$e = e_0\, \alpha\, f_0\, \rho\, e_1\, \alpha\, f_1 \dots e_n\, \alpha\, f_n = f, \quad \text{by Lemma 2.3.3;}$$

where, $e_k = (\ p_{\lambda_k i_k}{}^{-1};\ i_k, \lambda_k\ )$ and $f_k = (\ p_{\mu_k j_k}{}^{-1};\ j_k, \mu_k\ )$, $k = 0, 1, \dots, n$.

Since $f_k\ \rho\ e_{k+1}$ ($k = 0, 1, \dots, n\text{-}1$), then we have

(1) $p_{\mu_k j_k} = p_{\lambda_{k+1} i_{k+1}}$;

(2) $p_{\lambda j_k} = p_{\lambda i_{k+1}}$ for all $\lambda \in \Lambda$, since $j_k\, P_\rho\, i_{k+1}$;

(3) $p_{\mu_k j} = p_{\lambda_{k+1} j}$ for all $j \in I$, since $\mu_k\, \pi_\rho\, \lambda_{k+1}$.

Since $e_k\ \alpha\ f_k$ ($k = 0, 1, \dots, n$), then we have

(4) $p_{\lambda i_k} p_{\lambda j_k}{}^{-1} \in N_\alpha$ for all $\lambda \in \Lambda$;

(5) $p_{\lambda_k j} p_{\mu_k j}{}^{-1} \in N_\alpha$ for all $j \in I$.

Case 1: $\mathscr{V} = \mathscr{RG}$. To show that $\alpha \vee \rho$ is over $\mathscr{RG}$, it suffices to show that

$$ef = (\ p_{\lambda_0 i_0}{}^{-1};\ i_0, \lambda_0\ )(\ p_{\mu_n j_n}{}^{-1};\ j_n, \mu_n\ )$$
$$= (\ p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1};\ i_0, \mu_n\ )$$
$$\in E(S), \qquad \text{by Lemma 2.4.7;}$$

that is, $ef = (\ p_{\mu_n i_0}{}^{-1};\ i_0, \mu_n\ )$. Thus, it remains to show that $p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1} = p_{\mu_n i_0}{}^{-1}$.

For any $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i\, P_\alpha\, j$ and $\lambda\, \pi_\alpha\, \mu$, then $(\ p_{\lambda i}{}^{-1};\ i, \lambda\ )$, $(\ p_{\mu j}{}^{-1};\ j, \mu)$ $\in E(S)$, and so $(\ p_{\lambda i}{}^{-1};\ i, \lambda\ )\ \alpha\ (\ p_{\mu j}{}^{-1};\ j, \mu\ )$, by Corollary 3.1.4. Since α is over $\mathscr{RG}$, then $(\ p_{\lambda i}{}^{-1};\ i, \lambda\ )(\ p_{\mu j}{}^{-1};\ j, \mu\ ) = (\ p_{\lambda i}{}^{-1} p_{\lambda j} p_{\mu j}{}^{-1};\ i, \mu\ ) \in E(S)$, so that we have

(6) $p_{\lambda i}{}^{-1} p_{\lambda j} p_{\mu j}{}^{-1} = p_{\mu i}{}^{-1}$.

For $k \in \{\ 0, 1, \dots, n\ \}$. By (2), we have

$$p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_n} p_{\mu_k j_n}{}^{-1} = (\ p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0}\ ) \cdots (\ p_{\lambda_k i_{n-1}}{}^{-1} p_{\lambda_k j_{n-1}}\ )\ \{\ (\ p_{\lambda_k i_n}{}^{-1} p_{\lambda_k j_n}\ ) p_{\mu_k j_n}{}^{-1}\ \}$$
$$= (\ p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0}\ ) \cdots (\ p_{\lambda_k i_{n-1}}{}^{-1} p_{\lambda_k j_{n-1}}\ ) p_{\mu_k i_n}{}^{-1} \qquad \text{by (6)}$$
$$= (\ p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0}\ ) \cdots (\ p_{\lambda_k i_{n-2}}{}^{-1} p_{\lambda_k j_{n-2}}\ )(\ p_{\lambda_k i_{n-1}}{}^{-1} p_{\lambda_k j_{n-1}} p_{\mu_k j_{n-1}}{}^{-1}\ ) \qquad \text{by (2)}$$
$$= (\ p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0}\ ) \cdots (\ p_{\lambda_k i_{n-2}}{}^{-1} p_{\lambda_k j_{n-2}}\ ) p_{\mu_k i_{n-1}}{}^{-1} \qquad \text{by (6)}$$

$= \ldots\ldots$ (repeating the same procedure for n-2 times)

$= (p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0}) p_{\mu_k i_1}{}^{-1}$

$= p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_0} p_{\mu_k j_0}{}^{-1}$ by (2)

$= p_{\mu_k i_0}{}^{-1}$ by (6).

Thus, we have proved that

(7) $p_{\lambda_k i_0}{}^{-1} p_{\lambda_k j_n} p_{\mu_k j_n}{}^{-1} = p_{\mu_k i_0}{}^{-1}$ for all $k \in \{ 0, 1, \ldots , n \}$.

Hence, we have

$p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1} = \{ p_{\lambda_0 i_0}{}^{-1} (p_{\lambda_0 j_n} p_{\mu_0 j_n}{}^{-1}) \} (p_{\lambda_1 j_n} p_{\mu_1 j_n}{}^{-1}) \ldots (p_{\lambda_n j_n} p_{\mu_n j_n}{}^{-1})$ by (3)

$= p_{\mu_0 i_0}{}^{-1} (p_{\lambda_1 j_n} p_{\mu_1 j_n}{}^{-1}) \ldots (p_{\lambda_n j_n} p_{\mu_n j_n}{}^{-1})$ by (7) with $k = 0$

$= \{ p_{\lambda_1 i_0}{}^{-1} (p_{\lambda_1 j_n} p_{\mu_1 j_n}{}^{-1}) \} \ldots (p_{\lambda_n j_n} p_{\mu_n j_n}{}^{-1})$ by (3)

$= \ldots\ldots$ (repeating the same procedure for $k = 1, 2, \ldots , n-1$)

$= p_{\mu_{n-1} i_0}{}^{-1} (p_{\lambda_n j_n} p_{\mu_n i_n}{}^{-1})$

$= p_{\lambda_n i_0}{}^{-1} p_{\lambda_n j_n} p_{\mu_n j_n}{}^{-1}$ by (3)

$= p_{\mu_n i_0}{}^{-1}$ by (7).

The proof of Case 1 is completed.

Case 2: $\mathscr{V} = \mathscr{C}$. By Corollary 3.1.4, we have

(8) $N_\alpha p_{\lambda i}{}^{-1} = N_\alpha p_{\mu j}{}^{-1}$ for all $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i (P_\alpha \vee P_\rho) j$ and $\lambda (\pi_\alpha \vee \pi_\rho) \mu$.

To show that $\alpha \vee \rho$ is over $\mathscr{C}$, it suffices to show that

$$ef = (p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1}; i_0, \mu_n)$$

is in the centre of $H_{ef} \cap e(\alpha \vee \rho)$. Note that

$$H_{ef} \cap e(\alpha \vee \rho) = \{ (g; i_0, \mu_n) \mid g \in N_\alpha p_{\mu_n i_0}{}^{-1} \}.$$

Thus, it remains to show that

$$g p_{\mu_n i_0} (p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1}) = (p_{\lambda_0 i_0}{}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}{}^{-1}) p_{\mu_n i_0} g \quad \text{for all } g \in N_\alpha p_{\mu_n i_0}{}^{-1}.$$

For any $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i \, P_\alpha \, j$ and $\lambda \, \pi_\alpha \, \mu$, let $e' = (p_{\lambda i}^{-1}; i, \lambda)$ and $f' = (p_{\mu j}^{-1}; j, \mu)$. Then $e', f' \in E(S)$ and $e' \, \alpha \, f'$, by Corollary 3.1.4. Since α is over $\mathscr{C}$, then $e'f'$ lies in the centre $H_{e'f'} \cap e'(\alpha \vee \rho)$. It follows that

(9) $g p_{\mu i}(p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}) = (p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}) p_{\mu i} g \quad$ for all $g \in N_\alpha p_{\mu i}^{-1}$.

For $k \in \{0, 1, \ldots, n\}$ and $g \in N_\alpha p_{\mu_n i_0}^{-1}$. By (3), we have

$$\begin{aligned}
& g(p_{\mu_n i_k} p_{\lambda_0 i_k}^{-1}) p_{\lambda_0 j_k} \\
&= g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_1 i_k} p_{\lambda_1 i_k}^{-1})(p_{\mu_0 i_k} p_{\lambda_0 i_k}^{-1}) p_{\lambda_0 j_k} \\
&= \{ g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_1 i_k} p_{\lambda_1 i_k}^{-1}) \} p_{\mu_0 i_k}(p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_0 j_k}^{-1}) p_{\mu_0 j_k} \\
&= p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_0 j_k}^{-1} p_{\mu_0 i_k} g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_1 i_k} p_{\lambda_1 i_k}^{-1}) p_{\mu_0 j_k} \\
&\qquad \text{since } g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_1 i_k} p_{\lambda_1 i_k}^{-1}) \in N_\alpha p_{\mu_0 i_k}^{-1} \text{ and by (9)} \\
&= p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_0 j_k}^{-1} p_{\mu_0 i_k} \{ g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_2 i_k} p_{\lambda_2 i_k}^{-1}) \} p_{\mu_1 i_k} \\
&\qquad \cdot (p_{\lambda_1 i_k}^{-1} p_{\lambda_1 j_k} p_{\mu_1 j_k}^{-1}) p_{\mu_1 j_k} \qquad \text{since } p_{\mu_0 j_k} = p_{\lambda_1 j_k} \\
&= p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_0 j_k}^{-1} p_{\mu_0 i_k} p_{\lambda_1 i_k}^{-1} p_{\lambda_1 j_k} p_{\mu_1 j_k}^{-1} p_{\mu_1 i_k} g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \\
&\qquad \cdots (p_{\mu_2 i_k} p_{\lambda_2 i_k}^{-1}) p_{\mu_1 j_k} \\
&\qquad \text{since } g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_2 i_k} p_{\lambda_2 i_k}^{-1}) \in N_\alpha p_{\mu_1 i_k}^{-1} \text{ and by (9)} \\
&= p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_1 j_k}^{-1} p_{\mu_1 i_k} g(p_{\mu_n i_k} p_{\lambda_n i_k}^{-1}) \cdots (p_{\mu_2 i_k} p_{\lambda_2 i_k}^{-1}) p_{\mu_1 j_k} \\
&\qquad \text{since } p_{\mu_0 i_k} = p_{\lambda_1 i_k} \text{ and } p_{\mu_0 j_k} = p_{\lambda_1 j_k} \\
&= \ldots\ldots \quad (\text{repeating the same procedure n-1 times}) \\
&= p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_n j_k}^{-1} p_{\mu_n i_k} g p_{\mu_n j_k}.
\end{aligned}$$

Thus, we have proved that

(10) $g(p_{\mu_n i_k} p_{\lambda_0 i_k}^{-1}) p_{\lambda_0 j_k} = p_{\lambda_0 i_k}^{-1} p_{\lambda_0 j_k} p_{\mu_n j_k}^{-1} p_{\mu_n i_k} g p_{\mu_n j_k}$

for all $k \in \{0, 1, \ldots, n\}$ and $g \in N_\alpha p_{\mu_n i_0}^{-1}$.

For any $g \in N_\alpha p_{\mu_n i_0}^{-1}$, we now have

$$\begin{aligned}
& g p_{\mu_n i_0}(p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}^{-1}) \\
&= g p_{\mu_n i_0}(p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1} \\
&= g p_{\mu_n i_0}(p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0})(p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1}) \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1} \qquad \text{by (2)}
\end{aligned}$$

$$= \{ g(p_{\mu_n i_0} p_{\lambda_0 i_0}^{-1}) p_{\lambda_0 j_0} \} (p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1}) \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1}$$

$$= p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0} p_{\mu_n j_0}^{-1} p_{\mu_n i_0} g p_{\mu_n j_0} (p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1}) \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1}$$

by applying (10) with $k = 0$.

$$= (p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0}) \{ (p_{\mu_n j_0}^{-1} p_{\mu_n i_0}) g (p_{\mu_n i_1} p_{\lambda_0 i_1}^{-1}) p_{\lambda_0 j_1} \} \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1}$$

since $p_{\mu_n j_0} = p_{\mu_n i_1}$

$$= (p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0}) (p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1} p_{\mu_n j_1}^{-1} p_{\mu_n i_1}) (p_{\mu_n j_0}^{-1} p_{\mu_n i_0}) g p_{\mu_n j_1} (p_{\lambda_0 i_2}^{-1} p_{\lambda_0 j_2}) \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1}$$

since $(p_{\mu_n j_0}^{-1} p_{\mu_n i_0}) g \in N_\alpha p_{\mu_n i_0}^{-1}$, and by applying (10) with $k = 1$

$$= (p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0}) (p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1}) \{ (p_{\mu_n j_1}^{-1} p_{\mu_n i_1}) (p_{\mu_n j_0}^{-1} p_{\mu_n i_0}) g (p_{\mu_n i_2} p_{\lambda_0 i_2}^{-1}) p_{\lambda_0 j_2} \} \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) p_{\mu_n j_n}^{-1}$$

since $p_{\mu_n j_1} = p_{\mu_n i_2}$

$= \ldots\ldots$ (repeating the same procedure for $k = 2, \ldots, n$)

$$= (p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_0}) (p_{\lambda_0 i_1}^{-1} p_{\lambda_0 j_1}) \cdots (p_{\lambda_0 i_n}^{-1} p_{\lambda_0 j_n}) \{ (p_{\mu_n j_n}^{-1} p_{\mu_n i_n}) (p_{\mu_n j_{n-1}}^{-1} p_{\mu_n i_{n-1}}) \cdots (p_{\mu_n j_0}^{-1} p_{\mu_n i_0}) g \} p_{\mu_n j_n} p_{\mu_n j_n}^{-1}$$

$$= (p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}^{-1}) p_{\mu_n i_0} g \quad \text{by (2) and (3).}$$

The proof of Case 2 is completed. #

Notation 3.1.17. We now introduce one more subvariety of $\mathscr{CS}$:

$$\mathscr{D} = [\, ax^0a^0y^0a = ay^0a^0x^0a \,],$$

and point out the obvious fact that $\mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{D}$.

Lemma 3.1.18 [PR5]. The following conditions on a completely simple semigroup S are equivalent.

(i) $S \in \mathscr{D}$.

(ii) In every (respectively some) Rees matrix representation of S with a normalized sandwich matrix P the entries of P commute.

(iii) $[\, E(S) \,] \in \mathscr{A}$.

Lemma 3.1.19. Let S be a completely simple semigroup and let $\alpha \in \Lambda(S)$ be such that α is over $\mathscr{D}$. Then for any $k \geq 1$, the following two conditions are equivalent.

(i) For any $e, f \in E(S)$, $e\, \alpha\, f$ implies $(ef)^k = (ef)^0$.

(ii) For any $e \in E(S)$, subgroups of $[\, E(e\alpha)\,]$ lie in $\mathscr{A}_k$.

Proof. (i) implies (ii). For any $e \in E(S)$, we have $e\alpha \in \mathscr{D}$, since α is over $\mathscr{D}$. Let H be the maximal subgroup of $e\alpha$ containing e and let

$$I_e = \{\, e' \in E(e\alpha) \mid e'e = e' \,\},\quad \Lambda_e = \{\, f \in E(e\alpha) \mid ef = f \,\},$$

$P_e = (p_{fe'})$ where $p_{fe'} = fe'$. By Theorem 2.4.3, $e\alpha \cong \mathscr{M}(H; I_e, \Lambda_e; P_e)$, and so $[\, E(e\alpha)\,] \cong \mathscr{M}(< P_e >; I_e, \Lambda_e; P_e)$ by Lemma 2.4.6, where $< P_e >$ is the subgroup of H generated by the entries of P_e. From Lemma 3.1.18, $< P_e > \in \mathscr{AG}$, and so $< P_e > \in \mathscr{A}_k$ since $(fe')^k = (fe')^0 = e$ for all $f \in \Lambda_e$, $e' \in I_e$. Hence, subgroups of $[\, E(e\alpha)\,]$ lie in $\mathscr{A}_k$.

(ii) implies (i). This is straightforward. #

Proposition 3.1.20. Let S be a completely simple semigroup and let $\alpha, \rho \in \Lambda(S)$, with ρ idempotent pure and $\alpha \vee \rho$ is over $\mathscr{G}$. Then for any $k \geq 1$, the following two statements are equivalent.

(i) For any $e \in E(S)$, subgroups of $[\, E(e(\alpha \vee \rho))\,]$ lie in $\mathscr{A}_k$.

(ii) For any $e \in E(S)$, subgroups of $[\, E(e\alpha)\,]$ lie in $\mathscr{A}_k$.

Proof. (i) implies (ii). This is obvious.

(ii) implies (i). In terms of the discussion at the beginning of the proof of Proposition 3.1.16, we may take $S = \mathscr{M}(G; I, \Lambda; P)$ with normalized P, $\alpha = (N_\alpha; P_\alpha, \pi_\alpha)$ and $\rho = (\{e\}; P_\rho, \pi_\rho)$, so that $\alpha \vee \rho = (N_\alpha; P_\alpha \vee P_\rho, \pi_\alpha \vee \pi_\rho)$. For any $e, f \in E(S)$ with $e\,(\alpha \vee \rho)\,f$, there exist $e_m, f_m \in E(S)$ $(m = 0, 1, \ldots, n)$ such that

$$e = e_0\, \alpha\, f_0\, \rho\, e_1\, \alpha\, f_1 \ldots e_n\, \alpha\, f_n = f,$$

where $e_m = (p_{\lambda_m i_m}^{-1}; i_m, \lambda_m)$ and $f_m = (p_{\mu_m j_m}^{-1}; j_m, \mu_m)$, $m = 0, 1, \dots , n$; further, for $m \in \{ 0, 1, \dots , n-1 \}$, we have

(a) $p_{\lambda j_m} = p_{\lambda i_{m+1}}$ for all $\lambda \in \Lambda$ and $p_{\mu_m j} = p_{\lambda_{m+1} j}$ for all $j \in I$;

and for $m \in \{ 0, 1, 2, \dots , n \}$, we have

(b) $p_{\lambda i_m} p_{\lambda j_m}^{-1} \in N_\alpha$ for all $\lambda \in \Lambda$ and $p_{\lambda_m j} p_{\mu_m j}^{-1} \in N_\alpha$ for all $j \in I$.

By Lemma 3.1.19, it suffices to show that $(ef)^k = (ef)^0$. Since

$$(ef)^k = [(p_{\lambda_0 i_0}^{-1}; i_0, \lambda_0)(p_{\mu_n j_n}^{-1}; j_n, \mu_n)]^k$$
$$= ((p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}^{-1} p_{\mu_n i_0})^k p_{\mu_n i_0}^{-1}; i_0, \mu_n)$$

and $$(ef)^0 = (p_{\mu_n i_0}^{-1}; i_0, \mu_n),$$

it remains to show that $(p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}^{-1} p_{\mu_n i_0})^k p_{\mu_n i_0}^{-1} = p_{\mu_n i_0}^{-1}$, or equivalently $(p_{\lambda_0 i_0}^{-1} p_{\lambda_0 j_n} p_{\mu_n j_n}^{-1} p_{\mu_n i_0})^k = 1_G$, the identity of G.

For any $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i \, P_\alpha \, j$ and $\lambda \, \pi_\alpha \, \mu$, let $e' = (p_{\lambda i}^{-1}; i, \lambda)$ and $f' = (p_{\mu j}^{-1}; j, \mu)$. Then $e', f' \in E(S)$ and $e' \, \alpha \, f'$, by Corollary 3.1.4. By (ii) and Lemma 3.1.19, $(e'f')^k = (e'f')^0$. It follows that

(c) $(p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} p_{\mu i})^k = 1_G$, for any $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i \, P_\alpha \, j$ and $\lambda \, \pi_\alpha \, \mu$.

Since $\alpha \vee \rho$ is over $\mathscr{C}$, then from the proof of Case 2 of Proposition 3.1.16, we have

(d) For $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i \, (P_\alpha \vee P_\rho) \, j$ and $\lambda \, (\pi_\alpha \vee \pi_\rho) \, \mu$, then

$$g p_{\mu i} (p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}) = (p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}) p_{\mu i} g \qquad \text{for all } g \in N_\alpha p_{\mu i}^{-1}.$$

Further, we have

(e) For any $i, j \in I$ and $\lambda, \mu \in \Lambda$, with $i \, (P_\alpha \vee P_\rho) \, j$ and $\lambda \, (\pi_\alpha \vee \pi_\rho) \, \mu$, then

$$N_\alpha p_{\lambda i}^{-1} = N_\alpha p_{\mu j}^{-1}.$$

For $r, m \in \{ 0, 1, 2, \dots , n \}$, $t \in \{ 1, 2, \dots , k \}$ and $g \in N_\alpha$, then

$$[g(p_{\lambda_r i_m}^{-1} p_{\lambda_r j_m}) \cdots (p_{\lambda_r i_n}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}^{-1} p_{\mu_n i_0})]^t \, p_{\mu_n i_0}^{-1}$$
$$= g(p_{\lambda_r i_m}^{-1} p_{\lambda_r j_m} p_{\mu_r j_m}^{-1}) p_{\mu_r i_m} \{ (p_{\mu_r i_m}^{-1} p_{\mu_r j_m})(p_{\lambda_r i_{m+1}}^{-1} p_{\lambda_r j_{m+1}})$$
$$\cdots (p_{\lambda_r i_n}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}^{-1} p_{\mu_n i_0}) [g(p_{\lambda_r i_m}^{-1} p_{\lambda_r j_m})$$
$$\cdots (p_{\lambda_r i_n}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}^{-1} p_{\mu_n i_0})]^{t-1} \, p_{\mu_n i_0}^{-1} \}$$

noting that the products in { } belong to $N_\alpha p_{\mu_r i_m}{}^{-1}$, by (b) and (e)

$$= [\, g(p_{\mu_r i_m}{}^{-1} p_{\mu_r j_m})(p_{\lambda_r i_{m+1}}{}^{-1} p_{\lambda_r j_{m+1}}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]$$
$$\bullet [\, g(p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^{t-1} p_{\mu_n i_0}{}^{-1}$$
$$\bullet (p_{\mu_r i_m} p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m} p_{\mu_r j_m}{}^{-1}) \qquad \text{by (d),}$$

thus, by induction on t, we have

(f) For $r, m \in \{ 0. 1. 2, \dots, n \}$ and $g \in N_\alpha$,

$$[\, g(p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^k p_{\mu_n i_0}{}^{-1}$$
$$= [\, g(p_{\mu_r i_m}{}^{-1} p_{\mu_r j_m})(p_{\lambda_r i_{m+1}}{}^{-1} p_{\lambda_r j_{m+1}}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^k$$
$$\bullet\, p_{\mu_n i_0}{}^{-1} (p_{\mu_r i_m} p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m} p_{\mu_r j_m}{}^{-1})^k$$
$$= [\, g(p_{\mu_r i_m}{}^{-1} p_{\mu_r j_m})(p_{\lambda_r i_{m+1}}{}^{-1} p_{\lambda_r j_{m+1}}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})]^k p_{\mu_n i_0}{}^{-1}$$

since $(p_{\mu_r i_m} p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m} p_{\mu_r j_m}{}^{-1})^k$

$$= p_{\mu_r i_m} (p_{\lambda_r i_m}{}^{-1} p_{\lambda_r j_m} p_{\mu_r j_m}{}^{-1} p_{\mu_r i_m})^k p_{\mu_r i_m}{}^{-1}$$
$$= p_{\mu_r i_m} p_{\mu_r i_m}{}^{-1} \qquad \text{by (c)}$$
$$= 1_G.$$

Hence, for $r \in \{ 0, 1, 2, \dots, n \}$, we have

$$(p_{\lambda_r i_0}{}^{-1} p_{\lambda_r j_n} p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$
$$= [\, (p_{\lambda_r i_0}{}^{-1} p_{\lambda_r j_0})(p_{\lambda_r i_1}{}^{-1} p_{\lambda_r j_1}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^k p_{\mu_n i_0}{}^{-1}$$

by (a)

$$= [\, (p_{\mu_r i_0}{}^{-1} p_{\mu_r j_0})(p_{\lambda_r i_1}{}^{-1} p_{\lambda_r j_1}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^k p_{\mu_n i_0}{}^{-1}$$

by applying (f) with $g = 1_G$ and $m = 0$

$$= \{\, [p_{\mu_r i_0}{}^{-1} p_{\mu_r j_0}](p_{\mu_r i_1}{}^{-1} p_{\mu_r j_1}) \cdots (p_{\lambda_r i_n}{}^{-1} p_{\lambda_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,\}^k p_{\mu_n i_0}{}^{-1}$$

by applying (f) with $g = p_{\mu_r i_0}{}^{-1} p_{\mu_r j_0} \in N_\alpha$ and $m = 1$

= ...... (repeating the same procedure for $m = 2, \dots, n$)

$$= [\, (p_{\mu_r i_0}{}^{-1} p_{\mu_r j_0})(p_{\mu_r i_1}{}^{-1} p_{\mu_r j_1}) \cdots (p_{\mu_r i_n}{}^{-1} p_{\mu_r j_n})(p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})\,]^k p_{\mu_n i_0}{}^{-1}$$
$$= (p_{\mu_r i_0}{}^{-1} p_{\mu_r j_n} p_{\mu_n j_n}{}^{-1} p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

since $p_{\mu_r j_m} = p_{\mu_r i_{m+1}}$ $(m = 0, 1, 2, \dots, n-1)$, by (a).

So, we have proved that

$$\text{(g)}\quad (p_{\lambda_r i_0}{}^{-1}p_{\lambda_r j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1} = (p_{\mu_r i_0}{}^{-1}p_{\mu_r j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

for all $r \in \{0, 1, 2, \dots, n\}$.

Finally,

$$(p_{\lambda_0 i_0}{}^{-1}p_{\lambda_0 j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1} = (p_{\mu_0 i_0}{}^{-1}p_{\mu_0 j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

by applying (g) with $r = 0$

$$= (p_{\lambda_1 i_0}{}^{-1}p_{\lambda_1 j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

by (a)

$$= (p_{\mu_1 i_0}{}^{-1}p_{\mu_1 j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

by applying (g) with $r = 1$

= ...... (repeating the same procedure for $r = 2, 3, \dots, n$)

$$= (p_{\mu_n i_0}{}^{-1}p_{\mu_n j_n}p_{\mu_n j_n}{}^{-1}p_{\mu_n i_0})^k p_{\mu_n i_0}{}^{-1}$$

$$= p_{\mu_n i_0}{}^{-1},$$

which completes the proof of (ii) implies (i). #

We are now ready for the main result of this section. C.A. Vachuska also obtained this result in [V].

Theorem 3.1.21. For $\mathscr{V} \in [\mathscr{RB}, \mathscr{C}]$ and $S \in \mathscr{CS}$. Let $\alpha, \rho \in \Lambda(S)$, with α over $\mathscr{V}$ and ρ idempotent pure. Then $\alpha \vee \rho$ is over $\mathscr{V}$.

Proof. To show that $\alpha \vee \rho$ is over $\mathscr{V}$, it suffices to show that $e(\alpha \vee \rho) \in \mathscr{V}$ for all $e \in E(S)$.

Since $\mathscr{RB} \subseteq \mathscr{V}$, then by Theorem 3.1.15, $\mathscr{V}\zeta = (\mathscr{RB}, \mathscr{U}, \mathscr{W})$, where $\mathscr{U} \subseteq \mathscr{W} \in \mathscr{L}(\mathscr{G})$ and $\mathscr{U} \in \mathscr{L}(\mathscr{AG})$. From Lemma 3.1.14, we have $\mathscr{U} = \mathscr{AG}$ or $\mathscr{A}_k$ for some $k \geq 1$.

Let $e \in E(S)$. To show that $e(\alpha \vee \rho) \in \mathscr{V}$, and by Theorem 3.1.15, it suffices to show the following three statements:

(a) $e(\alpha \vee \rho)/_{\mathscr{H}} \in \mathscr{RB}$.

This is obvious, since $e(\alpha \vee \rho) \in \mathscr{CS}$.

(b) Subgroups of $[\, E(e(\alpha \vee \rho))\,]$ lie in $\mathscr{U}$.

Since α is over $\mathscr{V}$, and by Theorem 3.1.15, subgroups of $[\, E(f\alpha)\,]$ lie in $\mathscr{U}$ for all $f \in E(S)$. By Proposition 3.1.16, $\alpha \vee \rho$ is over $\mathscr{C}$, so that $e(\alpha \vee \rho) \in \mathscr{C}$; and thus by Lemma 3.1.11, subgroups of $[\, E(e(\alpha \vee \rho))\,]$ lie in $\mathscr{AG}$. It follows that (b) holds for $\mathscr{U} = \mathscr{AG}$. If $\mathscr{U} = \mathscr{A}_k$ for some $k \geq 1$, then by Proposition 3.1.20, subgroups of $[\, E(e(\alpha \vee \rho))\,]$ lie in $\mathscr{A}_k$, which completes the proof of (b).

(c) Subgroups of $e(\alpha \vee \rho)$ lie in $\mathscr{W}$.

Since α is over $\mathscr{V}$, then $e\alpha \in \mathscr{V}$. By Theorem 3.1.15, subgroups of S lie in $\mathscr{W}$, and so

$$H_e \cap e(\alpha \vee \rho) = H_e \cap e\alpha \qquad \text{by Lemma 3.1.5}$$
$$\in \mathscr{W},$$

it follows that, subgroups of $e(\alpha \vee \rho)$ lie in $\mathscr{W}$. This completes the proof of this theorem. #

The next corollary will be useful in Section 3.2.

Corollary 3.1.22. For $\mathscr{V} \in \mathscr{L}(\mathscr{C})$ and $S \in \mathscr{CS}$. Let $\alpha, \rho \in \Lambda(S)$, with α over $\mathscr{V}$ and ρ idempotent pure. Then $(\alpha \vee \rho)/_\rho$ is over $\mathscr{V}$ on $S/_\rho$.

Proof. If $\mathscr{V} \in [\, \mathscr{RB}, \mathscr{C}\,]$, then $(\alpha \vee \rho)/_\rho$ is over $\mathscr{V}$, by Theorem 3.1.21. Otherwise, $\mathscr{V} \in [\, \mathscr{LZ}, \mathscr{LG}\,] \cup [\, \mathscr{RZ}, \mathscr{RG}\,] \cup \mathscr{L}(\mathscr{G})$. We consider the following three cases:

(a) $\mathscr{V} \in [\, \mathscr{LZ}, \mathscr{LG}\,]$. By Lemma 3.1.7, $(\alpha \vee \rho)/_\rho$ is over $\mathscr{LG}$. For $\mathscr{V} \neq \mathscr{LG}$, it remains to show that $(e\rho)[\, (\alpha \vee \rho)/_\rho\,] \cap H_{(e\rho)} \in \mathscr{V}$ for all $e \in E(S)$. Let $e \in E(S)$. Since $\rho \cap \mathscr{H} = \varepsilon$, we then have

$$(e\rho)[\,{}^{(\alpha \vee \rho)}/_{\rho}\,] \cap H_{(e\rho)} \cong H_e \cap e(\alpha \vee \rho)$$

$$= H_e \cap e\alpha \qquad \text{by Lemma 3.1.5}$$

$$\in \mathcal{V} \qquad \text{since } \alpha \text{ is over } \mathcal{V}.$$

(b) $\mathcal{V} \in [\,\mathcal{RZ}, \mathcal{RG}\,]$. This is the dual of (a).

(c) $\mathcal{V} \in \mathcal{L}(\mathcal{G})$. Note that $\mathcal{V} = (\mathcal{LZ} \vee \mathcal{V}) \cap (\mathcal{RZ} \vee \mathcal{V})$. Combining (a) with (b), we have (c). #

Whether Corollary 3.1.22 holds for every variety $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$ we do not know.

§ 3.2 The Mal'cev product on $\mathcal{L}(\mathcal{CR})$

We now introduce the most important definition of this thesis, the Mal'cev product. This product has proved useful in many considerations concerning the lattice of subvarieties of a variety of algebras. The following is a specialization of Mal'cev's original definition [M].

Definition 3.2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be any classes of completely regular semigroups. The class of all completely regular semigroups S for which there exists a congruence ρ on S with the property that all idempotent ρ-classes are in $\mathcal{U}$ and $S/\rho \in \mathcal{V}$ is the *Mal'cev product* of $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \circ \mathcal{V}$.

Notation 3.2.2. For any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $S \in \mathcal{CR}$, a congruence ρ on S is an $\mathcal{V}$-congruence if $S/\rho \in \mathcal{V}$; $\sigma_{\mathcal{V}}$ will denote the least $\mathcal{V}$- congruence on S.

The next result is obvious.

Lemma 3.2.3. For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{U}\circ\mathcal{V} = \{ S \in \mathcal{CR} \mid \sigma_{\mathcal{V}}$ is over $\mathcal{U} \}$.

As we shall see in Lemma 3.2.6, the Mal'cev product of varieties $\mathcal{U}$ and $\mathcal{V}$ need not be a variety. However, it does have the following property.

Lemma 3.2.4 [M]. If $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, then $\mathcal{U}\circ\mathcal{V}$ is closed under direct products and completely regular subsemigroups.

Proof. Let $S_\alpha \in \mathcal{U}\circ\mathcal{V}$ for $\alpha \in I$, and for each $\alpha \in I$, let ρ_α be a congruence on S_α figuring in the definition of the Mal'cev product. On $S = \prod_{\alpha\in I} S_\alpha$ define a relation ρ by

$$(a_\alpha) \, \rho \, (b_\alpha) \quad \text{if} \quad a_\alpha \, \rho_\alpha \, a_\alpha \qquad \text{for all } \alpha \in I.$$

Then ρ is a congruence on S such that for all $(e_\alpha) \in E(S)$, $(e_\alpha)\rho \equiv \prod_{\alpha\in I} e_\alpha\rho_\alpha$ and $S/_\rho \equiv \prod_{\alpha\in I} S_\alpha/_{\rho_\alpha}$, so that $S \in \mathcal{U}\circ\mathcal{V}$.

Next let $S \in \mathcal{U}\circ\mathcal{V}$ with the corresponding congruence ρ, and let T be a completely regular subsemigroup of S. Then $\rho' = \rho \mid_T$ is a congruence on T which gives $T \in \mathcal{U}\circ\mathcal{V}$.

#

It is important to point out the fact that $\mathcal{U} \subseteq \mathcal{U}\circ\mathcal{V}$ and $\mathcal{V} \subseteq \mathcal{U}\circ\mathcal{V}$ for any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$. The general form of the next lemma is proven in [M]. We sketch the proof for completeness.

Lemma 3.2.5. For any $\mathcal{U}, \mathcal{V}, \mathcal{W}$ in $\mathcal{L}(\mathcal{CR})$, we have

$$\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W}) \subseteq (\mathcal{U}\circ\mathcal{V})\circ\mathcal{W}.$$

Proof. Let $S \in \mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$. Then there exists $\theta \in \Lambda(S)$ such that $S/_\theta \in \mathcal{V}\circ\mathcal{W}$ and $e\theta \in \mathcal{U}$ for each $e \in E(S)$. That $S/_\theta \in \mathcal{V}\circ\mathcal{W}$ implies that there exists $\rho \in \Lambda(S/_\theta)$ such that $(S/\theta)/_\rho \in \mathcal{W}$ and $(e\theta)\rho \in \mathcal{V}$ for each $e\theta \in E(S/_\theta)$. Define $\bar{\rho}$ on S by

$$a \; \bar{\rho} \; b \quad \text{if} \; (a\theta) \, \rho \, (b\theta).$$

Then $\bar{\rho}$ is a congruence on S such that $\theta \subseteq \bar{\rho}$, and $(S/\theta)/\rho \cong S/\bar{\rho} \in \mathcal{W}$. Moreover, for each $e \in E(S)$, $\theta|_{e\bar{\rho}}$ is a congruence on $e\bar{\rho}$ such that $e\bar{\rho}/\theta \cong (e\theta)\rho \in \mathcal{V}$ and $f\theta \in \mathcal{U}$ for each $f \in E(e\bar{\rho})$. Hence $S \in (\mathcal{U} \circ \mathcal{V}) \circ \mathcal{W}$, as required. #

Jones [J5] showed that the Mal'cev product is not associative on $[\mathcal{S}, \mathcal{CR}]$, even when all partial products are again varieties. In Corollary 4.6.12, we shall see that the Mal'cev product is not associative on $\mathcal{L}(\mathcal{CS})$ either.

Some important observations about the Mal'cev product on $\mathcal{L}(\mathcal{CR})$ are adapted from Jones [J5] and stated in the following lemma. The proof of this lemma is also included for completeness.

Lemma 3.2.6 [J5]. Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$.

(i) If $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$, then $\mathcal{U} \circ \mathcal{V} = (\mathcal{U} \cap \mathcal{CS}) \circ \mathcal{V}$ for any $\mathcal{U}$.

(ii) If $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$ and $\mathcal{U} \in [\mathcal{S}, \mathcal{CR}]$, then $\mathcal{U} \circ \mathcal{V}$ can't be a variety except in the degenerate instances when $\mathcal{U} \circ \mathcal{V} = \mathcal{U}$.

Proof. (i) If $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$, then on any completely regular semigroup S, $\sigma_{\mathcal{V}} \subseteq \mathcal{D} = \sigma_{\mathcal{S}}$. Thus if $S \in \mathcal{U} \circ \mathcal{V}$, so that $\sigma_{\mathcal{V}}$ is over $\mathcal{U}$, then $\sigma_{\mathcal{V}}$ is over $\mathcal{U} \cap \mathcal{CS}$.

(ii) Since $\mathcal{U} \in [\mathcal{S}, \mathcal{CR}]$, then $\mathcal{U}$ contains the two-element semilattice $Y = \{0, 1\}$, $0 < 1$. Suppose that $T \in \mathcal{U} \circ \mathcal{V}$, $T \notin \mathcal{U}$. Then $Y \times T \in \mathcal{U} \circ \mathcal{V}$ and consists of the two $\mathcal{D}$-classes $\{0\} \times T$ and $\{1\} \times T$. Now the Rees quotient A modulo the ideal $\{0\} \times T$ does not belong to $\mathcal{U}$, since T does not, and the only $\mathcal{V}$-congruence on A is the universal congruence. Thus $A \notin \mathcal{U} \circ \mathcal{V}$, and so $\mathcal{U} \circ \mathcal{V}$ is not a variety. #

It follows from the above lemma that only products whose first factor is in $\mathcal{L}(\mathcal{CS})$ are of interest. Thus we may restrict attention to $\mathcal{U} \in \mathcal{L}(\mathcal{CS})$. Under this restriction, Jones [J5] established a necessary and sufficient condition for the Mal'cev product to be a variety. We state this condition in the next lemma.

Lemma 3.2.7 [J5]. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CS})$ and $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Then $\mathscr{U} \circ \mathscr{V}$ is a variety if and only if for each $S \in \mathscr{U} \circ \mathscr{V}$, for each $\mathscr{D}$-class D of S, and for each congruence $\rho \subseteq \mathscr{D}$ whose restriction to D is idempotent pure, $(\sigma_{\mathscr{V}} \vee \rho)/_{\rho}$ is over $\mathscr{U}$ on $D/_{\rho}$.

In [J5] Jones showed that $\mathscr{U} \circ \mathscr{V}$ is indeed a variety in many important instances. He also raised the following question: whether $\mathscr{U} \circ \mathscr{V}$ is *always* a variety when $\mathscr{U} \in \mathscr{L}(\mathscr{CS})$. Corollary 3.1.22 enables us to extend Theorem 5.1 of Jones [J5]. This result was also obtained by C.A. Vachuska [V].

Theorem 3.2.8. If $\mathscr{U} \in \mathscr{L}(\mathscr{C})$ and $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, then $\mathscr{U} \circ \mathscr{V}$ is again a variety.

Proof. Let $S \in \mathscr{U} \circ \mathscr{V}$, put $\alpha = \sigma_{\mathscr{V}}\ (\subseteq \mathscr{D})$, let D be a $\mathscr{D}$-class of S and let $\rho \in \Lambda(S)$ be contained in $\mathscr{D}$ and idempotent pure on D. By Lemma 3.2.3, α is over $\mathscr{U}$, and so $(\alpha \vee \rho)/_{\rho}$ is over $\mathscr{U}$ on $D/_{\rho}$ by Corollary 3.1.22. Hence $\mathscr{U} \circ \mathscr{V}$ is a variety, by Lemma 3.2.7. #

Notation 3.2.9. For any $\mathscr{H} \in \mathscr{L}(\mathscr{G})$, we write

$$CS(\mathscr{H}) = \{ S \in \mathscr{CS} \mid \text{all subgroups of } S \text{ lie in } \mathscr{H} \}.$$

It is easily verified that $CS(\mathscr{H}) = \mathscr{H} \circ \mathscr{RB}$, and so $CS(\mathscr{H})$ is a variety.

The proof of Theorem 4.1 in [J5] motivated the following result.

Proposition 3.2.10. If $\mathscr{H} \in \mathscr{L}(\mathscr{G})$ and $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, then $CS(\mathscr{H}) \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$.

Proof. We apply Lemma 3.2.7. Let $S \in CS(\mathcal{H})\circ\mathcal{V}$, put $\alpha = \sigma_{\mathcal{V}} (\subseteq \mathcal{D})$, let D be a $\mathcal{D}$-class of S and let $\rho \in \Lambda(S)$ be contained in $\mathcal{D}$ and idempotent pure on D. It remains to show that $(\alpha \vee \rho)/_\rho$ is over $CS(\mathcal{H})$ on $D/_\rho$. For any $e \in E(D)$, we have

$$e(\alpha \vee \rho) \cap H_e = e\alpha \cap H_e \quad \text{by Lemma 3.1.5}$$
$$\in \mathcal{H} \quad \text{since } \alpha \text{ is over } CS(\mathcal{H}),$$

that is, $e(\alpha \vee \rho) \in CS(\mathcal{H})$. Thus $\alpha \vee \rho$ is over $CS(\mathcal{H})$ on D, whence $(\alpha \vee \rho)/_\rho$ is also over $CS(\mathcal{H})$ on $D/_\rho$, as required. #

As a consequence of Theorem 3.2.8, the next corollary will be useful in the sequel.

Corollary 3.2.11 [J5]. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, then $\mathcal{LG}\circ\mathcal{U}$, $\mathcal{RG}\circ\mathcal{U}$ and $\mathcal{G}\circ\mathcal{U}$ are again varieties.

Theorem 3.2.12 [J5]. If $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS})$, then $\mathcal{U}\circ\mathcal{V} \in \mathcal{L}(\mathcal{CS})$.

Proof. Let $\mathcal{H} = \mathcal{U} \cap \mathcal{G}$, then $\mathcal{H} \subseteq \mathcal{U} \subseteq CS(\mathcal{H})$ and $\mathcal{G} \cap CS(\mathcal{H}) = \mathcal{H}$, so by Theorem 2.7.6, we have $\mathcal{U} = (\mathcal{U} \vee \mathcal{G}) \cap CS(\mathcal{H})$, whence $\mathcal{U}\circ\mathcal{V} = (\mathcal{U} \vee \mathcal{G})\circ\mathcal{V} \cap CS(\mathcal{H})\circ\mathcal{V}$. The cases $\mathcal{U} \in \mathcal{L}(\mathcal{C})$ were treated in Theorem 3.2.8. By Proposition 3.2.10, we therefore may assume that $\mathcal{U}$ contains $\mathcal{RG}$.

Now if $\mathcal{LZ} \subseteq \mathcal{V}$ or $\mathcal{RZ} \subseteq \mathcal{V}$, then $\mathcal{U}\circ\mathcal{V} = \mathcal{RG}\circ\mathcal{V}$ or $\mathcal{LG}\circ\mathcal{V}$, so that $\mathcal{U}\circ\mathcal{V} \in \mathcal{L}(\mathcal{CS})$. Thus we may assume that $\mathcal{V} \in \mathcal{L}(\mathcal{G})$. We now apply Lemma 3.2.7. Let $S \in \mathcal{U}\circ\mathcal{V}$, put $\alpha = \sigma_{\mathcal{V}}$ and let ρ be an idempotent pure congruence on S. It remains to show that $(\alpha \vee \rho)/_\rho$ is over $\mathcal{U}$ on $S/_\rho$. Note that

$$\ker(\alpha \vee \rho) = \cup_{e \in E(S)} [H_e \cap e(\alpha \vee \rho)]$$
$$= \cup_{e \in E(S)} (H_e \cap e\alpha) \quad \text{by Lemma 3.1.5}$$
$$= \ker \alpha.$$

Thus for any $e \in E(S)$, we have $e\alpha = \ker \alpha = \ker(\alpha \vee \rho) = e(\alpha \vee \rho) \in \mathscr{U}$, since α is a group congruence over $\mathscr{U}$. Thus $\alpha \vee \rho$ is over $\mathscr{U}$ on S, and therefore $(\alpha \vee \rho)/_{\rho}$ is over $\mathscr{U}$ on $S/_{\rho}$, as required. #

§ 3.3 The operators T_l, T_r, T, K, T_l^* and T_r^*

In this section we present alternative descriptions of the operators in the title in terms of Mal'cev products and identities. Most of descriptions about these operators T_l, T_r, T and K are taken from Jones [J5], Pastijn [P], and Petrich and Reilly [**PR8**].

Lemma 3.3.1. Let $\mathscr{U} = [\, u_\alpha = v_\alpha \,]_{\alpha \in A} \in \mathscr{L}(\mathscr{CR})$. Then

$$\begin{aligned} \mathscr{U}^{T_l} &= \mathscr{LG} \circ \mathscr{U} \\ &= \{\, S \in \mathscr{CR} \mid S/_{\mathscr{L}^0} \in \mathscr{U} \,\} \\ &= [\, (xu_\alpha)^0 = (xu_\alpha x v_\alpha)^0,\ (xv_\alpha)^0 = (xv_\alpha x u_\alpha)^0 \,]_{\alpha \in A} \\ &= [\, xu_\alpha = xu_\alpha (xv_\alpha)^0,\ xv_\alpha = xv_\alpha (xu_\alpha)^0 \,]_{\alpha \in A}, \end{aligned}$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Proof. The equality $\mathscr{U}^{T_l} = \mathscr{LG} \circ \mathscr{U}$ was established in [**P**, the dual of Lemma 3]. Let $S \in \mathscr{LG} \circ \mathscr{U}$. Then $\sigma_{\mathscr{U}}$ is over $\mathscr{LG}$, and so $\sigma_{\mathscr{U}} \subseteq \mathscr{L}$. But then $\sigma_{\mathscr{U}} \subseteq \mathscr{L}^0$ and $S/_{\mathscr{L}^0}$ is a homomorphic image of $S/_{\sigma_{\mathscr{U}}}$ and thus $S/_{\mathscr{L}^0} \in \mathscr{U}$. Conversely, if $S/_{\mathscr{L}^0} \in \mathscr{U}$, then $S \in \mathscr{LG} \circ \mathscr{U}$ since $\mathscr{L}^0$ is evidently over $\mathscr{LG}$. This establishes the second equality. The third and fourth equalities in the statement of the lemma are simple consequences of Lemma 2.6.13. This completes the proof. #

The next lemma is the dual of Lemma 3.3.1.

Lemma 3.3.2. Let $\mathscr{U} = [\, u_\alpha = v_\alpha \,]_{\alpha \in A} \in \mathscr{L}(\mathscr{CR})$. Then

$$\mathcal{U}^{T_r} = \mathcal{RG} \circ \mathcal{U}$$
$$= \{ S \in \mathcal{CR} \mid S/_{\mathcal{R}^0} \in \mathcal{U} \}$$
$$= [(u_\alpha x)^0 = (v_\alpha x u_\alpha x)^0, (v_\alpha x)^0 = (u_\alpha x v_\alpha x)^0]_{\alpha \in A}$$
$$= [u_\alpha x = (v_\alpha x)^0 u_\alpha x, v_\alpha x = (u_\alpha x)^0 v_\alpha x]_{\alpha \in A},$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Lemma 3.3.3. Let $\mathcal{U} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathcal{L}(\mathcal{CR})$. Then

$$\mathcal{U}^T = \mathcal{G} \circ \mathcal{U}$$
$$= \{ S \in \mathcal{CR} \mid S/_\mu \in \mathcal{U} \}$$
$$= [u_\alpha^0 = v_\alpha^0, (x u_\alpha y)^0 = (x v_\alpha y)^0]_{\alpha \in A}$$
$$= [u_\alpha^{-1} (u_\alpha^0 x^0 u_\alpha^0)^0 u_\alpha = v_\alpha^{-1} (v_\alpha^0 x^0 v_\alpha^0)^0 v_\alpha]_{\alpha \in A},$$

where $x, y \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Proof. By Lemma 2.6.9, $\mathcal{U}^T = \mathcal{U}^{T_l} \cap \mathcal{U}^{T_r}$

$= \mathcal{LG} \circ \mathcal{U} \cap \mathcal{RG} \circ \mathcal{U}$ by Lemmas 3.3.1 and 3.3.2

$= \mathcal{G} \circ \mathcal{U}$ since $\mathcal{G} = \mathcal{LG} \cap \mathcal{RG}$,

this establishes the first equality. The same type of argument as in the second part of the proof of Lemma 3.3.1 yields the second equality in the statement of this lemma. The equality $\{ S \in \mathcal{CR} \mid S/_\mu \in \mathcal{U} \} = [u_\alpha^0 = v_\alpha^0, (x u_\alpha y)^0 = (x v_\alpha y)^0]_{\alpha \in A}$ was established in [**Reil**, Theorem 3.9]. The last equality in the statement of the lemma is a simple consequence of Lemma 2.6.11. The proof is completed. #

Lemma 3.3.4. Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ and $\mathcal{U} \vee \mathcal{S} = [u_\alpha = v_\alpha]_{\alpha \in A}$. Then

$$\mathcal{U}^K = \mathcal{B} \circ (\mathcal{U} \vee \mathcal{S})$$
$$= \mathcal{RB} \circ (\mathcal{U} \vee \mathcal{S})$$
$$= \{ S \in \mathcal{CR} \mid S/_\tau \in \mathcal{U} \}$$
$$= \{ S \in \mathcal{CR} \mid S/_{(\tau \cap \mathcal{D})} \in \mathcal{U} \}$$

$= [\ (xu_\alpha y)(xv_\alpha y)^{-1} \in E\ ]$,

where x, y $\notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$ and $w \in E$ means $w^2 = w$.

Proof. The equality $\mathcal{U}^K = \mathcal{B} \circ (\mathcal{U} \vee \mathcal{S})$ was eastablished in [J5, Proposition 7.2 (ii)]. The equality $\mathcal{B} \circ (\mathcal{U} \vee \mathcal{S}) = \mathcal{RB} \circ (\mathcal{U} \vee \mathcal{S})$ follows from Lemma 3.2.6 (i). The same type of argument as in the second part of the proof of Lemma 3.3.1 yields the third and fourth equalities in the statement of this lemma. The last equality in the statement of the lemma is a simple consequence of Lemma 2.6.12. #

Notation 3.3.5. For $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, let

$$\mathcal{U}^{T_l^*} = \mathcal{LX} \circ \mathcal{U} \quad \text{and} \quad \mathcal{U}^{T_r^*} = \mathcal{RX} \circ \mathcal{U}.$$

By Theorem 3.2.8, T_l^* and T_r^* are two operators on $\mathcal{L}(\mathcal{CR})$.

In order to describe T_l^* and T_r^*, we require some preliminary observations.

Lemma 3.3.6. Let $S \in \mathcal{CS}$, then

(i) $\tau \cap \mathcal{L} = \{\ (a, b) \in S \times S \mid xa = xb \text{ for all } x \in S\ \}$.

(ii) $\tau \cap \mathcal{R} = \{\ (a, b) \in S \times S \mid ax = bx \text{ for all } x \in S\ \}$.

Proof. (i) Let $a\ (\tau \cap \mathcal{L})\ b$ and $x \in S$. Then $xa\ \tau\ xb$ and $xa\ \mathcal{L}\ xb$. Thus $(xa)^0 = (xaxb)^0 = (xb)^0$ so that $xa\ \mathcal{H}\ xb$. Since $\tau \cap \mathcal{H} = \varepsilon_S$, hence $xa = xb$.

Conversely, suppose that $xa = xb$ for all $x \in S$. Then $(ab)^0 = (aa)^0 = a^0$ and $(ba)^0 = (bb)^0 = b^0$, so that $a\ \mathcal{L}\ b$. Let $x, y \in S$. If $xay \in E(S)$, then $xby = xay \in E(S)$. If $xa \in E(S)$, then $xb = xa \in E(S)$. Next assume that $ay \in E(S)$. By Lemma 2.3.3, ay is a right identity of $L_{ay} = L_{by}$, so that $(by)^2 = byby = byay = by$, that is, $by \in E(S)$. Hence $a\ \tau\ b$, and so $a\ (\tau \cap \mathcal{L})\ b$, as required.

(ii) This is the dual of (i). #

Lemma 3.3.7. If $\mathcal{U} = [\ u_\alpha = v_\alpha\ ]_{\alpha \in A} \in \mathcal{L}(\mathcal{CS})$, then

(i) $\mathcal{LZ} \circ \mathcal{V} = \mathcal{LZ} \vee \mathcal{V}$

$$= [\, xu_\alpha = xv_\alpha \,]_{\alpha \in A},$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

(ii) $\mathcal{RZ} \circ \mathcal{V} = \mathcal{RZ} \vee \mathcal{V}$

$$= [\, u_\alpha y = v_\alpha y \,]_{\alpha \in A},$$

where $y \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Proof. (i) Let $S \in \mathcal{LZ} \circ \mathcal{V}$. Then there exists $\rho \in \Lambda(S)$ such that ρ over $\mathcal{LZ}$ and $S/\rho \in \mathcal{V}$. Hence $\rho \cap \mathcal{R} = \varepsilon_S$. Thus S is a subdirect product of the left zero semigroup $S/\mathcal{R}$ and the semigroup S/ρ and therefore $S \in \mathcal{LZ} \vee \mathcal{V}$. Clearly $\mathcal{LZ} \vee \mathcal{V} \subseteq [\, xu_\alpha = xv_\alpha \,]_{\alpha \in A}$. Let $S \in [\, xu_\alpha = xv_\alpha \,]_{\alpha \in A}$. In views of Lemma 3.3.6 (i), we see that $S/(\tau \cap \mathcal{L})$ satisfies $u_\alpha = v_\alpha$ for all $\alpha \in A$ and hence $S/(\tau \cap \mathcal{L}) \in \mathcal{V}$ so that $S \in \mathcal{LZ} \circ \mathcal{V}$.

(ii) This is the dual of (i). #

Lemma 3.3.8. Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, then

$$\text{(i)}\quad \mathcal{V}^{T_l^*} = \begin{cases} \mathcal{LZ} \vee \mathcal{V} & \text{if } \mathcal{V} \in \mathcal{L}(\mathcal{CS}); \\ \mathcal{V}^{T_l} \cap \mathcal{V}^K & \text{otherwise.} \end{cases}$$

$$\text{(ii)}\quad \mathcal{V}^{T_r^*} = \begin{cases} \mathcal{RZ} \vee \mathcal{V} & \text{if } \mathcal{V} \in \mathcal{L}(\mathcal{CS}); \\ \mathcal{V}^{T_r} \cap \mathcal{V}^K & \text{otherwise.} \end{cases}$$

Proof. (i) For $\mathcal{V} \in \mathcal{L}(\mathcal{CS})$, the assertion follows from Lemma 3.3.7 (i). For $\mathcal{V} \in [\mathcal{S}, \mathcal{CR}]$, the assertion follows from [Rei5, the dual of Proposition 3.4].

(ii) This is the dual of (i). #

Remark 3.3.9. Combining the above lemmas one can derive a basis for the identities of $\mathcal{V}^{T_l^*}$ [$\mathcal{V}^{T_r^*}$] in terms of a basis for the identities of $\mathcal{V}$.

We complete this section by providing a basis for the identities of $\mathcal{RZ} \vee \mathcal{U}$, which will be useful in Section 5.2.

Lemma 3.3.10. Let $S \in \mathcal{CR}$. Define a relation λ on S by

$$a \,\lambda\, b \quad \Leftrightarrow \quad ax = bx \quad \text{for all } x \in S.$$

Then λ is the least right reductive congruence on S.

Proof. See [Pe2, the dual of exercise III. 7.6.7]. #

Lemma 3.3.11. $\mathcal{LG} \circ \mathcal{S} = \{ \; S \in \mathcal{CR} \mid E(S) \in \mathcal{LRB} \; \}$

$$= [\; x^0y^0x^0 = x^0y^0 \;].$$

Proof. See [Pe1, Theorem IV. 3.10]. #

Lemma 3.3.12. Let $S \in [\; x^0y^0x^0z = x^0y^0z \;]$. Define ρ on S by

$$a \,\rho\, b \quad \Leftrightarrow \quad ca = cb \qquad \text{for some } c \in S.$$

Then ρ is the least right group congruence on S.

Proof. To show that ρ is a congruence on S, it suffices to show that ρ is transitive and left compatible. Let $(a, b), (b, c) \in \rho$, then $xa = xb$ and $yb = yc$ for some $x, y \in S$, thus $x^0a = x^0b$ and $y^0b = y^0c$, and so

$$\begin{aligned} y^0x^0a &= y^0x^0b \\ &= y^0x^0y^0b \\ &= y^0x^0y^0c \\ &= y^0x^0c, \end{aligned}$$

whence $(a, c) \in \rho$ and ρ is transitive. Let $(a, b) \in \rho$ and $c \in S$, then $x^0a = x^0b$ for some $x \in S$, so that

$$\begin{aligned} (x^0c^{-1})ca &= x^0c^0a \\ &= x^0c^0x^0a \\ &= x^0c^0x^0b \end{aligned}$$

$= x^0c^0b$

$= (x^0c^{-1})cb,$

and so $(ca, cb) \in \rho$ and ρ is left compatible. Hence $\rho \in \Lambda(S)$.

For any a, b $\in$ S, we have $(a^0b^0)a^0b^0 = (a^0b^0)b^0$ so that $(a^0b^0, b^0) \in \rho$. By Lallement's lemma, $E(S/_\rho) \in \mathscr{RZ}$, and so $S/_\rho \in \mathscr{RG}$.

Let α be any right group congruence on S and let a, b $\in$ S such that $(a, b) \in \rho$, then $c^0a = c^0b$ for some $c \in S$, and so

$$\begin{aligned} a\alpha &= (a^0\alpha)(a\alpha) \\ &= (c^0\alpha)(a^0\alpha)(a\alpha) \quad \text{since } E(S/_\alpha) \in \mathscr{RZ} \\ &= (c^0a)\alpha \\ &= (c^0b)\alpha \\ &= b\alpha, \end{aligned}$$

whence $\rho \subseteq \alpha$ as required. #

Proposition 3.3.13. If $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$ and $\mathscr{U} = [u_\alpha = v_\alpha]_{\alpha \in A}$, then

$$\mathscr{RZ} \vee \mathscr{U} = [u_\alpha y = v_\alpha y]_{\alpha \in A},$$

where $y \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Proof. Clearly $\mathscr{RZ} \vee \mathscr{U} \subseteq [u_\alpha y = v_\alpha y]_{\alpha \in A}$. For the opposite inclusion, we consider in three cases separately.

Case 1. $\mathscr{RZ} \subseteq \mathscr{U}$. Straightforward.

Case 2. $\mathscr{U} \in \mathscr{L}(\mathscr{CS})$. This follows immediately from Lemma 3.3.7 (ii).

Case 3. $\mathscr{RZ} \not\subseteq \mathscr{U}$ and $\mathscr{S} \subseteq \mathscr{U}$. Then $\mathscr{U} \subseteq \mathscr{LG} \circ \mathscr{S}$ so that $[u_\alpha y = v_\alpha y]_{\alpha \in A} \subseteq [x^0y^0x^0z = x^0y^0z]$. Let $S \in [u_\alpha y = v_\alpha y]_{\alpha \in A}$, and let λ be the congruence on S defined in Lemma 3.3.10. Clearly $S/_\lambda \in \mathscr{U}$. For each $\mathscr{D}$-class D of S, let $F_D = \{ a \in S : D_a \geq D \}$. Then F_D is a completely regular subsemigroup of S, and so $F_D \in [u_\alpha y = v_\alpha y]_{\alpha \in A}$. Let

ρ_D be the congruence on F_D defined in Lemma 3.3.12. Since $[\, u_\alpha y = v_\alpha y\,]_{\alpha \in A} \cap \mathcal{G} = \mathcal{Z} \cap \mathcal{G}$, then we can easily see that $F_D/\rho_D \in \mathcal{RZ} \vee \mathcal{Z}$, and so does $(F_D/\rho_D)^0$. Define

$$\psi : S \longrightarrow \Pi_{D \in S/\mathcal{D}} (F_D/\rho_D)^0$$

$$\text{by } (s\psi)_D = \begin{cases} s\rho_D & \text{if } s \in F_D; \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that ψ is a homomorphism of S into $\Pi_{D \in S/\mathcal{D}} (F_D/\rho_D)^0$. Define

$$\Phi : S \longrightarrow S/\lambda \times \Pi_{D \in S/\mathcal{D}} (F_D/\rho_D)^0$$

by $s\Phi = (s\lambda, s\psi)$. Then Φ is a homomorphism of S into $S/\lambda \times \Pi_{D \in S/\mathcal{D}} (F_D/\rho_D)^0$. If s, $t \in S$ and $s\Phi = t\Phi$, then $s\lambda = t\lambda$ and $s\psi = t\psi$. Thus xs = xt for all $x \in S$, and so $D_s = D_t$ and sc = tc for some $c \in F_{D_s}$. Since D_s is the least $\mathcal{D}$-class of F_{D_s}, we have sc = tc for some $c \in D_s$. By Lemma 2.4.1 (iii), D_s is weakly cancellative, this implies that s = t. Hence Φ is injective, and whence $S \in \mathcal{RZ} \vee \mathcal{Z}$, as required. #

Obviously Proposition 3.3.13 has its left-right dual. We may use this dual result without further notice.

§ 3.4 CR-relational morphisms

The concept of relational morphism, introduced by Tilson [E, Chapters XI and XII], is a very useful and powerful tool in the study of Mal'cev products of pseudovarieties of finite semigroups (monoids). The reader is referred to Pin ([Pi1], [Pi2]) and Tilson [T] for the basic definitions and results on this subject.

In order to study the Mal'cev products of varieties of completely regular semigroups effectively, this section introduces the concept of CR-relational morphism for completely regular semigroups. It is an analogue of the concept of relational morphism for semigroups (monoids).

Definition 3.4.1. Let S, T $\in \mathcal{CR}$. A *relation* $\tau : S \longrightarrow T$ is a function from S into P(T), the set of subsets of T. The *graph* of the relation τ is the subset graph(τ) = $\{ (s, t) \mid t \in s\tau \}$ of $S \times T$. The *inverse* of τ is the relation $\tau^{-1} : T \longrightarrow S$ defined by $t\tau^{-1} = \{ s \in S \mid t \in s\tau \}$. The relations τ and τ^{-1} can be extended to functions from P(S) into P(T) [respectively from P(T) into P(S)] by setting

$$X\tau = \cup_{x\in X}\, x\tau \quad [\ X\tau^{-1} = \cup_{x\in X}\, x\tau^{-1}\ ].$$

Definition 3.4.2. Let S, T $\in \mathcal{CR}$. A *completely regular relational morphism* (*CR-relational morphism* for short) $\tau : S \longrightarrow T$ is a relation satisfying the following conditions:

(i) for every $s \in S$, $s\tau \neq \varnothing$,

(ii) for every s, $t \in S$, $(s\tau)(t\tau) \subseteq (st)\tau$,

(iii) graph(τ) is a completely regular subsemigroup of $S \times T$.

Equivalently, a CR-relational morphism $\tau : S \longrightarrow T$ is a relation such that graph(τ) is a completely regular subsemigroup of $S \times T$ and the projection of graph(τ) into S is a surjective homomorphism.

Notation 3.4.3. For S, T $\in \mathcal{CR}$, we denote the set of all CR-relational morphisms of S into T by CR(S, T).

Definition 3.4.4. Let S, T $\in \mathcal{CR}$ and let $\tau \in$ CR(S, T). Then τ is called *injective* if the condition $s\tau \cap t\tau \neq \varnothing$ implies s = t (or equivalently, if the relation τ^{-1} is a partial function). τ is called *surjective* if $t\tau^{-1} \neq \varnothing$ for every $t \in T$.

It is very important to point out the following property of CR-relational morphisms. Let $S, T \in \mathscr{CR}$ and let $\tau \in CR(S, T)$. Then its graph $graph(\tau) = \{ (s, t) \in S \times T \mid t \in s\tau \}$ is a completely regular subsemigroup of $S \times T$ and the projections $S \times T \longrightarrow S$ and $S \times T \longrightarrow T$ induce homomorphisms $\alpha : graph(\tau) \longrightarrow S$ and $\beta : graph(\tau) \longrightarrow T$ such that

(i) α is a surjective homomorphism,

(ii) $\tau = \alpha^{-1}\beta$.

The factorization $S \xrightarrow{\alpha^{-1}} graph(\tau) \xrightarrow{\beta} T$ is called the *canonical factorization* of τ.

Proposition 3.4.5. For $S, T \in \mathscr{CR}$ and let $\tau : S \longrightarrow T$ be a homomorphism of S into T. Then $\tau \in CR(S, T)$.

Proof. To show that $\tau \in CR(S, T)$, it suffices to show that

$$graph(\tau) = \{ (s, t) \in S \times T \mid t = s\tau \}$$

is a completely regular subsemigroup of $S \times T$. Clearly, $graph(\tau)$ is a subsemigroup of $S \times T$. For any $(s, t) \in graph(\tau)$, we have $s\tau = t$, and $s^{-1}\tau = t^{-1}$ by Lemma 2.5.6. Thus $(s, t)^{-1} = (s^{-1}, t^{-1}) \in graph(\tau)$ for any $(s, t) \in graph(\tau)$. It follows from Lemma 2.5.4 that $graph(\tau)$ is completely regular, so that $\tau \in CR(S, T)$. #

Proposition 3.4.6. For $S, T \in \mathscr{CR}$ and let $S \xrightarrow{\alpha^{-1}} graph(\tau) \xrightarrow{\beta} T$ be the canonical factorization of a CR-relational morphism $\tau : S \longrightarrow T$. Then

(i) τ is injective if and only if β is injective.

(ii) τ is surjective if and only if β is surjective.

Proof. (i) Suppose that β is injective and let $s_1, s_2 \in S$ be such that $s_1\tau \cap s_2\tau \neq \varnothing$. Then $s_1\alpha^{-1}\beta \cap s_2\alpha^{-1}\beta \neq \varnothing$, whence $s_1\alpha^{-1} \cap s_2\alpha^{-1} \neq \varnothing$ since β is injective. Since α is a function, it follows that $s_1 = s_2$.

Conversely, suppose that τ is injective and let $r_1, r_2 \in graph(\tau)$ be such that $r_1\beta =$

$r_2\beta$. Since $r_1 \in r_1\alpha\alpha^{-1}$ and $r_2 \in r_2\alpha\alpha^{-1}$ it follows that $r_1\alpha\tau \cap r_2\alpha\tau \neq \varnothing$, whence $r_1\alpha = r_2\alpha$ since τ is injective, but $r_1 = (r_1\alpha, r_1\beta)$ is therefore equal to $r_2 = (r_2\alpha, r_2\beta)$.

(ii) This is obvious. #

Proposition 3.4.7. Let S, T, R $\in \mathscr{CR}$. If $\tau_1 \in CR(S, T)$ and $\tau_2 \in CR(T, R)$, then $\tau_1\tau_2 \in CR(S, R)$. If in addition τ_1 and τ_2 are injective, so is $\tau_1\tau_2$.

Proof. To show that $\tau_1\tau_2 \in CR(S, R)$, it suffices to show that graph($\tau_1\tau_2$) $\in \mathscr{CR}$. Clearly, graph($\tau_1\tau_2$) is a subsemigroup of $S \times R$. For any $(s, r) \in$ graph($\tau_1\tau_2$), there exists $t \in T$ such that $t \in s\tau_1$ and $r \in t\tau_2$, that is, $(s, t) \in$ graph(τ_1) and $(t, r) \in$ graph(τ_2). Since graph(τ_1), graph(τ_2) $\in \mathscr{CR}$, then $(s, t)^{-1} = (s^{-1}, t^{-1}) \in$ graph(τ_1) and $(t, r)^{-1} = (t^{-1}, r^{-1}) \in$ graph(τ_2), so that $t^{-1} \in s^{-1}\tau_1$ and $r^{-1} \in t^{-1}\tau_2$. Thus $r^{-1} \in s^{-1}\tau_1\tau_2$, and whence $(s, r)^{-1} = (s^{-1}, r^{-1}) \in$ graph($\tau_1\tau_2$). Therefore graph($\tau_1\tau_2$) $\in \mathscr{CR}$, and the first assertion follows.

Suppose that τ_1 and τ_2 are injective. If $s_1\tau_1\tau_2 \cap s_2\tau_1\tau_2 \neq \varnothing$, there exist $t_1 \in s_1\tau_1$ and $t_2 \in s_2\tau_1$ such that $t_1\tau_2 \cap t_2\tau_2 \neq \varnothing$. From the injectivity of τ_2 we have $t_1 = t_2$ and therefore $s_1\tau_1 \cap s_2\tau_2 \neq \varnothing$, whence $s_1 = s_2$ from the injectivity of τ_1. #

Proposition 3.4.8. Let S, T $\in \mathscr{CR}$, and let $\tau \in CR(S, T)$ be surjective. If S' is a completely regular subsemigroup of S, then $S'\tau$ is a completely regular subsemigroup of T. If T' is a completely regular subsemigroup of T, then $T'\tau^{-1}$ is a completely regular subsemigroup of S.

Proof. For any $t_1, t_2 \in S'\tau$, there exist $s_1, s_2 \in S'$ such that $t_1 \in s_1\tau$ and $t_2 \in s_2\tau$, so that $t_1t_2 \in (s_1\tau)(s_2\tau) \subseteq (s_1s_2)\tau \subseteq S'\tau$, and therefore $S'\tau$ is a subsemigroup of T. For any $t \in S'\tau$, there exists $s \in S'$ such that $(s, t) \in$ graph(τ). Since graph(τ), S' $\in \mathscr{CR}$, then $(s, t)^{-1} = (s^{-1}, t^{-1}) \in$ graph(τ) and $s^{-1} \in S'$, and whence $t^{-1} \in S'\tau$. Therefore $S'\tau$ is completely regular.

For any $s_1, s_2 \in T'\tau^{-1}$, then there exist $t_1, t_2 \in T'$ such that $t_1 \in s_1\tau$ and $t_2 \in s_2\tau$. From this it follows as above that $t_1t_2 \in (s_1s_2)\tau$, whence $s_1s_2 \in (t_1t_2)\tau^{-1} \subseteq T'\tau^{-1}$ and $T'\tau^{-1}$ is a subsemigroup of S. For any $s \in T'\tau^{-1}$, there exists $t \in T'$ such that $(s, t) \in \text{graph}(\tau)$. Then $(s^{-1}, t^{-1}) \in \text{graph}(\tau)$ and $t^{-1} \in T'$, whence $s^{-1} \in T'\tau^{-1}$ and $T'\tau^{-1}$ is completely regular. #

Corollary 3.4.9. Let $S, T \in \mathcal{CR}$. Then the following two statements are equivalent:

(i) There exists an injective CR-relational morphism from S into T.

(ii) S is a homomorphic image of a completely regular subsemigroup of T.

Proof. (i) implies (ii). Let $\tau : S \longrightarrow T$ be an injective CR-relational morphism and let $\tau = \alpha^{-1}\beta$ be the canonical factorization of τ. By Proposition 3.4.6 (i), $\beta : \text{graph}(\tau) \longrightarrow T$ is an injective homomorphism, that is, graph(τ) is isomorphic to a completely regular subsemigroup of T. Moreover, S is a homomorphic image of graph(τ). Therefore, (ii) holds.

(ii) implies (i). Suppose that (ii) holds, then there exist a completely regular subsemigroup T' of T and a surjective homomorphism $\alpha : T' \longrightarrow S$. Let $\beta : T' \longrightarrow T$ be the embedding of T' into T. Then $\tau = \alpha^{-1}\beta$ is a CR-relational morphism from S into T, since $\text{graph}(\tau) \cong T' \in \mathcal{CR}$. By Proposition 3.4.6 (i), τ is also injective. #

§ 3.5 Varieties of the form $< \mathcal{U} \circ \mathcal{V} >$

In Lemma 3.2.6 we saw that the Mal'cev product need not be a variety in general. However, it is of interest and important to study the variety $< \mathcal{U} \circ \mathcal{V} >$ generated by the Mal'cev product $\mathcal{U} \circ \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$. This will be the focus of our investigations throughout the rest of this thesis. Our goal in this section is to establish the connection with the CR-relational morphisms.

The CR-relational morphisms enable us to introduce a new operation on $\mathscr{L}(\mathscr{CR})$ as follows.

Definition 3.5.1. For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$, let

$$CR(\mathscr{U}, \mathscr{V}) = \{ S \in \mathscr{CR} \mid \text{exist } T \in \mathscr{V} \text{ and } \tau \in CR(S, T) \text{ such that } \tau \text{ is surjective and } e\tau^{-1} \in \mathscr{U} \text{ for any } e \in E(T) \}.$$

Then we have the following fact.

Proposition 3.5.2. For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we have $CR(\mathscr{U}, \mathscr{V}) \in \mathscr{L}(\mathscr{CR})$.

Proof. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$. To show that $CR(\mathscr{U}, \mathscr{V})$ is a variety, it suffices to show that $CR(\mathscr{U}, \mathscr{V})$ is closed under direct products, completely regular subsemigroups and homomorphic images.

(i) $CR(\mathscr{U}, \mathscr{V})$ is closed under direct products.

Let $S_\alpha \in CR(\mathscr{U}, \mathscr{V})$ for $\alpha \in A$, and for each $\alpha \in A$, let $T_\alpha \in \mathscr{V}$ and $\tau_\alpha \in CR(S_\alpha, T_\alpha)$ figuring in Definition 3.5.1. Then $\tau = \Pi_{\alpha\in A}\tau_\alpha : \Pi_{\alpha\in A}S_\alpha \longrightarrow \Pi_{\alpha\in A}T_\alpha$ is a surjective CR-relational morphism such that for any $(e_\alpha)_{\alpha\in A} \in E(\Pi_{\alpha\in A}T_\alpha)$, $[(e_\alpha)_{\alpha\in A}]\tau^{-1} \cong \Pi_{\alpha\in A}e_\alpha\tau_\alpha^{-1} \in \mathscr{U}$ and $\Pi_{\alpha\in A}T_\alpha \in \mathscr{V}$, so that $\Pi_{\alpha\in A}S_\alpha \in CR(\mathscr{U}, \mathscr{V})$.

(ii) $CR(\mathscr{U}, \mathscr{V})$ is closed under completely regular subsemigroups.

Let $S \in CR(\mathscr{U}, \mathscr{V})$ with the corresponding $T \in \mathscr{V}$ and $\tau \in CR(S, T)$, and let S' be a completely regular subsemigroup of S. Let φ be the embedding of S' into S. By Proposition 3.4.5, $\varphi \in CR(S', S)$. Thus by Proposition 3.4.7, $\varphi\tau \in CR(S', T)$. Let $S' \xrightarrow{\alpha^{-1}} \text{graph}(\varphi\tau) \xrightarrow{\beta} T$ be the canonical factorization of $\varphi\tau$, and let $T' = [\text{graph}(\varphi\tau)]\beta$. Since $\text{graph}(\varphi\tau) \in \mathscr{CR}$ and β is a homomorphism, then T' is a completely regular subsemigroup of T, so that $T' \in \mathscr{V}$ and $\varphi\tau \in CR(S', T')$ is surjective. For any $e \in E(T')$, $e(\varphi\tau)^{-1} = e\tau^{-1} \cap S' \in \mathscr{U}$, since $e\tau^{-1} \in \mathscr{U}$. Hence $S' \in CR(\mathscr{U}, \mathscr{V})$.

(iii) $CR(\mathscr{U}, \mathscr{V})$ is closed under quotients.

Let $S \in CR(\mathscr{U}, \mathscr{V})$ with the corresponding $T \in \mathscr{V}$ and $\tau \in CR(S, T)$, and let $\varphi : S \longrightarrow S'$ be a homomorphism of S onto S'. Clearly, $\varphi^{-1} : S' \longrightarrow S$ is an injective and surjective CR-relational morphism. By Proposition 3.4.7, $\varphi^{-1}\tau \in CR(S',T)$ and $\varphi^{-1}\tau$ is surjective. For any $e \in E(T)$, $e(\varphi^{-1}\tau)^{-1} = (e\tau^{-1})\varphi \in \mathscr{U}$, since $e\tau^{-1} \in \mathscr{U}$. Hence $S' \in CR(\mathscr{U}, \mathscr{V})$. #

We are now ready for the desired result.

Theorem 3.5.3. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Then $\langle \mathscr{U} \circ \mathscr{V} \rangle = CR(\mathscr{U}, \mathscr{V})$.

Proof. Let $S \in CR(\mathscr{U}, \mathscr{V})$. Then there exist $T \in \mathscr{V}$ and a surjective CR-relational morphism $\tau : S \longrightarrow T$ such that $e\tau^{-1} \in \mathscr{U}$ for any $e \in E(T)$. Let $S \xrightarrow{\alpha^{-1}} graph(\tau) \xrightarrow{\beta} T$ be the canonical factorization of τ. Since $\beta : graph(\tau) \longrightarrow T$ is a surjective homomorphism of $graph(\tau)$ onto T such that, for any $e \in E(T)$, $e\beta^{-1} = \{ (s, e) \in S \times T \mid e \in s\tau \} \cong e\tau^{-1} \in \mathscr{U}$, thus $graph(\tau) \in \mathscr{U} \circ \mathscr{V}$. Moreover, $\alpha : graph(\tau) \longrightarrow S$ is a surjective homomorphism, so that $S \in \langle \mathscr{U} \circ \mathscr{V} \rangle$. Hence $CR(\mathscr{U}, \mathscr{V}) \subseteq \langle \mathscr{U} \circ \mathscr{V} \rangle$.

For the opposite inclusion, it suffices to show that $\mathscr{U} \circ \mathscr{V} \subseteq CR(\mathscr{U}, \mathscr{V})$. Let $S \in \mathscr{U} \circ \mathscr{V}$. Then there exists $\rho \in \Lambda(S)$ such that $S/\rho \in \mathscr{V}$ and $e\rho \in \mathscr{U}$ for any $e \in E(S)$. Let $\tau : S \longrightarrow S/\rho$ be defined by $s\tau = s\rho$. Then by Proposition 3.4.5, $\tau \in CR(S, S/\rho)$ and τ is surjective. For any $f \in E(S/\rho)$, we have $f\tau^{-1} = e\rho \in \mathscr{U}$ for some $e \in E(S)$. Thus $S \in CR(\mathscr{U}, \mathscr{V})$, and whence $\langle \mathscr{U} \circ \mathscr{V} \rangle \subseteq CR(\mathscr{U}, \mathscr{V})$, as required. #

Theorem 3.5.3 shows that CR-relational morphisms play an important role in the study of varieties of the form $\langle \mathscr{U} \circ \mathscr{V} \rangle$. This description will prove very useful in the sequel.

Corollary 3.5.4. For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$, $\mathscr{U} \vee \mathscr{V} \subseteq \langle \mathscr{U} \circ \mathscr{V} \rangle$.

Proof. Straightforward. #

The next corollary will be needed in the sequel.

Corollary 3.5.5. For any $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ in $\mathscr{L}(\mathscr{CR})$, we have

$$<\mathscr{U}\circ<\mathscr{V}\circ\mathscr{W}>> \subseteq <<\mathscr{U}\circ\mathscr{V}>\circ\mathscr{W}>.$$

Proof. It is enough to show that $\mathscr{U}\circ<\mathscr{V}\circ\mathscr{W}> \subseteq <<\mathscr{U}\circ\mathscr{V}>\circ\mathscr{W}>$. Let $S \in \mathscr{U}\circ<\mathscr{V}\circ\mathscr{W}>$. Then there exists $\rho \in \Lambda(S)$ such that $S/\rho \in <\mathscr{V}\circ\mathscr{W}>$ and $e\rho \in \mathscr{U}$ for all $e \in E(S)$. By Theorem 3.5.3, there exist $T \in \mathscr{W}$ and $\tau \in CR(S/\rho, T)$ such that τ is surjective and $f\tau^{-1} \in \mathscr{V}$ for all $f \in E(T)$. Let $\rho^{\#} : S \longrightarrow S/\rho$ be the surjective homomorphism defined by $s\rho^{\#} = s\rho$. It follows from Propositions 3.4.5 and 3.4.7 that $\rho^{\#}\tau \in CR(S, T)$ and $\rho^{\#}\tau$ is surjective. Let $f \in E(T)$. Define $\Phi_f : f(\rho^{\#}\tau)^{-1} \longrightarrow f\tau^{-1}$ by $s\Phi_f = s\rho$. Clearly, $\Phi_f \in CR(f(\rho^{\#}\tau)^{-1}, f\tau^{-1})$ and Φ_f is surjective. Moreover, for any $h \in E(f\tau^{-1})$, and by Lemma 2.2.5, we have $h = e\rho$ for some $e \in E(S)$, so that $h\Phi_f^{-1} = e\rho \in \mathscr{U}$. By Theorem 3.5.3, $f(\rho^{\#}\tau)^{-1} \in <\mathscr{U}\circ\mathscr{V}>$, and whence $S \in <<\mathscr{U}\circ\mathscr{V}>\circ\mathscr{W}>$, as required. #

As the following example shows, the opposite inclusion in Corollary 3.5.5 need not be true. For an alternative example, see [J5, Proposition 6.6].

Example 3.5.6. Let $\mathscr{U} = \mathscr{G}$, $\mathscr{V} = \mathscr{RB}$ and $\mathscr{W} = \mathscr{S}$. Then $<<\mathscr{U}\circ\mathscr{V}>\circ\mathscr{W}> = \mathscr{CS}\circ\mathscr{S} = \mathscr{CR}$ and $<\mathscr{U}\circ<\mathscr{V}\circ\mathscr{W}>> = \mathscr{G}\circ\mathscr{B} = \mathscr{BG}$, so that $<\mathscr{U}\circ<\mathscr{V}\circ\mathscr{W}>> \neq <<\mathscr{U}\circ\mathscr{V}>\circ\mathscr{W}>$. #

Corollary 3.5.5 enables us to provide an alternative description of the operator K.

Corollary 3.5.7. For any $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, we have $\mathscr{U}^K = <\mathscr{B}\circ\mathscr{U}>$.

Proof. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then we have

$$\begin{aligned} \mathscr{U}^K &= \mathscr{B}\circ(\mathscr{U}\vee\mathscr{S}) && \text{by Lemma 3.3.4} \\ &= \mathscr{RB}\circ(\mathscr{U}\vee\mathscr{S}) && \text{by Lemma 3.2.6 (i)} \\ &\subseteq \mathscr{RB}\circ<\mathscr{S}\circ\mathscr{U}> && \text{by Corollary 3.5.4} \end{aligned}$$

$\subsetneq << \mathscr{RB} \circ \mathscr{S} > \circ \mathscr{U} >$ by Corollary 3.5.5

$= < \mathscr{B} \circ \mathscr{U} >$ since $\mathscr{B} = \mathscr{RB} \circ \mathscr{S}$ [**Pe2**, Corollary II.1.2].

Clearly, $< \mathscr{B} \circ \mathscr{U} > \subsetneq \mathscr{B} \circ (\mathscr{U} \vee \mathscr{S}) = \mathscr{U}^K$. Hence $\mathscr{U}^K = < \mathscr{B} \circ \mathscr{U} >$, as required. #

Chapter 4

Varieties of The Form $<\mathcal{V}\circ\mathcal{G}>$

In this chapter we restrict our attention to varieties of the form $<\mathcal{V}\circ\mathcal{G}>$. We first study the least full and self-conjugate subsemigroup $C^*(S)$ of a completely regular semigroup S. This enables us to introduce the operator C^*, and characterize $<\mathcal{V}\circ\mathcal{G}>$. The operator C^* is considered in detail. As a consequence, we extend a result of Petrich and Reilly [**PR7**] by showing that the well known operator C is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$. By restricting the operator C^* to completely simple semigroup varieties, we show that the order of C^* is infinite and the Mal'cev product is not associative on $\mathcal{L}(\mathcal{CS})$. The semigroup generated by the operators C^* and C is determined here. We also describe $\mathcal{V}^{(C^*)^i}$, $\mathcal{V}\in[\mathcal{RB}, \mathcal{CS}]$ and $i \geq 0$, in terms of $\mathcal{E}$-invariant normal subgroups of the free group over a countably infinite set.

§ 4.1 The subsemigroup $C^*(S)$

Definition 4.1.1. Let $S\in\mathcal{CR}$. A subsemigroup T of S is *full* if $E(S) \subseteq T$; T is *self - conjugate* if $a^{-1}Ta \subseteq T$ for each $a\in S$.

Definition 4.1.2. For any $S\in\mathcal{CR}$, let $C^*(S)$ denote the least full and self-conjugate subsemigroup of S. Thus

$$C^*(S) = \cup_{i\geq 0} V_i\,,$$

where $V_0 = C(S)$ and, for $i \geq 0$

$$V_{i+1} = [\cup_{a\in S} a^{-1}V_i a].$$

Lemma 4.1.3. Let $S \in \mathcal{CR}$. Then $C^*(S) \in \mathcal{CR}$.

Proof. Since $V_i \subseteq V_{i+1}$, for $i \geq 0$ and $C^*(S) = \cup_{i\geq 0} V_i$, it suffices to show that by induction on i that each V_i is a completely regular subsemigroup of S. By Lemma 2.5.8, $V_0 = C(S) \in \mathcal{CR}$. Assume that $V_i \in \mathcal{CR}$, we are going to show that $V_{i+1} \in \mathcal{CR}$. By Lemma 2.5.4, it remains to show that $v^{-1} \in V_{i+1}$ for any $v \in V_{i+1}$. Let v be an arbitrary element of V_{i+1}. Then there exist $u_1, \dots, u_n \in V_i$ and $a_1, \dots, a_n \in S$ ($n \geq 1$) such that $v_j = a_j^{-1}u_ja_j$ ($j = 1, 2, \dots, n$) and $v = v_1 \dots v_n$. By induction again, this time on n, we are going to show that $v^{-1} \in V_{i+1}$. This will be done in the following two steps.

(i) If $n = 1$, then

$$\begin{aligned} v^{-1} &= v_1^{-1} = (a_1^{-1}u_1a_1)^{-1} \\ &= (a_1^{-1}u_1a_1)^0 a_1^{-1}(a_1^0u_1)^0(a_1^{-1}u_1)^{-1}(a_1^{-1}u_1a_1)^0 && \text{by Lemma 2.5.7} \\ &= v^0a_1^{-1}(a_1^0u_1)^0u_1^{-1}(u_1a_1^{-1})^0a_1(a_1^{-1}u_1)^0v^0 && \text{by Lemma 2.5.7} \\ &\in V_{i+1}, \quad \text{Since } u_1^{-1} \in V_{i+1} \text{ and } C(S) \subseteq V_i \subseteq V_{i+1}. \end{aligned}$$

(ii) Assume that $(v_1 \dots v_{n-1})^{-1} \in V_{i+1}$, then

$$\begin{aligned} v^{-1} &= (v_1 \dots v_{n-1}v_n)^{-1} \\ &= v^0v_n^{-1}(v_nv_1 \dots v_{n-1})^0(v_1 \dots v_{n-1})^{-1}v^0 && \text{by Lemma 2.5.7} \\ &\in V_{i+1}, \quad \text{since } v_n^{-1}, (v_1 \dots v_{n-1})^{-1} \in V_{i+1} \text{ and } C(S) \subseteq V_{i+1}. \end{aligned}$$

By induction, $v^{-1} \in V_{i+1}$ for all $v \in V_{i+1}$, that is, $V_{i+1} \in \mathcal{CR}$. Therefore, $C^*(S) = \cup_{i\geq 0} V_i \in \mathcal{CR}$, as required. #

Lemma 4.1.4 [RS]. If $S \in \mathcal{OG}$, then $C^*(S) = C(S) = E(S)$.

Proof. Since $S \in \mathcal{OG}$, then E(S) is a subsemigroup of S, so that C(S) = E(S). To show that $C^*(S) = E(S)$, it suffices to show that E(S) is closed under conjugation, that is, $a^{-1}E(S)a \subseteq E(S)$ for any $a \in S$. Let $e \in E(S)$ and $a \in S$. Then $a^0 = aa^{-1} \in E(S)$ and therefore $aa^{-1}e \in E(S)$. Thus $a^{-1}ea = (a^{-1}aa^{-1})ea = a^{-1}(aa^{-1}e)a = a^{-1}(aa^{-1}e)^2a = a^{-1}aa^{-1}eaa^{-1}ea = (a^{-1}ea)^2$, so that $a^{-1}ea \in E(S)$. Hence $C^*(S) = E(S)$. #

Lemma 4.1.5. Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup whose matrix P is normalized with respect to some $1 \in I$, $1 \in \Lambda$. Then $C^*(S) = \mathcal{M}(N; I, \Lambda; P)$ where N is the normal subgroup of G generated by the entries of P.

Proof. Let $H = \{ x \in G \mid (x; 1, 1) \in C^*(S) \}$, and let e denote the identity of G. Then H is a subgroup of G, since $C^*(S)$ is a completely simple subsemigroup of S. For any $(x; i, \lambda)$ with $x \in H$, we have

$$(x; i, \lambda) = (e; i, 1)(x; 1, 1)(e; 1, \lambda) \in C^*(S),$$

since $(e; i, 1), (e; 1, \lambda) \in E(S)$. Thus $\mathcal{M}(H; I, \Lambda; P) = \{ (x; i, \lambda) \mid x \in H, i \in I, \lambda \in \Lambda \} \subseteq C^*(S)$. By the same type of argument, we can show that $C^*(S) \subseteq \mathcal{M}(H; I, \Lambda; P)$, so that $C^*(S) = \mathcal{M}(H; I, \Lambda; P)$. For any $y \in G$,

$$(y; 1, 1)^{-1}\{ (x; 1, 1) \mid x \in H \}(y; 1, 1) = \{ (y^{-1}xy; 1, 1) \mid x \in H \} \subseteq C^*(S)$$

implies that $y^{-1}Hy \subseteq H$, and so H is a normal subgroup of G. By Lemma 2.4.6, H contains the normal subgroup of G generated by the entries of P, so that $\mathcal{M}(N; I, \Lambda; P) \subseteq C^*(S)$. For any $(y; i, \lambda) \in S$ and $(x; j, \mu) \in \mathcal{M}(N; I, \Lambda; P)$, we have

$$\begin{aligned}(y; i, \lambda)^{-1}(x; j, \mu)(y; i, \lambda) &= [(p_{\lambda i}yp_{\lambda i})^{-1}; i, \lambda](x; j, \mu)(y; i, \lambda) \\ &= (p_{\lambda i}^{-1}y^{-1}p_{\lambda i}^{-1}p_{\lambda j}xp_{\mu i}y; i, \lambda) \\ &\in \mathcal{M}(N; I, \Lambda; P),\end{aligned}$$

since $p_{\lambda i}^{-1}y^{-1}p_{\lambda i}^{-1}p_{\lambda j}xp_{\mu i}y \in N$; it follows that $\mathcal{M}(N; I, \Lambda; P)$ is closed under conjugation. By the definition of $C^*(S)$, $\mathcal{M}(N; I, \Lambda; P) = C^*(S)$, as required. #

Lemma 4.1.6. Let $S, T \in \mathcal{CS}$, and let $\phi : S \longrightarrow T$ be a surjective homomorphism of S onto T. Then $C^*(S)\phi = C^*(T)$.

Proof. Note that $C^*(S) = \cup_{n \geq 0}V_n$ and $C^*(T) = \cup_{n \geq 0}U_n$, where $V_0 = C(S)$ and $U_0 = C(T)$, for $n \geq 0$

$$V_{n+1} = [\cup_{a\in S} a^{-1}V_n a] \quad \text{and} \quad U_{n+1} = [\cup_{b\in T} b^{-1}U_n b].$$

To show that $C^*(S)\phi = C^*(T)$, it suffices to show that $V_n\phi = U_n$ for all $n \geq 0$. This will be done by induction on n. By Lemma 2.2.6, $V_0\phi = U_0$. So assume that $V_n\phi = U_n$, we are going to show that $V_{n+1}\phi = U_{n+1}$. For any $a \in S$, we have

$$\begin{aligned} \{ a^{-1}xa \mid x \in V_n \}\phi &= (a^{-1}\phi)\{ x\phi \mid x \in V_n \}(a\phi) \\ &= (a\phi)^{-1}V_n\phi(a\phi) && \text{by Lemma 2.5.6} \\ &= (a\phi)^{-1}U_n(a\phi). \end{aligned}$$

Moreover, for any $b \in T$ and by Lemma 2.5.6, there exists $a \in S$ such that $a\phi = b$ and $a^{-1}\phi = b^{-1}$, so that $\{ a^{-1}xa \mid x \in V_n \}\phi = b^{-1}U_nb$, and therefore $V_{n+1}\phi = U_{n+1}$, as required. #

As a consequence of Lemmas 4.1.4 and 4.1.6, we have

Corollary 4.1.7. Let $S \in \mathscr{CR}$ and $G \in \mathscr{G}$. If $\varphi : S \longrightarrow G$ is a homomorphism of S into G, then $C^*(S) \leq 1\varphi^{-1}$, where 1 is the identity of G.

The next lemma, adapted from Petrich and Reilly [**PR2**, Theorem 5.2], will be used in the proof of Lemma 4.1.9.

Lemma 4.1.8 [PR2]. Let $S \in \mathscr{CS}$, and let K be a subsemigroup of S. Then the following statements are equivalent.

(i) K is a full and self-conjugate completely simple subsemigroup of S.

(ii) K is the kernel of a group congruence on S.

Lemma 4.1.9. For any $S \in \mathscr{CS}$, there exist $G \in \mathscr{G}$ with identity 1 and a homomorphism $\varphi : S \longrightarrow G$ such that $C^*(S) = 1\varphi^{-1}$.

Proof. By Lemma 4.1.5, $C^*(S)$ is a self-conjugate completely simple subsemigroup of S, and so by Lemma 4.1.8, there exists a group congruence σ on S such that ker σ = $C^*(S)$. Let $G = S/_\sigma$ and $\varphi : S \longrightarrow G = S/_\sigma$ be the canonical epimorphism. Then $1\varphi^{-1}$ = ker $\sigma = C^*(S)$, where 1 is the identity of G. #

As a consequence of Lemma 4.1.9, we have

Corollary 4.1.10. For any $S \in \mathscr{CS}$, then $C^*(S)$ is the kernel of the least group congruence on S.

The next lemma, adapted from Ljapin [Lj, Chapter VII. Section 5.5], will be used in the proof of Proposition 4.1.12.

Lemma 4.1.11 [Lj]. If T is an ideal of a semigroup S and $\varphi : T \longrightarrow M$ is a homomorphism of T onto the monoid M. Then there exists a homomorphism $\psi : S \longrightarrow M$ of S onto M such that $\psi|_T = \varphi$.

The following proposition parallels Lemma 2.5.9.

Proposition 4.1.12. For any $S = \cup_{\alpha\in Y} S_\alpha \in \mathscr{CR}$, we have $C^*(S) = \cup_{\alpha\in Y} C^*(S_\alpha)$.

Proof. Clearly, $C^*(S) = \cup_{\alpha\in Y}[S_\alpha \cap C^*(S)]$. It remains to show that $S_\alpha \cap C^*(S) = C^*(S_\alpha)$ for any $\alpha \in Y$.

Let $\alpha \in Y$. Then $S_\alpha \in \mathscr{CS}$. By Lemma 4.1.9, there exist $G \in \mathscr{G}$ with identity 1 and a homomorphism $\varphi : S_\alpha \longrightarrow G$ of S_α onto G such that $C^*(S_\alpha) = 1\varphi^{-1}$. Let $F(S_\alpha) = \{ S_\beta : \beta \in Y \text{ and } \alpha \not\leq \beta \}$. It is easy to see that $F(S_\alpha)$ is an ideal of S, and so is $F(S_\alpha) \cup S_\alpha$. Extend φ to $\varphi : F(S_\alpha) \cup S_\alpha \longrightarrow G^0$, the group G with a zero adjoined, by $F(S_\alpha)\varphi = \{ 0 \}$. Thus φ is a homomorphism of $F(S_\alpha) \cup S_\alpha$ onto a monoid, G^0. By

Lemma 4.1.11, there exists a homomorphism $\psi : S \longrightarrow G^0$ of S onto G^0 such that $\psi \mid_{F(S_\alpha) \cup S_\alpha} = \varphi$. By Lemmas 4.1.4 and 4.1.6, $C^*(S)\psi = C^*(G^0) = \{0, 1\}$, so that $C^*(S) \leq \{0, 1\}\psi^{-1}$, and so $S_\alpha \cap C^*(S) \leq 1\varphi^{-1} = C^*(S_\alpha)$. Clearly, $C^*(S_\alpha) \subseteq S_\alpha \cap C^*(S)$, and therefore $S_\alpha \cap C^*(S) = C^*(S_\alpha)$, as required. #

Proposition 4.1.12 enables us to simplify the expression of $C^*(S)$ in Definition 4.1.2.

Corollary 4.1.13. For any $S \in \mathscr{CR}$, $C^*(S) = V_1$, where $V_1 = [\cup_{a \in S} a^{-1}C(S)a]$.

Proof. By Proposition 4.1.12, it suffices to show that $C^*(S) = V_1$ for any $S \in \mathscr{CS}$. Let $S \in \mathscr{CS}$. Without loss of generality, we assume that $S = \mathcal{M}(G; I, \Lambda; P)$ whose sandwich matrix P is normalized with respect to some $1 \in I$, $1 \in \Lambda$, by Theorem 2.4.3. Thus $C(S) = \mathcal{M}(\langle P \rangle; I, \Lambda; P)$ and $C^*(S) = \mathcal{M}(N; I, \Lambda; P)$, by Lemmas 2.4.6 and 4.1.5, respectively, where $\langle P \rangle$ [N] is the subgroup [normal subgroup] of G generated by the entries of P. Clearly $C(S) \leq V_1 \leq C^*(S)$. It is easily verified that $N = [\cup_{g \in G} g^{-1}\langle P \rangle g]$. From the proof of Lemma 4.1.3, $V_1 \in \mathscr{CR}$ so that V_1 is a completely simple subsemigroup of S. Thus $H = \{x \in G \mid (x; 1, 1) \in V_1\}$ is a subgroup of G, and $\langle P \rangle \subseteq H \subseteq N$. For any $(x; i, \lambda)$ with $x \in H$, we have

$$(x; i, \lambda) = (e; i, 1)(x; 1, 1)(e; 1, \lambda) \in V_1$$

since $(e; i, 1)$, $(e; 1, \lambda) \in C(S)$, where e is the identity of G; it follows that $V_1 = \{(x; i, \lambda) \mid x \in H, i \in I, \lambda \in \Lambda\} = \mathcal{M}(H; I, \Lambda; P)$. For any $g \in G$ and $x \in \langle P \rangle$, we have

$$\begin{aligned}(g; 1, 1)^{-1}(x; 1, 1)(g; 1, 1) &= (g^{-1}xg; 1, 1)\\ &\in (g; 1, 1)^{-1}C(S)(g; 1, 1)\\ &\subseteq V_1\end{aligned}$$

so that $g^{-1}xg \in H$, and therefore $N = [\cup_{g \in G} g^{-1}\langle P \rangle g] \subseteq H$. Hence $C^*(S) = V_1$, as required. #

§ 4.2 Some congruences related to $C^*(S)$

Notation 4.2.1. For $S = \cup_{\alpha \in Y} S_\alpha \in \mathscr{CR}$, denote by σ, β and γ the least group, semilattice of groups and orthogroup congruences on S and by σ_α , β_α and γ_α those on S_α, respectively, $\alpha \in Y$.

The next lemma extends a result of Reilly [**Rei1**, Lemma 2.9] for completely simple semigroups.

Lemma 4.2.2. Let $S \in \mathscr{SG}$. Then

$$\gamma = \{ (a, b) \in S \times S \mid a^0 = b^0 \text{ and } ab^{-1} \in C^*(S) \}$$

and $\ker \gamma = C^*(S)$.

Proof. Let $\alpha = \{ (a, b) \in S \times S \mid a^0 = b^0 \text{ and } ab^{-1} \in C^*(S) \}$. Clearly α is an equivalence relation on S. Now let $(a, b) \in \alpha$ and let x be any element of S. Then $a^0 = b^0$ so that $a \mathscr{H} b$, thus $ax \mathscr{H} bx$ and $xa \mathscr{H} xb$ so that $(ax)^0 = (bx)^0$ and $(xa)^0 = (xb)^0$, since $\mathscr{H}$ is a congruence on S. Since $a^0 = b^0$ and $ab^{-1} \in C^*(S) \cap H_a$, then $a = nb$ with $n = ab^{-1} \in C^*(S)$, so that

$$\begin{aligned} (ax)(bx)^{-1} &= ax(bx)^0x^{-1}(xb)^0b^{-1}(bx)^0 && \text{by Lemma 2.5.7} \\ &= nbx(bx)^0x^{-1}(xb)^0b^{-1}(bx)^0 \\ &\in C^*(S). \end{aligned}$$

Similarly, $(xa)(xb)^{-1} \in C^*(S)$. Hence $(ax, bx), (xa, xb) \in \alpha$, and therefore α is a congruence on S.

Now for any $e, f \in E(S)$, $ef \in C^*(S)$ so that $(e\alpha)(f\alpha) = (ef)\alpha = (ef)^0\alpha \in E(S/\alpha)$. Thus, by Lemma 2.2.5, S/α is an orthogroup, and so $\gamma \subseteq \alpha$. On the other hand, since S/γ is an orthogroup, by Lemmas 4.1.4 and 4.1.6, $C^*(S)\gamma^\# = C^*(S/\gamma) = E(S/\gamma)$, which implies that $(n, n^0) \in \gamma$ for any $n \in C^*(S)$. Hence for $(a, b) \in \alpha$, say $a =$

nb where $n = ab^{-1} \in C^*(S) \cap H_e$ and $e = a^0 = b^0$, $a = nb \gamma n^0 b = b$. Thus $\alpha \subseteq \gamma$, and therefore $\gamma = \alpha$. The equality $\ker \gamma = C^*(S)$ is straightforward. #

Notation 4.2.3. For $S = \cup_{\alpha \in Y} S_\alpha \in \mathscr{CR}$, denote by F [resp. F_α] the smallest subsemigroup T of S [resp. S_α] containing E(S) [resp. E(S_α)], and such that

$$a' T a \subseteq T \quad \text{for all } a \in S \text{ [resp. } a \in S_\alpha \text{], } a' \in V(a).$$

Lemma 4.2.4 [J2]. For $S = \cup_{\alpha \in Y} S_\alpha \in \mathscr{CR}$, we then have

(i) $\beta = \cup_{\alpha \in Y} \sigma_\alpha$.

(ii) $F = \ker \beta = \ker \gamma = \cup_{\alpha \in Y} \ker \sigma_\alpha = \cup_{\alpha \in Y} F_\alpha$.

Corollary 4.2.5. For any $S \in \mathscr{CR}$, we have $C^*(S) = F$.

Proof. Since $S \in \mathscr{CR}$, then $S = \cup_{\alpha \in Y} S_\alpha$. For each $\alpha \in Y$, by Lemmas 4.2.2 and 4.2.4 (ii), $C^*(S_\alpha) = \ker \gamma_\alpha = F_\alpha$. Thus $C^*(S) = \cup_{\alpha \in Y} C^*(S_\alpha) = \cup_{\alpha \in Y} F_\alpha = F$. #

Corollary 4.2.5 enables us to adopt the descriptions about these congruences σ, β and γ from Feigenbaum [F], LaTorre ([La1], [La2]) and Trotter ([Tr1], [Tr2], [Tr3]). For details or alternative descriptions of these congruences, the reader is referred to the above references and Pirnot [Pi].

Lemma 4.2.6 [F]. Let $S \in \mathscr{CR}$. Then

$$\sigma = \{ (a, b) \in S \times S \mid ax = yb \text{ for some } x, y \in C^*(S) \}.$$

Lemma 4.2.7 [La2]. Let $S \in \mathscr{CR}$. Then

$$\beta = \{ (a, b) \in S \times S \mid a \mathscr{D} b \text{ and } ax = yb \text{ for some } x, y \in C^*(D_\alpha) \}$$

$$= \{ (a, b) \in S \times S \mid a \mathscr{D} b \text{ and } ab^{-1} \in C^*(D_\alpha) \}$$

and $\ker \beta = C^*(S)$.

Let $S \in \mathscr{CR}$. Define a relation $\pi_{\mathscr{R}}$ on $E(S)$ as follows: $(e, f) \in \pi_{\mathscr{R}}$ if and only if there exist $y_i \in S$, $k_i \in E(S)$ and $h_i \in C^*(S)$, $1 \le i \le n$ for some n, where for each i

$$k_i h_i^0 = k_i = y_i^0 k_i, \quad e y_1^{-1} k_1 h_1 y_1 \cdots y_{i-1}^{-1} k_{i-1} h_{i-1} y_{i-1} \ \mathscr{L} \ y_i^{-1} k_i y_i$$

and

$$e y_1^{-1} k_1 h_1 y_1 \cdots y_n^{-1} k_n h_n y_n = f.$$

Dually define $\pi_{\mathscr{L}}$. Let $\pi = \pi_{\mathscr{R}} \circ \pi_{\mathscr{L}}$. We then have the following result.

Lemma 4.2.8 [Tr2]. Let $S \in \mathscr{CR}$. Then $\operatorname{tr} \gamma = \pi$ and $\ker \gamma = C^*(S)$ so that

$$\gamma = \{ (a, b) \in S \times S \mid a^0 \pi b^0 \text{ and } ab^{-1} \in C^*(S) \}.$$

§ 4.3 Varieties of the form $< \mathscr{V} \circ \mathscr{G} >$

Notation 4.3.1. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we write

$$\mathscr{V}^{C^*} = \{ S \in \mathscr{CR} \mid C^*(S) \in \mathscr{V} \}.$$

Lemma 4.3.2. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, $\mathscr{V}^{C^*} \in \mathscr{L}(\mathscr{CR})$.

Proof. To show that $\mathscr{V}^{C^*} \in \mathscr{L}(\mathscr{CR})$, it suffices to show that $\mathscr{V}^{C^*}$ is closed under direct products, completely regular subsemigroups and quotients.

Let $S_\alpha \in \mathscr{V}^{C^*}$ for $\alpha \in A$. On $S = \prod_{\alpha \in A} S_\alpha$, it is easy to see that $C^*(S) = \prod_{\alpha \in A} C^*(S_\alpha)$, so that $C^*(S) \in \mathscr{V}$, since $C^*(S_\alpha) \in \mathscr{V}$ for each $\alpha \in A$. Thus $S \in \mathscr{V}^{C^*}$.

Let $S \in \mathscr{V}^{C^*}$ and let T be a completely regular subsemigroup of S. Thus $C^*(T)$ is a completely regular subsemigroup of $C^*(S)$, by Lemma 4.1.3, and so $C^*(T) \in \mathscr{V}$. Hence $S \in \mathscr{V}^{C^*}$.

Let $S \in \mathscr{V}^{C^*}$ and let φ be a homomorphism of S onto T. Clearly, $T \in \mathscr{CR}$. By Lemma 4.1.6, $C^*(S)\varphi = C^*(T)$. Since $C^*(S) \in \mathscr{V}$, then $C^*(T) \in \mathscr{V}$, and whence $T \in \mathscr{V}^{C^*}$. #

Lemma 4.3.3. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, $< \mathscr{V} \circ \mathscr{G} > \subseteq \mathscr{V}^{C^*}$.

Proof. It suffices to show that $\mathscr{V} \circ \mathscr{G} \subseteq \mathscr{V}^{C^*}$. Let $S \in \mathscr{V} \circ \mathscr{G}$. Then there exists a group congruence σ on S such that $e\sigma = \ker \sigma \in \mathscr{V}$ for any $e \in E(S)$. Clearly, $C(S) \subseteq \ker \sigma$. For any $a \in S$ and $x \in \ker \sigma$, we have $(a^{-1}xa)\sigma = (a^{-1}\sigma)(x\sigma)(a\sigma) = a^0\sigma = \ker \sigma$, since $x\sigma = \ker \sigma$ is the identity of the group $S/_\sigma$. It follows that $a^{-1}xa \in \ker \sigma$. Thus $C^*(S) \leq \ker \sigma$, and so $C^*(S) \in \mathscr{V}$. Hence $S \in \mathscr{V}^{C^*}$, and the inclusion follows. #

The main objective of this section is to establish the equality in Lemma 4.3.3, which gives a description of the varieties of the form $< \mathscr{V} \circ \mathscr{G} >$. In order to do so, we need some preparation. First, we have the following direct consequence of Theorem 3.5.3.

Corollary 4.3.4. Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$ and $S \in \mathscr{CR}$. Then $S \in < \mathscr{V} \circ \mathscr{G} >$ if and only if there exists a CR-relational morphism $\tau : S \longrightarrow G$ for some $G \in \mathscr{G}$ with identity 1 such that $1\tau^{-1} \in \mathscr{V}$.

Lemma 4.3.5. Let $S \in \mathscr{CR}$. Then

$$C^*(S) \leq \cap \{ 1\tau^{-1} \mid G \in \mathscr{G} \text{ with identity 1 and } \tau \in CR(S, G) \}.$$

Proof. Let $G \in \mathscr{G}$ with identity 1 and $\tau \in CR(S, G)$. By Proposition 3.4.8, $1\tau^{-1}$ is a completely regular subsemigroup of S. Let $e \in E(S)$. Since the projection of τ into S is surjective, then $(e, g) \in \text{graph}(\tau)$ for some $g \in G$, and so $(e, g)^{-1} = (e, g^{-1}) \in \text{graph}(\tau)$; it follows that $1 = gg^{-1} \in (e\tau)(e\tau) \subseteq e\tau$, so that $e \in 1\tau^{-1}$. Thus $E(S) \subseteq 1\tau^{-1}$. For any $s \in 1\tau^{-1}$ and $x \in S$, then $(x, g) \in \text{graph}(\tau)$ for some $g \in G$, and so $(x, g)^{-1} = (x^{-1}, g^{-1}) \in \text{graph}(\tau)$. Thus

$$1 = g^{-1}1g \in (x^{-1}\tau)(s\tau)(x\tau) \subseteq (x^{-1}sx)\tau,$$

and so $x^{-1}sx \in 1\tau^{-1}$; it follows that $1\tau^{-1}$ is self-cojugate. From Definition 4.1.2, we have $C^*(S) \leq 1\tau^{-1}$. Since G and τ were chosen arbitrarily, the inequality follows. #

Our next goal is to establish the opposite inclusion in Lemma 4.3.5. The techniques used here are modified from Ash [A], Birget [BMR] and Pin [Pi2].

Definition 4.3.6. From now on, until Proposition 4.3.13, let $S \in \mathcal{CR}$ and let R be an $\mathcal{R}$-class of S. Define an equivalence relation $\equiv$ on R as follows:

$$a \equiv b \text{ if and only if } ax = b \text{ and } by = a \text{ for some } x, y \in C^*(S).$$

Lemma 4.3.7. Let $a, b \in R$ and $s \in S$ be such that $as, bs \in R$. Then $a \equiv b$ if and only if $as \equiv bs$.

Proof. Necessity. Since $b \,\mathcal{R}\, bs$, there exists $x \in S$ such that $b = bsx$, so that $b = b(sx)^0$. Also, $a \equiv b$ implies that $a = by$ for some $y \in C^*(S)$. Thus $as = bys = b(sx)^0ys = bsw$ for some $w = x(sx)^{-1}ys$. By Lemma 2.5.7,

$$\begin{aligned} w &= x(sx)^{-1}ys \\ &= x(sx)^0x^{-1}(xs)^0s^{-1}(sx)^0ys \\ &\in C^*(S). \end{aligned}$$

In a symmetric way one finds an element $w' \in C^*(S)$ such that $bs = asw'$. Thus $as \equiv bs$.

Sufficiency. Since $as \equiv bs$, there exists $x \in C^*(S)$ such that $as = bsx$. Also, $a \,\mathcal{R}\, as$ implies that $a = asy$ for some $y \in S$, so that $a = a(sy)^0$. Let $w = sxy(sy)^{-1}$, then

$$\begin{aligned} w &= sxy(sy)^{-1} \\ &= sxy(sy)^0y^{-1}(ys)^0s^{-1}(sy)^0 \quad \text{by Lemma 2.5.7} \\ &\in C^*(S), \end{aligned}$$

and $a = bw$. In a symmetric way one finds an element $w' \in C^*(S)$ such that $b = aw'$. Thus $a \equiv b$. #

Definition 4.3.8. For each $a \in R$, let $[a]$ be the equivalence class of a modulo $\equiv$ on R, and let

$$R/_{\equiv} = \{ [a] \mid a \in R \}.$$

For each $s \in S$, define $s^R : R/_{\equiv} \longrightarrow R/_{\equiv}$ by

$$[\, a \,]s^R = \begin{cases} [\, as \,] & \text{if } as \in R; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Lemma 4.3.9. For each $s \in S$, $s^R : R/_{\equiv} \longrightarrow R/_{\equiv}$ is a partial function which, in addition, is injective.

Proof. Clearly, s^R is well defined, so that s^R is a partial function. If $[\, a \,], [\, b \,] \in R/_{\equiv}$ are such that $[\, a \,]s^R = [\, b \,]s^R$, then $as, bs \in R$ and $as \equiv bs$. By Lemma 4.3.7, $a \equiv b$, that is, $[\, a \,] = [\, b \,]$, and therefore, s^R is injective. #

Notation 4.3.10. For each $a \in S$, let

$$\begin{aligned} \mathrm{Dom}(\, s^R \,) &= \{\ [\, a \,] \in R/_{\equiv} \mid [\, a \,]s^R \text{ is defined } \} \\ &= \{\ [\, a \,] \in R/_{\equiv} \mid as \in R\ \} \end{aligned}$$

be the domain of $s^R : R/_{\equiv} \longrightarrow R/_{\equiv}$.

Lemma 4.3.11. For any $s_1, s_2 \in S$, we have $\mathrm{Dom}(\, s_1^R s_2^R \,) = \mathrm{Dom}(\, (\, s_1 s_2 \,)^R \,)$ and $(\, s_1 s_2 \,)^R = s_1^R s_2^R$.

Proof. Let $[\, a \,] \in \mathrm{Dom}(\, s_1^R s_2^R \,)$, then $[\, a \,] \in \mathrm{Dom}(\, s_1^R \,)$ and $[\, a \,]s_1^R \in \mathrm{Dom}(\, s_2^R \,)$. Now $[\, a \,] \in \mathrm{Dom}(\, s_1^R \,)$ implies that $as_1 \in R$ and $[\, a \,]s_1^R = [\, as_1 \,]$. Also $[\, a \,]s_1^R = [\, as_1 \,] \in \mathrm{Dom}(\, s_2^R \,)$ implies that $as_1s_2 \in R$ and $[\, as_1 \,]s_2^R = [\, as_1s_2 \,]$. Thus $[\, a \,] \in \mathrm{Dom}((s_1s_2)^R)$ and $[\, a \,](\, s_1s_2 \,)^R = [\, as_1s_2 \,] = [\, a \,]s_1^R s_2^R$. Conversely, let $[\, a \,] \in \mathrm{Dom}(\, (\, s_1s_2 \,)^R \,)$, then $as_1s_2 \in R$ and $[\, a \,](\, s_1s_2 \,)^R = [\, as_1s_2 \,]$. Since $a \mathcal{R} as_1s_2$, then it is easy to see that $as_1 \mathcal{R} as_1s_2$, so that $as_1 \in R$. Thus $[\, a \,] \in \mathrm{Dom}(\, s_1^R \,)$ and $[\, a \,]s_1^R = [\, as_1 \,] \in \mathrm{Dom}(\, s_2^R \,)$, and so $[\, a \,] \in \mathrm{Dom}(\, s_1^R s_2^R \,)$. Therefore, $\mathrm{Dom}(\, s_1^R s_2^R \,) = \mathrm{Dom}(\, (\, s_1s_2 \,)^R \,)$ and $(\, s_1s_2 \,)^R = s_1^R s_2^R$. #

Lemma 4.3.12. If $a, s \in S$, then $a, as^{-1} \in R$ imply that $as^0 \in R$ and $[\, as^0 \,] = [\, a \,]$.

Proof. Since $as^{-1} = as^0s^{-1}$ and $as^0 = as^{-1}s$, then $as^0 \mathscr{R} as^{-1}$, so that $as^0 \in \mathscr{R}$. Also, $as^0 \mathscr{R} a$ implies that $a = as^0x$ for some $x \in S$, and so $a = a(\, s^0x \,)^0$. Thus $a = as^0w$ and $as^0 = a \cdot s^0$, where $s^0 \in C^*(S)$ and $w = x(\, s^0x \,)^{-1} = x(\, s^0x \,)^0x^{-1}(\, xs^0 \,)^0s^0(\, s^0x \,)^0 \in C^*(S)$, by Lemma 2.5.7; it follows that $as^0 \equiv a$. Hence $[\, as^0 \,] = [\, a \,]$. #

We are now ready for the desired result.

Proposition 4.3.13. For any $S \in \mathscr{CR}$, we have

$$C^*(S) = \cap \{\, 1\tau^{-1} \mid G \in \mathscr{G} \text{ with identity } 1 \text{ and } \tau \in CR(S, G) \,\}.$$

Proof. For convenience, let $K(S) = \cap\{\, 1\tau^{-1} \mid G \in \mathscr{G} \text{ and } \tau \in CR(S, G) \,\}$. By Lemma 4.3.5, it suffices to show that if $r \in S$, then $r \in K(S)$ implies that $r \in C^*(S)$.

Let $r \in K(S)$, and let R be the $\mathscr{R}$-class containing r. Let G be the group of all bijections on $R/_{\equiv}$. We define $\tau : S \longrightarrow G$ by

$$s\tau = \{\, \sigma \in G \mid \sigma|_{\mathrm{Dom}(\, s^R \,)} = s^R \,\} \qquad (\, s \in S \,).$$

Clearly, $s\tau \neq \varnothing$ for any $s \in S$ [where $s\tau = G$ if $\mathrm{Dom}(\, s^R \,) = \varnothing$]. For any $s_1, s_2 \in S$, and let $\sigma_1 \in s_1\tau$ and $\sigma_2 \in s_2\tau$, then $\sigma_1|_{\mathrm{Dom}(\, s_1{}^R \,)} = s_1{}^R$ and $\sigma_2|_{\mathrm{Dom}(\, s_2{}^R \,)} = s_2{}^R$. By Lemma 4.3.11, it is easy to see that $\sigma_1\sigma_2|_{\mathrm{Dom}(\, (\, s_1s_2 \,)^R \,)} = (\, s_1s_2 \,)^R$, so that $\sigma_1\sigma_2 \in (\, s_1s_2 \,)\tau$, and whence $(\, s_1\tau \,)(\, s_2\tau \,) \subseteq (\, s_1s_2 \,)\tau$. Let $s \in S$ and $\sigma \in G$ be such that $(\, s, \sigma \,) \in \mathrm{graph}(\tau)$, then $\sigma|_{\mathrm{Dom}(\, s^R \,)} = s^R$. For any $[\, a \,] \in \mathrm{Dom}[\, (\, s^{-1} \,)^R \,] = \{\, [\, a \,] \in R/_{\equiv} \mid as^{-1} \in R \,\}$, and by Lemma 4.1.12, $as^0 = as^{-1}s \in R$, so that $[\, as^{-1} \,] \in \mathrm{Dom}(\, s^R \,)$ and $[\, as^{-1} \,]\sigma = [\, as^{-1} \,]s^R = [\, as^{-1}s \,] = [\, as^0 \,] = [\, a \,]$, thus $[\, a \,]\sigma^{-1} = [\, as^{-1} \,] = [\, a \,](\, s^{-1} \,)^R$, and so $\sigma^{-1}|_{\mathrm{Dom}[\, (\, s^{-1} \,)^R \,]} = (\, s^{-1} \,)^R$. Then $\sigma^{-1} \in (\, s^{-1} \,)\tau$, so that $(\, s, \sigma \,)^{-1} = (\, s^{-1}, \sigma^{-1} \,) \in \mathrm{graph}(\tau)$. By Lemma 2.5.4, $\mathrm{graph}(\tau) \in \mathscr{CR}$. Therefore $\tau \in CR(S, G)$.

Since $r \in K(S)$, then $1 \in r\tau = \{\, \sigma \in G \mid \sigma|_{\mathrm{Dom}(\, r^R \,)} = r^R \,\}$, and so r^R is a partial identity on $R/_{\equiv}$. Let e be an idempotent of R. By Lemma 2.3.3, $er = r$, so that $[\, e \,] \in$

Dom(r^R). Thus [e]r^R = [e], and whence $r \equiv e$. Hence $r = ex$ for some $x \in C^*(S)$, so that $r \in C^*(S)$. Therefore $K(S) \leq C^*(S)$, and so $K(S) = C^*(S)$. #

We are now ready to prove the principal result of this section.

Theorem 4.3.14. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we have

$$< \mathscr{V} \circ \mathscr{G} > = \mathscr{V}^{C^*}$$
$$= \{ S \in \mathscr{CR} \mid C^*(S) \in \mathscr{V} \}.$$

Proof. By Lemma 4.3.3, it remains to show that $\mathscr{V}^{C^*} \subseteq < \mathscr{V} \circ \mathscr{G} >$. Let $S \in \mathscr{V}^{C^*}$, then $C^*(S) \in \mathscr{V}$. To show that $S \in < \mathscr{V} \circ \mathscr{G} >$, it suffices to show that there exists a CR-relational morphism $\tau : S \longrightarrow G$ for some $G \in \mathscr{G}$ with identity 1 such that $C^*(S) = 1\tau^{-1}$, by Corollary 4.3.4. Let $\{ \tau_\alpha : S \longrightarrow G_\alpha \}_{\alpha \in A}$ be all distinct CR-relational morphisms of S into groups. Set $G = \prod_{\alpha \in A} G_\alpha$ and define $\tau : S \longrightarrow G$ by setting

$$s\tau = \prod_{\alpha \in A} s\tau_\alpha \qquad \text{for any } s \in S.$$

It is easy to see that $G \in \mathscr{G}$ and $\tau \in CR(S, G)$, so that $1\tau^{-1} = \cap_{\alpha \in A} 1_\alpha \tau_\alpha^{-1} = C^*(S)$ [1 is the identity of G, and for each $\alpha \in A$, 1_α is the identity of G_α], by Proposition 4.3.13. This proves the theorem. #

The next corollary is essentially contained within the proof of Theorem 4.3.14.

Corollary 4.3.15. For every $S \in \mathscr{CR}$, there exist $G \in \mathscr{G}$ with identity 1 and $\tau \in CR(S, G)$ such that $C^*(S) = 1\tau^{-1}$.

In regard to the principal result of this section, it is interesting to note the following varieties of the form $< \mathscr{V} \circ \mathscr{G} >$ which are well known:

$$< \mathscr{S} \circ \mathscr{G} > = \mathscr{SG}$$
$$< \mathscr{LZ} \circ \mathscr{G} > = \mathscr{LG}$$

$$\langle \mathcal{RZ} \circ \mathcal{G} \rangle = \mathcal{RG}$$
$$\langle \mathcal{RB} \circ \mathcal{G} \rangle = \mathcal{RBG}$$
$$\langle \mathcal{B} \circ \mathcal{G} \rangle = \mathcal{OG}.$$

§ 4.4 Commutativity between operators

We begin this section with the following notation.

Notation 4.4.1. For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let
$$\mathcal{V}^C = \{ S \in \mathcal{CR} \mid C(S) \in \mathcal{V} \}.$$
Since HSP($\mathcal{V}^C$) = $\mathcal{V}^C$, as is easily verified, it follows that $\mathcal{V}^C \in \mathcal{L}(\mathcal{CR})$. Thus C is an operator on $\mathcal{L}(\mathcal{CR})$. Clearly $(\mathcal{V}^C)^C = \mathcal{V}^C$ so that $C^2 = C$. The operator C was considered in detail by Petrich and Reilly (**[PR6]**, **[PR7]**) and Polák **[Po3]**. In the rest of this chapter we shall see that the operator C^*, introduced in Section 4.3, is similar to C in many ways.

The next lemma is a direct consequence of Lemma 4.1.4.

Lemma 4.4.2. For any $\mathcal{V} \in \mathcal{L}(\mathcal{OG})$, $\mathcal{V}^{C^*} = \mathcal{V}^C$. Thus $C^* = C$ on $\mathcal{L}(\mathcal{OG})$.

However, as we shall see in Corollary 4.6.12, $C^* \neq C$ on $\mathcal{L}(\mathcal{CR})$.

Lemma 4.4.3. $C = C^*C = CC^*$.

Proof. Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$. Clearly $\mathcal{V}^{C^*} \subseteq \mathcal{V}^C$, so that $\mathcal{V}^{C^*C} \subseteq \mathcal{V}^{C^2} = \mathcal{V}^C$. Let $S \in \mathcal{V}^C$, then $C(S) \in \mathcal{V}$. Since $C^*(C(S)) = C(S)$, then $C(S) \in \mathcal{V}^{C^*}$, and so $S \in \mathcal{V}^{C^*C}$. Thus $\mathcal{V}^{C^*C} = \mathcal{V}^C$, and therefore $C = C^*C$. In a symmetric way one can show that $C = CC^*$. This completes the proof. #

Lemma 4.4.4. [PR6]. C commutes with K, T_l, T_r and T.

The main purpose of this section is to prove the next theorem, which is the analogue for C^* of Lemma 4.4.4.

Theorem 4.4.5. C^* commutes with K, T_l, T_r, T, T_l^* and T_r^*.

In order to prove the above theorem, we need some preparations.

Lemma 4.4.6. Let $S \in \mathscr{CR}$ and $C^* = C^*(S)$. Then

$$(\tau_S \cap \mathscr{D}) \mid_{C^*} = \tau_{C^*} \cap \mathscr{D}.$$

Proof. The proof is similar to the proof of [PR6, Lemma 5.1]. For convenience, let $\tau = (\tau_S \cap \mathscr{D}) \mid_{C^*}$ and $\sigma = \tau_{C^*} \cap \mathscr{D}$. Clearly $\tau \subseteq \sigma$. Since τ and σ are idempotent pure, it suffices to show that tr σ = tr τ. So let e, f $\in E(S)$ and e σ f. Let x, y $\in S^1$ be such that xey $\in E(S)$. Then

$$(eyx)^2 = ey(xey)x = ey(xey)^2x = (eyx)^3$$

so that, since eyx lies in a subgroup, eyx $\in E(S)$. Thus e(eyx) $\in E(C^*)$ and so, since e σ f and eyx $\in C^*$, we must have f(eyx) $\in E(C^*) = E(S)$. Hence

$$(yxfe)^3 = yx(feyx)(feyx)fe = yx(feyx)fe = (yxfe)^2$$

and yxfe $\in E(S)$. Thus (yxfe)e $\in E(C^*)$ and, again since e σ f and yxfe $\in C^*$, we have yxfef $\in E(C^*) = E(S)$. But e $\mathscr{D}$ f so that f $\mathscr{H}$ fef. Since e τ_{C^*} f and τ_{C^*} restricted to any $\mathscr{H}$-class is trivial we get fef = f. Therefore yxf $\in E(S)$ and

$$(xfy)^3 = xf(yxf)(yxf)y = xf(yxf)y = (xfy)^2.$$

Thus xfy $\in E(S)$. By symmetry, we have that, for all x, y $\in S^1$,

$$xey \in E(S) \quad \text{if and only if} \quad xfy \in E(S)$$

and therefore e τ f. Hence $\sigma = \tau$, as required. #

The next two lemmas will be needed in the proof of Lemma 4.4.9.

Lemma 4.4.7 [J3]. The mapping $\mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{CS}$ ($\mathcal{U} \in \mathcal{L}(\mathcal{CR})$) is an endomorphism of $\mathcal{L}(\mathcal{CR})$.

Lemma 4.4.8 [Pe1]. The following conditions on a completely regular semigroup $S = \cup_{\alpha \in Y} S_\alpha$ are equivalent.

(i) S is a normal band of groups.

(ii) $S \in \mathcal{CS} \vee \mathcal{S}$.

(iii) S is a subdirect product of completely simple semigroups with a zero possibly adjoined.

(iv) For any $\alpha, \beta \in Y$ with $\beta \leq \alpha$ and any $e \in E(S_\alpha)$ there exists a unique $f \in E(S_\beta)$ such that $f \leq e$.

Lemma 4.4.9. Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$. Then

$$< (\mathcal{U} \vee \mathcal{S}) \circ \mathcal{G} > = < \mathcal{U} \circ \mathcal{G} > \vee < \mathcal{S} \circ \mathcal{G} > = < \mathcal{U} \circ \mathcal{G} > \vee \mathcal{S}.$$

Proof. The proof is similar to the proof of [**PR6**, Lemma 5.2]. We clearly have

$$< \mathcal{U} \circ \mathcal{G} > \vee \mathcal{S} \subseteq < \mathcal{U} \circ \mathcal{G} > \vee < \mathcal{S} \circ \mathcal{G} > \subseteq < (\mathcal{U} \vee \mathcal{S}) \circ \mathcal{G} >.$$

It suffices to show that $< (\mathcal{U} \vee \mathcal{S}) \circ \mathcal{G} > \subseteq < \mathcal{U} \circ \mathcal{G} > \vee \mathcal{S}$. The claim is trivial if $\mathcal{S} \subseteq \mathcal{U}$. So suppose that $\mathcal{S} \nsubseteq \mathcal{U}$ so that $\mathcal{U} \in \mathcal{L}(\mathcal{CS})$. Let $S \in < (\mathcal{U} \vee \mathcal{S}) \circ \mathcal{G} >$. Then $C^*(S) \in \mathcal{U} \vee \mathcal{S}$ and thus $C^*(S)$ is a normal band of groups. By Lemma 4.4.8 (iv), S itself is a normal band of groups. If D is a $\mathcal{D}$-class of S, then by Proposition 4.1.12 and Lemma 4.4.7, we obtain

$$C^*(D) = C^*(S) \cap D \in (\mathcal{U} \vee \mathcal{S}) \cap \mathcal{CS} = (\mathcal{U} \cap \mathcal{CS}) \vee (\mathcal{S} \cap \mathcal{CS}) = \mathcal{U}.$$

Therefore $D \in < \mathcal{U} \circ \mathcal{G} >$ and so $D^0 \in < \mathcal{U} \circ \mathcal{G} > \vee \mathcal{S}$ for each completely simple component D of S. By Lemma 4.4.8 (iii), we have $S \in < \mathcal{U} \circ \mathcal{G} > \vee \mathcal{S}$. #

Proposition 4.4.10. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then $< \mathscr{B} \circ < \mathscr{U} \circ \mathscr{G} > > = < < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} >$.

Proof. First let $S \in \mathscr{CR}$ and define a mapping χ by

$$\chi : a(\tau_S \cap \mathscr{D}) \longrightarrow a(\tau_S \cap \mathscr{D}) \cap C^*(S) \qquad (a \in C^*(S)).$$

Lemma 4.4.6 asserts that $(\tau_S \cap \mathscr{D})|_{C^*} = \tau_{C^*} \cap \mathscr{D}$ which then implies that χ is a bijection of $C^*(S/_{(\tau_S \cap \mathscr{D})})$ onto $C^*(S)/_{(\tau_{C^*} \cap \mathscr{D})}$ by Lemma 4.1.6. It now follows by Lemma 4.4.6 that χ is also a homomorphism. Therefore $C^*(S/_{(\tau_S \cap \mathscr{D})}) \cong C^*(S)/_{(\tau_{C^*} \cap \mathscr{D})}$.

For $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, and by Corollary 3.5.5, it suffices to show that $< < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} > \subseteq < \mathscr{B} \circ < \mathscr{U} \circ \mathscr{G} > >$. For $S \in \mathscr{CR}$, we have

$$\begin{aligned}
S \in < < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} > \quad &\Rightarrow \quad C^*(S) \in < \mathscr{B} \circ \mathscr{U} > && \text{by Theorem 4.3.14} \\
&\Rightarrow \quad C^*(S)/_{(\tau_{C^*} \cap \mathscr{D})} \in \mathscr{U} \vee \mathscr{S} && \text{by Lemma 3.3.4} \\
& && \text{and Corollary 3.5.7} \\
&\Rightarrow \quad C^*(S/_{(\tau_S \cap \mathscr{D})}) \in \mathscr{U} \vee \mathscr{S} && \\
&\Rightarrow \quad S/_{(\tau_S \cap \mathscr{D})} \in < (\mathscr{U} \vee \mathscr{S}) \circ \mathscr{G} > && \text{by Theorem 4.3.14} \\
&\Rightarrow \quad S \in < \mathscr{B} \circ < (\mathscr{U} \vee \mathscr{S}) \circ \mathscr{G} > > && \text{by Lemma 3.3.4} \\
& && \text{and Corollary 3.5.7} \\
&\Rightarrow \quad S \in < \mathscr{B} \circ (< \mathscr{U} \circ \mathscr{G} > \vee \mathscr{S}) > && \text{by Lemma 4.4.9,}
\end{aligned}$$

so that $< < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} > \subseteq < \mathscr{B} \circ (< \mathscr{U} \circ \mathscr{G} > \vee \mathscr{S}) >$. Furthermore, we have

$$\begin{aligned}
< \mathscr{B} \circ (< \mathscr{U} \circ \mathscr{G} > \vee \mathscr{S}) > &\subseteq < \mathscr{B} \circ < \mathscr{S} \circ < \mathscr{U} \circ \mathscr{G} > > && \text{by Corollary 3.5.4} \\
&\subseteq < < \mathscr{B} \circ \mathscr{S} > \circ < \mathscr{U} \circ \mathscr{G} > > && \text{by Corollary 3.5.5} \\
&= < \mathscr{B} \circ < \mathscr{U} \circ \mathscr{G} > > && \text{since } < \mathscr{B} \circ \mathscr{S} > = \mathscr{B}.
\end{aligned}$$

Therefore $< \mathscr{B} \circ < \mathscr{U} \circ \mathscr{G} > > = < < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} >$. #

Lemma 4.4.11. Let $S \in \mathscr{CR}$ and $C^* = C^*(S)$. Then

$$\mathscr{L}^0|_{C^*} = (\mathscr{L}_{C^*})^0, \quad \mathscr{R}^0|_{C^*} = (\mathscr{R}_{C^*})^0, \quad \mu|_{C^*} = \mu_{C^*}.$$

Proof. The proof is similar to the proof of [**PR6**, Lemma 5.4]. Consider $\mathscr{L}^0$. Clearly $\mathscr{L}^0|_{C^*} \subseteq (\mathscr{L}_{C^*})^0$. So let a, b $\in C^*(S)$, a $(\mathscr{L}_{C^*})^0$ b and x, y $\in S^1$. Then $x^0ay^0 \mathscr{L}_{C^*} x^0by^0$, and $x^0ay \mathscr{L} x^0by$ since $\mathscr{L}$ is a right congruence on S. But clearly

$$xay \mathscr{L} x^0ay, \quad xby \mathscr{L} x^0by$$

so that xay $\mathscr{L}$ xby. Thus a $(\mathscr{L}^0|_{C^*})$ b and $\mathscr{L}^0|_{C^*} = (\mathscr{L}_{C^*})^0$. Similarly for $\mathscr{R}$ and $\mu = \mathscr{H}^0 = \mathscr{L}^0 \cap \mathscr{R}^0$. #

Proposition 4.4.12. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then

$$\mathscr{LG} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{LG} \circ \mathscr{U}) \circ \mathscr{G} \rangle$$

$$\mathscr{RG} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{RG} \circ \mathscr{U}) \circ \mathscr{G} \rangle$$

$$\mathscr{G} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{G} \circ \mathscr{U}) \circ \mathscr{G} \rangle.$$

Proof. Let $S \in \mathscr{CR}$. From Lemma 4.4.11, it follows that

$$C^*(S/\mathscr{L}^0) \cong C^*(S)/(\mathscr{L}_{C^*(S)})^0.$$

For any $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, we then obtain

$$\begin{aligned}
S \in \langle (\mathscr{LG} \circ \mathscr{U}) \circ \mathscr{G} \rangle &\Leftrightarrow C^*(S) \in \mathscr{LG} \circ \mathscr{U} && \text{by Theorem 4.3.14}\\
&\Leftrightarrow C^*(S)/(\mathscr{L}_{C^*(S)})^0 \in \mathscr{U} && \text{by Lemma 3.3.1}\\
&\Leftrightarrow C^*(S/\mathscr{L}^0) \in \mathscr{U} &&\\
&\Leftrightarrow S/\mathscr{L}^0 \in \langle \mathscr{U} \circ \mathscr{G} \rangle && \text{by Theorem 4.3.14}\\
&\Leftrightarrow S \in \mathscr{LG} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle && \text{by Lemma 3.3.1,}
\end{aligned}$$

and therefore $\mathscr{LG} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{LG} \circ \mathscr{U}) \circ \mathscr{G} \rangle$. The rest of the equalities can be proved similarly. #

Corollary 4.4.13. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then

$$\mathscr{LZ} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{LZ} \circ \mathscr{U}) \circ \mathscr{G} \rangle$$

$$\mathscr{RZ} \circ \langle \mathscr{U} \circ \mathscr{G} \rangle = \langle (\mathscr{RZ} \circ \mathscr{U}) \circ \mathscr{G} \rangle.$$

Proof. For $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, and by Corollary 3.5.5, we have $\mathscr{LX} \circ < \mathscr{U} \circ \mathscr{G} > \subseteq < (\mathscr{LX} \circ \mathscr{U}) \circ \mathscr{G} >$. For the opposite inclusion, let $S \in \mathscr{CR}$, we then obtain

$$
\begin{aligned}
S \in < (\mathscr{LX} \circ \mathscr{U}) \circ \mathscr{G} > &\Rightarrow C^*(S) \in \mathscr{LX} \circ \mathscr{U} \qquad \text{by Theorem 4.3.14} \\
&\Rightarrow C^*(S) \in \mathscr{LG} \circ \mathscr{U} \cap \mathscr{RB} \circ \mathscr{U} \\
&\Rightarrow C^*(S)/_{(\mathscr{L}_{C^*(S)})^0} \in \mathscr{U} \text{ and } C^*(S)/_{(\tau_{C^*} \cap \mathscr{D})} \in \mathscr{U} \\
&\qquad \text{by Lemmas 3.3.1 and 3.2.3} \\
&\Rightarrow C^*(S/_{\mathscr{L}^0}) \in \mathscr{U} \text{ and } C^*(S/_{(\tau_S \cap \mathscr{D})}) \in \mathscr{U} \\
&\qquad \text{from the proofs of Propositions 4.4.10 and 4.4.12} \\
&\Rightarrow S/_{\mathscr{L}^0} \in < \mathscr{U} \circ \mathscr{G} > \text{ and } S/_{(\tau_S \cap \mathscr{D})} \in < \mathscr{U} \circ \mathscr{G} > \\
&\Rightarrow S \in \mathscr{LG} \circ < \mathscr{U} \circ \mathscr{G} > \text{ and } S \in \mathscr{RB} \circ < \mathscr{U} \circ \mathscr{G} > \\
&\Rightarrow S \in \mathscr{LX} \circ < \mathscr{U} \circ \mathscr{G} >,
\end{aligned}
$$

so that $< (\mathscr{LX} \circ \mathscr{U}) \circ \mathscr{G} > \subseteq \mathscr{LX} \circ < \mathscr{U} \circ \mathscr{G} >$. Therefore $\mathscr{LX} \circ < \mathscr{U} \circ \mathscr{G} > = < (\mathscr{LX} \circ \mathscr{U}) \circ \mathscr{G} >$. Similarly for the second equality. #

Proof of Theorem 4.4.5. For any $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, we then obtain

$$
\begin{aligned}
\mathscr{U}^{C^*K} &= < \mathscr{B} \circ \mathscr{U}^{C^*} > && \text{by Corollary 3.5.7} \\
&= < \mathscr{B} \circ < \mathscr{U} \circ \mathscr{G} > > && \text{by Theorem 4.3.14} \\
&= < < \mathscr{B} \circ \mathscr{U} > \circ \mathscr{G} > && \text{by Proposition 4.4.10} \\
&= < \mathscr{U}^K \circ \mathscr{G} > \\
&= \mathscr{U}^{KC^*},
\end{aligned}
$$

and therefore $C^*K = KC^*$. Similarly for T_l, T_r, T, $T_l{}^*$ and $T_r{}^*$ by applying Proposition 4.4.12 and Corollary 4.4.13. #

Lemma 4.4.14 [PR6]. Let $S \in \mathscr{CR}$. Then $\mu |_{C(S)} = \mu_{C(S)}$.

Notation 4.4.15. For any $\mathscr{U} \in \mathscr{L}(\mathscr{B})$, we write

$$\mathscr{UBG} = \{ S \in \mathscr{BG} \mid S/_{\mathscr{K}} \in \mathscr{U} \}.$$

Clearly $\mathscr{U}\mathscr{H}\mathscr{G} = \mathscr{G}\circ\mathscr{U}$, so that $\mathscr{U}\mathscr{H}\mathscr{G} \in \mathscr{L}(\mathscr{C}\mathscr{R})$.

Corollary 4.4.16. For any $\mathscr{U} \in \mathscr{L}(\mathscr{B})$, we have

$$(\mathscr{U}\mathscr{H}\mathscr{G})^{C^*} = < \mathscr{U}\mathscr{H}\mathscr{G} \circ \mathscr{G} >$$
$$= (\mathscr{U}\mathscr{H}\mathscr{G})^{C}.$$

Proof. Let $\mathscr{U} \in \mathscr{L}(\mathscr{B})$. By Proposition 4.4.12, $\mathscr{G}\circ< \mathscr{U}\circ\mathscr{G} > = < \mathscr{U}\mathscr{H}\mathscr{G} \circ\mathscr{G} > = (\mathscr{U}\mathscr{H}\mathscr{G})^{C^*}$.

On the other hand,

$$\mathscr{G}\circ< \mathscr{U}\circ\mathscr{G} > = \{ S \in \mathscr{C}\mathscr{R} \mid S/_{\mu} \in < \mathscr{U}\circ\mathscr{G} > \} \quad \text{by Lemma 3.3.3}$$
$$= \{ S \in \mathscr{C}\mathscr{R} \mid C^*(S/_{\mu}) \in \mathscr{U} \} \quad \text{by Theorem 4.3.14}$$
$$= \{ S \in \mathscr{C}\mathscr{R} \mid C(S/_{\mu}) \in \mathscr{U} \}$$
$$\text{by Lemma 4.1.4 and since } < \mathscr{U}\circ\mathscr{G} > \subseteq \mathscr{O}\mathscr{G}$$
$$= \{ S \in \mathscr{C}\mathscr{R} \mid S/_{\mu_{C(S)}} \in \mathscr{U} \} \quad \text{by Lemma 4.4.14}$$
$$= \{ S \in \mathscr{C}\mathscr{R} \mid C(S) \in \mathscr{G}\circ\mathscr{U} \} \quad \text{by Lemma 3.3.3}$$
$$= \{ S \in \mathscr{C}\mathscr{R} \mid C(S) \in \mathscr{U}\mathscr{H}\mathscr{G} \}$$
$$= (\mathscr{U}\mathscr{H}\mathscr{G})^{C}.$$

Therefore $(\mathscr{U}\mathscr{H}\mathscr{G})^{C^*} = (\mathscr{U}\mathscr{H}\mathscr{G})^{C}$, as required. #

§ 4.5 The restriction of C to $\mathscr{L}(\mathscr{O}\mathscr{G})$

Recalling the fact that $C = C^*$ on $\mathscr{L}(\mathscr{O}\mathscr{G})$ from Section 4.4. The main purpose of this section is to show that C is a complete endomorphism of $\mathscr{L}(\mathscr{O}\mathscr{G})$.

Lemma 4.5.1 [PR7, Lemma 4.4]. Let $\mathscr{U} \in \mathscr{L}(\mathscr{O}\mathscr{G})$. Then

$$\mathscr{U}^{C} = \mathscr{U}^{C^*}$$
$$= (\mathscr{U} \cap \mathscr{B})^{C}$$
$$= \{ S \in \mathscr{O}\mathscr{G} \mid E(S) \in \mathscr{U} \cap \mathscr{B} \}.$$

In particular, $(\mathcal{OG})^C = (\mathcal{OG})^{C^*} = \mathcal{B}^C = \mathcal{B}^{C^*} = \mathcal{OG}$.

Lemma 4.5.2 [**PR7**, Lemma 4.5]. Let $\mathcal{V} \in \mathcal{L}(\mathcal{OG})$ and

$$\mathcal{V} \cap \mathcal{B} = [\ x^0 = x,\ u(x_1, \dots, x_n) = v(x_1, \dots, x_n)\ ].$$

Then $\mathcal{V}^C = \mathcal{V}^{C^*}$

$$= [\ x^0y^0 = (x^0y^0)^0,\ u(x_1^0, \dots, x_n^0) = v(x_1^0, \dots, x_n^0)\ ].$$

Definition 4.5.3. Let $S \in \mathcal{CR}$. Then S is *E-unitary* if $e \in E(S)$, $a \in S$, $ea \in E(S)$ imply $a \in E(S)$.

Lemma 4.5.4. The following conditions on a completely regular semigroup S are equivalent.

(i) S is E-unitary.

(ii) S is an orthogroup and $\ker \sigma = E(S)$.

(iii) $S \in \mathcal{B} \circ \mathcal{G}$.

Proof. (i) implies (ii). Let $e, f \in E(S)$. Then $ef(ef)^{-1} = (ef)^0 \in E(S)$. By hypothesis, $f(ef)^{-1} \in E(S)$, and so $(ef)^{-1} \in E(S)$. Thus $ef \in E(S)$, and therefore S is an orthogroup.

Clearly $E(S) \subseteq \ker \sigma$. By Lemma 4.1.4, $C^*(S) = E(S)$. Let $a \in \ker \sigma$, then $(a, e) \in \sigma$ for some $e \in E(S)$, it follows from Lemma 4.2.6 that $fa = ef'$ for some $f, f' \in E(S)$. Thus $f, fa \in E(S)$ so that $a \in E(S)$ by hypothesis. We conclude that $\ker \sigma = E(S)$.

(ii) implies (iii). Trivial.

(iii) implies (i). If $S \in \mathcal{B} \circ \mathcal{G}$, then there exists a congruence ρ on S such that $S/\rho \in \mathcal{G}$ and $e\rho = E(S)$ is the identity of S/ρ for any $e \in E(S)$. If $e, ea \in E(S)$, then $e\rho = (ea)\rho = (e\rho)(a\rho) = a\rho$, so that $a \in E(S)$. Hence S is E-unitary. #

Definition 4.5.5. Let $S, T \in \mathcal{CR}$. A homomorphism φ of S into T is *idempotent separating* if for any $e, f \in E(S)$, $e\varphi = f\varphi$ implies $e = f$. S is called *fundamental* if $\mu_S = \varepsilon$.

We now give an alternative proof of [PP1, Corollary 6.39] for orthogroups.

Proposition 4.5.6 [PP1]. Every orthogroup is an idempotent separating homomorphic image of an E-unitary orthogroup.

Proof. Let $S \in \mathcal{OG}$. By Lemma 4.5.1, $S \in \mathcal{B}^{C^*} = CR(\mathcal{B}, \mathcal{G})$, and so there exist $G \in \mathcal{G}$ with identity 1 and a surjective $\tau \in CR(S, G)$ such that $1\tau^{-1} \in \mathcal{B}$. Let $S \xrightarrow{\alpha^{-1}} \text{graph}(\tau) \xrightarrow{\beta} G$ be the canonical factorization of τ. Then $\beta : \text{graph}(\tau) \longrightarrow G$ is a homomorphism of graph(τ) onto G such that $1\beta^{-1} \cong 1\tau^{-1} \in \mathcal{B}$, which implies that graph(τ) $\in \mathcal{B} \circ \mathcal{G}$, and so by Lemma 4.5.4, graph(τ) is an E-unitary orthogroup. On the other hand, since $E(\text{ graph}(\tau)\ ) = \{\ (e, 1)\ \mid\ e \in E(S)\ \}$ and α is the projection of graph(τ) onto S, thus $\alpha : \text{graph}(\tau) \longrightarrow S$ is an idempotent separating homomorphism of graph(τ) onto S. This completes the proof. #

Corollary 4.5.7. Every E-unitary orthogroup is a subdirect product of a fundamental orthogroup and a group.

Proof. Let S be an E-unitary orthogroup. Then $\ker(\sigma \cap \mu) \subseteq \ker \sigma = E(S)$ by Lemma 4.5.4, and $\operatorname{tr}(\sigma \cap \mu) \subseteq \operatorname{tr} \mu = \varepsilon$, thus $\sigma \cap \mu = \varepsilon$. Hence, S is isomorphic to a subdirect product of S/μ and S/σ, where S/μ is a fundamental orthogroup and S/σ is a group.

#

Lemma 4.5.8. Let $S, T \in \mathcal{OG}$ and let φ be an idempotent separating homomorphism of S onto T. Then for all $a, b \in S$, $a \mu b$ if and only if $(a\varphi)\ \mu\ (b\varphi)$.

Proof. By Lemma 2.5.6, $(a\varphi)^{-1} = a^{-1}\varphi$ for all $a \in S$. The direct implication is obvious. Let $(a\varphi)\ \mu\ (b\varphi)$. Since $T \in \mathcal{OG}$ and by Lemma 2.6.11, we have $(a\varphi)^{-1}(e\varphi)(a\varphi) = (b\varphi)^{-1}(e\varphi)(b\varphi)$ for all $e \in E(S)$, so that $(a^{-1}ea)\varphi = (b^{-1}eb)\varphi$ for all $e \in E(S)$. Since φ is idempotent separating and $a^{-1}ea, b^{-1}eb \in E(S)$, we must have $a^{-1}ea = b^{-1}eb$ for all $e \in E(S)$. Hence, by Lemma 2.6.11, $a \mu b$. #

The following result will be needed in the proof of Proposition 4.5.10.

Lemma 4.5.9. For every $S \in \mathcal{OG}$ there exist a group G, a (completely regular semigroup) subdirect product T of S/μ and G, and an idempotent separating homomorphism φ of T onto S.

Proof. Let $S \in \mathcal{OG}$. By Proposition 4.5.6, there exist an E-unitary orthogroup T and an idempotent separating homomorphism $\varphi : T \longrightarrow S$ of T onto S. It follows from Corollary 4.5.7 that T is a subdirect product of T/μ and $G = T/\sigma$. By Lemma 4.5.8, $T/\mu \cong S/\mu$, and the assertion follows. #

Proposition 4.5.10. For any $\mathcal{U} \in \mathcal{L}(\mathcal{OG})$, we have

$$\begin{aligned} \mathcal{U} \vee \mathcal{G} &= \mathcal{U}^T \cap \mathcal{U}^C \\ &= \{ \ S \in \mathcal{OG} \ \mid \ S/\mu \in \mathcal{U} \text{ and } E(S) \in \mathcal{U} \cap \mathcal{B} \ \} \end{aligned}$$

Proof. Clearly $\mathcal{U} \vee \mathcal{G} \subseteq \mathcal{U}^T \cap \mathcal{U}^C$. For the opposite inclusion, let $S \in \mathcal{U}^T \cap \mathcal{U}^C$, then $S \in \mathcal{OG}$ and $S/\mu \in \mathcal{U}$. By Lemma 4.5.9, there exist an orthogroup T, which is a subdirect product of S/μ and a group G, and an idempotent separating homomorphism φ of T onto S. Thus $T \in \mathcal{U} \vee \mathcal{G}$, and so $S \in \mathcal{U} \vee \mathcal{G}$. This proves the first equality. The second equality is an immediate consequence of Lemmas 3.3.3 and 4.5.1. #

The lattice $\mathcal{L}(\mathcal{B})$ of subvarieties of the variety $\mathcal{B}$ of all bands is presented in Figure 4.1, as determined by Birjukov [Bi], Fennemore [Fe] and Gerhard [G]. However, the description of the vertices as Mal'cev products is due to Pastijn [P].

By simple inspection of Figure 4.1, the next lemma follows.

Lemma 4.5.11. Let $\mathcal{U}, \mathcal{V} \in [\mathcal{S}, \mathcal{B}]$. Then $\mathcal{U} \not\subseteq \mathcal{V}$ and $\mathcal{V} \not\subseteq \mathcal{U}$ imply that $\mathcal{U} \vee \mathcal{V} = \mathcal{LZ} \circ \mathcal{U} \cap \mathcal{RZ} \circ \mathcal{V}$ or $\mathcal{RZ} \circ \mathcal{U} \cap \mathcal{LZ} \circ \mathcal{V}$.

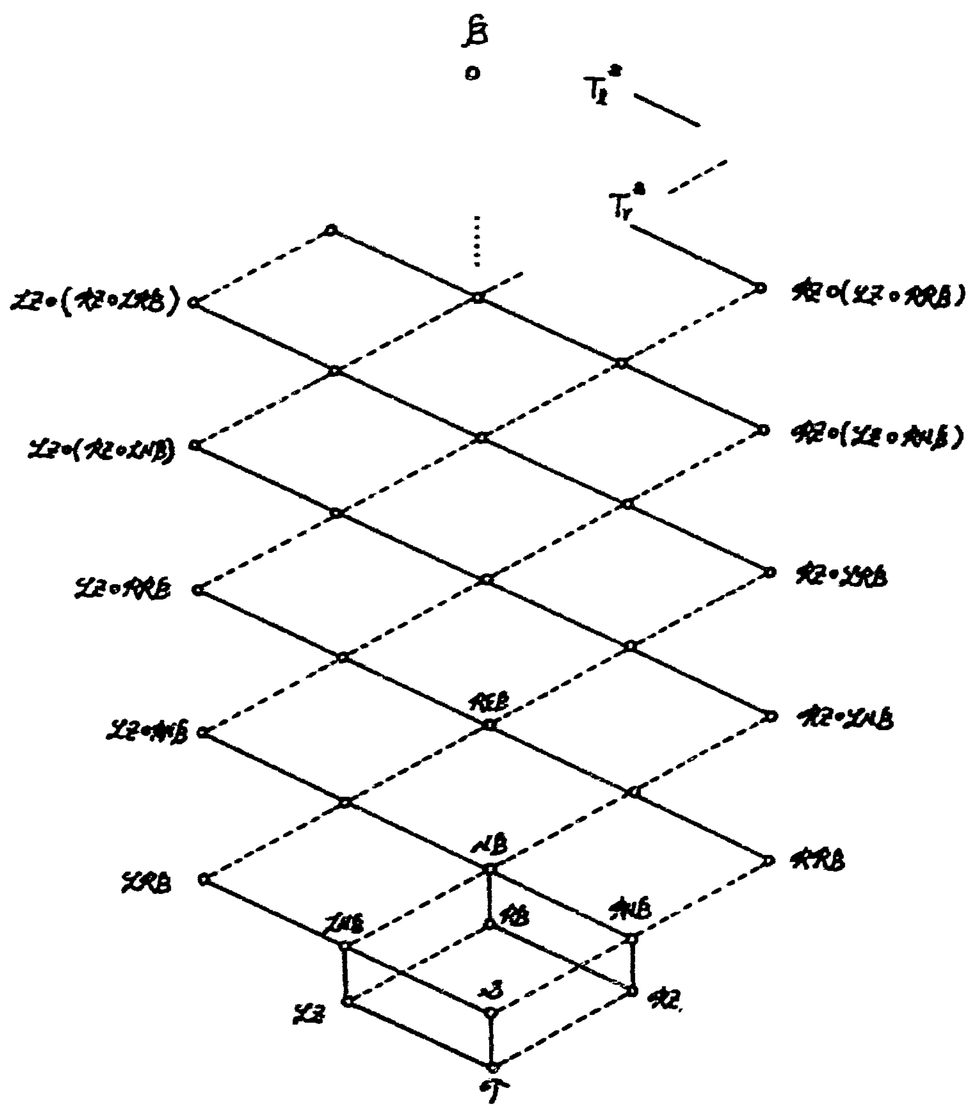

Figure 4.1.

Lemma 4.5.12. For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have $\mathcal{RZ} \circ \mathcal{U} \cap \mathcal{LZ} \circ \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{V}$.

Proof. Let $S \in \mathcal{RZ} \circ \mathcal{U} \cap \mathcal{LZ} \circ \mathcal{V}$. Then there exist congruences $\rho_1, \rho_2 \in \Lambda(S)$ such that $S/\rho_1 \in \mathcal{U}$, $S/\rho_2 \in \mathcal{V}$ and $e\rho_1 \in \mathcal{RZ}$, $e\rho_2 \in \mathcal{LZ}$ for any $e \in E(S)$. Let $\rho = \rho_1 \cap \rho_2$, then $\rho \in$

$A(S)$ and S/ρ is a subdirect product of S/ρ_1 and S/ρ_2, so that $S/\rho \in \mathscr{U} \vee \mathscr{V}$. On the other hand, $\ker \rho \subseteq \ker \rho_1 = E(S)$ and $\operatorname{tr} \rho = \operatorname{tr} \rho_1 \cap \operatorname{tr} \rho_2 = \varepsilon$, which imply that $\rho = \varepsilon$, so that $S \cong S/\rho \in \mathscr{U} \vee \mathscr{V}$, as required. #

Notation 4.5.13. We denote by $\mathscr{REB}$ the variety of all regular bands. Then $\mathscr{REB}$ = [$a^0 = a$, $axya = axaya$].

Lemma 4.5.14 [PR7, Lemma 4.6]. For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{REB})$, $(\mathscr{U} \vee \mathscr{V})^C = \mathscr{U}^C \vee \mathscr{V}^C$.

Proposition 4.5.15. For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{B})$, we have $(\mathscr{U} \vee \mathscr{V})^C = \mathscr{U}^C \vee \mathscr{V}^C$.

Proof. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{B})$. Clearly $\mathscr{U}^C \vee \mathscr{V}^C \subseteq (\mathscr{U} \vee \mathscr{V})^C$. For the opposite inclusion, it suffices to show that $(\mathscr{U} \vee \mathscr{V}) \circ \mathscr{G} \subseteq \mathscr{U}^C \vee \mathscr{V}^C$. There are three cases:

(a) $\mathscr{U} \vee \mathscr{V} = \mathscr{U}$ or $\mathscr{V}$. Trivial.

(b) $\mathscr{U} \vee \mathscr{V} \in \mathscr{L}(\mathscr{REB})$. Then

$$(\mathscr{U} \vee \mathscr{V}) \circ \mathscr{G} \subseteq (\mathscr{U} \vee \mathscr{V})^C \qquad \text{by Lemma 4.4.2}$$
$$= \mathscr{U}^C \vee \mathscr{V}^C \qquad \text{by Lemma 4.5.14}$$

(c) $\mathscr{U}, \mathscr{V} \notin \mathscr{L}(\mathscr{REB})$ and $\mathscr{U} \vee \mathscr{V} \neq \mathscr{U}$ or $\mathscr{V}$. Thus $\mathscr{U}, \mathscr{V} \in [\mathscr{S}, \mathscr{B}]$, and by Lemma 4.5.11, $\mathscr{U} \vee \mathscr{V} = \mathscr{LZ} \circ \mathscr{U} \cap \mathscr{RZ} \circ \mathscr{V}$ or $\mathscr{RZ} \circ \mathscr{U} \cap \mathscr{LZ} \circ \mathscr{V}$.

For the former case, we have

$$(\mathscr{U} \vee \mathscr{V}) \circ \mathscr{G} \subseteq (\mathscr{LZ} \circ \mathscr{U}) \circ \mathscr{G} \cap (\mathscr{RZ} \circ \mathscr{V}) \circ \mathscr{G}$$
$$\subseteq (\mathscr{LZ} \circ \mathscr{U}^{C^*}) \cap (\mathscr{RZ} \circ \mathscr{V}^{C^*}) \qquad \text{by Corollary 4.4.13}$$
$$\subseteq \mathscr{U}^{C^*} \vee \mathscr{V}^{C^*} \qquad \text{by Lemma 4.5.12.}$$
$$= \mathscr{U}^C \vee \mathscr{V}^C$$

For the latter case, we also have

$$(\mathscr{U} \vee \mathscr{V}) \circ \mathscr{G} \subseteq (\mathscr{RZ} \circ \mathscr{U}) \circ \mathscr{G} \cap (\mathscr{LZ} \circ \mathscr{V}) \circ \mathscr{G}$$
$$\subseteq (\mathscr{RZ} \circ \mathscr{U}^{C^*}) \cap (\mathscr{LZ} \circ \mathscr{V}^{C^*}) \qquad \text{by Corollary 4.4.13}$$
$$\subseteq \mathscr{U}^{C^*} \vee \mathscr{V}^{C^*} \qquad \text{by Lemma 4.5.12.}$$

$= \mathscr{U}^C \vee \mathscr{V}^C .$

Therefore, the equality follows. #

Lemma 4.5.16 [**PR6**, Lemma 7.6]. Let $\mathscr{U} \in \mathscr{L}(\mathscr{B})$.

(i) $\mathscr{U}^{T_l} \cap \mathscr{B} = (\mathscr{U} \cap \mathscr{B})^{T_l^*}$.

(ii) $\mathscr{U}^{T_r} \cap \mathscr{B} = (\mathscr{U} \cap \mathscr{B})^{T_r^*}$.

Lemma 4.5.17. For $\mathscr{U}_\alpha \in \mathscr{L}(\mathscr{B})$, $\alpha \in A$, then $\mathscr{B} = \vee_{\alpha \in A} \mathscr{U}_\alpha$ implies that $\mathscr{OG} = \vee_{\alpha \in A} (\mathscr{U}_\alpha)^C$.

Proof. In order to establish the assertion, it suffices to show that $F_{\mathscr{OG}}(k) \in \vee_{\alpha \in A} (\mathscr{U}_\alpha)^C$ for all positive integers k, where $F_{\mathscr{OG}}(k)$ is the free orthogroup on k generators (see [**GP**]). Then the same type of argument as in the last part of the proof of [**PR6**, Theorem 4.6] yields that $F_{\mathscr{OG}}(k) \in \mathscr{S}^{(T_l T_r)^k}$ for all positive integers k.

Let $k \geq 1$. From Lemma 4.5.16, it follows that $\mathscr{S}^{(T_l T_r)^k} \cap \mathscr{B} = \mathscr{S}^{(T_l^* T_r^*)^k}$, and so $C(F_{\mathscr{OG}}(k)) = E(F_{\mathscr{OG}}(k)) \in \mathscr{S}^{(T_l^* T_r^*)^k}$. Since $\mathscr{B} = \vee_{\alpha \in A} \mathscr{U}_\alpha$, and by inspection of Figure 4.1, there must exist $\beta \in A$ such that $\mathscr{S}^{(T_l^* T_r^*)^k} \subseteq \mathscr{U}_\beta$. Thus $C(F_{\mathscr{OG}}(k)) \in \mathscr{U}_\beta$, which then implies that $F_{\mathscr{OG}}(k) \in (\mathscr{U}_\beta)^C \subseteq \vee_{\alpha \in A} (\mathscr{U}_\alpha)^C$, as required. #

Corollary 4.5.18. The restriction of C to $\mathscr{L}(\mathscr{B})$ is a complete monomorphism of $\mathscr{L}(\mathscr{B})$ into $\mathscr{L}(\mathscr{OG})$.

Proof. Clearly $\mathscr{U} \longrightarrow \mathscr{U}^C$ is a complete $\cap$-homomorphism of $\mathscr{L}(\mathscr{B})$ into $\mathscr{L}(\mathscr{OG})$. Let $\mathscr{U}_\alpha \in \mathscr{L}(\mathscr{B})$ for $\alpha \in A$. By inspection of Figure 4.1, we then have $\vee_{\alpha \in A} \mathscr{U}_\alpha = \mathscr{B}$ or $\vee_{\alpha \in A} \mathscr{U}_\alpha = \vee_{i=1,n} \mathscr{U}_{\alpha_i}$ for some finite set $\{ \alpha_1, \dots, \alpha_n \} \subseteq A$. For the former case, it follows from Lemmas 4.5.1 and 4.5.17 that $(\vee_{\alpha \in A} \mathscr{U}_\alpha)^C = \vee_{\alpha \in A} (\mathscr{U}_\alpha)^C$. For the latter case, it follows from Proposition 4.5.15 and induction on i that $(\vee_{\alpha \in A} \mathscr{U}_\alpha)^C = (\vee_{i=1,n} \mathscr{U}_{\alpha_i})^C = \vee_{i=1,n} (\mathscr{U}_{\alpha_i})^C = \vee_{\alpha \in A} (\mathscr{U}_\alpha)^C$. Thus $\mathscr{U} \longrightarrow \mathscr{U}^C$ is also a complete $\vee$-homomorphism of $\mathcal{L}(\mathcal{B})$ into $\mathcal{L}(\mathcal{OG})$. The injectivity of C on $\mathcal{L}(\mathcal{B})$ follows from Lemma 4.5.1. Therefore the restriction of C to $\mathcal{L}(\mathcal{B})$ is a complete monomorphism of $\mathcal{L}(\mathcal{B})$ into $\mathcal{L}(\mathcal{OG})$. #

Lemma 4.5.19 [PR7]. The mapping

$$\mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{B} \qquad (\ \mathcal{U} \in \mathcal{L}(\mathcal{OG})\ )$$

is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$.

We are now ready for the main result of this section, which extends the last assertion of [**PR7, Theorem 4.1**].

Theorem 4.5.20. The restriction of C to $\mathcal{L}(\mathcal{OG})$ is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$.

Proof. For $\mathcal{U}_\alpha \in \mathcal{L}(\mathcal{OG})$, $\alpha \in A$, we get

$$\begin{aligned}
(\vee_{\alpha\in A}\mathcal{U}_\alpha)^C &= [\,(\vee_{\alpha\in A}\mathcal{U}_\alpha) \cap \mathcal{B}\,]^C && \text{by Lemma 4.5.1}\\
&= [\,\vee_{\alpha\in A}(\mathcal{U}_\alpha \cap \mathcal{B})\,]^C && \text{by Lemma 4.5.19}\\
&= \vee_{\alpha\in A}(\mathcal{U}_\alpha \cap \mathcal{B})^C && \text{by Corollary 4.5.18}\\
&= \vee_{\alpha\in A}(\mathcal{U}_\alpha)^C && \text{by Lemma 4.5.1}
\end{aligned}$$

and

$$\begin{aligned}
(\cap_{\alpha\in A}\mathcal{U}_\alpha)^C &= \{\ S \in \mathcal{CR} \mid C(S) \in \cap_{\alpha\in A}\mathcal{U}_\alpha\ \}\\
&= \cap_{\alpha\in A}\{\ S \in \mathcal{CR} \mid C(S) \in \mathcal{U}_\alpha\ \}\\
&= \cap_{\alpha\in A}(\mathcal{U}_\alpha)^C.
\end{aligned}$$

This concludes the proof. #

However, the operator **C** is not a $\vee$-homomorphism of $\mathcal{L}(\mathcal{CR})$, the reader is referred to Petrich and Reilly **[PR7]** for details.

§ 4.6 The restriction of C^* to $\mathscr{L}(\mathscr{CS})$

In this section we consider the operator C^* on $\mathscr{L}(\mathscr{CS})$ in detail. As consequences, we determine the semigroup generated by the operators C^* and C, and show that the Mal'cev product is not associative on $\mathscr{L}(\mathscr{CS})$.

Definition 4.6.1. Let $S \in \mathscr{CR}$ and let $(C^*)^0(S) = S$. Then for $n \geq 1$, define $(C^*)^n(S)$ by $(C^*)^n(S) = C^*((C^*)^{n-1}(S))$. This gives a sequence of subsemigroups of S :

$$S \geq C^*(S) \geq (C^*)^2(S) \geq \ldots\ldots .$$

We now have the following observations.

Lemma 4.6.2. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then

(i) $\mathscr{U}^{(C^*)^n} = \{ S \in \mathscr{CR} \mid (C^*)^n(S) \in \mathscr{U} \}$ for all $n \geq 0$,

where $\mathscr{U}^{(C^*)^0} = \mathscr{U}$ and $\mathscr{U}^{(C^*)^n} = [\mathscr{U}^{(C^*)^{n-1}}]^{C^*}$ for $n \geq 1$.

(ii) $\mathscr{U} \subseteq \mathscr{U}^{C^*} \subseteq \ldots \subseteq \mathscr{U}^{(C^*)^n} \subseteq \ldots \subseteq \mathscr{U}^C$,

so that $\vee_{n \geq 0} \mathscr{U}^{(C^*)^n} \subseteq \mathscr{U}^C$.

(iii) $[\vee_{n \geq 0} \mathscr{U}^{(C^*)^n}]^{C^*} = \vee_{n \geq 0} \mathscr{U}^{(C^*)^n}$.

Proof. (i) The straightforward verification is omitted.

(ii) Clearly $\mathscr{V} \subseteq \mathscr{V}^{C^*} \subseteq \mathscr{V}^C$ for any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, and so the ascending sequence

$$\mathscr{U} \subseteq \mathscr{U}^{C^*} \subseteq \ldots \subseteq \mathscr{U}^{(C^*)^n} \subseteq \ldots\ldots$$

follows. Moreover, $C(S) \leq (C^*)^n(S)$ for any $S \in \mathscr{CR}$ and $n \geq 0$, whence $\mathscr{U}^{(C^*)^n} \subseteq \mathscr{U}^C$ for all $n \geq 0$, so that $\vee_{n \geq 0} \mathscr{U}^{(C^*)^n} \subseteq \mathscr{U}^C$.

(iii) Clearly $\vee_{n \geq 0} \mathscr{U}^{(C^*)^n} \subseteq [\vee_{n \geq 0} \mathscr{U}^{(C^*)^n}]^{C^*}$. Let $S \in [\vee_{n \geq 0} \mathscr{U}^{(C^*)^n}]^{C^*}$, then $C^*(S) \in \vee_{n \geq 0} \mathscr{U}^{(C^*)^n}$. By (ii), $\vee_{n \geq 0} \mathscr{U}^{(C^*)^n} = \cup_{n \geq 0} \mathscr{U}^{(C^*)^n}$, so that $C^*(S) \in \mathscr{U}^{(C^*)^n}$ for some $n \geq 1$, thus $S \in \mathscr{U}^{(C^*)^{n+1}}$, and whence $S \in \vee_{n \geq 0} \mathscr{U}^{(C^*)^n}$. #

In Proposition 4.6.11, we shall see that the ascending sequence in Lemma 4.6.2 (ii) is strict in some instances. The next lemma follows easily from Corollary 4.1.13.

Lemma 4.6.3. If $\mathscr{U} = [\ u_\alpha(x_i) = v_\alpha(x_i)\ ]_{\alpha\in A} \in \mathscr{L}(\mathscr{CR})$, then

$$\mathscr{U}^{C^*} \subseteq [\ u_\alpha(y_i^{-1}x_i^0y_i) = v_\alpha(y_i^{-1}x_i^0y_i)\ ]_{\alpha\in A},$$

where for each x_i, $x_i \leftrightarrow y_i$ is one-to-one, and $y_i \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

For any subvariety of $\mathscr{C}$, the equality in Lemma 4.6.3 will be established in this section.

Lemma 4.6.4. $\mathscr{A}^{C^*} = \mathscr{C}^{C^*}$

$$= [\ (axb)^0 = (ab)^0,\ x^0y^{-1}b^0yx^{-1}a^0x = x^{-1}a^0xy^{-1}b^0yx^0\ ].$$

Proof. Denote the third class above by $\mathscr{U}$. Since $\mathscr{A} \subseteq \mathscr{C}$, then $\mathscr{A}^{C^*} \subseteq \mathscr{C}^{C^*}$. Let $S \in \mathscr{C}^{C^*}$. Then $S \in \mathscr{CS}$, without loss of generality, we may assume that $S = \mathscr{M}(G; I, \Lambda; P)$ where P is normalized. Let $< P >$ and N be the subgroup and the normal subgroup of G generated by the entries of P respectively. By Lemma 4.1.5, $C^*(S) = \mathscr{M}(N; I, \Lambda; P) \in \mathscr{C}$, it follows from Lemma 3.1.11 (ii) that all entries of P lie in the centre of N, so that $< P > \subseteq Z(N) \triangleleft N \triangleleft G$, where $Z(N)$ is the centre of N. By [Hu, Lemma II.7.13], $Z(N)$ is normal in G so that $N = Z(N) \in \mathscr{AG}$. Thus $C^*(S) \in \mathscr{A}$ and $S \in \mathscr{A}^{C^*}$. This proves the first equality. It remains to show that $\mathscr{A}^{C^*} = \mathscr{U}$. In the light of [Pe5, Lemma 3.7], $\mathscr{CS} = [\ (axb)^0 = (ab)^0\ ]$, it follows from Lemma 3.1.9 that $\mathscr{A} = [\ (axb)^0 = (ab)^0,\ a^0ba = aba^0\ ]$. Let $S \in \mathscr{A}^{C^*}$, then $S \in \mathscr{CS}$ so that S satisfies the identity $(axb)^0 = (ab)^0$. On the other hand, by Lemma 4.6.3, S satisfies the identity

$$(x^{-1}a^0x)^0(y^{-1}b^0y)(x^{-1}a^0x) = (x^{-1}a^0x)(y^{-1}b^0y)(x^{-1}a^0x)^0,$$

which implies that S satisfies the identity $x^0y^{-1}b^0yx^{-1}a^0x = x^{-1}a^0xy^{-1}b^0yx^0$. Thus $S \in \mathscr{U}$, and therefore $\mathscr{A}^{C^*} \subseteq \mathscr{U}$. For the opposite inclusion, let $S \in \mathscr{U}$. Then $S \in \mathscr{CS}$, by the same reason as above, and we assume that $S = \mathscr{M}(G; I, \Lambda; P)$ where P is normalized, so that $C^*(S) = \mathcal{M}(N; I, \Lambda; P)$. Let $i, j \in I$, $\lambda, \mu \in \Lambda$ and $g \in G$. Since S satisfies the identity $x^0y^{-1}b^0yx^{-1}a^0x = x^{-1}a^0xy^{-1}b^0yx^0$, we then have

$$(e; 1, \lambda)^0(g; 1, \mu)^{-1}(e; j, 1)^0(g; 1, \mu)(e; 1, \lambda)^{-1}(e; i, 1)^0(e; 1, \lambda)$$
$$= (e; 1, \lambda)^{-1}(e; i, 1)^0(e; 1, \lambda)(g; 1, \mu)^{-1}(e; j, 1)^0(g; 1, \mu)(e; 1, \lambda)^0,$$

where e is the identity of G, so that $(g^{-1}p_{\mu j}gp_{\lambda i}; 1, \lambda) = (p_{\lambda i}g^{-1}p_{\mu j}g; 1, \lambda)$, whence $p_{\lambda i}(g^{-1}p_{\mu j}g) = (g^{-1}p_{\mu j}g)p_{\lambda i}$. Taking inverse and conjugating by $p_{\lambda i}$, we have that $p_{\lambda i}(g^{-1}p_{\mu j}^{-1}g) = (g^{-1}p_{\mu j}^{-1}g)p_{\lambda i}$. It follows that $p_{\lambda i}$ lies in the centre of N, since $N = [\cup_{g\in G} g^{-1}\langle P\rangle g]$, and therefore all entries of P lie in the centre of N. By Lemma 3.1.11 (ii), $C^*(S) = \mathcal{M}(N; I, \Lambda; P) \in \mathscr{C}$ so that $S \in \mathscr{C}^{C^*} = \mathscr{A}^{C^*}$. Hence $\mathscr{U} \subseteq \mathscr{A}^{C^*}$, which completes the proof. #

Corollary 4.6.5. For any $\mathscr{U} \in \mathscr{L}(\mathscr{C})$, we have $\mathscr{U}^{C^*} = (\mathscr{U} \cap \mathscr{A})^{C^*}$.

Proof. This is immediate from Lemma 4.6.4. #

Lemma 4.6.6. If $\mathscr{U} = \mathscr{A} \cap [\,(x^0y^0)^k = (xy)^0\,]$ for some $k \geq 1$, then

$$\mathscr{U}^{C^*} = \mathscr{A}^{C^*} \cap [\,(x^0y^0)^k = (xy)^0\,].$$

Proof. Clearly $\mathscr{A}^{C^*} \cap [\,(x^0y^0)^k = (xy)^0\,] \subseteq \mathscr{U}^{C^*}$. On the other hand, we have

$$\mathscr{U}^{C^*} = \mathscr{A}^{C^*} \cap [\,(x^0y^0)^k = (xy)^0\,]^{C^*}$$
$$\subseteq \mathscr{A}^{C^*} \cap [\,((x_1^{-1}x^0x_1)^0(y_1^{-1}y^0y_1)^0)^k = (x_1^{-1}x^0x_1y_1^{-1}y^0y_1)^0\,]$$

by Lemma 4.6.3

$$= \mathscr{A}^{C^*} \cap [\,(x_1^0y_1^0)^k = (x_1y_1)^0\,]$$
$$= \mathscr{A}^{C^*} \cap [\,(x^0y^0)^k = (xy)^0\,].$$

Hence $\mathscr{U}^{C^*} = \mathscr{A}^{C^*} \cap [\,(x^0y^0)^k = (xy)^0\,]$. #

Lemma 4.6.7. If $\mathscr{U} = \mathscr{A} \cap [\,x^m = x^0\,]$ for some $m \geq 1$, then

$$\mathscr{U}^{C^*} = \mathscr{A}^{C^*} \cap [\,(y^{-1}x^0y)^m = y^0\,].$$

Proof. First, we have

$$\mathcal{U}^{C^*} = \mathcal{A}^{C^*} \cap [\ x^m = x^0\ ]^{C^*}$$
$$\subseteq \mathcal{A}^{C^*} \cap [\ (y^{-1}x^0y)^m = (y^{-1}x^0y)^0\ ] \quad \text{by Lemma 4.6.3}$$
$$= \mathcal{A}^{C^*} \cap [\ (y^{-1}x^0y)^m = y^0\ ].$$

Indeed, $\mathcal{U} = \mathcal{A}_m \circ \mathcal{RB}$. Let $S \in \mathcal{A}^{C^*} \cap [\ (y^{-1}x^0y)^m = y^0\ ]$. Without loss of generality, we may assume that $S = \mathcal{M}(G; I, \Lambda; P)$ where P is normalized, thus by Lemma 4.1.5, $C^*(S) = \mathcal{M}(N; I, \Lambda; P)$ where N is the normal subgroup of G generated by the entries of P. Since $C^*(S) \in \mathcal{A}$, then $N \in \mathcal{AG}$. To show that $C^*(S) \in \mathcal{U}$ it suffices to show that $N \in \mathcal{A}_m$. Since N is generated by the set $\{\ g^{-1}p_{\lambda i}g\ ,\ g^{-1}p_{\lambda i}^{-1}g \mid g \in G, i \in I$ and $\lambda \in \Lambda\ \}$, and noting that $(\ g^{-1}p_{\lambda i}^{-1}g\ )^m = [\ (\ g^{-1}p_{\lambda i}g\ )^m\ ]^{-1}$, it remains to show that $(\ g^{-1}p_{\lambda i}g\ )^m = e$, for all $g \in G$, $i \in I$ and $\lambda \in \Lambda$, where e is the identity of G. Let $g \in G$, $i \in I$ and $\lambda \in \Lambda$. Since S satisfies the identity $(\ y^{-1}x^0y\ )^m = y^0$, we then have

$$[\ (\ g;\ 1,\lambda\ )^{-1}(\ e;\ i,1\ )(\ g;\ 1,\lambda\ )\ ]^m = (\ g;\ 1,\lambda\ )^0,$$

and so $(\ g^{-1}p_{\lambda i}g;\ 1,\lambda\ )^m = [\ (\ g^{-1}p_{\lambda i}g\ )^m;\ 1,\lambda\ ]$

$$= (\ e;\ 1,\lambda\ ),$$

whence $(\ g^{-1}p_{\lambda i}g\ )^m = e$, as required. #

The next corollary is a simple consequence of Lemma 4.5.1.

Corollary 4.6.8. For any $\mathcal{U} \in \mathcal{L}(\mathcal{RBG})$, we have

$$\mathcal{U}^{C^*} = \begin{cases} \mathcal{G}, & \text{if } \mathcal{U} \in \mathcal{L}(\mathcal{G}); \\ \mathcal{LG}, & \text{if } \mathcal{U} \cap \mathcal{RB} = \mathcal{LZ}; \\ \mathcal{RG}, & \text{if } \mathcal{U} \cap \mathcal{RB} = \mathcal{RZ}; \\ \mathcal{RBG}, & \text{if } \mathcal{RB} \subseteq \mathcal{U}. \end{cases}$$

Rasin **[R2]** has obtained a description of the subvarieties of $\mathcal{A}$ (see also [**J1**, Theorem 4.2]). We state this result in the next theorem.

Theorem 4.6.9 [R2]. Let $\mathscr{U} \in \mathscr{L}(\mathscr{A})$. Then

either (i) $\mathscr{U} \in \mathscr{L}(\mathscr{AG} \vee \mathscr{RB})$,

or (ii) $\mathscr{U} = \mathscr{A}$,

or (iii) $\mathscr{U} = \mathscr{A} \cap [\ (x^0y^0)^k = (xy)^0\ ]$ for some $k \geq 1$,

or (iv) $\mathscr{U} = \mathscr{A} \cap [\ x^m = x^0\ ]$ for some $m \geq 1$,

or (v) $\mathscr{U} = \mathscr{A} \cap [\ (x^0y^0)^k = (xy)^0,\ x^m = x^0\ ]$ for some $m, k \geq 1$ with $k \mid m$.

Combining Theorem 4.6.9 with the above facts, we deduce easily the following.

Proposition 4.6.10. If $\mathscr{U} = [\ u_\alpha(x_i) = v_\alpha(x_i)\ ]_{\alpha \in A} \in \mathscr{L}(\mathscr{C})$, then

$$\mathscr{U}^{C^*} = [\ u_\alpha(y_i^{-1}x_i^0y_i) = v_\alpha(y_i^{-1}x_i^0y_i)\ ]_{\alpha \in A},$$

where for each x_i, $x_i \leftrightarrow y_i$ is one-to-one, and $y_i \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

The next proposition shows that the order of the operator C^* is infinite.

Proposition 4.6.11. The varieties $\mathscr{A}^{(C^*)^n}$, $n \geq 0$, form a strictly ascending sequence

$$\mathscr{A} \subset \mathscr{A}^{C^*} \subset \ldots \subset \mathscr{A}^{(C^*)^n} \subset \ldots \subset \mathscr{A}^C = \mathscr{D},$$

and $\vee_{n \geq 0} \mathscr{A}^{(C^*)^n} \subset \mathscr{D}$.

Proof. Let $S = \mathscr{M}(G; I, I; P)$, where $I = \{1, 2\}$, $G = S_3$ and $P = \begin{bmatrix} (1) & (1) \\ (1) & (12) \end{bmatrix}$. By Lemma 3.1.18, $S \in \mathscr{A}^C = \mathscr{D}$. Since the normal subgroup of S_3 generated by the set $\{(1), (12)\}$ is S_3 itself, it follows from Lemma 4.1.5 that $C^*(S) = S$, and so $(C^*)^n(S) = S$ for all $n \geq 0$. Thus $S \notin \mathscr{A}^{(C^*)^n}$ for all $n \geq 0$, since $S_3 \notin \mathscr{AG}$. By Lemma 4.6.2, $S \notin \vee_{n \geq 0} \mathscr{A}^{(C^*)^n}$, and therefore the last assertion follows.

Let $T = \mathscr{M}(S_3; I, I; P)$, where $I = \{1, 2\}$ and $P = \begin{bmatrix} (1) & (1) \\ (1) & (123) \end{bmatrix}$. Since the normal subgroup of S_3 generated by $\{(1), (123)\}$ is $N = \{(1), (123), (132)\}$, it follows from

Lemma 4.1.5 that $C^*(T) = \mathcal{M}(N; I, I; P)$, whence $C^*(T) \in \mathscr{A}$ so that $T \in \mathscr{A}^{C^*}$. But $T \notin \mathscr{A}$ since $S_3 \notin \mathscr{AG}$. Hence $\mathscr{A} \subset \mathscr{A}^{C^*}$.

For each $n \geq 1$, let $S_n = M(G_n; I, I; P)$, where $I = \{1, 2\}$, $G_n = D_{2^{n+1}} = \langle a, b : a^{2^{n+1}} = e = b^2, ba = a^{-1}b \rangle$ —— the dihedral group of order 2^{2+n} [**Hu**, Theorem I.6.13], and $P = \begin{bmatrix} e & e \\ e & b \end{bmatrix}$. The normal subgroup M_n of G_n generated by the entries of P is precisely the subgroup of G_n generated by the elements b and a^2, i.e., $M_n = \langle b, a^2 \rangle$. It is easily verified that $M_n \cong G_{n-1}$ if $n \geq 2$ and $M_1 = \langle a, b : a^2 = e = b^2, ab = ba \rangle$. By Lemma 4.1.5, $C^*(S_1) = \mathcal{M}(M_1; I, I; P) \in \mathscr{A}$ since $M_1 \in \mathscr{AG}$. For each $n \geq 2$, it follows that $C^*(S_n) = \mathcal{M}(M_n; I, I; P) \cong \mathcal{M}(G_{n-1}; I, I, P) = S_{n-1}$, thus

$$(C^*)^n(S_n) \cong C^*(S_1) \in \mathscr{A}$$

and

$$(C^*)^{n-1}(S_n) \cong S_2 \notin \mathscr{A}$$

since $G_1 \notin \mathscr{AG}$, and so $S_n \in \mathscr{A}^{(C^*)^n}$ and $S_n \notin \mathscr{A}^{(C^*)^{n-1}}$. Therefore, by Lemma 4.6.2, $\mathscr{A}^{(C^*)^{n-1}} \subset \mathscr{A}^{(C^*)^n}$ for all $n \geq 2$. #

Corollary 4.6.12. (i) $C^* \neq C$.

(ii) The Mal'cev product " $\circ$ " is not associative on $\mathcal{L}(\mathscr{CS})$.

Proof. (i) This is an immediate consequence of Proposition 4.6.11.

(ii) By Proposition 4.6.11, we have $(\mathscr{A} \circ \mathscr{G}) \circ \mathscr{G} = \mathscr{A}^{(C^*)^2} \neq \mathscr{A}^{C^*} = \mathscr{A} \circ \mathscr{G} = \mathscr{A} \circ (\mathscr{G} \circ \mathscr{G})$, so that $(\mathscr{A} \circ \mathscr{G}) \circ \mathscr{G} \neq \mathscr{A} \circ (\mathscr{G} \circ \mathscr{G})$, and therefore " $\circ$ " is not associative on $\mathcal{L}(\mathscr{CS})$. #

Corollary 4.6.13. Let $\mathscr{U} \in \mathcal{L}(\mathscr{CR})$. Then

(i) $\mathscr{U}^{C^*} = \mathscr{U} \quad \not\Rightarrow \quad \mathscr{U}^C = \mathscr{U}$.

(ii) $\mathscr{U}^{C^*} = \mathscr{U}^{(C^*)^2} \not\Rightarrow \mathscr{U}^C = \mathscr{U}^{C^*}$.

Proof. This follows easily from Lemma 4.6.2 (iii) and Proposition 4.6.11. #

Corollary 4.6.14. The semigroup generated by the operators C^* and C is isomorphic to the infinite cyclic semigroup with a zero adjoined.

Proof. This is a consequence of Lemma 4.4.3 and Proposition 4.6.11. #

The rest of this section is devoted to characterizations of $\mathcal{V}^{(C^*)^i}$, $\mathcal{V} \in [\mathcal{RB}, \mathcal{CS}]$ and $i \geq 0$, in terms of $\mathcal{E}$-invariant normal subgroups of the free group over a countably infinite set. We need some preparation.

Lemma 4.6.15 [R1]. Let $S = \mathcal{M}(G; I, \Lambda; P)$ where P is normalized. Let $\varphi \in \mathcal{T}_I$, $\omega \in$ End G, $\psi \in \mathcal{T}_\Lambda$ be such that

$$[\lambda, i]\omega = [1\psi, 1\varphi][\lambda\psi, 1\varphi]^{-1}[\lambda\psi, i\varphi][1\psi, i\varphi]^{-1} \quad (\lambda \in \Lambda, i \in I). \qquad (1)$$

Then $\theta = \theta(\omega; \varphi, \psi)$ defined by

$$(g; i, \lambda)\theta = ([1\psi, i\varphi]^{-1}(g\omega)[1\psi, 1\varphi][\lambda\psi, 1\varphi]^{-1}; i\varphi, \lambda\psi)$$

is an endomorphism of S. Conversely, every endomorphism of S can be so written uniquely.

The following is a construction of the Rees matrix representation of a free completely simple semigroup.

Lemma 4.6.16 ([C], [R1]). Let $X = \{x_i \mid i \in I\}$ be a nonempty set, fix $1 \in I$ and let $I' = I \setminus \{1\}$. Let

$$Z = \{q_i \mid i \in I\} \cup \{[j, k] \mid j, k \in I'\},$$

F_Z be the free group on Z, and let $P = ([j, k])$ with $[1, k] = [j, 1] = 1$, the identity of F_Z. Then

$$F = \mathcal{M}(F_Z; I, I; P)$$

is a free completely simple semigroup over X, with embedding $x_i \longrightarrow (q_i; i, i)$.

Notation 4.6.17. We fix a countably infinite set X, and in addition to the above notation, introduce

$$F_P = \langle [j, k] \mid j, k \in \Gamma' \rangle$$

the free subgroup of F_Z generated by the set $\{ [j, k] \mid j, k \in \Gamma' \}$.

By Theorem 3.1.1 and Lemma 3.1.3 (ii), we have the following .

Lemma 4.6.18. Let $S = \mathcal{M}(G; I, \Lambda; P)$ where P is normalized. If N is a normal subgroup of G, then ρ_N defined on S by

$$(g; i, \lambda) \rho_N (h; j, \mu) \Leftrightarrow gh^{-1} \in N, i = j, \lambda = \mu$$

is an idempotent separating congruence on S, and every such congruence is obtained in this way. Write $P/_N$ for the $\Lambda \times I$ matrix with the (j, k)-th entry equal to the (j, k)-th entry of P modulo N, $S/_{\rho_N}$ is isomorphic to $\mathcal{M}(G/_N; I, \Lambda; P/_N)$.

Notation 4.6.19. Let

$$\mathcal{E}(F_Z) = \{ \omega \in \text{End } F_Z \mid \text{there exist } \varphi, \psi \in \mathcal{T}_I \text{ such that (1) holds } \}.$$

Then $\mathcal{E}(F_Z)$ consists precisely of endomorphisms of F_Z that arise in association with endomorphisms of F.

Lemma 4.6.20 [R1]. Let N be a normal subgroup of F_Z. Then ρ_N is fully invariant if and only if $N\omega \subseteq N$ for all $\omega \in \mathcal{E}(F_Z)$.

Definition 4.6.21. A normal subgroup of F_Z is $\mathcal{E}$*-invariant* if it is invariant under all $\omega \in \mathcal{E}(F_Z)$. The set of all $\mathcal{E}$-invariant normal subgroups of F_Z will be denoted by $\mathcal{N}$.

It is clear that $\mathcal{N}$ is a sublattice of the lattice of all normal subgroups of F_Z.

Theorem 4.6.22 [Ri]. The interval [$\mathscr{RB}$, $\mathscr{CS}$] is anti-isomorphic to the lattice $\mathcal{N}$. In particular, $\mathscr{V} \in$ [$\mathscr{RB}$, $\mathscr{CS}$] if and only if $\rho_{\mathscr{V}}$ is idempotent separating and so is of the form ρ_N, $N \in \mathcal{N}$. The mapping $\rho_{\mathscr{V}} \longrightarrow N$, where $\rho_{\mathscr{V}} = \rho_N$, is an isomorphism of the lattice of fully invariant idempotent separating congruences on F onto $\mathcal{N}$.

Notation 4.6.23. For any subgroup H of F_Z, we will denote by $\hat{H}$ the normal closure of H in F_Z.

Notation 4.6.24. For any subgroups H and K of a group G, we denote by [H, K] the subgroup of G generated by the elements of the form $[h, k] = h^{-1}k^{-1}hk$, where $h \in H$, $k \in K$.

Definition 4.6.25. Let $M_0 = F_Z$. Then for $i \geq 1$, define M_i to be the normal closure of F_P in M_{i-1} . This gives a sequence of subgroups of F_Z, each normal in the preceding one :

$$F_Z = M_0 \geq M_1 \geq \ldots \geq M_i \geq \ldots \geq F_P .$$

Lemma 4.6.26. For $i \geq 0$, we have $M_i\omega \subseteq M_i$ for all $\omega \in \mathcal{E}(F_Z)$.

Proof. If i = 0, then $M_0 = F_Z$, so that $M_0\omega = F_Z\omega \subseteq F_Z = M_0$ for all $\omega \in \mathcal{E}(F_Z)$. Assume that $M_i\omega \subseteq M_i$ for all $\omega \in \mathcal{E}(F_Z)$. It is easy to see that $M_{i+1} = [\cup_{g \in M_i} g^{-1}F_P g]$, thus for any $\omega \in \mathcal{E}(F_Z)$, we have

$$\begin{aligned} M_{i+1}\omega &= [\cup_{g \in M_i} g^{-1}F_P g]\omega \\ &= [\cup_{g \in M_i} (g\omega)^{-1}F_P\omega(g\omega)] \\ &\subseteq [\cup_{g \in M_i} g^{-1}F_P g] \qquad \text{since } F_P\omega \subseteq F_P \text{ and } M_i\omega \subseteq M_i \\ &= M_{i+1}. \end{aligned}$$

By induction, the assertion follows. #

Lemma 4.6.27. For any $\theta \in \mathcal{E}(F_Z)$, there exist two unique endomorphisms θ_1 and θ_2 of F_Z such that

(i) $\theta_1 \mid_X = \theta \mid_X$ and $\theta_1 \mid_Y = \varepsilon_Y$;

(ii) $\theta_2 \mid_X = \varepsilon_X$ and $\theta_2 \mid_Y = \theta \mid_Y$;

(iii) $\theta_1, \theta_2 \in \mathcal{E}(F_Z)$ and $\theta = \theta_2\theta_1$;

where $X = \{ x_i \mid i \in I \}$ and $Y = \{ [j, k] \mid j, k \in I' \}$ as introduced in Lemma 4.6.16.

Proof. Since F_Z is the free group on $Z = X \cup Y$, thus the existence and uniqueness of θ_1 and θ_2 follows easily. Clearly θ_1 satisfies the condition (1) in Lemma 4.6.15 with $\varphi = \psi = \varepsilon_I$, and so $\theta_1 \in \mathcal{E}(F_Z)$. Since $\theta \in \mathcal{E}(F_Z)$, there exist $\varphi, \psi \in \mathcal{T}_I$ such that the condition (1) holds, thus θ_2 satisfies the condition (1) with the same φ and ψ, and whence $\theta_2 \in \mathcal{E}(F_Z)$. Clearly, $\theta \mid_Z = \theta_2\theta_1 \mid_Z$, which implies that $\theta = \theta_2\theta_1$. #

Definition 4.6.28. Let $N \in \mathcal{N}$. For any subgroup M of F_Z, we define

$$\mathcal{V}_N(M) = \langle \bigcup N\omega : \omega \in \mathcal{E}(F_Z) \text{ such that } F_Z\omega \subseteq M \text{ and } \omega \mid_{F_P} = \varepsilon_{F_P} \rangle.$$

Clearly $\mathcal{V}_N(M)$ is a subgroup of F_Z such that $\mathcal{V}_N(M) \subseteq N \cap M$.

We are now ready to prove the last main result of this section.

Theorem 4.6.29. For $\mathcal{V} \in [\mathcal{RB}, \mathcal{CS}]$ with $\rho_{\mathcal{V}} = \rho_N$, we have

$$\rho_{\mathcal{V}(C^*)^i} = \rho_{N_i} \qquad \text{for all } i \geq 0,$$

where $N_i = \widehat{\mathcal{V}_N(M_i)}$.

Proof. For $i = 0$, we clearly have that $N_0 = \widehat{\mathcal{V}_N(F_Z)} = N$, so that $\rho_{\mathcal{V}(C^*)^0} = \rho_{\mathcal{V}} = \rho_{N_0}$.

Let $i \geq 1$. To show that $N_i \in \mathcal{N}$ it suffices to show that $\mathcal{V}_N(M_i)\theta \subseteq \mathcal{V}_N(M_i)$ for all $\theta \in \mathcal{E}(F_Z)$. Let $\theta \in \mathcal{E}(F_Z)$, and let $\omega \in \mathcal{E}(F_Z)$ be such that $F_Z\omega \subseteq M_i$ and $\omega \mid_{F_P} = \varepsilon_{F_P}$. Thus $X\omega \subseteq M_i$, which implies that $X\omega\theta \subseteq M_i\theta \subseteq M_i$ by Lemma 4.6.26, so that

$X(\omega\theta)_1 \subseteq M_i$. On the other hand, $Y(\omega\theta)_1 \subseteq F_P \subseteq M_i$. Then $Z(\omega\theta)_1 = X(\omega\theta)_1 \cup Y(\omega\theta)_1 \subseteq M_i$, and so $F_Z(\omega\theta)_1 \subseteq M_i$. Hence

$$\begin{aligned}(N\omega)\theta &= N(\omega\theta) \\ &= N(\omega\theta)_2(\omega\theta)_1 && \text{by Lemma 4.6.27} \\ &\subseteq N(\omega\theta)_1 && \text{since } N(\omega\theta)_2 \subseteq N \\ &\subseteq \mathcal{V}_N(M_i) && \text{since } F_Z(\omega\theta)_1 \subseteq M_i \text{ and } (\omega\theta)_1 \mid_{F_P} = \varepsilon_{F_P},\end{aligned}$$

and whence $\mathcal{V}_N(M_i)\theta \subseteq \mathcal{V}_N(M_i)$. Therefore $N_i \in \mathcal{N}$.

To show that $\rho_{\mathcal{U}(C^*)^i} = \rho_{N_i}$, it suffices to show the following two statements:

(a) $F/\rho_{N_i} = \mathcal{M}(F_Z/N_i; I, I; P/N_i) \in \mathcal{U}(C^*)^i$.

In order to show that $F/\rho_{N_i} \in \mathcal{U}(C^*)^i$, it suffices to show that $(C^*)^i(F/\rho_{N_i}) = \mathcal{M}(M_i/(M_i \cap N_i); I, I; P/(M_i \cap N_i)) \in \mathcal{U}$ by Lemmas 4.1.5 and 4.6.2.

Let $\omega^* : X \longrightarrow M_i$ be a bijection; since both X and M_i are countably infinite, such a ω^* must exist. Extend ω^* to $\omega^* : Z \longrightarrow M_i$ by $[j, k]\omega^* = [j, k]$ for any $[j, k] \in Y$. Then there exists a unique homomorphism $\omega : F_Z \longrightarrow M_i$ of F_Z onto M_i such that

(i) $\omega \mid_Z = \omega^*$ and $\omega \mid_{F_P} = \varepsilon_{F_P}$;

(ii) $\omega \in \mathcal{E}(F_Z)$; since ω satisfies the condition (1) with $\varphi = \psi = \varepsilon_I$;

(iii) $N\omega \subseteq F_Z\omega = M_i$.

Thus $N\omega \subseteq \mathcal{V}_N(M_i) \subseteq N_i$, and so $N\omega \subseteq M_i \cap N_i$. Define

$$\Phi : F/\rho_N \longrightarrow \mathcal{M}(M_i/(M_i \cap N_i); I, I; P/(M_i \cap N_i))$$

by $(gN; i, j)\Phi = [(g\omega)(M_i \cap N_i); i, j]$. It is a straightforward verification that Φ is a homomorphism of F/ρ_N onto $\mathcal{M}(M_i/(M_i \cap N_i); I, I; P/(M_i \cap N_i))$, and whence $(C^*)^i(F/\rho_{N_i}) \in \mathcal{U}$, as required.

(b) If $N' \in \mathcal{N}$ with $F/\rho_{N'} \in \mathcal{U}(C^*)^i$, then $N_i \subseteq N'$.

Since $F/\rho_{N'} = \mathcal{M}(F_Z/N'; I, I; P/N') \in \mathcal{U}(C^*)^i$, then $T = (C^*)^i(F/\rho_{N'}) = \mathcal{M}(M_i/(N' \cap M_i); I, I; P/(N' \cap M_i)) \in \mathcal{U}$ by Lemmas 4.1.5 and 4.6.2. Define

$$\varphi : \mathcal{M}(M_i; I, I; P) \longrightarrow T$$

by $(g; i, j)\varphi = [g(N' \cap M_i); i, j]$. It is easily verified that φ is a homomorphism of $\mathcal{M}(M_i; I, I; P)$ onto T. For $\omega \in \mathcal{E}(F_Z)$ with $F_Z\omega \subseteq M_i$ and $\omega|_{F_P} = \varepsilon_{F_P}$, we define θ : $F \longrightarrow \mathcal{M}(M_i; I, I; P)$ by $(g; i, j)\theta = (g\omega; i, j)$. Then θ is a homomorphism of F into $\mathcal{M}(M_i; I, I; P)$. It follows that $\theta\varphi : F \longrightarrow T\ (\in \mathcal{V})$ is a homomorphism of F into T, so that $\rho_N \leq \overline{\theta\varphi}$. Let $g \in N$. Since $(g; 1, 1)\ \rho_N\ (1; 1, 1)$, thus $(g; 1, 1)\theta\varphi = (1; 1, 1,)\theta\varphi$, that is, $[(g\omega)(N' \cap M_i); 1, 1] = (N' \cap M_i; 1, 1)$, so that $g\omega \in N' \cap M_i$. Then $N\omega \subseteq N' \cap M_i \subseteq N'$ for all $\omega \in \mathcal{E}(F_Z)$ with $F_Z\omega \subseteq M_i$ and $\omega|_{F_P} = \varepsilon_{F_P}$, and whence $N_i \subseteq N'$ by the definition of N_i, as required. #

It is a simple consequence of Theorem 4.6.29 that

$$F/\rho_{N_i} = \mathcal{M}(F_Z/N_i; I, I; P/N_i)$$

is a relatively free object in $\mathcal{V}^{(C^*)^i}$, for all $\mathcal{V} \in [\mathcal{RB}, \mathcal{CS}]$ with $\rho_{\mathcal{V}} = \rho_N$ and $i \geq 0$.

Corollary 4.6.30. $\rho_{\mathcal{A}(C^*)^i} = \rho_{\widehat{[M_i, M_i]}}$ for all $i \geq 0$.

Proof. From [PR5, Proposition 7.2], $N = [F_Z, F_Z] \in \mathcal{N}$ and $\rho_{\mathcal{A}} = \rho_N$. By Theorem 4.6.29, it suffices to show that $N_i = \widehat{[M_i, M_i]}$ for all $i \geq 0$.

The case is trivial for $i = 0$. For $i \geq 1$, we then have

$$\begin{aligned} \mathcal{V}_N(M_i) &= \langle \cup [F_Z, F_Z]\omega : \omega \in \mathcal{E}(F_Z) \text{ with } F_Z\omega \subseteq M_i \\ &\qquad \text{and } \omega|_{F_P} = \varepsilon_{F_P} \rangle \\ &\subseteq \langle \cup [F_Z\omega, F_Z\omega] : \omega \in \mathcal{E}(F_Z) \text{ with } F_Z\omega \subseteq M_i \\ &\qquad \text{and } \omega|_{F_P} = \varepsilon_{F_P} \rangle \\ &\subseteq [M_i, M_i], \end{aligned}$$

so that $N_i = \widehat{\mathcal{V}_N(M_i)} \subseteq \widehat{[M_i, M_i]}$. On the other hand, from the proof of Theorem 4.6.29, there exists $\omega \in \mathcal{E}(F_Z)$ such that $F_Z\omega = M_i$ and $\omega|_{F_P} = \varepsilon_{F_P}$; this implies that for any a, b $\in M_i$, there exists u, v $\in F_Z$ such that a = uω and b = vω, thus $a^{-1}b^{-1}ab = (u^{-1}v^{-1}uv)\omega \in [F_Z, F_Z]\omega \subseteq \mathcal{V}_N(M_i)$, so that $[M_i, M_i] \subseteq \mathcal{V}_N(M_i)$, and whence $\widehat{[M_i, M_i]} \subseteq \widehat{\mathcal{V}_N(M_i)} = N_i$. Therefore, $N_i = \widehat{[M_i, M_i]}$ for all $i \geq 0$. #

Corollary 4.6.31. The subgroups M_i of F_Z, $i \geq 0$, forms a strictly descending sequence

$$F_Z = M_0 > M_1 > M_2 > \dots > M_i > \dots > F_P .$$

Proof. This is an immediate consequence of Proposition 4.6.11 and Corollary 4.6.30. #

§ 4.7 Concluding remarks on the operator C^*

We conclude this chapter by gathering together some supplementary facts about the operator C^* in the next result.

Theorem 4.7.1. The operator C^* is a complete $\cap$-endomorphism of $\mathcal{L}(\mathcal{CR})$. Its restriction to $\mathcal{L}(\mathcal{OG})$ is a complete endomorphism of $\mathcal{L}(\mathcal{OG})$.

Proof. For $\mathcal{U}_\alpha \in \mathcal{L}(\mathcal{CR})$ with $\alpha \in A$, and let $S \in \mathcal{CR}$, we then have

$$\begin{aligned} S \in (\cap_{\alpha\in A}\mathcal{U}_\alpha)^{C^*} &\Leftrightarrow C^*(S) \in \cap_{\alpha\in A}\mathcal{U}_\alpha \\ &\Leftrightarrow C^*(S) \in \mathcal{U}_\alpha \qquad \text{for all } \alpha \in A \\ &\Leftrightarrow S \in (\mathcal{U}_\alpha)^{C^*} \qquad \text{for all } \alpha \in A \\ &\Leftrightarrow S \in \cap_{\alpha\in A}(\mathcal{U}_\alpha)^{C^*}, \end{aligned}$$

whence C^* is a complete $\cap$-endomorphism of $\mathcal{L}(\mathcal{CR})$. The last assertion is a consequence of Lemma 4.4.2 and Theorem 4.5.20. #

It remains an open question whether or not the operator C^* is a (complete) $\vee$-homomorphism of $\mathcal{L}(\mathcal{CR})$.

Chapter 5

Varieties of The Forms $<\mathscr{U}\circ\mathscr{V}>$ With $\mathscr{V}\in\{\mathscr{LX},\mathscr{RX},\mathscr{RB}\}$

This chapter is devoted to study the varieties of the forms $<\mathscr{U}\circ\mathscr{V}>$ with $\mathscr{V}\in\{\mathscr{LX},\mathscr{RX},\mathscr{RB}\}$. We first provide descriptions of these varieties $<\mathscr{U}\circ\mathscr{LX}>$, $<\mathscr{U}\circ\mathscr{RX}>$ and $<\mathscr{U}\circ\mathscr{RB}>$. The operators L_l, L_r and L^* on $\mathscr{L}(\mathscr{CR})$ associated with these varieties are introduced and studied. We also obtain some general relationships between the operators L_l, L_r, L^* and the well known operator L.

§ 5.1 Varieties $<\mathscr{U}\circ\mathscr{V}>$ with $\mathscr{V}\in\{\mathscr{LX},\mathscr{RX},\mathscr{RB}\}$

In this section we give descriptions of the varieties of the forms $<\mathscr{U}\circ\mathscr{LX}>$, $<\mathscr{U}\circ\mathscr{RX}>$ and $<\mathscr{U}\circ\mathscr{RB}>$. We require some preliminary observations.

Lemma 5.1.1. If $S\in\mathscr{CR}$, then eS, Sf and eSf are members of $\mathscr{CR}$ for any $e, f\in E(S)$.

Proof. Let $e, f\in E(S)$. Clearly eS is a subsemigroup of S. To show that $eS\in\mathscr{CR}$, by Lemma 2.5.4 it suffices to show that $a^{-1}\in eS$ for any $a\in eS$. Let $a = ex\in eS$ with $x\in S$, then

$$\begin{aligned} a^{-1} &= (ex)^{-1} \\ &= (ex)^0x^{-1}(xe)^0e^{-1}(ex)^0 && \text{by Lemma 2.5.7} \\ &= e[x(ex)^{-1}x^{-1}(xe)^0e(ex)^0] \\ &\in eS, \end{aligned}$$

whence eS $\in \mathcal{CR}$. The case of Sf is symmetric and the case of eSf follows from these two cases, since eSf = eS $\cap$ Sf. #

The above lemma enables us to introduce three operators on $\mathcal{L}(\mathcal{CR})$ defined as follows : for any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$,

$$\mathcal{V}^{L_l} = \{ S \in \mathcal{CR} \mid eS \in \mathcal{V} \text{ for any } e \in E(S) \},$$

$$\mathcal{V}^{L_r} = \{ S \in \mathcal{CR} \mid Se \in \mathcal{V} \text{ for any } e \in E(S) \},$$

$$\mathcal{V}^{L^*} = \{ S \in \mathcal{CR} \mid eSf \in \mathcal{V} \text{ for any } e, f \in E(S) \}.$$

Lemma 5.1.2. Let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$. Then

(i) $\mathcal{V}^{L_l} \in \mathcal{L}(\mathcal{CR})$ and $< \mathcal{V} \circ \mathcal{LZ} > \subseteq \mathcal{V}^{L_l}$.

(ii) $\mathcal{V}^{L_r} \in \mathcal{L}(\mathcal{CR})$ and $< \mathcal{V} \circ \mathcal{RZ} > \subseteq \mathcal{V}^{L_r}$.

(iii) $\mathcal{V}^{L^*} \in \mathcal{L}(\mathcal{CR})$ and $< \mathcal{V} \circ \mathcal{RB} > \subseteq \mathcal{V}^{L^*}$.

Proof. (i) If S $\in \mathcal{V}^{L_l}$, T is a completely regular subsemigroup of S and e $\in$ E(T) then e $\in$ E(S) and eT is a completely regular subsemigroup of eS and so belongs to $\mathcal{V}$, whence T $\in \mathcal{V}^{L_l}$. If T is a homomorphic image of S, under ϕ, say, and e $\in$ E(T) then by Lemma 2.2.6, e = e'ϕ for some e' $\in$ E(S), whence eT = (e'S)ϕ $\in \mathcal{V}$. That $\mathcal{V}^{L_l}$ is closed under direct products is immediate upon noting that an element of a direct product of semigroups is idempotent if and only if each of its components is idempotent. Hence $\mathcal{V}^{L_l}$ is a variety, i.e., $\mathcal{V}^{L_l} \in \mathcal{L}(\mathcal{CR})$.

To see that $< \mathcal{V} \circ \mathcal{LZ} > \subseteq \mathcal{V}^{L_l}$ it suffices to show that $\mathcal{V} \circ \mathcal{LZ} \subseteq \mathcal{V}^{L_l}$. Let S $\in \mathcal{V} \circ \mathcal{LZ}$. Then there exists a congruence ρ on S such that $S/_\rho \in \mathcal{LZ}$ and eρ $\in \mathcal{V}$ for any e $\in$ E(S). Let e $\in$ E(S). For any a $\in$ S, we then have (es)ρ = (eρ)(sρ) = eρ since $S/_\rho$ is a left zero semigroup, so that eS $\leq$ eρ, which clearly implies that eS $\in \mathcal{V}$, and whence S $\in \mathcal{V}^{L_l}$. Therefore $< \mathcal{V} \circ \mathcal{LZ} > \subseteq \mathcal{V}^{L_l}$.

(ii) This is the dual of (i).

(iii) The proof of this part is entirely similar to that of (i). #

Let X be a nonempty set. For any $u \in X^+$, we denote by h(u) and t(u) respectively the first and the last variables which appear in u. We define three relations on X^+ as follows: for any $u, v \in X^+$,

$$u\, \beta_l\, v \iff h(u) = h(v),$$

$$u\, \beta_r\, v \iff t(u) = t(v),$$

$$u\, \beta\, v \iff h(u) = h(v) \text{ and } t(u) = t(v).$$

Clearly β_l, β_r and β are congruences on X^+ such that $\beta = \beta_l \cap \beta_r$, and we have the following observations.

Lemma 5.1.3. Let $w \in X^+$. Then

(i) $X^+/\beta_l \in \mathcal{LZ}$ and $w\beta_l = h(w)X^*$.

(ii) $X^+/\beta_r \in \mathcal{RZ}$ and $w\beta_r = X^*t(w)$.

(iii) $X^+/\beta \in \mathcal{RB}$ and $w\beta = h(w)X^* \cap X^*t(w)$.

Proof. (i) For any $u, v \in X^+$, we clearly have that h(uv) = h(u), which implies that $X^+/\beta_l \in \mathcal{LZ}$. For $w \in X^+$, $w\beta_l = h(w)X^*$ follows easily from the definition of β_l.

(ii) This is dual to (i).

(iii) This follows directly from (i), (ii) and the fact that $\beta = \beta_l \cap \beta_r$ #

The main purpose of this section is to establish the following result.

Theorem 5.1.4. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ we have

(i) $\mathcal{U}^{L_l} = \langle \mathcal{U} \circ \mathcal{LZ} \rangle$.

(ii) $\mathcal{U}^{L_r} = \langle \mathcal{U} \circ \mathcal{RZ} \rangle$.

(iii) $\mathcal{U}^{L^*} = \langle \mathcal{U} \circ \mathcal{RB} \rangle$.

Proof. (i) By Lemma 5.1.2 (i) it remains to show that $\mathscr{V}^{L_1} \subseteq \langle \mathscr{V} \circ \mathscr{LZ} \rangle$. Let $S \in \mathscr{V}^{L_1}$. Let X denote the alphabet whose letters are elements of S. There results the usual surjective homomorphism of semigroups

$$\alpha : X^+ \longrightarrow S$$

which maps each letter of X into itself. By Lemma 5.1.3 (i), $T = X^+/\beta_1 \in \mathscr{LZ}$ and $\rho : X^+ \longrightarrow T$ becomes a surjective homomorphism by defining $w\rho = w\beta_1$. Let $\tau = \alpha^{-1}\rho$. Clearly $\tau : S \longrightarrow T$ is a relational morphism of S onto T. For any $t \in T$, there exists $w \in X^+$ such that $t = w\beta_1$, and by Lemma 5.1.3 (i) we have that $w\beta_1 = h(w)X^*$, whence

$$\begin{aligned} t\tau^{-1} &= (w\beta_1)\tau^{-1} \\ &= (h(w)X^*)\alpha \\ &= sS \qquad \text{where } s = (h(w))\alpha \\ &= s^0S, \end{aligned}$$

and therefore $t\tau^{-1} \in \mathscr{V}$ since $S \in \mathscr{V}^{L_1}$. To complete the proof of this part, by Theorem 3.5.3 it suffices to show that $\text{graph}(\tau) \in \mathscr{CR}$. Let $(s, t) \in \text{graph}(\tau)$, then $s \in eS$ and $t\tau^{-1} = eS$, where $e = s^0$, it follows from Lemma 5.1.1 that $s^{-1} \in eS$, so that $(s, t)^{-1} = (s^{-1}, t^{-1}) = (s^{-1}, t) \in \text{graph}(\tau)$, and therefore $\text{graph}(\tau) \in \mathscr{CR}$, as required.

(ii) This is dual to (i).

(iii) The proof of this part is entirely similar to that of (i). #

The next corollary is a simple consequence of Theorem 5.1.4.

Corollary 5.1.5. If $\mathscr{V} = [\, u_\alpha(x_i) = v_\alpha(x_i) \,]_{\alpha \in A} \in \mathscr{L}(\mathscr{CR})$, then

(i) $\mathscr{V}^{L_1} = \langle \mathscr{V} \circ \mathscr{LZ} \rangle$

$$= [\, u_\alpha(xx_1, \dots, xx_n) = v_\alpha(xx_1, \dots, xx_n) \,]_{\alpha \in A}$$

$$= [\, u_\alpha(x^0x_1, \dots, x^0x_n) = v_\alpha(x^0x_1, \dots, x^0x_n) \,]_{\alpha \in A}.$$

(ii) $\mathscr{U}^{L_r} = < \mathscr{U} \circ \mathscr{RZ} >$

$$= [\ u_\alpha(x_1y, \dots , x_n y) = v_\alpha(x_1y , \dots , x_n y)\ ]_{\alpha \in A}$$

$$= [\ u_\alpha(x_1y^0, \dots , x_n y^0) = v_\alpha(x_1y^0, \dots , x_n y^0)\ ]_{\alpha \in A}.$$

(iii) $\mathscr{U}^{L^*} = < \mathscr{U} \circ \mathscr{RB} >$

$$= [\ u_\alpha(xx_1y, \dots , xx_n y) = v_\alpha(xx_1y , \dots , xx_n y)\ ]_{\alpha \in A}$$

$$= [\ u_\alpha(x^0x_1y^0, \dots , x^0x_n y^0) = v_\alpha(x^0x_1y^0, \dots , x^0x_n y^0)\ ]_{\alpha \in A}.$$

Where $x, y \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

The next corollary is essentially contained within the proof of Theorem 5.1.4.

Corollary 5.1.6. Let $S \in \mathscr{CR}$. Then

(i) There exist $T \in \mathscr{LZ}$ and $\tau \in CR(S, T)$ such that τ is surjective and for any $t \in T$, $t\tau^{-1} = eS$ for some $e \in E(S)$.

(ii) There exist $T \in \mathscr{RZ}$ and $\tau \in CR(S, T)$ such that τ is surjective and for any $t \in T$, $t\tau^{-1} = Se$ for some $e \in E(S)$.

(iii) There exist $T \in \mathscr{RB}$ and $\tau \in CR(S, T)$ such that τ is surjective and for any $t \in T$, $t\tau^{-1} = eSf$ for some $e, f \in E(S)$.

§ 5.2 The operators L_l and L_r

In this section we consider the operators L_l and L_r in detail. We only study the operator L_l instead of L_l and L_r, since L_r is dual to L_l.

Lemma 5.2.1. The operator L_l is a closure operator on $\mathscr{L}(\mathscr{CR})$.

Proof. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Clearly $\mathscr{U} \subseteq \mathscr{U}^{L_l}$, and $\mathscr{U} \subseteq \mathscr{V}$ implies that $\mathscr{U}^{L_l} \subseteq \mathscr{V}^{L_l}$. If $S \in \mathscr{U}^{(L_l)^2}$ and $e \in E(S)$ then $eS \in \mathscr{U}^{L_l}$. But $e \in E(eS)$ and so $eS \in \mathscr{U}$. Thus $S \in \mathscr{U}^{L_l}$, whence $\mathcal{U}^{L_1} = \mathcal{U}^{(L_1)^2}$. Therefore L_1 is a closure operator on $\mathcal{L}(\mathcal{CR})$, by Lemma 5.1.2 (i). #

Before proceeding, we require some preparation.

Definition 5.2.2 [PR7]. Let $\mathcal{A}$ be a class of completely regular semigroups. Call its members $\mathcal{A}$-semigroups. We will say that $\mathcal{A}$ is a *pre-image class* if it is closed under direct products and homomorphic images and has the following property :

$$\text{(P)} \qquad \begin{array}{l} \text{for any epimorphism } \theta : S \longrightarrow T, \text{ where } S \in \mathcal{CR} \text{ and } T \in \mathcal{A}, \\ \text{there is a completely regular subsemigroup } R \text{ of } S \text{ with } R \in \mathcal{A} \\ \text{and } R\theta = T. \end{array}$$

A subclass $\mathcal{K}$ of $\mathcal{A}$ is an $\mathcal{A}$ *-variety* if it is closed under the formation of direct products, homomorphic images and $\mathcal{A}$-subsemigroups. Denote the class of all $\mathcal{A}$-varieties by $\mathcal{L}_{\mathcal{A}}(\mathcal{CR})$.

Clearly, all $\mathcal{A}$-varieties are varieties if $\mathcal{A} = \mathcal{CR}$.

Lemma 5.2.3 [PR7, Proposition 2.2]. Let $\mathcal{A}$ be a pre-image class of completely regular semigroups.

(i) $\mathcal{L}_{\mathcal{A}}(\mathcal{CR})$ is a complete lattice.

(ii) The mapping

$$\theta_{\mathcal{A}} : \mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{A} \qquad (\mathcal{U} \in \mathcal{L}(\mathcal{CR}))$$

is a complete homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}_{\mathcal{A}}(\mathcal{CR})$.

The following result of a lattice theoretical nature will be useful.

Lemma 5.2.4 [**PP2**, Lemma 4.10]. Let ρ be a complete congruence on a complete lattice L. For each $x \in L$, let x^* be the least element of $x\rho$. Then for any $A \subseteq L$, we have $\bigvee_{x \in A} x^* = (\bigvee_{x \in A} x)^*$.

Notation 5.2.5. We denote by $\mathcal{LMCR}$ the class of all completely regular semigroups with left identity. Clearly, $\mathcal{LMCR} = \{ eS \mid S \in \mathcal{CR} \text{ and } e \in E(S) \}$. Also if θ is a mapping of a set A, then $\bar{\theta}$ denotes the equivalence on A induced by θ.

Theorem 5.2.6. (i) $\mathcal{LMCR}$ is a pre-image class.

(ii) The mapping

$$\theta_{\mathcal{LMCR}} : \mathcal{U} \longrightarrow \mathcal{U} \cap \mathcal{LMCR} \qquad (\mathcal{U} \in \mathcal{L}(\mathcal{CR}))$$

is a complete homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}_{\mathcal{LMCR}}(\mathcal{CR})$. Moreover, for any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, we have $\mathcal{U}\overline{\theta}_{\mathcal{LMCR}} = [< \mathcal{U} \cap \mathcal{LMCR} >, \mathcal{U}^{L_1}]$.

Proof. (i) Clearly $\mathcal{LMCR}$ is closed under direct products and homomorphic images. Now let $S \in \mathcal{CR}$, $T \in \mathcal{LMCR}$ and θ be an epimorphism of S onto T. Let $a \in S$ be such that $a\theta = e$, a left identity of T, and let $R = a^0S$. Then $R \in \mathcal{LMCR}$ and $R\theta = T$. Thus $\mathcal{LMCR}$ has property (P) so that (i) holds.

(ii) It follows immediately from (i) and Lemma 5.2.3 that $\theta_{\mathcal{LMCR}}$ is a complete homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}_{\mathcal{LMCR}}(\mathcal{CR})$.

Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$. It is easily verified that

$$< \mathcal{U} \cap \mathcal{LMCR} > \cap\, \mathcal{LMCR} = \mathcal{U} \cap \mathcal{LMCR} = \mathcal{U}^{L_1} \cap \mathcal{LMCR},$$

so that $< \mathcal{U} \cap \mathcal{LMCR} >, \mathcal{U}^{L_1} \in \mathcal{U}\overline{\theta}_{\mathcal{LMCR}}$. Next let $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ be such that $\mathcal{U} \cap \mathcal{LMCR} = \mathcal{V} \cap \mathcal{LMCR}$. Then $< \mathcal{U} \cap \mathcal{LMCR} > = < \mathcal{V} \cap \mathcal{LMCR} > \subseteq \mathcal{V}$. Also, for $S \in \mathcal{V}$, we have $eS \in \mathcal{V} \cap \mathcal{LMCR}$ for all $e \in E(S)$ so that $S \in \mathcal{U}^{L_1}$. It follows that $\mathcal{V} \subseteq \mathcal{U}^{L_1}$. Consequently $\mathcal{U}\overline{\theta}_{\mathcal{LMCR}} = [< \mathcal{U} \cap \mathcal{LMCR} >, \mathcal{U}^{L_1}]$, as required. #

Corollary 5.2.7. For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have

$$\mathcal{U}^{L_1} = \mathcal{V}^{L_1} \Leftrightarrow \mathcal{U} \cap \mathcal{LMCR} = \mathcal{V} \cap \mathcal{LMCR} \Leftrightarrow < \mathcal{U} \cap \mathcal{LMCR} > = < \mathcal{V} \cap \mathcal{LMCR} >.$$

Corollary 5.2.8. The relation L_1 which is given by

$$\mathcal{U}\, L_1 \mathcal{V} \quad \Leftrightarrow \quad < \mathcal{U} \circ \mathcal{LZ} > = < \mathcal{V} \circ \mathcal{LZ} >$$

is a complete congruence on $\mathcal{L}(\mathcal{CR})$. For $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ the L_1-class $\mathcal{U} L_1$ is an interval $[\mathcal{U}_{L_1}, \mathcal{U}^{L_1}]$, where $\mathcal{U}_{L_1} = < \mathcal{U} \cap \mathcal{LMCR} >$ and $\mathcal{U}^{L_1} = < \mathcal{U} \circ \mathcal{LZ} >$.

Corollary 5.2.9. The mapping

$$\mathcal{U} \longrightarrow \mathcal{U}_{L_1} \qquad (\ \mathcal{U} \in \mathcal{L}(\mathcal{CR})\ )$$

is a complete $\vee$-endomorphism of $\mathcal{L}(\mathcal{CR})$.

Proof. This follows directly from Theorem 5.2.6 and Lemma 5.2.4. #

Whether or not the mapping $\mathcal{U} \longrightarrow \mathcal{U}_{L_1}$ is a (complete) $\cap$-homomorphism of $\mathcal{L}(\mathcal{CR})$ we do not know.

Lemma 5.2.10. If $\mathcal{U} \in \mathcal{L}(\mathcal{CS})$, then $\mathcal{U}_{L_1} = < \mathcal{U} \cap \mathcal{LMCR} >$

$$= \mathcal{U} \cap \mathcal{RG}.$$

Proof. By Lemma 2.3.3, $\mathcal{RG} \subseteq \mathcal{LMCR}$ so that $\mathcal{U} \cap \mathcal{RG} \subseteq < \mathcal{U} \cap \mathcal{LMCR} >$. Let $S \in \mathcal{U} \cap \mathcal{LMCR}$, then $S \in \mathcal{CS}$, without loss of generality, we may assume $S = \mathcal{M}(G; I, \Lambda; P)$. Since $S \in \mathcal{LMCR}$, and let $(x; i, \lambda)$ be a left identity of S, then for any $(y; j, \mu) \in S$, we have

$$(x; i, \lambda)(y; j, \mu) = (x p_{\lambda j} y; i, \mu)$$

$$= (y; j, \mu),$$

so that $|I| = 1$, and so $S \in \mathcal{RG}$. Hence $\mathcal{U}_{L_1} = \mathcal{U} \cap \mathcal{RG}$. #

We now consider the commutativity between the operator L_l and the operators K, T_l, T_r, T, T_l^* and T_r^*.

Lemma 5.2.11 [H2, Corollary 6]. Let T be a regular subsemigroup of a regular semigroup S such that for any idempotents e, f in S with $f \leq e$, $e \in T$ implies $f \in T$. Then $\mu_T = \mu_S|_T$.

Lemma 5.2.12. Let $S \in \mathscr{CR}$ and $e \in E(S)$. Then

$$\mu_{eS} = \mu_S|_{eS}, \qquad (\mathscr{L}_{eS})^0 = \mathscr{L}^0|_{eS}.$$

Proof. Let T = eS. For any f, $h \in E(S)$ with $f \leq h$, $h \in T$, we then have f = hf = fh and h = eh, so that $f = hf = ehf = ef \in eS = T$. By Lemma 5.2.11, $\mu_{eS} = \mu_S|_{eS}$.

Clearly $\mathscr{L}^0|_{eS} \subseteq (\mathscr{L}_{eS})^0$. So let a, $b \in eS$, $a\,(\mathscr{L}_{eS})^0\,b$ and $x \in S$. First note that, for any $w \in S$,

$$\begin{aligned}(exe)^0(xe)^0(exe)w &= (exe)^{-1}(exe)(xe)^0 exew \\ &= (exe)w\end{aligned}$$

so that $(exe)w\ \mathscr{L}\ (xe)^0(exe)w$. Also

$$\begin{aligned} a\,(\mathscr{L}_{eS})^0\,b &\Rightarrow (ex)a\ \mathscr{L}_{eS}\ (ex)b \\ &\Rightarrow (exe)a\ \mathscr{L}_{eS}\ (exe)b && \text{since } a, b \in eS \\ &\Rightarrow (xe)^0(exe)a\ \mathscr{L}\ (xe)^0(exe)b && \text{by the above remark} \\ &\Rightarrow xea\ \mathscr{L}\ xeb \\ &\Rightarrow xa\ \mathscr{L}\ xb. \end{aligned}$$

Thus $a\,\mathscr{L}^0\,b$, since $\mathscr{L}$ is a right congruence. #

Proposition 5.2.13. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then

$$\mathscr{G}\circ\langle \mathscr{U}\circ\mathscr{LZ}\rangle = \langle(\mathscr{G}\circ\mathscr{U})\circ\mathscr{LZ}\rangle$$
$$\mathscr{LG}\circ\langle \mathscr{U}\circ\mathscr{LZ}\rangle = \langle(\mathscr{LG}\circ\mathscr{U})\circ\mathscr{LZ}\rangle.$$

Proof. From Lemma 5.2.12 it follows that

$$eS/\mu_{eS} \cong (e\mu)(S/\mu).$$

For any $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$, we then obtain

$$\begin{aligned}
S \in \langle (\mathscr{G} \circ \mathscr{U}) \circ \mathscr{LX} \rangle &\Leftrightarrow eS \in \mathscr{G} \circ \mathscr{U} && \text{for all } e \in E(S) \\
&\Leftrightarrow eS/\mu_{eS} \in \mathscr{U} && \text{for all } e \in E(S) \\
&\Leftrightarrow (e\mu)(S/\mu) \in \mathscr{U} && \text{for all } e \in E(S) \\
&\Leftrightarrow S/\mu \in \langle \mathscr{U} \circ \mathscr{LX} \rangle \\
&\Leftrightarrow S \in \mathscr{G} \circ \langle \mathscr{U} \circ \mathscr{LX} \rangle.
\end{aligned}$$

Thus $\mathscr{G} \circ \langle \mathscr{U} \circ \mathscr{LX} \rangle = \langle (\mathscr{G} \circ \mathscr{U}) \circ \mathscr{LX} \rangle$. That $\mathscr{LG} \circ \langle \mathscr{U} \circ \mathscr{LX} \rangle = \langle (\mathscr{LG} \circ \mathscr{U}) \circ \mathscr{LX} \rangle$ follows similarly. #

Corollary 5.2.14. L_1 commutes with T and T_1.

Proof. This is a direct consequence of Proposition 5.2.13. #

Lemma 5.2.15. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. Then

$$\mathscr{RG} \circ \langle (\mathscr{U} \vee \mathscr{RX}) \circ \mathscr{LX} \rangle = \langle [\mathscr{RG} \circ (\mathscr{U} \vee \mathscr{RX})] \circ \mathscr{LX} \rangle.$$

Proof. Let $\mathscr{U} = [\, u_\alpha(x_i) = v_\alpha(x_i) \,]_{\alpha \in A} \in \mathscr{L}(\mathscr{CR})$. By Proposition 3.3.13, $\mathscr{U} \vee \mathscr{RX} = [\, u_\alpha y = v_\alpha y \,]_{\alpha \in A}$, where $y \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$. We then have

$$\begin{aligned}
&\langle [\mathscr{RG} \circ (\mathscr{U} \vee \mathscr{RX})] \circ \mathscr{LX} \rangle \\
&= \langle (\mathscr{RG} \circ [\, u_\alpha y = v_\alpha y \,]_{\alpha \in A}) \circ \mathscr{LX} \rangle \\
&= \langle [\, (u_\alpha yz)^0 = (v_\alpha yzu_\alpha yz)^0, (v_\alpha yz)^0 = (u_\alpha yzv_\alpha yz)^0 \,]_{\alpha \in A} \circ \mathscr{LX} \rangle \\
&\qquad \text{by Lemma 3.3.2} \\
&= [\, (u_\alpha(xx_i)xyxz)^0 = (v_\alpha(xx_i)xyxzu_\alpha(xx_i)xyxz)^0, \\
&\qquad (v_\alpha(xx_i)xyxz)^0 = (u_\alpha(xx_i)xyxzv_\alpha(xx_i)xyxz)^0 \,]_{\alpha \in A} \\
&\qquad \text{by Corollary 5.1.5 (ii)} \\
&\subseteq [\, (u_\alpha(xx_i)xyz)^0 = (v_\alpha(xx_i)xyzu_\alpha(xx_i)xyz)^0, \\
&\qquad (v_\alpha(xx_i)xyz)^0 = (u_\alpha(xx_i)xyzv_\alpha(xx_i)xyz)^0 \,]_{\alpha \in A}
\end{aligned}$$

substituting the variable z by $y(xy)^{-1}z$

$= \mathscr{RG} \circ [\, u_\alpha(xx_i)xy = v_\alpha(xx_i)xy \,]_{\alpha \in A}$ by Lemma 3.3.2

$= \mathscr{RG} \circ \langle [\, u_\alpha y = v_\alpha y \,]_{\alpha \in A} \circ \mathscr{LZ} \rangle$ by Corollary 5.1.5 (ii)

$= \mathscr{RG} \circ \langle (\mathscr{U} \vee \mathscr{RZ}) \circ \mathscr{LZ} \rangle .$

The opposite inclusion follows from Corollary 3.5.5. #

Corollary 5.2.16. $L_l T_r = T_r L_l$ on $[\mathscr{RZ}, \mathscr{CR}]$.

Proof. This follows from Lemma 5.2.15. #

Lemma 5.2.17. $KL_l \neq L_l K$.

Proof. Since $\mathscr{G}^{L_l K} = \langle \mathscr{B} \circ \langle \mathscr{G} \circ \mathscr{LZ} \rangle \rangle$ by Corollaries 5.1.5 (ii) and 3.5.7

$= \langle \mathscr{B} \circ \mathscr{LG} \rangle.$

$= \langle \mathscr{B} \circ \langle \mathscr{LZ} \circ \mathscr{G} \rangle \rangle$

$\subseteq \langle \langle \mathscr{B} \circ \mathscr{LZ} \rangle \circ \mathscr{G} \rangle$ by Corollary 3.5.5

$= \langle \mathscr{B} \circ \mathscr{G} \rangle$

$= \mathscr{OG}$ by Lemma 4.5.1,

so that $\mathscr{CS} \not\subseteq \mathscr{G}^{L_l K}$. On the other hand, $\mathscr{G}^{KL_l} = \langle \langle \mathscr{B} \circ \mathscr{G} \rangle \circ \mathscr{LZ} \rangle \supseteq \langle \mathscr{RG} \circ \mathscr{LZ} \rangle = \mathscr{CS}$. Thus $\mathscr{G}^{L_l K} \neq \mathscr{G}^{KL_l}$, and so $KL_l \neq L_l K$. #

Theorem 5.2.18. The mapping

$$\mathscr{U} \longrightarrow \mathscr{U}^{L_l} \qquad (\, \mathscr{U} \in \mathscr{L}(\mathscr{CR}) \,)$$

is a complete $\cap$-endomorphism of $\mathscr{L}(\mathscr{CR})$ but is not a $\vee$-homomorphism.

Proof. For the first assertion of the theorem, let $\mathscr{U}_\alpha \in \mathscr{L}(\mathscr{CR})$ for $\alpha \in A$ and $S \in \mathscr{CR}$, we then have

$$S \in (\cap_{\alpha \in A} \mathscr{U}_\alpha)^{L_l} \Leftrightarrow eS \in \cap_{\alpha \in A} \mathscr{U}_\alpha \qquad \text{for all } e \in E(S)$$

$$\Leftrightarrow eS \in \mathscr{U}_\alpha \qquad \text{for all } e \in E(S) \text{ and all } \alpha \in A$$

$$\Leftrightarrow S \in (\mathscr{U}_\alpha)^{L_l} \qquad \text{for all } \alpha \in A$$

$$\Leftrightarrow \quad S \in \cap_{\alpha \in A} (\mathscr{U}_\alpha)^{L_1},$$

whence $(\cap_{\alpha\in A} \mathscr{U}_\alpha)^{L_1} = \cap_{\alpha\in A} (\mathscr{U}_\alpha)^{L_1}$.

For the second assertion of the theorem, consider

$$\begin{aligned} \mathscr{LG}^{L_1} \vee \mathscr{RX}^{L_1} &= \langle \mathscr{LG} \circ \mathscr{LX} \rangle \vee \langle \mathscr{RX} \circ \mathscr{LX} \rangle \\ &= \mathscr{LG} \vee \mathscr{LB} \\ &= \mathscr{RBG}, \\ (\mathscr{LG} \vee \mathscr{RX})^{L_1} &= \langle \mathscr{RBG} \circ \mathscr{LX} \rangle \\ &= \langle \mathscr{RG} \circ \mathscr{LX} \rangle \\ &= \mathscr{CS}, \end{aligned}$$

so the mapping $\mathscr{U} \longrightarrow \mathscr{U}^{L_1}$ is not a $\vee$-homomorphism. #

In the rest of this section, we restrict our attention to $\mathscr{L}(\mathscr{B})$. The behaviour of L_1 is determined exactly on $\mathscr{L}(\mathscr{B})$.

The following observations are elementary:

$$\mathscr{T}^{L_1} = \mathscr{LX}^{L_1} = \mathscr{LX}, \quad \mathscr{RX}^{L_1} = \mathscr{RB}^{L_1} = \mathscr{RB},$$
$$\mathscr{S}^{L_1} = \mathscr{LNB}^{L_1} = \mathscr{LNB}, \quad \mathscr{B}^{L_1} = \mathscr{B}.$$

Corollary 5.2.19. (i) L_1 commutes with T_l^* on $\mathscr{L}(\mathscr{B})$.

(ii) L_1 commutes with T_r^* on $[\mathscr{RX}, \mathscr{B}]$.

Proof. (i) Let $\mathscr{U} \in \mathscr{L}(\mathscr{B})$. Then

$$\begin{aligned} \mathscr{U}^{T_l^* L_1} &= (\mathscr{U}^{T_l} \cap \mathscr{B})^{L_1} && \text{by Lemma 4.5.16 (i)} \\ &= \mathscr{U}^{T_l L_1} \cap \mathscr{B}^{L_1} && \text{by Theorem 5.2.18} \\ &= \mathscr{U}^{L_1 T_l} \cap \mathscr{B} && \text{by Corollary 5.2.14} \\ &= \mathscr{U}^{L_1 T_l^*} && \text{by Lemma 4.5.16 (i),} \end{aligned}$$

so that $T_l^* L_1 = L_1 T_l^*$.

(ii) Similarly, $T_r^* L_1 = L_1 T_r^*$ on $[\mathscr{RX}, \mathscr{B}]$ follows from Corollary 5.2.16. #

Lemma 5.2.20. (i) $\mathcal{LRB} = \mathcal{LZ} \circ \mathcal{S} = [\, x^2 = x,\ axa = ax \,]$.

(ii) $\mathcal{LZ} \circ \mathcal{RNB} = [\, x^2 = x,\ axyay = axy \,]$.

Proof. (i) See [**Pe2,** the dual of Proposition II. 3.12].

(ii) See [**Pe2,** the dual of Proposition II. 3.8]. #

Surprisingly, we have

Lemma 5.2.21. (i) $\mathcal{RNB}^{L_1} = \mathcal{NB}^{L_1} = \mathcal{NB}$.

(ii) $\mathcal{LRB}^{L_1} = \mathcal{LRB}$.

(iii) $(\mathcal{LRB} \vee \mathcal{RZ})^{L_1} = \mathcal{LZ} \circ \mathcal{RNB}$.

(iv) $(\mathcal{RZ} \circ \mathcal{LRB})^{L_1} = \mathcal{RZ} \circ (\mathcal{LZ} \circ \mathcal{RNB})$.

Proof. (i) Clearly $\mathcal{NB} \subseteq \mathcal{RNB}^{L_1} \subseteq \mathcal{NB}^{L_1}$. It remains to show that $\mathcal{NB}^{L_1} \subseteq \mathcal{NB}$. Let $S \in \mathcal{NB}^{L_1}$, then $eS \in \mathcal{NB}$ for all $e \in E(S)$. Let $e, f, g \in E(S)$ be such that $f \leq e$, $g \leq e$ and $f \,\mathcal{D}\, g$, then $e, f, g \in E(eS)$. Since $eS \in \mathcal{NB}$, it follows from Lemma 4.4.8 (iv) that $f = g$. Hence $S \in \mathcal{NB}$, as required.

(ii)
$$
\begin{aligned}
\mathcal{LRB}^{L_1} &= (\mathcal{LZ} \circ \mathcal{S})^{L_1} \\
&= \mathcal{S}^{T_1^* L_1} \\
&= \mathcal{S}^{L_1 T_1^*} && \text{by Corollary 5.2.19 (i)} \\
&= \mathcal{LNB}^{T_1^*} \\
&= \mathcal{LRB}.
\end{aligned}
$$

(iii) From Figure 4.1, we have that $\mathcal{LRB} \vee \mathcal{RZ} = \mathcal{LZ} \circ \mathcal{RNB} \cap \mathcal{RZ} \circ \mathcal{LRB}$, so that

$$
\begin{aligned}
(\mathcal{LRB} \vee \mathcal{RZ})^{L_1} &\subseteq (\mathcal{LZ} \circ \mathcal{RNB})^{L_1} \\
&= \mathcal{RNB}^{T_1^* L_1} \\
&= \mathcal{RNB}^{L_1 T_1^*} && \text{by Corollary 5.2.19 (i)} \\
&= \mathcal{NB}^{T_1^*} && \text{by (i)} \\
&= \mathcal{LZ} \circ \mathcal{RNB}.
\end{aligned}
$$

On the other hand, we have

$$(\mathscr{LRB} \vee \mathscr{RZ})^{L_1} = ([x^2 = x, axa = ax] \vee \mathscr{RZ})^{L_1} \quad \text{by Corollary 5.2.20 (i)}$$

$$= [x^2 = x, axay = axy]^{L_1} \quad \text{by Proposition 3.3.13}$$

$$= [x^2 = x, zazxzazy = zazxzy] \quad \text{by Corollary 5.1.5 (i)}$$

$$= [x^2 = x, zazxzaz = zazxz] \vee \mathscr{RZ}$$

by Proposition 3.3.13

$$\supseteq [x^2 = x, axyay = axy] \vee \mathscr{RZ}$$

by straightforward verification

$$= (\mathscr{LZ} \circ \mathscr{RNB}) \vee \mathscr{RZ} \quad \text{by Corollary 5.2.20 (ii)}$$

$$= \mathscr{LZ} \circ \mathscr{RNB},$$

whence (iii) holds.

(iv) Since $\mathscr{RZ} \circ \mathscr{LRB} = \mathscr{RZ} \circ (\mathscr{LRB} \vee \mathscr{RZ})$, we then have

$$(\mathscr{RZ} \circ \mathscr{LRB})^{L_1} = [\mathscr{RZ} \circ (\mathscr{LRB} \vee \mathscr{RZ})]^{L_1}$$

$$= (\mathscr{LRB} \vee \mathscr{RZ})^{T_r^* L_1}$$

$$= (\mathscr{LRB} \vee \mathscr{RZ})^{L_1 T_r^*} \quad \text{by Corollary 5.2.19 (ii)}$$

$$= (\mathscr{LZ} \circ \mathscr{RNB})^{T_r^*} \quad \text{by (iii)}$$

$$= \mathscr{RZ} \circ (\mathscr{LZ} \circ \mathscr{RNB}),$$

whence (iv) holds. #

Theorem 5.2.22. The complete congruence L_1 on $\mathscr{L}(\mathscr{B})$ has the following properties :

(i) Each L_1-class is finite.

(ii) The circled elements in Figure 5.1 are exactly all the maximum elements from all the L_1-classes.

(iii) The set of all the maximum elements from all the L_1-classes is not a sublattice of $\mathscr{L}(\mathscr{B})$. However, the set of all the maximum elements above $\mathscr{RZ}$ is a sublattice of $\mathscr{L}(\mathscr{B})$.

This theorem follows by simple inspection of Figure 4.1, Corollary 5.2.19 and Lemma 5.2.21. Figure 5.1 is modified from Figure 4.1.

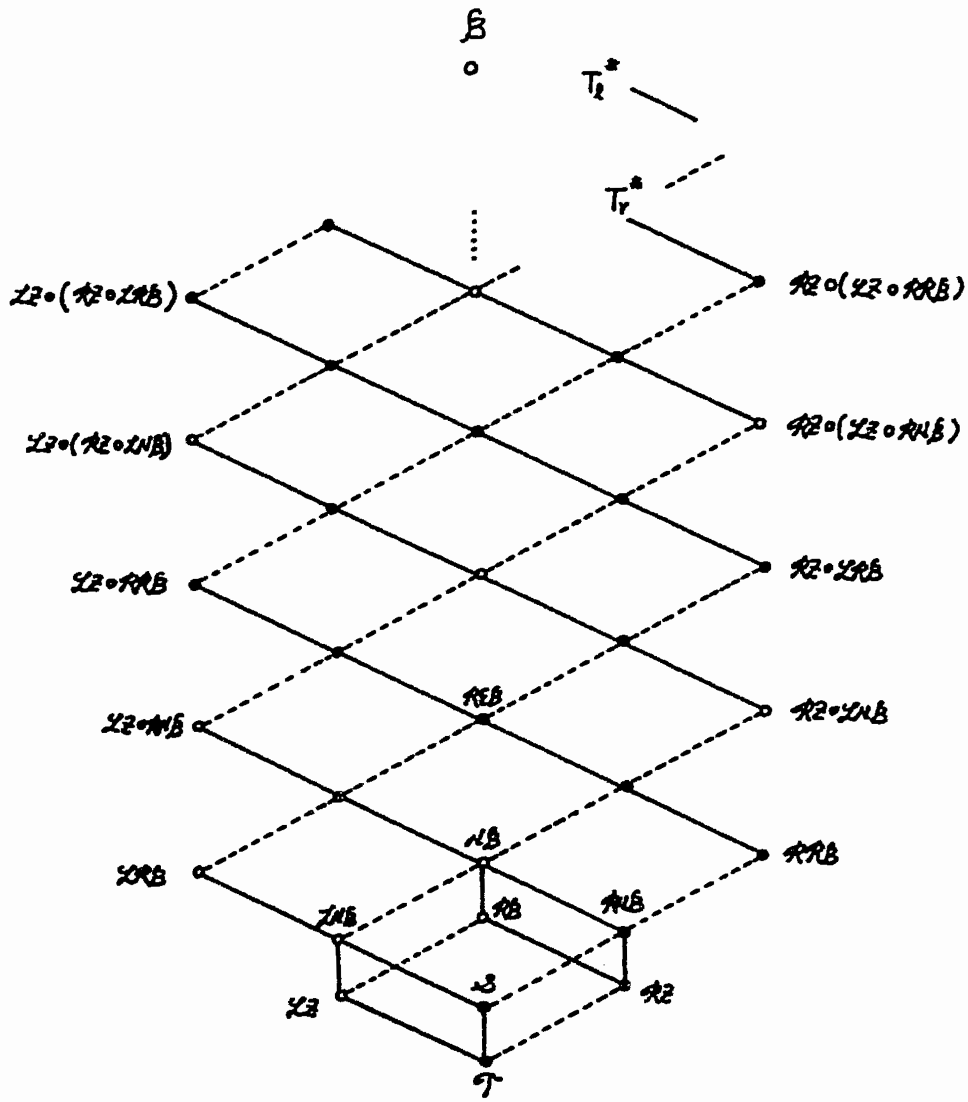

Figure 5.1.

Corollary 5.2.23. (i) The operator L_1 is not an endomorphism of $\mathscr{L}(\mathscr{B})$. Its restriction to [$\mathscr{RX}$, $\mathscr{B}$] is a complete homomorphism of [$\mathscr{RX}$, $\mathscr{B}$] onto [$\mathscr{RB}$, $\mathscr{B}$].

(ii) The mapping

$$\mathcal{U} \longrightarrow \mathcal{U}_{L_1} \qquad (\mathcal{U} \in \mathcal{L}(\mathcal{B}))$$

is a complete endomorphism of $\mathcal{L}(\mathcal{B})$.

Proof. (i) By Lemma 5.2.21, we obtain

$$\begin{aligned} \mathcal{LRB}^{L_1} \vee \mathcal{RX}^{L_1} &= \mathcal{LRB} \vee \mathcal{RB} \\ &\neq \mathcal{LX} \circ \mathcal{RNB} \\ &= (\mathcal{LRB} \vee \mathcal{RX})^{L_1}, \end{aligned}$$

so the first assertion follows. The second assertion follows from Theorem 5.2.22 (iii).

(ii) By simple inspection of Figure 5.1, we obtain that $\mathcal{U} \longrightarrow \mathcal{U}_{L_1}$ is a complete $\cap$-endomorphism of $\mathcal{L}(\mathcal{B})$. Combining this fact with Corollary 5.2.9, the required result follows. #

Corollary 5.2.24. $L_1T_r^* \neq T_r^*L_1$ and $L_1T_r \neq T_rL_1$ on $\mathcal{L}(\mathcal{CR})$.

Proof. Since
$$\begin{aligned} \mathcal{LRB}^{L_1T_r^*} &= \mathcal{LRB}^{T_r^*} \\ &= \mathcal{RX} \circ \mathcal{LRB} \\ &\neq \mathcal{RX} \circ (\mathcal{LX} \circ \mathcal{RNB}) \\ &= (\mathcal{RX} \circ \mathcal{LRB})^{L_1} \\ &= \mathcal{LRB}^{T_r^*L_1}, \end{aligned}$$

so $L_1T_r^* \neq T_r L_1$. By this fact we then have

$$\begin{aligned} \mathcal{LRB}^{L_1T_r} \cap \mathcal{B} &= \mathcal{LRB}^{L_1T_r^*} \\ &\neq \mathcal{LRB}^{T_r^*L_1} \\ &= \mathcal{LRB}^{T_rL_1} \cap \mathcal{B}, \end{aligned}$$

whence $\mathcal{LRB}^{L_1T_r} \neq \mathcal{LRB}^{T_rL_1}$, and so $L_1T_r \neq T_rL_1$. #

Obviously the notions and results obtained so far in this section have their left-right duals. We shall use these dual results in the sequel.

§ 5.3 The operator L

Definition 5.3.1. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, let

$$\mathcal{U}^L = \{ S \in \mathcal{CR} \mid eSe \in \mathcal{U} \text{ for all } e \in E(S) \}.$$

In the context of varieties of completely regular semigroups this operator was introduced by Hall and Jones [HJ]; see [**HJ**, Proposition 4.1, where the notation P is used] where it is shown that $\mathcal{U}^L \in \mathcal{L}(\mathcal{CR})$ and that $(\mathcal{U}^L)^L = \mathcal{U}^L$ or $L^2 = L$. It was discussed in greater depth by Polák [Po3], Petrich and Reilly ([PR6], [PR7]).

First we recall some results about L.

Lemma 5.3.3 [Pe5]. If $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ and $\mathcal{U} = [\, u_\alpha(x_i) = v_\alpha(x_i)\,]_{\alpha \in A}$, then

$$\mathcal{U}^L = [\, u_\alpha(x^0 x_i x^0) = v_\alpha(x^0 x_i x^0)\,]_{\alpha \in A}$$
$$= [\, u_\alpha(x x_i x) = v_\alpha(x x_i x)\,]_{\alpha \in A},$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ for all $\alpha \in A$.

Lemma 5.3.3 [Rei1]. If $\mathcal{U} \in \mathcal{L}(\mathcal{CS})$, then

$$\mathcal{U}^L = \{ S \in \mathcal{CS} \mid \text{all subgroups of } S \text{ lie in } \mathcal{U} \}.$$

Notation 5.3.4. We denote by $\mathcal{MCR}$ the class of all completely regular monoids. Clearly $\mathcal{MCR} = \{ eSe \mid S \in \mathcal{CR} \text{ and } e \in E(S) \}$.

Lemma 5.3.5 [PR7]. For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have

$$\mathcal{U}^L = \mathcal{V}^L \iff \mathcal{U} \cap \mathcal{MCR} = \mathcal{V} \cap \mathcal{MCR} \iff \langle \mathcal{U} \cap \mathcal{MCR} \rangle = \langle \mathcal{V} \cap \mathcal{MCR} \rangle.$$

Theorem 5.3.6 [PR7]. The relation L which is given by

$$\mathcal{U}\, L\, \mathcal{V} \iff \mathcal{U}^L = \mathcal{V}^L$$

is a complete congruence on $\mathcal{L}(\mathcal{CR})$. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ the L-class $\mathcal{U}L$ is an interval $[\mathcal{U}_L, \mathcal{U}^L]$, where $\mathcal{U}_L = \langle \mathcal{U} \cap \mathcal{MCR} \rangle$.

Theorem 5.3.7 [PR7]. The mapping

$$\mathcal{U} \longrightarrow \mathcal{U}_L \qquad (\mathcal{U} \in \mathcal{L}(\mathcal{CR}))$$

is a complete endomorphism of $\mathcal{L}(\mathcal{CR})$.

To know how to obtain a basis of identities for $\mathcal{U}_L$ from one for $\mathcal{U}$, the reader is referred to [**PR7**, Proposition 5.4].

Theorem 5.3.8 [Po3]. (i) $\mathcal{OG}^L = \mathcal{CS}^K$.

(ii) The operator L is a complete $\cap$-endomorphism of $\mathcal{L}(\mathcal{CR})$. Its restriction to $\mathcal{L}(\mathcal{OG}^L)$ is a complete endomorphism of $\mathcal{L}(\mathcal{OG}^L)$.

It remains an open question whether or not the operator L is a (complete) $\vee$-endomorphism of $\mathcal{L}(\mathcal{CR})$.

Theorem 5.3.9 [PR6]. (i) $(\mathcal{U} \vee \mathcal{S})^L = \mathcal{U}^L \vee \mathcal{S}$ for any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$.

(ii) L commutes with K, T, T_l and T_r.

Corollary 5.3.10. L commutes with T_l^* and T_r^*.

Proof. This follows from Lemma 3.3.8 and Theorem 5.3.9. #

Lemma 5.3.11. Let $S \in \mathcal{CR}$ and let $e, f \in E(S)$ be such that $fe = f$. Then

$$\phi : \; fSe \longrightarrow eSe$$

defined by $x\phi = ex$ is an embedding.

Proof. Clearly ϕ is well-defined. For any $x, y \in fSe$, we then have

$$(x\phi)(y\phi) = exey = exy = (xy)\phi,$$

so that ϕ is a homomorphism. If $x, y \in fSe$ and $x\phi = y\phi$, then $ex = ey$ and $fex = fey$, thus $x = fx = fy = y$, and so ϕ is one-to-one. Hence ϕ is an embedding. #

The main purpose of this section is to establish the following result. In the draft of this thesis, it was left open whether or not $L = L^*$ on $\mathscr{L}(\mathscr{CR})$. However, P. R. Jones pointed out that $L = L^*$ indeed. His proof of $L = L^*$ is included here.

Theorem 5.3.12. $L = L_l L_r = L_r L_l = L^*$.

Proof. Let $\mathscr{U} \in \mathscr{L}(\mathscr{CR})$. By Theorem 5.1.4, we then have

$$\begin{aligned} \mathscr{U}^{L_l L_r} &= (\mathscr{U}^{L_l})^{L_r} \\ &= \{ S \in \mathscr{CR} \mid Se \in \mathscr{U}^{L_l} \text{ for all } e \in E(S) \} \\ &= \{ S \in \mathscr{CR} \mid fSe \in \mathscr{U} \text{ for all } e \in E(S) \text{ and } f \in E(Se) \} \\ &= \{ S \in \mathscr{CR} \mid fSe \in \mathscr{U} \text{ for all } e, f \in E(S) \text{ with } fe = f \}. \end{aligned}$$

Clearly $\mathscr{U}^{L_l L_r} \subseteq \mathscr{U}^L$. For the opposite inclusion, let $S \in \mathscr{U}^L$. Then $eSe \in \mathscr{U}$ for all $e \in E(S)$. Let $e, f \in E(S)$ be such that $fe = f$. By Lemma 5.3.11, $eSe \in \mathscr{U}$ implies that $fSe \in \mathscr{U}$, and so $S \in \mathscr{U}^{L_l L_r}$. Thus $\mathscr{U}^L = \mathscr{U}^{L_l L_r}$, and therefore $L = L_l L_r$. Similarly one can show that $L = L_r L_l$ by using the dual of Lemma 5.3.11.

We now show that $L = L^*$. Let $\mathscr{U} = [\, u_\alpha(x_i) = v_\alpha(x_i) \,] \in \mathscr{L}(\mathscr{CR})$. Clearly $\mathscr{U}^{L^*} \subseteq \mathscr{U}^L$. For the opposite inclusion, let $S \in \mathscr{U}^L$, that is, $eSe \in \mathscr{U}$ for all $e \in E(S)$. To show that $S \in \mathscr{U}^{L^*}$, and by Theorem 5.1.4 (iii), it suffices to show that $eSf \in \mathscr{U}$ for all $e, f \in E(S)$. Let $e, f \in E(S)$, we consider in three cases separately.

Case 1. $\mathscr{RZ} \subseteq \mathscr{V}$. Let $\alpha \in A$, and let $u_\alpha(x_i) = u_\alpha(x_1, \ldots, x_n)$ and $v_\alpha(x_i) = v_\alpha(x_1, \ldots, x_n)$. For any $a_1, \ldots, a_n \in eSf$, we then have $a_ie \in eSe$ and $a_i^{-1}e = (a_ie)^{-1} \in eSe$, for all $i \in \{1, 2, \ldots, n\}$. Now

$$\begin{aligned} u_\alpha(a_1, \ldots, a_n)e &= u_\alpha(a_1e, \ldots, a_ne) && \text{by the above remarks} \\ &= v_\alpha(a_1e, \ldots, a_ne) && \text{since } eSe \in \mathscr{V} \\ &= v_\alpha(a_1, \ldots, a_n)e. \end{aligned}$$

But $u_\alpha(x_1, \ldots, x_n)$ and $v_\alpha(x_1, \ldots, x_n)$ end in the same variable x_i, say, since $\mathscr{RZ} \subseteq \mathscr{V}$; thus

$$\begin{aligned} u_\alpha(a_1, \ldots, a_n) &= u_\alpha(a_1, \ldots, a_n)ea_i^0 \\ &= v_\alpha(a_1, \ldots, a_n)ea_i^0 \\ &= v_\alpha(a_1, \ldots, a_n), \end{aligned}$$

and whence $eSf \in \mathscr{V}$.

Case 2. $\mathscr{LZ} \subseteq \mathscr{V}$. The dual of Case 1.

Case 3. $\mathscr{RZ}, \mathscr{LZ} \not\subseteq \mathscr{V}$. Thus $\mathscr{V} \subseteq \mathscr{SG}$. Now let $a, b \in E(eSf)$, we have $ae, be \in E(eSe) \in \mathscr{S}$, so that

$$ab = abb = (ae)(be)b = (be)(ae)b = bab.$$

Similarly, using $fa, fb \in E(fSf)$, $ba = bab$. Then $ab = ba$, and whence $eSf \in \mathscr{SG}$. It follows from Lemma 2.5.12 that eSf is a subdirect product of a group and a semilattice. On the other hand, any maximal subgroup of eSf is a maximal subgroup of gSg for some $g \in E(S)$, and whence $eSf \in \mathscr{V}$ since $\mathscr{S} \subseteq \mathscr{V}$ if and only if $\mathscr{S} \subseteq \mathscr{V}^L$. Hence $S \in \mathscr{V}^{L^*}$ so that $\mathscr{V}^L = \mathscr{V}^{L^*}$, and therefore $L = L^*$. #

Corollary 5.3.13. If $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, then

$$\begin{aligned} \mathscr{V}^L &= \{ S \in \mathscr{CR} \mid fSe \in \mathscr{V} \text{ for all } e, f \in E(S) \text{ with } fe = f \} \\ &= \{ S \in \mathscr{CR} \mid eSf \in \mathscr{V} \text{ for all } e, f \in E(S) \text{ with } ef = f \} \\ &= \{ S \in \mathscr{CR} \mid eSf \in \mathscr{V} \text{ for all } e, f \in E(S) \}. \end{aligned}$$

Proof. This is an immediate consequence of Theorem 5.3.12. #

In conclusion, we given the relationships between L, L_l and L_r on $\mathscr{L}(\mathscr{B})$.

Lemma 5.3.14. (i) $L = L_l$ on $[\mathscr{R}\mathscr{X}, \mathscr{B}]$.

(ii) $L = L_r$ on $[\mathscr{L}\mathscr{X}, \mathscr{B}]$.

(iii) $L = L_l = L_r$ on $[\mathscr{R}\mathscr{B}, \mathscr{B}]$.

Proof. (i) This follows easily by combining Theorem 5.2.22 with [**Rei1**, Theorem 6.2] and by simple inspection of Figure 5.1 and Diagram 1 in [**Rei1**, Section 6].

(ii) This is the dual of (i).

(iii) This is a combination of (i) and (ii). #

Chapter 6

Some Consequences

As consequences of results obtained in the previous chapters, we give descriptions of varieties of the forms $<\mathcal{U}\circ\mathcal{V}>$ with $\mathcal{V}\in\{\mathcal{LG}, \mathcal{RG}, \mathcal{RBG}, \mathcal{SG}, \mathcal{LNB}, \mathcal{RNB}, \mathcal{NB}\}$ in this chapter.

§ 6.1 Varieties of the forms $<\mathcal{U}\circ\mathcal{V}>$ with $\mathcal{V}\in\{\mathcal{LG}, \mathcal{RG}, \mathcal{RBG}, \mathcal{SG}\}$

Proposition 6.1.1. For any $\mathcal{U}\in\mathcal{L}(\mathcal{CR})$ we have

(i) $<\mathcal{U}\circ\mathcal{LG}>$

$= << \mathcal{U}\circ\mathcal{LZ} >\circ\mathcal{G}>$

$= \{ S\in\mathcal{CR} \mid eC^*(S)\in\mathcal{U} \text{ for all } e\in E(S) \}.$

(ii) $<\mathcal{U}\circ\mathcal{RG}>$

$= << \mathcal{U}\circ\mathcal{RZ} >\circ\mathcal{G}>$

$= \{ S\in\mathcal{CR} \mid C^*(S)e\in\mathcal{U} \text{ for all } e\in E(S) \}.$

(iii) $<\mathcal{U}\circ\mathcal{RBG}>$

$= << \mathcal{U}\circ\mathcal{RB} >\circ\mathcal{G}>$

$= \{ S\in\mathcal{CR} \mid eC^*(S)e\in\mathcal{U} \text{ for all } e\in E(S) \}.$

Proof. Let $\mathcal{U}\in\mathcal{L}(\mathcal{CR})$, and we denote the third class in (i) by A. From Theorems 4.3.14 and 5.1.4 (i), it follows that

$$<< \mathcal{U}\circ\mathcal{LZ} >\circ\mathcal{G}> = \{ S\in\mathcal{CR} \mid eC^*(S)\in\mathcal{U} \text{ for all } e\in E(S) \}$$

$$= A.$$

On the other hand, $< \mathcal{U} \circ \mathcal{LG} > = < \mathcal{U} \circ < \mathcal{LZ} \circ \mathcal{G} > >$

$\subseteq < < \mathcal{U} \circ \mathcal{LZ} > \circ \mathcal{G} >$ by Corollary 3.5.5

It remains to show that $A \subseteq < \mathcal{U} \circ \mathcal{LG} >$. Let $S \in A$. It follows from Corollary 4.3.15 that there exist $G \in \mathcal{G}$ and $\tau_1 \in CR(S, G)$ such that τ_1 is surjective and $C^*(S) = 1\tau_1^{-1}$, where 1 is the identity of G. By Corollary 5.1.6 (i) there exist $T \in \mathcal{LZ}$ and $\tau_2 \in CR(S, T)$ such that τ_2 is surjective and for any $t \in T$, $t\tau_2^{-1} = eS$ for some $e \in E(S)$. Clearly $\tau_1 \times \tau_2 \in CR(S, G \times T)$ and $G \times T \in \mathcal{LG}$. Since $E(G \times T) = \{ (1, t) \mid t \in T \}$, then for any $(1, t) \in E(G \times T)$ we have

$$(1, t)(\tau_1 \times \tau_2)^{-1} = 1\tau_1^{-1} \cap t\tau_2^{-1}$$

$$= C^*(S) \cap eS \quad \text{for some } e \in E(S)$$

$$= eC^*(S)$$

$$\in \mathcal{U} \quad \text{since } S \in A;$$

it follows from Theorem 3.5.3 that $S \in < \mathcal{U} \circ \mathcal{LG} >$, and whence $A \subseteq < \mathcal{U} \circ \mathcal{LG} >$.

(ii) This is dual to (i).

(iii) The proof of this part is entirely similar to that of (i). #

Reilly **[Rei1]** has completely determined the varieties of the form $< \mathcal{U} \circ \mathcal{S} >$. We now recall this result from **[Rei1]**.

Notation 6.1.2. For any identity u = v in the variables $x_1, \ldots, x_n$ (so that each x_i appears either in u or v or both) let $u^* = v^*$ denote the identity $u(x_1^*, \ldots, x_n^*) = v(x_1^*, \ldots, x_n^*)$ where $x_i^* = e_i x_i e_i$, $e_i = (x_i x_1 x_2 \ldots x_n x_i)^0$.

Lemma 6.1.3 [**Rei1**, Theorem 4.4]. If $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ and $\mathcal{U} = [u_\alpha = v_\alpha]_{\alpha \in I}$, then

(i) $\mathcal{U} \circ \mathcal{S} \in \mathcal{L}(\mathcal{CR})$.

(ii) $\mathcal{U} \circ \mathcal{S} = \{ S \in \mathcal{CR} \mid \text{all } \mathcal{D}\text{-classes of } S \text{ belong to } \mathcal{U} \}$

$= [u_\alpha^* = v_\alpha^*]_{\alpha \in I}$.

Proposition 6.1.4. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, we have

$$\mathcal{U} \circ \mathcal{SG} = < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{G} >$$
$$= \{ S \in \mathcal{CR} \mid \text{all } \mathcal{D}\text{-classes of } C^*(S) \text{ belong to } \mathcal{U} \}$$
$$= \{ S \in \mathcal{CR} \mid \text{for each } \mathcal{D}\text{-class } D \text{ of } S,\ C^*(D) \in \mathcal{U} \}.$$

Proof. Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$. By Corollary 3.5.5, we have

$$\mathcal{U} \circ \mathcal{SG} = \mathcal{U} \circ < \mathcal{S} \circ \mathcal{G} >$$
$$\subseteq << \mathcal{U} \circ \mathcal{S} > \circ \mathcal{G} >$$
$$= < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{G} >.$$

Combining Theorem 4.3.14 with Lemma 6.1.3, we obtain that

$$< (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{G} > = \{ S \in \mathcal{CR} \mid \text{all } \mathcal{D}\text{-classes of } C^*(S) \text{ belong to } \mathcal{U} \}.$$

It follows from Proposition 4.1.12 that

$$\{ S \in \mathcal{CR} \mid \text{all } \mathcal{D}\text{-classes of } C^*(S) \text{ belong to } \mathcal{U} \}$$
$$= \{ S \in \mathcal{CR} \mid \text{for each } \mathcal{D}\text{-class } D \text{ of } S,\ C^*(D) \in \mathcal{U} \}.$$

It remains to show that $\{ S \in \mathcal{CR} \mid \text{all } \mathcal{D}\text{-classes of } C^*(S) \text{ belong to } \mathcal{U} \} \subseteq \mathcal{U} \circ \mathcal{SG}$. Let $S \in \mathcal{CR}$ be such that all $\mathcal{D}$-classes of $C^*(S)$ belong to $\mathcal{U}$. By Lemma 4.2.7, $C^*(S) = \ker \beta$ and $e\beta = C^*(S) \cap D_e$ for any $e \in E(S)$. Thus $e\beta \in \mathcal{U}$ for any $e \in E(S)$, and whence $S \in \mathcal{U} \circ \mathcal{SG}$, as required. #

Corollary 6.1.5. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{U} \circ \mathcal{SG}$ is a variety.

Proof. This is an immediate consequence of Proposition 6.1.4. #

§ 6.2 Varieties of the forms $< \mathcal{U} \circ \mathcal{V} >$ with $\mathcal{V} \in \{ \mathcal{LNB}, \mathcal{RNB}, \mathcal{NB} \}$

Let X be a nonempty set. As introduced in § 2.7, we denote the free unary semigroup over X by U. The following result is well known.

Lemma 6.2.1. For u, v ∈ U we have

$$u\ \sigma_{\mathcal{LNB}}\ v \iff c(u) = c(v),\ h(u) = h(v),$$

$$u\ \sigma_{\mathcal{RNB}}\ v \iff c(u) = c(v),\ t(u) = t(v),$$

$$u\ \sigma_{\mathcal{NB}}\ v \iff c(u) = c(v),\ h(u) = h(v),\ t(u) = t(v).$$

Proof. See, e.g., II. 3 of [**Pe2**]. #

The main result of this section is the following.

Proposition 6.2.2. For any $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ we have

(i) $< \mathcal{U} \circ \mathcal{LNB} >$

$= < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{LZ} >$

$= \{ S \in \mathcal{CR} \mid$ for each $e \in E(S)$, all $\mathcal{D}$-classes of eS belong to $\mathcal{U} \}$.

(ii) $< \mathcal{U} \circ \mathcal{RNB} >$

$= < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{RZ} >$

$= \{ S \in \mathcal{CR} \mid$ for each $e \in E(S)$, all $\mathcal{D}$-classes of Se belong to $\mathcal{U} \}$.

(iii) $< \mathcal{U} \circ \mathcal{NB} >$

$= < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{RB} >$

$= \{ S \in \mathcal{CR} \mid$ for each $e \in E(S)$, all $\mathcal{D}$-classes of eSe belong to $\mathcal{U} \}$.

Proof. (i) Let $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, and we denote the third class in (i) by A. Combining Theorem 5.1.4 (i) with Lemma 6.1.3 (ii), we obtain that

$< (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{LZ} > = \{ S \in \mathcal{CR} \mid$ for each $e \in E(S)$, all $\mathcal{D}$-classes of eS belong to $\mathcal{U} \}$.

$= A.$

On the other hand, $< \mathcal{U} \circ \mathcal{LNB} > = < \mathcal{U} \circ < \mathcal{S} \circ \mathcal{LZ} > >$

$\subseteq < < \mathcal{U} \circ \mathcal{S} > \circ \mathcal{LZ} >$ by Corollary 3.5.5

$= < (\mathcal{U} \circ \mathcal{S}) \circ \mathcal{LZ} >$ by Lemma 6.1.3 (i).

It remains to show that $A \subseteq \langle \mathscr{V} \circ \mathscr{LNB} \rangle$. Let $S \in A$. Let X denote the alphabet whose letters are elements of S. Then we have the usual surjective homomorphism of unary semigroups

$$\alpha : U \longrightarrow S$$

which maps each letter of X into itself (see, e.g., the proof of Theorem 2.8 of [Rei2]). Let $T = U/\sigma_{\mathscr{LNB}}$. We define $\rho : U \longrightarrow T$ by $u\rho = u\sigma_{\mathscr{LNB}}$. Then ρ is a surjective homomorphism of U onto T. Let $\tau = \alpha^{-1}\rho$. Clearly $\tau : S \longrightarrow T$ is a relational morphism of S onto T. For any $t \in T$, we have $t = u\sigma_{\mathscr{LNB}}$ for some $u \in U$. Let $u = u(x_1, \dots, x_n)$, $x_i \in X$ ($i = 1, 2, \dots, n$). Then

$$\begin{aligned} t\tau^{-1} &= (u\sigma_{\mathscr{LNB}})\tau^{-1} \\ &= \{ v \in U \mid c(v) = c(u) \text{ and } h(v) = h(u) \}\alpha && \text{by Lemma 6.2.1} \\ &= \{ v \in U \mid c(v) = \{ x_1, \dots, x_n \} \text{ and } h(v) = h(u) \}\alpha . \end{aligned}$$

Let $e = [h(u)\alpha]^0$. Thus $e \in E(S)$ and $t\tau^{-1} \leq eS$. It is easy to see that $\{ v \in U \mid c(v) = c(u) \text{ and } h(v) = h(u) \}$ is a unary subsemigroup of U. Since $\alpha : U \longrightarrow S$ is a surjective homomorphism of unary semigroups, and by Lemma 2.5.4, $t\tau^{-1}$ is a completely regular subsemigroup of eS. If $s_1, s_2 \in t\tau^{-1}$, then there exist $v_i \in U$ such that $c(v_i) = \{ x_1, \dots, x_n \}$, $h(v_i) = h(u)$ and $s_i = v_i\alpha$, for $i = 1, 2$. Since $\mathscr{D}$ is a semilattice congruence on S, then $s_1\mathscr{D} = (v_1\alpha)\mathscr{D} = (x_1\alpha) \dots (x_n\alpha)\mathscr{D} = (v_2\alpha)\mathscr{D} = s_2\mathscr{D}$, i.e., $s_1 \mathscr{D} s_2$. Thus $t\tau^{-1}$ is a completely regular subsemigroup of some $\mathscr{D}$-class of eS, and so $t\tau^{-1} \in \mathscr{V}$, since $S \in A$. Further, since $T \in \mathscr{LNB}$ and $t\tau^{-1} \in \mathscr{CR}$ for any $t \in T$, then by Lemma 2.5.4, graph(τ) $\in \mathscr{CR}$, so that $\tau \in CR(S, T)$. From Theorem 3.5.3, it follows that $S \in \langle \mathscr{V} \circ \mathscr{LNB} \rangle$, as required.

(ii) This is dual to (i).

(iii) The proof of this part is entirely similar to that of (i). #

Remark 6.2.3. Combining Proposition 6.2.2 with Lemma 6.1.3 (ii) and Corollary 5.1.5 one can derive a basis for the identities of $<\mathscr{U}\circ\mathscr{LNB}>$ ($<\mathscr{U}\circ\mathscr{RNB}>$, $<\mathscr{U}\circ\mathscr{NB}>$) in terms of a basis for the identities of $\mathscr{U}$.

REFERENCES

[A] Ash, C.J., Finite semigroups with commuting idempotents, J. Austral, Math. Soc. (Series A), 43 (1987), 81-90.

[BMR] Birget, J.C., Margolis, S.W. and Rhodes, J., Semigroups whose idempotents form a subsemigroup, Bull. Austral. Math. Soc., 41 (1990), 161-184.

[Bi] Birjukov, A.P., Varieties of idempotent semigroups, Algebra i Logika, 9 (1970), 255-273 (Russian), AMS Translation.

[BS] Burris, S. and Sankappanavar, H.P., *A Course In Universal Algebra*, Springer-Verlag, New York, 1981.

[C] Clifford, A.H., The free completely regular semigroup on a set, J. Algebra, 59 (1979), 434-451.

[CP] Clifford, A.H. and Preston, G.B., *The Algebraic Theory of Semigroups*, Math. Survey No.7, Amer. Math. Soc., Providence, R.I., Vol. I (1961), Vol. II (1967).

[E] Eilenberg, S., *Automata, Languages and Machines*, Vol. B, Academic Press, New York, 1976.

[Ev] Evans, T., The lattice of semigroup varieties, Semigroup Forum, 2 (1971), 1-43.

[F] Feigenbaum, R., Regular semigroup congruences, Semigroup Forum, 17 (1979), 373-377.

[Fe] Fennemore, C.F., All varieties of bands, Math. Nachr., 48 (1971), I: 237-252, II: 253-262.

[Fi] Fitz-Gerald, D.G., On inverse of products of idempotents in regular semigroups, J. Austral. Math. Soc., 13 (1972), 335-337.

[G] Gerhard, J.A., The lattice of equational classes of idempotent semigroups, J. Algebra, 15 (1970), 195-224.

[GP] Gerhard, J.A. and Petrich, M., The word problem for orthogroups, Can. J. Math., Vol. XXXIII, No. 4 (1981), 893-900.

[Gr1] Grätzer, G., *General Lattice Theory*, Academic Press, New York, 1978.

[Gr2] Grätzer, G., *Universal Algebra*, 2nd ed., Springer-Verlag, New York, 1979.

[Gre] Green, J.A., On the structure of semigroups, Annals of Math., 54 (1951), 163-172.

[H1] Hall, T.E., Congruences and Green's relations on regular semigroups, Glasgow Math. J., 13 (1972), 167-175.

[H2] Hall, T.E., On regular semigroups, J. Algebra, 24 (1973), 1-24.

[H3] Hall, T.E., Some properties of local subsemigroups inherited by larger subsemigroups, Semigroup Forum, 25 (1982), 35-49.

[HJ] Hall, T.E. and Jones, P.R., On the lattice of varieties of bands of groups, Pacific J. Math., 91 (1980), 327-337.

[How1] Howie, J.M., *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[How2] Howie, J.M., Idempotents in completely 0-simple semigroups, Glasgow Math. J., 19 (1978), 109-113.

[HL] Howie, J.M. and Lallement, G., Certain fundamental congruences on a regular semigroups, Proc. Glasgow Math. Assoc., 7 (1966), 145-159.

[Hu] Hungerford, T.W., *Algebra*, Springer-Verlag, New York, 1980.

[J1] Jones, P.R., Completely simple semigroups: free products, free semigroups and varieties, Proc. Roy. Soc. Edinburgh, 88A (1981), 293-313.

[J2] Jones, P.R., The least inverse and orthodox congruences on a completely regular semigroups, Semigroup Forum, 27 (1983), 390-392.

[J3] Jones, P.R., On the lattice of varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A), 35 (1983), 227-235.

[J4] Jones, P.R., Joins and meets of congruences on a regular semigroup, Semigroup Forum, 30 (1984), 1-16.

[J5] Jones, P.R., Mal'cev products of varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A), 42 (1987), 227-246.

[L1] Lallement, G., A note on congruences on Rees matrix semigroups, Semigroup Forum, 8 (1974), 89-92.

[L2] Lallement, G., *Semigroups and Combinatorial Applications*, Wiley, New York, 1979.

[La1] LaTorre, D.R., Group congruences on regular semigroups, Semigroup Forum, 24 (1982), 327-340.

[La2] LaTorre, D.R., The least semilattice of groups congruence on a regular semigroup, Semigroup Forum, 27 (1983), 319-329.

[Lj] Ljapin, E.S., *Semigroups*, English translation (3rd edition) (Amer. Math. Soc., Providence, R. I., 1974).

[M] Mal'cev, A.I., *The Metamathematics of Algebraic Systems*, North-Holland, Amsterdam, 1971.

[Mc] McAlister, D.B., Regular semigroups, fundamental semigroups and groups, J. Austral. Math. Soc. (Series A), 29 (1980), 475-503.

[MMT] McKenzie, R.N., McNulty, G.F. and Taylor, W.F., *Algebras, Lattices, Varieties*, Vol.1, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987.

[Mi] Mitsch, H., Congruences on regular semigroups, in Seminario di Algebra non Commutativa, Lecce, 1989, 23-52.

[N] Neumann, H., *Varieties of Groups*, Springer-Verlag, New York, 1967.

[P] Pastijn, F., The lattice of completely regular semigroup varieties, J. Austral. Math. Soc. (Series A), 49 (1990), 24-42.

[PP1] Pastijn, F. and Petrich, M., *Regular Semigroups As Extensions*, Research Notes in Mathematics, Vol. 136 (1985), Pitman Advanced Publishing Program, London.

[PP2] Pastijn, F. and Petrich, M., Congruences on regular semigroups, Tran. Amer. Math. Soc., 295 (1986), 607-633.

[PT1] Pastijn, F. and Trotter, P.G., Lattices of completely regular semigroup varieties, Pacific J. Math., 119 (1985), 191-214.

[PT2] Pastijn, F. and Trotter, P.G., Residual finiteness in completely regular semigroup varieties, Semigroup Forum, 37 (1988), 127-147

[Pe1] Petrich, M., *Introduction to Semigroups*, Merrill, Columbus, 1973.

[Pe2] Petrich, M., *Lectures in Semigroups*, Akademie-Verlag, Berlin, 1977.

[Pe3] Petrich, M., *Inverse Semigroups*, John wiley & Sons, Now York, 1984.

[Pe4] Petrich, M., The structure of completely regular semigroups, Trans. Amer. Math. Soc., 189 (1974), 211-236.

[Pe5] Petrich, M., On the varieties of completely regular semigroups, Semigroup Forum, 25 (1982), 153-169.

[PR1] Petrich, M. and Reilly, N.R., Varieties of groups and of completely simple semigroups, Bull. Austral. Math. Soc., 23 (1981), 339-359.

[PR2] Petrich, M. and Reilly, N.R., The kernel-trace approach to congruences on completely simple semigroups, Studia Scientiarum Mathematicarum Hungarica, 16 (1981), 103-114.

[PR3] Petrich, M. and Reilly, N.R., Bands of groups with universal properties, Monatshefte für Mathematik, 94 (1982), 45-67.

[PR4] Petrich, M. and Reilly, N.R., All varieties of central completely simple semigroups, Trans. Amer. Math. Soc., 280 (1983), 623-636.

[PR5] Petrich, M. and Reilly, N.R., Near varieties of idempotent generated completely simple semigroups, Algebra Universalis, 16 (1983), 83-104.

[PR6] Petrich, M. and Reilly, N.R., Semigroups generated by certain operators on varieties of completely regular semigroups, Pacific J. Math., 132 (1988), 151-175.

[PR7] Petrich, M. and Reilly, N.R., Operators related to idempotent generated and monoid completely regular semigroups, J. Austral. Math. Soc. (Series A), 49 (1990), 1-23.

[PR8] Petrich, M. and Reilly, N.R., Operators related to E-disjunctive and fundamental completely regular semigroups, J. Algebra, 134 (1990), 1-27.

[Pi1] Pin, J.E., *Varieties of Formal Languages*, North Oxford Academic, London and Plenum, New York, 1986.

[Pi2] Pin, J.E., On a conjecture of Rhodes, Semigroup Forum, 30 (1989), 1-15.

[Pir] Pirnot, T.L., Certain congruences on a completely regular semigroups, Glasgow Math. J., 15 (1974), 109-120.

[Po1] Polák, L., On varieties of completely regular semigroups I, Semigroup Forum, 32 (1985), 97-123.

[Po2] Polák, L., On varieties of completely regular semigroups II, Semigroup Forum, 36 (1987), 253-284.

[Po3] Polák, L., On varieties of completely regular semigroups III, Semigroup Forum, 37 (1988), 1-30.

[R1] Rasin, V.V., Free completely simple semigroups, Math. Zapiski Ural. Univ., 11 (1979), 140-151 (Russian).

[R2] Rasin,V.V., On the lattice of varieties of completely simple semigroups, Semigroup Forum, 17 (1979), 113-122.

[R3] Rasin, V.V., On the varieties of Cliffordian semigroups, Semigroup Forum, 23 (1981), 201-220

[Re] Rees, D., On semi-groups, Proc. Cambridge Phil. Soc., 36 (1940), 387-400.

[Rei1] Reilly, N.R., Varieties of completely regular semigroups, J. Austral. Math. Soc. (Series A), 38 (1985), 372-393.

[Rei2] Reilly, N.R., Free objects in certain varieties of semigroups, in 2° Seminario di Algebra non Commutativa, Seina, 1987, 39-99.

[Rei3] Reilly, N.R., Structure, congruences and varieties of completely regular semigroups, in Seminario di Algebra non Commutativa, Lecce, 1989, 55-85.

[Rei4] Reilly, N.R., Completely regular semigroups, in Lattices, Semigroups, and Universal Algebra (Proc. of the Inter. Conf. on Algebra, Lisbon, 1988), Plenum Press, New York (1990), 225-242.

[Rei5] Reilly, N.R., The Rhodes expansion and free objects in varieties of completely regular semigroups, J. Pure and Appl. Algebra, 69 (1990), 89-109.

[RS] Reilly, N.R. and Scheiblich, H.E., Congruences on regular semigroups, Pacific J. of Math., 23 (1967), 349-360.

[S] Straubing, H., Finite semigroup varieties of the form V*D, J. Pure and Appl. Algebra, 36 (1985), 53-94.

[Su] Suzuki, M., *Group Theory* I, Springer-Verlag, New York, 1982.

[T] Tamura, T., Decompositions of a completely simple semigroup, Osaka Math. J., 12 (1960), 269-275.

[Ti] Tilson, B., Categories as algebra: an essential ingredient in the theory of monoids, J. Pure and Appl. Algebra, 48 (1987), 83-198.

[Tr1] Trotter, P.G., Congruences on regular and completely regular semigroups, J. Austral. Math. Soc., 32 (1982), 388-398.

[Tr2] Trotter, P.G., The least orthodox congruence on a completely regular semigroup, Indian J. Math., 25 (1983), 125-133.

[Tr3] Trotter, P.G., Congruence extensions in regular semigroups, J. Algebra, 137 (1991), 166-179.

[V] Vachuska, C., Contributions on completely regular semigroups, Ph. D. Dissertation, Marquette University, Milwaukee, WI, 1990.