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MATROIDS WITH THE CIRCUIT COVER PROPERTY

by

Xudong Fu

B.Sc., Wuhan University, Wuhan, China, 1986

A THESIS SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

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Matroids with the Circuit Cover Property

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Abstract

A circuit cover of a weighted binary matroid (M,p) is a multiset of circuits in M such that every element e is contained in exactly $p(e)$ circuits in the multiset. A non-negative integer-valued weight function p is admissible if the total weight of any cocircuit is even, and no element has more than half the total weight of any cocircuit containing it. A binary matroid M has the circuit cover property if (M,p) has a circuit cover for every admissible weight function p. In this thesis Seymour's conjecture, a binary matroid has the circuit cover property if and only if it contains no minor which is isomorphic to F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ or $\mathcal{M}(P_{10})$, has been proved.

Acknowledgements

I would like to thank my supervisor, Luis Goddyn, for all his suggestions and financial support.

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Chapter 1

Introduction

1.1 The Circuit Cover and Bond Cover of a Graph

A *cycle* (or *even subgraph*) in a graph $G = (V, E)$ is a subset of edges $F \subseteq E$ such that each vertex of G is incident with an even number of edges in F. A *circuit* is a minimal non-empty cycle.

For any subset S of vertices of G, the set of edges $\delta(S) = [S, V - S]$ which have exactly one endvertex in S is called an *edge-cut* (or *cocycle*) of G. A *bond* is a minimal non-empty edge-cut.

Proposition 1.1.1 *If C is an arbitrary circuit and D is an arbitrary bond in a graph G then the number of common edges, $|C \cap D|$, of C and D is even.*

Let (G, p) be an edge-weighted graph (with loops and multiple edges allowed) where $p : E(G) \rightarrow \mathcal{Z}^+$. We say that (G, p) has a circuit cover if there exists a multiset (or list) L of circuits in G such that each edge e is covered exactly $p(e)$ times by circuits in L. More precisely, we say that (G, p) has a circuit cover provided the following holds:

(1.1) There exists a vector of non-negative integer coefficients $(\lambda_C : C \in \mathcal{C})$ such that

$$\sum_{C\in\mathcal{C}} \lambda_C \chi^C = p.$$

Here, $\mathcal{C}$ denotes the collection of circuits in G and (λ_C) is the multiplicity vector for the circuit cover L, and for any subgraph H of G, χ^H denotes the $\{0,1\}$-characteristic function of the edge set of H. We use the convention that χ^L means $\sum \chi^{H_i}$, where $L=\{H_1, H_2, ..., H_n\}$.

Seymour [11] gave necessary conditions for an arbitrary weighted graph (G,p) to have a circuit cover:

(1.2) (i) for every bond D and $e \in D$, $p(e) \leq p(D\backslash e)$ (that is, p is *balanced*),

(ii) for every bond D, $p(D)$ is even (that is, p is *eulerian*), and

(iii) p is non-negative integer valued.

(We use the convention that $p(F)$ means $\sum_{e\in F} p(e)$, for any $F \subseteq E$.) These conditions follow easily from the fact that any circuit in a graph intersects any bond in an even number of edges. The conditions in (1.2) are collectively called *admissibility conditions*, and p is said to be *admissible* if it satisfies (1.2).

Definition 1 A graph G has the *circuit cover property* if (G,p) has a circuit cover for every admissible weight p.

The following classic result of P. D. Seymour was proved in [11].

Theorem 1.1.1 *Every planar graph has the circuit cover property.*

Several authors [11,12] observed:

Proposition 1.1.2 *Petersen's graph does not have the circuit cover property.*

Let P_{10} denote the graph in Fig. 1 and let a weight p of P_{10} take the value 1 on some 2-factor of P_{10}, and the value 2 on the complementary 1-factor. Then (P_{10},p) is admissible, but (P_{10},p) has no circuit cover.

If $e \in E(G)$ then $G\backslash e$ denotes the graph obtained from G by *deleting* e, and G/e denotes the graph obtained from G by *contracting* e (that is, identifying the endvertices of e, then

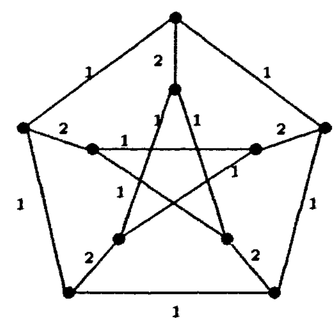


Fig. 1: Petersen's graph

deleting e). Loops and multiple edges (other than e) which arise from a contraction are not deleted. Any graph obtained from G by successive deletions and contractions is called a *minor* of G. An H-*minor* of graph G is a minor of G which is isomorphic to H. The following nice result was proved by B. Alspach, L. Goddyn and C. Q. Zhang in [2].

Theorem 1.1.2 (B. Alspach, L. Goddyn and C. Q. Zhang) *A graph has the circuit cover property if and only if it has no* P_{10} *– minor.*

Let (G,p) be an edge-weighted graph (with loops and multiple edges allowed) where $p : E(G) \rightarrow Z^+$. We say that (G,p) has a bond cover if there exist a multiset (or list) L of bonds in G such that each edge e is covered exactly $p(e)$ times by bonds in L. More precisely, we say that (G,p) has a bond cover provided the following holds:

(1.3) There exists a vector of non-negative integer coefficients $(\lambda_D : C \in \mathcal{C}^*)$ such that

$$\sum_{D \in \mathcal{C}^*} \lambda_D \chi^D = p.$$

Here, $\mathcal{C}^*$ denotes the collection of bonds in G and (λ_D) is the multiplicity vector for the bond cover L.

Analogously to the circuit case, the following conditions are necessary for an arbitrary weighted graph (G,p) to have a bond cover:

(1.4) (i) for every circuit C and $e \in C$, $p(e) \leq p(C \backslash e)$ (that is, p is *balanced*),

(ii) for every circuit C, $p(C)$ is even (that is, p is *eulerian*), and

(iii) p is non-negative integer valued.

The conditions in (1.4) are also collectively called *admissibility conditions*, and p is said to be *admissible* if it satisfies (1.4).

Definition 2 A graph G has the *bond cover property* if (G, p) has a bond cover for every admissible weight p.

Not every graph has the bond cover property.

Proposition 1.1.3 *K_5 does not have the bond cover property.*

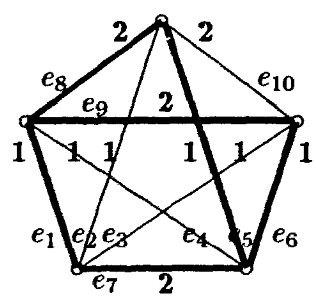


Fig. 2

Proof: Let a weight p of K_5 be as in Fig. 2.

Since all edges with weight 1 form a bond and every circuit intersects each bond in an even number of edges, every circuit contains an even number of edges with weight 1. Hence p is eulerian. Since every circuit has at least 3 edges, but for any edge e, $1 \leq p(e) \leq 2$, p is balanced. Thus p is admissible.

Suppose (K_5, p) has a bond cover. Let us find the bond D covering edge e_1 and remove D. In triangle $\{e_1, e_2, e_8\}$, since $p(e_8) = 2$, then $e_8 \in D$ and $e_2 \notin D$ because if not, then after removing D, the weights of e_1 and e_2 are 0, and the weight of e_8 is 2, so the triangle is unbalanced. The same situation occurs in triangles $\{e_1, e_3, e_9\}$ and $\{e_1, e_4, e_7\}$, so $e_9 \in D$, $e_3 \notin D$ and $e_7 \in D$, $e_4 \notin D$. Thus e_5 and e_6 must be in D, otherwise D will not be a bond. Therefore after removing D, the triangle $\{e_5, e_6, e_{10}\}$ is unbalanced. Hence K_5 does not have the bond cover property. □

Since the dual of a planar graph is still a planar graph and the circuits in a planar graph correspond to the bonds in the planar dual, from Theorem 1.1.1 we have:

Corollary 1.1.1 *Planar graphs have the bond cover property.*

We shall later see that, in fact, a graph has the bond cover property if and only if it has no K_5-minor.

1.2 Matroids and Binary Matroids

All the results in this section can be found in Welsh [20].

A *matroid* M=$M(S,\mathcal{I})$ is a finite set S and a collection $\mathcal{I}$ of subsets of S (called *independent sets*) such that (I1)-(I3) are satisfied.

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(I3) If U, V are members of $\mathcal{I}$ with $|U| = |V| + 1$ there exists $x \in U \setminus V$ such that $V \cup x \in \mathcal{I}$.

A subset of S not belonging to $\mathcal{I}$ is called *dependent.* An element $x \in S$ is called a *loop* if $\{x\} \notin \mathcal{I}$. A *circuit* in M is a minimal dependent subset of S.

One can show that a collection $\mathcal{C}$ of subsets of S is the set of *circuits* of a matroid on S if and only if condition (C1) and (C2) are satisfied.

(C1) If $X \neq Y \in \mathcal{C}$, then $X \not\subseteq Y$.

(C2) If C_1, C_2 are distinct members of $\mathcal{C}$ and $z \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus z$.

Every proper subset of a circuit is independent.

A matroid is determined by its set of circuits since $X \subseteq S$ is independent if and only if X contains no circuit.

Let $\mathcal{C}$ be the collection of circuits of a matroid M. Then a collection $\mathcal{C}^*$ of subsets of S is a set of *cocircuits* of M if and only if for every $X \in \mathcal{C}^*$ the conditions below are satisfied.

(C*1) $X \neq \emptyset$.

(C*2) $|X \cap Y| \neq 1$ for every $Y \in \mathcal{C}$.

(C*3) X is minimal with respect to these properties.

A matroid M^* on S is said to be the *dual matroid* of matroid M on S if the collection of circuits of M^* is the collection of cocircuits in M.

The element $x \in S$ is a *coloop* of the matroid $M{=}M(S,\mathcal{I})$ if $\{x\}$ is a cocircuit of M. This happens if and only if x is a loop in M^*.

Proposition 1.2.1 *An element x is a loop (coloop) in M if and only if no cocircuit (circuit) in M contains x.*

If M is a matroid on S and $x \in S$ then define $\mathcal{I}'$ such that for $X \subseteq S - \{x\}$, $X \in \mathcal{I}'$ if and only if $X \in \mathcal{I}$ (that is, $\mathcal{I}'$ contains those independent subsets of M which are disjoint from $\{x\}$). Then $\mathcal{I}'$ is the collection of independent sets of a matroid M' on $S - \{x\}$. This matroid is denoted by $M\backslash x$ and is called the *deletion* of x from M.

If M is a matroid on S and $x \in S$, then define $\mathcal{I}'$ so that if x is a loop then for $X \subseteq S - \{x\}$ let $X \in \mathcal{I}'$ if and only if $X \in \mathcal{I}$ (that is, consider those independent subsets of M which are disjoint from $\{x\}$), if x is not a loop then for $X \subseteq S - \{x\}$ let $X \in \mathcal{I}'$ if and only if $X \cup \{x\} \in \mathcal{I}$. Then $\mathcal{I}'$ is the collection of independent sets of a matroid M' on $S - \{x\}$. This matroid will be denoted by M/x and called the *contraction* of x from M.

By deleting or contracting the elements of S, many new matroids can be obtained from an original matroid M on S. The result of a sequence of deletions and contractions is called a *minor* of **M**. As the order of deletions and contractions is immaterial, we use $M/A \setminus B$ to denote $M/a_1/a_2.../a_r \setminus b_1 \setminus b_2... \setminus b_s$ when $A = \{a_1, a_2, ...a_r\}$, $B = \{b_1, b_2, ...b_s\}$.

Let T be an arbitrary field, $V[T]$ be a vector space over T and S be a set of vectors from this vector space. This set leads to a matroid $M=M(S,\mathcal{I})$ as follows: $X \subseteq S$ is independent (denoted by $X \in \mathcal{I}$) if and only if the vectors belonging to X are linearly independent over T.

A matroid $M=M(S,\mathcal{I})$ is called *representable* over a field T if suitable vectors from a vector space over T can play the role of S in the above construction.

A matroid is said to be *regular* if it is representable over every field.

Proposition 1.2.2 *If a matroid is representable over a field then so is its dual and its minors.*

A matroid is said to be *binary* if it is representable over $GF(2)$.

A *cycle* is any disjoint union of circuits (thus the empty set is a cycle).

Let the *symmetric difference* $X \triangle Y$ of two sets X, Y be defined as $(X - Y) \cup (Y - X)$. One can prove that "$\triangle$" is an associative, commutative binary operation on the set of cycles of a binary matroid.

Proposition 1.2.3 *The following statements about a matroid* M *are equivalent.*

(i) M *is binary.*

(ii) For any circuit C *and cocircit* C^*, $|C \cap C^*|$ *is even.*

(iii) The symmetric difference of any two cycles of M *is a cycle of* M.

(iv) If C_1, C_2 *are distinct circuits of* M, *then* $C_1 \triangle C_2$ *contains a circuit* C.

Graphs are a rich source of binary matroids. A *graphic matroid* (or *polygon matroid of graphs*) $\mathcal{M}(G)$ and a *cographic matroid* $\mathcal{M}^*(G)$ are defined on the edge set $E(G)$ of the graph G and $X \subseteq E$ is independent in $\mathcal{M}(G)$ or in $\mathcal{M}^*(G)$ if and only if X, as a subgraph of G, is a forest or contains no bond, respectively, in G. The circuits of $\mathcal{M}(G)$ are just the circuits of G. The circuits of $\mathcal{M}^*(G)$ are just the bond of G. A loop and a coloop in $\mathcal{M}^*(G)$ are a loop and a bridge in G, respectively. It is clear that minor of $\mathcal{M}(G)$ correspond to the minors of G. A matroid is called *graphic* or *cographic* if it arises as the graphic matroid or cographic matroid of some graph. Graphic matroids and cographic matroids are regular.

Chapter 2

Main Theorem

2.1 Introduction

Let M be a binary matroid. Let $S = S(M)$ denote the set of elements of M, and let $\mathcal{C} = \mathcal{C}(M)$ denote the set of all circuits C of M. Let

$$p : S \to \mathcal{Z}^+.$$

We say that (M, p) has a circuit cover if there exists a multiset (or list) L of circuits in M such that each element e is covered exactly $p(e)$ times by circuits in L. More precisely, we say that (M, p) has a circuit cover provided the following holds:

(2.1) There exists a vector of non-negative integer coefficients $(\lambda_C : C \in \mathcal{C})$ such that

$$\sum_{C \in \mathcal{C}} \lambda_C \chi^C = p.$$

Here, (λ_C) is the multiplicity vector for the circuit cover L, and for any subset H of $S(M)$, χ^H denotes the $\{0, 1\}$-characteristic function of H. We use the convention that χ^L means $\sum \chi^{H_i}$, where L=$\{H_1, H_2, ..., H_n\}$.

As in the graphic case we have the following necessary conditions for an arbitrary weighted binary matroid (M, p) to have circuit cover:

(2.2) (i) for every cocircuit D and $e \in D$, $p(e) \leq p(D \backslash e)$ (that is, p is *balanced*),

(ii) for every cocircuit D, $p(D)$ is even (that is, p is *eulerian*), and

(iii) p is non-negative integer.

(Again we use the convention that $p(F)$ means $\sum_{e \in F} p(e)$, for any $F \subseteq S(M)$.) As before these conditions follow easily from the fact that any circuit in a binary matroid intersects any cocircuit in an even number of elements. The conditions in (2.2) are collectively called *admissibility conditions*, and p is said to be *admissible* if it satisfies (2.2).

Definition 3 M has the *circuit cover property* if (2.1) and (2.2) are equivalent for all admissible weights p.

An N-minor of matroid M is a minor of M which isomorphic to N.

Here we restate Proposition 1.1.3, and Theorems 1.1.1 and 1.1.2.

Corollary 2.1.1 *$\mathcal{M}^*(K_5)$ does not have the circuit cover property.*

Theorem 2.1.1 (Seymour) *Every graphic matroid of a planar graph has the circuit cover property.*

Theorem 2.1.2 (B.Alspach, L.Goddyn and C.Q.Zhang) *A graphic matroid has the circuit cover property if and only if it has no $\mathcal{M}(P_{10})$-minor.*

The main result of this thesis (Theorem 2.3.1) is an extension of this result to binary matroids.

2.2 Some Special Matroids

We introduce here two special binary matroids which, like $\mathcal{M}(P_{10})$ and $\mathcal{M}^*(K_5)$, do not have the circuit cover property.

Fig. 3 represents a special binary matroid on a 7-element set (the points). The circuits consist of any 3 points which lie on a line, and also any 4 points not containing a line (a

4-arc). The cocircuits are precisely the 4-arcs in Fig. 2. This matroid called, the *Fano matroid*, is denoted by F_7.

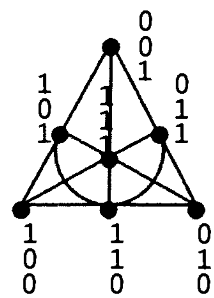


Fig. 3

The matroid F_7 represented by the seven non-zero 3-tuples over $GF(2)$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 3 shows a correspondence between the points and the 3-tuples.

F_7^* is the dual matroid of F_7, so every circuit of F_7^* is a 4-arc and every cocircuit of F_7^* is a line or a 4-arc in Fig. 3 above.

Proposition 2.2.1 *F_7^* does not have the circuit cover property.*

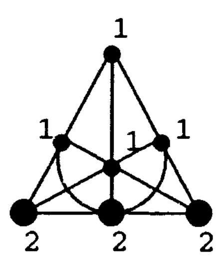


Fig. 4

Proof: Let a weight p of F_7^* take the value 1 on some 4-arc and the value 2 on the complement of the 4-arc.

Since the 4-arc with weights 1 is a circuit of F_7^* and every cocircuit intersects a circuit in an even number of elements, it follows that each cocircuit contains an even number of the elements with weight 1, so p is eulerian. Since every cocircuit has 3 or 4 elements, and $1 \leq p(e) \leq 2$ for every $e \in F_7$, p is balanced. Thus p is an admissible weight.

Suppose (F_7^*, p) has circuit cover. Consider the circuit covering e_1. There are only 4 circuits in F_7^* containing e as shown in Fig. 5 (the circuits containing e are denoted by black dots). But removing any one of the 4 circuits will cause some cocircuits (dotted lines) to become unbalanced.

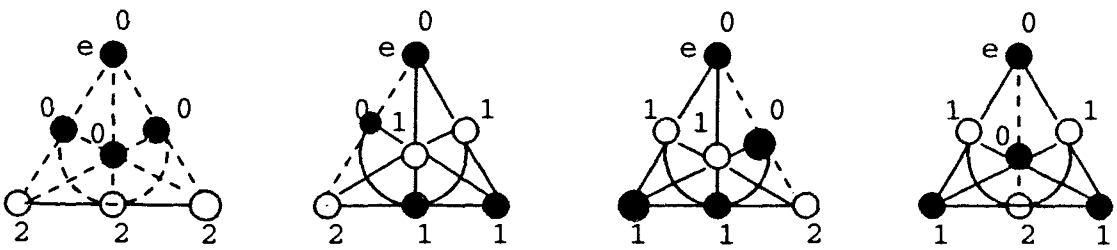


Fig. 5

Alternatively, (F_7^*, p) has total weight 10, and each circuit in F_7^* has size 4. However, 4 does not divide 10.

Therefore F_7^* does not have the circuit cover property. □

Let R_{10} denote the matroid represented over $GF(2)$ by the ten 5-tuples with three 1s and two 0s. A totally unimodular representation of R_{10} is given below. As this matrix represents R_{10} over any field, R_{10} is a regular matroid. One can check that R_{10} is isomorphic to its dual (although not self-dual).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

It is possible to identify the ten elements of R_{10} with the edges of K_5 in such a way that the collection of circuits in R_{10} is the collection of the (graphical) 4-circuits in K_5 and their complements (which form 6-circuits in R_{10}). The collection of cocircuits in R_{10} corresponds to the collection of bonds and their complements in K_5. See Fig. 6 and Fig. 7.

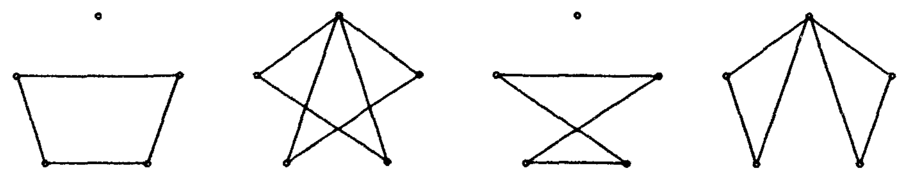


Fig. 6: The typical circuits in R_{10}

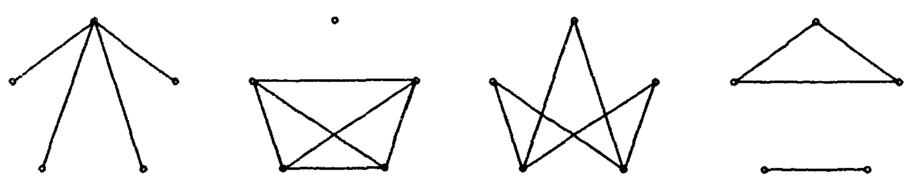


Fig. 7: The typical cocircuits in R_{10}

Like F_7^*, R_{10} does not have the circuit cover property.

Proposition 2.2.2 R_{10} *does not have the circuit cover property.*

Proof: Let a weight p of R_{10} be as in Fig. 8. Since each cocircuit in R_{10} has 4 or 6 elements, p is even, and since $1 \leq p(e) \leq 3$ for every $e \in R_{10}$, p is also balanced.

Suppose (R_{10}, p) has a circuit cover. Consider the circuit C covering e_1. Considering cocircuit $\{e_1, e_2, e_8, e_6\}$, $e_8 \in C$ and $e_2 \notin C$, $e_6 \notin C$ because if not, then after removing C, the weights of e_1 and one of e_2 and e_6 are 0, and the weight of e_8 is 3, so the cocircuit is unbalanced. The same situation occurs in cocircuits $\{e_1, e_5, e_6, e_{10}\}$ and $\{e_1, e_3, e_5, e_9\}$, so $e_{10} \in D$, $e_5 \notin D$ $e_9 \in D$ and $e_3 \notin D$. Then C must be $\{e_1, e_4, e_7, e_8, e_9, e_{10}\}$. But after removing C, the cocircuit $\{e_1, e_4, e_7, e_{10}\}$ is unbalanced. Hence R_{10} does not have the circuit cover property. □

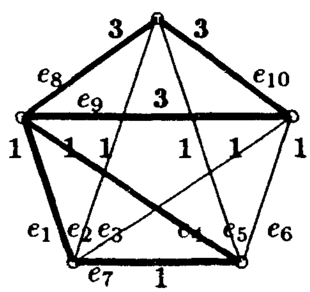


Fig. 8

2.3 The Main Theorem

Seymour [10] proposed a forbidden minor characterization of binary matroids with the circuit cover property. As the main result of this thesis we prove his conjecture.

Theorem 2.3.1 *A binary matroid M has the circuit cover property if and only if M has no F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ or $\mathcal{M}(P_{10})$ minor.*

We know that F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ and $\mathcal{M}(P_{10})$ do not have the circuit cover property, so by Lemma 3.1.1 we know that if a binary matroid M has either a F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ or $\mathcal{M}(P_{10})$ minor then M does not have the circuit cover property.

In the following chapters we shall introduce a decomposition theorem (Corollary 3.2.1) which says that any binary matroid with no F_7^*, R_{10}, or $\mathcal{M}^*(K_5)$ minor may be obtained by means of certain sum operations from graphic matroids and copies of two special matroids, F_7 and $\mathcal{M}^*(V_8)$. We shall prove that the sum operations preserve the circuit cover property, and that F_7 and $\mathcal{M}^*(V_8)$ each have the circuit cover property. These results, together with Theorem 2.1.2 (which deals with the graphic case) imply Theorem 2.3.1.

By the fact that none of F_7^*, R_{10} and $\mathcal{M}(P_{10})$ is cographic, Theorem 2.3.1 implies the following.

Corollary 2.3.1 *A cographic matroid has the circuit cover property if and only if it has no $\mathcal{M}^*(K_5)$ minor.*

Restating this in graphical terms gives the following.

Corollary 2.3.2 *A graph has the bond cover property if and only if it has no K_5 minor.*

Chapter 3

Minors and Decomposition Theorems

3.1 Minors and the Circuit Cover Property

The concept of a minor was introduced in Section 2.1. Here we prove that the circuit cover property of binary matroids is closed under minors.

Lemma 3.1.1 *If a binary matroid $M(S)$ has the circuit cover property, then any minor of $M(S)$ also has the circuit cover property.*

Proof: Suppose $M(S)$ has the circuit cover property. It is sufficient to show that for any $f \in S$, both $M\backslash f$ and M/f also have the circuit cover property.

First we consider $M\backslash f$. Let $p : S'(M\backslash f) \rightarrow \mathcal{Z}^+$ be admissible.We define $p' : S(M) \rightarrow \mathcal{Z}^+$ by

$$p'(e) = \begin{cases} p(e) & (e \neq f) \\ 0 & (e = f). \end{cases}$$

It is easy to see that (M, p') is admissible and, by hypothesis, has a circuit cover. Clearly this circuit cover for (M, p') is also a circuit cover for $(M\backslash f, p)$. Thus $M\backslash f$ has the circuit cover property.

Now we prove that M/f has the circuit cover property. Assume f is not a loop, since $M/f \cong M\backslash f$ if f is loop.

Let $p : S'(M/f) \to \mathcal{Z}^+$ be admissible. We define $p' : S(M) \to \mathcal{Z}^+$ as follows.

Since f is not a loop of M, there is a cocircuit of M containing f. Choose such a cocircuit D with

$$p'(D - \{f\})$$

minimum. We define p' by

$$p'(e) = p(e)\ (e \neq f) \text{ and}$$
$$p'(f) = p'(D - \{f\}).$$

We claim that (M, p') is admissible.

Any cocircuit D_1 not containing f in M is also a cocircuit in M/f, so $p'(D_1) = p(D_1)$ is even and balanced.

For every cocircuit D' containing f in M, by the definition of $p'(f)$ we have

$$p'(f) \leq p'(D' - \{f\}).$$

Now $D' \triangle D$ has even intersection with every cycle of M/f, and so is a disjoint union of cocircuits of M/f. Thus $p(D' \triangle D)$ is even. But $p'(D)$ is even, and

$$p(D' \triangle D) \equiv p'(D') + p'(D) \pmod 2$$

so that $p'(D')$ is even.

We now show that p' is balanced on the cocircuit D'. For any $e \in D' \cap D$,

$$p'(e) \leq p'(D - \{e\}) \leq p'(D' - \{e\})$$

For any $e \in D' - D$, we have $e \in D' \triangle D$, and

$$\begin{aligned} p'(e) = p(e) &\leq p(D \triangle D' - \{e\}) \leq p(D' - \{f\}) + p(D - \{f\}) - p(e) \\ &= p'(D' - \{f\}) + p'(f) - p'(e) = p'(D' - \{e\}). \end{aligned}$$

Thus (M, p') is admissible as claimed.

By hypothesis, there are collections of circuits L in M such that $\chi^{L} = p'$ (recall that χ^{F} denotes $\sum \chi^{F_i}$ when $F = \{F_1, F_2, ... F_n\}$). Let $W = \{C_1, C_2, ..., C_{p'(f)}\}$ be the circuits containing f in M. Clearly, $W/f := \{C_1 - f, C_2 - f, ..., C_{p'(f)} - f\}$ and $L - W$ are collections of circuits in M/f and

$$\chi^{W/f \cup (L-W)} = p.$$

Thus, M/f has the circuit cover property. □

3.2 Decomposition of Binary Matroids

Let M_1, M_2 be binary matroids with element sets S_1, S_2, respectively, where S_1 and S_2 may intersect. We define a new binary matroid $M_1 \triangle M_2$ to be the matroid with element set $S_1 \triangle S_2$ and with cycles all subsets of $S_1 \triangle S_2$ of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i $(i = 1, 2)$. (For sets S_1, S_2, $S_1 \triangle S_2$ denotes $(S_1 - S_2) \cup (S_2 - S_1)$. Recall from Section 1.2 that a *cycle* of a binary matroid is a subset of the elements expressible as a disjoint union of circuits. It is easy to see that if C, C' are cycles, then $C \triangle C'$ is a cycle.)

We are only concerned with three special cases of this operation, as follows.

(i) When $S_1 \cap S_2 = \emptyset$ and $|S_1|, |S_2| < |S_1 \triangle S_2|$ (that is, $S_1, S_2 \neq \emptyset$), then $M_1 \triangle M_2$ is a *1-sum* (or *disjoint union*) of M_1 and M_2.

(ii) When $|S_1 \cap S_2| = 1$, $S_1 \cap S_2 = \{f\}$, f is not a loop or coloop of M_1 or M_2, and $|S_1|, |S_2| < |S_1 \triangle S_2|$ (that is, $S_1, S_2 \geq 3$), then $M_1 \triangle M_2$ is a *2-sum* of M_1 and M_2.

(iii) When $|S_1 \cap S_2| = 3$, $S_1 \cap S_2 = Z$, Z is a circuit of size 3 of both M_1 M_2, Z includes no cocircuit of either M_1 or M_2, and $|S_1|, |S_2| < |S_1 \triangle S_2|$ (that is, $S_1, S_2 \geq 7$), then $M_1 \triangle M_2$ is a *3-sum* of M_1 and M_2.

(iii)* The *dual* form of 3-sum

When $|S_1 \cap S_2| = 3$, $S_1 \cap S_2 = Z$, Z is a cocircuit of size 3 of both M_1 M_2, Z includes no circuit of either M_1 or M_2, and $|S_1|, |S_2| < |S_1 \triangle S_2|$ (that is, $S_1, S_2 \geq 7$), then

$M_1 \triangle M_2$ is a *dual 3-sum* of M_1 and M_2

It is helpful to visualize these operations in terms of polygon matroids of graphs. For $k = 1, 2, 3$, a k-sum of two polygon matroids corresponds to taking two graphs, choosing a k-clique from each, identifying the vertices in the cliques pairwise and deleting the edges in the cliques. If M is the k-sum of M_1 and M_2, then M_1 and M_2 are minors of M.

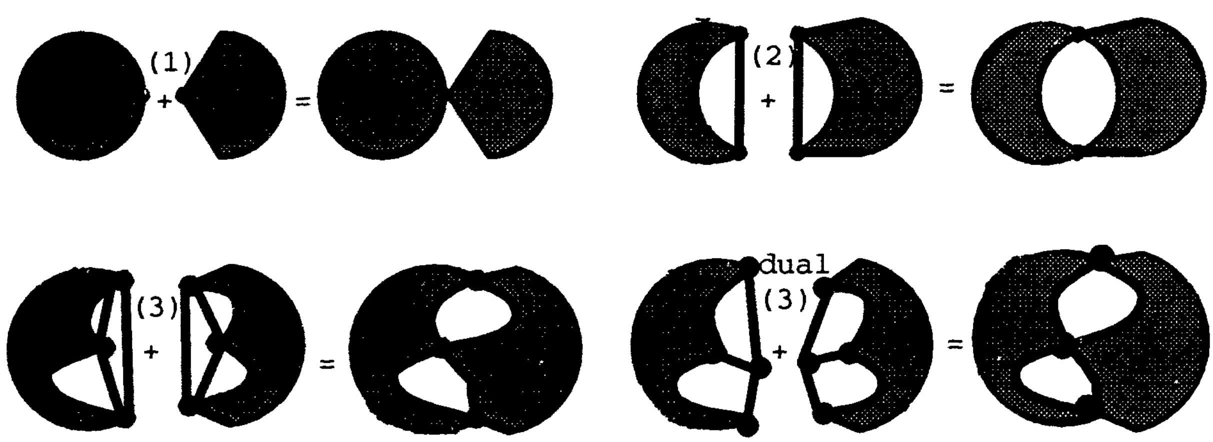


Fig. 9

There is no need to introduce dual 1- and 2-sums since these two operations are self-dual. That is, if M is a 1- or 2-sum of matroids M_1 and M_2, then M^* is a 1- or 2-sum of M_1^* and M_2^*. However, 3-sum is not self dual, since if $S(M_1) \cap S(M_2)$ is a circuit of size 3 in M_i (i=1 or 2) then by Proposition 1.2.3, $S(M_1) \cap S(M_2)$ is not a cocircuit of M_i. In fact, if M is a 3-sum of matroids M_1 and M_2, then M^* is a dual 3-sum of M_1^* and M_2^*.

We shall need several theorems which assert that binary matroids without certain minors may be obtained by means of these three sum operations, starting from a simpler class of matroids.

The following three results were proved by Seymour [14].

Theorem 3.2.1 (Seymour) *Every binary matroid with no F_7^* minor may be obtained by means of 1- and 2-sums from regular matroids and copies of F_7.*

Theorem 3.2.2 (Seymour) *Every binary matroid with no F_7 minor may be obtained by means of 1- and 2-sums from regular matroids and copies of F_7^*.*

Theorem 3.2.3 (Seymour) *Every regular matroid with no R_{10} minor may be obtained by means of 1-,2- and 3-sums from graphic and cographic matroids.*

From Propsition 1.2.2 we have that the dual of a binary matroid is binary and the dual of a regular matroid is regular. Also, R_{10} is isomorphic to its dual. Clearly Theorem 3.2.1 and Theorem 3.2.2 are dual forms of each other and we may restate Theorem 3.2.3 in the dual form below.

Theorem 3.2.4 *Every regular matroid with no R_{10} minor may be obtained by means of 1-,2- and dual 3-sums from graphic and cographic matroids.*

The well-known Kuratowski Theorem states that a graph is planar if and only if it has no K_5 or $K_{3,3}$ minor. The next result is a generalization proved by Wagner [19].

Theorem 3.2.5 *Every graphic matroid with no $\mathcal{M}(K_5)$ minor may be obtained by means of 1-,2- and 3-sums from polygon matroids of planar graphs and copies of $\mathcal{M}(V_8)$.*

See Fig. 10 in Page 30 for the picture of V_8. We shall use the dual form of this theorem.

Theorem 3.2.6 *Every cographic matroid with no $\mathcal{M}^*(K_5)$ minor may be obtained by means of 1-,2- and dual 3-sums from polygon matroids of planar graphs and copies of $\mathcal{M}^*(V_8)$.*

The following corollary follows from Theorem 3.2.1, Theorem 3.2.3 and Theorem 3.2.6.

Corollary 3.2.1 *Every binary matroid with no* F_7^**,* R_{10} *or* $\mathcal{M}^*(K_5)$ *minor may be obtained by means of 1-,2- and dual 3-sums from graphic matroids, copies of* F_7 *and copies of* $\mathcal{M}^*(V_8)$.

Chapter 4

Sums and the Circuit Cover Property

Our object in this section is to show that the three matroid sum operations described in Section 3.2 preserve the circuit cover property. All the matroids in this chapter are binary matroids.

Lemma 4.0.1 *If M is the 1-sum of M_1 and M_2, and M_1, M_2 both have the circuit cover property, then so has M.*

Proof: Let $p : E(M) \to \mathcal{Z}^+$ such that p is admissible. Write $S = S(M), S_i = S_i(M_i)$ $(i = 1, 2)$.

Define $p_1 : S_1 \to \mathcal{Z}^+$ and $p_2 : S_2 \to \mathcal{Z}^+$ by $p_1(e) = p(e)$ $(e \in S_1)$ and $p_2(e) = p(e)$ $(e \in S_2)$.

By the definition of 1-sum, we know that p_1 and p_2 are admissible. Therefore by hypothesis, there are collections of circuits L_1 in M_1 and L_2 in M_2, such that $\chi^{L_1} = p_1$ and $\chi^{L_2} = p_2$. Clearly L_1 and L_2 are also the collections of circuits in M and $\chi^{L_1 \cup L_2} = p$ as required. □

Lemma 4.0.2 *If M is the 2-sum of M_1 and M_2, and M_1, M_2 both have the circuit cover property, then so has M.*

Proof: Let $p : E(M) \to \mathcal{Z}^+$ such that p is admissible. Write $S = S(M)$, $S_i = S_i(M_i)$ $(i = 1,2)$, and let $S_1 \cap S_2 = \{f\}$. By the definition of the matroid 2-sum, f is not a loop of M_i, so there is a cocircuit of M_i containing f. For $i = 1,2$, choose such a cocircuit D_i in M_i with

$$p(D_i - \{f\})$$

minimum and let these numbers be n_i $(i = 1,2)$. Then put $n = \min\{n_1, n_2\}$. Choose $j \in \{1,2\}$ such that $n_j = n$.

Define $p_1 : S_1 \to \mathcal{Z}^+$ and $p_2 : S_2 \to \mathcal{Z}^+$ by

$$\begin{aligned} p_1(e) &= p(e) \quad (e \neq f), & p_2(e) &= p(e) \quad (e \neq f), \\ p_1(f) &= n, & p_2(f) &= n. \end{aligned}$$

We shall now show that each p_i is an admissible weight for M_i.

For every cocircuit D of M_i $(i = 1,2)$ not containing f, D has even intersection with every cycle of M. Thus D is a disjoint union of cocircuits of M, implying $p_i(D) = p(D)$ is even and

$$p_i(e) = p(e) \leq p(D - \{e\}) = p_i(D - \{e\})$$

for every $e \in D$. Thus D is balanced and eulerian.

We now show that every cocircuit D containing f in M_i $(i = 1,2)$ is balanced and eulerian. By the definition of $p_i(f)$ we have

$$p_i(f) \leq p_i(D - \{f\})$$

Since $D_j \triangle D$ has even intersection with every cycle of M, it is a disjoint union of cocircuits of M. Thus $p(D_j \triangle D)$ is even. But $p_j(D_j)$ is even, and

$$p(D_j \triangle D) \equiv p_j(D_j) + p_i(D) \pmod 2$$

so that $p_i(D)$ is even.

For any $e \in D_j \cap D$, we have

$$p_i(e) = p_j(e) \leq p_j(D_j - \{e\}) \leq p_i(D - \{e\})$$

For any $e \in D - D_j$, we have $e \in D_1 \triangle D$, and

$$\begin{aligned} p_i(e) = p(e) &\leq p(D_1 \triangle D - \{e\}) \leq p(D - \{f\}) + p(D_1 - \{f\}) - p(e) \\ &= p_i(D - \{f\}) + p_i(f) - p_i(e) = p_i(D - \{e\}) \end{aligned}$$

Thus p_i is admissible.

By hypothesis, there are collections of circuits L_1 in M_1 and L_2 in M_2 such that $\chi^{L_1} = p_1$ and $\chi^{L_2} = p_2$ and there are exactly n cycles containing f in M_1 and M_2. Let $W_1 = \{c_1, c_2, ..., c_n\}$ and $W_2 = \{d_1, d_2, ..., d_n\}$ be such circuits in M_1 and M_2, respectively. Let $W = \{c_1 \triangle d_1, c_2 \triangle d_2, ..., c_n \triangle d_n\}$. Clearly, W, $L_1 - W_1$, $L_2 - W_2$ are collections of circuits in M and

$$\chi^{W \cup (L_1 - W_1) \cup (L_2 - W_2)} = p$$

as required. □

Lemma 4.0.3 *If M is the dual 3-sum of M_1 and M_2, and M_1, M_2 both have the circuit cover property, then so has M.*

Proof: Let $p : S(M) \rightarrow Z^+$ be such that p is admissible. Put $S(M_i) = S_i$ $(i = 1, 2)$, and $S_1 \cap S_2 = Z = \{z_1, z_2, z_3\}$, where Z is a cocircuit of both M_1 and M_2.

For $1 \leq i \leq 2, 1 \leq j \leq 3$, since Z is a cocircuit in M_i and Z contains no circuit in M_i, z_j is not a loop in $M_i/(Z - \{z_j\})$. Thus there is a cocircuit in $M_i/(Z - \{z_j\})$ containing z_j. By the definition of contraction, this cocircuit is also a cocircuit in M_i.

Hence, let d_{ij} be the minimum of

$$p(D - z_j)$$

taken over all cocircuits D of M_i with $D \cap Z = z_j$. Let D_{ij} be a cocircuit of M_i attaining equality. For $1 \le j \le 3$, put $n_j = \min\{d_{1j}, d_{2j}\}$. Let D_j be a cocircuit in $\{D_{1j}, D_{2j}, D_{3j}\}$ such that $p(D_j - z_j) = n_j$. Now $D_1 \triangle D_2 \triangle D_3 \triangle Z$ is a cocycle of M and so $p(D_1 \triangle D_2 \triangle D_3 \triangle Z)$ is even. Thus $n := n_1 + n_2 + n_3 = p(D_1) + p(D_2) + p(D_3) - p(Z) \equiv p(D_1 \triangle D_2 \triangle D_3 \triangle Z)$ (mod 2) is even.

Define $p_i : S_i \to \mathcal{Z}^+ (i = 1, 2)$ by

$$p_i(e) = p(e) \ (e \notin Z)$$
$$p_i(z_j) = \min(n_j, n - n_j), j = 1, 2, 3.$$

Let D be any cocircuit of either M_1 or M_2, say M_i. We shall show that D is eulerian and balanced in (M_i, p_i). We have 4 cases depending on $|Z \cap D|$.

Case $|Z \cap D| = 3$: Here $D = Z$ and the cocircuit Z is eulerian and balanced by the definition of p_i.

Case $|Z \cap D| = 0$: As D has even intersection with every circuit of M, D is a disjoint union of cocircuits of M. Thus D is eulerian and balanced.

Case $|Z \cap D| = 1$: Suppose without loss of generality, $Z \cap D = \{z_1\}$.
If $p_i(z_1) = n_1$, then by the same argument as in previous lemma, D is eulerian and balanced. Suppose that $p_i(z_1) < n_1$, so that $n_1 > n_2 + n_3$, $p_i(z_1) = n_2 + n_3, p_i(z_2) = n_2, p_i(z_3) = n_3$ and $p_i(D - z_1) \ge n_2 + n_3 = p_i(z_1)$. We claim that in this case, neither M_1 nor M_2 can contain both D_2 and D_3. Otherwise, $D_2 \triangle D_3 \triangle Z$ will be a cocycle in M_1 or M_2 and $(D_2 \triangle D_3 \triangle Z) \cap Z = \{z_1\}$. Thus there is a cocircuit $D'_1 \subseteq D_2 \triangle D_3 \triangle Z$ in M_1 or M_2 and $D'_1 \cap Z = \{z_1\}$. Thus, for k=1 or 2 we have,

$$p_k(D'_1 - \{z_1\}) \le p_k(D_2 - \{z_2\}) + p_k(D_3 - \{z_3\}) = n_2 + n_3 < n_1 \ (k = 1 \text{ or } 2).$$

This contradicts the minimality of n_1, proving our claim.

Hence exactly one of D_2, D_3, say D_2, belongs to M_i. Now $D_3' = D_2 \triangle D \triangle Z$ is a cocycle of M_i and $D_3' \cap Z = \{z_3\}$. Thus $p_i(D_3')$ is even. But $p_i(D_2)$ and $p_i(Z)$ are even, and

$$p_i(D_3') \equiv p_i(D) + p_i(Z) + p_i(D_2) \pmod 2$$

so that $p_i(D)$ is even.

We now show D is balanced. For any $e \in D \cap D_2$, we have $e \in D_2$ and

$$\begin{aligned} p_i(e) &\leq p_i(D_2 - \{e\}) = p_i(D_2 - \{z_2\}) + p(z_2) - p_i(e) \\ &= n_2 + n_2 - p_i(e) \leq 2(n_2 + n_3) - p_i(e) = 2p_i(z_1) - p_i(e) \\ &\leq p_i(D - \{z_1\}) + p_i(z_1) - p_i(e) = p_i(D - \{e\}). \end{aligned}$$

The last inequality follows from the definition of $p_i(z_1)$.

For any $e \in D - D_2$, we have $e \in D_2 \triangle D \triangle Z$, and $(D_2 \triangle D \triangle Z) \cap Z = \{z_3\}$, so $D_2 \triangle D \triangle Z$ is eulerian and balanced. Therefore

$$\begin{aligned} p_i(e) &\leq p_i(D_2 \triangle D \triangle Z - \{e\}) \leq p_i(D - \{z_1\}) + p_i(D_2 - \{z_2\}) + p_i(z_3) - p_i(e) \\ &= p_i(D - \{z_1\}) + p_i(z_1) - p_i(e) = p_i(D - \{e\}) \end{aligned}$$

Thus D is eulerian and balanced.

Case $|Z \cap D| = 2$: Without loss of generality, let $D \cap Z = \{z_1, z_2\}$ so that $D \triangle Z$ is a cocycle of M_i and $(D \triangle Z) \cap Z = \{z_3\}$. By the previous case, $p_i(D \triangle Z)$ is even. But $p_i(Z)$ is even, and

$$p_i(D \triangle Z) \equiv p_i(D) + p_i(Z) \pmod 2$$

so that $p_i(D)$ is even.

For any $e \in D - \{z_1, z_2\}$, we have $e \in D \triangle Z$. Since $D \triangle Z$ is balanced,

$$\begin{aligned} p_i(e) &\leq p_i(D \triangle Z - \{e\}) \\ &= p_i(D) - p_i(z_1) - p_i(z_2) + p_i(z_3) - p_i(e) \leq p_i(D - \{e\}) \end{aligned}$$

and

$$p_i(z_1) \leq p_i(z_2) + p_i(z_3) \leq p_i(z_2) + p_i(D - \{z_1, z_2\}) = p_i(D - \{z_1\}).$$

Similarly, we have

$$p_i(z_2) \leq p_i(D - \{z_2\}).$$

Thus D is eulerian and balanced. Therefore p_1 and p_2 are admissible.

By hypothesis, there are collections of circuits $\boldsymbol{L}_1$ in $\boldsymbol{M}_1$ and $\boldsymbol{L}_2$ in $\boldsymbol{M}_2$ such that $\chi^{\boldsymbol{L}_1} = p_1$ and $\chi^{\boldsymbol{L}_2} = p_2$. But since Z is a cocircuit, every cycle which contains any of z_1, z_2, z_3 contains exactly two of them. Thus there are

$0 \leq m_1 = 1/2(p_i(z_2) + p_i(z_3) - p_1(z_1))$ cycles containing $\{z_2, z_3\}$,

$0 \leq m_2 = 1/2(p_i(z_1) + p_i(z_3) - p_1(z_2))$ cycles containing $\{z_1, z_3\}$, and

$0 \leq m_3 = 1/2(p_i(z_1) + p_i(z_2) - p_1(z_3))$ cycles containing $\{z_1, z_2\}$

($i = 1$ or 2) in $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$. Let

$R^i = \{R^i_1, R^i_2, ..., R^i_{m_1}\}$ be the m_1 cycles containing $\{z_2, z_3\}$,

$S^i = \{S^i_1, S^i_2, ..., S^i_{m_2}\}$ be the m_2 cycles containing $\{z_1, z_3\}$, and

$T^i = \{T^i_1, T^i_2, ..., T^i_{m_3}\}$ be the m_3 cycles contain $\{z_1, z_2\}$

in $\boldsymbol{M}_i (i = 1, 2)$. Let

$$\begin{aligned} R &= \{R^1_1 \triangle R^2_1, R^1_2 \triangle R^2_2, ...R^1_{m_1} \triangle R^2_{m_1}\}, \\ S &= \{S^1_1 \triangle S^2_1, S^1_2 \triangle S^2_2, ...S^1_{m_2} \triangle S^2_{m_2}\}, \text{ and} \\ T &= \{T^1_1 \triangle T^2_1, S^1_2 \triangle T^2_2, ...T^1_{m_3} \triangle T^2_{m_3}\}. \end{aligned}$$

Clearly, R, S, T, $\boldsymbol{L}_1 - (R^1 \cup S^1 \cup T^1)$ and $\boldsymbol{L}_2 - (R^2 \cup S^2 \cup T^2)$ are collections of cycles in M and

$$\chi^{R \cup S \cup T \cup (\boldsymbol{L}_1 - (R^1 \cup S^1 \cup T^1)) \cup (\boldsymbol{L}_2 - (R^2 \cup S^2 \cup T^2))} = p$$

as required. □

Chapter 5

The Circuit Cover Property of F_7

Proposition 5.0.1 *Every two distinct elements of F_7 are in a unique 3-circuit.*

Proposition 5.0.2 *$F_7 \setminus i \cong \mathcal{M}(K_4)$ for every element i of F_7.*

Definition 4 Let $p : E(F_7) \rightarrow \mathcal{Z}^+$ be an admissible weight of F_7 and p be positive. Let C be circuit of F_7. Define a new weight p_C by $p_C := p - \chi^C$. That is

$$p_C(e) = \begin{cases} p(e) - 1 & (e \in C) \\ p(e) & (e \notin C) \end{cases}$$

If (F_7, p_C) is still admissible, then say that C is *removable*.

Removing a circuit C means reducing the weights of the elements in C by 1.

Lemma 5.0.4 *Let (F_7, p) be admissible and p be positive. Let l_1 and l_2 be any two heaviest weighted elements of F_7. That is, $min(p(l_1), p(l_2)) \geq p(e)$ for every $e \in S(F_7) - \{l_1, l_2\}$. Then the unique 3-circuit C containing l_1 and l_2 is removable.*

Proof: Since both p and χ^C are eulerian, so is $p_C = p - \chi^C$. Also, p_C is non-negative valued since p is positive. It remains to show p_C is balanced.

For any cocircuit D of F_7, we have $|C \cap D| = 0$ or 2, For any cocircuit D where $|D \cap C| = 0$, we have, since p is balanced,

$$p_C(e) = p(e) \leq p(D - \{e\}) = p_C(D - \{e\})$$

for every $e \in D$.

For any cocircuit D and $|D \cap C| = 2$, then at least one of l_1 and l_2 is in D. Let l_i in D ($i = 1$ or 2).

For each $e \in D - C$, since p positive, $p(e) \leq \min(p(l_1), p(l_2)) \leq p(l_i) < p(D - \{e\})$, so that $p(e) \leq p(D - \{e\}) - 1$. But $p(D)$ is even, so $p(e) \leq p(D - \{e\}) - 2$. Therefore

$$p_C(e) = p(e) \leq p(D - \{e\}) - 2 = p_C(D - \{e\}) + 2 - 2 = p_C(D - \{e\})$$

For each $e \in D \cap C$,

$$p_C(e) = p(e) - 1 = p(D - \{e\}) - 1 = p_C(D - \{e\}) + 1 - 1 = p_C(D - \{e\})$$

Hence $p_C(D)$ is balanced. Therefore C is removable. □

Lemma 5.0.5 *F_7 has the circuit cover property.*

Proof: Let $p : E(F_7) \to \mathcal{Z}^+$ and p be admissible. If $p(i) = 0$ for some i, $0 \leq i \leq 6$, we delete i from F_7 and obtain $F_7 \setminus i \cong \mathcal{M}(K_4)$.
We define $p\prime : E(M(K_4)) \to \mathcal{Z}^+$ by

$$p\prime(e) = p(e)\ (e \neq i).$$

Clearly $p\prime$ is admissible, Therefore by Corollary 1.1.1 there is a collection of circuits $\mathbf{L}$ in K_4 such that $\chi^{\mathbf{L}} = p\prime$, but $\mathbf{L}$ is also a collection of circuits in F_7 and $\chi^{\mathbf{L}} = p$ as required. We assume that $p(i) > 0 (0 \leq i \leq 6)$ and prove the result by finding a removable circuit C, removing circuit C and using induction on the new weighted (F_7, p_C). By Lemma 5.0.4 we can always find a removable circuit. □

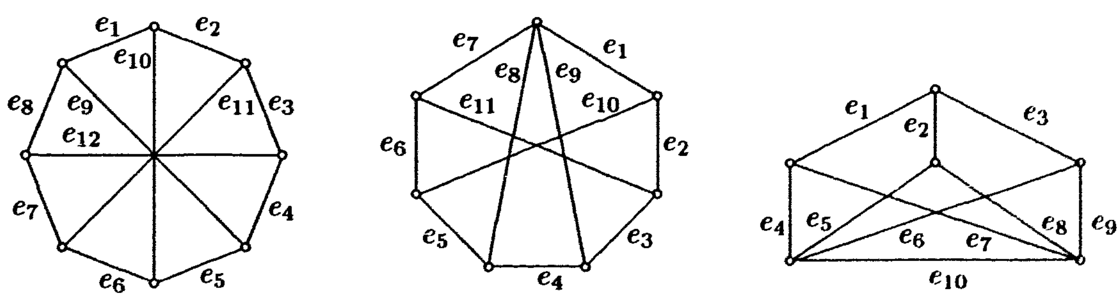
Chapter 6

The Bond Cover Property of V_8

In this chapter we show that $\mathcal{M}^*(V_8)$ has the circuit cover property by showing that V_8 has the bond cover property. This is a key step towards the main theorem (Theorem 2.3.1).

6.1 Introdution

V_8 is a graph of 8 vertices with 8 rim-edges and 4 spokes. In Fig. 10 below, e_1, e_2,...,e_8 are the rim-edges, and e_9, e_{10}, e_{11}, e_{12} are spokes. If any one of the 4 spokes is contracted,



V_8 V_7 V_6

Fig. 10

a planar graph is obtained. If any one of the 8 rim-edges is contracted, then we obtain a non-planar graph which we call V_7.

In V_7 in Fig 10, if e_4 is contracted, then identifying e_8 and e_9 we obtain $K_{3,3}$. If e_3 or e_5

is contracted, we obtain the non-planar graph denoted by V_6. If any one of the other edges contracted, then a planar graph is obtained.

If any one of the edges in V_6 and in $K_{3,3}$ is contracted, then a planar graph is obtained.

The concept and definition of bond cover and the bond cover property have been introduced in Section 1.1.

Proposition 6.1.1 *Let (G,p) admissible. Let $e \in E(G)$ such that $p(e) = 0$. If G/e has the bond cover property, then (G,p) has a bond cover.*

Proof: Define $p' : E(G/e) \to \mathcal{Z}^+$ by

$$p'(f) = p(f)\ (f \neq e)$$

Clearly $(G/e, p')$ is admissible. Thus by hypothesis, there is a collection L of bonds in G/e such that $\chi^L = p'$. L is also a collection of bonds in G and $\chi^L = p$ as required. □

Proposition 6.1.2 *A graph with multiple edges has the bond cover property if and only if its underline simple graph has the bond cover property.*

6.2 Preliminaries

In this section we prove some results which are key to the proof of the bond cover property of V_8.

Definition 5 Let (G,p) be admissible. A *tight circuit* is a circuit C with $p(l) = p(C - \{l\})$ for some $l \in C$, and l is called a *leader* of C. For an edge e, if there is some tight circuit C such that e is the leader of C, then e is said to be a *leader* in (G,p). If there is no such tight circuit in which e is a leader, then e is said to be a *follower* in (G,p).

In a non-tight circuit C of admissible (G,p), $p(e) \leq p(C - \{e\}) - 1$ for all $e \in C$. Since $p(C)$ is even, we have $p(e) \leq p(C - \{e\}) - 2$. In the following proofs, we assume every circuit has cardinality at least 3. If p is positive, then in each tight circuit C exactly one edge is a leader of C.

Lemma 6.2.1 *Let (G,p) be admissible and p be positive, and let l_1 be the leader of circuit C_1. If circuit C_2 is tight, $l_1 \in C_2$ and $|C_1 \cap C_2| \geq 2$, then l_1 is also the leader of C_2.*

Proof: Let $A = C_1 \cap C_2 - \{l_1\}$. Then

$$p(l_1) = p(A) + p(C_1 - C_2). \tag{6.1}$$

Suppose l_1 is not the leader of C_2. Let l_2 be the leader of C_2, so that $l_2 \notin A$, and

$$p(l_2) = p(l_1) + p(A) + p(C_2 - C_1) - p(l_2). \tag{6.2}$$

From (6.1) and (6.2), we have

$$2p(l_2) = 2p(A) + p(C_1 \triangle C_2). \tag{6.3}$$

But $C_1 \triangle C_2$ is a cycle, and $l_2 \in C_1 \triangle C_2$, so that

$$2p(l_2) \leq p(C_1 \triangle C_2) \tag{6.4}$$

Now (6.3) and (6.4) imply $p(A) \leq 0$. But $|A| \geq 1$ and p is positive, so that $p(A) > 0$, which is a contradiction. Hence l_1 must be the leader of C_2. □

Definition 6 Let D be a bond of admissible (G,p) and p be positive. Define p_D : $E(G) \to \mathcal{Z}^+$ by

$$p_D(e) = \begin{cases} p(e) - 1 & (e \in D) \\ p(e) & (e \notin D). \end{cases}$$

If (G, p_D) is still admissible, then D is said to be *removable.*

Removing bond D means reducing the weights of the edges in D by 1.

Lemma 6.2.2 *Let (G,p) be admissible and p be positive, and let $D = \{e_1, e_2, e_3\}$ be an arbitrary 3-bond. If at least two edges of D are leaders in (G,p), then D is removable.*

Proof: Since D is a 3-bond, $|D \cap C| = 0$ or 2 for any circuit C in G. For any non-tight circuit C of (G,p), $p_D(C) = p(C)$, or $p(C) - 2$, so $p_D(C)$ is even, and

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C$. Thus $p_D(C)$ is also balanced.

For any tight circuit C, if $C \cap D = \emptyset$, then $p_D(C) = p(C)$ is even and balanced. Otherwise, let $C \cap D = \{e_1, e_2\}$, and consider the following two cases.

Case 1: If one of e_1 and e_2 is a leader, then e_3 is a leader. Without loss of generality let e_1 be the leader, e_2 be the follower. Now we prove that e_1 is the leader of C. Suppose e_1 is not the leader of C. Let e be the leader of C so that $e \notin D$. Let C_1 and C_2 be the tight circuits in which e_1 and e_3 are the leaders, respectively. Then by Lemma 6.2.1, $e_2 \notin C_1$, so $D \cap C_1 = \{e_1, e_3\}$. By Lemma 6.2.1, $C \cap C_1 = \{e_1\}$, therefore again by Lemma 6.2.1, $e_1 \notin C_2$. Thus $D \cap C_2 = \{e_2, e_3\}$ and by Lemma 6.2.1, $C_1 \cap C_2 = \{e_3\}$. Thus e is also the leader of the tight circuit $C \triangle C_1$. But $C_2 \cap (C \triangle C_1) \supseteq \{e_2, e_3\}$, and by Lemma 6.2.1, e_3 is also the leader of $C \triangle C_1$, which is impossible.

Hence e_1 must be the leader of C, and

$$p_D(e_1) = p(e_1) - 1 \leq p(C - \{e_1\} - 1 = p_D(C - \{e_1\} + 1 - 1 = p_D(C - \{e_1\}) \text{ while}$$

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C - \{e_1\}$. Thus $p_D(C)$ is balanced and even.

Case 2: e_1 and e_2 are both leaders. Now we prove that one of e_1 and e_2 must be the leader of C. Suppose not, and let e be the leader of C so that $e \notin D$. Let C_1 and C_2 be the tight circuits in which e_1 and e_2 are the leaders, respectively. By Lemma 6.2.1, $C \cap C_1 = \{e_1\}$ and $C \cap C_2 = \{e_2\}$. Thus, $e_3 \in C_1 \cap C_2$, and e is also the leader of the tight circuit $C \triangle C_1$. But $C_2 \cap (C \triangle C_1) \supset \{e_2, e_3\}$, and by Lemma 6.2.1, e_2 is also the leader of $C \triangle C_1$, which is impossible.

Hence e_1 or e_2 is the leader of C, and by the same argument as that in Case 1, $p_D(C)$ is balanced and even. □

A *k-star bond* is a bond of size k with all its edges incident to a given point.

Lemma 6.2.3 *Let $(\mathbf{G}, p)$ be admissible and p be positive, and let $D = \{e_1, e_2, e_3, e_4\}$ be a 4-star bond. If at least three edges of D are leaders in $(\mathbf{G}, p)$, then D is removable.*

Proof: Since D is a star bond, then $|D \cap C| = 0$ or 2 for any circuit C in $\mathbf{G}$. For any non-tight circuit C of $(\mathbf{G}, p)$, $p_D(C) = p(C)$ or $p(C) - 2$, so $p_D(C)$ is even, and

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C$. Thus $p_D(C)$ is also balanced. For any tight circuit C, if $C \cap D = \emptyset$, then $p_D(C) = p(C)$ is even and balanced. Otherwise let $C \cap D = \{e_1, e_2\}$, and we consider the following two cases.

Case 1. Only one of e_1 and e_2 is a leader, so that e_3 and e_4 are leaders. Without loss of generality let e_1 be the leader and e_2 be the follower. Now we prove that e_1 is the leader of C. Suppose e_1 is not the leader of C. Let e be the leader of C, so that $e \notin D$. Let C_1, C_2 and C_3 be the tight circuits in which e_1, e_3 and e_4 are the leaders, respectively. By Lemma 6.2.1, $C \cap C_1 = \{e_1\}$, and $e_2 \notin C_1$. Without loss of generality, let $e_3 \in C_1$, implying e is also the leader of the tight circuit $C \triangle C_1$, and $\{e_2, e_3\} \subset C_1 \triangle C$. Since $e_1 \in C_1$, $e_2 \in C_1 \triangle C$, and C_1, $C_1 \triangle C$ are tight circuits, then by Lemma 6.2.1, e_1 and e_2 are not in C_2. Thus $e_4 \in C_2$ and by Lemma 6.2.1, $C_1 \triangle C_2 = \{e_3\}$, $(C_1 \triangle C) \cap C_2 = \{e_3\}$, so that e is the leader of the tight circuit $C_1 \triangle C \triangle C_2$, and $\{e_2, e_4\} \subset C_1 \triangle C \triangle C_2$. Since $e_2 \in C_1 \triangle C \triangle C_2$ and $e_3 \in C_2$, e_2 and e_3 are not in C_3, and therefore $e_1 \in C_3$. But $C_1 \cap C_2 = \{e_3\}$, so e_1 is the leader of the tight circuit $C_1 \triangle C_2$ and $\{e_1, e_4\} \subset C_1 \triangle C_2$. Thus $\{e_1, e_4\} \subseteq (C_1 \triangle C_2) \cap C_3$, and by

Lemma 6.2.1, e_1 is also the leader of C_3, which is impossible. Hence e_1 is the leader of C, and

$$p_D(e_1) = p(e_1) - 1 \leq p(C - \{e_1\} - 1 = p_D(C - \{e_1\} + 1 - 1 = p_D(C - \{e_1\}) \text{ while}$$

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C - \{e_1\}$. Thus $p_D(C)$ is balanced and even.

Case 2: e_1 and e_2 are both leaders. Now we prove that one of e_1 and e_2 must be the leader of C. Suppose not, and let e be the leader of C, so that $e \notin D$. Let C_1 and C_2 be the tight circuits in which e_1 and e_2 are the leaders respectively. By Lemma 6.2.1, $C \cap C_1 = \{e_1\}$ and $e_2 \notin C_1$, so e_3 or $e_4 \in C_1$. Without loss of generality, let $e_3 \in C_1$, so that e is the leader of the tight circuit $C \triangle C_1$, and $\{e_2, e_3\} \subset C \triangle C_1$. Since $\{e_1, e_2\} \subset C$ and $\{e_3, e_2\} \subset C \triangle C_1$, by Lemma 6.2.1, e_1 and e_3 are not in C_2, implying $e_4 \in C_2$. By Lemma 6.2.1, $C_2 \cap C = \{e_2\}$ and $C_2 \cap (C \triangle C_1) = \{e_2\}$. Therefore e is the leader of the tight circuit $C \triangle C_1 \triangle C_2$ and $\{e_3, e_4\} \subset C \triangle C_1 \triangle C_2$. Without loss of generality let e_3 be the leader and C_3 be the tight circuit in which e_3 is the leader. Then $(C \triangle C_1) = \{e_3, e_2\}$ and $(C \triangle C_1 \triangle C_2) = \{e_3, e_4\}$, so by Lemma 6.2.1, e_2 and e_4 are not in C_3. Therefore $e_1 \in C_3$. But $C_1 \cap C_3 = \{e_1, e_3\}$, by Lemma 6.2.1, e_1 is also the leader of C_3, which is impossible. Hence e_1 or e_2 is the leader of C, and by the same argument as that in **Case 1**, $p_D(C)$ is balanced and even. □

Lemma 6.2.4 *Let $(\mathbf{G}, p)$ be admissible, p be positive, and D be a star bond such that all the possible leaders of $(\mathbf{G}, p)$ are in D. Then D is removable.*

Proof: Since D is a star bond, then $|D \cap C| = 0$ or 2 for any circuit C in $\mathbf{G}$. If $C \cap D = \emptyset$, then $p_D(C) = p(C)$ is even and balanced. If $C \cap D \neq \emptyset$, then $|C \cap D| = 2$, and $p_D(C) = p(C) - 2$ is even. We need to consider two cases.

First, if C is non-tight, then

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C$. Thus $p_D(C)$ is also balanced.

Second, if C is tight, let l be the leader of C. Then $l \in D$ and $p(l) = p(C - \{l\})$ and $p(e) \leq p(C - \{e\}) - 2$ for all $e \in C - \{l\}$. Thus

$$p_D(l) = p(l) - 1 = p(C - \{l\}) - 1 = p_D(C - \{l\}) + 1 - 1 = p_D(C - \{l\}) \text{ while}$$

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\}).$$

for all $e \in C - \{l\}$. Hence p_D is admissible and D is removable. □

An edge $e = \{x, y\}$ is a *chord* of the circuit C if $e \notin E(C)$ yet $x, y \in V(C)$ are met.

Lemma 6.2.5 *Let $(\mathbf{G}, p)$ be admissible and p be positive, then every chord of a tight circuit C is a leader in $(\mathbf{G}, p)$.*

Proof: Let e be a chord of $C = P_1 \cup P_2$, where P_1 and P_2 are the two parts of C split by e, and let the leader l of C in P_1. Then

$$2p(l) = p(P_1) + p(P_2).$$

Since l is in circuit $P_1 \cup \{l\}$,

$$2p(l) \leq p(P_1) + p(e).$$

Therefore $p(e) \geq p(P_2)$. But e is in the circuit $P_2 \cup \{e\}$, so

$$p(e) \leq p(P_2).$$

Hence $p(e) = p(P_2)$ and e is the leader of tight circuit $P_2 \cup \{e\}$. □

Lemma 6.2.6 *Let $(\mathbf{G}, p)$ be admissible and p be positive. Let C be a circuit with chords such that at least one of the chords is a follower. For any $l \in C$, if $p(l) = p(C - \{l\}) - 2$, then l is a leader.*

Proof: Let e be a chord which is follower, let $C = P_1 \cup P_2$, where P_1 and P_2 are the two parts of C split by e, and let $l \in P_1$. Then

$$2p(l) = p(P_1) + p(P_2) - 2.$$

Since l is in circuit $P_1 \cup \{l\}$,

$$2p(l) \leq p(P_1) + p(e).$$

Therefore $p(e) \geq p(P_2) - 2$. But e is a follower in the circuit $P_2 \cup \{e\}$, so

$$p(e) \leq p(P_2) - 2,$$

and we have $p(e) = p(P_2) - 2$. Thus we have $2p(l) = p(P_1) + p(e)$, and therefore l is the leader of the tight circuit $P_1 \cup \{e\}$. □

From Lemma 6.2.5 and Lemma 6.2.6, we have the following corollary.

Corollary 6.2.1 *Let (G, p) be admissible p be positive and C be a circuit with chords such that at least one of the chords is a follower. Then C is non-tight, for every leader l in C, $p(l) \leq p(C - \{l\}) - 2$, and for every follower f in C, $p(f) \leq p(C - \{f\}) - 4$.*

Lemma 6.2.7 *Let (G, p) be admissible, p be positive, D be a bond such that all the leaders of (G, p) are in D, and C be a circuit such that $|C \cap D| \leq 2$. Then after removing D, $p_D(C)$ is still even and balanced.*

Proof: If $C \cap D = \emptyset$, then $p_D(C) = p(C)$ is even and balanced. If $|C \cap D| = 2$, then $p_D(C) = p(C) - 2$ is even. We need to consider two cases.

First, if C is non-tight, then

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C$. Thus $p_D(C)$ is also balanced.

Second, if C is tight, let l be the leader of C. Then $l \in D$, $p(l) = p(C - \{l\})$ and $p(e) \leq p(C - \{e\}) - 2$ for all $e \in C - \{l\}$. Thus

$$p_D(l) = p(l) - 1 = p(C - \{l\}) - 1 = p_D(C - \{l\}) + 1 - 1 = p_D(C - \{l\}) \text{ while}$$

$$p_D(e) \leq p(e) \leq p(C - \{e\}) - 2 \leq p_D(C - \{e\}) + 2 - 2 = p_D(C - \{e\})$$

for all $e \in C - \{l\}$. Hence p_D is admissible. □

Lemma 6.2.8 *Let (G, p) be admissible, p be positive, D be a bond, C be a non-tight circuit such that $|C \cap D| \leq 4$, and if $p(f) = p(C - \{f\}) - 2$ then $e \in C \cap D$ for every $e \in C$. Then after removing D, $p_D(C)$ is still even and balanced.*

Proof: Since C is circuit and D is bond, then $|C \cap D|$ is even, so $p_D(C)$ is even. For every $e \in C \cap D$, we have

$$p_D(e) = p(e) - 1 \leq p(C - \{e\}) - 2 - 1 \leq p_D(C - \{e\}) + 3 - 3 = p_D(C - \{e\}).$$

For every $f \in C - D$, we have

$$p_D(f) \leq p(f) \leq p(C - \{f\}) - 4 \leq p_D(C - \{f\}) + 4 - 4 = p_D(C - \{f\}).$$

Hence $p_D(C)$ is balanced. □

From Corollary 6.2.1 and Lemma 6.2.8, we have the following corollary.

Corollary 6.2.2 *Let (G, p) be admissible, p be positive, D be a bond such that all the leaders of (G, p) are in D, C be a circuit such that at least one of its chords is a follower, and $|C \cap D| \leq 4$. Then after removing D, $p_D(C)$ is still even and balanced.*

From Lemma 6.2.8 we have the following result.

Corollary 6.2.3 *Let (G,p) be admissible, p be positive, D be a bond, C be a 4-circuit, and $C \subseteq D$. If C is not tight, then after removing D, $p_D(C)$ is still even and balanced.*

6.3 The Bond Cover Property of $K_{3,3}$

Lemma 6.3.1 *$K_{3,3}$ has the bond cover property.*

Proof: Let $p: E(K_{3,3}) \to \mathcal{Z}^+$ be admissible. If there is an edge $e \in E(K_{3,3})$ with $p(e) = 0$, then we contract e to obtain a planar graph. By Corollary 1.1.1 and Proposition 6.1.1, $(K_{3,3}, p)$ has a bond cover.

We assume that p is positive and prove the result by finding a removable bond D, removing bond D and using induction on the new weighted $(K_{3,3}, p_D)$. If there are no edges which are leaders in $(K_{3,3}, p)$, then by Lemma 6.2.4 an arbitrary star bond is removable. If there is only one edge which is a leader in $(K_{3,3}, p)$, then by Lemma 6.2.4 the star bond containing this leader is removable. If there is a star bond which contains at least two leaders, then by Lemma 6.2.2 this star bond is removable.

So we assume that $K_{3,3}$ has at least two leaders no two of which are adjacent. By symmetry we have to check following two cases. In the following cases, we try to find a bond D and prove D is removable by proving $p_D(C)$ is balanced and even for every circuit C of $K_{3,3}$. Since all the possible leaders of $(K_{3,3}, p)$ are in the removable bond, by Lemma 6.2.7, we don't have to check 3-circuits and 4-circuits unless all the edges of a 4-circuit are in the

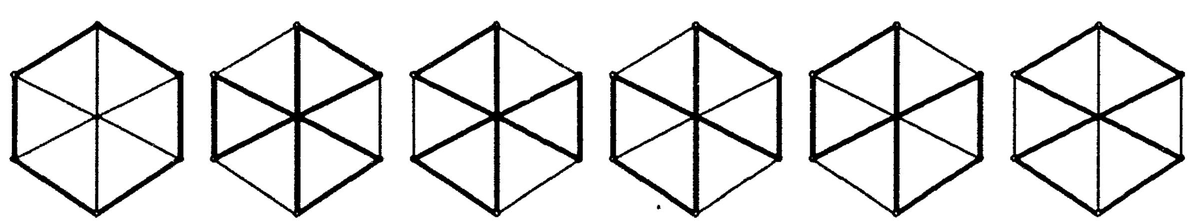


Fig. 11

removable bond. If so then by Corollary 6.2.3, we need to prove this 4-circuit is non-tight.

Fig. 11 above shows all the 6-circuits of $K_{3,3}$. Notice that each 6-circuit has 3 chords.

Case 1. $(K_{3,3}, p)$ **has exactly two non-adjacent leaders.** Let e_6 and e_8 be the leaders. Then a bond $D = \{e_2, e_6, e_7, e_8\}$ is removable.

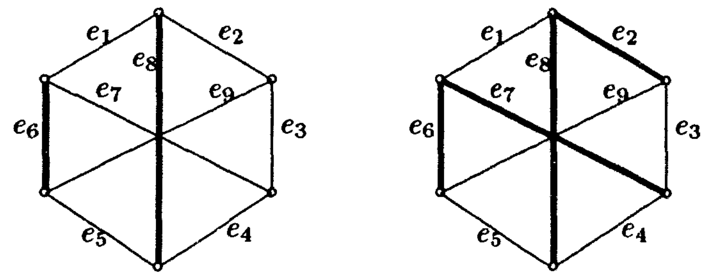


Fig. 12

Proof: For any 6-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$ then no leader can be a chord of C, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. Thus D is removable.

Case 2. $(K_{3,3}, p)$ **has exactly three non-adjacent leaders.**

Let e_3, e_6 and e_8 be the leaders. Then a bond $D = \{e_3, e_6, e_7, e_8, e_9\}$ is removable.

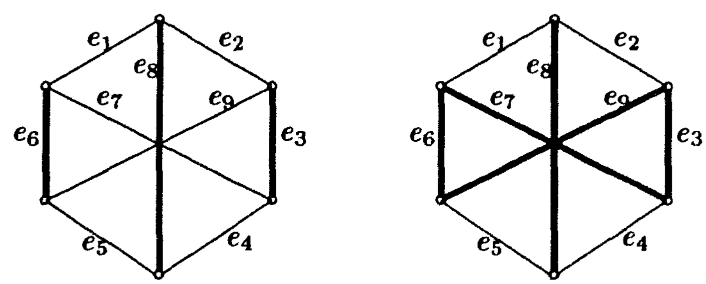


Fig. 13

Proof: For any 6-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C. But C has three chords, so at least two chords of C are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

By Corollary 6.2.3, it remains to show that the 4-circuit $C = \{e_3, e_6, e_7, e_9\}$ is non-tight. Suppose C is tight. Without loss of generality, let e_6 be the leader of C. Let C_1 be the tight circuit in which e_3 is its leader. By lemma 6.2.1, C_1 can only be $\{e_8, e_3, e_2, e_4\}$, and e_6 is the leader of tight tight circuit $C \triangle C_1$. But $C \triangle C_1$ is a 6-circuit with follower chords e_1 and e_5, and by Lemma 6.2.5, it can not be tight. Therefore C is non-tight, and by Corollary 6.2.3, $p_D(C)$ is even and balanced. Hence D is removable. □

6.4 The Bond Cover Property of V_6

Lemma 6.4.1 *V_6 has the bond cover property.*

Proof: Let $p : E(V_6) \to \mathcal{Z}^+$ be admissible. If there is an edge $e \in E(V_6)$ with $p(e) = 0$, then we contract e to obtain a planar graph. By Corollary 1.1.1 and Proposition 6.1.1, (V_6, p) has a bond cover.

We assume that p is positive and prove the result by finding a removable bond D, removing bond D and using induction on the new weighted (V_6, p_D). If there are no edges which are leaders in (V_6, p), then by Lemma 6.2.4 an arbitrary star bond is removable. If there is only one edge which is a leader in (V_6, p), then by Lemma 6.2.4 the star bond containing this leader is removable. If there is a 3-star bond which contains at least two leaders, then by Lemma 6.2.2 this star bond is removable. If there is a 4-star bond which contains at least three leaders, then by Lemma 6.2.3 this star bond is removable.

So we assume that V_6 has at least two leaders and no vertex of degree 3 adjacent to more than one leader, no vertex of degree 4 adjacent to more than two leaders, and by symmetry we have to check the following cases according to the number of leaders among the six edges $\{e_4, e_5, e_6, e_7, e_8, e_9\}$ in Fig. 10 in Page 30, of which at most two edges can be the leaders.

In the following cases, we try to find a bond D and prove D is removable by proving $p_D(C)$ is balanced and even for every circuit C of V_6. Since all the possible leaders of (V_6, p)

are in the removable bond, by Lemma 6.2.7, we don't have to check 3-circuits and 4-circuits unless all the edges of a 4-circuit are in the removable bond. If so then by Corollary 6.2.3, we need to prove this 4-circuit is non-tight.

The figures below are all the 5-circuits of V_6. Notice that each one has 2 chords.

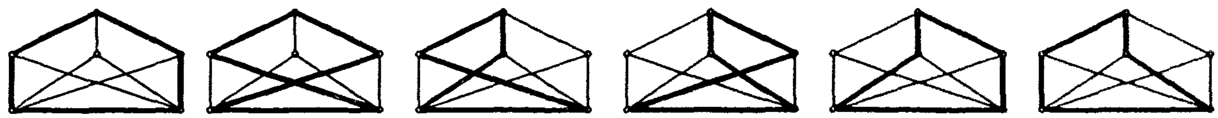


Fig. 14

The figures below are all the 6-circuits of V_6 and each one has 4 chords.

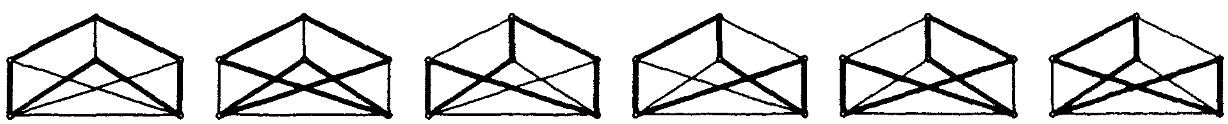


Fig. 15

Case 1. No edges in the six edges $\{e_4, e_5, e_6, e_7, e_8, e_9\}$ are leaders.

Then e_{10} is a possible leader, and only one of e_1, e_2 and e_3 can be a leader. Since they are identical, we let e_1 be the possible leader. A removable bond for this case is $D = \{e_1, e_5, e_6, e_{10}\}$.

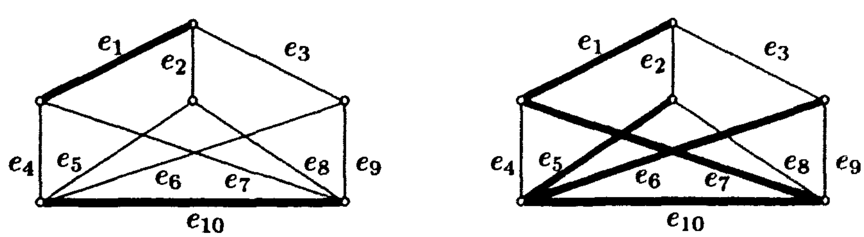


Fig. 16

Proof: For every 5- or 6-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C. But C has at least two chords, so at least one chord of C is a follower, by Corollary 6.2.2, $p_D(C)$ is even and balanced. Therefore D is removable.

Case 2. Exactly one edge of the six edges $\{e_4, e_5, e_6, e_7, e_8, e_9\}$ is the leader.

By symmetry let e_4 be the leader, implying that e_{10} is the possible leader and only one of e_2 and e_3 can be a leader, again by symmetry let e_3 be the possibe leader. A removable bond for this case is $D = \{e_3, e_4, e_5, e_9, e_{10}\}$.

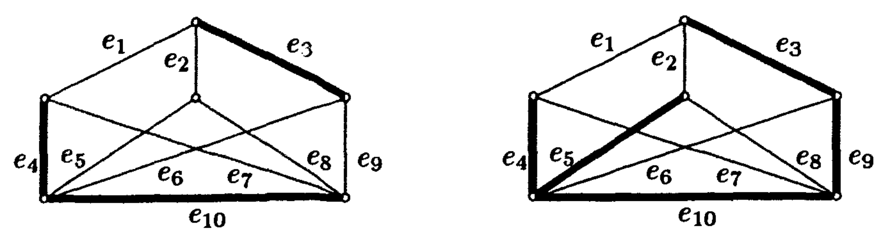


Fig. 17

The proof is the same as that in **Case 1**.

Case 3. Exactly two edges of the six edges $\{e_4, e_5, e_6, e_7, e_8, e_9\}$ are leders.

Case 3.1. The two leaders are adjacent.

By symmetry let e_4 and e_5 be the leaders, implying that e_3 is the only possible leader. A removable bond is $D = \{e_3, e_4, e_5, e_9, e_{10}\}$.

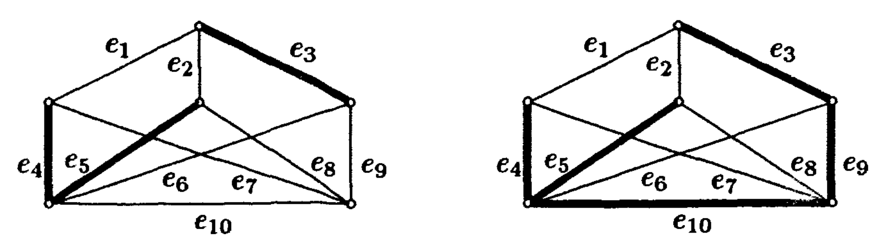


Fig. 18

The proof is the same as that in **Case 1**.

Case 3.2. The two leaders are not adjacent.

By symmetry let e_4 and e_9 be the leaders. Then only e_2 and e_{10} can be possible leaders. We need to consider the following two subcases.

Case 3.2.1. If e_2 is not a leader, then a removable bond is $D = \{e_3, e_4, e_5, e_9, e_{10}\}$.

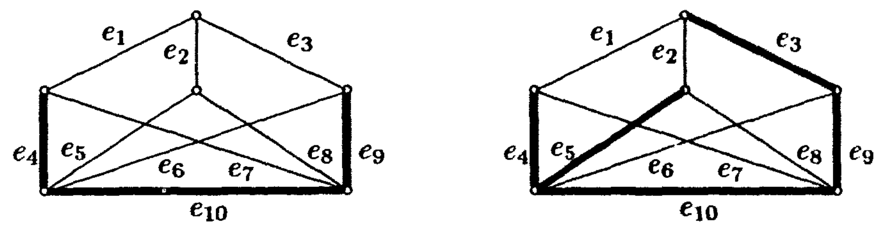


Fig. 19

The proof is the same as that in **Case 1.**

Case 3.2.2. If e_2 is a leader, then a removable bond is $D = \{e_1, e_2, e_4, e_5, e_9, e_{10}\}$.

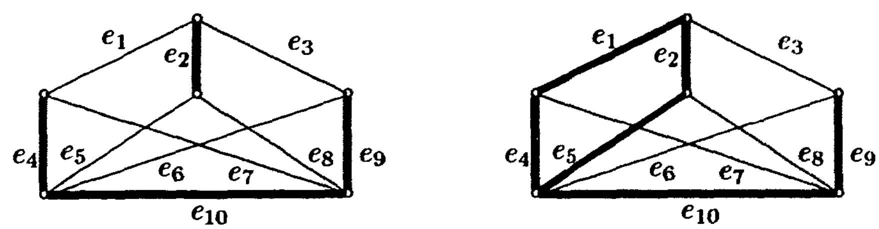


Fig. 20

Proof: Since $|D| = 6$ and D contains a 4-circuit, $|D \cap C| \leq 4$ for any circuit C in V_6.

For every 6-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most two leader can be chords of C. However, C has four chords, so at least two chords of C are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Except for the 5-circuit $C_1 = \{e_1, e_3, e_6, e_7, e_{10}\}$, every 5-circuit C has a follower chord. By Corollary 6.2.2, $p_D(C)$ is even and balanced. But $|C_1 \cap D| = 2$, and by Lemma 6.2.7, $p_D(C_1)$ is even and balanced.

Now we have to prove that the 4-circuit $C_2 = \{e_1, e_2, e_4, e_5\}$ is non-tight. Suppose C_2 is tight, we let e_4 be the leader of C_2 (if e_2 is the leader of C_2, the proof follows the same idea). Let C_3 be the tight circuit in which e_2 is the leader.

The 6-circuit $C_4 = \{e_1, e_3, e_5, e_6, e_7, e_8\}$ contains no leaders, and every other 6-circuit has a follower chord, so they can not be tight, thus C_3 can not be a 6-circuit. Except for the 5-circuit $C_1 = \{e_1, e_3, e_6, e_7, e_{10}\}$, every 5-circuit has a follower chord, so they can not be tight. Also, $e_2 \notin C_1$ and thus C_3 can not be a 5-circuit. By Lemma 6.2.1, C_3 can only be the 4-circuit $\{e_2, e_3, e_8, e_9\}$, and therefore e_4 is also the leader of the tight circuit $C_2 \triangle C_3 = \{e_1, e_3, e_4, e_5, e_8, e_9\}$. However, $C_2 \triangle C_3$ is a 6-circuit, so it can not be tight. Hence C_2 is non-tight, and by Corollary 6.2.3 $p_D(C_2)$ is even and balanced. Therefore D is removable. □

6.5 The Bond Cover Property of V_7

Lemma 6.5.1 *V_7 has the bond cover property.*

Proof: Let $p : E(V_7) \to \mathcal{Z}^+$ be admissible. If there is an edge $e \in E(V_7)$ with $p(e) = 0$, then we contract e and obtain either a planar graph, $K_{3,3}$ or V_6. By Corollary 1.1.1 and Lemma 6.3.1 or Lemma 6.4.1 and Proposition 6.1.1, (V_7, p) has a bond cover.

We assume that p is positive and prove the lemma by finding a removable bond D, removing bond D and using induction on the new weighted (V_7, p_D). If there are no edges which are leaders in (V_7, p), then by Lemma 6.2.4 an arbitrary star bond is removable. If there is only one edge which is leader in (V_7, p), then by Lemma 6.2.4 the star bond containing this leader is removable. If there is a 3-star bond which contains at least two leaders, then by Lemma 6.2.2 this star bond is removable. If there is a 4-star bond which contains at least three leaders, then by Lemma 6.2.3 this star bond is removable.

So we assume that V_7 has at least two leaders, no vertex of degree 3 adjacent to more than one leader and no vertex of degree 4 adjacent to more than two leaders. Therefore by symmetry we have to check the following cases according to the number of leaders among the four edges $\{e_2, e_6, e_{10}, e_{11}\}$ in V_7 in Fig. 10 in Page 30. At most two edges of them can be the leaders and the two leaders can not be adjacent.

The figures below are all the 5-circuits of V_7. The first four have one chord and the last two have no chord.

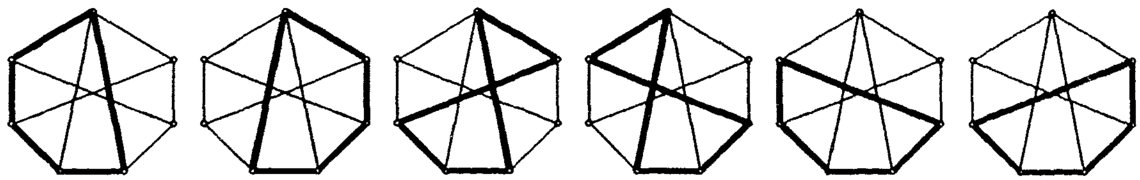


Fig. 21

The figures below are all the 6-circuits of V_7, each of which has two chords.

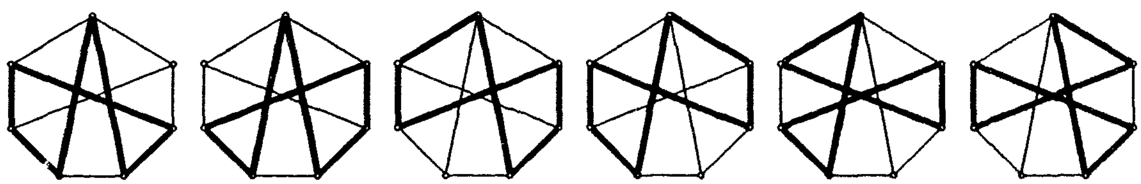


Fig. 22

The figures below are all the 7-circuits of V_7, each of which has four chords.

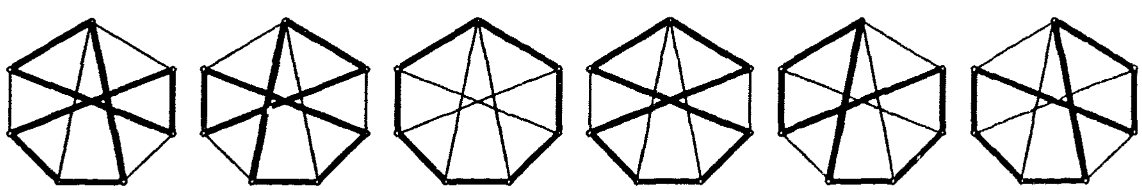


Fig. 23

In the following cases, we try to find a bond D and prove D is removable by proving $p_D(C)$ is balanced and even for every circuit C of V_7. Since all the possible leaders of (V_7, p) are in the removable bond, by Lemma 6.2.7, we don't have to check 3-circuits and 4-circuits unless all the edges of a 4-circuit are in the removable bond, and if so then by Corollary 6.2.3 we need to prove this 4-circuit is non-tight.

Case 1. No edges in $\{e_2, e_6, e_{10}, e_{11}\}$ are leaders.

Case 1.1. Let e_4 be the leader. Then e_1 and e_7 are possible leaders. A removable bond is $D = \{e_1, e_4, e_5, e_7, e_9\}$.

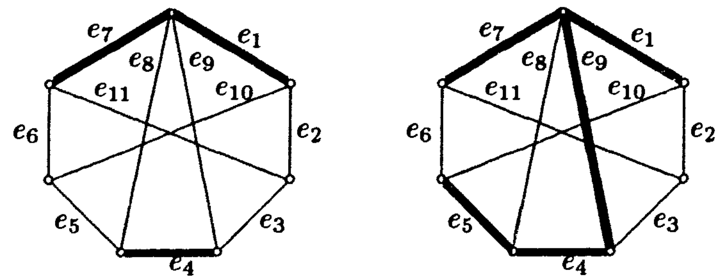


Fig. 24

Proof: Except for the 5-circuits $C_1 = \{e_7, e_9, e_4, e_5, e_6\}$ and $C_2 = \{e_1, e_{10}, e_5, e_4, e_9\}$, any 5-circuit C stisfies $|C \cap D| \leq 2$ and by Lemma 6.2.7, $p_D(C)$ is even and balanced. But $|C_1 \cap D| = 4$, $|C_2 \cap D| = 4$ and C_1, C_2 have a follower chord e_8, so by Corollary 6.2.2, $p_D(C_1)$ and $p_D(C_2)$ are even and balanced.

For any 6- or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C. But C has at least two chords, so at least one chord is a follower. By Corollary 6.2.2, $p_D(C)$ is even and balanced.

Case 1.2. e_4 is not a leader. We consider the following three subcases.

Case 1.2.1. If e_3 and e_5 are not leaders, then at most two of e_1, e_7, e_8 and e_9 can be leaders. By Lemma 6.2.4, the star bond $D = \{e_1, e_7, e_8, e_9\}$ is removable.

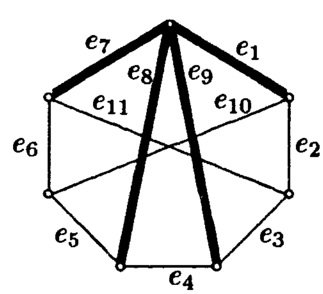


Fig. 25

Case 1.2.2. Only one of e_3 and e_5 is a leader. Since e_3 and e_5 are identical, let e_5 be the leader. Then e_1, e_7 and e_9 are possible leaders. A removable bond is $D = \{e_1, e_4, e_5, e_7, e_9\}$.

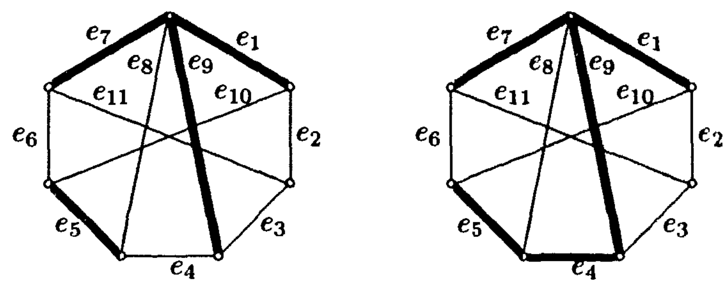


Fig. 26

The proof is the same as that in Case 1.1.

Case 1.2.3 If both of e_3 and e_5 are leaders, then e_1, and e_7 are possible leaders. A removable bond is $D = \{e_1, e_3, e_5, e_7\}$.

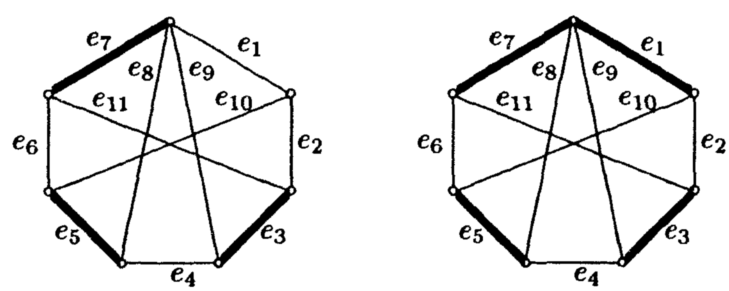


Fig. 27

Proof: For any 5-circuit C, $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced.

For any 6-circuit or any 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then no leader can be a chord of C, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Case 2. Exactly one of the edges $\{e_2, e_6, e_{10}, e_{11}\}$ is a leader. Let e_2 be the leader.

Case 2.1. If e_4 is the leader, then only e_7 can be a possible leader. A removable bond is $D = \{e_2, e_4, e_6, e_7, e_9\}$.

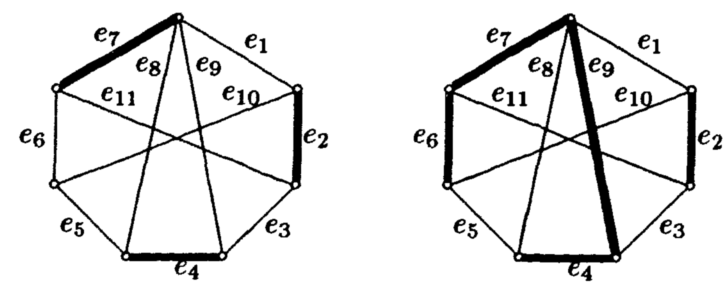


Fig. 28

Proof: Except for the 5-circuit $C_1 = \{e_6, e_7, e_9, e_4, e_5\}$, every 5-circuit C satisfies $|C \cap D| \leq 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. However $|C_1 \cap D| = 4$ and C_1 has a follower chord e_8, so by Corollary 6.2.2, $p_D(C_1)$ is even and balanced.

For any 6- or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C, but C has at least two chords, so at least one chord is a follower. By Corollary 6.2.2, $p_D(C)$ is even and balanced. Hence D is removable.

Case 2.2. The edge e_4 is not a leader. We consider the following two subcases.

Case 2.2.1. If e_5 is not a leader, only two of e_7, e_8 and e_9 can be leaders. A removable bond is $D = \{e_2, e_7, e_8, e_9, e_{10}\}$.

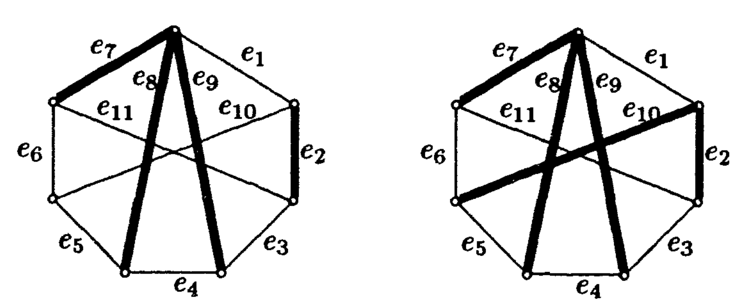


Fig. 29

The proof follows the same as that in **Case 1.2.3.**

Case 2.2.2 If e_5 is a leader, then only e_7 and e_9 are possible leaders.

Case 2.2.2.1. If e_7 is not a leader, then a removable bond is $D = \{e_2, e_5, e_8, e_9, e_{11}\}$.

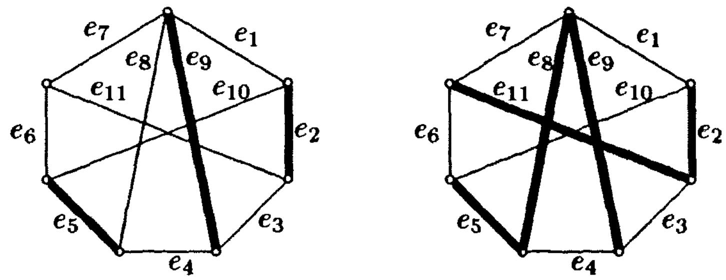


Fig. 30

The proof follows the same as that in **Case 1.2.3**.

Case 2.2.2.2. If e_7 is a leader, then a removable bond is $D = \{e_2, e_5, e_6, e_7, e_8, e_9\}$.

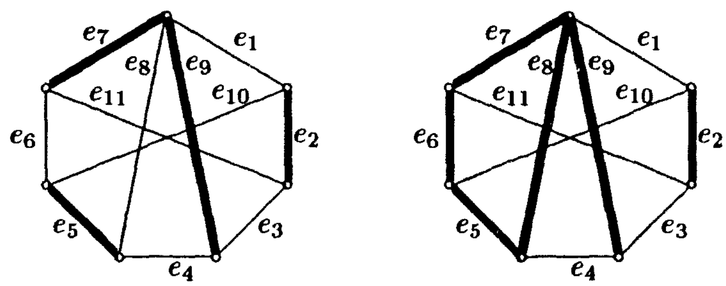


Fig. 31

Proof: Since $|D| = 6$ and D contains a 4-circuit, then $|C \cap D| \leq 4$ for every circuit C of V_7. Except for the 5-circuit $C_1 = \{e_4, e_5, e_6, e_7, e_9\}$, any other 5-circuit C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. But C_1 has a follower chord e_8, so by Corollary 6.2.2, $p_D(C_1)$ is even and balanced.

Except for the 6-circuit $C_2 = \{e_6, e_{11}, e_{10}, e_9, e_1, e_3\}$, any other 6-circuit C has a follower chord , so they are non-tight, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. But $|C_2 \cap D| = 2$, and by Lemma 6.2.7, $p_D(C_2)$ is even and balanced.

For any 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most two leaders can be a chord of C, but C has four chords, so at least two chords are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Now we have to prove that the 4-circuit $C_3 = \{e_5, e_6, e_7, e_8\}$ is non-tight. Suppose C_3 is tight. We let e_7 be the leader of C_3 (if e_5 is the leader of C_3, the proof follows the same idea). Let C_4 be the tight circuit in which e_5 is the leader. Only the 6-circuit $\{e_1, e_3, e_6, e_9, e_{10}, e_{11}\}$ can possibly be tight, but $e_5 \notin \{e_1, e_3, e_6, e_9, e_{10}, e_{11}\}$, and thus C_4 can not be a 6-circuit. Only the 7-circuit $\{e_1, e_3, e_4, e_6, e_8, e_{10}, e_{11}\}$ can possibly be tight, but $e_5 \notin \{e_1, e_3, e_4, e_6, e_8, e_{10}, e_{11}\}$, and thus C_4 can not be a 7-circuit. By Lemma 6.2.1, C_4 can only be one of the 5-circuits $\{e_2, e_3, e_4, e_5, e_{10}\}$ and $\{e_4, e_5, e_1, e_{11}, e_9\}$, but $\{e_4, e_5, e_1, e_{11}, e_9\}$ has a follower chord e_8 and can not be tight. Thus C_4 can only be $\{e_2, e_3, e_4, e_5, e_{10}\}$. Then e_7 is also the leader of the tight circuit $C_3 \triangle C_4$, but $C_3 \triangle C_4$ has follower chords e_1, e_{11} and e_{10}, and it is non-tight. Hence C_3 is non-tight, and by Corollary 6.2.3, $p_D(C_3)$ is even and balanced. Therefore D is removable.

Case 3. Only two of the edges $\{e_2, e_6, e_{10}, e_{11}\}$ are leaders. Let e_2 and e_6 be the two leaders.

Case 3.1. If e_4 is a leader, then there are no other leaders. A removable bond is $D = \{e_2, e_4, e_6, e_7, e_9\}$.

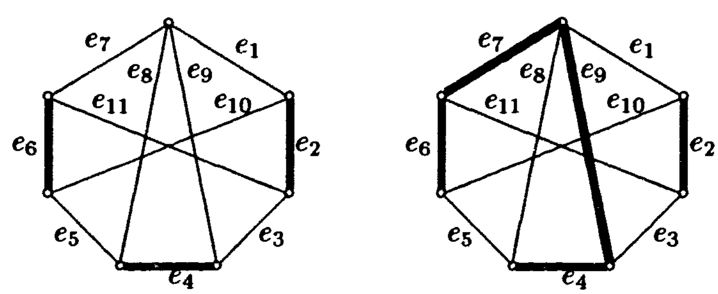


Fig. 32

The proof is the same argument as that in Case 2.1.

Case: 3.2 If e_4 is not a leader, then only e_8 and e_9 are possible leaders. A removable bond is $D = \{e_2, e_6, e_8, e_9, e_{10}, e_{11}\}$.

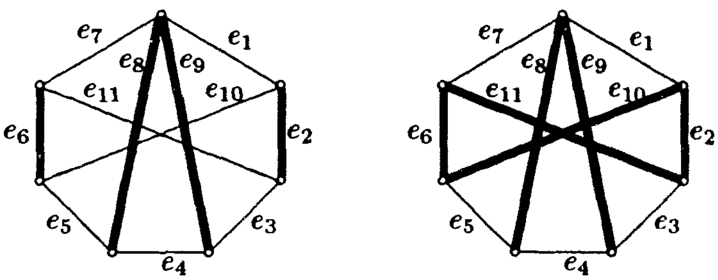


Fig. 33

Proof: Since $|D| = 6$ and D contains a 4-circuit, then $|C \cap D| \leq 4$ for every circuit C of V_7.

Any 5-circuit C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. Every 6-circuit C has a follower chord , so they are non-tight, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. For any 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most two leaders can be a chord of C, but C has four chords, so at least two chords are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Now we have to prove that the 4-circuit $C_1 = \{e_2, e_6, e_{10}, e_{11}\}$ is non-tight. Suppose C_1 is tight. We let e_2 be the leader of C_1 (if e_6 is the leader of C_1, the proof follows the same idea). Let C_2 be the tight circuit in which e_6 is the leader.
Every 6-circuit has follower chord, they can not be tight, thus C_3 can not be a 6-circuit. The 7-circuit $\{e_1, e_3, e_4, e_5, e_7, e_{10}, e_{11}\}$ contains no leaders, and every other 7-circuit has follower chord, so they can not be tight, thus C_2 can not be a 7-circuit. By Lemma 6.2.1, C_2 can only be the 4-circuit $C_3 = \{e_5, e_6, e_7, e_8\}$ or the 5-circuit $C_4 = \{e_4, e_5, e_6, e_7, e_9\}$. Therefore e_2 is also the leader of the tight circuit $C_1 \triangle C_3 = \{e_5, e_7, e_2, e_8, e_{10}, e_{11}\}$ or $C_1 \triangle C_4 = \{e_4, e_5, e_7, e_2, e_9, e_{10}, e_{11}\}$. However $C_1 \triangle C_3$ and $C_1 \triangle C_4$ are a 6-circuit and a 7-circuit respectively, and they can not be tight. Hence C_1 is non-tight, and by Corollary 6.2.3, $p_D(C_1)$ is even and balanced. Therefore D is removable. □

6.6 The Bond Cover Property of V_8

Lemma 6.6.1 *V_8 has the bond cover property.*

Proof: Let $p : E(V_8) \to \mathcal{Z}^+$ be admissible.
If there is an edge $e \in E(V_8)$ with $p(e) = 0$, then we contract e to obtain a planar graph or V_7. Therefore, by Corollary 1.1.1 or Lemma 6.5.1 and Proposition 6.1.1, (V_8, p) has a bond cover.

We assume that p is positive and prove the result by finding a removable bond D and removing bond D then using induction on the new weighted (V_8, p_D). If there are no edges which are leaders in (V_8, p), then by Lemma 6.2.4 an arbitrary star bond is removable. If there is only one edge which is a leader in (V_8, p), then by Lemma 6.2.4 the star bond containing this leader is removable. If there is a 3-star bond which contains at least two leaders, then by Lemma 6.2.2 this star bond is removable.

So we assume then that V_8 has at least two leaders and no vertex of degree 3 adjacent to more than one leader. Therefore, we have to check following cases.

In the following cases, we try to find a bond D and prove D is removable by proving $p_D(C)$ is balanced and even for every circuit C of V_8. Since all the possible leaders of (V_8, p) are in the removable bond, by Lemma 6.2.7, we don't have to check 3-circuits and 4-circuits unless all the edges of a 4-circuit are in the removable bond. If so then by Corollary 6.2.3, we need to prove this 4-circuit is non-tight.

Fig. 34 below is the collection of all the 5-circuits of V_8. They have no chords.

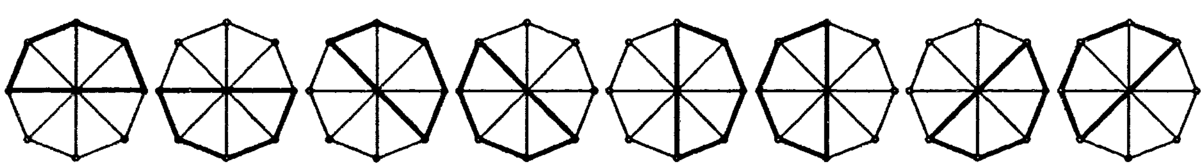


Fig. 34

Fig. 35 below is the collection of all the 6-circuit of V_8. Each has one chord.

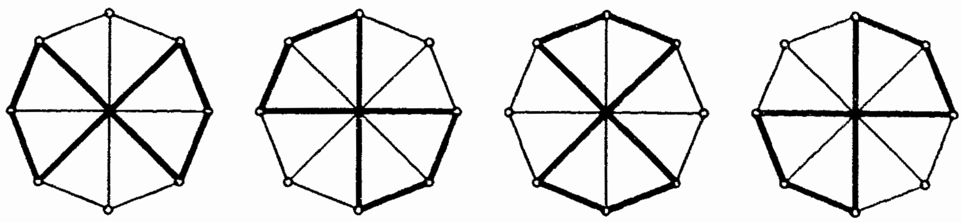


Fig. 35

Fig. 36 below is the collection of all the 7-circuit of V_8. Each has two chords.

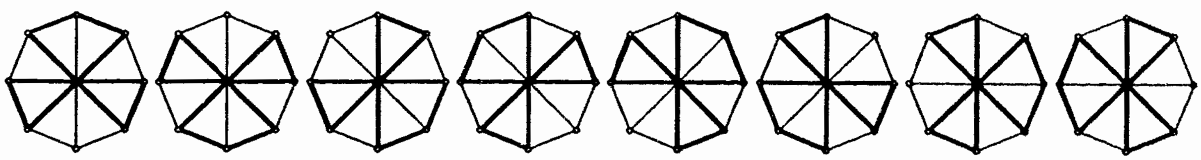


Fig. 36

Fig. 37 below is the collection of all the 8-circuit of V_8. Each has four chords.

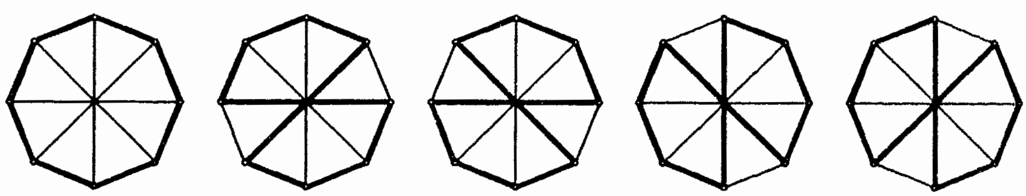


Fig. 37

(1) Exactly one leader on the rim.

By symmetry, we need only check one case.

Case 1.1: Let e_1 be the leader, then e_{11} and e_{12} are the possible leaders. A removable bond is $D = \{e_1, e_6, e_9, e_{11}, e_{12}\}$.

Proof: For any 8-circuit or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C, but C has at least two chords, implying that at least one chord is a follower, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

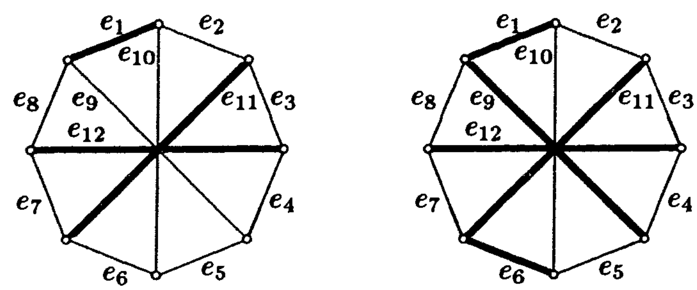


Fig. 38

Every 6-circuit C, other than $C_1 = \{e_1, e_2, e_5, e_6, e_9, e_{11}\}$, satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. But C_1 has a follower chord e_{10}, and by Corollary 6.2.2, $p_D(C_1)$ is even and balanced.

Since every 5-circuit C can use only one spoke and there are three spokes in D, $|C \cap D| \leq 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. Therefore D is removable.

(2) Exactly two leaders on the rim.

By symmetry, we have to check the following three cases.

Case 2.1. Let e_1 and e_7 be the leaders, so there are no other leaders. A removable bond is $D = \{e_1, e_3, e_5, e_7\}$.

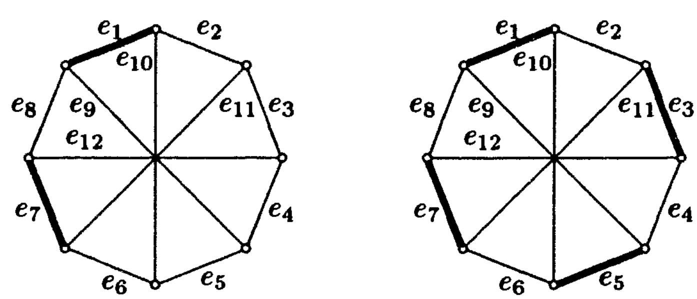


Fig. 39

Proof: For any 8-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then no leader can be a chord of C. But C has four chords, so the four chords are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Each 7-circuit C contains 3 spokes, 2 adjacent rim edges and 2 rim edges adjacent to each end of a spoke and on one side of this spoke. But D contains no such four rim edges, so $|C \cap D| \leq 2$, and thus by Lemma 6.2.7, $p_D(C)$ is even and balanced.

Each 6-circuit C contains 2 spokes and 2 pairs of 2 adjacent rim edges. But D contains no such four rim edges, so $|C \cap D| \leq 2$, and thus by Lemma 6.2.7, $p_D(C)$ is even and balanced.

Each 5-circuit C contains a spoke and all rim edges on one side of the spoke. There are only two edges of D on each side of each spoke, so $|C \cap D| = 2$, and thus by Lemma 6.2.7, $p_D(C)$ is even and balanced. Therefore D is removable.

Case 2.2. Let e_1 and e_5 be the leaders. Then e_{11} and e_{12} are possible leaders. A removable bond is $D = \{e_1, e_5, e_9, e_{10}, e_{11}, e_{12}\}$.

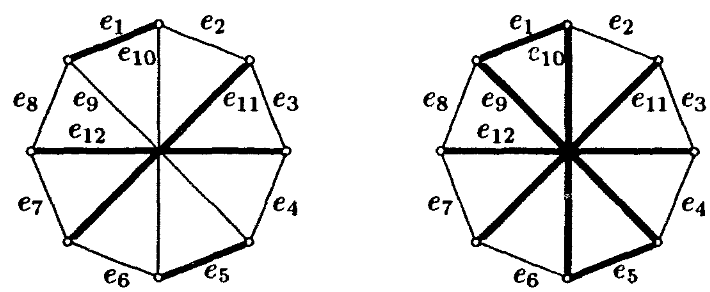


Fig. 40

Proof: Since $|D| = 6$ and D contains a 4-circuit, then $|C \cap D| \leq 4$ for every circuit C of V_6. For any 8-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most two leaders can be a chord of C. However, C has four chords, so at least two chords are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. The edges e_{11} and e_{12} can not be the chords of a 7-circuit, and e_1 and e_5 can not the chords of a 7-circuit at the same time. Thus every 7-circuit C has at least one follower chord, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. Except for the 6-circuits $C_1 = \{e_1, e_2, e_5, e_6, e_9, e_{11}\}$ and $C_2 = \{e_1, e_4, e_5, e_8, e_{10}, e_{12}\}$, any other two 6-circuits C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. But C_1 and C_2

have follower chords e_{10} and e_9, respectively. By Corollary 6.2.2, $p_D(C_1)$ and $p_D(C_2)$ are even and balanced. The edges e_1 and e_5 are not in the same side of any spoke, so for any 5-circuit C, $|C \cap D| = 2$. By Lemma 6.2.7, $p_D(C)$ is even and balanced.

Now we have to prove that the 4-circuit $C_3 = \{e_1, e_5, e_9, e_{10}\}$ is non-tight. Suppose C_3 is tight. We let e_1 be the leader of C_3 (if e_5 is the leader of C_3, the proof follows the same idea). Let C_4 be the tight circuit in which e_5 is the leader. Only the 8-circuit $\{e_2, e_3, e_4, e_6, e_7, e_8, e_9, e_{10}\}$ has no follower chords, but $\{e_2, e_3, e_4, e_6, e_7, e_8, e_9, e_{10}\}$ contains no leaders. Hence every 8-circuit is non-tight, and C_4 can not be a 8-circuit. Every 7-circuit has follower chord, so they are non-tight. Therefore C_4 can not be a 7-circuit. Also, e_5 is only in two 6-circuits C_1 and C_2, but they are not tight, so C_4 can not be a 6-circuit. By Lemma 6.2.1, C_4 can only be the 5-circuit $\{e_3, e_4, e_5, e_6, e_{11}\}$ or $\{e_4, e_5, e_6, e_7, e_{12}\}$. Then e_1 is also the leader of the tight circuit $C_3 \triangle C_4$ which is a 7-circuit. Thus it can not be tight. Hence C_3 is non-tight, and by Corollary 6.2.3, $p_D(C_3)$ is even and balanced. Thus D is removable.

Case 2.3. Let e_1 and e_4 be the leaders, so that e_{11} is the possible leader. A removable bond is $D = \{e_1, e_4, e_{10}, e_{11}, e_{12}\}$.

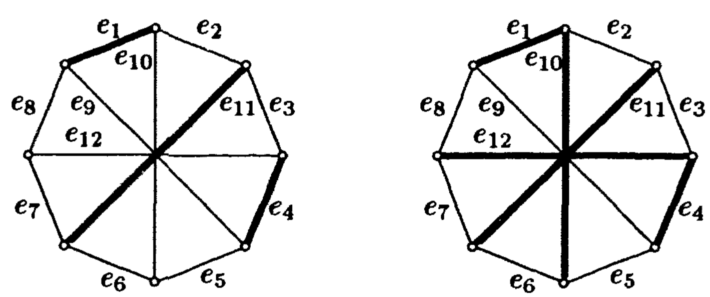


Fig. 41

Proof: For any 8-circuit or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C. However, C has at least two chords, so at least one chord is a follower, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

Except for the 6-circuit $C_1 = \{e_1, e_2, e_5, e_6, e_9, e_{11}\}$, any other 6-circuit C has a follower chord, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. But $|C_1 \cap D| = 2$, and by Lemma 6.2.7, $p_D(C_1)$ is even and balanced.

The edges e_1 and e_4 are in only one side of spoke e_9. As $e_9 \notin D$, every 5-circuit C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced. Therefore D is removable.

(3) Exactly three leaders on the rim.

By symmetry, we need to consider only two cases.

Case 3.1. Let e_1, e_3 and e_7 be the leaders, so there are no other leaders. A removable bond is $D = \{e_1, e_3, e_5, e_7\}$.

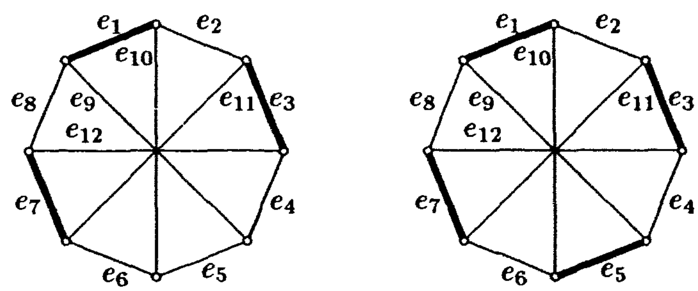


Fig. 42

The proof is the same as that in Case 2.1.

Case 3.2. Let e_1, e_4 and e_7 be the leaders, so there are no other leaders. A removable bond is $D = \{e_1, e_4, e_5, e_7, e_{12}\}$.

Proof: For any 8-circuit or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C. But C has at least two chords, so at least one chord is a follower, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. Since no spokes are leaders, every 6-circuit C has a follower chord, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. Except for the 5-circuit $C_1 = \{e_4, e_5, e_6, e_7, e_{12}\}$, every 5-circuit C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C)$ is even and balanced.

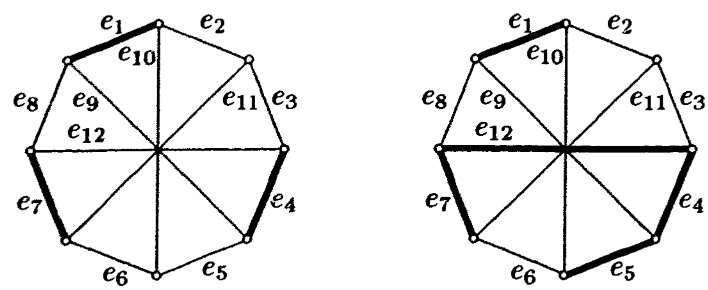


Fig. 43

Now we have to prove that C_1 is non-tight and $p(e_6) \leq p(C_1 - \{e_6\}) - 4$. Suppose C_1 is tight and let e_4 be the leader of C_1 (if e_7 is the leader of C_1, the prove follows the same idea). Let C_2 be the tight circuit in which e_7 is the leader. Since there are only three leaders, but every 8-circuit has 4 chords, at least one chord is a follower. Then every 8-circuit is non-tight, and thus C_2 can not be an 8-circuit. Also, C_2 can not be a 6-circuit. The two 7-circuits $\{e_2, e_3, e_5, e_8, e_9, e_{10}, e_{12}\}$ and $\{e_3, e_5, e_6, e_8, e_9, e_{11}, e_{12}\}$ contain no leaders and every other 7-circuit has follower chord. Hence they can not be tight and C_2 can not be a 7-circuit. By Lemma 6.2.1, C_2 can not be a 4-circuit or the 5-circuits $\{e_4, e_5, e_6, e_7, e_{12}\}$, $\{e_5, e_6, e_7, e_8, e_9\}$ and $\{e_1, e_{10}, e_6, e_7, e_8\}$. Thus C_2 can only be $\{e_1, e_2, e_{11}, e_7, e_8\}$, so that e_4 is the leader of the tight circuit $C_1 \triangle C_2$. Since $C_1 \triangle C_2$ is a 8-circuit, it can not be tight. Therefore C_1 is non-tight. Thus $p(e) = p(C_1 - \{e\}) - 2$ for every $e \in C_1$.

Suppose $p(e_6) = p(C_1 - \{e_6\}) - 2$. Let C_3 be a tight circuit in which e_4 is the leader. Then $p(e_4) = p(C_3 - \{e_4\})$. If $e_6 \in C_3$, then

$$p(e_6) < p(e_4) \leq p(C_1 - \{e_4\}) - 2 < p(C_1 - \{e_6\}) - 2.$$

This is a contradiction implying that $e_6 \notin C_3$. Therefore $e_6 \in C_1 \triangle C_3$.

Let $C_1 \cap C_3 = A \cap \{e_4\}$. Then

$$\begin{aligned} p(e_6) &\leq p(C_1 \triangle C_3 - \{e_6\}) - 2 \\ &= p(C_3) + p(C_1) - 2p(A) - 2p(e_4) - p(e_6) - 2 \\ &= p(C_1 - \{e_6\}) - 2 - 2p(A). \end{aligned}$$

Thus, $2p(A) \leq 0$, and since $p(A) \not< 0$, $A = \emptyset$. Hence C_3 can only be $\{e_1, e_2, e_3, e_4, e_9\}$.

Let C_4 be the tight circuit in which e_7 is the leader. Similar to above C_4 can only be $\{e_1, e_2, e_7, e_8, e_{11}\}$. Therefore $e_1 \in C_3$ and $e_1 \in C_4$. Any circuit C_5 containing e_1 must contain at least one of e_2, e_8 and e_9. Therefore $|C_5 \cap C_3| \geq 2$ or $|C_5 \cap C_4| \geq 2$, and by Lemma 6.2.1, e_1 can not be a leader, which is a contradiction. Hence, $p(e_6) \leq p(C_1 - \{e_6\}) - 4$. Thus, $p_D(C_1)$ is even and balanced. and D is removable.

(4) Exactly 4 leaders on the rim.

Case 4.1. Let e_1, e_3, e_5 and e_7 be the leaders. A removable bond is $D = \{e_1, e_3, e_5, e_7\}$.

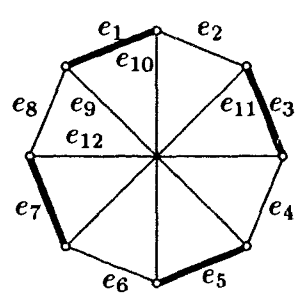


Fig. 44

The proof is the same as that in Case 2.1.

(5) No leaders on the rim.

Case 5.1. All the 4 spokes are leaders. A removable bond is $D = \{e_3, e_7, e_9, e_{10}, e_{11}, e_{12}\}$

Proof: Since $|D| = 6$ and D contains a 4-circuit, $|C \cap D| \leq 4$ for every circuit C of V_8. For any 8-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most two leaders can be a chord of C. As C has four chords, at least two chords are followers, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. Since spokes can not be the chords of a 7-circuit, every chord of any 7-circuit C is a follower, and by Corollary 6.2.2, $p_D(C)$ is even and balanced.

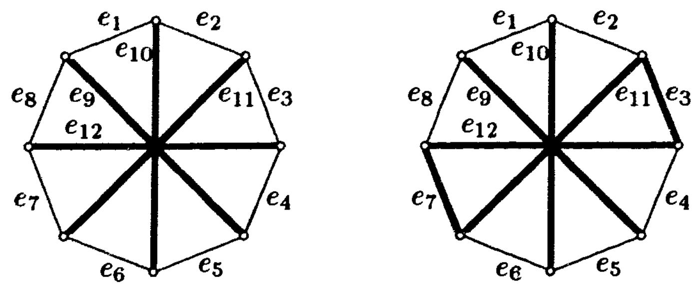


Fig. 45

Now we prove that every 6-circuit is not tight. Without loss of generality suppose $C_1 = \{e_2, e_3, e_6, e_7, e_{10}, e_{12}\}$ is tight, and e_{10} is the leader of C_1. Let C_2 be the circuit in which e_{12} is the leader. Now C_2 can not be an 8- or 7-circuit since every 8- and 7-circuit is non-tight. By Lemma 6.2.1, C_2 can not be the other 6-circuit which contains e_{12} or the two 5-circuits which contain e_{12}. Thus C_2 can only be the 4-circuit $\{e_4, e_8, e_9, e_{12}\}$, and then e_{10} is the leader of the tight circuit $C_1 \triangle C_2$ which is an 8-circuit, and can not be tight. Therefore, C_1 is not tight.

Here we prove that every 4-circuit is not tight. Without loss of generality suppose $C_3 = \{e_3, e_7, e_{11}, e_{12}\}$ is tight and e_{11} is the leader of C_3, and let C_4 be the tight circuit in which e_{12} is the leader. Then C_4 can not be an 8-,7-, or 6-circuit. By Lemma 6.2.1, C_4 can not be the two 5-circuits which contain e_{12}, so C_4 can only be the 4-circuit $\{e_4, e_8, e_9, e_{12}\}$. Then e_{11} is the leader of tight circuit $C_3 \triangle C_4$. As $C_3 \triangle C_4$ is a 6-circuit, it can not be tight. Therefore C_3 is not tight. Therefore by Corollary 6.2.3, $p_D(C_3)$ is even and balanced. Except for the 6-circuits C_1 and $C_5 = \{e_3, e_4, e_7, e_8, e_9, e_{11}\}$, any other 6-circuit C satisfies $|C \cap D| = 2$, and by Lemma 6.2.7, $p_D(C_3)$ is even and balanced.

Now we have to prove:

$$
\begin{aligned}
p(e_6) &\leq p(C_1 - \{e_6\}) - 4, \\
p(e_2) &\leq p(C_1 - \{e_2\}) - 4, \\
p(e_8) &\leq p(C_5 - \{e_8\}) - 4, \text{ and}
\end{aligned}
$$

$$p(e_4) \leq p(C_5 - \{e_4\}) - 4.$$

Here we only prove $p(e_6) \leq p(C_1 - \{e_6\}) - 4$. The other inequalities may be proved similarly. Let B_1 and B_2 be the tight circuits in which e_{10} and e_{12} are the leaders, respectively. If e_6 is in B_1 or B_2, then $p(e_6) < p(e_{10}) \leq p(C_1 - \{e_{10}\}) - 2 < p(C_1 - \{e_6\}) - 2$ or $p(e_6) < p(e_{12}) \leq p(C_1 - \{e_{12}\}) - 2 < p(C_1 - \{e_6\}) - 2$. Therefore, $p(e_6) \leq p(C_1 - \{e_6\}) - 4$ as required.

So now suppose e_6 is not in B_1 or B_2. Then B_1 can only be $\{e_2, e_3, e_4, e_5, e_{10}\}$ and B_2 can only be $\{e_1, e_2, e_3, e_8, e_{12}\}$.

If $p(e_6) = p(C_1 - \{e_6\} - 2$, then

$$\begin{aligned} p(e_6) &= p(e_2) + p(e_3) + p(e_7) + p(e_{10}) + p(e_{12}) - 2, \\ p(e_{12}) &= p(e_1) + p(e_2) + p(e_3) + p(e_8), \text{ and} \\ p(e_{10}) &= p(e_2) + p(e_3) + p(e_4) + p(e_5). \end{aligned}$$

Thus, $p(e_6) = p(e_1) + p(e_2) + p(e_3) + p(e_4) + p(e_5) + p(e_7) + p(e_8) + 2(p(e_2) + p(e_3)) - 2$. Since $e_6 \in \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$,

$$p(e_6) \leq p(e_1) + p(e_2) + p(e_3) + p(e_4) + p(e_5) + p(e_7) + p(e_8) - 2.$$

Therefore, $2(p(e_2) + p(e_3)) \leq 0$, but $p(e_2) + p(e_3) > 0$, which is a contradiction. Hence, $p(e_6) \leq p(C_1 - \{e_6\}) - 4$, so $p_D(C_1)$ and $p_D(C_5)$ are even and balanced as required.

Case 5.2. Three spokes are leaders. Let e_9, e_{10} and e_{11} be the leaders. A removable bond is $D = \{e_3, e_8, e_9, e_{10}, e_{11}\}$.

Proof: For any 8-circuit or 7-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then at most one leader can be a chord of C, but C has at least two chords, so at least one chord is a follower. By Corollary 6.2.2, $p_D(C)$ is even and balanced.

Except for the 6-circuit $C_1 = \{e_3, e_4, e_7, e_8, e_9, e_{11}\}$, any other 6-circuit C satisfies $|C \cap D| = 2$. By Lemma 6.2.7, $p_D(C)$ is even and balanced. But C_1 has a follower chord e_{12}, and

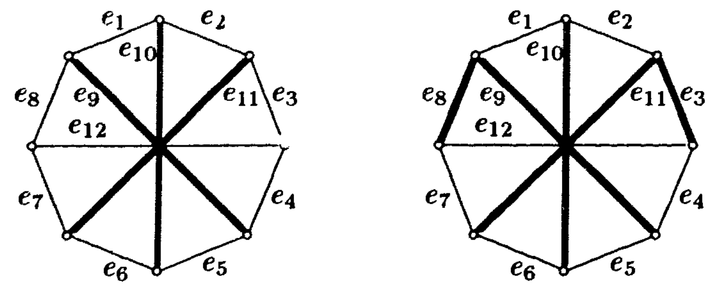


Fig. 46

by Corollary 6.2.2, $p_D(C_1)$ is even and balanced. The edges e_3 and e_8 are in only one side of spoke e_{12}, but $e_{12} \notin D$, so that for every 5-circuit C, $|C \cap D| \leq 2$. Then by Lemma 6.2.7, $p_D(C)$ is even and balanced. Therefore D is removable.

Case 5.3. Two neighbor spokes are leaders. Then e_9, and e_{10} are leaders. A removable bond is $D = \{e_2, e_8, e_9, e_{10}\}$.

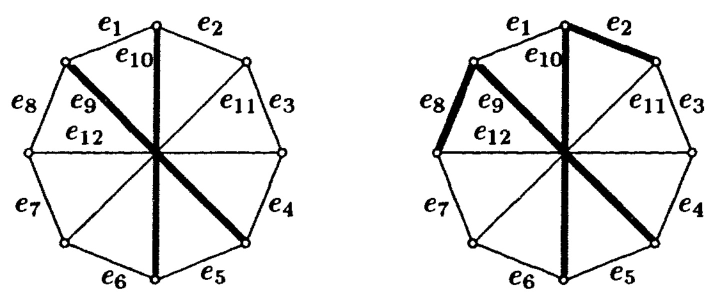


Fig. 47

Proof: For any 8-circuit, 7-circuit or 6-circuit C, if $|C \cap D| \leq 2$, then by Lemma 6.2.7, $p_D(C)$ is even and balanced. If $|C \cap D| = 4$, then no leader can be a chord of C, so every chord of C is a follower, and by Corollary 6.2.2, $p_D(C)$ is even and balanced. For any 5-circuit C, $|C \cap D| \leq 2$. By Lemma 6.2.7, $p_D(C)$ is even and balanced. Hence D is removable.

Case 5.4. Two non-neighbor spokes are leaders. Let e_{11} and e_9 be the leaders. A removable bond $D = \{e_1, e_6, e_9, e_{11}, e_{12}\}$.

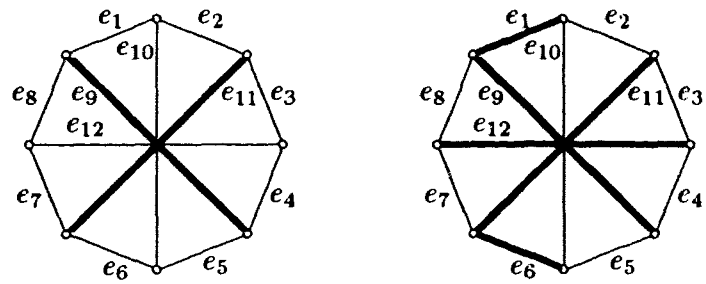


Fig. 48

The proof is the same as that in Case 1.1. □

Here we restate the dual form of Lemma 6.6.

Corollary 6.6.1 *$\mathcal{M}^*(V_8)$ has the circuit cover property.*

6.7 Summary

Propositions 1.1.2, 2.1.1, 2.2.1 and 2.2.2, together with Lemma 3.1.1 imply that if a binary matroid M has F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ or $\mathcal{M}(P_{10})$ as a minor, then it does not have the circuit cover property. Corollary 3.2.1, Lemmas 4.0.1, 4.0.2, 4.0.3, 5.0.5, and 6.6.1 imply that if a binary matroid M has no F_7^*, R_{10}, $\mathcal{M}^*(K_5)$ or $\mathcal{M}(P_{10})$ minor, then it has the circuit cover property. Thus we complete the proof.

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