THREE-DIMENSIONAL STEADY-STATE NORMAL INDENTATION PROBLEM FOR A GENERAL VISCOELASTIC MATERIAL

by

Qiang Lan
M.Sc., Fudan University, Shanghai, 1987

THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics and Statistics

SIMON FRASER UNIVERSITY
June, 1991

National Library of Canada
Acquisitions and Bibliographic Services Branch
395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale du Canada
Direction des acquisitions et des services bibliographiques
395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

ISBN 0-315-78215-3

Canada

Approval

Name : Qiang Lan

Degree : Master of Science

Title of Thesis : Three-Dimensional Steady-State Normal Indentation Problem for a General Viscoelastic Material

Examining committee :

Chairman: *Dr. A.H.Lachlan*

Dr. G.A.C.Graham

Senior Supervisor

Dr. R.W.Lardner

Dr. M.Singh

Dr. T.Tang

External Examiner
Department of Mathematics and Statistics
Simon Fraser University

Date Approved : July 25, 1991

Abstract

In this thesis, the three-dimensional steady-state normal contact problem for a general linear viscoelastic material is studied.

In Chapter 1, a brief review of the history of this topic is given and the specific problem to be solved is described.

In Chapter 2, solutions of the normal elastic contact problem due to Sneddon and Popov are presented. These solutions are used for solving the viscoelastic problem in Chapter 3.

Chapter 3, the main part of the thesis, consists of four sections. Section 1 describes viscoelastic material behaviour and states the Correspondence Principle. In Section 2, the viscoealstic contact problem is reduced to solutions of six integral equations; and the steady-state limit is derived. These integral equations have kernels that are infinite series of multiple integrals involving creep and relaxation functions for the material. In the case of a standard linear material, evaluation of these kernels can be reduced to summation of geometric series. This is done in Section 3. For more general material behaviour this method breaks down. However, in that case a method, previously used for crack problems, remains valid. That method, which expresses the kernels as solutions of other integral equations, is used in the present work to derive specific information on the solution of contact problems, for materials more general than the standard linear material. Section 4 contains the analytical details of this development. A specific model ($N = 2$) is studied in detail and the results of numerical calculations are presented. Results for the standard linear model (N=1), given in Section 3, are recovered as a special case.

Acknowledgements

I am deeply grateful to my supervisor, Dr. G.A.C. Graham, for suggesting the topic of this thesis and for all his ideas and guidance; without that this thesis would not have been written.

I would like to express my thanks to Mrs. S. Holmes and Mrs. M. Fankboner for their help, and to Simon Fraser University and the Department of Mathematics and Statistics for giving me the opportunity to study here.

Financial support received from Dr. Graham's research grant is also appreciated.

Table of Contents

List of Figures

Chapter 1

Introduction

In this paper, normal contact problems for general viscoelastic solids are studied.

Contact problems are sometimes called Hertz problems. In this paper, we consider the following Hertz problem: a rigid indentor(or punch) of axisymmetric curved form $S(r)$, pressed into a lubricated viscoelastic half-space, occupying the region $z > 0$, by a time-dependent normal load $W(t)$, where r and z are the usual cylindrical coordinates.

To solve this problem means to find the relationship among the total load $W(t)$, pressure distribution under the punch $p(\vec{r}, t)$ and the contact area $C(t)$. For the axisymmetric problem, we use $p(r, t)$ to express the pressure distribution and use $a(t)$ to specify the radius of circular contact area.

Contact problems, in some sense, are inherently nonlinear, even in linear elasticity; since for an indentor of curved profile, the contact area is unknown before the solution is known. They have provided a challenge to applied mathematicians ever since the work of Heinrich Hertz in 1880's. In 1882, Hertz successfully treated a static contact problem in elasticity. He considered the equilibrium of two curved elastic bodies pressed together and assumed that these two bodies can be replaced by two elastic half-spaces which are in contact only over the contact area C lying in the common tangent plane of these two bodies. Then he used Boussinesq's solution to obtain valid formulas for the pressure and indentation, confirming the assumption that the contact area is an ellipse. We can find additional details of Hertz' theory in a survey up to 1980 by Gladwell[2]. Since then many contributions to this field have been made. Up to 1960's, the two-dimensional elastic contact theory was well-established by Russian scholars. Muskhelishvili[16, 17] developed a systematic method to solve the two-dimensional contact problem. By using complex potentials and conformal maps, one of the most powerful method of mathematics, he cast them in the form of a Hilbert problem. Then he solved this problem by using the properties of Cauchy integrals. Galin [1] also considered a wide range of problems by casting them in the form of a Riemann-Hilbert Problem. Three-dimensional contact problems are more complicated. In fact, the only contact problem that allows an explicit analytic solution is the one with an elliptical contact area [1, 15, 18, 19, 21]. In Chapter 2, the details of Sneddon's work are given. From his work, we can see how the integral transform method, another powerful mathematical method, can be applied to the three-dimensional contact problem. The result for spherical indentor with large radius will be used for the viscoelastic case in Chapter 3. Popov [18, 19] established a relationship between the boundary displacement and the pressure distribution in terms of Lengendre polynomials. An account of his results is also given in Chapter 2 (see [2] for details).

As we know, the classical elastic problems can be divided into two kinds. For the first kind, either the forces or the dispacements are specified at all points of the boundary surfaces. The second kind consists of the mixed boundary value problems for which forces are given on part of the boundary surface(referred as B_1), while over the complementary boundary surface (referred as B_2) the displacement is given.

In viscoelasticity, the second kind boundary value problems are subdivided into two further classes depending on whether B_1 and B_2 are time-independent or time-dependent. An example of the problem with time-independent boundary regions is the indentation problem of a half-space by a flat-ended punch of circular section. An example of the problem with time-dependent boundary region is the contact problem for a spherical punch. The first kind of viscoelstic boundary value problems together with the second kind with *time-independent* boundary regions can be solved by employing the classical correspondence principle. We can reduce the time-dependent viscoelastic equations to a standard elastic form by taking Fourier tranforms over time t. Therfore the difficulties of this kind of problem are no more than that for the corresponding elastic ones. But for the problems involving time-dependent regions(such problems are called "essential viscoelastic problems" by Hunter [11]), we cannot use the classical correspondence principle. This is because there will be some points of the boundary at which the entire history of one type of boundary conditions is unavailable. This precludes the taking of Fourier transforms of the boundary conditions.

The "essential viscoelastic problems" remained untouched until 1960's. The problem of a rigid spherical indentor was first treated by Lee and Radok [13], In that solution, the radius $a(t)$ of the contact area is restricted to be a monotonically increasing function of time t. Later Hunter [10] extended this solution to the case in which $a(t)$ increases monotonically to a maximum and then decreases to zero monotonically. The idea for the case where $a(t)$ has any number of maxima and minima came from the works by Graham [7] and Ting [24, 25]. The strategy of their methods is to reduce the viscoelastic problems to a form analogous to the elastic equations by using a special decomposition, and then use the familiar elastic solutions to determine the viscoelastic quantities.

Recently, Golden and Graham [3] gave the steady-state solution to the problem of a rigid indentor, subject to normal periodic loading, on a viscoelastic half-space. Detailed solutions were given for the case where the half-space was assumed to respond as a standard linear solid. In that paper, only the plane strain problem was considered. The corresponding three-dimensional problem was discussed subsquently in [6, 8]. In these papers, it was shown that the contact pressure and indentation obeyed integral equations where the kernels are infinite sums of terms involving integrals of the viscoelastic functions. For a standard linear solid, the summations can be carried out without difficulty to give closed formula. For this case, considerable analytical progress can be made before resorting to numerical techniques of solution.

For more general materials, the infinite summations cannot be carried out in an elementary manner. However, in [5], this question was addressed in the context of a different problem, namely that of a fixed length crack in an infinite body under sinusoidal loading at infinity. In that paper, the authors showed that the kernels obey certain integral equations, whose solutions can be determined in closed form, at least for discrete spectrum models. The solution of these equations amounts to summing the infinite series. This method is extended to the contact problem in this thesis.

There are two parts in this paper. The first part Chapter 2 is devoted to the elastic problem, which was studied by several scholars. Here, I give the solution due to Sneddon. First the three-dimensional problem is reduced to a bihamonic equation subject to some boundary conditicns by introducing Love's strain function. Then we use Hankel transforms to transform this into a set of dual integral equations, which are solved by using some properties of Hankel and Fourier transforms. At the end of this chapter, an alternative solution to the same problem due to Popov is provided.

The second part, Chapter 3 of the thesis, deals with the viscoelastic contact problem. First of all in Section 3.1, viscoelastic material behaviour described by spectrum models is given and the classical correspondence principle for viscoelsticity is stated. In Section 3.2, the viscoealstic contact problem is reduced to the solution of six integral equations; and the steady-state limit is derived. These integral equations have kernels that are infinite series of multiple integrals involving creep and relaxation functions for the material. In the case of a standard linear material, evaluation of these kernels can be reduced to the summation of geometric series. This was done by Golden and Graham [3] and is given in Section 3.3. For more general material behaviour this method does not work. However, in that case a method, previously used for crack problems[5], remains valid. That method, which expresses the kernels as the solutions of other integral equations, is used in this work to derive specific information on the solution of contact problems, for materials more general than the standard linear material. Section 3.4 contains the analytical details of this development. A specific model ($N = 2$) is studied in detail and the results of numerical calculations are presented. Some general results and discussions are also given in Section 3.4.

Chapter 2

Three-Dimensional Contact Problems in Elasticity

Governing equations of linear isotropic and homogeneous elasticity are [23]

$$\varepsilon_{ij}(\vec{r}) = \frac{1}{2}(u_{i,j}(\vec{r}) + u_{j,i}(\vec{r})), \tag{2.0.1}$$

$$\sigma_{ij}(\vec{r}) = 2\mu\varepsilon_{ij}(\vec{r}) + \lambda\varepsilon_{kk}(\vec{r})\delta_{ij}, \tag{2.0.2}$$

and

$$\sigma_{ij,j}(\vec{r}) = 0, \tag{2.0.3}$$

with boundary conditions

$$\sigma_{ij}(\vec{r})n_j(\vec{r}) = T_i(\vec{r}), \quad \vec{r} \in B_1, \tag{2.0.4}$$

$$u_i(\vec{r}) = u_i^o(\vec{r}), \quad \vec{r} \in B_2, \tag{2.0.5}$$

in the Cartesian coordinates x_i, if there are no body forces, inertial forces are neglected and the summation convention is in force. Here λ and μ are Lame's constants, $\vec{r} = (x_1, x_2, x_3)$, u_i denotes the displacements, ε_{ij} and σ_{ij} are strain and stress components, respectively. $T_i(\vec{r})$ and $u_i^0(\vec{r})$ are applied tractions and displacements, respectively.

We only consider axisymmetric problems. By introducing Love's strain function $\Phi(r,z)$ [14] and using cylindrical coordinates (r,θ,z), we can express displacements by

$$u_r(r,z) = -\frac{\lambda+\mu}{\mu}\Phi_{rz}, \tag{2.0.6}$$

$$u_\theta(r,z) = 0, \tag{2.0.7}$$

$$u_z(r,z) = \frac{\lambda+2\mu}{\mu}\nabla^2\Phi - \frac{\lambda+\mu}{\mu}\Phi_{zz}. \tag{2.0.8}$$

Then strains are determined by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = -\frac{\lambda+\mu}{\mu}\Phi_{rrz}, \tag{2.0.9}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = -\frac{\lambda+\mu}{\mu}\frac{\Phi_{rz}}{r}, \tag{2.0.10}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\lambda+2\mu}{\mu}\nabla^2\Phi_z - \frac{\lambda+\mu}{\mu}\Phi_{zzz}, \tag{2.0.11}$$

$$\varepsilon_{r\theta} = \varepsilon_{z\theta} = 0, \tag{2.0.12}$$

$$\varepsilon_{rz} = \frac{1}{2}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}) = -\frac{\lambda+\mu}{\mu}\Phi_{rzz} + \frac{\lambda+2\mu}{\mu}\frac{\partial}{\partial r}(\nabla^2\Phi). \tag{2.0.13}$$

The corresponding stresses are obtained by direct substitution in equation (2.0.2):

$$\sigma_{rr} = \lambda\nabla^2\Phi_z - 2(\lambda+\mu)\Phi_{rrz}, \tag{2.0.14}$$

$$\sigma_{zz} = (3\lambda+4\mu)\nabla^2\Phi_z - 2(\lambda+\mu)\Phi_{zzz}, \tag{2.0.15}$$

$$\sigma_{\theta\theta} = \lambda\nabla^2\Phi_z - \frac{2}{r}(\lambda+\mu)\Phi_{zr}, \tag{2.0.16}$$

$$\sigma_{rz} = (\lambda+2\mu)\nabla^2\Phi_r - 2(\lambda+\mu)\Phi_{zzr}, \tag{2.0.17}$$

$$\sigma_{r\theta} = \sigma_{z\theta} = 0. \tag{2.0.18}$$

It may be shown that the first equation of (2.0.3) in cylindrical coordintes

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} + \frac{\partial\sigma_{rz}}{\partial z} = 0 \tag{2.0.19}$$

is automatically satisfied by such a selection of $\Phi(r,z)$ and the second one holds because of the axisymmetry while the third one

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{2.0.20}$$

is equivalent to

$$\nabla^4 \Phi = 0. \tag{2.0.21}$$

Then the problem is reduced to finding a biharmonic function $\Phi(r, z)$ which satisfies certain boundary conditions.

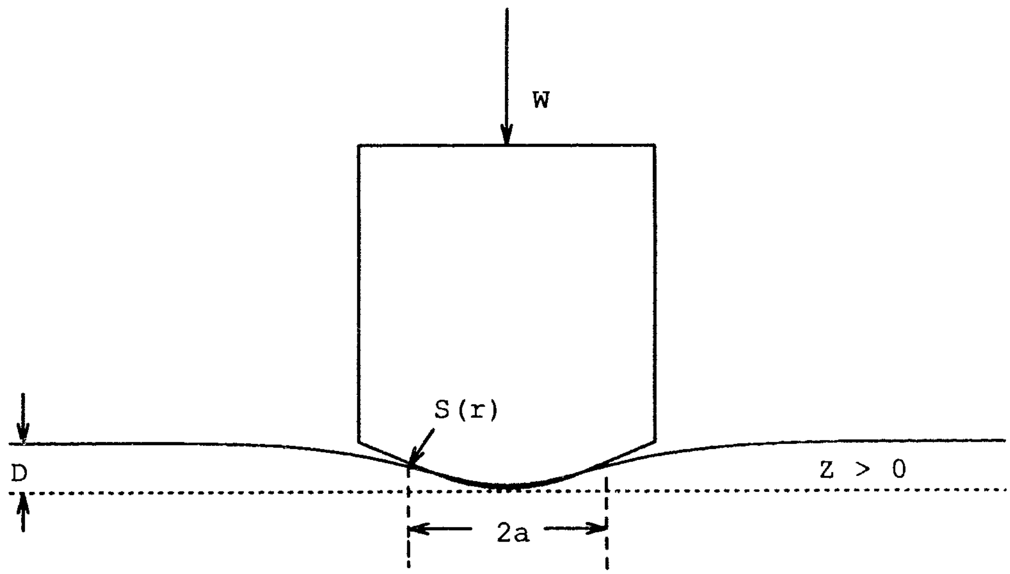

Figure 2.1: Contact problem for an axisymmetric punch with curved profile $S(r)$.

For the elastic contact problem to be considered, a smooth indentor of axisymmetric form $S(r)$ being pressed against the half-space $z > 0$ by a normal force W (see Fig.2.1), the boundary conditions are

$$u_z(r, 0) = D - S(r), \quad r \leq a, \tag{2.0.22}$$

$$\sigma_{rz}(r, 0) = 0, \quad r \geq 0, \tag{2.0.23}$$

$$\sigma_{zz}(r, 0) = 0, \quad r \geq a, \tag{2.0.24}$$

where a is the radius of the contact area, D is the indentation at $r = 0$ and $S(r)$ characterizes the shape of the punch, which satisfies $S(0) = 0$ and $S'(0) = 0$. In addition there must be no tensile stress under the punch and no contact when $r > a$.

Taking zero order Hankel Transforms of the both sides of equation (2.0.21) we obtain the ordinary differential equation [20](see Appendix for details)

$$(\frac{d^2}{dz^2} - s^2)^2\bar{\Phi}(s,z) = 0 \tag{2.0.25}$$

where $\bar{\Phi}(s,z)$[1] is the zero-order Hankel Tranform of the function $\Phi(r,z)$ given by

$$\bar{\Phi}(s,z) = \int_0^\infty rJ_0(sr)\Phi(r,z)dr. \tag{2.0.26}$$

The solution of (2.0.25) is

$$\bar{\Phi}(s,z) = (A + Bz)e^{sz} + (E + Fz)e^{-sz} \tag{2.0.27}$$

where A,B,E,F are functions of s.

Then, if we multiply both sides of (2.0.6) by $rJ_0'(sr)$ and integrate over r from 0 to ∞, we find

$$\int_0^\infty ru_r(r,z)J_1(sr)dr = \frac{\lambda+\mu}{\mu}s\frac{d\bar{\Phi}}{dz}, \tag{2.0.28}$$

after using equation (A.10) and integration by parts. Hankel inverse transform of the above equation gives us

$$u_r(r,z) = \frac{\lambda+\mu}{\mu}\int_0^\infty s^2\frac{d\bar{\Phi}}{dz}J_1(sr)ds. \tag{2.0.29}$$

Similarly we obtain the quantities of interest

$$u_z(r,z) = \int_0^\infty s(\frac{d^2\bar{\Phi}}{dz^2} - \frac{\lambda+2\mu}{\mu}s^2\bar{\Phi})J_0(sr)ds, \tag{2.0.30}$$

$$\sigma_{zz}(r,z) = \int_0^\infty s[(\lambda+2\mu)\frac{d^3\bar{\Phi}}{dz^3} - (3\lambda+4\mu)s^2\frac{d\bar{\Phi}}{dz}]J_0(sr)ds, \tag{2.0.31}$$

$$\sigma_{rz}(r,z) = \int_0^\infty s^2[\lambda\frac{d^2\bar{\Phi}}{dz^2} + (\lambda+2\mu)s^2\bar{\Phi}]J_1(sr)ds. \tag{2.0.32}$$

Considering that all the components of stress and displacement tend to zero as $z \to \infty$, and $\sigma_{rz}(r,0) = 0$ for all r, we have $A(s) = B(s) = 0$, and $E(s) = \frac{\lambda}{(\lambda+\mu)}F(s)/s$. Therefore equation (2.0.27) becomes to

$$\bar{\Phi}(s,z) = \frac{F(s)}{s}(\frac{\lambda}{\lambda+\mu} + sz)e^{-sz}. \tag{2.0.33}$$

[1] Here we are using a slightly different notation from that in Appendix.

Using the above equation and putting $z = 0$ in equations (2.0.30) and (2.0.31), gives

$$u_z(r,0) = -\frac{\lambda + 2\mu}{\mu} \int_0^\infty s^2 F(s) J_0(sr) ds, \tag{2.0.34}$$

$$\sigma_{zz}(r,0) = 2(\lambda + \mu) \int_0^\infty s^3 F(s) J_0(sr) ds. \tag{2.0.35}$$

Then the boundary conditions (2.0.22) and (2.0.24) yield the dual integral equations

$$-\frac{\lambda + 2\mu}{\mu} \int_0^\infty s^2 F(s) J_0(sr) ds = u_z(r,0), \;\; 0 \le r \le a, \tag{2.0.36}$$

$$\int_0^\infty s^3 F(s) J_0(sr) ds = 0, \;\; r \ge a. \tag{2.0.37}$$

These equations may be transformed into the form

$$\int_0^\infty \psi(p) J_0(px) dx = u_z(x,0) = D - S(x), \;\; 0 \le x \le 1, \tag{2.0.38}$$

$$\int_0^\infty p\psi(p) J_0(px) dx = 0, \;\; x \ge 1, \tag{2.0.39}$$

by introducing the new variables

$$x = r/a, \tag{2.0.40}$$

$$p = sa, \tag{2.0.41}$$

$$\psi(p) = -\frac{\lambda + 2\mu}{a^3 \mu} p^2 F(\frac{p}{a}). \tag{2.0.42}$$

Sneddon [22] solved this set of dual integral equations by using some relations between Hankel and Fourier transforms, which are listed in Appendix A.

Taking note of (A.20), equation (2.0.39) is automatically satisfied, if

$$\psi(p) = \int_0^1 X(t) cos(pt) dt, \tag{2.0.43}$$

and equation (2.0.38) is equivalent to the Abel integral equation

$$\int_0^x \frac{X(t)dt}{\sqrt{x^2 - t^2}} = D - S(x), \;\; 0 \le x \le 1. \tag{2.0.44}$$

The solution of this equation is given by(see Appendix)

$$X(t) = \frac{2D}{\pi} - \frac{2}{\pi}\frac{d}{dt} \int_0^t \frac{xS(x)dx}{\sqrt{t^2 - x^2}}. \tag{2.0.45}$$

Integrating by parts and then taking a derivative, one deduces that

$$\begin{aligned} X(t) &= \frac{2D}{\pi} - \frac{2}{\pi}\frac{d}{dt}\left[-S(0)t - \int_0^t \sqrt{t^2-x^2}S'(x)dx\right] \\ &= \frac{2D}{\pi} - \frac{2}{\pi}\left[S(0) + \int_0^t \frac{S'(x)tdx}{\sqrt{t^2-x^2}}\right]. \end{aligned} \tag{2.0.46}$$

From (2.0.35), we know

$$\begin{aligned} \sigma_{zz}(r,0) &= 2(\lambda+\mu)\int_0^\infty s^3F(s)J_0(sr)ds \\ &= -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)a}\int_0^\infty p\psi(p)J_0(px)dp \\ &= -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)a}\int_0^1 X(t)dt\int_0^\infty pcos(pt)J_0(px)dp. \end{aligned} \tag{2.0.47}$$

With the aid of (A.11), we obtain

$$\begin{aligned} \int_0^\infty pcos(pt)J_0(px)dp &= \frac{1}{x}\frac{d}{dx}\left[x\int_0^\infty cos(pt)J_1(px)dp\right] \\ &= \frac{1}{x}\frac{d}{dx}\int_x^1 \frac{tX(t)dt}{\sqrt{t^2-x^2}}. \end{aligned} \tag{2.0.48}$$

Here use has been made of equation (A.22). As before, integrating by parts and taking a derivative with respect to x and then substituting into equation (2.0.47), we get

$$\sigma_{zz}(r,0) = -\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)a}\left\{\frac{X(1)}{\sqrt{1-x^2}} - \int_x^1 \frac{X'(t)dt}{\sqrt{t^2-x^2}}\right\}. \tag{2.0.49}$$

To assure $\sigma_{zz}(r,0)$ is finite under the punch we must take

$$X(1) = 0. \tag{2.0.50}$$

Therefore, we have from (2.0.46) that

$$D = \int_0^1 \frac{S'(x)dx}{\sqrt{1-x^2}}, \tag{2.0.51}$$

considering that $S(0) = 0$. This formula relates D the depth of penetration of the tip of the indentor into the half-space, to a the radius of the circular area of contact. Now we can find, from (2.0.49), that the total load acting on the punch is

$$\begin{aligned} W &= -2\pi\int_0^a r\sigma_{zz}(r,0)dr \\ &= \frac{4\mu(\lambda+\mu)a\pi}{(\lambda+2\mu)}\int_0^1 xdx\int_0^\infty p\psi(p)J_0(px)dp. \end{aligned} \tag{2.0.52}$$

Changing the order of integration and noting that

$$xJ_0(x) = \frac{d}{dx}(xJ_1(x)), \tag{2.0.53}$$

we get

$$W = \frac{4\mu(\lambda+\mu)\pi a}{(\lambda+2\mu)} \int_0^\infty \psi(p)J_1(p)dp. \tag{2.0.54}$$

With the aid of (A.21) and (2.0.43), we can change the above equation to

$$W = \frac{4\mu(\lambda+\mu)\pi a}{(\lambda+2\mu)} \int_0^1 X(p)dp. \tag{2.0.55}$$

For the spherical indentor of large radius R, we have

$$S(r) = \frac{r^2}{2R} = \frac{a^2x^2}{2R}. \tag{2.0.56}$$

By using (2.0.51), one gets

$$D = \frac{a^2}{R}, \tag{2.0.57}$$

while (2.0.46) gives us

$$X(t) = \frac{a^2}{\pi R}(1-t^2). \tag{2.0.58}$$

Therefore, one deduces from equations (2.0.49) and (2.0.55) that

$$\begin{aligned} \sigma_{zz}(r,0) &= -\frac{8\mu(\lambda+\mu)}{\pi(\lambda+2\mu)R}\sqrt{a^2-r^2}, && (2.0.59) \\ W &= \frac{16\mu(\lambda+\mu)a^3}{3(\lambda+2\mu)R}. && (2.0.60) \end{aligned}$$

This is the solution given by Sneddon [22]. In that paper, equation (6.15) for the general spherical indentor should be written as

$$P = \frac{\mu}{1-\eta}\left[(a^2+R^2)log\left(\frac{R+a}{R-a}\right) - 2aR\right], \tag{2.0.61}$$

or

$$W = \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\left[(a^2+R^2)log\left(\frac{R+a}{R-a}\right) - 2aR\right], \tag{2.0.62}$$

in our notation here, instead of

$$P = \frac{\mu}{1-\eta}\left[(a^2+R^2)log\left(\frac{R+a}{R-a}\right) - aR\right]. \tag{2.0.63}$$

For the case when $R >> a$, equation (2.0.62) and

$$D = \frac{a}{2} log\left(\frac{R+a}{R-a}\right) \tag{2.0.64}$$

reduce to

$$D = \frac{a}{2}\left(\frac{2a}{R} + O(\frac{2a^3}{3R^3})\right), \tag{2.0.65}$$

$$W = \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{8a^3}{3R} + O(\frac{a^5}{R^3})\right) \tag{2.0.66}$$

by using Taylor's expansion and keeping the highest order terms. This corrects formula (30.41) in [21]. From equations (2.0.65) and (2.0.66), we can see that equation (2.0.57) and (2.0.60) are special cases of equations (2.0.62) and (2.0.64).

An alternative solution was given by Popov[18]. By using Boussinesq's formula he obtained the relationship between the boundary quantities

$$2\pi u_z(r,0) = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_C \frac{p(r')ds'}{|\vec{r'} - \vec{r}|_{z=0}}, \quad r \leq a \tag{2.0.67}$$

and showed that if the specified surface displacement is

$$u_z(r,0) = \frac{\pi(\lambda+2\mu)}{4\mu(\lambda+\mu)} a \sum_{n=0}^{N} a_n [P_{2n}(0)]^2 P_{2n}[(1-r^2/a^2)^{1/2}], \quad r \leq a \tag{2.0.68}$$

then the normal stress distribution is

$$p(r) = -\sigma_{zz}(r,0) = (1-r^2/a^2)^{-1/2} \sum_{n=0}^{N} a_n P_{2n}[(1-r^2/a^2)^{1/2}], \quad r \leq a \tag{2.0.69}$$

while the total load is

$$W = 2\pi \int_0^a sp(s)ds = 2\pi a^2 a_0 \tag{2.0.70}$$

where $P_n(x)$ are Legendre polynomials, which are defined by

$$P_0 = 1, \quad P_1 = x \tag{2.0.71}$$

and the recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{2.0.72}$$

For the spherical punch of large R, we know that

$$\begin{aligned} u_z(r,0) &= D - \frac{r^2}{2R} \\ &= \frac{\pi a(\lambda+2\mu)}{4\mu(\lambda+\mu)}\left\{a_0[P_0(0)]^2 + a_1[P_2(0)]^2 P_2[(1-r^2/a^2)^{1/2}]\right\} \\ &= \frac{\pi a(\lambda+2\mu)}{4\mu(\lambda+\mu)}\left\{a_0 + \frac{a_1}{4} - \frac{3a_1 r^2}{8a^2}\right\}, \quad 0 \le r \le a. \end{aligned} \tag{2.0.73}$$

This gives us that

$$\frac{\pi a(\lambda+2\mu)}{4\mu(\lambda+\mu)}a_0 + \frac{a^2}{3R} = D, \tag{2.0.74}$$

$$a_1 = \frac{16\mu(\lambda+\mu)a}{3\pi(\lambda+2\mu)R}. \tag{2.0.75}$$

Noting that $p(a) = 0$, one gets from (2.0.69)

$$a_0 - \frac{1}{2}a_1 = 0. \tag{2.0.76}$$

Then equations (2.0.74) ,(2.0.69) and (2.0.70) will give us the same result as (2.0.57), (2.0.59) and (2.0.60), which will be used later for the viscoelastic solution.

Chapter 3

Three-Dimensional Contact Problems in Viscoelasticity

3.1 Viscoelastic Material Behaviour and the Correspondence Principle

In viscoelasticity, all the field quantities depend on time t and the governing equations take the same form as the corresponding elastic ones[12]

$$\varepsilon_{ij}(\vec{r},t) = \frac{1}{2}(u_{i,j}(\vec{r},t) + u_{j,i}(\vec{r},t)), \tag{3.1.1}$$

$$\sigma_{ij,j}(\vec{r},t) = 0 \tag{3.1.2}$$

except for the constitutive equations. The latter are

$$\sigma_{ij}(\vec{r},t) = 2\int_{-\infty}^{t} dt'\mu(t-t')\varepsilon_{ij}(\vec{r},t') + \delta_{ij}\int_{-\infty}^{t} dt'\lambda(t-t')\varepsilon_{kk}(\vec{r},t') \tag{3.1.3}$$

where $\lambda(t)$ and $\mu(t)$ are related to the relaxation moduli in bulk and shear of an isotropic and homogeneous material, respectively. Taking time Fourier tranforms of these governing equations and the given boundary conditions

$$\sigma_{ij}(\vec{r},t)n_j(\vec{r}) = T_i(\vec{r},t), \quad \vec{r} \in B_1, \tag{3.1.4}$$

$$u_i(\vec{r},t) = u_i^o(\vec{r},t), \quad \vec{r} \in B_2, \tag{3.1.5}$$

for fixed regions B_1 and B_2, we obtain

$$\hat{\varepsilon}_{ij}(\vec{r},\omega) = \frac{1}{2}[\hat{u}_{i,j}(\vec{r},\omega) + \hat{u}_{j,i}(\vec{r},\omega)], \tag{3.1.6}$$

$$\hat{\mu}(\omega)\hat{u}_{i,jj}(\vec{r},\omega) + [\hat{\mu}(\omega) + \hat{\lambda}(\omega)]\hat{u}_{j,ji}(\vec{r},\omega) = 0, \tag{3.1.7}$$

$$\hat{\sigma}_{ij}(\vec{r},\omega) = 2\hat{\mu}(\omega)\hat{\varepsilon}_{ij}(\vec{r},\omega) + \delta_{ij}\hat{\lambda}(\omega)\hat{\varepsilon}(\vec{r},\omega), \tag{3.1.8}$$

and

$$\hat{\sigma}_{ij}(\vec{r},\omega) = \hat{T}_i(\vec{r},\omega), \quad \vec{r} \in B_1, \tag{3.1.9}$$

$$\hat{u}_i(\vec{r},\omega) = \hat{u}_i^0(\vec{r},\omega), \quad \vec{r} \in B_2, \tag{3.1.10}$$

where $\hat{\varepsilon}(\vec{r},\omega)$ denotes the Fourier transform of $\varepsilon(\vec{r},t)$, etc. The transformed equations have the same form as the governing equations of the corresponding elastic problem. If solutions to the elastic problem are known, one can obtain the solution of the viscoelastic problem by replacing the elastic constants by the complex moduli in the expressions for the displacements and stresses and then calculating the inverse transforms. For viscoelastic problems with time-independent boundary region, it is easy to do this. In this sense, viscoelastic boundary value problems are no more difficult than the corresponding elastic ones. This is the content of the classical correspondence principle(e.g. see [12]).

For convenience, in this paper, we adopt the proportionality assumption[4]

$$\lambda(t) = \frac{2\nu}{1-2\nu}\mu(t), \tag{3.1.11}$$

where ν is a a constant that plays the role of Poisson's ratio. Therefore sometimes this model is referred as unique Poisson's ratio model.

Later on we will use $l(t)$, a singular function, which is related to the relaxition moduli $\mu(t)$ by

$$l(t) = \frac{\mu(t)}{1-\nu}. \tag{3.1.12}$$

Its inverse $k(t)$ is chosen to satisfy

$$\int_{t_1}^{t_2} dt' l(t_2 - t')k(t' - t_1) = \int_{t_1}^{t_2} dt' k(t_2 - t')l(t' - t_1) = \delta(t_2 - t_1) \tag{3.1.13}$$

for any t_1, t_2 where $t_2 \geq t_1$, or in a more compact form

$$\int_0^t dt' l(t - t')k(t') = \int_0^t dt' k(t - t')l(t') = \delta(t). \tag{3.1.14}$$

Here $k(t)$ is also a singular functions, which is closely related to the creep functions of the material.

For the discrete spectrum model of a general linear viscoelastic material, $l(t)$ and $k(t)$ take the form

$$l(t) = l_0\delta(t) + \sum_{i=1}^{N} l_i e^{-\alpha_i t}, \tag{3.1.15}$$

$$k(t) = k_0\delta(t) + \sum_{i=1}^{N} k_i e^{-\beta_i t}. \tag{3.1.16}$$

In order to satisfy (3.1.14), the coefficients $l_i, k_i,\ i = 1, 2, \ldots, N$ must be related by[4]

$$l_0 k_0 = 1 \tag{3.1.17}$$

$$l_0 + \sum_{i=1}^{N} \frac{l_i}{\alpha_i - \beta_j} = 0,\ j = 1, 2, \ldots, N; \tag{3.1.18}$$

$$k_0 - \sum_{i=1}^{N} \frac{k_i}{\alpha_j - \beta_i} = 0,\ j = 1, 2, \ldots, N; \tag{3.1.19}$$

$$l_i = -\{\sum_{j=1}^{N} \frac{k_j}{(\alpha_i - \beta_j)^2}\}^{-1},\ i = 1, 2, \ldots, N; \tag{3.1.20}$$

$$k_i = -\{\sum_{j=1}^{N} \frac{l_j}{(\alpha_j - \beta_i)^2}\}^{-1},\ i = 1, 2, \ldots, N. \tag{3.1.21}$$

The case $N = 1$ corresponds to the standard linear viscoelastic solid

$$l(t) = l_0\delta(t) + l_1 e^{-\alpha t}, \tag{3.1.22}$$

$$k(t) = k_0\delta(t) + k_1 e^{-\beta t} \tag{3.1.23}$$

for which

$$l_0 k_0 = 1, \tag{3.1.24}$$

$$k_1 = -l_1 k_0^2, \tag{3.1.25}$$

$$\beta = \alpha + l_1 k_0 = \alpha - k_1/k_0. \tag{3.1.26}$$

3.2 Formulation of the Contact Problem in Viscoelasticity

In this thesis, we only consider certain viscoelastic contact problems that belong to the so called essential viscoelastic problem. Unfortunately the classical correspondence principle given in the last section cannot be applied to this kind of problem. To solve this problem, we use the ideas of Graham[7] and Ting[24] and try to change the surface quantity relationship into a form that is the same as the elastic one. Then the familiar elastic result given in Chapter 2 is used to solve the problem.

Using the proportionality assumption described in last section and replacing $u_i(\vec{r},t)$ in the viscoelastic equations (3.1.1) to (3.1.3) by the pseudodisplacements

$$v_i(\vec{r},t) = \int_{-\infty}^{t} dt' l(t-t') u_i(\vec{r},t), \tag{3.2.1}$$

we find that the viscoelastic equations are same as the elastic equations for a material with $\frac{\lambda+2\mu}{2\mu(\lambda+\mu)} = 1$. Therefore we obtain the viscoelastic analogue of the elastic result (2.0.67)

$$v(r,t) = \frac{1}{2\pi}\int_{C(t)} ds' \frac{p(r',t)}{|\vec{r}' - \vec{r}|}, \quad r \le a(t), \tag{3.2.2}$$

where

$$v(r,t) = \int_{-\infty}^{t} dt' l(t-t') u(r,t') \tag{3.2.3}$$

for our axisymmetric contact problem. Here $v(r,t)$ denotes the normal pseudodisplacement on the boundary surface $v_3(\vec{r},t)|_{z=0}$, $u(r,t)$ express the normal surface displacement $u_3(\vec{r},t)|_{z=0}$, and $\vec{r} = (x_1, x_2, 0)$, $r = \sqrt{x_1^2 + x_2^2}$, $\vec{r}' = (x_1', x_2', 0)$, $r' = \sqrt{x_1'^2 + x_2'^2}$ and

$p(r',t) = -\sigma_{zz}(r',t)$. Later on we will use all these notations except where stated otherwise. This displacement-traction relationship on the boundary will form the basis of the considerations of the thesis.

Now the displacement boundary condition, instead of (2.0.22), takes the form

$$u(r,t) = D(t) - S(r), \quad r \leq a(t). \tag{3.2.4}$$

The inverse of (3.2.3) can be written as

$$u(r,t) = \int_{-\infty}^{t} dt' k(t-t') v(r,t'). \tag{3.2.5}$$

We are interested in the case where the applied load is oscillating in magnitude, so that the contact area radius $a(t)$ will pass through a series of maxima and minima before the current time t.

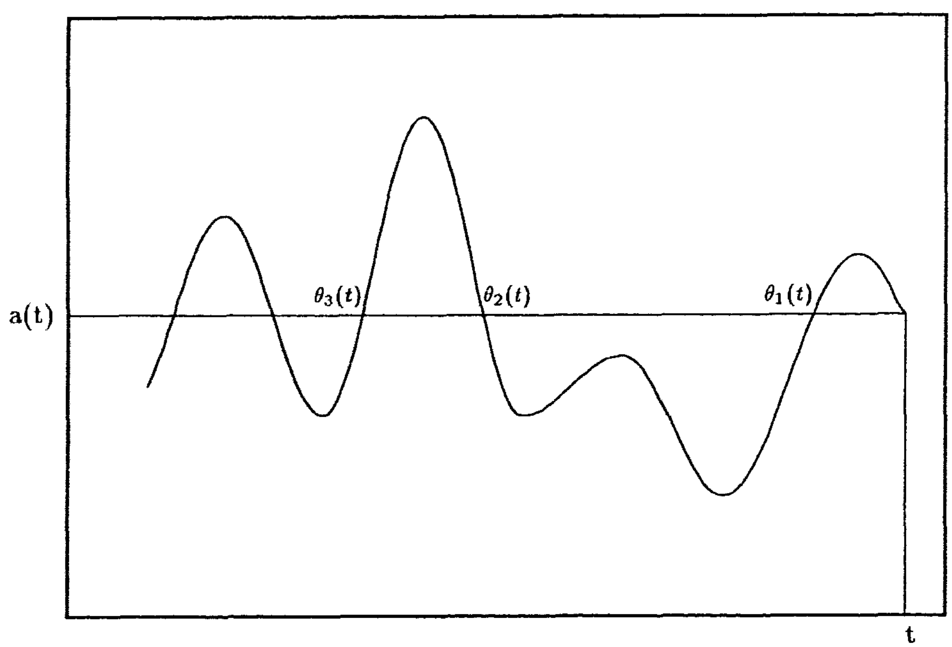

Figure 3.1: Typical distribution of $\theta_i(t)$, $i = 1, 2...$ when $a(t)$ is decreasing.

Now we consider first case in which the contact area is shrinking at the current time t, as shown in Fig.3.1. Let $S_G(t)$ be the set of all these time t' such that $C(t') \supseteq C(t)$, while $S_L(t)$ is its complement in $(-\infty, t]$. Using the method given by Golden and Graham[3], we can decompose $v(r,t)$ into integrals over these two sets. Consider times $\theta_i(t)$, $i = 1,2,3,...$ such that $\theta_1(t) > \theta_2(t) > \theta_3(t)...$ and

$$a(\theta_i(t)) = a(t). \tag{3.2.6}$$

Then we can write (3.2.3) as

$$v(r,t) = \int_{\theta_1}^{t} dt' l(t-t')u(r,t') + \int_{-\infty}^{\theta_1} dt' l(t-t')u(r,t') \tag{3.2.7}$$

where the second term can be rewritten in the form

$$\int_{-\infty}^{\theta_1} dt'' l(t-t'') \int_{-\infty}^{t''} dt' k(t" - t')v(r,t') = \int_{-\infty}^{\theta_1} dt' T_1(t,t')v(r,t'), \tag{3.2.8}$$

$$T_1(t,t') = \int_{t'}^{\theta_1} dt'' l(t-t'')k(t''-t'). \tag{3.2.9}$$

The same processure can be applied to this second term, where now the split is into an integral over $[\theta_2(t), \theta_1(t)]$ and $(-\infty, \theta_2(t)]$. This can be done repeatly to give the final decomposition

$$v(r,t) = \int_{S_G(t)} dt' \Pi_G(t,t')u(r,t') + \int_{S_L(t)} dt' \Pi_L(t,t')v(r,t') \tag{3.2.10}$$

where

$$\Pi_C(t,t') = T_0(t,t')R(t';\theta_1(t),t) + T_2(t,t')R(t';\theta_3(t),\theta_2(t)) + ..., \tag{3.2.11}$$

$$\Pi_L(t,t') = T_1(t,t')R(t';\theta_2(t),\theta_1(t)) + T_3(t,t')R(t';\theta_4(t),\theta_3(t)) + ... \tag{3.2.12}$$

Here functions $T_i(t,t')$ are defined as follows

$$T_0(t,t') = l(t-t'), \tag{3.2.13}$$

$$T_i(t,t') = \int_{t'}^{\theta_i(t)} dt'' T_{i-1}(t,t'')k(t''-t'), \quad i \ odd, \tag{3.2.14}$$

$$= \int_{t'}^{\theta_i(t)} dt'' T_{i-1}(t,t'')l(t''-t'), \quad i \ even, \tag{3.2.15}$$

while function $R(t;t_2,t_1)$ is defined as

$$R(t;t_2,t_1) = \begin{cases} 1, & t \in [t_2,t_1] \\ 0, & t \notin [t_2,t_1] \end{cases} \tag{3.2.16}$$

for all t_2,t_1,t.

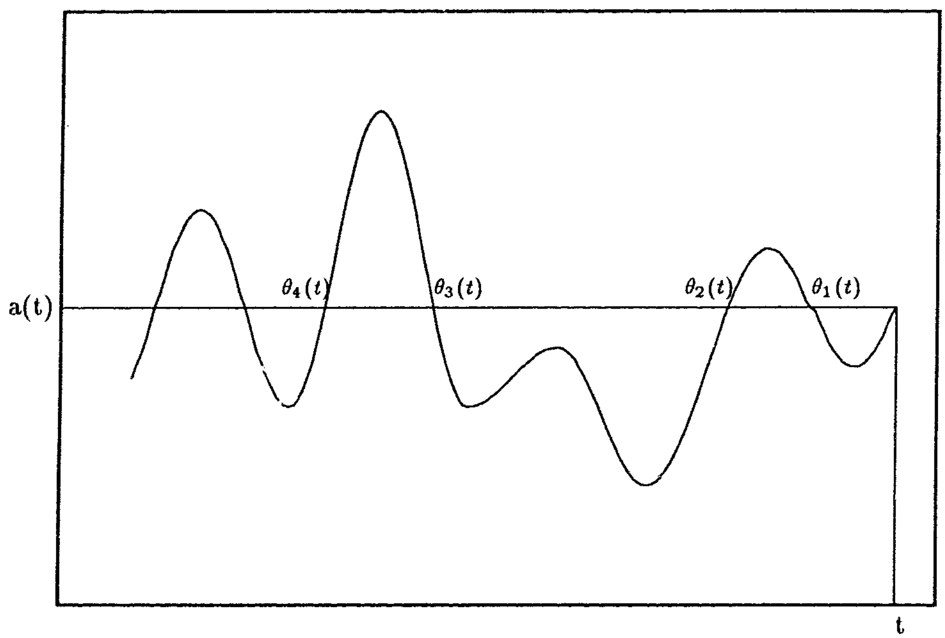

Figure 3.2: Typical distribution of $\theta_i(t)$, $i = 1,2...$ when $a(t)$ is increasing.

If $a(t)$ is increasing at time t(Fig.3.2), we obtain in a similar manner the decomposition of equation (3.2.5)

$$u(r,t) = \int_{S_G(t)} dt'\Gamma_G(t,t')u(r,t') + \int_{S_L(t)} dt'\Gamma_L(t,t')v(r,t') \tag{3.2.17}$$

where

$$\Gamma_L(t,t') = N_0(t,t')R(t';\theta_1(t),t) + N_2(t,t')R(t';\theta_3(t),\theta_2(t)) + ..., \tag{3.2.18}$$
$$\Gamma_G(t,t') = N_1(t,t')R(t';\theta_2(t),\theta_1(t)) + N_3(t,t')R(t';\theta_4(t),\theta_3(t)) + ... \tag{3.2.19}$$

and functions $N_i(t,t')$ are given by

$$\begin{aligned}
N_0(t,t') &= k(t-t'), && (3.2.20)\\
N_i(t,t') &= \int_{t'}^{\theta_i(t)} dt'' N_{i-1}(t,t'')l(t''-t'), \quad i \ odd, && (3.2.21)\\
&= \int_{t'}^{\theta_i(t)} dt'' N_{i-1}(t,t'')k(t''-t'), \quad i \ even. && (3.2.22)
\end{aligned}$$

Consider equation (3.2.10) for the time when $a(t)$ is decreasing. According to the definition of $S_G(t)$, we know that r is in the contact area for any time $t' \in S_G(t)$ if it is there at time t. Therefore $u(r,t')$ is known to be $D(t') - S(r)$ for any time $t' \in S_G(t)$ and the first term in equation (3.2.10) now becomes

$$\begin{aligned}
\int_{S_G(t)} dt'\Pi_G(t,t')u(r,t') &= \int_{S_G(t)} dt'\Pi_G(t,t')(D(t') - S(r))\\
&= D_c(t) - S(r)\Pi_G(t) \qquad (3.2.23)
\end{aligned}$$

for r belonging to the contact area at time t, where

$$\begin{aligned}
D_c(t) &= \int_{S_G(t)} dt'\Pi_G(t,t')D(t'), && (3.2.24)\\
\Pi_G(t) &= \int_{S_G(t)} dt'\Pi_G(t,t'). && (3.2.25)
\end{aligned}$$

Furthermore, for time $t' \in S_L(t)$, the contact region $C(t')$ is always contained in $C(t)$. Hence we can interchange the time and space integration to put the second integral of equation (3.2.10) in the form

$$\int_{S_L(t)} dt'\Pi_L(t,t')v(r,t') = \frac{1}{2\pi}\int_{C(t)} ds' \frac{q_c(r',t)}{|\vec{r'} - \vec{r}|}, \quad r \le a(t), \qquad (3.2.26)$$

where

$$q_c(r',t) = \int_{S_L(t)} dt'\Pi_L(t,t')p(r,t'). \qquad (3.2.27)$$

Therefore equation (3.2.10) becomes

$$v(r,t) = v_c(r,t) + \frac{1}{2\pi}\int_{C(t)} ds' \frac{q_c(r',t)}{|\vec{r'} - \vec{r}|}, \quad r \le a(t), \qquad (3.2.28)$$

where

$$v_c(r,t) = D_c(t) - S(r)\Pi_G(t). \tag{3.2.29}$$

Substituting (3.2.2) into (3.2.28) gives us

$$v_c(r,t) = \frac{1}{2\pi}\int_{C(t)} ds' \frac{p(r,t) - q_c(r',t)}{|\vec{r'} - \vec{r}|}, \quad r \le a(t). \tag{3.2.30}$$

If there is a quantity $D_e(t)$ (we will see its physical meaning later) such that

$$D_c(t) = \int_{S_G(t)} dt' \Pi_G(t,t') D(t') = \Pi_G(t) D_e(t), \tag{3.2.31}$$

then we have

$$v_c(r,t) = \Pi_G(t)(D_e(t) - S(r)), \tag{3.2.32}$$

and equation (3.2.30) takes the form

$$\Pi_G(t)(D_e(t) - S(r)) = \frac{1}{2\pi}\int_{C(t)} ds' \frac{p(r,t) - q_c(r',t)}{|\vec{r'} - \vec{r}|}, \quad r \le a(t). \tag{3.2.33}$$

Comparing this equation with (2.0.67) and recalling the elastic solution, we get

$$p(r,t) = q_c(r,t) + k_e \Pi_G(t) p_e(r,t) \tag{3.2.34}$$

where $p_e(r,t)$ is the pressure distribution on an elastic medium characterized by $k_e = \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}$. For the viscoelastic problem considered here, k_e is free to choose but $k_e p_e(r,t)$ is fixed. Here k_e is chosen to be [4]

$$k_e = \int_0^\infty k(t')dt' = k_0 + \sum_{i=1}^{N} \frac{k_i}{\beta_i} = \left\{ l_0 + \sum_{i=1}^{N} \frac{l_i}{\alpha_i} \right\}^{-1}. \tag{3.2.35}$$

Such a choice of k_e makes $p_e(r,t)$ and $W_e(t)$ have special meaning (we can see this later in Section 3.4.2). Now, it is clear that $D_e(t)$ in equation (3.2.31) is the elastic indentation corresponding to the pressure distribution $p_e(r,t)$.

For the case in which $C(t)$ is expanding, we begin with the decomposition (3.2.17) and get

$$\int_{S_L(t)} dt' \Gamma_L(t,t') p(r,t') = k_e(1 - \Gamma_G(t)) p_e(r,t), \tag{3.2.36}$$

$$D(t) = \int_{S_G(t)} dt' \Gamma_G(t,t') D(t') + D_e(t)(1 - \Gamma_G(t)) \tag{3.2.37}$$

where

$$\Gamma_G(t) = \int_{S_G(t)} dt' \Gamma_G(t,t'). \tag{3.2.38}$$

Integrating (3.2.34) and (3.2.36) gives us integral equations for the total load

$$W(t) = \int_{C(t)} ds q_c(r',t) + k_e \Pi_G(t) W_e(t), \tag{3.2.39}$$

$$\int_{S_L(t)} dt' \Gamma_L(t,t') W(t') = k_e(1 - \Gamma_G(t)) W_e(t). \tag{3.2.40}$$

The viscoelastic contact problem has therefore been reduced to solutions of six integral equations (3.2.31), (3.2.34), (3.2.36), (3.2.37), (3.2.39) and (3.2.40), which involve solutions of the corresponding elastic problem. For a spherical indentor of large radius R, we have, from the results in Chapter 2, that

$$D_e(t) = \frac{a^2(t)}{R}, \tag{3.2.41}$$

$$k_e p_e(r,t) = \frac{4}{\pi R}(a^2(t) - r^2)^{1/2}, \tag{3.2.42}$$

$$k_e W_e(t) = \frac{8a^3(t)}{3R}. \tag{3.2.43}$$

Let us now consider the steady-state limit of these integral equations. If the loading varies periodically with time, we would expect the response of the half-space to reflect this periodicity after a sufficiently long time. We therefore set

$$a(t) = a(t+\Delta), \tag{3.2.44}$$

$$D(t) = D(t+\Delta), \tag{3.2.45}$$

$$p(r,t) = p(r,t+\Delta), \tag{3.2.46}$$

$$W(t) = W(t+\Delta) \tag{3.2.47}$$

for any time t, where Δ is the period of the applied load. We choose $t \in [\Delta_1, \Delta_2]$ where $\Delta_2 - \Delta_1 = \Delta$ and Δ_1, Δ_2 are times when the contact region $C(t)$ is maximum. Also, we assume $C(t)$ is minimum at time t_0.

First we consider the contracting phase when $t \in [\Delta_1, t_0]$. Let $t_1(t)$ be the solution to equation $a(t_1(t)) = a(t)$ in $[t_0, \Delta_2]$. This function is determined by the shape of $a(t)$. In terms of t and $t_1(t)$, we have

$$\theta_1(t) = t_1(t) - \Delta, \tag{3.2.48}$$

$$\theta_2(t) = t - \Delta, \tag{3.2.49}$$

$$\theta_3(t) = t_1(t) - 2\Delta \tag{3.2.50}$$

and so on. It follows that $p(r,t)$, $W(t)$ and $D(t)$ in the decreasing phase $[\Delta_1, t_0]$, satisfy the following integral equations

$$p(r,t) = \int_t^{t_1(t)} dt' \Pi_L{}^{(p)}(t,t') p(r,t') + k_e \Pi_G{}^{(p)}(t) p_e(r,t), \tag{3.2.51}$$

$$W(t) = \int_t^{t_1(t)} dt' \Pi_L{}^{(p)}(t,t') W(t') + k_e \Pi_G{}^{(p)}(t) W_e(t), \tag{3.2.52}$$

$$\int_{t_1(t)-\Delta}^{t} dt' \Pi_G{}^{(p)}(t,t') D(t') = \Pi_G{}^{(p)}(t) D_e(t), \tag{3.2.53}$$

where

$$\Pi_L{}^{(p)}(t,t') = \sum_{k=1}^{\infty} T_{2k-1}(t, t' - k\Delta), \tag{3.2.54}$$

$$\Pi_G{}^{(p)}(t,t') = \sum_{k=0}^{\infty} T_{2k}(t, t' - k\Delta), \tag{3.2.55}$$

$$\Pi_G{}^{(p)}(t) = \int_{t_1(t)-\Delta}^{t} dt' \Pi_G{}^{(p)}(t,t'). \tag{3.2.56}$$

In the expanding phase, i.e. $t \in [t_0, \Delta_2]$, we get from (3.2.36), (3.2.37) and (3.2.40) that

$$\int_{t_1(t)}^{t} dt' \Gamma_L{}^{(p)}(t,t') p(\vec{r},t') = k_e (1 - \Gamma_G{}^{(p)}(t)) p_e(\vec{r},t), \tag{3.2.57}$$

$$\int_{t_1(t)}^{t} dt' \Gamma_L{}^{(p)}(t,t') W(t') = k_e (1 - \Gamma_G{}^{(p)}(t)) W_e(t), \tag{3.2.58}$$

$$D(t) = \int_{t-\Delta}^{t_1(t)} dt' \Gamma_G{}^{(p)}(t,t') D(t') + (1 - \Gamma_G{}^{(p)}(t)) D_e(t), \tag{3.2.59}$$

where

$$\Gamma_L{}^{(p)}(t,t') = \sum_{k=0}^{\infty} N_{2k}(t, t' - k\Delta), \tag{3.2.60}$$

$$\Gamma_G{}^{(p)}(t,t') = \sum_{k=0}^{\infty} N_{2k+1}(t, t'-k\Delta), \tag{3.2.61}$$

$$\Gamma_G{}^{(p)}(t) = \int_{t-\Delta}^{t_1(t)} dt' \Gamma_G{}^{(p)}(t,t'). \tag{3.2.62}$$

Here we have used equations

$$\begin{aligned} \theta_1(t) &= t_1(t), \\ \theta_2(t) &= t-\Delta, \\ \theta_3(t) &= t_1(t)-\Delta \end{aligned} \tag{3.2.63}$$

and so on and $t_1(t)$ is the solution of $a(t_1(t)) = a(t)$ in $[\Delta_1, t_0]$. The steady-state contact problem is thus expressed in terms of the six integral equations (3.2.51)-(3.2.53) and (3.2.57)-(3.2.59), four of them independent.

3.3 Standard Linear Model

In the last section, we have reduced the steady-state contact problem to the solution of six integral equations. Six kernels in these equations are infinite sums of terms involving integrals of the viscoelastic functions. In this section, we present formulas for these kernels for the standard linear solid given by Golden and Graham [3].

Consider first a time t when the contact area is decreasing. According to the definition of $T_i(t,t')$ and the standard linear model given by (3.1.22) and (3.1.23), we have

$$T_0(t,t') = l_0\delta(t-t') + l_1 e^{-\alpha(t-t')}, \tag{3.3.1}$$

$$T_1(t,t') = l_1 k_0 e^{-\alpha(t-\theta_1)-\beta(\theta_1-t')}, \tag{3.3.2}$$

$$T_2(t,t') = l_1 e^{-\alpha(t-\theta_1)-\beta(\theta_1-\theta_2)-\alpha(\theta_2-t')} \tag{3.3.3}$$

and so on. It is easy to verify that

$$\begin{aligned} T_{i+2}(t,t'-\Delta) &= T_i(t,t')e^{-\beta(\theta_i-\theta_{i+1})-\alpha(\theta_{i+1}-\theta_{i+2})}, i \ odd \qquad (3.3.4) \\ &= T_i(t,t')e^{-\alpha(\theta_i-\theta_{i+1})-\beta(\theta_{i+1}-\theta_{i+2})}, i \ even \qquad (3.3.5) \end{aligned}$$

for $t \neq t'$. The latter restriction is included to exclude the delta function in $T_0(t,t')$. By using (3.2.48)-(3.2.50), we see that

$$\theta_i - \theta_{i+1} = t_1(t) - t, \tag{3.3.6}$$

$$\theta_{i+1} - \theta_{i+2} = t - t_1(t) + \Delta, \tag{3.3.7}$$

for odd i, while for even i we have

$$\theta_i - \theta_{i+1} = t - t_1(t) + \Delta, \tag{3.3.8}$$

$$\theta_{i+1} - \theta_{i+2} = t_1(t) - t. \tag{3.3.9}$$

This gives that for all i

$$T_{i+2}(t, t' - \Delta) = T_i(t,t')E(t), \tag{3.3.10}$$

where

$$E(t) = e^{(t-t_1(t))(\beta-\alpha)-\alpha\Delta}. \tag{3.3.11}$$

It is also easy to see that $0 < E(t) < 1$, so that

$$\begin{aligned} \Pi_L^{(p)}(t,t') &= T_1(t,t'-\Delta)\sum_{n=0}^{\infty}[E(t)]^n \\ &= \frac{l_1 k_0}{1-E(t)} e^{-\alpha(t-t_1+\Delta)+\beta(t'-t_1)}. \end{aligned} \tag{3.3.12}$$

By the same method, we obtain

$$\Pi_G{}^{(p)}(t,t') = l_0\delta(t-t') + \frac{l_1}{1-E(t)}e^{-\alpha(t-t')}, \tag{3.3.13}$$

and therefore, from equation (3.2.56), we get

$$\Pi_G^{(p)}(t) = l_0 + \frac{l_1}{\alpha(1-E(t))}[1 - e^{-\alpha(t-t_1+\Delta)}]. \tag{3.3.14}$$

Similarly we obtain the kernels

$$\Gamma_L^{(p)}(t,t') = k_0\delta(t-t') + \frac{k_1 e^{-\beta(t-t')}}{1-E(t_1)}, \tag{3.3.15}$$

$$\Gamma_G{}^{(p)}(t,t') = \frac{k_1 l_0}{1-E(t_1)}e^{-\beta(t-t_1)-\alpha(t_1-t')}, \tag{3.3.16}$$

$$\Gamma_G^{(p)}(t) = \frac{k_1 l_0}{\alpha(1-E(t_1))}e^{-\beta(t-t_1)}[1 - e^{\alpha(t-t_1-\Delta)}] \tag{3.3.17}$$

for times t when the contact area is increasing.

Using these expressions, we can replace the six integral equations, which involve quantities in both expanding and contracting phases with other integral equations which contain quantities only in one phase. These equations then can be reduced to ordinary differential equations. Here we give the derivation for $p(r,t)$. We can get the equation for $W(t)$ simply by integrating the equation of $p(r,t)$ over the contact area. The ordinary differential equation for $D(t)$ is given in paper[8].

From (3.3.15) and (3.3.17), we can write (3.2.57) as

$$p(r,t) = C_1(t)\int_{t_1}^{t} dt' e^{\beta t'} p(r,t') + D_1(t)p_e(r,t), \quad t \in [t_0, \Delta_2]. \tag{3.3.18}$$

From (3.3.12) and (3.3.13), write (3.2.51) as

$$p(r,t_1) = C_2(t)\int_{t_1}^{t} dt' e^{\beta t'} p(r,t') + D_2(t)p_e(r,t_1), \quad t \in [t_0, \Delta_2]. \tag{3.3.19}$$

The functions $C_1(t)$, $C_2(t)$, $D_1(t)$, $D_2(t)$ are calculated in [3]. Considering that $p_e(r,t) = p_e(r,t_1)$ we get the following relationship between the pressure function in expanding and contracting phase, by eliminating the integral terms in (3.3.18) and (3.3.19)

$$p(r,t_1) = \eta(t)p_e(r,t) + \varepsilon(t)p(r,t), \quad t \in [t_0, \Delta_2]. \tag{3.3.20}$$

Using this relationship, we reduce (3.2.57) to a integral equation for $p(r,t)$ in the expansion phase only, and then to a ordinary differential equation

$$\dot{p}(r,t) + \alpha p(r,t) = b(r,t), \quad t \in [t_0, \Delta_2] \tag{3.3.21}$$

where $b(r,t)$ is also given in [3]. Equations (3.3.20) and (3.3.21) were solved for $t_1(t)$ and $p(r,t)$ by an iteration method in [3]. The contact problem of standard linear material was discussed extensively by Golden and Graham for three modes in [6](see Section 3.4.3 for the definition of these three modes). In that paper, they also calculated the rates of energy loss and gave a simpler numerical technique. Instead of solving equations (3.3.20) and (3.3.21), they determined $t_1(t)$ by directly solving a first order differential equation, namely (21) for the stress-controlled mode and (34) for the strain-controlled mode, in [6]. Once $t_1(t)$ is known, $p(r,t)$ is obtained from (3.3.21). For area-controlled mode, $t_1(t) = -t$.

3.4 General Viscoelastic Model

In this section, solutions to the contact problem for general viscoelastic materials, which correspond to the case that N can be any integer number in (3.1.15) or (3.1.16), are given. Obviously $N = 1$ reduces the case to the standard linear material. As we mentioned before, the infinite summations of the kernels cannot be carried out in an elememtary manner for the case when $N \geq 2$. However, in [5], this question was addressed in the context of a different problem, namely that of a fixed length crack in an infinite body under sinusoidal loading. In that paper, the authors showed that the kernels obey certain integral equations, whose solutions can be determined in closed form, at least for discrete spectrum models. The solution of these equations amounts to summing the infinite series. Here, this method is extended to the contact problem.

3.4.1 Integral Equations for Kernels

We shall now show that the kernels of (3.2.51) – (3.2.53) and (3.2.57) – (3.2.59) obey certain integral equations, which at least for the discrete spectrum models (3.1.15) and (3.1.16) can be solved in closed form. Let us consider the kernel

$$\Pi_L{}^{(p)}(t,t') = \sum_{k=1}^{\infty} T_{2k-1}(t, t' - k\Delta) \tag{3.4.1}$$

of equation (3.2.51) first. According to the definition of $T_n(t,t')$, we have

$$T_n(t,t') = \int_{t'}^{\theta_n(t)} dt''' \int_{t'''}^{\theta_{n-1}(t)} dt'' T_{n-2}(t,t'') l(t''-t''') k(t'''-t') \tag{3.4.2}$$

for odd numbers $n \geq 3$. The integral over t'' can be extended at the lower limit to t' since $l(t''-t''')$ vanishes over this interval. This allows the order of integration to be interchanged without difficulty and one has (omitting explicit mention of the t dependence of θ_n)

$$T_n(t,t') = \int_{t'}^{\theta_{n-1}} dt'' T_{n-2}(t,t'') G_n(t'',t'), \tag{3.4.3}$$

$$G_n(t'',t') = \int_{t'}^{\theta_n} dt''' l(t''-t''') k(t'''-t'). \tag{3.4.4}$$

Using the inverse relationship (3.1.13) between $k(t)$ and $l(t)$, one deduces that

$$G_n(t'',t') = \delta(t''-t'), \; t'' \le \theta_n. \tag{3.4.5}$$

Therefore

$$T_n(t,t') = T_{n-2}(t,t') + \int_{\theta_n}^{\theta_{n-1}} dt'' T_{n-2}(t,t'') G_n(t'',t'), \; t' \le \theta_n. \tag{3.4.6}$$

Making the subscript explicitly odd, we can write

$$\begin{aligned} T_{2i-1}(t,t'-i\Delta) &= T_{2i-3}(t,t'-i\Delta) \\ &+ \int_{t_1-i\Delta}^{t-(i-1)\Delta} dt'' T_{2i-3}(t,t'') G_{2i-1}(t'',t'-i\Delta), \end{aligned} \tag{3.4.7}$$

by using $\theta_{2n-1} = t_1 - n\Delta$ and $\theta_{2n-2} = t-(n-1)\Delta$, where

$$\begin{aligned} G_{2i-1}(t'',t'-i\Delta) &= \int_{t'-i\Delta}^{t_1-i\Delta} dt''' l(t''-t''') k(t'''-t'+i\Delta) \\ &= \int_{t'-\Delta}^{t_1-\Delta} du\, l(t''-u+(i-1)\Delta) k(u-t'+\Delta) \\ &= G(t''+(i-1)\Delta, t'-\Delta). \end{aligned} \tag{3.4.8}$$

Here the transformation of variables $u = t''' + (i-1)\Delta$ is employed. The function

$$G(t'',t') = \int_{t'}^{\theta_1} du\, l(t''-u) k(u-t') \tag{3.4.9}$$

has the same functional form as $T_1(t,t')$ with t'' replacing t but $\theta_1(t)$ left untouched. Using (3.4.7) and (3.4.8), we have

$$\begin{aligned} T_{2i-1}(t,t'-i\Delta) &= T_{2i-3}(t,t'-i\Delta) \\ &+ \int_{t_1-\Delta}^{t} du\, T_{2i-3}(t,u-(i-1)\Delta) G(u,t'-\Delta). \end{aligned} \tag{3.4.10}$$

Therefore ${\Pi_L}^{(p)}(t,t')$, given by (3.2.54) or (3.4.1), obeys the equation

$$ {\Pi_L}^{(p)}(t,t') = T_1(t,t'-\Delta) + {\Pi_L}^{(p)}(t,t'-\Delta) + \int_{t_1-\Delta}^{t} du {\Pi_L}^{(p)}(t,u) G(u,t'-\Delta). \quad (3.4.11) $$

Thus

$$ \begin{aligned} {\Pi_L}^{(p)}(t,t'-\Delta) &= T_1(t,t'-2\Delta) + {\Pi_L}^{(p)}(t,t'-2\Delta) \\ &+ \int_{t_1-\Delta}^{t} du {\Pi_L}^{(p)}(t,u) G(u,t'-2\Delta) \end{aligned} \quad (3.4.12) $$

and so on. Repeated substitution of (3.4.12) and its sucessors into (3.4.11) together with the assumption, which will be justified later, that

$$ \lim_{n\to\infty} {\Pi_L}^{(p)}(t,t'-n\Delta) = 0, \quad (3.4.13) $$

finally gives an integral equation for ${\Pi_L}^{(p)}(t,t')$ of the form

$$ {\Pi_L}^{(p)}(t,t') = K(t,t') + \int_{\theta_1}^{t} du {\Pi_L}^{(p)}(t,u) K(u,t'), \quad (3.4.14) $$

where

$$ K(u,t') = \sum_{n=1}^{\infty} G(u,t'-n\Delta). \quad (3.4.15) $$

We recall that $G(t,t'-n\Delta) = T_1(t,t'-n\Delta)$. In a similar way, it is found that ${\Pi_G}^{(p)}(t,t')$ obeys the integral equation

$$ \begin{aligned} {\Pi_G}^{(p)}(t,t') &= l_0\delta(t-t') + \sum_{i=1}^{N} \frac{l_i}{1-e^{-\alpha_i\Delta}} e^{-\alpha_i(t-t')} \\ &+ \int_{\theta_2}^{\theta_1} du {\Pi_G}^{(p)}(t,u) L(u,t'), \end{aligned} \quad (3.4.16) $$

provided

$$ \lim_{n\to\infty} {\Pi_G}^{(p)}(t,t'-n\Delta) = 0 \quad (3.4.17) $$

where

$$ L(u,t') = \sum_{n=1}^{\infty} H(u,t'-n\Delta), \quad (3.4.18) $$

$$ H(t'',t') = \int_{t'}^{\theta_2} du k(t''-u) l(u-t'). \quad (3.4.19) $$

Furthermore, by comparing the definition of $\Pi_G{}^{(p)}(t,t')$ and $\Gamma_L{}^{(p)}(t,t')$, we find that $\Gamma_L{}^{(p)}(t,t')$ may be obtained from $\Pi_G{}^{(p)}(t,t')$ by interchanging the roles of $l(t)$ and $k(t)$. Therefore $\Gamma_L{}^{(p)}(t,t')$ satisfies an equation obtained from (3.4.16) by interchanging $l(t)$ with $k(t)$, provided it satisfies a relation analogous to (3.4.17). Finally we see that $\Gamma_G{}^{(p)}(t,t'-\Delta)$ satisfies an integral equation obtained from (3.4.14) by interchanging $l(t)$ and $k(t)$ provided an analogue of (3.4.13) is satisfied.

3.4.2 Solutions of Integral Equations for Kernels

We now solve these integral equations for the kernels. First of all we consider (3.4.14). For discrete spectrum models (3.1.15) and (3.1.16), we obtain from (3.4.9) that

$$G(u,t') = \sum_{i,j=1}^{N} \frac{l_i k_j}{\alpha_i - \beta_j} e^{-\alpha_i(u-\theta_1)-\beta_j(\theta_1-t')}, \quad u > \theta_1(t). \tag{3.4.20}$$

Thus, equation (3.4.15) takes the form

$$K(u,t') = \sum_{i,j=1}^{N} K_{ij} e^{-\alpha_i(u-\theta_1)-\beta_j(\theta_1-t')}, \tag{3.4.21}$$

where

$$K_{ij} = \frac{l_i k_j}{\alpha_i - \beta_j} \frac{e^{-\beta_j \Delta}}{1 - e^{-\beta_j \Delta}}. \tag{3.4.22}$$

To solve (3.4.14), we make the ansatz for $\Pi_L{}^{(p)}(t,t')$ of the form

$$\Pi_L{}^{(p)}(t,t') = \sum_{i,j=1}^{N} P_{ij}(t) e^{\beta_j t'}, \tag{3.4.23}$$

which clearly obeys (3.4.13). Substitution into (3.4.14) gives

$$\begin{aligned} \sum_{i=1}^{N} P_{ij} &= \sum_{i=1}^{N} K_{ij} e^{-\alpha_i(t-\theta_1)-\beta_j\theta_1} \\ &+ \sum_{i,m,n=1}^{N} \frac{P_{im} K_{nj}}{\beta_m - \alpha_n} \{ e^{\beta_m t - \alpha_n(t-\theta_1)-\beta_j\theta_1} - e^{(\beta_m-\beta_j)\theta_1} \}. \end{aligned} \tag{3.4.24}$$

This algebraic equation will certainly be satisfied if a stronger condition is imposed that cancellation takes place term by term in the variable i.This gives us the matrix equation

$$P = K_1 + PAK_2 \tag{3.4.25}$$

where P is a square matrix formed by P_{ij} while

$$(K_1)_{ij} = K_{ij}e^{-\alpha_i(t-\theta_1)-\beta_j\theta_1}, \tag{3.4.26}$$

$$(K_2)_{nj} = K_{nj}e^{-\beta_j\theta_1}, \tag{3.4.27}$$

$$A_{mn} = \frac{e^{-\alpha_n(t-\theta_1)+\beta_m t} - e^{\beta_m\theta_1}}{\beta_m - \alpha_n}. \tag{3.4.28}$$

The formal solution of (3.4.25) is

$$P = K_1(I - AK_2)^{-1}. \tag{3.4.29}$$

Similarly, we get the solution to (3.4.16) of the form

$$\Pi_G{}^{(p)}(t,t') = l_0\delta(t-t') + \sum_{i,j=1}^{N} Q_{ij}(t)e^{\alpha_j t'}, \tag{3.4.30}$$

where $Q_{ij}(t)$ is a square matrix given by

$$Q = L_1(I - BL_2)^{-1}, \tag{3.4.31}$$

and

$$(L_1)_{ij} = \frac{\delta_{ij}l_j e^{-\alpha_j t}}{1-e^{-\alpha_j\Delta}}, \tag{3.4.32}$$

$$(L_2)_{ij} = \frac{k_i l_j e^{-\alpha_j\theta_2}}{\beta_i - \alpha_j}\frac{e^{-\alpha_j\Delta}}{1-e^{-\alpha_j\Delta}}, \tag{3.4.33}$$

$$B_{ij} = \frac{e^{(\alpha_i-\beta_j)\theta_1+\beta_j\theta_2} - e^{\alpha_i\theta_2}}{\alpha_i - \beta_j}. \tag{3.4.34}$$

Clearly, $\Pi_G{}^{(p)}(t,t')$ satisfies assumption (3.4.17). Using the observation after (3.4.19), we can write

$$\Gamma_L{}^{(p)}(t,t') = k_0\delta(t-t') + \sum_{i,j=1}^{N} \hat{Q}_{ij}(t)e^{\beta_j t'}, \tag{3.4.35}$$

$$\hat{Q} = \hat{L}_1(I - \hat{B}\hat{L}_2)^{-1}, \tag{3.4.36}$$

where $\hat{L}_1$,$\hat{B}$ and $\hat{L}_2$ are obtained from (3.4.32)-(3.4.34) by interchanging the role of $l(t)$ and $k(t)$. They are

$$(\hat{L}_1)_{ij} = \frac{\delta_{ij}k_j e^{-\beta_j t}}{1-e^{-\beta_j\Delta}}, \tag{3.4.37}$$

$$(\hat{L}_2)_{ij} = \frac{l_i k_j e^{-\beta_j \theta_2}}{\alpha_i - \beta_j} \frac{e^{-\beta_j \Delta}}{1 - e^{-\beta_j \Delta}}, \tag{3.4.38}$$

$$\hat{B}_{ij} = \frac{e^{(\beta_i - \alpha_j)\theta_1 + \alpha_j \theta_2} - e^{\beta_i \theta_2}}{\beta_i - \alpha_j}. \tag{3.4.39}$$

Also from that observation, we have

$$\Gamma_G{}^{(p)}(t, t') = \sum_{i,j=1}^{N} \hat{P}_{ij}(t) e^{\alpha_j (t' + \Delta)}, \tag{3.4.40}$$

$$\hat{P} = \hat{K}_1 (I - \hat{A}\hat{K}_2)^{-1} \tag{3.4.41}$$

where the components of $\hat{K}_1, \hat{A}, \hat{K}_2$ are

$$(\hat{K}_1)_{ij} = \hat{K}_{ij} e^{-\beta_i (t - \theta_1) - \alpha_j \theta_1}, \tag{3.4.42}$$

$$(\hat{K}_2)_{nj} = \hat{K}_{nj} e^{-\alpha_j \theta_1}, \tag{3.4.43}$$

$$\hat{A}_{mn} = \frac{e^{-\beta_n (t - \theta_1) + \alpha_m t} - e^{\alpha_m \theta_1}}{\alpha_m - \beta_n}, \tag{3.4.44}$$

and

$$\hat{K}_{ij} = \frac{k_i l_j}{\beta_i - \alpha_j} \frac{e^{-\alpha_j \Delta}}{1 - e^{-\alpha_j \Delta}}. \tag{3.4.45}$$

Equation (3.4.30) together with (3.2.56) gives

$$\Pi_G{}^{(p)}(t) = l_0 + \sum_{i,j=1}^{N} Q_{ij} \frac{e^{\alpha_j t} - e^{\alpha_j \theta_1}}{\alpha_j}, \tag{3.4.46}$$

while (3.4.40) combined with (3.2.62) gives

$$\Gamma_G{}^{(p)}(t) = \sum_{i,j=1}^{N} \hat{P}_{ij} \frac{e^{\alpha_j (\theta_1 + \Delta)} - e^{\alpha_j (\theta_2 + \Delta)}}{\alpha_j}. \tag{3.4.47}$$

Now the steady-state contact problem equations (3.2.51)-(3.2.53) and (3.2.57)-(3.2.59) take the form

$$p(r,t) = \sum_{i,j=1}^{N} P_{ij}(t) \int_t^{t_1(t)} dt' e^{\beta_j t'} p(r,t') \quad + \quad k_e \left\{ l_0 + \sum_{i,j=1}^{N} Q_{ij}(t) \frac{e^{\alpha_j t} - e^{\alpha_j (t_1 - \Delta)}}{\alpha_j} \right\} p_e(r,t),$$
$$for \;\; t \in [\Delta_1, t_0], \tag{3.4.48}$$

$$k_0 p(r,t) + \sum_{i,j=1}^{N} \hat{Q}_{ij}(t) \int_{t_1(t)}^{t} dt' e^{\beta_j t'} p(r,t') \;\; = \;\; k_e \left\{ 1 - \sum_{i,j=1}^{N} \hat{P}_{ij}(t) \frac{e^{\alpha_j (\Delta + t_1)} - e^{\alpha_j t}}{\alpha_j} \right\} p_e(r,t),$$
$$for \;\; t \in [t_0, \Delta_2], \tag{3.4.49}$$

$$W(t) = \sum_{i,j=1}^{N} P_{ij}(t) \int_{t}^{t_1(t)} dt' e^{\beta_j t'} W(t') \quad + \quad k_e \left\{ l_0 + \sum_{i,j=1}^{N} Q_{ij}(t) \frac{e^{\alpha_j t} - e^{\alpha_j (t_1 - \Delta)}}{\alpha_j} \right\} W_e(t),$$
$$for \ \ t \in [\Delta_1, t_0], \tag{3.4.50}$$

$$k_0 W(t) + \sum_{i,j=1}^{N} \hat{Q}_{ij}(t) \int_{t_1(t)}^{t} dt' e^{\beta_j t'} W(t') \quad = \quad k_e \left\{ 1 - \sum_{i,j=1}^{N} \hat{P}_{ij}(t) \frac{e^{\alpha_j (\Delta + t_1)} - e^{\alpha_j t}}{\alpha_j} \right\} W_e(t),$$
$$for \ \ t \in [t_0, \Delta_2], \tag{3.4.51}$$

$$l_0 D(t) + \sum_{i,j=1}^{N} Q_{ij}(t) \int_{t_1 - \Delta}^{t} dt' e^{\alpha_j t'} D(t') \quad = \quad \left\{ l_0 + \sum_{i,j=1}^{N} Q_{ij}(t) \frac{e^{\alpha_j t} - e^{\alpha_j (t_1 - \Delta)}}{\alpha_j} \right\} D_e(t),$$
$$for \ \ t \in [\Delta_1, t_0], \tag{3.4.52}$$

$$D(t) = \sum_{i,j=1}^{N} \hat{P}_{ij}(t) e^{\alpha_j \Delta} \int_{t-\Delta}^{t_1} dt' e^{\alpha_j t'} D(t') \quad + \quad \left\{ 1 - \sum_{i,j=1}^{N} \hat{P}_{ij}(t) \frac{e^{\alpha_j (t_1 + \Delta)} - e^{\alpha_j t}}{\alpha_j} \right\} D_e(t),$$
$$for \ \ t \in [t_0, \Delta_2], \tag{3.4.53}$$

where the matrices P,Q are determined by (3.4.29),(3.4.31) and $\hat{Q}$,$\hat{P}$ given by (3.4.36) and (3.4.41).

Some general conclusions can be deduced from these equations. We will demonstrate that $p(r,t)$, $W(t)$, $D(t)$ satisfying the above equations are continuous at time t_0, and equal at Δ_1 and Δ_2.

Putting $t = t_0$ in (3.4.50), we have

$$W(t_0) = k_e \left\{ l_0 + \sum_{i,j=1}^{N} Q_{ij}(t_0) \frac{e^{\alpha_j t_0} - e^{\alpha_j (t_0 - \Delta)}}{\alpha_j} \right\} W_e(t_0). \tag{3.4.54}$$

From (3.4.34) we know that $B_{ij}(t_0) = 0$, since $\theta_1 = \theta_2 = t_0 - \Delta$ for $t = t_0$. Therefore it follows from (3.4.31) that

$$Q_{ij}(t_0) = \{L_1(t_0)\}_{ij} = \frac{\delta_{ij} l_j e^{-\alpha_j t_0}}{1 - e^{-\alpha_j \Delta}}, \tag{3.4.55}$$

so that

$$W(t_0) = k_e \left(l_0 + \sum_{i=1}^{N} l_i / \alpha_i \right) W_e(t_0) = W_e(t_0), \tag{3.4.56}$$

after using equation (3.2.35). This confirms the results given in [9]. Equation (3.4.51) at $t = t_0$ gives the same result by noting the fact that

$$\hat{P}_{ij}(t_0) = \frac{k_i l_j}{\beta_i - \alpha_j} \frac{e^{-\alpha_j(\Delta + t_0)}}{1 - e^{-\alpha_j \Delta}} \tag{3.4.57}$$

and relation (3.1.19). Therefore $W(t)$ is continous at time t_0. Equation (3.4.50) at $t = \Delta_1$ and equation (3.4.51) at $t = \Delta_2$ take the form

$$W(\Delta_1) = \sum_{ij=1}^{N} P_{ij}(\Delta_1) \int_{\Delta_1}^{\Delta_2} dt' e^{\beta_j t'} W(t') + k_e l_0 W_e(\Delta_1), \tag{3.4.58}$$

$$W(\Delta_2) = -\frac{1}{k_0} \sum_{ij=1}^{N} \hat{Q}_{ij}(\Delta_2) \int_{\Delta_1}^{\Delta_2} dt' e^{\beta_j t'} W(t') + k_e l_0 W_e(\Delta_2). \tag{3.4.59}$$

From (3.2.48)-(3.2.50), we know

$$t_1(t) = \Delta_2, \ \theta_1(t) = \Delta_1 \tag{3.4.60}$$

for time $t = \Delta_1$. Therefore, from (3.4.28), one deduces

$$A_{ij}(\Delta_1) = 0. \tag{3.4.61}$$

Thus we can write (3.4.29) as

$$P_{ij}(\Delta_1) = (K_1)_{ij}(\Delta_1) = \frac{l_i k_j}{\alpha_i - \beta_j} \frac{e^{-\beta_j \Delta_2}}{1 - e^{-\beta_j \Delta}}. \tag{3.4.62}$$

Similarly, we obtain

$$\hat{Q}_{ij}(\Delta_2) = (\hat{L}_1)_{ij}(\Delta_2) = \frac{\delta_{ij} k_j e^{-\beta_j \Delta_2}}{1 - e^{-\beta_j \Delta}} \tag{3.4.63}$$

considering $\theta_1 = \theta_2 = t_1 = \Delta_1$ and $\hat{B}_{ij} = 0$ for $t = \Delta_2$. Therefore

$$\sum_{i=1}^{N} P_{ij}(\Delta_1) = -\frac{1}{k_0} \sum_{i=1}^{N} \hat{Q}_{ij}(\Delta_2), \tag{3.4.64}$$

by virtue of equation (3.1.18). Noting $W_e(\Delta_1) = W_e(\Delta_2)$, we have, from equations (3.4.58) and (3.4.59), that $W(\Delta_1) = W(\Delta_2)$. This completes the verification of the periodicity of $W(t)$.

Now we consider D(t). It is clear that $D(\Delta_1) = D(\Delta_2)$ because (3.4.52) at $t = \Delta_1$ gives

$$D(\Delta_1) = D_e(\Delta_1), \tag{3.4.65}$$

equation (3.4.53) at $t = \Delta_2$ yields

$$D(\Delta_2) = D_e(\Delta_2), \tag{3.4.66}$$

and we know that $D_e(\Delta_1) = D_e(\Delta_2)$. Equations (3.4.65),(3.4.66) are in agreement with results given in [9] in a more general context.

When $t = t_0$, we have, from (3.4.52) and (3.4.53)

$$\begin{aligned} D(t_0) &= -k_0 \sum_{i=1}^{N} Q_{ij}(t_0) \int_{t_0-\Delta}^{t_0} dt' e^{\alpha_j t'} D(t') \\ &+ k_0 \left\{ l_0 + \sum_{i=1}^{N} Q_{ij}(t_0) \frac{e^{\alpha_j t_0} - e^{\alpha_j (t_0-\Delta)}}{\alpha_j} \right\} D_e(t_0), \qquad (3.4.67) \\ D(t_0) &= \sum_{ij=1}^{N} \hat{P}_{ij}(t_0) e^{\alpha_j \Delta} \int_{t_0-\Delta}^{t_0} dt' e^{\alpha_j t'} D(t') \\ &+ \left\{ 1 - \sum_{ij=1}^{N} \hat{P}_{ij}(t_0) e^{\alpha_j \Delta} \frac{e^{\alpha_j t_0} - e^{\alpha_j (t_0-\Delta)}}{\alpha_j} \right\} D_e(t_0). \qquad (3.4.68) \end{aligned}$$

To prove $D(t)$ is continuous at $t = t_0$, it is sufficient to know that

$$-k_0 \sum_{i=1}^{N} Q_{ij}(t_0) = \sum_{i=1}^{N} \hat{P}_{ij}(t_0) e^{\alpha_j \Delta}. \tag{3.4.69}$$

This is an immediate consequence of (3.4.55) and (3.4.57) with the aid of equation (3.1.19). The verification of the continuity and periodicity of $p(r,t)$ are the same as that of $W(t)$.

Actually, it is possible to prove the more general results:

$$\begin{aligned} (I - AK_2)(t) &= (I - \hat{B}\hat{L}_2)(t_1(t)), \quad for \ t \in [\Delta_1, t_0], \qquad (3.4.70) \\ (I - BL_2)(t) &= (I - \hat{A}\hat{K}_2)(t_1(t)), \quad for \ t \in [\Delta_1, t_0]. \qquad (3.4.71) \end{aligned}$$

These relations are very useful for the numerical calculation in the next section.

Also, we can show that the solutions for the kernels lead to the results for standard linear material given in Section 3 . If $N = 1$, we can write (3.4.26)-(3.4.28) as

$$(K_1)_{11} = \frac{l_1 k_1}{\alpha - \beta} \frac{e^{-\alpha(t-\theta_1)-\beta(\theta_1+\Delta)}}{1 - e^{-\beta\Delta}}, \tag{3.4.72}$$

$$(K_2)_{11} = \frac{l_1 k_1}{\alpha - \beta} \frac{e^{-\beta(\theta_1+\Delta)}}{1 - e^{-\beta\Delta}}, \tag{3.4.73}$$

$$A_{11} = \frac{e^{-\alpha(t-\theta_1)+\beta t} - e^{\beta\theta_1}}{\beta - \alpha}. \tag{3.4.74}$$

Then from (3.4.29), we obtain

$$\begin{aligned} P_{11} &= \frac{(K_1)_{11}}{1 - A_{11}(K_2)_{11}} \\ &= \frac{l_1 k_0}{1 - e^{-\alpha(\Delta+t-t_1)+\beta(t-t_1)}} e^{-\alpha(t-t_1+\Delta)-\beta t_1}, \end{aligned} \tag{3.4.75}$$

after using equations (3.1.24)-(3.1.26). By virtue of (3.4.23) and (3.3.11), one deduces

$$\Pi_L^{(p)}(t,t') = \frac{l_1 k_0}{1 - E(t)} e^{-\alpha(t-t_1+\Delta)+\beta(t'-t_1)}. \tag{3.4.76}$$

This formula is identical to one obtained in Section 3 by a different method. Similarly, we can show that the other three kernels $\Pi_G^{(p)}(t,t')$, $\Gamma_L^{(p)}(t,t')$ and $\Gamma_G^{(p)}(t,t')$ are same as before.

3.4.3 Numerical Results

In this section, numerical solutions to integral equations (3.4.48)-(3.4.53) obtained by the quadrature method are given for the following three cases

(i) where the normal load is specified (stress-controlled mode);

(ii) where the indentation is given (strain-controlled mode);

(iii) where the area of contact is specified (area-controlled mode).

Here we consider the spherical indentor of large radius R, for which the elastic solutions are known. All the numerical calculations are carried out for the case $N = 2$ and the dimensionless quantities $ca(t)$, $cD(t)$, $c^2k_0W(t)$, $k_0p(r,t)$ in terms of the dimensionless parameters $t' = \omega t$, $\beta_i' = \beta_i/\omega$, $k_i' = k_i/(\omega k_0)$, $\alpha_i' = \alpha_i/\omega$, $l_i' = l_ik_0/\omega$ and $k = k_e/k_0$, where $c = 1/(2R)$. This method can be applied to the cases when $N > 2$ without any difficulty.

(i) Stress-Controlled Mode

In this case the applied load is assumed to have the simple sinusoidal form

$$W(t) = K(d - cos\omega t),\ d \geq 1, \Delta = \frac{2\pi}{\omega}. \qquad (3.4.77)$$

To solve for $a(t), p(r,t), D(t)$, we need to know t_0 and Δ_1, Δ_2. According to [9], $t_0 = 0$. Setting $t = \Delta_1$ in (3.4.50), one obtains

$$W(\Delta_1) = -l_0 \sum_{j=1}^{N} \frac{k_j e^{-\beta_j \Delta_2}}{1 - e^{-\beta_j \Delta}} \int_{\Delta_1}^{\Delta_2} dt' e^{\beta_j t'} W(t') + l_0 k_e W_e(\Delta_1), \qquad (3.4.78)$$

while equation (3.4.51) at $t = \Delta_2$ gives the same equation.

According to (4.21) in [9], we have, at $t = \Delta_2$, that

$$\max_{t' \leq t} \frac{1}{k_e} \int_{-\infty}^{t'} dt'' k(t' - t'') W(t'') = W_e(t). \qquad (3.4.79)$$

Differentiating this equation at $t' = \Delta_2$ gives us

$$k_0 \dot{W}(\Delta_2) + W(\Delta_2) \sum_{i=1}^{N} k_i - \sum_{i=1}^{N} k_i \beta_i e^{-\beta_i \Delta_2} \int_{-\infty}^{\Delta_2} dt' e^{\beta_i t'} W(t') = 0. \qquad (3.4.80)$$

Considering that

$$\begin{aligned} \int_{-\infty}^{\Delta_2} dt' e^{\beta_i t'} W(t') &= \sum_{n=0}^{\infty} \int_{\Delta_2 - (n+1)\Delta}^{\Delta_2 - n\Delta} dt' e^{\beta_i t'} W(t') \\ &= \frac{1}{1 - e^{-\beta_i \Delta}} \int_{\Delta_1}^{\Delta_2} dt' e^{\beta_i t'} W(t'), \end{aligned} \qquad (3.4.81)$$

we obtain

$$\dot{W}(\Delta_2) = -\frac{1}{k_0} \sum_{i=1}^{N} k_i W(\Delta_2) + \sum_{i=1}^{N} \frac{k_i \beta_i e^{-\beta_i \Delta_2}}{k_0(1 - e^{-\beta_i \Delta})} \int_{\Delta_1}^{\Delta_2} dt' e^{\beta_i t'} W(t'). \qquad (3.4.82)$$

Substituting (3.4.77) into the above equation gives

$$tan(\omega\Delta_2) = \left\{\sum_{i=1}^{N} \frac{k_i\omega^2}{\omega^2+\beta_i^2}\right\} / \left\{\omega k_0 + \sum_{i=1}^{N} \frac{\omega k_i \beta_i}{\omega^2+\beta_i^2}\right\} \tag{3.4.83}$$

where use of $\Delta_2 - \Delta_1 = \Delta$ has been made. This agrees with a general result in [9] in a different form involving loss tangent. For the standard linear model $N = 1$, the above equation reduces to

$$tan(\omega\Delta_2) = \frac{\omega\beta f}{\beta^2 + \omega^2(1-f)} \tag{3.4.84}$$

where $f = 1 - \beta/\alpha$, which agrees with (4.56) in [3]. For this special model $k = 1/(1-f)$.

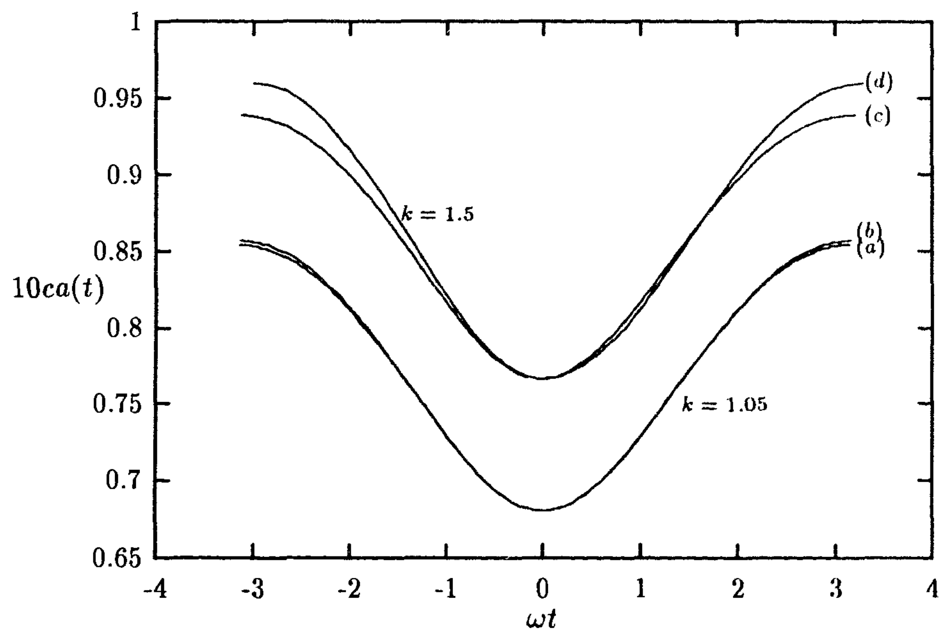

Figure 3.3: Contact area radius for the stress-controlled mode. This gives the dimensionless contact area radius $ca(t)$ over a complete cycle under stress-controlled condition for cases (a): $k_1' = 0.0096$, $k_2' = 0.003$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.05$; (b): $k_1' = 0.09$, $k_2' = 0.03$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.05$;(c): $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.5$ *and* (d): $k_1' = 0.8$, $k_2' = 0.6$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.5$ with $c^2Kk_0 = 0.0008, d = 3$.

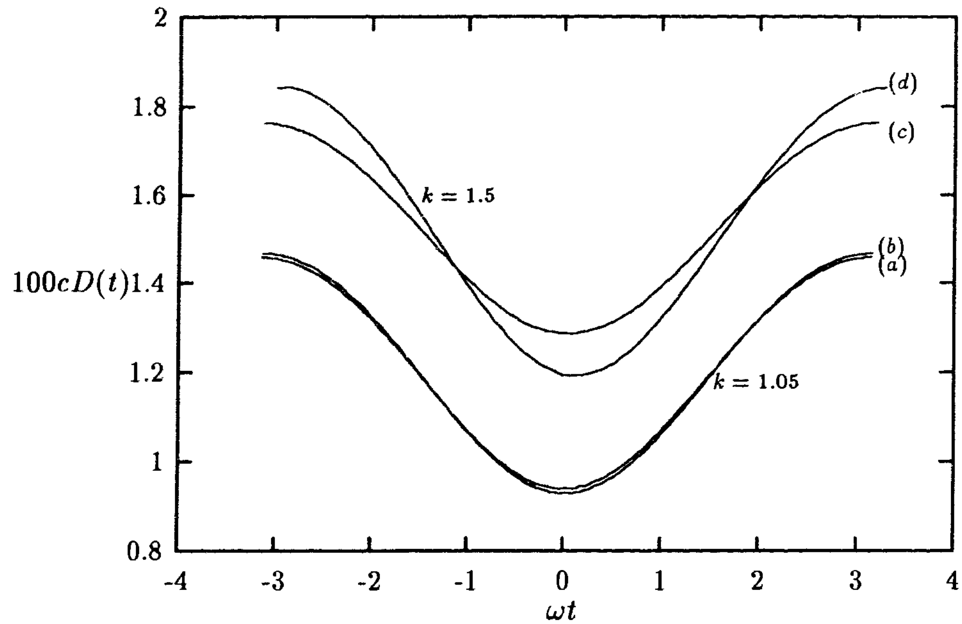

Figure 3.4: Indentation for the stress-controlled mode. This gives the dimensionless indentation $cD(t)$ over a complete cycle under stress-controlled condition for the same cases as those in Fig.3.3. (a): $k_1' = 0.0096$, $k_2' = 0.003$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.05$; (b): $k_1' = 0.09$, $k_2' = 0.03$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.05$;(c): $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.5$ *and* (d): $k_1' = 0.8$, $k_2' = 0.6$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.5$ with $c^2 K k_0 = 0.0008, d = 3$.

Once t_0 and Δ_1, Δ_2 are known, Newton's iteration method is employed to solve a functional equation, obtained by eliminating $W_e(t) = W_e(t_1(t))$ from (3.4.50) and (3.4.51), for $t_1(t)$. Then $W_e(t)$ can be got from one of these equations. If $W_e(t)$ is known, the contact radius $a(t)$, elastic indentation $D_e(t)$ and elastic pressure $p_e(r,t)$ can be found by using the relations (3.2.41) to (3.2.43). Therefore we can get $D(t)$ and $p(r,t)$. For example, indentation $D(t)$ can be obtained by solving equation (3.4.52) starting at $t = \Delta_1$ and equation (3.4.53) beginning with $t = \Delta_2$, using marching method. Considering that all the kernels are smooth functions, we use trapezoidal rule for the numerical computations. Results for the stress-controlled mode are presented in Fig.3.3-Fig.3.5.

From Fig.3.3 and Fig.3.4, we can see that $ca(t)$ and $cD(t)$ increase with the value of k and there is very little variation with β_i and k_i if k is close to 1. Note that, from Fig.3.3, the minimum contact area is only dependent on the value of k and independent of the individual values of α_i' and l_i' or β_i' and k_i'. Fig.3.3 and Fig.3.4 also confirm the result that the contact area and indentation achieve minimum a little bit later than load does for the stress-controlled mode.

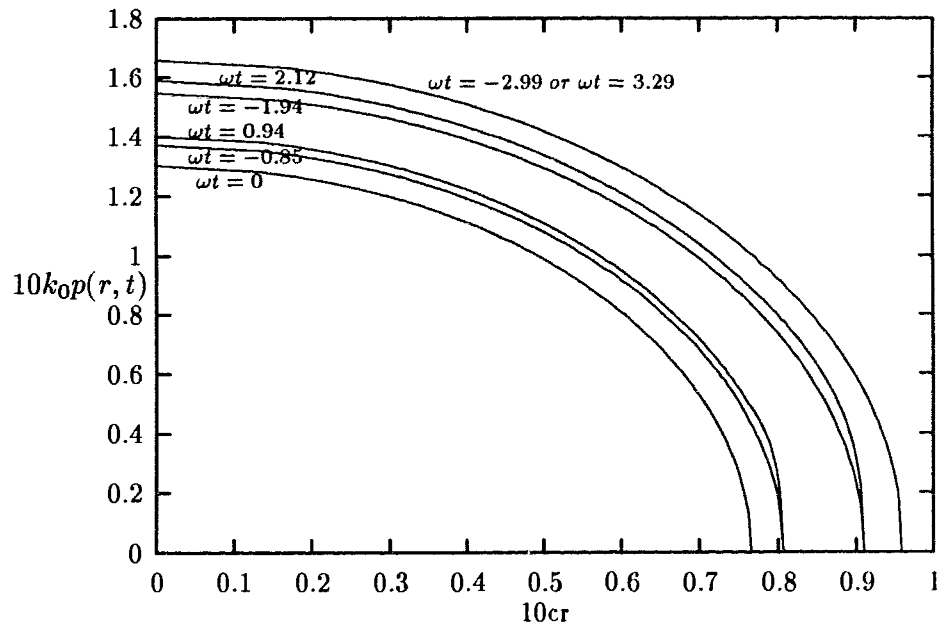

Figure 3.5: Pressure distribution for the stress-controlled mode. This picture gives the dimensionless pressure distribution $p(r,t)$ at various times during the cycle for the stress-controlled mode with $k_1' = 0.8$, $k_2' = 0.6$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.5$ and $c^2Kk_0 = 0.0008$. The lines on the top and bottom are for the times when the contact area is maximum and minimum, respectively. The pairs of lines which meet on the horizontal axis are for times, when the contact area radii are same.

(ii) Strain-Controlled Mode

For this case, indentation is taken to be

$$D(t) = N(b - cos\omega t), \ \Delta = \frac{2\pi}{\omega}, \ b \geq g \geq 1. \tag{3.4.85}$$

Here g is a constant related to the viscoelastic material and its definition is given in [6]. According to (3.4.65)(see also [9]), the maximum value of $D_e(t)$ and of $a(t)$ occurs at the same time as that of $D(t)$. Therefore

$$\Delta_2 = -\Delta_1 = \frac{\pi}{\omega}. \tag{3.4.86}$$

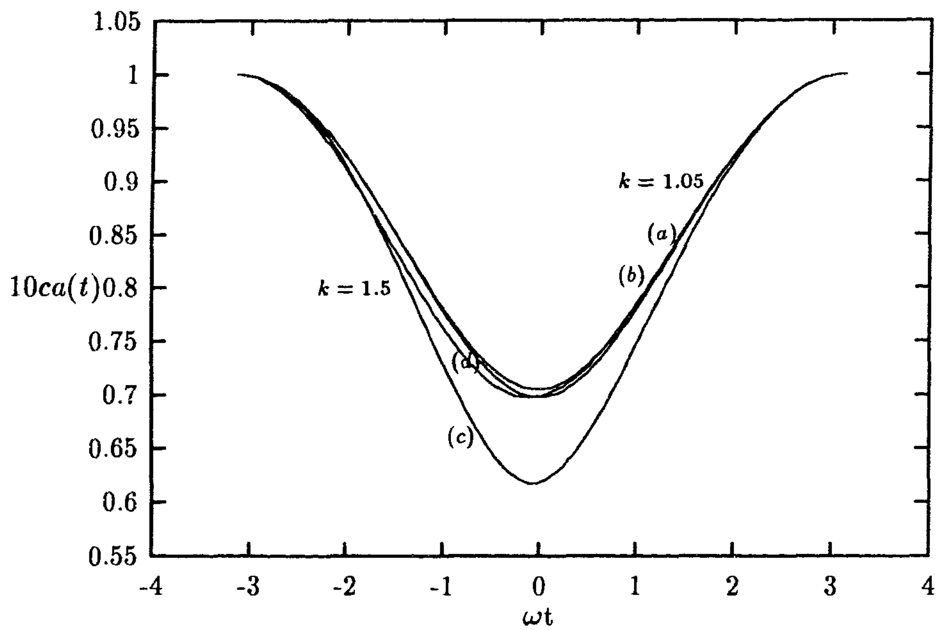

Figure 3.6: Contact area radius for strain-controlled mode. This shows the dimensionless contact area radius $ca(t)$ over a complete cycle in the strain-controlled mode for cases (a): $k'_1 = 0.0096, k'_2 = 0.003$, $\beta'_1 = 0.2, \beta'_2 = 1.5$, $k = 1.05$; (b): $k'_1 = 0.09$, $k'_2 = 0.03$, $\beta'_1 = 2$, $\beta'_2 = 6$, $k = 1.05$;(c): $k'_1 = 0.09$, $k'_2 = 0.075$, $\beta'_1 = 0.2, \beta'_2 = 1.5$, $k = 1.5$ *and* (d): $k'_1 = 0.8, k'_2 = 0.6$, $\beta'_1 = 2, \beta'_2 = 6$, $k = 1.5$ with $cN = 0.005$ and $b = 3$.

To find t_0, we use (4.11) in [9] and the same method as used to derive equation (3.4.83) to obtain

$$tan(\omega t_0) = \left\{ \sum_{i=1}^{N} \frac{l_i \omega^2}{\omega^2 + \alpha_i{}^2} \right\} / \left\{ \omega l_0 + \sum_{i=1}^{N} \frac{l_i \omega \alpha_i}{\omega^2 + \alpha_i{}^2} \right\}. \tag{3.4.87}$$

This equation is a special case of (4.25) in [9] involving loss angle of the viscoelastic material. Equation (3.4.52) and (3.4.53) give a functional equation for $t_1(t)$ when $D_e(t) = D_e(t_1(t))$ is eliminated. We can find $t_1(t)$ by solving this functional equation using iteration method. When $t_1(t)$ is known, $D_e(t)$ can be determined by either (3.4.52) or (3.4.53). Then $a(t)$,$W(t)$, $p(r,t)$ can be obtained by the same procedure as before. Numerical results for this case are presented in Fig.3.6-Fig.3.8.

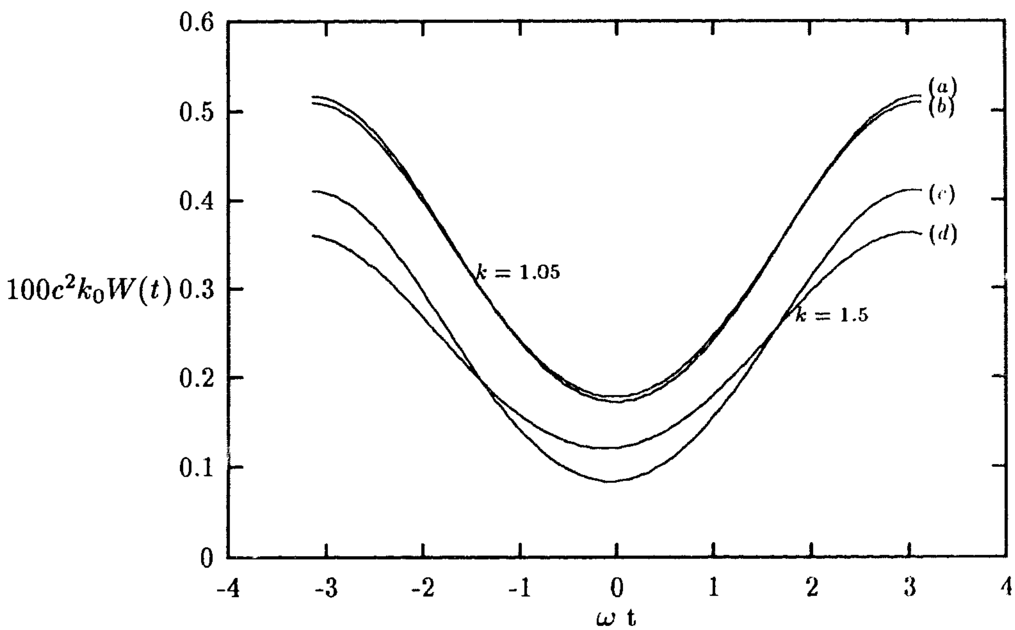

Figure 3.7: Total load for strain-controlled mode. It provides the dimensionless total load $c^2k_0W(t)$ over a complete cycle in the strain-controlled mode for the same four cases as Fig.3.6. (a): $k_1' = 0.0096$, $k_2' = 0.003$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.05$; (b): $k_1' = 0.09$, $k_2' = 0.03$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.05$;(c): $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.5$ *and* (d): $k_1' = 0.8, k_2' = 0.6$, $\beta_1' = 2, \beta_2' = 6$, $k = 1.5$ with $cN = 0.005$ and $b = 3$.

Fig.3.6 and Fig.3.7 indicate that there is very little variation with k_i' and β_i' when k is close to 1 and $ca(t)$ and $c^2k_0W(t)$ decrease with increasing values of k. From Fig.3.6 we can see that the maximum value of the contact area is independent of parameters, that is also an immediate consequence of equation (3.4.52). Furthermore we note, from these two graphs, that $ca(t)$ and $c^2k_0W(t)$ achieve their minimum values a little bit earlier than $cD(t)$ does. This follows from equation (3.4.87), noting that α_i, β_i, k_i are positive and l_i are negative.

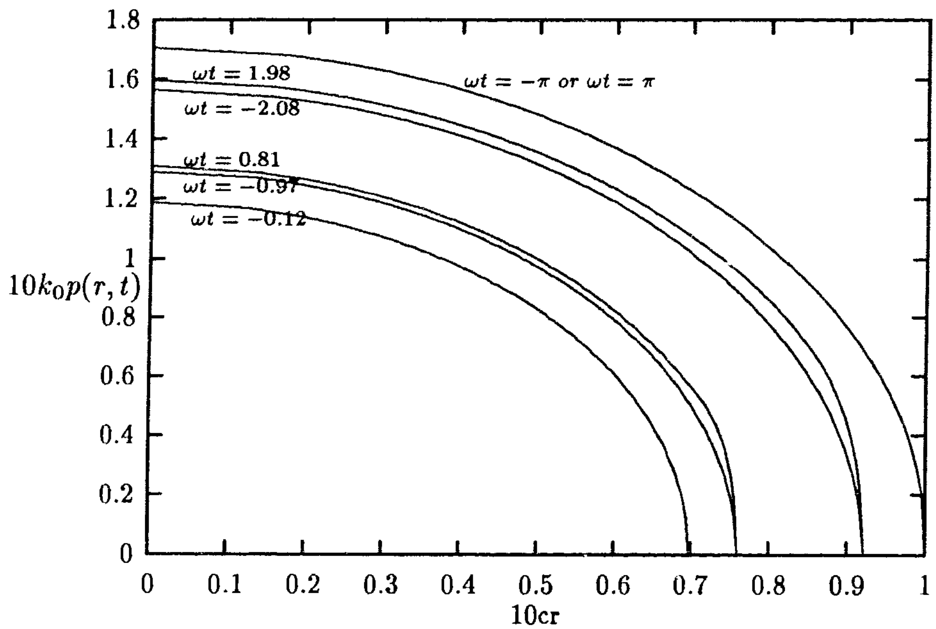

Figure 3.8: Pressure distribution for the strain-controlled mode This gives the dimensionless pressure distribution at different times in a cycle of strain-controlled mode with parameters $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2, \beta_2' = 1.5$, $k = 1.5, cN = 0.005$ and $b = 3$. The lines on the top and bottom are for the times when the contact area is maximum and minimum, respectively. The pairs of lines which meet on the horizotal axis are for times, when the contact area radii are equal.

(iii) Area-Controlled Mode

In this case we have

$$a(t) = M(a_0 - cos\omega t),\ a_0 \geq 1,\ \Delta = \frac{2\pi}{\omega}, \tag{3.4.88}$$

and

$$t_0 = 0,\ \Delta_2 = \Delta_1 = \frac{2\pi}{\omega},\ t_1(t) = -t. \tag{3.4.89}$$

Therefore $D(t)$, $W(t)$, $p(r,t)$ are easier to determine than in the other two cases. From equations (3.2.41)-(3.2.43), we can find $D_e(t)$, $k_e p_e(r,t)$ and $k_e W_e(t)$. Then equations (3.4.48)-(3.4.53) are solved numerically by trapezoidal rule to give $D(t)$, $p(r,t)$ and $W(t)$. Results are presented in Fig.3.9-Fig.3.11.

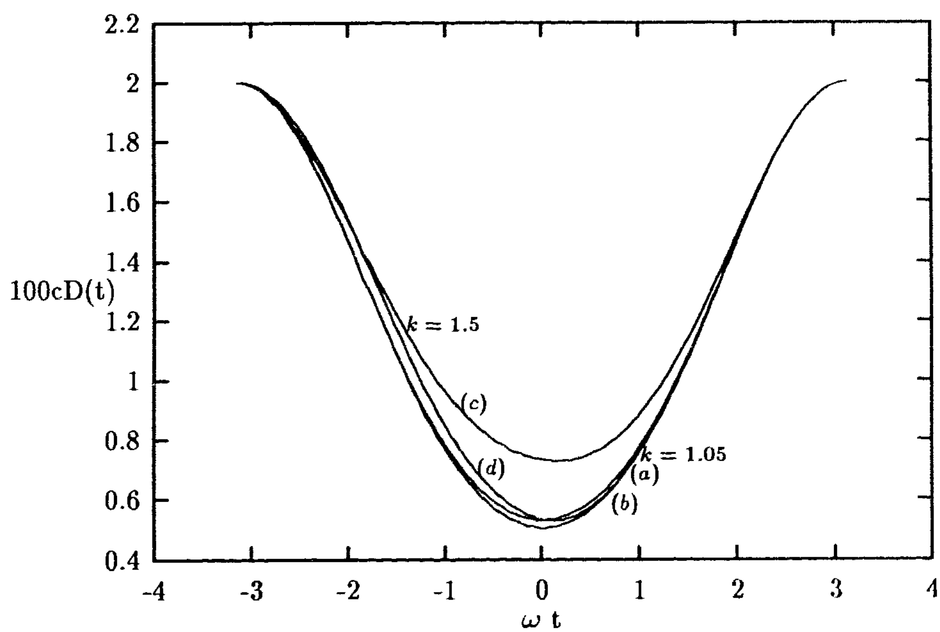

Figure 3.9: Indentation for the area-controlled mode. It shows the dimensionless indentation over a complete cycle under the area-controlled condition with parameters (a): $k_1' = 0.0096, k_2' = 0.003,\ \beta_1' = 0.2, \beta_2' = 1.5,$ $k = 1.05$; (b): $k_1' = 0.09,\ k_2' = 0.03,\ \beta_1' = 2,\ \beta_2' = 6,\ k = 1.05$;$(c)$: $k_1' = 0.09,\ k_2' = 0.075,\ \beta_1' = 0.2, \beta_2' = 1.5,\ k = 1.5$ *and* (d): $k_1' = 0.8, k_2' = 0.6,$ $\beta_1' = 2, \beta_2' = 6,\ k = 1.5, cM = 0.025$ and $a_0 = 3$.

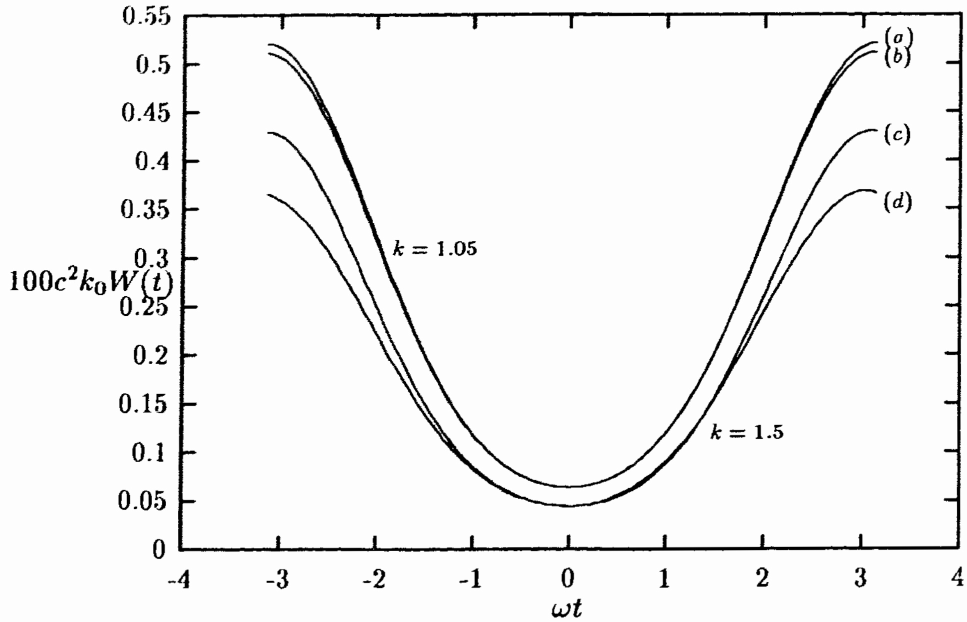

Figure 3.10: Total load for the area-controlled mode. This provides the total load over a complete cycle under the area-controlled condition with same parameters as those for Fig.3.9. (a): $k_1' = 0.0096$, $k_2' = 0.003$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.05$; (b): $k_1' = 0.09$, $k_2' = 0.03$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.05$;(c): $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.5$ *and* (d): $k_1' = 0.8$, $k_2' = 0.6$, $\beta_1' = 2$, $\beta_2' = 6$, $k = 1.5$,$cM = 0.025$ and $a_0 = 3$.

It is clear, from Fig.3.9, that the maximum indentation is independent of k_i', β_i' and k for the area-controlled case. Actually we can see this from (3.4.52) or (3.4.65). Also we notice, from Fig.3.10, that total load decreases with increase of k and the minimum value of the total load is only dependent on k.

From Fig.3.5, Fig.3.8 and Fig.3.11, the pressure distributions for the three modes, we can see that the effects of viscosity: the pressures are different even for the same contact area, or more exactly the pressures in the loading phase are larger than those at the corresponding times in the unloading phase. Note that the tendency of the pressure to develop a hump followed by a sharp decline, that was remarked in [3, 6] for positive time t, is also demonstrated.

All these results are quite similar to those given by Golden and Graham [6] for the standard linear material, but now we have five parameters k_1, k_2, β_1, β_2 and k_e (or l_1, l_2, α_1, α_2, and l_e), four of them are independent, instead of three parameters α, β and f(two of them are independent) in [6].

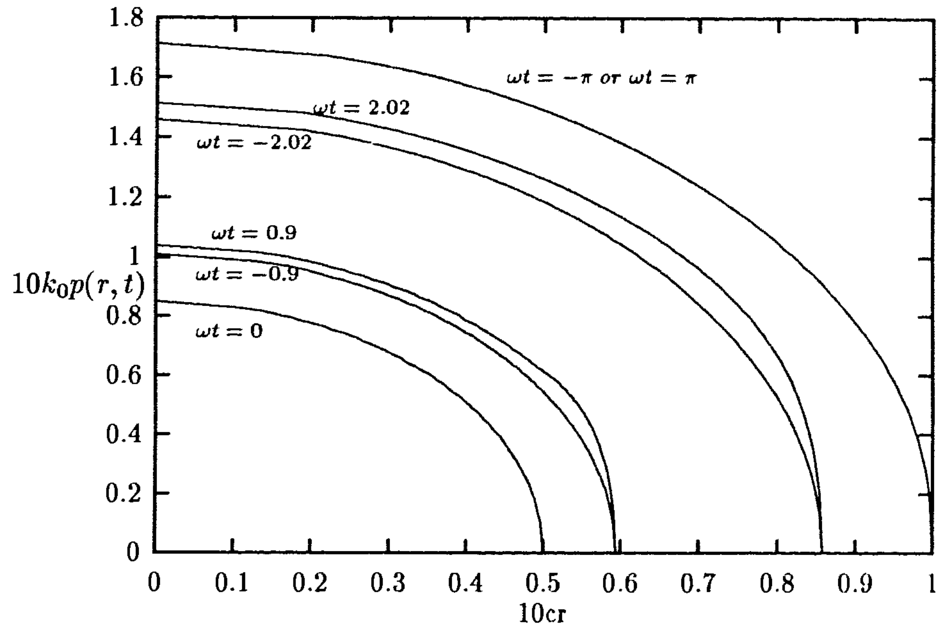

Figure 3.11: Pressure distribution of the area-controlled mode. This gives the dimensionless pressure at several times during a period under area-controlled condition for the case in which $k_1' = 0.09$, $k_2' = 0.075$, $\beta_1' = 0.2$, $\beta_2' = 1.5$, $k = 1.5$, $a_0 = 3$ and $cM = 0.025$. The lines on the top and bottom are for the times when the contact area is maximum and minimum, respectively. The pairs of lines which meet on the horizotal axis are for times, when the contact area radii are same.

Appendix

Hankel transforms of order n of a function $f(r)$ is defined as

$$\bar{f}^n(s) = H_n\{f(r); s\} = \int_0^\infty r f(r) J_n(sr) dr \tag{A.1}$$

where $J_n(x)$ is the first kind Bessel function of order n. Hankel inverse transform is

$$f(r) = \int_0^\infty s \bar{f}^n(s) J_n(sr) ds. \tag{A.2}$$

Bessel functions satisfy

$$J_n(x) = (-1)^n J_{-n}(x) \tag{A.3}$$

and

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x). \tag{A.4}$$

Fourier cosine tranform is

$$F_c\{f(t); s\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) cos(st) dt. \tag{A.5}$$

Fourier sine tranform is

$$F_s\{f(t); s\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) sin(st) dt. \tag{A.6}$$

Hankel transforms satisfy

$$H_n\left\{r^{n-1}\frac{\partial}{\partial r}[r^{1-n} f(r)]; s\right\} = -s H_{n-1}\{f(r); s\}, \tag{A.7}$$

$$\bar{f}'^n(s) = -s\left[\frac{n+1}{2n}\bar{f}^{n-1}(s) - \bar{f}^{n+1}(s)\right], \quad n \neq 0. \tag{A.8}$$

Now we can deduce the ordinary differential equation

$$(\frac{d^2}{dz^2} - s^2)^2 \bar{\Phi}^0(s, z) = 0. \tag{A.9}$$

Letting $n = 1$ in (A.3) and $n = 0$ in (A.4), we get

$$J_0'(x) = J_{-1}(x) = -J_1(x), \tag{A.10}$$

while equation (A.7) at $n = 0$ yields

$$H_0\left[\frac{1}{r}\frac{\partial}{\partial r}(rf(r)), s\right] = -sH_{-1}[f(r), s] \tag{A.11}$$

Using these twice and noting that

$$\nabla^4\Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \Phi, \tag{A.12}$$

we get equation (A.9). By virtue of the well-known integrals

$$\int_0^\infty J_0(sr)cos(st)ds = \frac{H(r-t)}{\sqrt{r^2 - t^2}}, \tag{A.13}$$

$$\int_0^\infty J_0(sr)sin(st)ds = \frac{H(t-r)}{\sqrt{t^2 - r^2}}, \tag{A.14}$$

we have

$$\begin{aligned} H_0\{s^{-1}F_c\{g(t), s\}; r\} &= \int_0^\infty J_0(sr)ds\sqrt{\frac{2}{\pi}}\int_0^\infty g(t)cos(t)dt \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(t)cos(t)dt\int_0^\infty J_0(sr)ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{g(t)H(r-t)dt}{\sqrt{r^2-t^2}} \\ &= \sqrt{\frac{2}{\pi}}\int_0^r \frac{g(t)dt}{\sqrt{r^2-t^2}}, \end{aligned} \tag{A.15}$$

and

$$H_0\{s^{-1}F_s\{g(t), s\}; r\} = \sqrt{\frac{2}{\pi}}\int_r^\infty \frac{g(t)dt}{\sqrt{t^2-r^2}}. \tag{A.16}$$

Considering

$$F_s\{g'(t), s\} = -sF_c\{g(t), s\}, \tag{A.17}$$

one deduces

$$\begin{aligned} H_0\{F_c\{g(t), s\}; r\} &= -H_0\{s^{-1}F_s\{g'(t); s\}; r\} \\ &= -\sqrt{\frac{2}{\pi}}\int_r^\infty \frac{g'(t)dt}{\sqrt{t^2-r^2}}. \end{aligned} \tag{A.18}$$

Replacing $g(t)$ by $X(t)H(1-t)$ in equations (A.16) and (A.18) yields

$$H_0\left\{s^{-1}\int_0^1 X(t)cos(st)dt; r\right\} = \int_0^r \frac{X(t)dt}{\sqrt{r^2-t^2}}, \quad 0 \le r \le 1, \tag{A.19}$$

$$H_0\left\{\int_0^1 X(t)cos(st)dt; r\right\} = 0, \quad r \ge 1. \tag{A.20}$$

Another integral we used in this paper is

$$\int_0^\infty J_1(s)cos(st)ds = 1, \;\; for \;\; 0 \le t \le 1, \tag{A.21}$$

or in a more general form

$$\int_0^\infty xJ_1(sx)cos(st)ds = \begin{cases} 1, & if \;\; 0 \le t \le x, \\ 1 - \frac{t}{\sqrt{t^2-x^2}}, & if \;\; x \le t \le 1. \end{cases} \tag{A.22}$$

Finally, let us look at the Abel's integral equation (2.0.44). Multiplying both sides of this equation by $x/\sqrt{y^2-x^2}$ and integrating over x from 0 to y, we get

$$\int_0^y \frac{xdx}{\sqrt{y^2-x^2}} \int_0^x \frac{X(t)dt}{\sqrt{x^2-t^2}} = \int_0^y \frac{x(D-S(x))dx}{\sqrt{y^2-x^2}}. \tag{A.23}$$

Changing the order of the integration on the left hand side, we have

$$\int_0^y X(t)dt \int_t^y \frac{xdx}{\sqrt{(y^2-x^2)(x^2-t^2)}} = \int_0^y \frac{x(D-S(x))dx}{\sqrt{y^2-x^2}}. \tag{A.24}$$

This equation takes the form

$$\int_0^y X(t)dt = \frac{2}{\pi} \int_0^y \frac{x(D-S(x))dx}{\sqrt{y^2-x^2}}. \tag{A.25}$$

Here use has been made of the following integral

$$\int_t^y \frac{xdx}{\sqrt{(y^2-x^2)(x^2-t^2)}} = \frac{\pi}{2}. \tag{A.26}$$

Taking derivative of equation (A.25) gives us the solution

$$\begin{aligned} X(t) &= \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{x(D-S(x))dx}{\sqrt{t^2-x^2}} \\ &= \frac{2D}{\pi} - \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{xS(x))dx}{\sqrt{t^2-x^2}}. \end{aligned} \tag{A.27}$$

Bibliography

[1] Galin, L.A., Contact Problems in the Theory of Elasticity, English translation, Edited by I.N. Sneddon, Department of Mathematics,North Carolina State College, Raleigh, 1961.

[2] Gladwell, G.M.L., Contact Problems in the Classical Theory of Elasticity, Sijhoff and Noordhoff, Alphen aan den Rijn,1980.

[3] Golden, J.M. and G.A.C. Graham, The Steady-State Plane Normal Viscoelastic Contact Problem, International Journal of Engineering Science, Vol.25, pp277-291, 1987.

[4] Golden, J.M. and G.A.C. Graham, Boundary Value Problems in Linear Viscoelasticity, Springer-Verlag, Berlin, Heidelberg, 1988.

[5] Golden, J.M. and G.A.C. Graham, A Fixed Length Crack in a Sinosoidally Loaded General Viscoelastic Medium, Continuum Mechanics and Its Applications, Edited by G.A.C. Graham and S.K. Malik, pp160-177, Hemisphere, Washington,D.C., 1989.

[6] Golden, J.M. and G.A.C. Graham, Stress, Strain and Area-Controlled Modes for the Steady-State Normal Viscoelastic Contact Problem, Ocean Waves Mechanics, Computational Fluid Dynamics and Mathematical Modelling, Edited by M. Rahman, Computational Mechanics Publications, Southampton, pp739-753, 1990.

[7] Graham, G.A.C., The Contact Problem in the Linear Thoery of Viscoelasticity When the Time Dependent Contact Area Has Any Number of Maximum and Minima, International Journal of Engineering Science, Vol.5, pp 495-514, 1967.

[8] Graham, G.A.C. and J.M. Golden, The Three-Dimensional Steady-State Viscoelastic Indentation problem, International Journal of Engineering Science, Vol.26, pp121-126, 1988.

[9] Graham, G.A.C. and J.M. Golden, The Generalized Partial Correspondence Principle in Linear Viscoelasticity, Quarterly of Applied Mathematics, Vol.46, pp527-538, 1988; Vol.49, p397, 1991.

[10] Hunter, S.C., The Hertz Problem for a Rigid Spherical Indentor and a Viscoelastic Half-space, Journal of the Mechanics and Physics of Solids, Vol.8, pp219-234, 1960.

[11] Hunter, S.C., The Solution of Boundary Value Problems in Linear Viscoelasticity, in Mechanics and Chemistry of Solid Propellants, Proceedings of the 4th Symposium on Naval Structural Mechanics, Edited by A.C. Eringen, H. Liebowitz, S.L. Koh, J.M. Crowly, Pergamon, Oxford, pp 257-295, 1966.

[12] Hunter, S.C., Mechanics of Continuous Media, Wiley, New York, 1983.

[13] Lee, E.H. and J.R.M. Radok, The Contact Problem for Viscoelastic Bodies, Journal of Applied Mechanics, Vol.27, pp438-444, 1960.

[14] Leipholz, H., Theory of Elasticity, Noordhoff International Publishing, Leyden, Netherland, 1976.

[15] Lur'e, A.I., Three-Dimensional Problems of the Theory of Elasticity, Edited by J.R.M. Radok, Interscience, New York, 1964.

[16] Muskhelishvili, N.I., Singular Integral Equations, 2nd ed., English translation by J.R.M. Radok, Noordhoff, Groningen, 1953.

[17] Muskhelishvili,N.I.,Some Basic Problems of the Mathematical Theory of Elasticity, 4th ed., English translation by J.R.M. Radok, Noordhoff, Groningen, 1963.

[18] Popov, G.Ia., Some Properties of Classical Polynomials and Their Application to Contact Problems, Journal of Applied Mathematics and Mechanics(PMM), Vol.27, pp1255-1271, 1963.

[19] Popov, G.Ia., On the Method of Orthogonal Polynomials in Contact Problems of the theory of Elasticity, Journal of Applied Mathematics and Mechanics(PMM), Vol.33, pp503-517, 1969.

[20] Sneddon, I.N., Fourier Transforms, McGraw-Hill, New York, 1951.

[21] Sneddon, I.N., The Use of the Transform Methods in Elasticity, Department of Mathematics, North Carolina State College, Raleigh, 1964.

[22] Sneddon, I.N., The Relation Between Load and Penetration in the Axisymmetric Boussinesq Problem for a Punch of Arbitrary Profile, International Journal of Engineering Science, Vol.3, pp47-57, 1965.

[23] Sokolnikoff, I. S., Mathematical Theory of Elasticity, 2nd ed., McGraw-Hill, New York, 1956.

[24] Ting, T.C.T., The Contact Stresses Between a Rigid Indentor and a Viscoelastic Half-Space, Journal of Applied Mechanics, Vol.33, pp845-854, 1966.

[25] Ting, T.C.T., Contact Problems in the Linear Theory of Viscoelasticity, Journal of Applied Mechanics, Vol.35, pp248-254, 1968.