TREE DECOMPOSITIONS OF COMPLETE GRAPHS

by

Min-li Yu

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Name: Min-li Yu

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Examining Committee:

Chairman: Dr. A. Lachlan

Dr. K. Heinrich, Professor, Senior Supervisor

Dr. B. Alspach, Professor

Dr. P. Hell, Professor

Dr. T. C. Brown, Professor

Dr. D. R. Stinson, Professor,
Department of Computing Science & Engineering
University of Nebraska,
External Examiner

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Author:

(signature)

Yu, Min-Li

(name)

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(date)

ABSTRACT

This thesis is a study of tree-factorizations of complete graph with particular emphasis on path factorizations. In the first two chapters of this thesis, we present a survey of results on G-decompositions and G-factorizations of complete graphs. In addition, we introduce some of the basic techniques which will be used to prove the main results in this thesis; in particular, the major construction lemma is presented.

In Chapter 3, we show that necessary and sufficient conditions for λK_n to have a T_k-factorization, where T_k is a tree with k vertices and satisfying certain additional assumptions, are $n \equiv 0 \pmod{k}$ and $\lambda(n-1) \equiv 0 \pmod{2(k-1)}$. Specializing these results gives necessary and sufficient conditions under which K_n has a cp-factorization, where cp is a caterpillar with an odd number of vertices (implying that the star factorization problem is completely resolved), and under which λK_n has a P_k-factorization. Previously only partial results were known in these cases.

In Chapter 4, we show that necessary and sufficient conditions for the existence of an almost resolvable P_k-factorization of λK_n are $n \equiv 1 \pmod{k}$ and $\lambda nk/2 \equiv 0 \pmod{k-1}$, and in Chapter 5, we show that necessary and sufficient conditions for λK_n to have a $(P_2(s), P_k(t))$-factorization are $n \equiv 0 \pmod{2}$, $n \equiv 0 \pmod{k}$ and $ks + 2t(k-1) = \lambda k(n-1)$.

Finally, in the last chapter, we present partial results on path factorizations of complete multipartite graphs. We show that when $n \equiv 0 \pmod{k}$ or $r \equiv 0 \pmod{k}$, $\lambda K(n, r)$ has a P_k-factorization if and only if $\lambda(r-1)nk \equiv 0 \bmod 2(k-1))$.

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Chapter 1. Introduction

Let G be a graph. A *spanning* subgraph H of G is a subgraph for which V(H) = V(G) and E(H) $\subseteq$ E(G). An H-*factor* of G is a spanning subgraph of G in which each component is isomorphic to H, and an *almost* H-*factor* of G is an H-factor of G-v for some vertex v. An H-*decomposition* of a graph G is defined to be a partition of E(G) into a set of edge-disjoint subgraphs each of which is isomorphic to H. If this set of graphs can be partitioned into H-factors (respectively almost H-factors), then we say G has a *resolvable* (respectively *almost resolvable*) H-decomposition or G has an H-*factorization* (respectively *almost* H-*factorization*). If G has an H-decomposition, then we write H | G. Similarly, we write $H \mid_R G$ if G has a resolvable H-decomposition, and $H \mid_{AR} G$ if G has an almost resolvable H-decomposition.

One natural question to ask is "Given graphs G and H, what are the necessary and sufficient conditions for the existence of an H-decomposition (an H-factorization or an almost H-factorization) of G?" In the remainder of this chapter, we will give a brief survey of some of the main results concerning these problems and also an overview of the work presented in the rest of the thesis. (All undefined terminologies are given in the appendix.)

Little work has been done when G is an arbitrary graph. Alon [1] showed that G has a t-matching decomposition if and only if $|E(G)| \equiv 0 \pmod t$ and $\Delta(G) \leq |E(G)|/t$ with only a finite number of exceptions. Caro and Schonheim [11] proved that G has a P_3-decomposition if and only if $|E(G)| \equiv 0 \pmod 2$. In general, given graphs G and H, the problem of determining if G has an H-decomposition (or a H-factorization) is very hard. Most results to date have been obtained when G = λK_n; but even in this case not a lot is known.

Suppose $|V(H)| = k$ and $n \geq k$. Necessary conditions for the existence of an H-decomposition of λK_n are:

- $\lambda n(n-1)/2 \equiv 0 \pmod{|E(H)|}$ and
- $\lambda(n-1) \equiv 0 \pmod{\gcd(d_1, d_2, ..., d_k)}$,

where d_i, $i = 1, ..., k$, are the degrees of the vertices in H. (1.1)

The first condition in (1.1) follows from the fact that the total number of edges in λK_n must be divisible by the number of edges in H. The second follows as the degree of a vertex in λK_n is the sum of some of the d_i's.

When H is a complete graph, showing the existence of an H-decomposition of λK_n is equivalent to showing the existence of a balanced incomplete block design BIBD(n, k, λ). The hundreds of papers dealing with the construction of balanced incomplete block designs testify to the interest in this problem. We will mention only the most basic results for small k and the asymptotic results of Wilson. The first concerns the so-called Steiner triple systems. There are many proofs of this result (see for example [39]).

1.1 Theorem. $K_3 \mid K_n$ if and only if $n \equiv 1, 3 \pmod 6$.

For small values of k ($k \leq 5$) K_k-decompositions were constructed by Hanani.

1.2 Theorem [15] [16]. For k = 3, 4 or 5 and every positive integer λ, $K_k \mid \lambda K_n$ if and only if $\lambda(n-1) \equiv 0 \pmod{k-1}$ and $\lambda n(n-1) \equiv 0 \pmod{k(k-1)}$, unless $(n, k, \lambda) = (15, 5, 2)$ in which case no such decomposition exists.

When $k > 5$, there are many partial results and we will not discuss them. For general k, Wilson [40] proved that the necessary conditions in (1.1) are asymptotically sufficient. We are going to state a more general version of his result; this is one of the most general theorems concerning the graph decomposition problem.

1.3 Theorem [29]. For any graph H, every sufficiently large complete graph K_n is the edge-disjoint union of copies of H, where $|V(H)| = k$, provided that

(1) $|E(H)|$ divides $n(n-1)/2$ and

(2) $n \equiv 1 \pmod{\gcd(d_1, d_2, ..., d_k)}$ where d_i, $i = 1, ..., k$, are the degrees of the vertices in H.

However, the problem of determining "exactly" when the necessary conditions are sufficient for the existence of an H-decompositions of K_n remains.

For $G = K_n$, it was shown by Harary, Robinson and Wormald [18] that the necessary condition for the decomposition of K_n into t isomorphic edge-disjoint subgraphs is sufficient, namely, that $n(n-1)/2 \equiv 0 \pmod{t}$. However, their proof does not specify exactly what the subgraph is. Thus, it is quite natural to ask this decomposition question for specified families of subgraphs H (and in fact this has been done).

Huang and Rosa [23] provided necessary and sufficient conditions for the existence of H-decompositions of K_n for all "small trees H"; that is, trees with 9 or fewer vertices. Tarsi [33] [34] gave necessary and sufficient conditions for the existence of P_k- and $K_{1,k-1}$- decompositions of λK_n (that is, path decompositions and star decompositions). In particular, Ringel [31] conjectured that for any tree T with $|E(T)| = n$, $T \mid K_{2n+1}$. Kotzig strengthened this by conjecturing that every complete graph K_{2n+1} has a cyclic decomposition into trees isomorphic to T, where $|E(T)| = n$. This is equivalent to asserting that every tree T is graceful, that is, that there exists a one to one labelling $\varphi: V(T) \rightarrow \{0, 1, ..., E(T)\}$ such that all the values $| \varphi(i) - \varphi(j) |$, where $ij \in E(T)$, are distinct. Although this problem is still unresolved, the conjecture has stimulated numerous papers dealing with various special cases. A discussion of much of this work can be found in the recent survey paper [12].

When $H = C_k$, it has long been conjectured that the conditions in (1.1) are sufficient. This problem has attracted a lot of attention. Cases for which (1.1) is sufficient include the following:

(a) $k = p^r$ or $2p^r$ for some prime p [3],

(b) $k \leq 31$ and k is odd, and $k \leq 18$ and k is even [7][9],

(c) $n \equiv 1 \pmod{2k}$ [25], and

(d) $n \equiv k \pmod{2k}$, where k is odd [24].

Details of this problem and related results are discussed in [32].

We next consider the question of the existence of resolvable H-decompositions of λK_n. Since we require the decompostions to be resolvable, then there are three obvious necessary conditions:

- $n \equiv 0 \pmod{k}$,
- $\lambda k(n-1)/2 \equiv 0 \pmod{|E(H)|}$ and
- there exists integers $x_1, x_2, ..., x_k$, such that $x_1d_1 + x_2d_2 + ... + x_kd_k = \lambda(n-1)$ and $x_1 + x_2 + ... + x_k = \lambda k(n-1)/(2(k-1))$, where $(d_1, d_2, ..., d_k)$ is the degree sequence of H. (1.2)

The first follows by observing that an H-factor is a spanning subgraph, the second follows as $|E(\lambda K_n)|$ must be a multiple of the number of edges in an H-factor, and the third follows from degree requirements (x_i is the number of factors in which a given vertex has degree d_i).

When $H = K_2$, such a decomposition is known as a 1-factorization and it is well known that λK_n has a 1-factorization if and only if $n \equiv 0 \pmod 2$. When $H = K_3$, the decomposition is known as a Kirkman triple system. Ray-Chaudhuri and Wilson [28] proved that K_n has a K_3-factorization if and only if $n \equiv 3 \pmod 6$. They also gave necessary and sufficient conditions for the existence of a resolvable K_4-decomposition of K_n. Recently, L. Zhu [42] and M. Greig [13] proved that (1.2) is sufficient when $k = 5$ and $k = 8$, respectively, except for about one hundred possible values of n in each case. (Note that when $H = K_k$, such a decomposition is equivalent to the existence of a resolvable balanced incomplete block design.)

The well known Oberwolfach problem (first formulated by Ringel and first mentioned in [14]) in the uniform case asks for a C_k-factorization of K_n. For a complete solution to this problem when $k \geq 4$ ($k = 3$ is the Kirkman triple system) see [2] and [20]. Note that this is one of the few factorization problems to be completely solved.

The case when $H = P_n$ was solved many years ago by Walecki [27]. (This is also known as a Hamilton path decomposition.) The first step towards a general solution for path factorization was made by J. Horton [21] who proved the following result:

1.4 Theorem [21]. $P_3 \mid_R \lambda K_n$ if and only if $n \equiv 0 \pmod 3$ and $\lambda(n-1) \equiv 0 \pmod 4$.

Using a result of Ray-Chaudhuri and Wilson [28], Horton also showed that the necessary conditions for the existence of a P_k-factorization of K_n are asymptotically sufficient; that is, if n is large enough and n and k satisfy the necessary conditions in (1.2), then there is a P_k-factorization of K_n. For even λ and even k the existence of P_k-factorizations of λK_n was completely resolved in [41] where it was shown that conditions in (1.2) are both necessary and sufficient. In Chapter 3, we will settle this problem completely.

Another family of trees H to be considered are stars. For $\lambda = 1$, Huang [22] proved that if k is even, then a resolvable $K_{1,k-1}$-decomposition of K_n does not exist for any n, and when k is odd she proved that the necessary conditions in (1.2) are asymptotically sufficient. Recently, Lonc [26] used similar techniques to prove that if T is a graceful tree with $|V(T)| = k$, where k is odd, then the necessary conditions in (1.2) are also asymptotically sufficient. This generalizes both the results of Horton and Huang as all paths and stars are graceful. In Chapter 3, we will also show that (1.2) is both necessary and sufficient for some other classes of trees. In particular, we give necessary and sufficient conditions for $P_k \mid_R \lambda K_n$, and also for $H \mid_R K_n$ and $H \mid_R \lambda K_n$, where H is an odd order caterpillar and λ is even. When λ is odd and $\lambda > 1$, we have a similar result but with a finite number of possible exceptions for n when k and λ are fixed. These results yield a complete answer to the question of the existence of $K_{1,k-1}$-factorizations of K_n (and so generalize Huang's result). We also extend this work to the directed case, where we consider the existence of an oriented tree factorization of a complete symmetric directed graph K_n^*.

Finally, we consider the question of the existence of almost H-factorizations of λK_n. Again we easily obtain necessary conditions for almost resolvable decompositions, namely

$$n \equiv 1 \pmod{k} \text{ and } nk\lambda/2 \equiv 0 \pmod{|E(H)|}. \tag{1.3}$$

When $H = K_2$, an almost K_2-factorization is known as a near 1-factorization, and λK_n has a near 1-factorization if and only if $n \equiv 1 \pmod{2}$. The only other family of graphs H to have been considered prior to this thesis are cycles. When $H = C_k$, from the necessary conditions, we know that λ must be even and hence it is enough to solve this problem for $\lambda = 2$. Burling and Heinrich [10] showed that there is an almost C_k-factorization of λK_n when k is even and conditions (1.3) hold. For the case k odd, Bennett and Sotteau [5] showed that the conditions of (1.3) are sufficient when $k = 3$ (these are known as almost resolvable Kirkman triple systems) and Heinrich, Lindner and Rodger [19] proved that when $k \geq 5$, the conditions of (1.3) are also sufficient. In the same way as we can think of the question of the existence of a P_k-factorization of λK_n as a generalization of that of the existence of a 1-factorization of λK_n, we can analogously view an almost P_k-factorziaton of λK_n as a generalization of an almost 1-factorization of λK_n. In Chapter 4, we will prove that $P_k \mid_{AR} \lambda K_n$ if and only if $n \equiv 1$ (mod k) and $nk\lambda/2 \equiv 0$ (mod k-1).

One generalization of the above factorization problem is what we call an $(H_1(s), H_2(t))$-factorization of λK_n. This is defined to be a partition of λK_n into s H_1-factors and t H_2-factors, where H_1, and H_2 are two given graphs. Very little is known for such factorizations. Rees [30] gave necessary and sufficient conditions for $(P_2(s), C_3(t))$-factorizations of λK_n. When $H_1 = P_2$, and $H_2 = P_3$, or $H_2 = P_4$, necessary and sufficient conditions for $(H_1(s), H_2(t))$-factorizations of λK_n are given in [41]. In Chapter 5, we will show that λK_n has a $(P_2(s), P_k(t))$-factorization if and only if $n \equiv 0$ (mod 2), $n \equiv 0$ (mod k) and $ks + 2t(k-1) = \lambda k(n-1)$.

Thinking of λK_r as a special case of $\lambda K(n, r)$ (the complete r-partite graph with part size n) leads us to the general question of necessary and sufficient conditions under which $\lambda K(n, r)$ has an H-decomposition (or a H-factorization) for a given graph H. Ushio, Tazawa and Yamamoto [38] gave necessary and sufficient conditions for $\lambda K(n, r)$ to have a $K_{1,s}$-decomposition and later Ushio [37] presented a similar result in which he also asked that the decomposition to be balanced (each vertex is required to belong to same number of $K_{1,s}$). Auerbach and Laskar [4] proved that $K(n, r)$ has a Hamilton cycle decomposition if and only if $n(r-1)$ is even.

We will consider P_k-factorizations of $\lambda K(n,r)$. It is not difficult to see that for $\lambda K(n, r)$ to have a P_k-factorization, necessary conditions are

$$nr \equiv 0 \pmod{k} \text{ and } \lambda(r-1)kn \equiv 0 \ (k-1). \qquad (1.4)$$

Ushio [36] proved that when k = 3, these conditions are sufficient. In Chapter 6, we will show that when $n \equiv 0 \pmod{k}$ or $r \equiv 0 \pmod{k}$, (1.4) is sufficient for the existence of a P_k-factorization of $\lambda K(n, r)$. (Note that this provides necessary and sufficient conditions for the existence of a P_k-factorization of $\lambda K(n, r)$ whenever k is prime.) As corollaries, we also show that these conditions are sufficient for all k when r = 2, 3.

Chapter 2. Quotient graphs and building blocks

2.1. The quotient graph

In this section we define the quotient graph of a graph, a fundamental concept essential to all the results in this thesis.

2.1.1 Definition. Let G be a k-partite graph with $V(G) = \bigcup_{i=1}^{k} X_i$. We call G *compressible* if for all i and j, $1 \leq i < j \leq k$, $|X_i| = |X_j|$ and the bipartite subgraph on vertex-set $X_i \cup X_j$ with bipartition (X_i, X_j) is $\tau(\{i, j\})$-regular, where τ is a mapping from the set $\{\{i, j\} : 1 \leq i \neq j \leq k\}$ to the non-negative integers.

2.1.2 Definition. Let G be a compressible graph. Then the *quotient graph*, Q(G), has $V(Q(G)) = \{1, 2, ..., k\}$ and the edge ij has multiplicity $\tau(\{i, j\})$, $1 \leq i < j \leq k$.

2.1.3 Remark. Suppose n = kq. We write $V(\lambda K_n) = \{(i, j) : 1 \leq i \leq q, 1 \leq j \leq k\} = \bigcup_{i=1}^{q} H_i = \bigcup_{j=1}^{k} V_j$, where $H_i = \{(i, j) : 1 \leq j \leq k\}$ and $V_j = \{(i, j) : 1 \leq i \leq q\}$. Let X be a subgraph of λK_n. If X is compressible with respect to the vertex-partition $\bigcup_{j=1}^{k} V_j$, we denote the quotient of X by $Q_V(X)$ and call it the V-quotient. If X is compressible with respect to the vertex-partition $\bigcup_{j=1}^{q} H_j$, then we call the quotient the H-quotient and denote it by $Q_H(X)$.

The concept of quotient graphs (both H- and V-quotients) will serve as a major tool in our proof. Their importance in tree-factorizations is seen in the following lemma.

2.1.4 Lemma. If G is a compressible multipartite graph and Q(G) has a factorization into r tree-factors $S^1, S^2, ..., S^r$, where S^i is a T^i-factor and T^i is a tree, then so does G.

Proof. Suppose that $V(G) = \bigcup_{j=1}^{k} V_j$ and Q(G) has a tree-factorization with tree-factors $S^1, S^2, ..., S^r$, where S^i is a T^i-factor and T^i is a tree. To each edge $pq \in E(S^i)$, associate a 1-factor F^i_{pq} from the $\tau(\{p,q\})$-regular bipartite subgraph with vertex-set $V_p \cup V_q$. Do this in such a way that the 1-factors associated with a given edge form a 1-factorization of the corresponding bipartite subgraph.
Clearly $\bigcup_{pq \in E(S^i)} F^i_{pq}$ is a T^i-factor of G. ∎

Notice that this result implies that if both $\lambda K_n - G$ and Q(G) have tree-factorizations for a given family of trees, then so too does λK_n. This is exactly the strategy we will use to prove our main results. For example, in order to show that λK_n has a $(P_2(s), T_k(t))$-factorization we will find a compressible graph G such that both λK_n - G and Q(G) have easily constructed $(P_2(s), T_k(t))$-factorizations.

2.2 Building blocks

The following basic lemmas will be used often in the rest of thesis in determining required factorizations of quotient graphs.

2.2.1 Lemma. Let G be a graph with V(G) = {1, 2, ..., k} and X = (1, 2, ..., k) be a k-cycle of G. Then

(a) the graph $G = \lambda X \cup (\bigcup_{j=1}^{N} P^j)$, where $P^1, P^2, ..., P^N$ are N vertex-disjoint paths of X with lengths $k_1, k_2, ..., k_N$ respectively, $k_i \geq 0$ and $\lambda k + k_1 + ... + k_N \equiv 0 \pmod{k-1}$.

(b) the graph $G = \lambda_1 X \cup \lambda_2 Y$, where $\lambda_1 + \lambda_2 \equiv 0 \pmod{k-1}$, k is odd, λ_2 is even and Y is the k-cycle (1, (k+1)/2, k, (k−1)/2, k−1, (k−3)/2, k−2, ..., (k+3)/2),

then G has a P_k-factorization.

Proof. **(a)** If $\lambda = 0$, then $N = 1$ and hence $G \cong P_k$ and the claim is trivial. Therefore, we assume λ is a positive integer. If all the k_i are zero, then $\lambda k = t(k-1)$ and we construct the following t P_k-factors in G:

$P(i) = [\ 1+ i(k-1), 2+ i(k-1), ..., k+ i(k-1)\ ], 0 \le i \le t-1.$

If precisely one of the k_i is not zero, we may assume $k_1 \neq 0$ and $P^1 = [1, 2, ..., k_1+1]$. Then $\lambda k + k_1 = t(k-1)$ and G has a P_k-factorization with factors:

$P(i) = [\ 1+ i(k-1), 2+ i(k-1), ..., k+ i(k-1)\ ], 0 \le i \le t-1.$

For the general case when $\lambda k+k_1+...+k_N = t(k-1)$ ($\lambda > 0$, $N \ge 0$), we apply double induction on t and N. It is not difficult to see that $t \ge 2$ and when $t = 2$ the factorization is trivial as $\lambda = 1$ and $k_1+...+k_N = k-2$. For $t > 2$ and assuming that $P^1 = [1, 2, ..., k_1+1]$, we delete the k-path [1, 2, ..., k] from G. The new graph G' has $t' = t-1$ and $N' = N$ or $N-1$. Applying the induction assumption to the resulting graph, we obtain a P_k-factorization of G.

(b) If one of λ_1 and λ_2 is zero, the P_k-factorization follows as in (a) when all k_i equal zero. Thus we assume $\lambda_1\lambda_2 \neq 0$. Let $\lambda_2 k = (k-1)p+x$, $0 < x \le k-2$ (Note that if $x = 0$, then both $\lambda_1 X$ and $\lambda_2 Y$ have P_k-factorizations). Since λ_2 and k-1 are even, then x must be even. Let $P = [1, (k+1)/2, k, (k-1)/2, k-1, (k-3)/2, ..., (k-x+3)/2, k - (x-2)/2]$ which is an (x+1)-path of Y. By (a), $\lambda_2 Y - P$ is P_k-factorable. Let $P(1) = [1, 2, ..., (k-x+1)/2]$ and $P(2) = [(k+3)/2, (k+5)/2, ..., k-(x-2)/2]$. By the definition of Y, $P(1) \cup P \cup P(2)$ is a k-path. Again by (a), $\lambda_1 X - P(1) - P(2)$ is P_k-factorable. ∎

2.2.2 Lemma. (a) If λk is even, and $k \ge 3$, then λK_k has a P_k-factorization.

(b) If λk is odd, and $k \ge 3$, then $\lambda K_k - N$, where N is a set of $(k-1)/2$ independent edges, has a P_k-factorization.

Proof. The results follow immediately from the well-known facts that K_k has a P_k-factorization when k is even, and that K_k - N has a P_k-factorization when k is odd. (To prove (b) one also needs to observe that (when k is odd) every path of length k-1 in K_k is the union of two disjoint sets of (k-1)/2 independent edges.) For completeness we now give the factorizations of K_k and K_k - N, where $V(K_k) = V(K_k - N) = \{1, 2, ..., k\}$. When k is even the paths are P(i) = [i, 1+i, k-1+i, 2+i, k-2+i, ..., k/2+2+i, k/2-1+i, k/2+1+i, k/2+i], $1 \le i \le k/2$, and when k is odd they are Q(i) = [i, 1+i, k-1+i, 2+i, k-2+i, ..., (k-1)/2-1+i, (k+1)/2+1+i, (k-1)/2+i, (k+1)/2+i], $1 \le i \le (k-1)/2$. Note that we have the freedom to choose the near 1-factors in each of the λ copies of K_k so that they form $\lfloor \lambda/2 \rfloor$ paths of length k-1 and $\lambda - 2\lfloor \lambda/2 \rfloor$ near 1-factors. When λ is odd that near 1-factor is N. ∎

2.2.3 Lemma. Let T_k be a tree on k vertices and assume that λK_k has a T_k-factorization. Then $\lambda K_{k,k} - \lambda F$, where F is a 1-factor of $K_{k,k}$, has a T_k-factorization.

Proof. We need only consider the case $\lambda = 1$. Let $V(K_{k,k}) = X \cup Y$, where $X = \{x_1, x_2,..., x_k\}$ and $Y = \{y_1, y_2,..., y_k\}$, and let $V(K_k) = \{1, 2,..., k\}$.

Assume K_k has a T_k-factorization. Let T be one of the T_k-factors in such a factorization. In $K_{k,k}$ we define the T_k-factor $\{x_iy_j, x_jy_i : ij \in E(T)\}$. Repeating for each factor in the T_k-factorization of K_k we obtain a T_k-factorization of $K_{k,k}$ - F, where $F = \{x_1y_1, x_2y_2,..., x_ky_k\}$. ∎

Notice that by relabelling vertices in Lemma 2.2.3 the 1-factor F can be chosen arbitrarily.

2.2.4 Lemma. Let k be odd.

(a) If $k \geq 3$, K_{k+1} is the union of $(k+1)/2$ k-paths and a $(k+3)/2$-path.

(b) If $k \geq 5$, then $K_{k,k}$ is the union of $(k+1)/2$ P_k-factors and one edge.

Proof. (a) Suppose $V(K_{k+1}) = \{1, 2, ..., k+1\}$. Let $S_i = [\, k+1+i, 1+i, k+i, 2+i, k+i-1, 3+i, ..., (k-1)/2+i, (k+3)/2+i \,]$, where $0 \leq i \leq (k-1)/2$. It is easy to see that the paths S_i, $0 \leq i \leq (k-1)/2$, form a set of edge-disjoint k-paths. Furthermore, the remaining edges of K_{k+1} comprises the $(k+3)/2$-path: $[k+1, k, ..., (k+1)/2]$.

(b) Assume $V(K_{k,k}) = \{u_1, u_2, ..., u_k\} \cup \{v_1, v_2, ..., v_k\}$. Let $k= 2x+1$ and let G^* be the subgraph of $K_{2x+1,2x+1}$ with edge set: $\{u_iv_{i+1}, u_{i+1}v_i, u_iv_i : i \in \{1, 2, ..., 2x+1\}\} \cup \{u_iv_{2x+2-i}, v_iu_{2x+2-i} : i \in \{2, ..., x\}\}$. (Note that subscripts are reduced modulo k.) First we are going to show that G^* is the union of a Hamilton path and a P_{2x+1}-factor.

When $x = 2$, the Hamilton path is $[u_5, v_5, u_1, v_1, u_2, v_4, u_3, v_2, u_4, v_3]$ and the P_5-factor is $[u_1, v_2, u_2, v_3, u_3] \cup [v_1, u_5, v_4, u_4, v_5]$.

When $x > 2$, the Hamilton path is $[v_{x+1}, u_{x+2}, v_x, u_{x+1}, v_{x+2}, u_x, v_{x-1}, u_{x+3}, v_{x+3}, ..., u_3, v_2, u_{2x}, v_{2x}, u_2, v_1, u_1, v_{2x+1}, u_{2x+1}]$ and the P_{2x+1}-factor is $[u_1, v_2, u_2, v_3, ..., u_x, v_{x+1}, u_{x+1}] \cup P$, where $P = [v_1, u_{2x+1}, v_{2x}, u_{2x-1}, ..., u_{x+5}, v_{x+4}, u_{x+3}, v_{x+2}, u_{x+2}, v_{x+3}, u_{x+4}, v_{x+5}, ..., v_{2x-1}, u_{2x}, v_{2x+1}]$ when x is even, and $[v_1, u_{2x+1}, v_{2x}, u_{2x-1}, ..., u_{x+4}, v_{x+3}, u_{x+2}, v_{x+2}, u_{x+3}, v_{x+4}, ..., v_{2x-1}, u_{2x}, v_{2x+1}]$ when x is odd.

Now we show that $K_{2x+1,2x+1}$ can be decomposed into two parts, so that one has a P_{2x+1}-factorization and the other is isomorphic to G^*. It is easy to see that this implies the claim of the lemma.

Let $V(K_{2x+1}) = \{\infty, 1, 2, ..., 2x\}$. The complete graph K_{2x+1} is the union of x edge-disjoint Hamilton paths and a set of x independent edges which we will specify.

When $x \equiv 0 \pmod 2$, let the Hamilton paths be $H_i = [1+i, x+i, 2+i, x-1+i, ..., (x/2)+i, (x/2)+1+i, \infty, (3x/2)+1+i, 3x/2 + i, ..., 2x-1+i, x+2+i, 2x+i, x+1+i]$, where $0 \le i \le x-1$, and the x independent edges be $N = \{(1+i)(x+1+i) : 0 \le i \le x-1\}$.

When $x \equiv 1 \pmod 2$, let the Hamilton paths be $H_i = [1+i, x+i, 2+i, x-1+i, 3+i, ..., (x+3)/2 + i, (x+1)/2 + i, \infty, (3x+1)/2 + i, (3x+3)/2 + i, ..., 2x-1+i, x+2+i, 2x+i, x+1+i]$ where $0 \le i \le x-1$, and the x independent edges be $N = \{(1+i)(x+1+i) : 0 \le i \le x-1\}$.

We now let $V(K_{2x+1,2x+1}) = \{u_\infty, u_1, ..., u_{2x}\} \cup \{v_\infty, v_1, ..., v_{2x}\}$ and let G_1 be the subgraph of $K_{2x+1,2x+1}$ induced by the edge-set $\bigcup_{i=1}^{x-1} \{u_p v_q, u_q v_p : pq \in H_i\}$, and G_2 be the subgraph induced by the edge-set $\{u_p v_q, u_q v_p : pq \in H_0 \cup N\} \cup \{u_p v_p : p \in \{\infty, 1, 2, ..., 2x\}\}$. From the definition of G_2 we can see that G_2 is isomorphic to G^*, where G^* is the union of one Hamilton path and one P_{2x+1}-factor. We claim the G_1 is P_{2x+1}-factorable since the subgraph of $K_{2x+1,2x+1}$ induced by the edge-set $\{u_p v_q, u_q v_p : pq \in H_i\}$ is a P_{2x+1}-factor. Finally, observing that deletion of the appropriate edge in a Hamilton path of $K_{2x+1,2x+1}$ yields a P_{2x+1}-factor, the proof is completed. ∎

Notice that the single edge remaining in (b) of Lemma 2.2.4 can be chosen arbitrarily.

2.2.5 Lemma. Let k be even and $k \ge 4$. The graph K_{k+1} is the union of k/2 k-paths and a k-cycle.

Proof. Let $V(K_{k+1}) = \{\infty, 1, 2, ..., k\}$. We define k/2 k-paths to be

$S_0 = [k, \infty, 1, k\text{-}1, 2, k\text{-}2, 3, ..., k/2\text{-}1, k/2\text{+}1]$, and

$S_i = [k/2 + i, \infty, 1+ i, k\text{-}1+ i, 2+ i, k\text{-}2+ i, 3+ i, k\text{-}3+ i, ..., k/2 - 1+ i, k/2 + 1 + i]$,

where $1 \leq i \leq k/2\text{-}1$.

It is not difficult to verify that if we delete the k-cycle (1, 2, ..., k) from K_{k+1}, then the remaining graph is $\bigcup_{i=0}^{k/2-1} S_i$. ∎

Chapter 3. Resolvable tree decompositions

3.1 Even tree factorizations of λK_n and tree factorizations of $2\mu K_n$.

From Chapter 1, we know that necessary conditions for $T_k \mid_R \lambda K_n$ are $n \equiv 0 \pmod{k}$, $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$ and the existence of integers $x_1, x_2, ..., x_k$, such that $x_1d_1 + x_2d_2 + ... + x_kd_k = \lambda(n-1)$ and $x_1 + x_2 + ... + x_k = \lambda k(n-1)/(2(k-1))$, where $(d_1, d_2, ..., d_k)$ is the degree sequence of T_k. We believe that they are sufficient for all trees and a goal of this chapter is to provide support for this belief.

We begin this section by considering the existence of tree factorizations of λK_n when λ is even or the tree has even order. Corollaries to the main theorem of this section provide complete answers for some interesting classes of trees. Thoughout, a tree with k vertices will be denoted by T_k.

3.1.1 Definition. A *double* 1*-factor* of the graph λG, $\lambda \geq 2$, is determined by taking a 1-factor of G and giving each edge in that 1-factor multiplicity two.

We remark that the trees we consider in this chapter satisfy either $T_k \mid K_k$ or $T_k \mid 2K_k$. It is easy to check that in both cases the third necessary condition (as stated above) is trivially satisfied when $k\lambda$ is even.

3.1.2 Theorem. Suppose that $k\lambda$ is even.

(1) If $T_k \mid K_k$, then $T_k \mid_R \lambda K_n$ if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

(2) If $T_k \mid 2K_k$, then $T_k \mid_R 2\mu K_n$ if and only if $n \equiv 0 \pmod{k}$ and $\mu k(n-1) \equiv 0 \pmod{k-1}$.

Proof. Before starting the proof, observe that the assumption $T_k \mid K_k$ implies that k is even. Let $\lambda^* \in \{1, 2\}$. We show that if $T_k \mid \lambda^* K_k$, then for $\lambda \equiv 0 \pmod{\lambda^*}$,

$T_k \,|_R\, \lambda K_n$ if and only if $n \equiv 0 \pmod k$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$. It is easy to see that this statement implies both (1) and (2). The necessity of each of the conditions can be established easily by applying counting arguments to vertices and edges. For the sufficiency, we will show that $T_k \,|_R\, \lambda K_n$, where $\lambda \equiv 0 \pmod{\lambda^*}$ by constructing a compressible subgraph G of λK_n such that both Q(G) and $\lambda K_n - G$ have a T_k-factorization. Let $n = kq$. The given congruence conditions imply $\lambda(q-1) \equiv 0$ (mod $2(k-1)$) when k is odd (then λ must be even) and $\lambda(q-1) \equiv 0 \pmod{k-1}$ when k is even (since $\lambda k(n-1) = \lambda k(kq-1) = \lambda k(k(q-1) + k-1)$).

Let $V(\lambda K_n) = \{(i, j) : 1 \le i \le q, 1 \le j \le k\} = \bigcup_{i=1}^{q} H_i = \bigcup_{j=1}^{k} V_j$, where $H_i = \{(i, j) : 1 \le j \le k\}$ and $V_j = \{(i, j) : 1 \le i \le q\}$. We will use the fact that $\lambda K_n = \lambda(K_q \otimes K_k)$. (See appendix for the definition of $H \otimes G$.)

The proof of the theorem is divided into two parts depending on the parity of q.

Case 1: q even.

When q is even K_q admits a 1-factorization $\{F_1, F_2, ..., F_{q-1}\}$. To each 1-factor F_i, $1 \le i \le q-1$, there corresponds in λK_n a subgraph which is the vertex disjoint union of q/2 copies of $\lambda K_{k,k}$. By Lemma 2.2.3, $\lambda K_{k,k}$ is the union of $\lambda k/2$ T_k-factors and λ 1-factors. Notice that when λ is odd, λ different 1-factors can be used, and if λ is even the $\lambda/2$ double 1-factors can be chosen independently. Furthermore, each of the subgraphs λK_k of λK_n associated with the vertices of K_q has a T_k-factorization consisting of $\lambda k/2$ T_k-factors. Removing these $(q-1)\lambda k/2 + \lambda k/2 = \lambda kq/2$ T_k-factors from λK_n leaves a subgraph which we will denote by R.

Notice that R is not uniquely determined in the sense that we have considerable freedom in arranging the 1-factors remaining in each subgraph K_{H_i,H_j}, $1 \le i < j \le q$. We need to show that they can be chosen so as to produce a subgraph R which has a T_k-factorization. This is done by choosing R so that $Q_V(R)$ has a T_k-factorization.

We first consider the case when k is odd. By assumption $T_k \mid 2K_k$. In R there are $\lambda/2$ double 1-factors in K_{H_i,H_j}. Since $\lambda(q-1) \equiv 0 \pmod{2(k-1)}$ and k is odd, then $\lambda(q-1)/2$ is even and so (as q is even) $\lambda \equiv 0 \pmod 4$. It is not difficult to see that R is compressible and $Q_H(R) = (\lambda/2)(2K_q)$ and $(\lambda/2)K_q$ has a Hamilton cycle factorization with cycles $W_1, W_2, ..., W_{\lambda(q-1)/4}$. Each edge ij of W_x corresponds to a double 1-factor in K_{H_i,H_j}. Since k is odd, K_k also has a Hamilton cycle factorization with cycles $Y_1, Y_2, ..., Y_{(k-1)/2}$. We show that for a given Y_i, we can use any W_j to construct a subgraph of R with $2Y_i$ as its V-quotient. Assuming this it follows that for any fixed set of (k-1)/2 Hamilton cycles in $(\lambda/2)K_q$, we can construct a subgraph of R with $Q_V(R) = 2K_k$.

Without loss of generality, assume $W_j = (1, 2, ..., q)$ and $Y_i = (y_1, y_2, ..., y_k)$. In $K_{H_s,H_{s+1}}$, $1 \le s \le q$, we choose the double 1-factor $\{(s, y_1)(s+1, y_2), (s, y_1)(s+1, y_2), (s, y_2)(s+1, y_3), (s, y_2)(s+1, y_3), ..., (s, y_{k-1})(s+1, y_k),(s, y_{k-1})(s+1, y_k), (s, y_k)(s+1, y_1), (s, y_k)(s+1, y_1),\}$. Let this subgraph of R be R'. Then there is a double 1-factor between V_{y_j} and $V_{y_{j+1}}$, where $1 \le j \le k$. Therefore R' is compressible and has $2Y_i$ as its V-quotient.

Since $\lambda(q-1) \equiv 0 \pmod{2(k-1)}$, we know that $\lambda(q-1)/4 \equiv 0 \pmod{(k-1)/2}$. Assume $\lambda(q-1)/4 = m(k-1)/2$ and construct R with $Q_V(R) = 2mK_k$ which, by assumption, is T_k-factorable. Note that the edges of λK_n-R contribute $\lambda kq/2$ T_k-factors and those of R yield a further $\lambda k(q-1)/(2(k-1))$ for a total of $\lambda k(n-1)/(2(k-1))$ T_k-factors. So we have all the T_k-factors.

We now consider the case when k is even. The graph K_k admits a 1-factorization with 1-factors $f_1, f_2, ..., f_{k-1}$. (Recall that $\{F_1, F_2, ..., F_{q-1}\}$ is a

1-factorization of K_q.) We will show that for f_i, $1 \leq i \leq k-1$, we can use any F_j, $1 \leq j \leq q-1$, to construct a subgraph of R which has f_i as its V-quotient. We will then use k-1 of the 1-factors F_j, $1 \leq j \leq q-1$, to construct a subgraph of R such that $Q_V(R) = K_k$.

Without loss of generality, assume $f_i = \{y_1y_2, y_3y_4, ..., y_{k-1}y_k\}$ and $F_j = \{12, 34, ..., (q-1)q\}$. In (H_{2s-1}, H_{2s}), choose the 1-factor $\{(2s-1, y_1)(2s, y_2), (2s-1, y_2)(2s, y_1), (2s-1, y_3)(2s, y_4), (2s-1, y_4), (2s, y_3), ..., (2s-1, y_{k-1})(2s, y_k), (2s-1, y_k)(2s, y_{k-1})\}$, where $s \in \{1, 2, ..., q/2\}$. Let this subgraph of R be R'. There is a 1-factor between V_{y_j} and $V_{y_{j+1}}$, where $1 \leq j \leq k$, and therefore R' is compressible and $Q_V(R') = f_i$.

When $\lambda^* = 1$, there are λ 1-factors of K_{H_i,H_j} in R which can be chosen independently. (Notice that λK_q is the union of $\lambda(q-1)$ edge disjoint 1-factors.) Since $\lambda(q-1) \equiv 0 \pmod{k-1}$, let $m = \lambda(q-1)/(k-1)$ and choose R such that $Q_V(R) = mK_k$ which is T_k-factorable.

When $\lambda^* = 2$, there are $\lambda/2$ double 1-factors of K_{H_i,H_j} each of which can be choosen independently. From the condition that $\lambda(q-1) \equiv 0 \pmod{(k-1)}$ and the fact that k and λ are both even we have $\lambda(q-1)/2 \equiv 0 \pmod{k-1}$. Let $m = \lambda(q-1)/(2(k-1))$. It is not difficult to see that we can choose R such that $Q_V(R) = 2mK_k$.

In either case we obtain a T_k-factorization of λK_n with $\lambda k(n-1)/(2(k-1))$ T_k-factors.

Case 2: q is odd

In this case K_q admits a near 1-factorization with near 1-factors $NF_1, NF_2, ..., NF_q$. To each NF_i, there corresponds, in λK_n, a subgraph which is the vertex-disjoint union of (q-1)/2 copies of $\lambda K_{k,k}$ and one copy of λK_k. By Lemma 2.2.3 and the assumption that $T_k \mid_R \lambda K_k$, each near 1-factor produces $\lambda k/2$ T_k-factors. Removing these $\lambda kq/2$ T_k-factors from λK_n, leaves a subgraph S consisting of λ 1-factors in K_{H_i,H_j}, where $1 \le i < j \le q$. We will show that S can be chosen so that it has a T_k-factorization. Again we consider separately the cases k odd and k even.

If k is odd, then λ is even and both $(\lambda/2)(2K_q)$ (which we can think of as the H-quotient of R) and K_k have Hamilton cycle factorizations. We use the same method as in the case when q was even to achieve a factorization.

If k is even, then K_k admits a 1-factorization with 1-factors $f_1, f_2, ..., f_{k-1}$. Since q is odd, K_q has a Hamilton cycle decomposition. We show that for a given f_i, we can use any one of the Hamilton cycles, say C, in this decomposition of K_q, say H, to construct a subgraph of S such that its V-quotient is $2f_i$. Then we can use the same method as before to construct a subgraph of S with quotient $2K_k$.

Without loss of generality, assume that $f_i = \{y_1y_2, y_3y_4, ..., y_{k-1}y_k\}$ and $C = (1, 2, ..., q)$. We choose in $K_{H_s,H_{s+1}}$ the 1-factor $\{(s, y_1)(s+1, y_2)$, $(s, y_2)(s+1, y_1)$, $(s, y_3)(s+1, y_4)$, $(s, y_4)(s+1,y_3)$,..., $(s, y_{k-1})(s+1, y_k)$, $(s, y_k)(s+1, y_{k-1})\}$, where $1 \le s \le q$. Let this subgraph of S be S'. Then the induced graph on vertex set $V_{y_j} \cup V_{y_{j+1}}$, where $1 \le j \le k$, is a 2-factor. Therefore S' is compressible and $Q_V(S') = 2f_i$.

When $\lambda^* = 1$, $\lambda(q-1) \equiv 0 \pmod{k-1}$ is equivalent to $\lambda(q-1) \equiv 0 \pmod{2(k-1)}$ as k is even. In this case there are λ 1-factors of K_{H_i,H_j} in S which can be chosen independently. (Notice that λK_q is the union of $\lambda(q-1)/2$ Hamilton cycles.) Assume $2m(k-1) = \lambda(q-1)$. Then we can construct S such that $Q_V(S) = 2mK_k$. (Note that for each Hamilton cycle in K_q, we obtain in $Q_V(S)$ a copy of $2f_i$ for some i.) By assumption $T_k \mid 2mK_k$.

When $\lambda^* = 2$, there are $\lambda/2$ double 1-factors of K_{H_i,H_j} in S which can be chosen independently. Notice that $(\lambda/2)K_q$ is the union of $\lambda(q-1)/4$ Hamilton cycles. It is easy to see that as k, λ and q-1 are even, $\lambda(q-1) \equiv 0 \pmod{k-1}$ implies $\lambda(q-1) \equiv 0 \pmod{4(k-1)}$. Assuming $4m(k-1) = \lambda(q-1)$, we can construct an S such that $Q_V(S) = 4mK_k$ which by assumption is T_k-factorable. (Notice that in this case, for each Hamilton cycle in K_q, we obtain in $Q_V(S)$ a copy of $4f_i$ for some i.) ∎

We now give some classes of trees for which either we have a tree-factorization of K_k or of $2K_k$.

3.1.3 Corollary: When k is even, $P_k \mid_R \lambda K_n$ if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

Proof. We know that when k is even, K_k has a P_k-decomposition. Hence the claim follows immediately from Theorem 3.1.2(a). ∎

There are also families of trees T_k, k even, for which $T_k \mid K_k$. Several examples are given in [23]. (These include trees with a certain symmetry property.) Hence for these trees we have necessary and sufficient conditions for the existence of tree factorizations of λK_n.

When λ is even we can obtain necessary and sufficient conditions for tree factorizations of λK_n for another family of trees; namely graceful trees, which we have already defined in Chapter 1.

3.1.4 Corollary: Let T_k be a graceful tree. Then $T_k \mid_R 2\mu K_n$ if and only if $n \equiv 0 \pmod{k}$ and $\mu k(n-1) \equiv 0 \pmod{k-1}$.

Proof. Since T_k is graceful, $T_k \mid_R 2K_k$ (a short proof can be found in [26]). Hence the claim follows immediately from Theorem 3.1.2(b). ∎

3.1.5 Corollary: When λ is even, $P_k \mid_R \lambda K_n$ if and only if $n \equiv 0 \pmod k$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

Proof. We know that P_k is a graceful tree [12]. Hence the claim follows immediately from Corollary 3.1.4. ∎

We complete this section by looking briefly at a directed analogue of the tree factorizations. Let K_n^* be the complete symmetric digraph on n vertices. Let di-P_k be a directed path of length k-1, i-$K_{1,k-1}$ be a directed star with all arcs directed towards the centre and o-$K_{1,k-1}$ be a directed star with all arcs directed away from the centre. Let $A = \{\text{di-}P_k, \text{i-}K_{1,k-1}, \text{o-}K_{1,k-1}: k = 2, 3, ...\}$.

It is not difficult to see that the techniques used above can be used to prove the following results.

3.1.6 Theorem: Let DT_k be an oriented tree obtained by assigning an orientation to each edge of T_k. Under the assumption the $DT_k \mid K_k^*$, $DT_k \mid_R \lambda K_n^*$ if and only if $n \equiv 0 \pmod k$ and $\lambda k(n-1) \equiv 0 \pmod{k-1}$.

3.1.7 Corollary: Let $X \in A- \{\text{di-}P_3, \text{di-}P_5\}$ and $k = |V(X)|$. Then $X \mid_R \lambda K_n^*$ if and only if $n \equiv 0 \pmod k$ and $\lambda k(n-1) \equiv 0 \pmod{k-1}$.

Proof. It is easy to see that i-$K_{1,k-1} \mid K_k^*$ and o-$K_{1,k-1} \mid K_k^*$, and that when k is even, di-$P_k \mid K_k^*$. When k is odd, Tillson [35] showed that di-$P_k \mid K_k^*$ provide $k \geq 7$. It is easy to show that di-P_k does not factor K_k^*, $k \in \{3, 5\}$. Therefore the claim follows immediately from Theorem 3.1.6. ∎

3.2 Odd tree factorizations of K_n

As before, T_k denotes a tree with k vertices. In this section, we give necesary and sufficient conditions for K_n to have a T_k-factorization when k is odd and T_k has certain properties. Recall that necessary conditions for the existence of a T_k-factorization of K_n are $n \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{2(k-1)}$. Letting $n = km$, we see that as k is odd, m must be odd as well. We will show that under the assumptions that $T_k \mid 2K_k$, where k is odd and T_k is bipartite spreadable (which we define next), we can construct T_k-factorizations of K_{mk} for all admissible m. But first we introduce a definition.

3.2.1 Definition. Let T_k be a tree on k vertices with $V(T_k) = \{1, 2, ..., k\}$. We call T_k *bipartite spreadable* if for some i, $1 \le i \le k$, T_k has bipartite representation: $\{a_1, a_2, ..., a_i\} \cup \{b_{i+1}, b_{i+2}, ..., b_k\} = \{1, 2, ..., k\}$, so that $\{b_p - a_q \pmod{k}: a_q b_p \in E(T_k)\} = \{1, 2, ..., k-1\}$.

Let $m = 2t+1$ and $V(K_{2t+1}) = \{\infty, 1, 2, ..., 2t\}$. Let Z_i be the Hamilton cycle of K_{2t+1} described by $Z_i = (\infty, i+1, i+2, i+2t, i+3, i+2t-1, ..., i+t, i+t+2, i+t+1)$, $0 \le i \le t-1$, where calculations are modulo 2t on the residues 1, 2, ..., 2t. We define $\mathfrak{I} = \{Z_0, Z_1, ..., Z_{t-1}\}$ and observe that $\mathfrak{I}$ is a Hamilton cycle factorization of K_{2t+1}. Finally, for convenience we write $Z_i = (a_{0,i}, a_{1,i}, ..., a_{2t,i})$ where $a_{0,i} = \infty$, $a_{2p-1,i} = i+2t+2-p$ and $a_{2p,i} = i+p+1$, $1 \le p \le t$.

Let $k = 2s+1$ and $V(K_n) = \bigcup_{j=1}^{2s+1} H_j = (\bigcup_{i=1}^{m-1} V_i) \cup V_\infty$ where $V_i = \{(1, i), (2, i), ..., (2s+1, i)\}$, for $i \in \{\infty, 1, ..., m-1\}$, and $H_j = \{(j, \infty), (j, 1), (j, 2), ..., (j, m-1)\}$ for $j \in \{1, 2, ..., 2s+1\}$. Let $Z = (x_1, x_2, ..., x_m)$ be an m-cycle of K_m. We define M(Z), a subgraph of K_n, to be $\overline{K}_{2s+1} \otimes Z$, and hence $(u, v)(p, q) \in E(M(Z))$ if and only if $\{v, q\} = \{x_i, x_{i+1}\}$ for some i. Clearly, we can view K_n as the union of m vertex-disjoint copies of K_{2s+1} on the vertex-sets V_∞, V_1, ... , V_{m-1}, and the (m-1)/2 edge-disjoint subgraphs isomorphic to $M(Z_j) = \overline{K}_{2s+1} \otimes Z_j$, where $Z_j \in \mathfrak{Z}$. We now present several technical lemmas which are essential for the proof of the main theorem. The first of our lemmas investigates properties of the subgraph M(Z).

3.2.2 Lemma. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. Let Z be an m-cycle of K_m. Then M(Z), is the edge-disjoint union of 2s+1 T_{2s+1}-factors and a subgraph S which consists of 2s+1 edge-disjoint m-cycles. If $m \equiv 0 \pmod{2s+1}$, S can be chosen to be a compressible graph such that $Q_H(S) \cong C_{2s+1}$.

Proof. Let $Z = (x_0, x_1, ..., x_{m-1})$. By the definition of bipartite spreadable, T_{2s+1} has bipartite representation (A', B'), where $A' \cap B' = \emptyset$ and $A' \cup B' = \{1, 2, ..., 2s+1\}$. Then 2s+1 edge-disjoint T_{2s+1}-factors of M(Z) are $G_i = \bigcup_{j=0}^{m-1} \{(a_r + i, x_j)(b_t + i, x_{j+1}) : a_r \in A', b_t \in B' \text{ and } a_rb_t \in E(T_{2s+1})\}$, $0 \le i \le 2s$. The edges of $S = M(Z) - \bigcup_{i=0}^{2s} G_i$ are made up of 2s+1 edge-disjoint m-cycles; as $S = \bigcup_{i=1}^{2s+1} \{((i, x_0), (i, x_1), ..., (i, x_{m-1}))\}$. If $m \equiv 0 \pmod{2s+1}$, we can relabel the vertices of M(Z) so that $S = \bigcup_{i=1}^{2s+1} \{((i, x_0), (i+1, x_1), ..., (i+m-1, x_{m-1}))\}$. Then S is a compressible graph and $Q_H(S) \cong C_{2s+1}$. ∎

We wish to use a similar idea in the case when $m \not\equiv 0 \pmod{2s+1}$. To do this we need the notion of a y-variation cycle.

3.2.3 Definition. Let $V(K_n)$ be defined as above, where n = mk. An m-cycle C of K_n is a y-*variation* cycle if

(1) $V(C) \cap V_i \neq \emptyset$, $i \in \{\infty, 1, ..., m-1\}$, and

(2) if C* is the directed cycle obtained from C, then C* has y A-arcs and m-y B-arcs, or m-y A-arcs and y B-arcs, where $((x_1, y_1), (x_2, y_2))$ is an A-arc if $x_2 = x_1+1$, and a B-arc if $x_2 = x_1-1$. (Note that the first coordinate is reduced to modulo k to the residues 1, 2, ..., k.)

3.2.4 Remark: Assume $m \not\equiv 0 \pmod{2s+1}$, m is odd, and there exists a positive even integer y such that $m-2y \equiv 0 \pmod{2s+1}$ and $m - 2y \geq 0$. Then we can construct a y-variation m-cycle in M(Z), where Z is an m-cycle of K_m. For example, if $Z = (\infty, 1, 2, ..., m-1)$, then $C = ((1, \infty), (2, 1), ..., (y, y-1), (y+1, y), (y, y+1), (y-1, y+2), ..., (3, m-2), (2, m-1))$ is a y-variation m-cycle. Thus we need to know when a suitable value for y exists.

The next lemma shows that if $m \geq 6s+1$, such y always exists.

3.2.5 Lemma. Let $m \geq 6s+1$ and m is odd. Then there exists a positive even integer y such that $m - 2y \equiv 0 \pmod{2s+1}$ and $0 \leq y \leq 4s$.

Proof. First we show that there exists a positive even integer y such that $m - 2y \equiv 0 \pmod{2s+1}$. If $m \equiv 0 \pmod{2s+1}$ it suffices to choose $y = 0$. So we assume $m \not\equiv 0 \pmod{2s+1}$. If $m - (2s+1) \equiv 0 \pmod 4$, then we put $y = (m-(2s+1))/2$. Otherwise, $m - 3(2s+1) \equiv 0 \pmod 4$ and $y = (m - 3(2s+1))/2$. We now show that y can be chosen between 0 and 4s. Observe first that $y \not\equiv 0 \pmod{2s+1}$. If $y > 4s$, then write $y = (4s+2)p+q$, where $0 < q \leq 4s$. Since y is even q must also be even and hence we can replace y by q. ∎

3.2.6 Lemma. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. Assume $m \not\equiv 0 \pmod{2s+1}$, and let $y \geq 4$ be an even integer such that $m-2y \equiv 0 \pmod{2s+1}$, $m - 2y \geq 0$ and $y \leq 4s$. Then $M(Z_0) \cup M(Z_1) \cup M(Z_{y/2})$ is the edge-disjoint union of $3(2s+1)$ T_{2s+1}-factors and a subgraph S where $Q_H(S) \cong 3C_{2s+1}$.

Proof. Using Remark 3.2.4, we construct three y-variation m-cycles in K_n corresponding to Z_0, Z_1 and $Z_{y/2}$ respectively, as follows:

$B_1 = ((1, a_{0,0}), (2, a_{1,0}), ..., (y, a_{y-1,0}), (y+1, a_{y,0}), (y, a_{y+1,0}), (y-1, a_{y+2,0}), \ldots,$

$(3, a_{m-2,0}), (2, a_{m-1,0})),$

$B_2 = ((1, a_{y-2,1}), (2, a_{y-1,1}), \ldots, (y, a_{2y-3,1}), (y+1, a_{2y-2,1}), (y, a_{2y-1,1}),$

$(y-1, a_{2y,1}), \ldots, (3, a_{y-4,1}), (2, a_{y-3,1})),$

$B_3 = ((1, a_{0,y/2}), (2s+1, a_{1,y/2}), (2s, a_{2,y/2}), \ldots, (2s-y+3, a_{y-1,y/2}),$

$(2s-y+2, a_{y,y/2}), (2s-y+3, a_{y+1,y/2}), (2s-y+4, a_{y+2,y/2}), \ldots, (2s, a_{m-2,y/2}),$

$(2s+1, a_{m-1,y/2})).$

By Lemma 3.2.2, we know that $M(Z_0)$ is the union of 2s+1 T_{2s+1}-factors and a subgraph S_1, where S_1 is a collection of 2s+1 edge-disjoint m-cycles. We now use B_1 to determine S_1 and define

$$E(S_1) = \bigcup_{j=0}^{2s} \{(u+j, v)(w+j, z): (u,v)(w,z) \in E(B_1)\}.$$

Similarly we can define S_2 and S_3 corresponding to B_2 and B_3, respectively:

$$E(S_2) = \bigcup_{j=0}^{2s} \{(u+j, v)(w+j, z): (u,v)(w,z) \in E(B_2)\} \text{ and}$$
$$E(S_3) = \bigcup_{j=0}^{2s} \{(u+j, v)(w+j, z): (u,v)(w,z) \in E(B_3)\}.$$

We delete the 3(2s+1) T_{2s+1}-factors from $M(Z_0) \cup M(Z_1) \cup M(Z_{y/2})$ and what remains is the subgraph S which consists of the 3m(2s+1) edges of S_1, S_2 and S_3. We will show that the subgraph of S with vertex-set $H_i \cup H_j$ is 3-regular if j = i+1 or j= i-1, and empty otherwise. From this it follows that S is compressible and then $Q_H(S) \cong 3C_{2s+1}$.

By the definition of S_i, $1 \le i \le 3$, we see there is no edge in S from H_i to H_j if $j \neq i+1$ or $j \neq i-1$. Now we consider the subgraph of S with vertex-set $H_i \cup H_{i+1}$, $1 \le i \le 2s+1$, and determine the degree of vertices (i, j) and (i+1, j) where $j \in \{\infty, 1, \ldots, m-1\}$.

We will denote by $\deg_{S_j}(v)$ the degree of vertex v in $H_i \cup H_{i+1}$, which is contributed from S_j, $1 \le j \le 3$.

$\deg_{S_1}((i, a_{y,0})) = \deg_{S_1}((i+1, a_{0,0})) = 0$
$\deg_{S_1}((i, a_{0,0})) = \deg_{S_1}((i+1, a_{y,0})) = 2$
$\deg_{S_1}((i, a_{j,0})) = \deg_{S_1}((i+1, a_{j,0})) = 1 \quad$ if $j \neq 0, y$.

$\deg_{S_2}((i, a_{2y-2,1})) = \deg_{S_2}((i+1, a_{y-2,1})) = 0$
$\deg_{S_2}((i, a_{y-2,1})) = \deg_{S_2}((i+1, a_{2y-2,1})) = 2$
$\deg_{S_2}((i, a_{j,1})) = \deg_{S_2}((i+1, a_{j,1})) = 1 \quad$ if $j \neq y-2, 2y-2$.

$\deg_{S_3}((i, a_{0,y/2})) = \deg_{S_3}((i+1, a_{y,y/2})) = 0$
$\deg_{S_3}((i, a_{y,y/2})) = \deg_{S_3}((i+1, a_{0,y/2})) = 2$
$\deg_{S_3}((i, a_{j,y/2})) = \deg_{S_3}((i+1, a_{j,y/2})) = 1 \quad$ if $j \neq 0, y$.

Recall that we defined $a_{0,0} = a_{0,y/2} = \infty$, $a_{y,0} = a_{y-2,1} = 1 + y/2$ and $a_{2y-2,1} = a_{y,y/2} = y+1$. Hence $\deg_S((i, j)) = \deg_S((i+1, j)) = 3$, where $j \in \{\infty, 1, \dots, m-1\}$. Therefore $Q_H(S)$ is indeed a (2s+1)-cycle with multiplicity 3. This completes the proof. ∎

Notice that $M(Z_0) \cup M(Z_1) \cup M(Z_{y/2}) \cong M(Z_i) \cup M(Z_{i+1}) \cup M(Z_{i+y/2})$, where i is any positive integer. Also, by relabelling if necessary, we can assume $C_{2s+1} = (1, 2, \dots, 2s+1)$.

3.2.7 Lemma. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. If $m = 6s+1$, then $M(Z_0) \cup M(Z_s) \cup M(Z_{2s})$ is the edge-disjoint union of 3(2s+1) T_{2s+1}-factors and a compressible subgraph S where $Q_H(S) \cong 3C_{2s+1}$.

Proof. Observe that $6s+1 - 2(2s) \equiv 0 \pmod{2s+1}$ and so by Remark 3.2.4, corresponding to Z_0, Z_s and Z_{2s}, respectively, we can construct three y variation m-cycles B_1, B_2, and B_3 in K_n, where

$B_1 = ((1, a_{0,0}), (2, a_{1,0}), \ldots, (2s, a_{2s-1,0}), (2s+1, a_{2s,0}), (2s, a_{2s+1,0}),$

$(2s-1, a_{2s+2,0}), \ldots, (3, a_{m-2,0}), (2, a_{m-1,0})),$

$B_2 = ((1, a_{1,s}), (2, a_{2,s}), \ldots, (2s, a_{2s,s}), (2s+1, a_{2s+1,s}), (2s, a_{2s+2,s}),$

$(2s-1, a_{2s+3,s}), \ldots, (3, a_{m-1,s}), (2, a_{0,s}))$, and

$B_3 = ((1, a_{0,2s}), (2s+1, a_{1,2s}), (2s, a_{2,2s}), \ldots, (3, a_{4s,2s}), (2, a_{4s+1,2s}),$

$(3, a_{4s+2,2s}), (4, a_{4s+3,2s}), \ldots, (2s, a_{m-2,2s}), (2s+1, a_{m-1,2s})).$

By Lemma 3.2.2, we know that we can delete 3(2s+1) T_{2s+1}-factors from $M(Z_0) \cup M(Z_s) \cup M(Z_{2s})$ and, if we call the remaining subgraph S and note that $a_{0,0} = a_{0,2s} = \infty$, $a_{2s,0} = a_{1,s} = 1+s$ and $a_{2s+1,s} = a_{4s+1,2s} = 1$, then, as in Lemma 3.2.6 we can show that the subgraph of S with bipartition (H_i, H_j) is 3-regular if $j = i+1$ or $j = i-1$, and empty otherwise. From this it follows that S is compressible and $Q_H(S) \cong 3C_{2s+1}$. ∎

Again note that by suitably relabelling we can choose C_{2s+1} to be an arbitrarily (2s+1)-cycle in K_{2s+1}.

Observe that in Lemma 3.2.6, we require $y \geq 4$. However, in the proof of the main theorem, we will need a similar result for $y = 2$. The following lemma serves this purpose.

3.2.8 Lemma. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. If $m-4 \equiv 0 \pmod{2s+1}$, then $M(Z_0) \cup M(Z_1) \cup M(Z_2)$ is the edge-disjoint union of 3(2s+1) T_{2s+1}-factors and a subgraph S where $Q_H(S) \cong 3C_{2s+1}$.

Proof. Observe that $a_{0,1} = a_{0,2} = \infty$, $a_{4,0} = a_{2,1} = 3$ and $a_{6,0} = a_{2,2} = 4$. We use the same construction as in Lemmas 3.2.6 and 3.2.7 to achieve the desired factorization. ∎

3.2.9 Lemma. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. Assume $m \geq 6s+5$. Then both $M(Z_0) \cup M(Z_1)$ and $M(Z_0) \cup M(Z_2)$ are the edge-disjoint union of

$2(2s+1)$ T_{2s+1}-factors and subgraphs S and S' , respectively, where $Q_H(S) \cong Q_H(S') \cong 2C_{2s+1}$.

Proof. Observe that $a_{1,0} = a_{3,1} = 1$ and $a_{2j+1,0} = a_{2j+3,1} = m - j$ if $1 \le 2j+1 \le m-4$ (or $0 \le 2j \le m-5$). Let y be a positive even integer such that $m - 2y \equiv 0 \pmod{2s+1}$ and $0 \le y \le 4s$; from which it follows that $0 \le y \le m-5$. According to Remark 3.2.4, corresponding to Z_0 and Z_1 respectively, we construct two y-variation m-cycles B_1 and B_2 in K_n where

$$B_1 = ((1, a_{1,0}), (2, a_{2,0}), ..., (y, a_{y,0}), (y+1, a_{y+1,0}), (y, a_{y+2,0}), \dots, (3, a_{m-1,0}), (2, a_{0,0})) \text{ and}$$

$$B_2 = ((1, a_{3,1}), (2s+1, a_{4,1}), (2s, a_{5,1}), \dots, (2s-y+3, a_{y+2,1}), (2s-y+2, a_{y+3,1}), (2s-y+3, a_{y+4,1}), \dots, (2s, a_{1,1}), (2s+1, a_{2,1})).$$

Using Lemma 3.2.2, we know that we can delete $2(2s+1)$ T_{2s+1}-factors from $M(Z_0) \cup M(Z_1)$ such that the remaining graph is $S = S_1 \cup S_2$, where

$$E(S_1) = \bigcup_{j=0}^{2s} \{(u+j, v)(w+j, z): (u,v)(w,z) \in E(B_1)\} \text{ and}$$

$$E(S_2) = \bigcup_{j=0}^{2s} \{(u+j, v)(w+j, z): (u,v)(w,z) \in E(B_2)\}.$$

Arguing as in Lemma 3.2.6, we can show that the subgraph of S with bipartition (H_i, H_j) is 2-regular if $i = j+1$ or $i = j-1$, and empty otherwise. From this it follows that S is compressible and $Q_H(S) \cong 2C_{2s+1}$.

Similarly if we consider $M(Z_0) \cup M(Z_2)$, then $a_{1,0} = a_{5,2} = 1$ and $a_{2j+1,0} = a_{2j+5,2} = m-j$ if $0 \le 2j \le m-7$. By using the same technique as above we can show that $M(Z_0) \cup M(Z_2)$ is the union of $2(2s+1)$ edge-disjoint T_{2s+1}-factors and a subgraph S' where $Q_H(S') \cong 2C_{2s+1}$. ∎

We are now ready to state and prove the main theorem of this section.

3.2.10 Theorem. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices so that $T_{2s+1} \mid 2K_{2s+1}$. Then $T_{2s+1} \mid_R K_n$ if and only if $n \equiv 0 \pmod{2s+1}$ and $n \equiv 1 \pmod{4s}$.

Proof. The necessity of the conditions is obvious. We now show their sufficiency.

Let $n = (2s+1)m$. From the second of the necessary conditions we can show that $m = 4sp+2s+1$ for some positive integer p. Let $p = (2s+1)q+i$, $0 \le i \le 2s$.

If $i = 0$, then $m = (2s+1)(4sq+1)$ and $n = (2s+1)^2(4sq+1)$. Let $\mathfrak{Z} = \{Z_0, ..., Z_{(m-3)/2}\}$ be as defined in the beginning of this section. As $m \equiv 0 \pmod{2s+1}$, it follows from Lemma 3.2.2 that $M(Z_j)$ is the edge-disjoint union of 2s+1 T_{2s+1}-factors and a subgraph S_j with $Q_H(S_j) \cong C_{2s+1}$, $0 \le j \le (m-3)/2$. Thus in K_n we obtain $(2s+1)(m-1)/2$ T_{2s+1}-factors. The subgraphs S_j, $0 \le j \le (m-3)/2$, can be chosen so that $Q_H(\bigcup_{j=0}^{(m-3)/2} S_j) \cong (4sq+2q+1)K_{2s+1}$ as each C_{2s+1} can be chosen independently. On deleting those $(2s+1)(m-1)/2$ T_{2s+1}-factors, the subgraph remaining in K_n is the union of $\bigcup_{j=0}^{(m-3)/2} S_j$ and m vertex-disjoint copies of K_{2s+1} (on the vertex sets $V_1, V_2, ..., V_m$). The H-quotient of this subgraph is $2(2sq+q+1)K_{2s+1}$ which by assumption is T_{2s+1}-factorable. By Lemma 2.1.4, we have a T_{2s+1}-factorization of K_n. (Note that the total number of T_{2s+1}-factors is $(2s+1)(2sq+q+1) + (2s+1)(m-1)/2 = (2s+1)(n-1)/(4s)$.)

If $i \neq 0$, then $m = 4s(2s+1)q+4si+2s+1 = 2t+1$. If $\{0, 1, ..., t-1\}$ can be partitioned into s 3-sets A_u, $u = 1, 2, ..., s$, and $(t-3s)/2$ 2-sets B_v, $v = 1, 2, ..., (t-3s)/2$, such that $\bigcup_{j \in A_u} M(Z_j)$ is the union of $3(2s+1)$ T_{2s+1}-factors and a subgraph with H-quotient $3C_{2s+1}$, and $\bigcup_{j \in B_v} M(Z_j)$ is the union of $2(2s+1)$ T_{2s+1}-factors and a subgraph with H-quotient $2C_{2s+1}$, then we can achieve the desired factorizations as follows. Arrange the s H-quotients $3C_{2s+1}$ so that their union is $3K_{2s+1}$. By including the edges of the V_j, $1 \le j \le m$, we have a subgraph with H-quotient $4K_{2s+1}$. Since (t-

3s)/2 = s((2s+1)q+i-1), we can arrange the (t-3s)/2 copies of $2C_{2s+1}$ so that their union is $2((2s+1)q + i-1)K_{2s+1}$. Since $T_{2s+1} \mid_R 2K_{2s+1}$, we have a T_{2s+1}-factorization.

The remainder of the proof will be spent on showing how to partition $\mathfrak{Z}$. For convenience, we define a *triplet* X to be a 3-set of integers such that $\bigcup_{j \in X} M(Z_j)$ is the union of 2s+1 T_{2s+1}-factors and a subgraph with H-quotient $3C_{2s+1}$ and define a *doublet* Y to be a pair of integers such that $\bigcup_{j \in Y} M(Z_j)$ is the union of 2s+1 T_{2s+1}-factors and a subgraph with H-quotient $2C_{2s+1}$. Let $I_x = \{0, 1, 2, ..., x-1\}$. All we need to show is that I_{t-1} can be partitioned into s triplets and (t-3s)/2 doublets. We first deal with the case when $q \neq 0$.

Case 1. $q \neq 0$.

When i is odd, y = 2s-i+1 is a positive even integer solution of $m-2y \equiv 0 \pmod{2s+1}$. Clearly $m \not\equiv 0 \pmod{2s+1}$ and $m \geq 2y$. Assume first that $y \geq 4$. Then by Lemma 3.2.6, {0, 1, s - (i-1)/2} is a triplet. We will locate s disjoint triplets in I_{t-1} such that on removing them, the remainder can be partitioned into (t-3s)/2 doublets.

When y/2 = s - (i-1)/2 is even and greater than 2, then {0, 1, ..., s - (i-1)/2} = {0, 1, s - (i-1)/2} ∪ X, where X = {2, 3}∪{4, 5}∪...∪{s - (i+3)/2, s - (i+1)/2}. By Lemma 3.2.9, X is partitioned into doublets. Now as s(s-(i-3)/2) < t = 2s(2s+1)q+2si+s, this implies that we can partition I_{t-1} into s triplets and (t-3s)/2 doublets.

When y/2 = s - (i-1)/2 is odd, then {0, 1, ..., s - (i-1)/2, s - (i-3)/2} = {0, 1, s - (i-1)/2} ∪ X, where X = {2, 3}∪{4, 5}∪...∪{s - (i+5)/2, s - (i+3)/2}∪ {s - (i+1)/2, s - (i-3)/2} and by applying Lemma 3.2.9, this, together with s(s-(i-3)/2 +1)) < t, implies a similar conclusion to that above.

When $y = 2$, we use Lemma 3.2.8 instead of Lemma 3.2.6 which means that we will group $\{0, 1, 2\}$ instead of $\{0, 1, s-(i-1)/2\}$, since $\{0, 1, 2\}$ is a triplet in this case.

If i is even, then $y = 4s+2-i$ is a positive even integer solution of $m-2y \equiv 0 \pmod{2s+1}$ and $m - 2y \geq 0$. We can use the same method as when i is odd to achieve the partition.

Case 2. q = 0.

The method used in Case 1 will not work here since $t = s(2i+1)$ is now considerably smaller. But using Lemmas 3.2.6- 3.2.9 and a new strategy, we can still achieve the required partition. We know $m = 4si+2s+1$ and consider separately the cases i odd and i even.

Suppose that i is odd. When $i = 1$, then $m = 6s+1$, K_{6s+1} has 3s Hamilton cycles, and $\{0, 1, ..., 3s-1\} = \bigcup_{j=0}^{s-1}\{j, j+s, j+2s\}$. By Lemma 3.2.7, $\{j, s+j, 2s+j\}$ is a triplet.

When $3 \leq i \leq 2s-5$, it follows that $m \geq 2(2s+1-i)$ and $2s+1-i$ is a positive even integer solution of $m-2y \equiv 0 \pmod{2s+1}$. By Lemma 3.2.6, $\{0, 1, s - (i-1)/2\}$ is a triplet.

Suppose first that $s - (i-1)/2$ even. Let $S = \{0, 2, 4, ..., s- (i+3)/2\}$ and $R = I_{2s-i} - \bigcup_{j\in S} \{j, 1+j, s - (i-1)/2 + j\}$. (The set R is obtained by removing $(s - (i-1)/2)/2$ disjoint triplets from I_{2s-i}.)

If $|R|$ is even, then I_{2s-i} can be partitioned into $(s - (i-1)/2)/2$ triplets and $(s - (i+3)/2)/4$ doublets. It is easy to see that $i(2s-i) \leq s(2i+1)=t$. Also $i(s-(i-1)/2)-2s = ((2s-i)(i-2)-i)/2 \geq (5(i-2)-i)/2 = 2i-5 > 0$, which implies that

$i(s-(i-1)/2)/2 > s$. Therefore, as $i(2s-i) \le t$, we can write $I_{t-1} = I_{2s-i} \cup (I_{2s-i} + (2s-i)) \cup \ldots \cup (I_{2s-i} + (i-1)(2s-i)) \cup I_{t-i(2s-i)} + i(2s-i)$. We can locate $(s-(i-1)/2)/2$ triplets in each $I_{2s-i}+j(2s-i+1)$, $0 \le j \le i-1$, and can always find s disjoint triplets as $i(s-(i-1)/2)/2 > s$. As $|R|$ is even, the remainder of I_{t-1} can be partitioned into doublets, (to see this note we can partition the remainder into pairs of the form $\{x, x+1\}$ or $\{x, x+2\}$; and by Lemma 3.2.9 they are doublets.

If $|R|$ is odd, consider I_{2s-i+1} in which we can locate $(s - (i-1)/2)/2$ disjoint triplets as before. Then $R' = I_{2s-i+1} - \bigcup_{j \in S} \{j, 1+j, s - (i-1)/2 + j\}$ and $|R'|$ is even. Thus R' can be partitioned into doublets of the type described in Lemma 3.2.9. In this case we also require $i(2s-i+1) \le s(2i+1) = t$ which is obviously true.

On the other hand, if $s-(i-1)/2$ is odd let $S = \{0, 2, \ldots, s - (i+5)/2\}$ and $R = I_{2s-i-1} - \bigcup_{j \in S} \{j, 1+j, s - (i-1)/2 + j\}$. This shows that we can locate $(s - (i+1)/2)/2$ triplets in I_{2s-i-1}. If $|R|$ is even, then I_{2s-i-1} can be partitioned into $(s - (i+1)/2)/2$ triplets and $(s - (i+1)/2)/4$ doublets, where the pairs are chosen according to Lemma 3.2.9. Since $i(2s-i-1) + s-(i-3)/2 \le s(2i+1) = t$, we can write I_{t-1} as in the last case. Thus we can choose $i(s - (i+1)/2)/2 + 1$ triplets in I_{t-1}. All that remains is to show that $i(s-(i+1)/2)/2 + 1 \ge s$. This follows as $i(s-(i+1)/2) - 2(s-1) = ((2s-i)(i-2) - (3i-4))/2 \ge (5(i-2) - (3i-4))/2 = i - 3 \ge 0$. (Recall that $3 \le i \le 2s-5$.) If $|R|$ is odd, then proceed as before but use I_{2s-i} instead of I_{2s-i-1}. Again we need to show $i(2s-i) + s-(i-3)/2 \le s(2i+1) = t$ and $i(s-(i+1)/2)/2 + 1 \ge s$; both of which are easily verified.

This leaves only two possibilities for odd i.

When $i = 2s-3$, $m = 8s^2-10s+1$. Then $m - 8 \equiv 0 \pmod{2s+1}$ $(y=4)$ and by Lemma 3.2.6, $\{j, 1+j, 2+j\}$ is a triplet.

When $i = 2s-1$, $m = 8s^2-2s+1$. Then $m - 4 \equiv 0 \pmod{2s+1}$ and by Lemma 3.2.8, $\{j, 1+j, 2+j\}$ is a triplet.

Finally, we consider the case i even.

Suppose $2 \leq i \leq 2s$. Then $y = 4s-i+2$ is a positive even integer solution of $m - 2y \equiv 0 \pmod{2s+1}$, $m \geq 2y$ and $m = 4si+2s+1$. By Lemma 3.2.6, $\{0, 1, 2s+1 - i/2\}$ is a triplet.

Suppose first that $2s+1 - i/2$ is even. Let $S = \{0, 2, ..., 2s -1 - i/2\}$ and $R = I_{4s-i+1} - \bigcup_{j \in S} \{j, 1+j, 2s +1- i/2 +j\}$, noting that we have removed $(2s+1 - i/2)/2$ disjoint triplets from I_{4s-i+1}.

If $|R|$ is even, then R can be partitioned into doublets. Clearly $(4s+1-i)(i/2) \leq s(2i+1) = t$ and $(i/2)(2s+1 - i/2) - 2s = (2s - i/2)(i/2 -1) \geq 0$, when $2 \leq i \leq 2s$, which implies $(i/2)(2s+1-i/2)/2 \geq s$. Thus we can locate s disjoint triplets in I_{t-1} and the remainder can be partitioned into doublets. If $|R|$ is odd, we use $R' = I_{4s-i+2} - \bigcup_{j \in S} \{j, 1+j, 2s +1- i/2 +j\}$. Then to complete the proof we require $(4k+2-i)i/2 \leq s(2i+1) = t$ and $(i/2)(2s+1-i/2)/2 \geq s$, when $2 \leq i \leq 2s$. Both inequalities can be verified easily.

Suppose then that $2s+1- i/2$ is odd (which implies $i \geq 4$). Let $S = \{0, 2, ..., 2s -2 - i/2\}$ and $R = I_{4s-i} - \bigcup_{j \in S} \{j, 1+j, 2s+1- i/2 +j\}$. If $|R|$ is even, then as before we only need to show $(4s-i)i/2 \leq s(2i+1)=t$ and $(i/2)(s-i/4) \geq s$, when $4 \leq i \leq 2s$. As $i(s - i/4) - 2s = (i/2 -1)(2s - i/2) - i/2 \geq 0$, when $i \geq 4$, the second follows. If $|R|$ is odd, then we consider I_{4s-i+1} and require the inqualities $(4s-i+1)i/2 \leq s(2i+1) = t$ and $(i/2)(s-i/4) \geq s$, where $2 \leq i \leq 2s$; which clearly hold.

This completes the proof. ∎

There are some interesting trees which are bipartite spreadable and also have the property $T_k \mid 2K_k$, for example, paths and stars.

3.2.11 Corollary. If k is odd, then $P_k \mid_R K_n$ if and only if $n \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{2(k-1)}$.

Proof. It is obvious that P_k is bipartite spreadable and $P_k \mid 2K_k$. Hence the claim follows immediately from Theorem 3.2.10. ∎

3.2.12 Corollary. If k is odd, then $K_{1,k-1} \mid_R K_n$ if and only if $n \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{2(k-1)}$.

Proof. It is obvious that $K_{1,k-1}$ is bipartite spreadable and $K_{1,k-1} \mid 2K_k$. Hence the claim follows immediately from Theorem 3.2.10. ∎

We know that when k is even, there does not exist n such that $K_{1,k-1} \mid_R K_n$. Hence we can completely solve the problem for the existence of star factorizations of K_n.

We now exhibit another interesting class of bipartite spreadable trees. Let $P_r = [v_1, v_2, ..., v_r]$ be a path on r vertices. The *caterpillar* $cp(k_1, k_2, ..., k_r)$ is the tree obtained from P_r by adding to P_r $k_1+k_2+...+k_r$ additional vertices $\{v_{i,j} : 1 \le i \le r$ and $1 \le j \le k_r\}$, and the additional edges $\{v_{i,j}v_i : \text{for } 1 \le i \le r \text{ and } 1 \le j \le k_r\}$.

3.2.13 Lemma. The caterpillar $cp(k_1, k_2, ..., k_r)$ is bipartite spreadable.

Proof. Let $T = cp(k_1, k_2, ..., k_r)$ and $k = r+k_1+k_2+...+k_r$. Assign k to v_1, $1,2,...k_1$ to $v_{1,1}, ..., v_{1,k_1}$, k_1+1 to v_2, $k-1, k-2, ... k-k_2$ to $v_{2,1}, ..., v_{2,k_2}$, $k-k_2-1$ to v_3 and $k_1+2, k_1+3, ..., k_1+k_3+1$ to $v_{3,1}\ v_{3,2}, ..., v_{3,k_3}$ and so on. It is easy to check that this labelling indeed satisfies the requirement. (Actually, this is just the well known graceful labelling of caterpillars [12].) ∎

3.2.14 Corollary. If $k = r+k_1+k_2+...+k_r$ is odd, then $cp(k_1, k_2, ..., k_r) \mid_R K_n$ if and only if $n \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{2(k-1)}$.

Proof. This claim follows immediately form Lemma 3.2.13 and Theorem 3.2.10 and the fact that all caterpillars are graceful which implies that $cp(k_1, k_2, ..., k_r) \mid 2K_k$. ∎

It is also easy to give a class of trees T_k which are bipartite spreadable (but not caterpillars) and which also have the property that $T_k \mid 2K_k$. The following example can be extended to an infinite class of trees: $V(T_8) = \{1, 2, ..., 8\}$ and $E(T_8) = \{35, 54, 47, 71, 18, 72, 26\}$. We can build a T_9 from T_8 by adding the vertex 9 and the edge 91, T_{10} by adding to T_9 the vertex 10 and the edge, (10)1 and so on. Of course, this idea can be extended to construct an infinite family from any bipartite spreadable tree.

A natural question is to ask if we can extend Theorem 3.2.10 to values of λ other than 1. Recall that necessary conditions for the existence of a T_k-factorization of λK_n are $n \equiv 0 \pmod{k}$ and $\lambda(n-1) \equiv 0 \pmod{2(k-1)}$. The results of the last section answered the question for even λ. A careful study of the proof of Theorem 3.2.10 yields the following result which we state without proof.

3.2.15 Theorem. Assume λ is odd and $\lambda > 1$. Let T_{2s+1} be a bipartite spreadable tree on 2s+1 vertices. Assume $T_{2s+1} \mid 2K_{2s+1}$. Then $T_{2s+1} \mid_R \lambda K_n$ if and only if $n \equiv 0$ (mod 2s+1) and $\lambda(n-1) \equiv 0 \pmod{4s}$ with only finitely many possible exceptions.

The "finitely many" of the theorem can be expressed more specifically as follows: For fixed k and λ, the claim holds for all $n = m(2s+1)$ where $m \geq \max\{6s+1, 1 + 4s^2/\lambda\}$.

3.3 Resolvable P_k-decomposition of λK_n

In this section, we are interested in determining necessary and sufficient conditions for the existence of a P_k-factorization of λK_n. We already have such conditions in the following cases.

(1) $k = 3$ (Theorem 1.4),

(2) k even or λ even (Corollaries 3.1.3 and 3.1.5),

(3) k odd and $\lambda = 1$ (Corollary 3.2.11).

The purpose of this section is to provide necessary and sufficient conditions in the case λk odd, $\lambda > 1$. Combined with the results we mentioned above the question of the existence of P_k-factorizations of λK_n will be completely resolved. We state the complete result as follows.

3.3.1 Theorem. When $k \geq 3$, λK_n has a P_k-factorization if and only if $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

As in previous cases, we begin with a lemma which will be the "building block" of the proof of our main result.

3.3.2 Lemma. **(a)** If k is odd and $k \geq 3$, then $\lambda K_{k,k} - W(\lambda)$, where $W(\lambda)$ is the union of λ subgraphs of $K_{k,k}$ each consisting of $(k-1)/2$ vertex disjoint cycles of length 4 and an independent edge, has a P_k-factorization.

(b) If k is odd, $k \geq 3$ and λ is even, then $\lambda K_{k,k} - C(\lambda/2)$, where $C(\lambda/2)$ is the union of $\lambda/2$ Hamilton cycles in $K_{k,k}$, has a P_k-factorization.

Proof. In (a) we need only consider the case $\lambda = 1$, and in (b) the case $\lambda = 2$. Let $V(K_{k,k}) = V(2K_{k,k}) = X \cup Y$ where $X = \{x_1, x_2, ..., x_k\}$ and $Y = \{y_1, y_2, ..., y_k\}$, and let $V(K_k) = \{1, 2, ..., k\}$.

(a) We know by Lemma 2.2.2(b) that K_k - N has a P_k-factorization (recall that N is an almost 1-factor). Let P be one of the k-paths in such a factorization. From P we define in $K_{k,k}$ the P_k-factor $\{x_iy_j, x_jy_i : ij \in E(P)\}$. Repeating for each k-path in the P_k-factorization of K_k - N we obtain a P_k-factorization of $K_{k,k}$ - W(1), where W(1) = $\{x_iy_j, x_jy_i : ij \in E(N)\} \cup \{x_1y_1, x_2y_2, ..., x_ky_k\}$.

(b) In this case a direct construction will be given. First observe that if $k = 2s+1$, then the 2k edges $E = \{x_iy_{i+s-1}, x_iy_{i+s+1} : 1 \leq i \leq k\}$ form a Hamilton cycle in $2K_{k,k}$. We consider separately the cases $k = 4t+1$ and $k = 4t+3$, and denote the k P_k-factors of $2K_{k,k}$ - C(1) by P(1), P(2),..., P(k). In each case we give P(1) from which P(i+1), $1 \leq i \leq k-1$, is obtained as follows: $x_{i+j}y_{i+s} \in E(P(1+i))$ if and only if $x_jy_s \in E(P(1))$

When $k = 4t+1$, $P(1) = \{[x_1, y_1, x_{2t}, y_2, x_{2t-1}, y_3, ..., y_t, x_{t+1}, y_{3t+1}, x_{3t+2}, y_{3t}, x_{3t+3}, ..., y_{2t+2}, x_{4t+1}], [y_{2t}, x_2, y_{2t-1}, x_3, ..., x_t, y_{t+1}, x_{3t+1}, y_{3t+2}, x_{3t}, ..., y_{4t+1}, x_{2t+1}, y_{2t+1}]\}$, and when $k = 4t+3$, $P(1) = \{[x_1, y_1, x_{2t+1}, y_2, x_{2t}, y_3, ..., x_{t+2}, y_{t+1}, x_{3t+3}, y_{3t+2}, x_{3t+4}, y_{3t+1}, ..., y_{2t+3}, x_{4t+3}], [y_{2t+1}, x_2, y_{2t}, x_3, ..., y_{t+2}, x_{t+1}, y_{3t+3}, x_{3t+2}, y_{3t+4}, x_{3t+1}, ..., x_{2t+3}, y_{4t+3}, x_{2t+2}, y_{2t+2}]\}$. ▮

3.3.3 Remark. Observe that in the construction given in Lemma 3.3.2(b) all "vertical" edges (that is, edges x_iy_i, $1 \leq i \leq k$) are contained in paths of the factorization. It is not difficult to show that in Lemma 3.3.2(a), provided $k \geq 5$, we can permute the vertices of Y so that here also all vertical edges are in paths of the factorization. When k = 5 and k = 7 permute the vertices so that W(1) has the form shown in Figure 3.3.1 with vertex bipartition (A, B), where $A = \{a_1, ..., a_k\}$ and B =

$\{b_1, ..., b_k\}$. Induction then takes care of all other values of k, as is shown in the cases k = 9 and k = 11. Observe that if we identify the vertices a_i and b_i, $1 \leq i \leq k$, (as shown in Figure 3.3.2), then the resulting multigraph is the union of a Hamilton cycle and a Hamilton path.

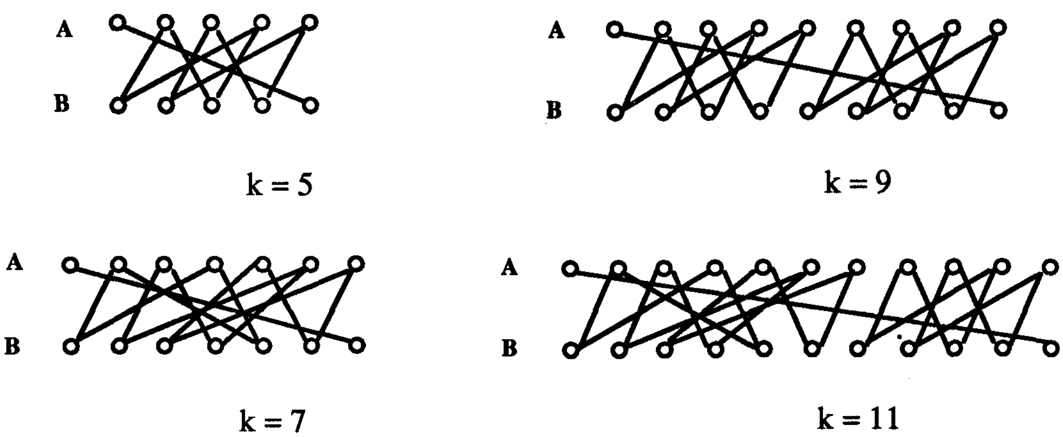

Figure 3.3.1

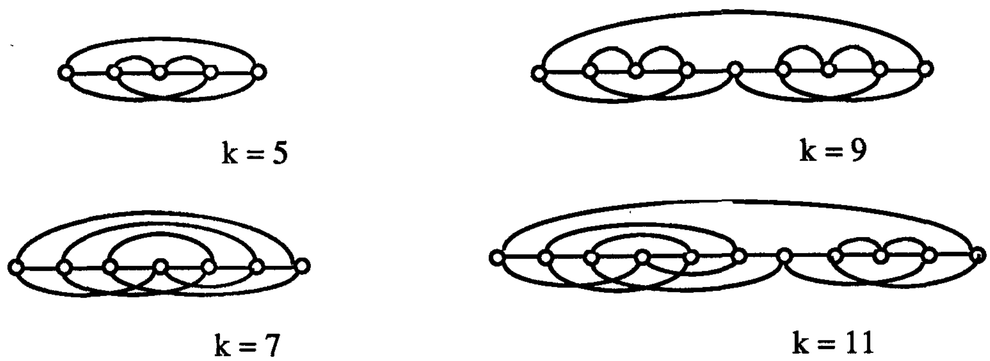

Figure 3.3.2

We now give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1.

As we have stated several times, the conditions $n \equiv 0 \pmod{k}$ and $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$ are necessary for the existence of the factorization. To show they are also sufficient, all that remains is the case when λk is odd and $\lambda \neq 1, k \neq 3$ (the other cases have been done as we mentioned in the begining of this section).

The necessary conditions imply that $n = kr$ and $2(k-1) | \lambda k(kr-1)$. The divisibility condition implies that $(k-1) | \lambda(r-1)$ when λk is odd. Notice that λk odd implies r odd.

Let $V(\lambda K_n) = \{(i, j) : 1 \leq i \leq r,\ 1 \leq j \leq k\} = \bigcup_{i=1}^{r} H_i = \bigcup_{j=1}^{k} V_j$, where $H_i = \{(i, j) :\ 1 \leq j \leq k\}$ and $V_j = \{(i, j) : 1 \leq i \leq r\}$. Note that $\lambda K_n = \lambda(K_r \otimes K_k)$.

We shall later define a subgraph S of λK_n. The edges of S will be given by $E(S) = (\bigcup_{1\leq i<j\leq r} W_{ij}(\lambda)) \cup (\bigcup_{i=1}^{r} N_i)$ where $W_{ij}(\lambda)$ is the union of λ subgraphs of K_{H_i,H_j} each isomorphic to the graph W(1) described in Remark 3.3.3, and N_i is a set of $(k-1)/2$ independent edges on the vertex set H_i.

We first show that the graph λK_n - S has a P_k-factorization. Since r is odd, K_r has a near 1-factorization with near 1-factors $M_1, M_2, ..., M_r$, and to each of these there corresponds in λK_n a subgraph which is the vertex-disjoint union of $(r-1)/2$ copies of $\lambda K_{k,k}$ and one copy of λK_k. Then Lemma 3.3.2(a) yields $\lambda(k-1)/2$ P_k-factors in each $\lambda K_{k,k}$. By Lemma 2.2.2(b) we have $(\lambda k-1)/2$ P_k-factors in λK_k. Notice that in the copy of λK_k only $\lambda(k-1)/2$ of a possible $(\lambda k-1)/2$ P_k-factors are used. So for each near 1-factor we obtain $\lambda(k-1)/2$ P_k-factors. On each vertex set V_i, $1 \leq i \leq r$, there remains a subgraph consisting of $(\lambda-1)/2$ paths of length k-1 and a set of $(k-1)/2$ independent edges. Together (that is, over all i) these paths constitute a further $(\lambda-1)/2$ P_k-factors of λK_n. When all these $\lambda(k-1)r/2 + (\lambda-1)/2 = (\lambda r(k-1)+(\lambda-1))/2$ P_k-factors are deleted from λK_n what remains is the subgraph S.

All that remains is to prove that there is a subgraph S which is compressible and $Q_V(S)$ has a P_k-factorization.

We begin with a 2-factorization of K_r and then direct each of the cycles. If the edge $ij \in E(K_r)$ is directed (i,j), let $W_{ij}(\lambda) = \lambda W(1)$, where $W(1)$ is as described in Figures 3.3.1 and 3.3.2 and $A = H_i$ and $B = H_j$. Let $N_i = \{(i,2j)(i,2j+1) : 1 \leq j \leq (k-1)/2\}$. This has now defined our subgraph S. Then $Q_V(S)$ is the union of a path of length k-1 with edge multiplicity $\lambda(r-1)/2$ (corresponding to the Hamilton paths of the W_{ij}) and of a cycle of length k in which the edges $2j(2j+1)$, $1 \leq j \leq (k-1)/2$, have multiplicity $\lambda(r-1)/2 + 1$ and the others have multiplicity $\lambda(r-1)/2$ (these cycles correspond to the Hamilton cycles of the W_{ij} and N_i). Since $\lambda(r-1) = s(k-1)$, for some integer s, then $\lambda(r-1)/2 = (k-1)(s-1)/2 + (k-1)/2$ and $\lambda(r-1)/2 + 1 = (k-1)(s-1)/2 + (k+1)/2$. (Note that it is only at this point that the construction fails for $k = 3$.) We obtain $\lambda(r-1)/2$ P_k-factors from the $\lambda(r-1)/2$ k-paths in $Q_V(S)$ and by applying Lemma 2.2.1(a) to the remainder of $Q_V(S)$ we obtain further $(sk+1)/2$ P_k-factors. (Note that s is odd and this can be shown as follows: Let $\lambda k(r(k-1) + r-1) = 2(k-1)q$. Then $s(k-1)k = \lambda k(r-1) = (k-1)(2q - \lambda kr)$ which is equivalent to $sk = 2q - \lambda kr$ implying that s is odd.)

As a final check we observe that there are $\lambda(r-1)/2 + (sk+1)/2$ P_k-factors in S, and adding these to the previously found P_k-factors we have a total of $\lambda k(rk-1)/(2k-2)$ P_k-factors as required. ∎

Chapter 4. Almost Resolvable P_k-decompositions of λK_n

In this chapter we give necessary and sufficient conditions for the existence of almost resolvable P_k-decompositions of λK_n. A special case of this main theorem is dealt with in the following lemma.

4.1 Lemma. Let k be even and $k \geq 4$. The graph λK_{2k+1} has an almost P_k-factorization if and only if $\lambda(2k+1) \equiv 0 \pmod{k-1}$.

Proof. Counting edges results in the necessary condition $\lambda(2k+1) \equiv 0 \pmod{k-1}$. We now construct factorizations when this condition is met. Let $V(\lambda K_{2k+1}) = \{1, 2, ..., 2k+1\}$. *Suppose first that* $\lambda \equiv 0 \pmod{k-1}$. *We only need to show that* $(k-1)K_{2k+1}$ has an almost P_k-factorization. Let $G(1,i) = G(2,i) = ... = G(k/2 -1,i) = [1+i, k+i, 2+i, k-1+i, ..., k/2 +2+i, k/2 + i, 3k/2 +1 +i] \cup [2k+1+i, k+2+i, 2k+i, k+3+i, ..., 3k/2+i, 3k/2 +2+i, k/2 +1+i]$ and $G(k/2, i) = [1+i, k+1+i, 2+i, k+i, ..., k/2 +i, k/2+2+i] \cup [k+2+i, k+3+i, ..., 2k+1+i]$, $0 \leq i \leq 2k$. Each $G(j, i)$ is an almost P_k-factor and it is not difficult to verify that

$$\bigcup_{j=1}^{k/2} \bigcup_{i=0}^{2k} G(j,i) = (k-1)K_{2k+1}.$$

On the other hand, if $\lambda \not\equiv 0 \pmod{k-1}$, then $\gcd(2k+1, k-1)=3$. Let $2k+1 = 3p$ and $k-1 = 3q$, where $\gcd(p, q) = 1$. Hence $\lambda \equiv 0 \pmod{q}$ and we only need to show that qK_{2k+1} has an almost P_k-factorization.

When $q = 1$, let $G(0, 3j) = [2+3j, 9+3j, 3+3j, 8+3j] \cup [1+3j, 6+3j, 4+3j, 7+3j]$ and $G(1, 3j) = [4+3j, 2+3j, 5+3j, 1+3j] \cup [6+3j, 7+3j, 8+3j, 9+3j]$, where $0 \leq j \leq 2$.

When $q > 1$ (and noting that q must be odd), for $0 \leq s \leq 2$, $0 \leq t \leq 2k$, let $P(s, t) = [2+s+t, 2k+1+s+t, 3+s+t, 2k+s+t, ..., k/2 +s+t, 3k/2 +3+s+t, k/2 +1+s+t, 3k/2 +2+s+t] \cup [1+s+t, k+2+s+t, 3k+1+s+t, k+3+s+t, 3k+s+t, ..., 3k/2 +s+t, 5k/2 +3+s+t, 3k/2 +1+s+t]$.

First let us look at an example. Assume $q = 3$. Then $k = 10$ and $p = 7$. It is easy to see that if we can partition $E(3K_{21})$ into 35 almost P_{10}-factors, then we are done. We construct them as follows. Let $G(0, 3j) = P(0, 3j)$, $G(1, 3j) = P(1, 3j)$, $G(2, 3j) = G(3, 3j) = P(2, 3j)$ and $G(4, 3j) = [3+3j, 1+3j, 4+3j, 3j, ..., 7+3j, 18+3j] \cup [8+3j, 9+3j, ..., 16+3j, 17+3j]$, where $0 \leq j \leq 6$. It is easy to see that $\bigcup_{j=0}^{6} \bigcup_{i=0}^{2} P(i, 3j) = 2K_{2k+1} - 2C$, where $C = (1, 2, ..., 21)$. Therefore, $\bigcup_{j=0}^{6} \bigcup_{i=0}^{4} G(i, 3j) = 3K_{21}$.

In general, we will use the same idea. Let $G(i, 3j) = P(0, 3j)$, $0 \leq i \leq (q-3)/2$, $G(i, 3j) = P(1, 3j)$, $(q-1)/2 \leq i \leq q-2$, $G(i, 3j) = P(2, 3j)$, $q-1 \leq i \leq (3q-3)/2$, and $G((3q-1)/2, 3j) = [3+3j, 2k+2+3j, 4+3j, 2k+1+3j, ..., k/2 +2+3j, 3k/2 +3+3j] \cup [k/2 +3+3j, k/2+4+3j, ..., 3k/2 +1+3j, 3k/2 +2+3j]$, where $0 \leq j \leq p-1$. (Notice that $(3q+1)/2 = k/2$.)

Again it can be shown that $\bigcup_{j=0}^{p-1} \bigcup_{i=0}^{k/2-1} G(i, 3j) = qK_{2k+1}$. Notice that $\bigcup_{j=0}^{p-1} \bigcup_{i=0}^{2} P(i, 3j) = 2K_{2k+1} - 2C$, where $C = (1, 2, ..., 2k+1)$. The rest of the proof follows easily from this.

∎

We next specify certain subgraphs of λK_n which will play important roles in the proof of the main theorem.

4.2 Definition. Let m be even, let $V= \{1, 2, ..., k\}$, and let $V(K_{mk+1}) = \{\infty\} \cup (\bigcup_{i=1}^{m} V_i)$, where $V_i = V \times \{i\}$. Let C be the $(m+1)$-cycle, $C = (1, 2, ..., m+1)$ and P be the m-path, $P = [1, 2, ..., m]$.

(a) Let $\Omega = \{G_1, G_2, \ldots, G_{m+1}\} \otimes C$, where $V(G_i) = V_i$, $1 \leq i \leq m$, $V(G_{m+1}) = \{\infty\}$, $G_i \cong \overline{K_k}$, for $2 \leq i \leq m-1$, $G_m \cong G_1 \cong K_k$ and $G_{m+1} \cong K_1$. (Note that since K_{m+1} has a C_{m+1}-factorization, it is easy to see that K_{mk+1} can be decomposed into m/2 isomorphic copies of Ω. We now define certain subgraphs of $\Omega - \{\infty\} = \{G_1, G_2, \ldots, G_m\} \otimes P$.

(b) When k is odd, we let $A \subseteq \Omega - \{\infty\}$ be the subgraph induced by the edge-set:

$$\bigcup_{j=1}^{k} \{ \bigcup_{i=1}^{m-1} \{(j, i)(j+1, i+1), (j, i+1)(j+1, i)\}$$

$$\cup \{(j, 1)(j + (k-1)/2, 1), (j, 1)(j+1, 1)\}$$

$$\cup \{(j, m)(j + (k-1)/2, m), (j, m)(j+1, m)\}\}$$ (see Figure 4.1)

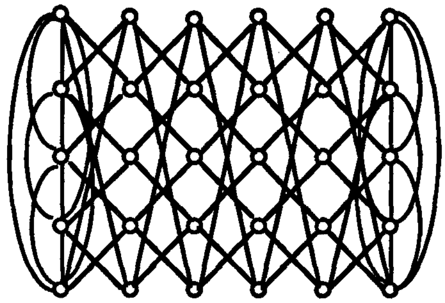

Figure 4.1 k = 5, m = 6

(c) When k is even and $m \geq 4$, let $B \subseteq \Omega - \{\infty\}$ be the subgraph induced by the edge-set:

$$\bigcup_{i=1}^{k/2} \{\{(2i-1, j)(2i, j+1), (2i, j)(2i-1, j+1) : j = 1, 3, ..., m-1\}$$

$$\cup \{(2i, j)(2i+1, j+1), (2i+1, j)(2i, j+1) : j = 2, 4, ..., m-2\}\}$$ (see Figure. 4.2)

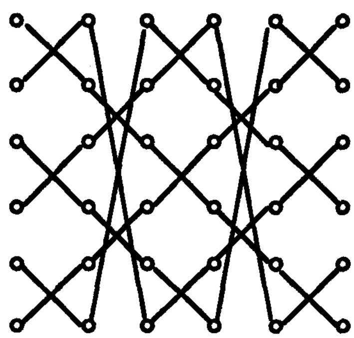

Figure 4.2 k = 6, m = 6

Next we define a family of graphs $\mu(j)$, $1 \le j \le m$, by $\mu(j) = \{ B \cup C(1, j) \cup C(m, j) :$ C(1, j) and C(m, j) are k-cycles in V_1 and V_m, respectively, where C(1, j) contains the edge (j,1)(j+1,1), and C(m, j) contains the edge (j,m)(j+1,m)$\}$.

(d) If $m \ge 4$, let $X^* \subseteq \Omega - \{\infty\}$ be the subgraph induced by the edge set:

$$\bigcup_{i=1}^{k}\{ \bigcup_{j=1}^{m-1} \{(i, j)(i, j+1)\} \cup \{(i,1)(i+1, 1), (i, m)(i+1, m)\}\} \quad \text{(see Figure 4.3)}$$

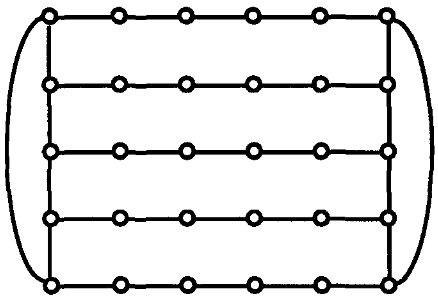

Figure 4.3 k = 5, m = 6

Before we begin to study these subgraphs we need another definition.

Definition 4.3. Let $K_{s,s}$ have an ordered bipartition (U, V), where U = {(1,u), (2,u), ..., (s,u)} and V = {(1,v), (2,v), ..., (s,v)}. The *distance* of the edge e = (i,u)(j,v) is defined to be j - i (mod s). Observe that the set of edges with distance i form a 1-factor and we say that this 1-factor has distance i. Let $X = [x_1, x_2, ..., x_p]$ be a p-path of $K_{s,s}$. The distance sequence $ds(X) = \langle d_1, d_2, ...d_{p-1}\rangle$ is the sequence of distances of the corresponding edges; that is, d_i is the distance of the edge x_ix_{i+1}. Note that X is uniquely determined by its first vertex and its distance sequence. So we can write $X = [x_1 : \langle d_1, d_2, ..., d_{p-1}\rangle]$.

Lemma 4.4. For even k, the graph Ω - X* is almost P_k-factorable.

Proof. We will construct an almost P_k-factorization of Ω - X*. Let X be the subgraph obtained from X* by deleting the two k-cycles on V_1 and V_m.

Case 1. k ≡ 0 (mod 4).

First we give the construction for k = 4 which illustrates the technique used in the general case, even though the general construction does not cover the case k = 4.

For $0 \leq i \leq 3$, let P(i, 1) = [(2+i,1), (1+i, 2), (4+i,1), (2+i, 2)];
let P(i, j) = [(3+i, j), (2+i, j+1), (4+i, j), (1+i, j+1)], where $2 \leq j \leq m-2$; and
let P(i, m-1) = [(3+i, m-1), (1+i, m), (4+i, m-1), (3+i, m)].

Notice that ds(P(i, j)) = <3, 1, 2>. The vertices of Ω which are not covered by $\bigcup_{j=1}^{m-1} P(i, j)$ are (1+i, 1), (3+i,1), (2+i, m) and (4+i, m) and ∞.

Let C(i) = [(3+i, 1), (1+i, 1), ∞, (2+i, m)], where $0 \leq i \leq 1$, and D(j) = [(1+j,1), ∞, (2+j, m), (4+j, m)], $2 \leq j \leq 3$. It is not difficult to see that C(0)∪C(1)∪D(2)∪D(3) will use all edges of the form ∞(i, 1) and ∞(i, m), and all edges of G_1 and G_m (recall the definition of Ω) except the two 4-cycles ((1,1), (2, 1), (3, 1), (4, 1)) and ((1, m), (2, m), (3, m), (4, m)). Therefore we obtain four almost P_4-factors of Ω - X*: C(i) ∪ $(\bigcup_{j=1}^{m-1} P(i, j))$, i = 0, 1, and D(i) ∪ $(\bigcup_{j=1}^{m-1} P(i, j)))$, i = 2, 3. These form an almost P_4-factorization of Ω - X*.

We now move to the more general case, k > 4. We construct the following k-paths and note their distance sequences. We remark that in the rest of the proof, we always assume that the edge (x, i)(y, i+1) has distance y - x (mod k).

For $0 \leq i \leq k-1$:

let P(i,1) = [(3k/4 +i, 1), (3k/4 +1+i, 2), ..., (k/2 +2+i, 1), (k-1+i, 2), (k+i, 1), (k/2 +i, 2), (k-1+i, 1), ..., (3k/4 +1+i, 1), (3k/4 -1+i, 2), (k/4 +1+i, 1), (3k/4+i, 2)], and hence ds(P(i,1)) = <1, 2, ..., k/2 -3, k-1, k/2, k/2 +1, ..., k-2, k/2 -2, k/2 -1>;

let P(i, j) = [(k, j), (k-1, j+1), (1, j), ..., (k/2 -1, j), (k/2, j+1)], and hence ds(P(i, j))= <k-1, k-2, ..., 2, 1>, $2 \leq j \leq m-2$; and

let P(i, m-1) = [(k/4 +i, m-1), (3k/4 +1+i, m), (k/4 -1+i, m-1), (k/4 +1+i, m), ..., (k/2 -1+i, m), (k+i, m-1), (k/2 +i, m), (k/2 -1+i, m-1), (2+i, m), ..., (k/4 +1+i, m-1), (k/4 +i, m)] and hence ds(P(i, m-1)) = <k/2 +1, k/2 +2, 2, 3, ..., k/2, 1, k/2 +3, k/2 +4, ..., k-1>.

The vertices of Ω which are not covered by $\bigcup_{j=1}^{m-1} P(i, j)$ are $\{\infty\} \cup \{$(1+i,1), (2+i, 1), ..., (k/4 +i, 1), (k/4 +2+i, 1), (k/4 +3+i, 1), ..., (k/2 +1+i, 1)$\} \cup \{$(1+i, m), (k/2 +1+i, m), (k/2 +2+i, m), ..., (3k/4 +i, m), (3k/4 +2+i, m), (3k/4 +3+i, m), ..., (k+i, m)$\}$.

Let C(i) = [(k/2 +1+i, 1), (1+i, 1), (k/2 +i, 1), (2+i, 1), ..., (k/4 +2+i, 1), (k/4 +i, 1), ∞, (3k/4 +i, m), (3k/4 +2+i, m), (3k/4 -1+i, m),..., (k+i, m), (k/2 +1+i, m)], $0 \leq i \leq k/2 -1$, and D(i) = [(1+i, 1), (k/2 +i, 1), (2+i, 1), ..., (k/4 +2+i, 1), (k/4 +i, 1), ∞, (3k/4 +i, m), (3k/4 +2+i, m), (3k/4 -1+i, m),..., (k+i, m), (k/2 +1+i, m), (1+i, m)], $k/2 \leq i \leq k-1$.

It is not difficult to see that $(\bigcup_{i=0}^{k/2-1} C(i)) \cup (\bigcup_{j=k/2}^{k-1} D(j))$ will use all edges of the forms ∞(i, 1) and ∞(i, m), and all edges of G_1 and G_m except for the two k-cycles ((1,1), (2,1), ..., (k,1)) and ((1, m), (2, m), ..., (k, m)). Therefore we obtain k almost P_k-factors of Ω - X*: C(i) $\cup$ $(\bigcup_{j=1}^{m-1} P(i, j))$, i = 0, 1, ..., k/2 -1, and D(i) $\cup$ $(\bigcup_{j=1}^{m-1} P(i, j))$, i = k/2, k/2 +1, ..., k-1. These form an almost P_k-factorization of Ω - X*.

Case 2. k $\equiv$ 2 (mod 4).

Again, for $0 \le i \le k-1$:

let $P(i, 1) = [(k/2 +2+i, 1), (k+i, 2), \ldots, ((3k+2)/4 +i, 1), ((3k+6)/4 +i, 2), ((k+6)/4 +i, 1), ((3k+2)/4 +i, 2), ((3k+6)/4, 1), ((3k-2)/4, 2), ((3k+10)/4, 1), \ldots, (k, 1), (k/2 +1, 2)]$, and $ds(P(i, 1)) = \langle (k/2)-2, (k/2)-3, \ldots, 1, k/2, (k/2)-1, k-1, k-2, \ldots, (k/2)+1 \rangle$;

let $P(i, j) = [(1+i, j), (k+i, j+1), (2+i, j), \ldots, (k/2+i, j), (k/2 +1+i, j+1)]$, and $ds(P(i, j) = \langle k-1, k-2, \ldots, 1 \rangle$, $2 \le j \le m-1$; and

let $P(i, m-1) = [(1+i, m-1), (k/2 +i, m), \ldots, ((k+6)/4 +i, m), ((k+2)/4 +i, m-1), ((3k+6)/4 +i, m), ((k+6)/4 +i, m-1), ((k+2)/4 +i, m), \ldots, (k/2 +i, m-1), (2+i, m)]$, and $ds(P(i, m-1)) = \langle k/2 -1, k/2 -2, \ldots, 1, k/2 +1, k/2, k-1, k-2, \ldots, k/2 +2 \rangle$.

As before, the vertices of Ω which are not covered by $\bigcup_{j=1}^{m-1} P(i, j)$ are $\{\infty\} \cup \{(1+i,1), (2+i, 1), \ldots, ((k+2)/4 +i, 1), ((k+10)/4 +i, 1), \ldots, (k/2 +1+i,1)\} \cup \{(k/2 +1+i, m), ((k/2 +2+i, m), \ldots, ((3k+2)/4 +i, m), ((3k+10)/4 +i, m),\ldots, (k+1+i, m)\}$. We now use them to construct a k-path and an isolated vertex.

Let $C(i) = [(1+i, 1), (k/2 +1+i, 1), (2+i, 1), (k/2 +i, 1), \ldots, ((k+10)/4 +i, 1), ((k+2)/4 +i, 1), \infty, ((3k+2)/4 +i, m), ((3k+10)/4 +i, m), \ldots, (k+i, m), (k/2 +2+i, m), (1+i, m)]$, $0 \le i \le k/2 -1$, and let $D(i) = [(k/2 +1+i, 1), (2+i, 1), (k/2 +i, 1), \ldots, ((k+10)/4 +i, 1), ((k+2)/4 +i, 1), \infty, ((3k+2)/4 +i, m), ((3k+10)/4 +i, m), \ldots, (k+i, m), (k/2 +2+i, m), (1+i, m), (k/2 +1+i, m)]$, $k/2 \le i \le k-1$.

It is not difficult to see that $(\bigcup_{i=0}^{k/2-1} C(i)) \cup (\bigcup_{j=k/2}^{k-1} D(j))$ will use all edges of the forms $\infty(i, 1)$ and $\infty(i, m)$ and all edges of G_1 and G_m except two k-cycles ((1,1), (2,1), ..., (k,1)) and ((1, m), (2, m), ..., (k, m)). Therefore we obtain k almost P_k-factors of Ω - X*: $C(i) \cup (\bigcup_{j=1}^{m-1} P(i, j))$, $i = 0, 1, \ldots, k/2 -1$, and $D(i) \cup (\bigcup_{j=1}^{m-1} P(i, j))$,

$i = k/2, k/2+1, \ldots, k-1$. These form an almost P_k-factorization of $\Omega - X^*$. Thus we complete the proof. ∎

4.5 Lemma. Let Ω, A and $\mu(j)$ be defined as in Definition 4.2.

(a) If k is odd, then the graph $2\Omega - A$ has an almost P_k-factorization.

(b) If k is even, then the graph $\Omega - b(j)$ has an almost P_k-factorization, for some $b(j) \in \mu(j)$. (Recall that $\mu(j)$ was defined in Definition 4.2 (c).)

Proof (a) First we partition the edges of A into $A_1 \cup A_2$, where $A_1 \cong A_2$ as follows:

Let A_1 have edge-set

$$\{(i, j)(i+1, j+1): 1 \le j \le m-1, 1 \le i \le k\} \cup \{(i, m)(i+1, m), (i, m)(i+(k-1)/2, m): i = 1, 2, \ldots, k\}.$$

and let A_2 have edge-set

$$\{(i+1, j)(i, j+1): 1 \le j \le m-1, 1 \le i \le k\} \cup \{(i, 1)(i+1, 1), (i, 1)(i+(k-1)/2, 1): i = 1, 2, \ldots, k\}.$$

Since $2\Omega - A$ can be partitioned into $(\Omega - A_1) \cup (\Omega - A_2)$, and $\Omega - A_1 \cong \Omega - A_2$, once we show that $\Omega - A_1$ is almost P_k-factorable we will be done. We now describe an almost P_k-factorization of $\Omega - A_1$.

Let $P(i, j) = [((k+1)/2 + j + i, j+1), ((k+1)/2 + j + i, j), ((k-1)/2 + j + i, j+1), ((k+3)/2 + j + i, j), \ldots, (k-1+j+i, j), (1+j+i, j+1)]$, where $0 \le i \le k-1$ and $1 \le j \le m-1$. Notice that $\bigcup_{i=0}^{k-1} P(i, j)$ uses all edges of $K_{V_j, V_{j+1}}$ except for the 1-factor $\bigcup_{x=1}^{k} \{(x, j)(x+1, j+1)\}$ (since $ds(P(i, j)) = <0, k-1, k-2, \ldots, 2>$). It is not difiicult to see that $\bigcup_{j=1}^{m-1} P(i, j)$ is a set of m-1 vertex-disjoint k-paths. There are k+1 vertices of $V(\Omega)$ which are not covered by $\bigcup_{j=1}^{m-1} P(i, j)$; they are $\{\infty, (1+i, 1), (2+i, 1), \ldots, ((k+1)/2 + i, 1)\} \cup \{(m + (k+1)/2 + i, m), (m + (k+3)/2 + i, m), \ldots, (m-1+i, m)\}$. On these k+1 vertices we must define a k-path and an isolated vertex. This contruction is divided into two parts (examples of each are given in Figure 4.4).

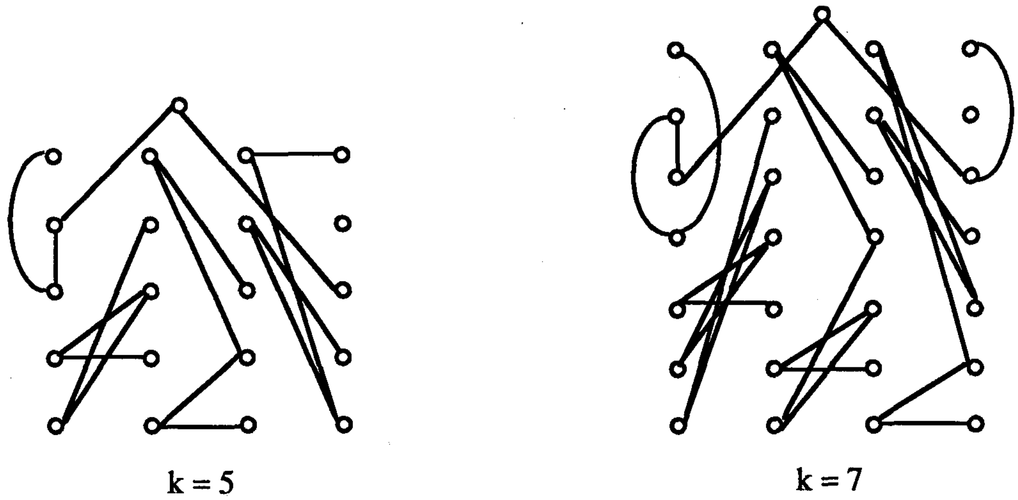

Figure 4.4

Case 1: k ≡ 3 (mod 4).

Let X(i) = [(1+i, 1), ((k+1)/2+i, 1), (2+i, 1), ((k-1)/2+i, 1), (3+i, 1), ((k-3)/2+i, 1), ..., ((k+9)/4 + i, 1), ((k+1)/4 + i, 1), ((k+5)/4 + i, 1), ∞, ((k+1)/2 + y + i, m), (2+y+i, m), ((k-1)/2 + y + i, m), (3+y+i, m), ..., ((k+1)/4 + y + i, m)], where y = m + (k-3)/2 and i = 0, 1, 2, ..., k-1.

Case 2: k ≡ 1 (mod 4).

Let X(i) = [(1+i, 1), ((k+1)/2+i, 1), (2+i, 1), ((k-1)/2+i, 1), ..., ((k+1)/4 + i, 1), ((k+7)/4 + i,1), ((k+3)/4 + i, 1), ∞, ((k+1)/2 + y + i, m), (2+y+i, m), ((k-1)/2 + y + i, m), ..., ((k+7)/4 + y + i, m)], where y = m + (k-3)/2 and i = 0, 1, 2, ..., k-1.

In both cases, $\bigcup_{i=0}^{k-1}$ X(i) uses all edges of Ω of the form ∞(j,1) and ∞(j, m) and all edges in V_1 and V_m except for the two k-cycles: ((1, m), (2, m), ..., (k, m)) and ((1, m), ((k+1)/2, m), (k, m), ((k-1)/2, m), ..., ((k+3)/2, m)). Thus X(i) ∪ ($\bigcup_{j=1}^{m-1}$ P(i, j)) is an almost P_k-factor of $\Omega - A_1$ and hence $\bigcup_{i=0}^{k-1} (X(i) \cup (\bigcup_{j=1}^{m-1} P(i, j)))$ is an almost P_k-factorization of $\Omega - A_1$.

(b) First we show that X^* is isomorphic to one of the elements in $\mu(1)$ (see Definition 4.2(c)). Let $B^* = B \cup \{(1,1)(2,1), (1,m)(2,m)\}$ (recall Definition 4.2). We know that B is the union of k disjoint m-paths and it is not difficult to see that the vertices of each V_i can be permuted so that B^* is isomorphic to one of $X_1 = X\cup\{(1,1)(2,1), (1,m)(2,m)\}$, $X_2 = X \cup\{(1,1)(2,1), (2,m)(3,m)\}$ or $X_3 = X\cup\{(1,1)(2,1), (3,m)(4,m)\}$, where $X = \bigcup_{i=1}^{k} \{[(i,1), (i,2), ..., (i,m)]\}$. We can obtain X^* from X_1, X_2 and X_3, by adding edges. This implies that X^* is isomorphic to one of the graphs in $\mu(1)$.

It is not difficult to see that we can use the same method to show that X^* is isomorphic to some element of $\mu(j)$, for any j, $1 \leq j \leq k$. Thus for each j, $1 \leq j \leq k$, there is an element b(j) of $\mu(j)$ such that $b(j) \cong X^*$ and so by Lemma 4.4, Ω - b(j) has an almost P_k-factorization. ∎

We now state and prove the main theorem.

4.6. Theorem. λK_n has an almost resolvable P_k-decomposition if and only if $n \equiv 1 \pmod k$ and $\lambda kn/2 \equiv 0 (\mathrm{mod}\ k-1)$.

Proof. The necessity of the conditions can be easily obtained by applying counting argument on vertices and edges. We now show their sufficiency. We first give a proof in the case when k = 3 and then prove the result for general k. In the general case we divide the proof into two parts according to the parity of m, where n = km+1. Throughout the proof, we will use the following technique: find a subgraph G of λK_n which contains an isolated vertex v, such that λK_n - G has an almost P_k-factorization and G - {v} is compressible and has a P_k-factorization..

Case 1. k = 3.

Here the conditions given in the statement of the theorem reduce to $n \equiv 1 \pmod 3$ and $\lambda n \equiv 0 \pmod 4$. The following three cases exhaust all possibilities for n and λ.

Case 1.1. $n \equiv 4 \pmod{12}$ and all values of λ.

In this case it suffices to show that K_n has an almost resolvable P_3-decomposition. Let $n = 4(3x+1)$ and $V(K_n) = \{(i, j) : 1 \leq i \leq 4, 1 \leq j \leq 3x+1\}$. Let $V_j = \{(i, j) : 1 \leq i \leq 4\}$, $j = 1, 2, ..., 3x+1$ and $H_j = \{(i, j) : 1 \leq j \leq 3x+1\}$, $i = 1, 2, 3, 4$. Then $K_n = K_4 \otimes K_{3x+1}$. Let $G = \overline{K}_4 \otimes K_{3x+1} \subseteq K_n$. Clearly G is compressible with a V-quotient $4K_{3x+1}$. We know that $2K_{3x+1}$ admits an almost resolvable C_3-decomposition [5], and hence $4K_{3x+1}$ can be decomposed into 3x+1 isomorphic copies of 2H, where H is the union of an isolated vertex and x vertex-disjoint K_3's. Note that the edge ij in H corresponds to a 1-factor in K_{V_i,V_j} and the isolated vertex corresponds to a K_4. Furthermore, the edge sets of both $2K_3$ and K_4 are the union of three 3-paths. Therefore the subgraph of K_n corresponding to 2H has an almost resolvable P_3-decomposition and which implies that K_n has an almost resolvable P_3-decomposition.

Case 1.2. $n \equiv 10 \pmod{12}$ and $\lambda \equiv 2 \pmod 4$.

Here it suffices to show that $2K_n$ has an almost resolvable P_3-decomposition. Let $n = 3(4x+3)+1$, $V(2K_n) = \{\infty\} \cup \{(i, j) : 1 \leq i \leq 3, 1 \leq j \leq 4x+3\}$, $V_j = \{(i, j) : 1 \leq i \leq 3\}$, where $1 \leq j \leq 4x+3$ and $H_i = \{(i, j) : 1 \leq j \leq 4x+3\}$, where $1 \leq i \leq 3$. Clearly $2K_n = \{2K_3, ..., 2K_3, K_1\} \otimes 2K_{4x+4}$ (see its definition in the appendix). We know that K_{4x+4} has a 1-factorization and in $2K_n$ each one factor corresponds to 2x+1 vertex-disjoint $2K_{3,3}$ and one $2K_4$ with vertex set $\{\infty, (1,i), (2,i), (3,i)\}$. It is not difficult to see that $2K_4$ is the union of four 3-paths and the subgraph T_i, where $E(T_i) = \{(1,i)(3,i), (1,i)(3,i), (2,i)(3,i), (1,i)(2,i)\}$, and $2K_{3,3}$ is the union of four P_3-factors and one $2K_2$.

On deleting four almost P_3-factors in the subgraph corresponding to each 1-factor in K_{4x+4}, what remains is a subgraph, R* of $2K_n$ in which ∞ is an isolated vertex. In R* the subgraph R_{ij} induced by (V_i,V_j) is $2K_2$, and the subgraph induced by V_i is T_i. It is easy to see that R* is not uniquely determined because of the freedom in choosing each of the R_{ij}. We will show that the R_{ij} can be chosen so that the resulting R* has an almost resolvable P_3-decomposition.

Let R = R*- {∞} and observe that we only need show that R has a P_3-factorization. Let $V(K_{4x+3}) = \{1, 2, ..., 4x+3\}$. We know that K_{4x+3} has a Hamilton cycle decomposition with cycles $h_1, ..., h_{2x+1}$. Assign an orientation to each h_i to create a directed cycle. If ij is an arc from i to j of h_k, $1 \leq k \leq x$, let $E(R_{ij}) = \{(1,i)(2,j), (1,i)(2,j)\}$, and if $x+1 \leq k \leq 2x+1$, let $E(R_{ij}) = \{(2,i)(3,j), (2,i)(3,j)\}$. Under this arrangement, $Q_H(R)$ is a 3-cycle in which one edge (13) has multiplicity 2, the second (12) has multiplicity 2x+1 and the third (13) has multiplicity 2x+3. Since $Q_H(R)$ is P_3-factorable, we can apply Lemma 2.1.4 to conclude that R is P_3-factorable.

Case 1.3. n ≡ 1 (mod 3) and λ ≡ 0 (mod 4).

In this case it is enough to show that $4K_n$ has an almost resolvable P_3-decomposition. Let n = 3x+1. Since $2K_n$ admits an almost resolvable C_3-decomposition, $4K_n$ can be decomposed into n isomorphic copies of 2H, where H is the union of x vertex-disjoint $2K_3$'s and an isolated vertex. As $2K_3$ is the union of three 3-paths, 2H has an almost resolvable P_3-decomposition and so too does $4K_n$.

Case 2. $k \geq 4$.

Since $n \equiv 1 \pmod{k}$ we write $n = mk+1$. Let $V(\lambda K_n) = \{\infty\} \cup \{(i, j) : 1 \leq i \leq k, 1 \leq j \leq m\}$, $V_j = \{(i, j) : 1 \leq i \leq k\}$, for $1 \leq j \leq m$, and $H_i = \{(i, j) : 1 \leq j \leq m\}$, for $1 \leq i \leq k$. Observe that $\lambda K_n = \{\lambda K_k, ..., \lambda K_k, K_1\} \otimes \lambda K_{m+1}$.

Case 2.1. m is odd.

Let R be a subgraph of λK_n with vertex set $V(\lambda K_n) - \{\infty\}$. For even k, let the edge-set E(R) be $(\bigcup_{1 \leq i<j \leq m} F_{ij}(\lambda)) \cup (\bigcup_{i=1}^{m} N_i(\lambda))$, where $F_{ij}(\lambda)$ is the union of λ 1-factors in the subgraph $\lambda K_{V_i,V_j}$ and $N_i(\lambda)$ is the union of λ k-cycles in the subgraph λK_{V_i}. When k is odd, let $E(R) = (\bigcup_{1 \leq i<j \leq m} E_{ij}(\lambda)) \cup (\bigcup_{i=1}^{m} M_i(\lambda))$, where $E_{ij}(\lambda)$ is the union of λ edges in the subgraph $\lambda K_{V_i,V_j}$ and $M_i(\lambda)$ is the union of λ edge-disjoint $(k+3)/2$-paths in the subgraph λK_{V_i}.

We will show that the graph $\lambda K_n - R$ has an almost resolvable P_k-decomposition. Since m is odd, K_{m+1} admits a 1-factorization with factors $f_1, f_2, ..., f_m$. To each factor f_i there corresponds, in λK_n, a subgraph H_i which is the vertex-disjoint union of $(m-1)/2$ copies of $\lambda K_{k,k}$ and one copy of λK_{k+1}. By Lemma 2.2.3, when k is even $\lambda K_{k,k} - F(\lambda)$, where $F(\lambda)$ is the union of λ 1-factors of $\lambda K_{k,k}$, has a P_k-factorization, and by Lemma 2.2.5 $\lambda K_{k+1} - N(\lambda)$, where $N(\lambda)$ is the union of λ k-cycles of λK_{k+1}, has a P_k-decomposition. By Lemma 2.2.4, when k is odd $\lambda K_{k,k} - E(\lambda)$, where $E(\lambda)$ is the union of λ edges of $\lambda K_{k,k}$, has a P_k-factorization, and $\lambda K_{k+1} - M(\lambda)$, where $M(\lambda)$ is the union of λ $(k+3)/2$-paths of λK_{k+1}, has a P_k-decomposition. Therefore H_i contains $\lambda k/2$ edge-disjoint almost P_k-factors of λK_n when k is even, and $\lambda(k+1)/2$ when k is odd. Remove these almost P_k-factors from each H_i, and denote the remaining subgraph of λK_n by R^*. It is not difficult to see (from Lemmas 2.2.4 and

2.2.5) that $M(\lambda)$ and $N(\lambda)$ can be chosen so that in R^*, ∞ is an isolated vertex and $R = R^* - \{\infty\}$. Thus $\lambda K_n - R$ has an almost resolvable P_k-decomposition.

All that remains is to prove that there is a graph R as described above which has a P_k-factorization. We consider two cases depending on the parity of k.

Case 2.1.1. k is even.

Since m is odd, K_m has a Hamilton cycle decomposition. Assign an orientation to each cycle to create (m-1)/2 directed cycles. If ij is an arc from i to j in one of the resulting directed cycles, let $F_{ij} = \{(1,i)(2,j), (2,i)(3,j), ..., (k,i)(1,j)\}$ be a 1-factor of $\lambda K_{V_i,V_j}$ and $F_{ij}(\lambda) = \lambda F_{ij}$. Let $N_i(\lambda)$ be the k-cycle ((1,i), (2,i), ..., (k,i)) with multiplicity λ. The Hamilton cycle decomposition of K_m guarantees both that in R the subgraph induced by the bipartition (H_i, H_{i+1}) is $(m+1)\lambda/2$-regular, and R has H-quotient $(\lambda(m+1)/2)C_k$. Since $n\lambda k/2 \equiv 0 \pmod{k-1}$ (one of the necessary conditions) and $n\lambda k/2 = (km+1)\lambda k/2 = \lambda(mk+k-(k-1))k/2 = \lambda k^2(m+1)/2 - \lambda k(k-1)/2$, then $\lambda k(m+1)/2 \equiv 0 \pmod{k-1}$. Lemma 2.2.1(a) gives a P_k-factorization of $Q_H(R)$ and thus by Lemma 2.1.4, R has a P_k-factorization.

Case 2.1.2. k is odd.

Write $\lambda(k+1)/2 = kx_1+y_1$, $0 \le y_1 \le k-1$. Let $M_i(\lambda)$ be the union of the k-cycle ((1, i), (2, i), ..., (k, i)) with multiplicity x_1 and the (y_1+1)-path [(1, i), (2, i), ..., (y_1+1, i)]. (By Lemma 2.2.4, this is possible by properly arranging the λ (k+3)/2-paths.) As m is odd, λK_m has a Hamilton cycle decomposition with cycles $h_1, h_2, ..., h_{\lambda(m-1)/2}$. Assign an orientation to each of these cycles to create $\lambda(m-1)/2$ directed cycles. Assume $\lambda(m-1)/2 = kx_2+y_2$, $0 \le y_2 \le k-1$. Let ij be an arc from i to j in h_p. If $1 \le p \le kx_2$, let $\{(p, i)(p+1, j)\} \in E_{ij}(\lambda)$ and if $p > kx_2$, let $\{(p-kx_2+y_1, i)(p-kx_2+y_1+1, j)\} \in E_{ij}(\lambda)$. Let $y_1+y_2 = kx_3+y_3$, $0 \le x_3 \le 1$, $0 \le y_3 \le k-1$. In R the subgraph on vertex-set

$H_i \cup H_{i+1}$ is bipartite and $(x_1+x_2+x_3+1)$-regular if $1 \le i < y_3+1$, and is bipartite and $(x_1+x_2+x_3)$-regular if $y_3+1 \le i \le k$. It is not difficult to see that R is compressible and $Q_H(R)$ is the union of a k-cycle with multiplicity $x_1+x_2+x_3$ and a (y_3+1)-path. Since $(x_1+x_2+x_3)k+y_3 = (\lambda(k+1)/2 - y_1)+(\lambda(m-1)/2 - y_2)+(y_1 + y_2 - y_3) +y_3 = \lambda(k+1)/2 + \lambda(m-1)/2 = \lambda(k+m)/2$, and $n\lambda/2 = (km+1)\lambda/2 = ((k-1)(m-1)+k+m)\lambda/2 \equiv 0 \pmod{k-1}$ implies $\lambda(k+m)/2 \equiv 0 \pmod{k-1}$, then $(x_1+x_2+x_3)k+y_3 \equiv 0 \pmod{k-1}$. By Lemma 2.2.1(a) $Q_H(R)$ has a P_k-factorization and hence R has a P_k-factorization.

Case 2.2. m is even.

Case 2.2.1. k is odd. As in the case when m is odd, we begin by defining a subgraph S of $\lambda K_n - \{\infty\}$. Let $E(S) = (\bigcup_{1\le i<j\le m} W_{ij}(\lambda/2)) \cup (\bigcup_{i=1}^{m} N_i(\lambda/2) \cup M_i(\lambda/2))$, where $W_{ij}(\lambda/2)= (\lambda/2)W_{ij}$, and $W_{ij} = \{(p,i)(p+1,j), (p+1,i)(p,j): p=1, ..., k\}$, $N_i(\lambda/2)$ is the k-cycle ((1,i), (2,i), ..., (k,i)) with multiplicity $\lambda/2$, and $M_i(\lambda/2)$ is the k-cycle ((1, i), ((k+1)/2, i), (k, i), ((k-1)/2, i), (k-1, i), ..., ((k+3)/2, i)) with multiplicity $\lambda/2$. (The necessary conditions imply that λ is even.)

We claim that the graph λK_n-S has an almost resolvable P_k-decomposition. Recalling the definitions of Ω and A (Definition 4.2), we know that λK_{mk+1} can be decomposed into m/2 copies of $\lambda\Omega$ (using Hamilton cycle decompositions of K_{m+1}) and that S is the union of m/2 copies of $(\lambda/2)A$. Thus λK_{mk+1} - S is the union of $m\lambda/4$ copies of 2Ω - A. By Lemma 4.5(a), 2Ω - A is almost P_k-factorable and therefore λK_n - S has an almost resolvable P_k-decomposition.

As in the previous cases, all that remains is to show that S has a P_k-factorization. By the definition of S, it is compressible and $Q_H(S)$ is the union of the k-cycle (1, 2, ..., k) with multiplicity $(m-1)\lambda/2 + \lambda/2$ and the k-cycle (1, (k+1)/2, k, (k-1)/2, k-1, (k-3)/2, k-2, ..., (k+3)/2) with multiplicity $\lambda/2$. Since k is odd, we have $n\lambda/2$

$\equiv 0 \pmod{k-1}$. Also, $n\lambda/2 = \lambda(mk+1)/2 = \lambda((m+1)k - (k-1))/2 = \lambda(m+1)k/2 - \lambda(k-1)/2$ and hence $\lambda(m+1)/2 \equiv 0 \pmod{k-1}$ (remember, λ is even). Applying Lemma 2.2.1(b), $Q_H(S)$ has a P_k-factorization and hence so too does S.

Case 2.2.2. k is even. The case m = 2 was dealt with in Lemma 4.1. We may now assume that $m \geq 4$. We first construct a compressible subgraph S of K_{mk+1} with V(S) $= V(K_n) - \{\infty\}$ such that $K_{mk+1} - S$ is almost P_k-factorable and $Q_H(S)$ is the union of the single edge j(j+1), where $j \in \{1, 2, ..., k\}$, and the k-cycle, (1, 2, ..., k), in which edges alternately have multiplicities m/2 and m/2 -1.

We know that K_{mk+1} can be decomposed into m/2 isomorphic copies of Ω, say $\Omega_1, \Omega_2, ..., \Omega_{m/2}$. For each Ω_i, define $\mu(j)_i$ in the same way that $\mu(j)$ was defined for Ω in Definition 4.2. Similarly we define B_i, $C(i_1,j)_i$ and $C(i_m,j)_i$ where $\{i_1, i_m\} = \{x, y\}$ if ∞ is adjacent to V_x and V_y in Ω_i. By Lemma 4.5(b), $\Omega_i - b(j)_i$ is almost P_k-factorable, where $b(j)_i \in \mu(j)_i$ and thus $K_{mk+1} - \bigcup_{i=1}^{m/2} b(j)_i$ is almost P_k-factorable. Let P $= \bigcup_{i=1}^{m/2} \{C(i_1,j)_i - (j,i_1)(j+1,i_1), C(i_m,j)_i - (j,i_m)(j+1,i_m)\}$. Obviously $P \cup \{\infty\}$ is an almost P_k-factor of K_{mk+1}. Let $S = \bigcup_{i=1}^{m/2} b(j)_i - P$. Then $K_{mk+1} - S$ has an almost P_k-factorization. To see that S is compressible and $Q_H(S)$ is as stated, we first recall the structure of B (as in Definition 4.2). Note that each vertex of the subgraph induced by (H_{2i-1}, H_{2i}) $1 \leq i \leq k/2$, has degree m/2 (there is a contribution of 1 from each Ω_t) and that each vertex (2i,x) or (2i+1,x) of the subgraph induced by (H_{2i}, H_{2i+1}), where $1 \leq i \leq k/2-1$, has degree m/2 - 1 (a contribution of 1 from all Ω_t except the one containing the edge ∞x). Finally, there are the edges $\{(j,i)(j+1,i): 1 \leq i \leq m\}$. Clearly S is compressible and $Q_H(S)$ is as described.

Define C^* to be the k-cycle (1, 2, ..., k) in which edges alternately have multiplicities m/2 on edges {12, 34, ..., (k-1)k} and m/2 -1 on the others. Similarly C_* is the k-cycle (1, 2, ..., k) in which the edge of multiplicity m/2 are {23, 45, ..., k1} and the others have multiplicity m/2 -1. (Although $Q_H(S)$ is the union of the k-cycle C^* and the edge j(j+1), it is not difficult to see that we could also define an S^* so that $Q_H(S^*)$ is the union of the k-cycle C_* and the edge j(j+1).)

The graph λK_n has a factorization into λ copies of K_n. For the ith copy, $1 \leq i \leq \lambda$, we will construct a compressible subgraph S_i so that both $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is P_k-factorable and $K_n - S_i$ is almost P_k-factorable. We divide the proof into two cases according to the parity of λ.

Case 2.2.2.1 : λ is even. We define S_i for the ith copy of K_n as follows. If $1 \leq i \leq \lambda/2$, choose S_i so that its H-quotient graph is the union of C^* and the edge i(i+1), and if $\lambda/2 < i \leq \lambda$ choose S_i so that $Q_H(S_i)$ is the union of C_* and the edge i(i+1). With this definition and letting $\lambda = xk+y$, $0 \leq y \leq k-1$, $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is the union of a k-cycle (1, 2, ..., k) with multiplicity $x + \lambda(m-1)/2$ and the (y+1)-path [1, 2, ..., y+1]. Since $k(x + \lambda(m-1)/2) + y = kx+y + \lambda k(m-1)/2 = \lambda((mk+1) - (k-1))/2 \equiv 0 \pmod{k-1}$, by Lemma 2.2.1(a), this quotient graph is P_k-factorable.

Case 2.2.2.2 : λ is odd. As in Case 2.2.2.1 we begin by defining all S_i $1 \leq i \leq \lambda$.

If $\lambda < k/2$, choose S_i, $1 \leq i \leq (\lambda-1)/2$, so that its H-quotient graph is the union of C^* and the edge (2i-1)2i, and if $(\lambda+1)/2 \leq i \leq \lambda$ choose it so that the H-quotient is the union of C_* and the edge (2i-1)2i. Under this arrangement, $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is the union of the k-cycle (1, 2, ..., k) with multiplicity $(\lambda-1)(m-1)/2 + m/2 - 1$, a $2(\lambda+1)$-path and $(k-2\lambda-2)/2$ independent edges (the path and the independent edges are edge-disjoint subgraphs of (1, 2, ..., k)). Since $k((\lambda-1)(m-1)/2 + m/2 -1) + 2\lambda +1 + (k-2\lambda-2)/2 = \lambda((mk+1) -(k-1))/2 \equiv 0 \pmod{k-1}$, then by Lemma 2.2.1(a), $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is P_k-factorable.

If $\lambda \geq k/2$, we will choose $(\lambda-1)/2$ of the S_i so that $Q_H(S_i)$ is the union of C^* and a single edge, and the remaining $(\lambda+1)/2$ of S_i so that $Q_H(S_i)$ is the union of C_* and a single edge. All the single edges in the first $k/2$ of the S_i are $\{12, 34, ..., (k-1)k\}$ and the rest of them are to be arranged as the union of the k-cycle $(1, 2, ..., k)$ with multiplicity x and the (y+1)-path $[1, 2, ..., y+1]$, where $\lambda - k/2 = kx + y$, $0 \leq y \leq k-1$. Now $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is the union of a k-cycle with multiplicity $(\lambda-1)(m-1)/2 + m/2 - 1 + (1+x)$ and the (y+1)-path $[1, 2, ...y+1]$. Since $((\lambda-1)(m-1)/2 + m/2 + x)k + y = \lambda((km+1) - (k-1))/2 \equiv 0 \pmod{k-1}$, then again by Lemma 2.2.1(a), $Q_H(\bigcup_{i=1}^{\lambda} S_i)$ is P_k-factorable. The proof is complete. ∎

We showed in Chapter 3 that for λK_n to have a P_k-factorization the obvious necessary conditions determined by simple counting on edges and vertices are sufficient. We have now shown that similar conditions are necessary and sufficient for an almost P_k-factorization of λK_n. These are both special cases of a more general question: what conditions other than those obtained by counting must be imposed on m, k, r and λ so that λK_{mk+r}, $0 \leq r < k$, has a factorization in which each factor consists of m vertex disjoint paths P_k and r isolated vertices (note that instead of r isolated vertices we might also ask for a path of length r-1). We feel that those simple conditions obtained by counting are also sufficient but expect that it will be difficult to show this.

Chapter 5. Resolvable mixed path decomposition of λK_n

In this chapter we are interested in the construction of factorizations of λK_n in which each factor is one of two types. As defined in Chapter 1, a (G(s), H(t))-factorization of λK_n is a factorization in which s of the factors are G-factors and t are H-factors. Our interest in this chapter is in finding necessary and sufficient conditions for the existence of a $(P_2(s), P_k(t))$-factorization of λK_n. In view of earlier results we will assume that $st \neq 0$ and that $k \geq 3$. The goal is to prove the following result. (Observe that simple counting, first on vertices and second on edges, yields the necessary conditions of the theorem.)

5.1 Theorem For $k \geq 2$ the complete multigraph λK_n has a factorization into s+t spanning subgraphs ($st \neq 0$), s of which are 1-factors and t of which are P_k-factors (that is, a $(P_2(s), P_k(t))$-factorization) if and only if $n \equiv 0 \pmod 2$, $n \equiv 0 \pmod k$ and $ks + 2t(k-1) = \lambda k(n-1)$.

As usual, we begin with some basic constructions, and then go on to use them to prove the main theorem.

5.2 Lemma. Let k be odd.

(a) $K_{2k} - P_2(1)$ has a P_k-factorization, where $P_2(1)$ is a 1-factor.

(b) $K_{2k,2k} - C_4(1)$ has a P_k-factorization, where $C_4(1)$ is a C_4-factor.

Proof. (a) Let $V(K_{2k}) = \{1, 2, ..., 2k\}$. Consider a P_{2k}-factorization of K_{2k}. Each path $P(i) = [i, 1+i, 2k-1+i, 2+i, 2k-2+i, ..., k+2+i, k-1+i, k+1+i, k+i]$, $1 \leq i \leq k$, of the factorization is the union of two paths of length k and the edge $((3k+1)/2+i)((k+1)/2+i)$. Observe that these k edges are in fact the edges of a 1-factor in K_{2k}. Deleting them from the paths yields a P_k-factorization of $K_{2k} - P_2(1)$.

(b) Let $V(K_{2k,2k}) = X \cup Y$, where $X = \{x_1, ..., x_{2k}\}$ and $Y = \{y_1, ..., y_{2k}\}$. We know that $K_{2k} - P_2(1)$ has a P_k-factorizaiton. If P is one of these P_k-factors, then $\{x_iy_j, x_jy_i : ij \in E(P)\}$ is a P_k-factor of $K_{2k,2k}$. On deleting the resulting k P_k-factors from $K_{2k,2k}$, what remains is $\{x_iy_j, x_jy_i : ij \in E(P_2(1))\} \cup \{x_iy_i : 1 \leq i \leq 2k\}$, which is a C_4-factor. ▮

5.3 Remark. It is not difficult to see that in Lemma 5.2(b) we can permute the vertices of $K_{2k,2k}$ in such a way that if the vertex bipartition is (A, B), where $A = \{a_1, ..., a_{2k}\}$ and $B = \{b_1, ..., b_{2k}\}$, then $C_4(1)$ consists of the k 4-cycles $C(i) = (a_{2i+1}, b_{2i+3}, a_{2i+2}, b_{2i+4})$, $1 \leq i \leq k$. We define T to be the graph obtained from $C_4(1)$ by identifying the vertices a_i and b_i, $1 \leq i \leq 2k$. If the vertices of T are labelled $V(T) = \{v_1, v_2, ..., v_{2k}\}$, then T is the union of the four 1-factors:

$F_1 = \{v_{2i}v_{2i+1}: 1 \leq i \leq k\}$,

$F_2 = \{v_1v_3\} \cup \{v_4v_6\} \cup \{v_{2i+3}v_{2i+6} : 1 \leq i \leq k\text{-}2\}$,

$F_3 = \{v_1v_4\} \cup \{v_{2i+4}v_{2i+6}, v_{2i+1}v_{2i+3} : i \in \{1, 3, ..., k\text{-}2\}\}$ and

$F_4 = \{v_3v_6\} \cup \{v_{2i+6}v_{2i+8}, v_{2i+3}v_{2i+5} : i \in \{1, 3, ..., k\text{-}2\}\}$. ▮

5.4 Lemma. Let k be odd and $V(G) = \{1, 2, ..., 2k\}$. If $G = ((k-1)/2)T$ (where T is described in Remark 5.3), then G has a P_k-factorization.

Proof. Let $V(G) = \{1, 2, ..., 2k\}$. First select the P_k-factors

P(i) = {[2i+1, 2i+4, 2i+5, 2i+8, 2i+9, ..., 2i+2k-5, 2i+2k-2, 2i+2k-1],

[2i+2, 2i+3, 2i+6, 2i+7, 2i+10, ..., 2i+2k-4, 2i+2k-3, 2i+2k]},

$1 \leq i \leq (k-1)/2$, and

R(i) = {[2i+k, 2i+k+2, 2i+k+4, ..., 2i+k-4, 2i+k-2],

[2i+k+1, 2i+k+3, 2i+k+5, ..., 2i+k-3, 2i+k-1]}, $1 \leq i \leq (k-1)/2$.

The edges remaining form the last P_k-factor which is

$\{[1, 4, 5, 8, 9, ..., k-4, k-1, k, k+2, k+4, ..., 2k-1],$

$[2, 3, 6, 7, 10, ..., k-3, k-2, k+1, k+3, k+5, ..., 2k]\}$ if $k \equiv 1 \pmod 4$, and

$\{[1, 4, 5, 8, 9, ..., k-3, k-2, k+1, k+3, ..., 2k],$

$[2, 3, 6, 7, 10, ..., k-4, k-1, k, k+2, ..., 2k-1]\}$ if $k \equiv 3 \pmod 4$. ∎

We now prove the main theorem of this section.

Proof of Theorem 5.1.

The necessity has already been addressed. To show the sufficiency of the conditions, we will divide the proof into two cases according to the parity of k.

Case 1. k is even.

From the first two necessary conditions we know that $n = kr$, and from the condition $ks + 2(k-1)t = \lambda k(n-1)$ we obtain $s \equiv \lambda(r-1) \pmod{k-1}$.

Let $V(\lambda K_n) = \{(i, j) : 1 \le i \le r, 1 \le j \le k\} = \bigcup_{i=1}^{r} H_i = \bigcup_{j=1}^{k} V_j$ where

$H_i = \{(i, j) : 1 \le j \le k\}$ and $V_j = \{(i, j) : 1 \le i \le r\}$, so that $\lambda K_n = \lambda(K_r \otimes K_k)$.

To each edge ij of λK_r associate in λK_n a 1-factor F_{ij} of K_{H_i,H_j}. Let R be the subgraph of λK_n consisting of the union of these 1-factors. Each vertex in R has degree $\lambda(r-1)$. First we will show that the 1-factors can be chosen so that R has a $(P_2(s_1),P_k(t_1))$-factorization for $0 \le s_1 \le \lambda(r-1)$ and $s_1 \equiv \lambda(r-1) \pmod{k-1}$. Let $s_1 = \lambda(r-1) - q(k-1)$.

Let the 1-factor F_{ij} be either the 1-factor

$C_{ij} = \{(i, 2m-1)(j, 2m), (i, 2m)(j, 2m-1) : 1 \le m \le k/2\}$ or the 1-factor

$D_{ij} = \{(i, 2m-1)(j, 2m-2), (i, 2m-2)(j, 2m-1) : 1 \leq m \leq k/2\}$. Call the first of these 1-factors a type C 1-factor and the second type D.

We claim that the 1-factors F_{ij} can be chosen in such a way that each vertex belongs to at least $q(k/2-1)$ 1-factors of type C and at least $q(k/2)$ of type D.

If r is even take a 1-factorization of λK_r. To each edge of λK_r determined by $s_1 + q(k/2-1)$ of the 1-factors associate a type C 1-factor and to the edges from remaining $qk/2$ 1-factors associate a type D 1-factor. If r is odd, then $\lambda(r-1)$ is even and λK_r has a 2-factorization. If $s_1 + q(k/2-1)$ is even (and consequently so is $q(k/2)$), then to each edge of λK_r determined by $(s_1 + q(k/2-1))/2$ of the 2-factors associate a type C 1-factor and to the remaining edges associate a type D 1-factor. If $s_1 + q(k/2-1)$ is odd (and consequently so is $q(k/2)$), then to each edge of λK_r determined by $(s_1 - 1 + q(k/2-1))/2$ of the 2-factors associate a type C 1-factor and to the remaining edges associate a type D 1-factor. (Note that $s_1 \geq 1$ since if $s_1 = 0$, then $\lambda(r-1) = q(k-1)$ which is impossible as k-1, q and r are all odd.)

It is not difficult to see that R is compressible and $Q_V(R)$ consists of the edge-disjoint union of s_1 1-factors and q cycles of length k in which alternate edges have multiplicities $k/2-1$ and $k/2$. By Lemmas 2.2.1(a) and 2.1.4 the graph R has a $(P_2(s_1), P_k(t_1))$-factorization.

Next we show that $\lambda K_n - R$ has a $(P_2(s_2), P_k(t_2))$-factorization for any s_2, $0 \leq s_2 \leq \lambda r(k-1)$ and $s_2 \equiv 0 \pmod{k-1}$.

If r is even, λK_r admits a 1-factorization with 1-factors $F_1, F_2, ..., F_{\lambda(r-1)}$, and to each 1-factor there corresponds in $\lambda K_n - R$ a $(K_{k,k} - P_2(1))$-factor. Thus $\lambda K_n - R$ has a $((K_{k,k} - P_2(1))(\lambda(r-1)), K_k(\lambda))$-factorization.

If r is odd λK_r admits a near 1-factorization with near 1-factors $M_1, M_2, ..., M_{\lambda r}$ and each near 1-factor M_i corresponds in λK_n - R to a Y-factor, where Y consists of the vertex-disjoint union of (r-1)/2 copies of $K_{k,k} - P_2(1)$ and one copy of K_k. Thus λK_n - R has a Y-factorization.

We note that since k is even, Lemmas 2.2.2 and 2.2.3 assure us that each of $K_{k,k}$-$P_2(1)$, K_k and Y has a P_k-factorization consisting of k/2 P_k-factors. But these three graphs also have 1-factorizations made up of k-1 1-factors. So in each of the $((K_{k,k}-P_2(1))(\lambda(r-1)), K_k(\lambda))$-factorizations of λK_n - R (r even), and the Y-factorization of λK_n - R (r odd), we replace $s_2/(k-1)$ of the factors by 1-factors and the remainder by P_k-factors.

The theorem then follows by letting $s_1 = s$ and $s_2 = 0$ if $s \leq \lambda(r-1)$, and $s_1 = \lambda(r-1)$ and $s_2 = s - \lambda(r-1)$ if $s \geq \lambda(r-1)$.

Case 2. k is odd.

From the first two necessary conditions we know that n = 2kr, and from the condition $ks + 2(k-1)t = \lambda k(n-1)$ we obtain $s \equiv \lambda(2r-1) \pmod{2(k-1)}$. The construction to be presented is quite similar to that given when k is even.

Let $V(\lambda K_n) = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq 2k\} = \bigcup_{i=1}^{r} H_i = \bigcup_{j=1}^{2k} V_j$, where $H_i = \{(i, j) : 1 \leq j \leq 2k\}$ and $V_j = \{(i, j) : 1 \leq i \leq r\}$ and again note that $\lambda K_n = \lambda(K_r \otimes K_{2k})$.

To each edge ij of λK_r associate in λK_n a C_4-factor C_{ij} of K_{H_i,H_j}. To each vertex i of λK_r associate λ 1-factors, H_i^{ε}, $1 \leq \varepsilon \leq \lambda$, of the graph λK_{2k} with vertex-set H_i. Let R be the $\lambda(2r-1)$-regular subgraph of λK_n consisting of the union of these C_4-factors and 1-factors. As in the previous case, we begin by showing that these factors can be chosen so that R has a $(P_2(s_1), P_k(t_1))$-factorization for $0 \le s_1 \le \lambda(2r-1)$ and $s_1 \equiv \lambda(2r-1) \pmod{2(k-1)}$. Let $s_1 = \lambda(2r-1) - 2q(k-1)$.

Suppose $\lambda(r-1)$ is even. Then λK_r has a 2-factorization. We arbitrarily direct the cycles in the 2-factorization so yielding a directed λK_r in which each vertex has both in- and out-degree $\lambda(r-1)/2$. If the edge ij is directed from i to j (that is, it becomes the arc (i, j)), then let C_{ij} be a copy of $C_4(1)$ as described in Remark 5.3 but with $A = H_i$ and $B = H_j$. Thus $Q_V(\bigcup_{i \neq j} C_{ij}) = (\lambda(r-1)/2)T$. For each i choose the H_i^ε, where $1 \le \varepsilon \le \lambda$, so that:

$\lfloor \lambda/4 \rfloor$ of them are $F_1^i = \{(i, 2j)(i, 2j+1) : 1 \le j \le k\}$;

$\lfloor \lambda/4 \rfloor$ are $F_2^i = \{(i,1)(i, 3), (i, 4)(i, 6)\} \cup \{(i, 2j+3)(i, 2j+6) : 1 \le j \le k-2\}$;

$\lfloor \lambda/4 \rfloor$ are $F_3^i = \{(i, 1)(i, 4)\} \cup \{(i, 2j+4)(i, 2j+6), (i, 2j+1)(i, 2j+3): j \in \{1, 3, ..., k-2\}\}$;

$\lfloor \lambda/4 \rfloor$ are $F_j^i = \{(i, 3)(i, 6)\} \cup \{(i, 2j+6)(i, 2j+8), (i, 2j+3)(i, 2j+5) : j \in \{1, 3, ..., k-2\}\}$

(where F_4^i is analogous to F_j as given in Remark 5.3) and the remaining $\lambda' = \lambda - 4\lfloor \lambda/4 \rfloor$ are chosen arbitrarily. Thus $Q_V(\bigcup_{i=1}^{r} \bigcup_{\varepsilon=1}^{\lambda} H_i^\varepsilon)$ consists of $\lfloor \lambda/4 \rfloor$ copies of T and λ' 1-factors and therefore $Q_V(R)$ consists of $\lambda(r-1)/2 + \lfloor \lambda/4 \rfloor = \lfloor \lambda(2r-1)/4 \rfloor$ edge-disjoint copies of T and λ' 1-factors. We use Lemma 5.4 to determine a P_k-factorization of $(q(k-1)/2)T$; and since $2q(k-1) < \lambda(2r-1)$ there are $q(k-1)/2$ copies of T available. Each of the remaining copies of T in $Q_V(R)$ has a 1-factorization. This now yields a $(P_2(s_1), P_k(t_1))$-factorization of R.

We next consider the case when $\lambda(r-1)$ is odd (and hence λ is odd). In this case $\lambda K_r - F$, where F is a 1-factor, has a 2-factorization. Proceed to define R as in the previous case using the 2-factorization of $\lambda K_r - F$. To the remaining edges ij of λK_r (those of the deleted 1-factor F) associate the C_4-factor $\{((i, 2p), (j, 2p), (i, 2p+1), (j, 2p+1)): 1 \le p \le k\}$. Again choose the λ 1-factors H_i^ε so that $Q_V(\bigcup_{i=1}^{r} \bigcup_{\varepsilon=1}^{\lambda} H_i^\varepsilon)$ contains $\lfloor \lambda/4 \rfloor$ copies of T and λ' 1-factors. Furthermore, if $\lambda' = 3$, choose those λ' 1-

factors to be F_2^i, F_3^i and F_4^i as given previously. Let us now analyse the subgraph R. It consists of a 1-factor $F' = \{(i, 2p)(j, 2p), (i, 2p+1)(j, 2p+1) : 1 \le p \le k, ij \in F\}$ and a subgraph R'. If $\lambda' = 3$, $Q_V(R')$ consists of $(\lambda(2r-1)-1)/4$ edge-disjoint copies of T, and if $\lambda' = 1$, $Q_V(R')$ consists of $(\lambda(2r-1)-3)/4$ edge-disjoint copies of T and two 1-factors. In each case there are at least $q(k-1)/2$ copies of T available and by Lemma 5.4 we have a P_k-factorization of $(q(k-1)/2)T$. Applying Lemma 2.1.4 $Q_V(R')$, (and therefore R') has a $(P_2(s_1-1), P_k(t_1))$-factorization. So R has a $(P_2(s_1), P_k(t_1))$-factorization as required.

The final step, in which we show that λK_n - R has a $(P_2(s_2), P_k(t_2))$-factorization for any s_2, $0 \le s_2 \le 2\lambda r(k-1)$ and $s_2 \equiv 0 \pmod{2(k-1)}$, is quite straightforward.

If r is even we use a 1-factorization of K_r to obtain a $((K_{2k,2k} - C_4(1))(\lambda(r-1)), (K_{2k} - P_2(1))(\lambda))$-factorization of λK_n - R, and if r is odd we use a near 1-factorization of K_r to obtain a Z-factorization of λK_n - R, where Z is the vertex-disjoint union of $(r-1)/2$ copies of $K_{2k,2k} - C_4(1)$ and one copy of K_{2k} - $P_2(1)$. By Lemma 5.2 the graphs $K_{2k,2k} - C_4(1)$ and $K_{2k} - P_2(1)$ have P_k-factorizations, each with k P_k-factors. In addition, they both have 1-factorizations with 2k-2 1-factors. So on $s_2/2(k-1)$ occasions we choose the 1-factorization and on the remaining occasions the P_k-factorization.

The theorem is now completed by letting $s_1 = s$ and $s_2 = 0$ if $s \le \lambda(2r-1)$, and $s_1 = \lambda(2r-1)$ and $s_2 = s - \lambda(2r-1)$ if $s \ge \lambda(2r-1)$. ∎

Chapter 6. P_k-factorizations of $\lambda K(n,r)$

Necessary conditions for the existence of a P_k-factorization of $\lambda K(n,r)$ are $nr \equiv 0 \pmod{k}$ (as each factor is a union of disjoint paths on k vertices) and $\lambda(r-1)nk \equiv 0 \bmod 2(k-1))$ (as $|E(\lambda K(n,r))|$ must be divisible by the number of edges in a P_k-factor). We would like to show that these conditions are also sufficient. As we mentioned in Chapter 1 Ushio [36] proved that when k = 3 the conditions are sufficient, and Bermond [6] later gave a short proof of this. In this section, we extend the result for k > 3 and show that the two conditions are sufficient if $n \equiv 0 \pmod{k}$ or $r \equiv 0 \pmod{k}$. (This implies, for example, that they are sufficient if k is prime.) We will also show that they are sufficient when r = 2 and r = 3. In general, however, this problem remains unresolved.

Let $V(\overline{K}_k \otimes C_r) = \{1, ..., k\} \times \{1, ..., r\}$, $H_i = \{(i, j) : 1 \leq j \leq r\}$, where $1 \leq i \leq k$, and $V_j = \{(i, j) : 1 \leq i \leq r\}$, where $1 \leq j \leq k$.

Once again we begin with a technical lemma.

6.1 Lemma. Let k be a positive integer, $k \geq 4$, and r be odd. The graph $\overline{K}_k \otimes C_r$ is the union of k P_k-factors and a subgraph S such that $Q_H(S) \cong C_k$.

Proof. We will construct k P_k-factors of $\overline{K}_k \otimes C_r$ so that on their deletion, the remaining subgraph is induced by one of the following two edge-sets: $\{(i, j)(i+1, j+1): 1 \leq i \leq k, 1 \leq j \leq r\}$ or $\{(i, j)(i-1, j+1): 1 \leq i \leq k, 1 \leq j \leq r\}$. Denoting these induced graphs by G_1 and G_2, respectively, it is not difficult to see that $Q_H(G_1) \cong Q_H(G_2) \cong C_k$. We divide the proof into four cases.

Case 1. $k \equiv 0 \pmod 4$.

When $k = 4$, for $0 \le j \le 3$, let

$P(1, j) = [(1+j, 2), (1+j, 1), (2+j, 2), (4+j, 1)]$,

$P(2, j) = [(2+j, 3), (4+j, 2), (4+j, 3), (3+j, 2)]$,

$P(2t+1, j) = [(2+j, 2t+2), (1+j, 2t+1), (3+j, 2t+2), (3+j, 2t+1)]$, $1 \le t \le (r-1)/2$,

and $P(2t, j) = [(4+j, 2t+1), (4+j, 2t), (2+j, 2t+1), (1+j, 2t)]$, $2 \le t \le (r-1)/2$.

Then $\bigcup_{i=1}^{r} P(i, j)$ is a P_4-factor and $E((\overline{K}_4 \otimes C_r) - \bigcup_{j=0}^{3} \bigcup_{i=1}^{r} P(i, j)) =$

$\{(i, j)(i-1, j+1) : 1 \le i \le 4, 1 \le j \le r\}$. (See Figure 6.1.)

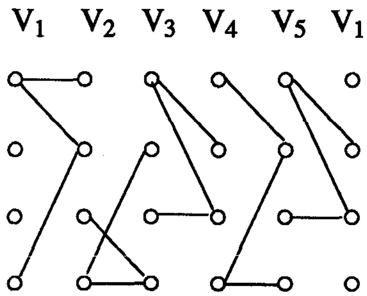

Figure 6.1

When $k \ge 8$, as in the case $k = 4$, we let $P(i, j)$ be a path in the bipartite subgraph of $\overline{K}_k \otimes C_r$ on vertex set (V_i, V_{i+1}). We will use the notation of definition 4.3. and we will use the convention that the distance of the edge $(s, i)(t, i+1)$ is $t-s$.

For $0 \le j \le k-1$, put

$P(1, j) = [(1+j, 2): \langle 0,1, \ldots, k-2\rangle]$,

$P(2, j) = [(k/2 +j, 3): \langle k/2, k/2 +1, \ldots, k-2, 0, 1, \ldots, k/2 -1\rangle]$,

$P(2t+1, j) = [(k/4 +1+ j, 2t+1): \langle 0, k/2 +1, k/2 +2, 2, 3, \ldots, k/2, 1, k/2 +3, k/2 +4, \ldots, k-2\rangle]$, $1 \le t \le (r-1)/2$ and

$P(2t, j) = [(3k/4 +1+j, 2t+1): \langle 0, 1, 2, \ldots, k/2 -2, k-2, k/2 -1, k/2, \ldots, k-3\rangle]$,

$2 \leq t \leq (r-1)/2$. (See Figure 6.2 which illustrates the case k=8 and r=5.)

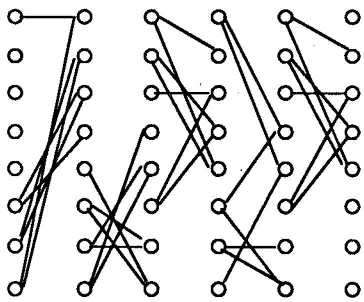

Figure 6.2

Case 2. k ≡ 1 (mod 4).

When $k \geq 5$, for $0 \leq j \leq k-1$, let

P(1, j) = [((k+3)/2 + j, 2): <0, k-2, k-3, ..., 1>],

P(2, j) = [(2+j, 3): <2, 3, ..., k-2, 0, 1>],

P(3, j) = [(1 + j, 4): <0, 1, ..., k-2)],

P(2t, j) = [((k+3)/2 + j, 2t+1): <0, k-2, k-3, ..., 1>], $2 \leq t \leq (r-1)/2$ and

P(2t+1, j) = [((k+7)/4 + j, 2t+2): <(k+3)/2, (k+5)/2, ..., k-2, 0, (k+1)/2, 1, 2, ..., (k-1)/2>], where $2 \leq t \leq (r-1)/2$.

(In Figure 6.3 the case when k = 9 and r = 5 is given.)

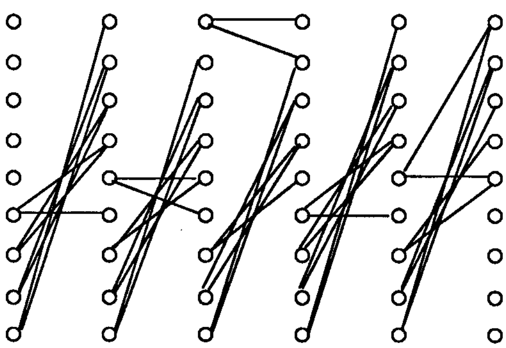

Figure 6.3

Case 3. $k \equiv 2 \pmod 4$.

When $k = 6$, for $0 \le j \le 5$ let

$P(1, j) = [(1+j, 2): \langle 0, 1, 2, 3, 4 \rangle]$,

$P(2, j) = [(6+j, 3): \langle 2, 1, 0, 4, 3 \rangle]$,

$P(2t+1, j) = [(4+j, 2t+2): \langle 3, 2, 1, 0, 4 \rangle]$, where $1 \le t \le (r-1)/2$ and

$P(2t, j) = [(6+j, 2t+1): \langle 1, 0, 4, 2, 3 \rangle]$, where $2 \le t \le (r-1)/2$.

When $k \ge 10$, for $0 \le j \le k-1$, let

$P(1, j) = [(1+j, 2): \langle 0, 1, 2, \ldots, k-2 \rangle]$,

$P(2, j) = [(k/2 +1 +j, 3): \langle k/2 +1, k/2 +2, \ldots, k-2, 0, 1, \ldots, k/2 -2,$
$k/2 -1, k/2 \rangle]$,

$P(2t+1, j) = [(2+ j, 2t+2): \langle k/2 +2, k/2 +3, \ldots, k-2, k/2, k/2 +1, 0,1, \ldots,$
$k/2 -1 \rangle]$, $1 \le t \le (r-1)/2$ and,

$P(2t, j) = [((3k+6)/4 + j, 2t+1): \langle 1, 2, \ldots, k/2 -1, 0, k/2, k/2 +1, \ldots,$
$k-2 \rangle]$, where $2 \le t \le (r-1)/2$.

(Shown in Figure 6.4 in the case $k = 10$ and $r = 5$.)

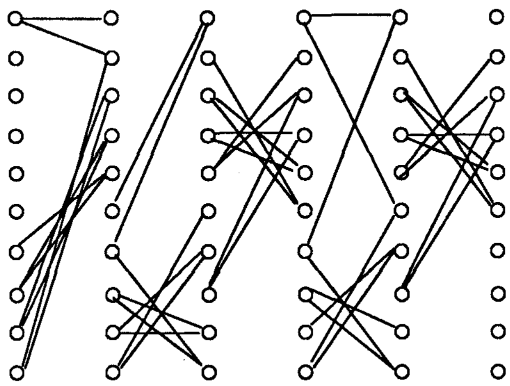

Figure 6.4

Case 4. k ≡ 3 (mod 4).

In this case the general pattern covers all cases. (Recall that $k \geq 4$.)

For $0 \leq j \leq k-1$, let

P(1, j) = [((k+1)/4 + j, 1): <1, 2, ..., (k-1)/2, 0, (k+1)/2,
(k+3)/2, ..., k-2>],

P(2, j) = [(3(k+1)/4 + j, 2): <0, (k+1)/2, (k+3)/2, 2, 3, ..., (k-1)/2, 1, (k+5)/2,
(k+7)/2, ..., k-2>],

P(3, j) = [(2 + j, 3): <k-2, k-3, ..., 1, 0>],

P(2t, j) = [((k+1)/4 + j, 2t): <1, 2, ..., (k-1)/2, 0, (k+1)/2,
(k+3)/2, ..., k-2>], $2 \leq t \leq (r-1)/2$ and

P(2t+1, j) = [((k+3)/2 + j, 2t+1): <(k-3)/2, (k-5)/2, ..., 1, 0, (k+1)/2,
(k-1)/2, k-2, k-3, ..., (k+3)/2>], $2 \leq t \leq (r-1)/2$.

(See Figure 6.5 for the case k = 7 and r = 5.)

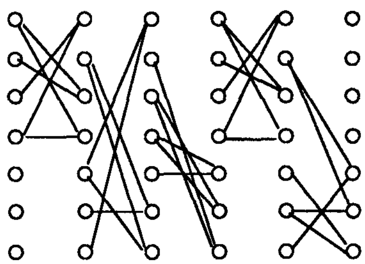

Figure 6.5

∎

We now state and prove the main theorem of this section.

Theorem 6.2. If $\lambda kn(r-1) \equiv 0 \pmod{2(k-1)}$ and $r \equiv 0 \pmod{k}$ or $n \equiv 0 \pmod{k}$, then $P_k \mid_R \lambda K(n,r)$.

Proof. Let $V(\lambda K(n,r)) = \{1, ..., n\} \times \{1, ..., r\}$, $H_i = \{(i, j) : 1 \le j \le r\}$, where $1 \le i \le n$ and $V_j = \{(i, j) : 1 \le i \le n\}$, where $1 \le j \le r$. Suppose $r \equiv 0 \pmod k$. It is easy to see that $\lambda K(n, r)$ is compressible with respect to vertex-partition $\bigcup_{i=1}^{n} V_i$ and hence $Q_V(\lambda K(n,r)) = n\lambda K_r$. By Theorem 3.3.1 $P_k|_R\, n\lambda K_r$ if and only if $r \equiv 0 \pmod k$ and $n\lambda k(r-1) \equiv 0 \pmod{2(k-1)}$ and hence by Lemma 2.1.4, $P_k |_R\, \lambda K(n,r)$. In this case, we are done.

We now consider the case when $\lambda kn(r-1) \equiv 0 \pmod{2(k-1)}$ and $n \equiv 0 \pmod k$. Let $n = km$. We first show that if $P_k |_R\, \lambda mK(k,r)$, then $P_k |_R\, \lambda K(n,r)$.

Let $X_{u,v}$ be a subset of $V(\lambda K(mk,r)) = \{1, 2, ..., mk\} \times \{1, 2, ..., r\}$, where $X_{u,v} = \{((u-1)m +1, v), ((u-1)m+2, v), ..., (um, v)\}$, $1 \le u \le k$ and $1 \le v \le r$. Let $P(1), P(2), ..., P(s)$ be the P_k-factors of a P_k-factorization of $\lambda mK(k,r)$, where $s = \lambda m(r-1)k^2/(2(k-1))$. Corresponding to each $P(i)$, we construct a P_k-factor $p(i)$ of $\lambda K(mk, r)$ as follows: With each edge $(u,v)(p,q) \in E(P(i))$, associate a 1-factor $F((u,v)(p,q))$ from $K_{X_{u,v}, X_{p,q}}$. Clearly, the induced subgraph with edge-set $\{e : e \in F((u,v)(p,q))$, where $(u,v)(p,q) \in E(P(i))\}$ is a P_k-factor of $\lambda K(mk,r)$. Since $\lambda K_{X_{u,v}, X_{p,q}}$ has a 1-factoriation with λm 1-factors it is easy to see that this method does indeed give a P_k-factorization of $\lambda K(n,r)$.

To complete the proof it only remains to show that $P_k |_R\, \lambda mK(k,r)$. The proof is divided into two parts according to the parity of r.

Case 1. r odd.

The graph $\mu K(k,r)$, where $\mu = \lambda m$, can be decomposed into $\mu(r-1)/2$ isomorphic copies of $\overline{K_k} \otimes C_r$. By Lemma 6.1 $\overline{K_k} \otimes C_r$ is the union of k P_k-factors and a subgraph with H-quotient C_k. Hence we can delete $\mu(r-1)k/2$ P_k-factors from

$\mu K(k,r)$ so that the remaining graph has H-quotient $((r-1)\mu/2)C_k$, which by Lemma 2.2.1(a) and the fact that $(r-1)\mu/2 \equiv 0 \pmod{k-1}$ is P_k-factorable.

Case 2. r even.

First consider k to be even. Let $R = \bigcup_{1 \le i < j \le k} F_{ij}(\mu)$, where $F_{ij}(\mu)$ is the union of μ 1-factors in K_{V_i,V_j}. (Notice that R is not uniquely determined.)

We claim $\mu K(n,r) - R$ has a P_k-factorzation. To see this begin by observing that as r is even, the graph μK_r has a 1-factorization $f_1, f_2, \ldots, f_{\mu(r-1)}$. In $\mu K(n,r)$, each 1-factor corresponds to r/2 vertex-disjoint copies of $K_{k,k}$. By Lemma 2.2.3, $K_{k,k} - F$, where F is a 1-factor, has a P_k-factorization. By chosing F appropriately we can delete $k\mu(r-1)/2$ P_k-factors from $\mu K(n,r)$ so that the subgraph remaining is R.

We now show that there exists such an R which is also P_k-factorable. For each edge $xy \in E(f_i)$, $1 \le i \le \lfloor \mu(r-1)/2 \rfloor$, let F_{xy} be the 1-factor of $\mu K_{V_x,V_y}$ defined by $F_{xy} = \{(1, x)(2, y), (2, x)(1, y), (3, x)(4, y), (4, x)(3, y), \ldots, (k-1, x)(k, y), (k, x), (k-1, y)\}$. Otherwise, if $\lfloor \mu(r-1)/2 \rfloor + 1 \le i \le \mu(r-1)$, let $F_{xy} = \{(2,x)(3,y), (3, x)(2,y), (4, x)(5, y), (5, x)(4, y), \ldots, (1, x)(k, y), (k, x), (1, y)\}$. Let $R = \bigcup_{xy} F_{x,y}$. Then $Q_H(R)$ is a k-cycle in which each edge has multiplicity $\mu(r-1)/2$ when $\mu(r-1)$ is even, and a k-cycle in which edges alternately have multiplicities $(\mu(r-1)-1)/2$ and $(\mu(r-1)+1)/2$ when $\mu(r-1)$ is odd. Since $\mu(r-1)k/2 \equiv 0 \pmod{k-1}$ we can apply Lemma 2.2.1(a) to show that in either case $Q_H(R)$ is P_k-factorable.

When k is odd, Lemma 2.2.3 states that $2K_{k,k} - 2F$, where F is an arbitrary 1-factor of $K_{k,k}$, has a P_k-factorization. Let $R = \bigcup_{1 \le i < j \le r} 2F_{ij}(\mu/2)$, where $F_{ij}(\mu/2)$ is the union of $\mu/2$ 1-factors of K_{V_i,V_j}. As before, we can show $\mu K(n, r) - R$ has a P_k-factorizaton. Then we will show that there exists such an R which also has a P_k-factorization. Observe that $\mu = \lambda m \equiv 0 \pmod 4$ as k is odd and r is even. In this case, we let $F_{ij}(\mu/2) = (\mu/4)P$, where $P = \{(s, i)(s+1, j), (s+1, i)(s, j) : 1 \leq s \leq k\}$. It is not difficult to see that $Q_H(R) \cong (\mu(r-1)/2)C_k$. Since $\mu(r-1)k/2 \equiv 0 \pmod{k-1}$, by Lemma 2.2.1, $Q_H(R)$ has a P_k-factorization. Therefore, the proof is complete. ∎

We now use Theorem 6.2 to prove two more results. But we first state a result due to Auerbach and Laskar [4].

6.3 Theorem. [4]. If (r-1)n is even, then K(n,r) has a C_{nr}-decomposition

6.4 Corollary. $P_k \mid_R \lambda K(n,2)$ if and only if $2n \equiv 0 \pmod{k}$ and $\lambda nk \equiv 0 \pmod{2(k-1)}$.

Proof. The necessity follows immediately from applying a counting argument on vertices and edges. For the sufficiency, we suppose that $2n \equiv 0 \pmod{k}$ and $\lambda nk \equiv 0 \pmod{2(k-1)}$. If k is odd, then $n \equiv 0 \pmod{k}$ and by Thorem 6.2 we are done.

If $k = 2m$, then $n \equiv 0 \pmod{m}$ and the second condition becomes $\lambda n \equiv 0$ (mod 2m-1) which implies $q\lambda \equiv 0 \pmod{2m-1}$, where $n = mq$. As in the proof of Theorem 6.2 we only need to show that $q\lambda K_{m,m}$ has a P_k-factorization. Let $V(q\lambda K_{m,m}) = \{a_1, a_2, ..., a_m\} \cup \{b_1, b_2, ..., b_m\}$.

When m is even, $K_{m,m}$ has a C_{2m}-factorization (Theorem 6.3) and by Lemma 2.2.1(a) $q\lambda C_{2m}$ has a P_{2m}-factorization since $q\lambda \equiv 0 \pmod{2m-1}$.

When m is odd, $K_{m,m} - F$ has a C_{2m}-factorization. We divide the remaining λq 1-factors into p groups with 2m-1 in each (assuming $\lambda q = p(2m-1)$). Fix a group made up of, say, $f_1, f_2, ..., f_{2m-1}$, where $f_1 = f_2 = ... = f_m = \{a_ib_i : i = 1, 2, ..., m\}$ and $f_{m+1} = ... = f_{2m-1} = \{a_ib_{i+1}, i = 1, 2, ..., m\}$. Then $f_i \cup f_{m+i} - a_ib_{i+1}$, $i = 1, 2, ...m-1$, is a P_{2m}-factor, as is $f_m \cup \{a_ib_{i+1}, i = 1, ..., m-1\}$. Hence, $q\lambda K_{m,m}$ has a P_k-factorziation and so does $\lambda K(n, r)$ ∎

6.5 Corollary. $P_k \mid_R \lambda K(n,3)$ if and only if $3n \equiv 0 \pmod{k}$ and $3\lambda nk \equiv 0 \pmod{k-1}$.

Proof. The necessity follows immediately on applying counting argument to vertices and edges. For sufficiency, if $k \equiv 1$ or $2 \pmod 3$, then $n \equiv 0 \pmod k$ and by Theorem 6.2 we are done.

When $k \equiv 0 \pmod 3$, we let $n = kq$ and show that $P_k \mid_R \lambda qK(k,3)$. By Theorem 6.3, $C_{3k} \mid_R K(k,3)$. From the given conditions, $3k\lambda q = 3\lambda n \equiv 0 \pmod{k-1}$ or $\lambda q \equiv 0 \pmod{k-1}$. We only need to show λqC_{3k} has a P_k-factorization. Since $(k-1)C_{3k}$ has a P_k-factorization with factors $\{[(ik+j+1), (ik+j+2),\ldots, (ik+j+k-1), (ik+j+k)] : 0 \le i \le 2\}, 0 \le j \le k-1$, then the result follows immediately. ∎

It is not difficult to see that by using the quotient technique, we can obtain many tree factorization results for $\lambda K(n, r)$. We suspect that the necessary conditions (obtained by counting arguments) for the existence of a tree factorization of $\lambda K(n, r)$ are sufficient.

Chapter 7. Summary

At this stage, we see that the concept of the quotient graph of a graph plays a very important role in the construction of factorizations of λK_n and $\lambda K(n, r)$. This is a technique which should be further exploited.

Several of the problems we mentioned in this thesis can be easily generalized. For example, we can ask the following questions:

1. What are necessary and sufficient conditions for λK_n to have a $(P_s(x), P_t(y))$-factorization?

2. Are the necessary conditions for λK_n to have an almost H-factorization given in Chapter 1 sufficient when H is a tree other than a path?

3. Can we get some similar factorization results when H is a directed graph and we are factorizing the complete symmetric digraph?

Another interesting problem is the following: What are necessary and sufficient conditions for an almost resolvable H-decomposition of λK_n to be balanced? (Let $V(H) = \{v_1, v_2, ..., v_k\}$. An H-decomposition is called balanced if there exist integers $a_1, a_2, ..., a_k$, where $a_1 + a_2 + ... + a_k$ is the total number of factors, so that each vertex of λK_n plays the role of v_i in a_i of the H-factors, $1 \leq i \leq k$.) It is easy to see that all resolvable decompositions are balanced.

Finally, we state once again the particularly interesting question: For what even k does $T_k \mid K_k$? At present there seem to be no known techniques other than that of searching for a cyclic decomposition.

Appendix

xy : an edge joining vetex x to vertex y.

(x, y) : an arc directed from vertex x towards vertex y.

λG : a multigraph obtained by assigning each edge of G multiplicity λ.

K_n^* : the complete symmetric digraph on n vertices.

K_n : the complete graph on n vertices in which each pair of vertices is joined by exactly one edge.

K(n,r) : the complete r-partite graph in which each part has size n.

$K_{A,B}$: the complete bipartite graph with bipartition (A, B).

K_A : the complete graph with vertex set A.

$K_{1,k-1}$: a star with k vertices.

P_k (or k-path) : a path with k vertices.

C_k (or a k-cycle) : a cycle with k vertices.

t-matching : a set of t independent edges.

1-factor of a graph G : a spanning subgraph of G which is the union of |V(G)|/2-matching.

near 1-factor of a graph G : a spanning subgraph of G which is the union of a (|V(G)|-1)/2 -matching and an isolated vertex.

$\overline{G}$: the complement of G.

$A \cup B$: the graph induced by the edge-set $E(A) \cup E(B)$.

A - B : the graph induced by the edge set E(A) - E(B) if B is a subgraph of A.

G - {v} : the graph obtained from G by deleting the vertex v and all edges incident with v.

$F \otimes G$: Let G be a graph with V(G) = {1, 2, ..., x}, and let $F = \{S_1, S_2, ..., S_x\}$ be a family of graphs. $F \otimes G$ is defined to be the graph obtained by replacing vertex i of G by S_i, $1 \le i \le x$, and inserting all possible edges between S_i and S_j with multiplicity λ exactly when the edge ij in G has multiplicity λ. When all S_i are isomorphic to S, we will write $S \otimes G$.

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