



National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file *Votre référence*

Our file *Notre référence*

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

FACTORS AND FACTOR EXTENSIONS

by

Qinglin Yu

M. Sc., Shandong University, China, 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics & Statistics

© Qinglin Yu, 1991

SIMON FRASER UNIVERSITY

February 1991

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.



National Library
of Canada

Bibliothèque nationale
du Canada

Acquisitions and
Bibliographic Services Branch

Direction des acquisitions et
des services bibliographiques

395 Wellington Street
Ottawa, Ontario
K1A 0N4

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file *Votre référence*

Our file *Notre référence*

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-78299-4

Canada

APPROVAL

Name: Qinglin Yu
Degree: Doctor of Philosophy
Title of Thesis: Factors and Factor Extensions

Examining Committee:

Chairman: Dr. Alistair H. Lachlan

Dr. Katherine Heinrich, Professor, Senior Supervisor

Dr. Brian R. Alspach, Professor

Dr. Luis Goddyn, Assistant Professor

Dr. Pavol Hell, Professor

Dr. Peter Horak, Associate Professor,
Slovak Technical University, External Examiner

Date Approve: Feb. 06, 1991


PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis, project or extended essay (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this work for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this work for financial gain shall not be allowed without my written permission.

Title of Thesis/Project/Extended Essay

Factors and Factor Extensions

Author: _____

(signature) 

QINGLIN YU

(name)

March 19, 1991

(date)

ABSTRACT

Given a graph G , a **perfect matching** of G is a set of independent edges which together cover all the vertices of G . We define G to be **n -extendable** if it contains a set of n independent edges and every set of n independent edges can be extended to a perfect matching of G .

In Chapter 1, having surveyed important results in the history of factor theory and presented a brief background to most of the problems I deal with here, I then present some frequently used definitions and notations, and several preliminary results.

In Chapter 2 tree-factor covered graphs are discussed. A **tree-factor** of a graph G is a spanning subgraph of G each component of which is a tree. A necessary and sufficient condition is obtained for a graph to have the property that every subgraph $K_{1,k}$ can be extended to a tree-factor. The main technique used for this problem is the augmenting path method.

In Chapter 3, I study the effect of deleting edges from n -extendable graphs and prove that a conjecture of Saito is true for bipartite graphs. For general graphs, in light of a recent counter-example of Györi, I give what is, in some sense, a best possible result with respect to this conjecture. Further generalizations of n -extendability are introduced and graphs with these properties are characterized.

In Chapter 4, I consider the extendability of products of graphs. These results give an easy way to construct a large family of n -extendable graphs. Two-extendable generalized Petersen graphs and two-extendable Cayley graphs on abelian groups are also classified. The former classification confirms a conjecture of Cammack and Schrag.

In the last chapter, I count the number of star-factors in graphs and also discuss the extendability of powers of graphs. A Nordhaus-Gaddum type of result for matchings is obtained.

To Jenny Qin, my wife,
and also to my mother

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my gratitude to my Senior Supervisor Dr. Katherine Heinrich for her encouragement and help-academic, financial and emotional-during the preparation of this thesis. Her tireless and critical reading for the draft of this thesis is greatly appreciated. Thanks also due to my wife for her support over the years and friends, faculty and staff members of the Department who made my stay here so enjoyable.

Thanks are also offered to the many persons who contributed references, advice, suggestions and discussion for my research. Prominent among these people are B. Alspach, P. Horak, P. Hell, J.P. Liu, M. Plummer, A. Saito and M.L. Yu.

Finally, financial support from Simon Fraser University is much appreciated.

TABLE OF CONTENTS

	Page
APPROVAL	ii
Abstract	iii
Dedication	v
Acknowledgement	vi
Table of Contents	vii
Table of Figures	ix
Chapter 1. Introduction	
§1.1. Background	1
§1.2. Terminology and notations	4
§1.3. Preliminary results	7
Chapter 2. On tree-factor covered graphs	
§2.1. Introduction	9
§2.2. Characterization of tree-factor k -covered graphs	10
Chapter 3. On n -extendable graphs	
§3.1. Introduction	19
§3.2. On Saito's n -extendable conjecture	21
§3.3. Some generalizations related to n -extendability	28
Chapter 4. Classifications of some families of n -extendable graphs	
§4.1. Introduction	37
§4.2. The extendability of product of graphs	39
§4.3. On 2-extendable Abelian Cayley graphs	62
§4.4. Classification of 2-extendable generalized Petersen graphs	78
Chapter 5. Some results about factors	
§5.1. Introduction	90

§5.2. Graphs with a unique $S(k)$ -factor	91
§5.3. The number of $S(k)$ -factors in an r -regular graph	101
§5.4 Miscellaneous results about perfect matchings	103
References	112

TABLE OF FIGURES

	Page
Figure 1.1	6
Figure 2.1	9
Figure 2.2	14
Figure 2.3	15
Figure 2.4	16
Figure 3.1	19
Figure 3.2	23
Figure 3.3	33
Figure 3.4	34
Figure 4.1	38
Figure 4.2	39
Figure 4.3	40
Figure 4.4	41
Figure 4.5	42
Figure 4.6	42
Figure 4.7	43
Figure 4.8	47
Figure 4.9	48
Figure 4.10	48
Figure 4.11	49
Figure 4.12	55
Figure 4.13	57
Figure 4.14	58
Figure 4.15	60

Figure 4.16	60
Figure 4.17	82
Figure 4.18	82
Figure 4.19	83
Figure 4.20	84
Figure 4.21	84
Figure 4.22	85
Figure 4.23	85
Figure 4.24	86
Figure 4.25	86
Figure 4.26	87
Figure 4.27	87
Figure 4.28	88
Figure 4.29	88
Figure 4.30	88
Figure 5.1	94
Figure 5.2	96
Figure 5.3	105
Figure 5.4	106

Chapter 1. Introduction.

§1.1. Background.

Let us look back at the brief history of graph factor theory. The first paper we are aware of was written by Petersen [46] in 1891. He proved that a graph G is 2-factorable if and only if G is an even regular graph and pointed out that the factorization of graphs with odd regular degree is much more complicated than the even case. He also showed that any connected 3-regular graph having no more than two cut-edges can be decomposed into a 1-factor and a 2-factor. Forty years later, König [30] studied 1-factors in bipartite graphs and proved that every k -regular bipartite graph is the union of k edge-disjoint perfect matchings. The importance of König's theorem (the so-called König Edge Colouring Theorem) is that it established the relationship between edge colourings of graphs and factorizations. The König Edge Colouring Theorem says that if G is a bipartite graph, then the chromatic index of G is equal to the maximum degree of the vertices of G . A characterization of bipartite graphs with a perfect matching was obtained by Hall [21] in 1935. However, this result was implied by the earlier work of König [30], and consequently it has come to be known as the König-Hall Theorem (or the Marriage Theorem).

A big step in factor theory was the establishment of criterion, both necessary and sufficient, for a graph to have a 1-factor (perfect matching). This result, which may be considered as one of the most fundamental results in graph theory, was found by Tutte [54] in 1947. Later, in the early 1950's, Tutte [55] proved his so-called "f-factor Theorem" which is a characterization of graphs with f -factors. A short and elegant proof of this theorem (given later by Tutte [56]) involved transforming the f -factor problem into a 1-factor problem. The main technique he used is now referred to as an alternating trails argument, and was also used by Gallai [19] to study the k -

factors in regular graphs. It has become a powerful method in present-day graph theory. A necessary and sufficient conditions for the existence of a (g, f) -factor in a graph was obtained by Lovász [39] in 1970 and since then the theory of graph factors has grown rapidly. Although many types of factors have been considered, research has mainly concentrated on two types; factors with constraints related to degree, so called degree-factors, and the factors which have each of their components isomorphic to one of a given set of subgraphs, referred as component-factor. One of earliest papers, we aware of which , deals with component-factors is that of Kirkpatrick and Hell [29]. The interested reader can also refer to [1], [37] and [44] for more information on graph factors.

In graph theory, much research has been concerned with the problem of extending subgraphs with certain properties in a graph G to spanning subgraphs of G with the same properties. For instance, Hendry [22] studied graphs in which every nonhamiltonian cycle can be extended to a cycle with one more vertex; Liu [34, 35] characterized graphs in which every edge can be extended to an $[a, b]$ -factor or an f -factor; Yu and Chen [59] gave a necessary and sufficient condition for a graph to have the property that every claw subgraph can be extended to a tree-factor; and Kano [27] obtained several sufficient conditions for an r -edge-connected graph to have the property that a given edge-set can be extended to a perfect matching. However, a great deal of such work has focused on the property of n -extendability. (A graph is said to be **n -extendable** if it contains n independent edges and any set of n independent edges can be extended to a perfect matching.) The concept of n -extendability seems to have its early roots in a paper of Heteyi [26] who studied it for bipartite graphs, and papers of Kotzig (see [44]) who used it to develop a decomposition theory for graphs with perfect matchings. In this early paper Heteyi obtained three different characterizations of 1-extendable bipartite graphs. Later Lovász and Plummer [42] gave another characterization which they referred to as an

"Ear Structure Theorem". Necessary and sufficient conditions for a graph to be 1-extendable were given by Little, Grant and Holton [33]. An analogous characterization of n -extendable graphs has recently been obtained by the author and is presented in Chapter 3 of this thesis.

There are two good reasons for studying n -extendable graphs. These are the desire to know more about the structure of graphs with perfect matchings, and the desire to determine good lower bounds on the number of different perfect matchings in a graph. Motivated by these problems, Lovász [40] began to develop a new structure theory for graphs with perfect matchings, and two important new families of graphs - bicritical graphs and elementary graphs - were introduced. Lovász showed that in a certain sense any graph with a perfect matching could be constructed using only elementary bipartite graphs and bicritical graphs as building blocks. This decomposition can be pushed one step further by decomposing bicritical graphs into 3-connected bicritical graphs (also called bricks). The new decomposition is referred to as Brick Decomposition and for a 1-extendable graph it is uniquely determined (up to isomorphism and the multiplicity of edges). (Brick Decomposition has also turned out to be very useful in the study of the matching lattice (Lovász [41]).) Subsequent to this early work, the study of these two classes of graphs was continued by Lovász and Plummer [42, 43]. Today much attention is still focused on understanding the structure of 3-connected bicritical graphs as, unfortunately, their structure is still quite unclear.

In 1980, Plummer [47] studied the properties of n -extendable graphs and showed that every 2-extendable graph is either bipartite or a brick. Motivated by this result he [49, 50] further looked at the relationship between n -extendability and other graphic parameters (e.g., degree, connectivity, genus, toughness). Recently, Schrag and Cammack [52] and Yu [62] classified the 2-extendable generalized Petersen graphs, and Chan, Chen and Yu [11] classified the 2-extendable Cayley graphs on

abelian groups. Further results concerning n -extendability will be discussed in this thesis.

The organization of the thesis is as follows. In Chapter 1, having surveyed important results in the history of factor theory and presented a brief background to most of the problems we deal with here, we then present some frequently used definitions and notations, and several preliminary results.

In Chapter 2 we discuss tree-factor covered graphs. A tree-factor of a graph G is a spanning subgraph of G each component of which is a tree. We obtain a necessary and sufficient condition for a graph to have the property that every subgraph $K_{1,k}$ can be extended to a tree-factor. The main technique used for this problem is the augmenting path method.

In Chapter 3, we study the effect of deleting edges from n -extendable graphs and prove that a conjecture of Saito is true for bipartite graphs. For general graphs, in light of a recent counterexample of Györi, we give what is, in some sense, a best possible result with respect to this conjecture. We introduce further generalizations of n -extendability and characterize graphs with these properties.

In Chapter 4, we consider the extendability of products of graphs. These results give us an easy way to construct a large family of n -extendable graphs. We also classify 2-extendable generalized Petersen graphs and 2-extendable Cayley graphs on abelian groups. The former classification confirms a conjecture of Cammack and Schrag [9].

In the last Chapter, we count the number of star-factors in graphs and also discuss the extendability of powers of graphs.

§1.2. Terminology and notations.

All graphs in this thesis are finite and have no loops or multiple edges.

For a graph G , we denote the vertex-set and the edge-set by $V(G)$ and $E(G)$, respectively. The **order** of a graph G is $|V(G)|$. For any set $S \subseteq V(G)$, we denote by $G-S$ the subgraph of G obtained by deleting the vertices of S together with their incident edges, and by $G[S]$ the subgraph of G induced by S . For $T \subseteq E(G)$ we denote by $G-T$ the graph obtained by deleting the edges of T from G and by $G \cup T$ the graph obtained by adding the edges of T to G . If $T = \{e\}$, we write $G \cup T$ as $G \cup e$. Denote the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. The **neighbourhood-set** of S in G is denoted by $N_G(S)$ and is the set of all vertices in G which have a neighbour in S . We use $o(G)$ to denote the number of odd components in a graph G , and $I(G)$ denotes the set of isolated vertices in G ($i(G) = |I(G)|$). The **complement** of G , denoted by \bar{G} , is that graph having the same vertex-set as G , but in which two vertices are adjacent if and only if they are not adjacent in G . For a graph G , if $S, T \subseteq V(G)$, the set of edges with one end-vertex in S and the other in T is expressed by $E_G(S, T)$, and we let $e_G(S, T) = |E_G(S, T)|$. If x and y are two vertices of a graph G , we denote by $d_G(x, y)$ the distance between x and y in G .

A **perfect matching**, or **1-factor**, of a graph G is a set of independent edges which together cover all the vertices of G . For a positive integer t , a **t-matching** of G is a set of t independent edges of G . We call a graph G **t-matching covered** if every edge of G belongs to a t -matching. A graph G is **n-extendable** if it contains a set of n independent edges and every set of n independent edges can be extended to a perfect matching of G . We call G **0-extendable** if it has a perfect matching. An edge of the graph G is **allowed** if it lies in some perfect matching of G . A graph is **elementary** if its allowed edges form a connected subgraph. A graph G is said to be **bicritical** if for every pair of distinct vertices u and v in $V(G)$ $G - \{u, v\}$ has a perfect matching (so bicritical graphs are 1-extendable). A 3-connected bicritical graph is called a **brick**. A graph G is said to be **factor-critical** if $G - v$ has a perfect matching for every $v \in V(G)$.

The cycle, the path, the complete graph and the independent set with n vertices will be denoted by C_n , P_n , K_n and \bar{K}_n , respectively. If $V(C_n) = V(P_n) = \{v_1, v_2, \dots, v_n\}$ we write $C_n = v_1v_2\dots v_nv_1$, where $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ and $P_n = v_1v_2\dots v_n$, where $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

For disjoint graphs G_1 and G_2 , the **sum** G_1+G_2 is the graph which has vertex-set $V(G_1)\cup V(G_2)$ and edge-set $E(G_1)\cup E(G_2)\cup\{xy \mid x\in V(G_1), y\in V(G_2)\}$.

The **cartesian product** $G_1 \times G_2$ of G_1 and G_2 (also called the cartesian sum) has vertex-set $V(G_1) \times V(G_2)$ and the vertex (u_1, u_2) is adjacent to (v_1, v_2) if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 , or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1 .

The **wreath product** $G_1 \otimes G_2$ of G_1 and G_2 (also called the composition, tensor product, or lexicographic product) is the graph with vertex-set $V(G_1) \times V(G_2)$ and an edge joining (u_1, u_2) to (v_1, v_2) if and only if either u_1 is adjacent to v_1 in G_1 , or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 .

By the definitions of the cartesian and the wreath products, it is easy to check that both products are associative and cartesian product is commutative, but the wreath product is not. These definitions are illustrated in Figure 1.1 when $G_1 = P_2$ and $G_2 = P_3$.

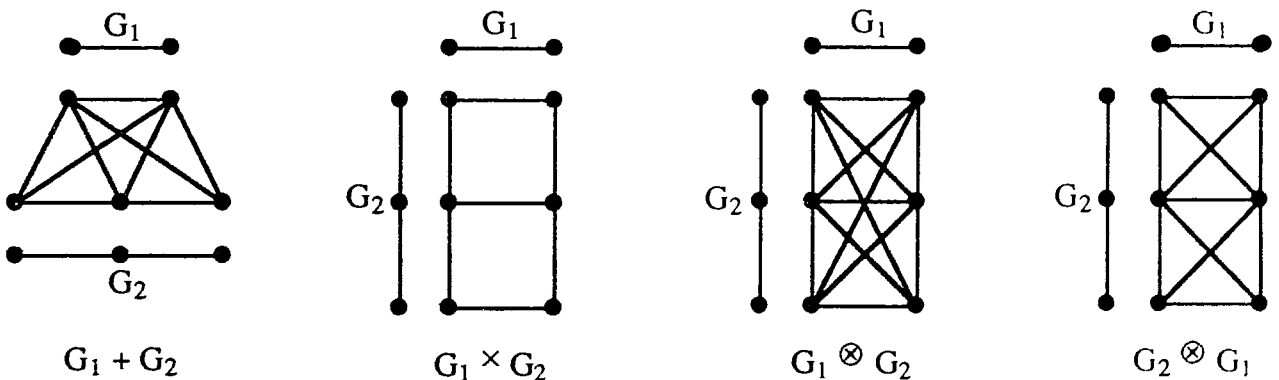


Figure 1.1

We have only listed the definitions and notations most frequently used in the thesis. Some special terminologies will be introduced in the separate sections. All notations used but not defined in this thesis can be found in [6].

§1.3. Preliminary results.

In this section, we list some results which will be used very often in the rest of the thesis.

Theorem 1.3.1 (Tutte's Theorem [54]) A graph G has a perfect matching if and only if $o(G-S) \leq |S|$, for all $S \subseteq V(G)$.

Theorem 1.3.2 (Little, Grant and Holton [33]) Let G be a graph of even order. Then G is 1-extendable if and only if for all $S \subseteq V(G)$,

- (1) $o(G-S) \leq |S|$ and
- (2) $o(G-S) = |S|$ implies that S is an independent set.

Theorem 1.3.3 (See [44]) A graph G is factor-critical if and only if G has an odd number of vertices and $o(G-S) \leq |S|$, for all $\emptyset \neq S \subseteq V(G)$.

Theorem 1.3.4 (Plummer [47]) Let n and p be positive integers with p even and $p \geq 2n+2$. If G is a graph with p vertices, then the following claims hold.

- (1) If G is n -extendable, then G is also $(n-1)$ -extendable.
- (2) If G is a connected n -extendable graph, then G is $(n+1)$ -connected.
- (3) If $p \geq 4$ and $\delta(G) \geq \frac{p}{2} + n$, then G is n -extendable.

Theorem 1.3.5 (Plummer [47]) Let G be a 2-extendable graph with at least six vertices. Then G is either bicritical or elementary and bipartite.

Theorem 1.3.6 (Hall's Theorem, see [6]) Let $B(X, Y)$ be a bipartite graph. Then $B(X, Y)$ has a matching of X into Y if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Chapter 2. On tree-factor covered graphs.

§2.1. Introduction.

For a given set \mathcal{F} of graphs, an \mathcal{F} -**subgraph** of a graph G is a subgraph M of G each component of which is isomorphic to one of the subgraphs in the set \mathcal{F} . Moreover, if M is a spanning \mathcal{F} -subgraph, then M is called an \mathcal{F} -**factor** of G . An \mathcal{F} -subgraph M of G is said to be **maximum**, if G has no \mathcal{F} -subgraph M' with $|V(M)| < |V(M')|$.

In particular, if $\mathcal{F} = S(n) = \{K_{1,k} \mid 1 \leq k \leq n\}$, then an \mathcal{F} -factor of G is also called a **star-factor**, or an $S(n)$ -**factor**. If $\mathcal{F} = T(n)$, the set of all trees with at least one and no more than n edges, then an \mathcal{F} -factor of G is also called a **tree-factor**, or a $T(n)$ -**factor**.

Let a and b be integers such that $0 \leq a \leq b$. We say that H is an $[a, b]$ -**graph** if $a \leq d_G(x) \leq b$, for all $x \in V(H)$. If a spanning subgraph H of a graph G is also an $[a, b]$ -graph, then we call H an $[a, b]$ -**factor**.

A graph G is \mathcal{F} -**factor** k -**covered**, $1 \leq k \leq \Delta(G)$, if for every subgraph $K_{1,k}$ of G there exists an \mathcal{F} -factor of G containing it. An example of a $\{K_{1,1}, K_{1,2}\}$ -factor 2-covered graph is shown in Figure 2.1.

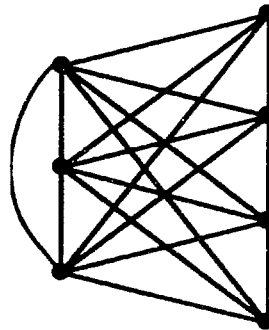


Figure 2.1

In [32] Little introduced the concept of a 1-extendable graph, which in our terminology is a $\{K_2\}$ -factor 1-covered graph and gave a criterion for classifying 1-extendable graphs. (In fact, Little called these graphs factor-covered graphs; the term 1-extendable being introduced by Plummer [47].) Later Little, Grant and Holton [33] generalized Little's result to t -matchings, and showed that a graph G is t -matching 1-covered if and only if it has a t -matching and each subset S of $V(G)$, for which $G-S$ has precisely $|S|+|V(G)|-2t$ odd components, is an independent set. In this section we generalize these earlier ideas and consider what we call tree-factor k -covered graphs. These are graphs with the property that every subgraph $K_{1,k}$ (or k -claw) lies in a tree-factor; that is, if the graph has $n+1$ vertices, then it is $T(n)$ -factor k -covered. We will give a criterion for a graph to be tree-factor k -covered. This is a generalization of the characterization of a graph having tree-factor.

If H is a graph, recall that $I(H)$ denotes the set of isolated vertices of H , and $i(H) = |I(H)|$.

§2.2. Characterization of tree-factor k -covered graphs.

The following theorem is proved by Las Vergnas [31] in 1972.

Theorem 2.2.1 (Las Vergnas [31]) Let G be a graph. Then G has a $[1, n]$ -factor, $n \geq 2$, if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

Las Vergnas [31, Remark 3.5] observed that G has a $[1, n]$ -factor if and only if it has an $S(n)$ -factor. In fact, Las Vergnas claimed that with respect to edges, an $S(n)$ -factor is a minimal $[1, n]$ -factor and there is a polynomial algorithm to produce

an $S(n)$ -factor from a given $[1, n]$ -factor. These results lead to a characterization of graphs with an $S(n)$ -factor.

Theorem 2.2.2 Let G be a graph. Then G has an $S(n)$ -factor, $n \geq 2$, if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

Theorem 2.2.2 was also proven independently by Hell and Kirkpatrick [23], and Amahashi and Kano [2]. Moreover, the above results are also explicitly mentioned in [25].

Since a $T(n)$ -factor is a $[1, n]$ -factor and a union of spanning trees of the components of a $[1, n]$ -factor is a $T(n)$ -factor, we obtain the following result from Theorem 2.2.1.

Theorem 2.2.3 The graph G has a $T(n)$ -factor, $n \geq 2$, if and only if $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$.

Theorem 2.2.3 gives a criterion for the existence of a $T(n)$ -factor in a graph. So, in order to provide a characterization of $T(n)$ -factor k -covered graphs, we need only to add more conditions to this. For this, we require more definitions and lemmas. This work is joint with C. P. Chen.

Let G be a graph and $A \subseteq V(G)$. If there exists a $T(n)$ -subgraph of G which spans A , then A is called **$T(n)$ -saturated**. Let M be a $T(n)$ -subgraph of G and let $x, y \in V(G)$ ($x \neq y$). If x and y belong to the same component of M , then we say that x **matches** y under M .

For a graph G , $\text{defect}(G) = \max_{S \subseteq V(G)} \{i(G-S) - n|S|\}$ is the **defect** of G .

Geometrically, $\text{defect}(G)$ is the number of vertices missing from any maximum $T(n)$ -

subgraph of G . Put $D(G) = \{S \mid S \subseteq V(G) \text{ and } i(G-S) - n|S| = \text{defect}(G)\}$. Clearly, $\text{defect}(G) \geq 0$ for any graph G (put $S = \emptyset$), and by Theorem 2.2.3 G has a $T(n)$ -factor if and only if $\text{defect}(G) = 0$.

We will need the following three lemmas. Lemma 2.2.4 has been proven independently by Las Vergnas [31], Hell and Kirkpatrick [23] and Yu [60].

Lemma 2.2.4 For every maximum $S(n)$ -subgraph M of a graph G

$$|V(M)| = |V(G)| - \text{defect}(G).$$

Lemma 2.2.5 For every maximum $T(n)$ -subgraph M of a graph G ,

$$|V(M)| = |V(G)| - \text{defect}(G).$$

Proof: Given a maximum $T(n)$ -subgraph, we can delete edges to get an $S(n)$ -subgraph on the same number of vertices. The $S(n)$ -subgraph is maximum for if not, since an $S(n)$ -subgraph is a $T(n)$ -subgraph, M would not be maximum. \square

Theorem 2.2.6 Let G be a graph without a $T(n)$ -factor ($n \geq 2$), and let $S_0 \in D(G)$. Then there exists a set V_0 , $V_0 \subseteq I(G-S_0)$ and $|V_0| = \text{defect}(G)$, such that $G-V_0$ has a $T(n)$ -factor.

Proof: As $S_0 \in D(G)$, $i(G-S_0) = n|S_0| + \text{defect}(G)$. Also, in any $T(n)$ -subgraph of G each vertex in S_0 matches with at most n vertices in $I(G-S_0)$. Therefore, any $T(n)$ -subgraph of G leaves at least $\text{defect}(G)$ unsaturated vertices in $I(G-S_0)$.

Let M be a maximum $T(n)$ -subgraph of G , and V_0 be the set of all vertices in $I(G-S_0)$ unsaturated under M . Then $|V_0| \geq \text{defect}(G)$. But, by Lemma 2.2.5, $\text{defect}(G) = |V(G) - V(M)| \geq |V_0| \geq \text{defect}(G)$ and so $|V_0| = \text{defect}(G)$. Therefore M is a $T(n)$ -factor of $G-V_0$. \square

The following theorem is fundamental to the proof of our main theorem.

Theorem 2.2.7 Let G be a graph, and K a $K_{1,k}$ subgraph of G , where $1 \leq k \leq n$. Then G has a $T(n)$ -factor containing K , $n \geq 2$, if and only if

- (1) $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$, and
- (2) $i(G^*-S) \leq n|S|+(n-k)$ for every $S \subseteq V(G^*)$, where $G^* = G-V(K)$.

Proof: We first prove the necessity of the conditions. Let M be a $T(n)$ -factor in G which contains K . Denote by C the component of M which contains K . Let $A = V(C)-V(K)$. Since $C \in T(n)$, we have $|V(C)| \leq n+1$. Moreover, $|V(K)| = k+1$ and thus $|A| \leq n-k$. Set $G^* = G-V(K)$. Because $G-V(C) = G^*-A$ has a $T(n)$ -factor, then by Theorem 2.2.3 we have $i(G^*-A-S^*) \leq n|S^*|$ for every $S^* \subseteq V(G^*)-A$. Let $S \subseteq V(G^*)$. Then $S-A \subseteq V(G^*)-A$ and $i(G^*-A-(S-A)) \leq n|S-A|$. Therefore,

$$i(G^*-S) \leq i(G^*-A-(S-A))+|A| \leq n|S-A|+|A| \leq n|S|+(n-k).$$

Consequently, condition (2) holds. Condition (1) holds by Theorem 2.2.3.

It now remains to prove the sufficiency of the conditions. We will do this by giving an augmenting path procedure to construct a $T(n)$ -factor containing K .

Suppose that conditions (1) and (2) hold. Condition (1) simply tells us that G has a $T(n)$ -factor.

If G^* has a $T(n)$ -factor F , then $F \cup K$ is a $T(n)$ -factor of G containing K . Suppose then that G^* has no $T(n)$ -factor. From this and condition (2) we have $n-k \geq \text{defect}(G^*) > 0$. Let $S_0 \in D(G^*)$. Then $i(G^*-S_0)-n|S_0| = \text{defect}(G^*)$. It follows from Lemma 2.2.6 that there exists a set V_0 , $V_0 \subseteq I(G^*-S_0)$, so that $|V_0| = \text{defect}(G^*)$ and G^*-V_0 has a $T(n)$ -factor M_0 ; that is, M_0 is a $T(n)$ -factor in $G-V(K)-V_0$. Notice that M_0 consists of n -stars and each n -star has its centre in S_0 and n leaves in $I(G^*-S_0)-V_0$.

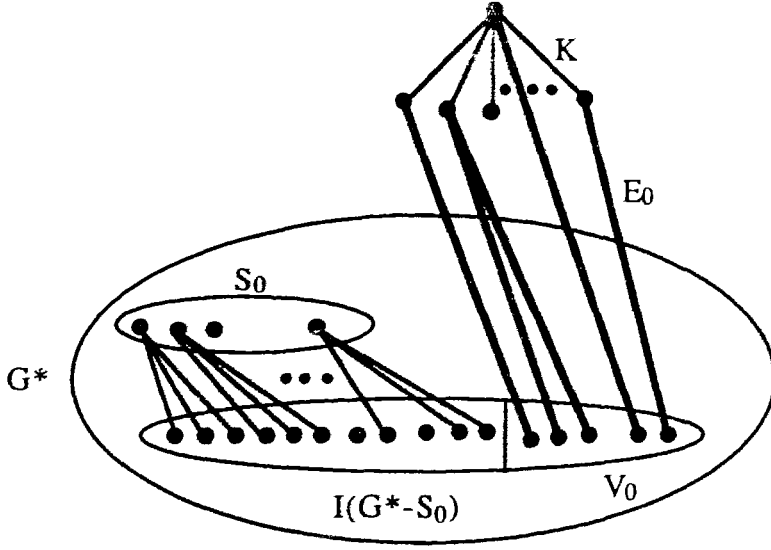


Figure 2.2

Suppose that for every $v \in V_0$, $N_G(\{v\}) \cap V(K) \neq \emptyset$. For each $v \in V_0$ choose a vertex $v' \in N_G(\{v\}) \cap V(K)$, and let E_0 be the set of edges vv' . So $|E_0| = \text{defect}(G^*)$ and $T_0 = G[E_0 \cup E(K)]$ is a spanning tree containing K in $G[V_0 \cup V(K)]$ (see Figure 2.2). Since

$$|E(T_0)| = \text{defect}(G^*) + |E(K)| \leq (n-k) + k = n,$$

T_0 is a $T(n)$ -factor containing K in $G[V(K) \cup V_0]$, and $M_0 \cup T_0$ is a $T(n)$ -factor of G containing K .

Otherwise, there exists a vertex v , $v \in V_0$, such that $N_G(v) \cap V(K) = \emptyset$. For every $A \subseteq S_0$, recall that $N_I(A)$ is the neighbourhood of A in $I(G^* - S_0)$. Denote by $\Gamma_{M_0}(A)$ the set of vertices which are matched under M_0 with vertices in A .

Set

$$S_1 = \{x \mid x \in S_0 \text{ and } xv \in E(G)\}$$

$$S_2 = \{x \mid x \in S_0 - S_1 \text{ and } N_I(\{x\}) \cap \Gamma_{M_0}(S_1) \neq \emptyset\}$$

$$S_3 = \{x \mid x \in S_0 - (S_1 \cup S_2) \text{ and } N_I(\{x\}) \cap \Gamma_{M_0}(S_2) \neq \emptyset\}$$

...

$$S_m = \{x \mid x \in S_0 - (\bigcup_{j=1}^{m-1} S_j) \text{ and } N_1(\{x\}) \cap \Gamma_{M_0}(S_{m-1}) \neq \emptyset\},$$

where m is the maximum integer satisfying the above properties.

Since G has a $T(n)$ -factor, G has no isolated vertices. This implies that $S_1 \neq \emptyset$. Since $S_0 \in D(G^*)$, the subset $\bigcup_{j=1}^m S_j$ of S_0 satisfies

$$|\Gamma_{M_0}(\bigcup_{j=1}^m S_j)| = n \mid \bigcup_{j=1}^m S_j \mid.$$

We claim that $\Gamma_{M_0}(\bigcup_{j=1}^m S_j) \cap N_1(V(K)) \neq \emptyset$. Since S_m is the last set defined, there is

no edge from $S_0 - \bigcup_{j=1}^m S_j$ to $\Gamma_{M_0}(\bigcup_{j=1}^m S_j)$. If $\Gamma_{M_0}(\bigcup_{j=1}^m S_j) \cap N_1(V(K)) = \emptyset$, then $I(G -$

$\bigcup_{j=1}^m S_j) \supseteq \Gamma_{M_0}(\bigcup_{j=1}^m S_j)$ (see Figure 2.3, where dotted lines indicate the edges of M_0).

Since $N_G(v) \subseteq S_1$, then $v \in I(G - \bigcup_{j=1}^m S_j)$ and hence

$$i(G - \bigcup_{j=1}^m S_j) \geq n \mid \bigcup_{j=1}^m S_j \mid + 1,$$

which is contrary to (1).

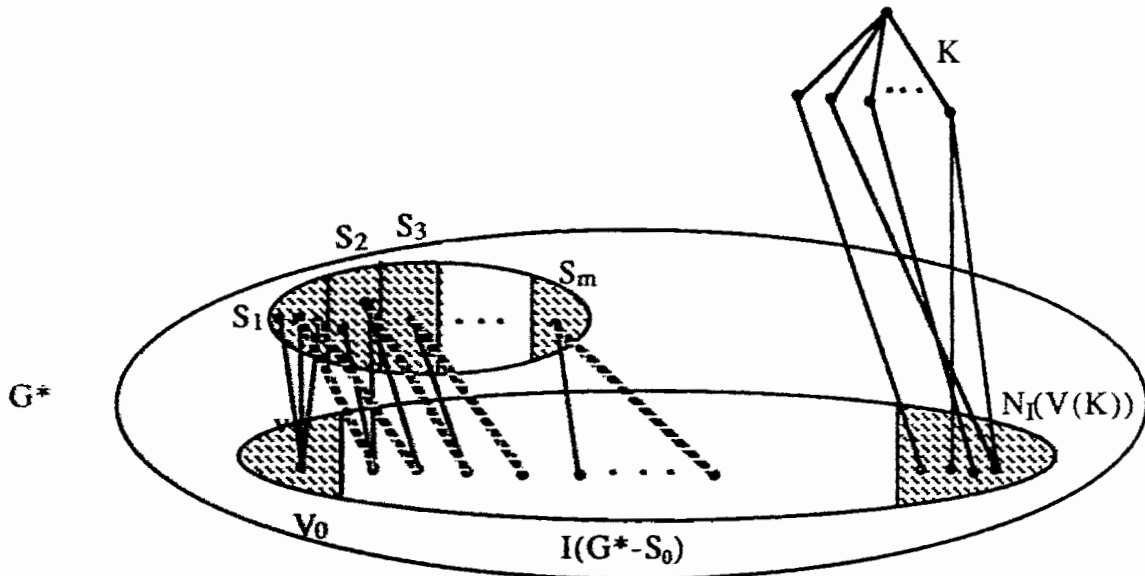


Figure 2.3

Let y_r be a vertex of $(\Gamma_{M_0}(S_r) - S_r) \cap N_I(V(K))$ and x_r be the vertex of S_r matching y_r under M_0 . For $2 \leq i \leq r$, let y_{i-1} be the vertex which is adjacent to x_i in $\Gamma_{M_0}(S_{i-1})$ and x_{i-1} the vertex of S_{i-1} matching y_{i-1} under M_0 . Let x_1 be a vertex matching y_1 in S_1 .

Set

$$M_1 = M_0 - \{x_1 y_1, \dots, x_r y_r\} \cup \{x_1 v, x_2 y_1, \dots, x_r y_{r-1}\},$$

and $V_1 = (V_0 - \{v\}) \cup \{y_r\}$. Then, from the construction of S_j , M_1 is a $T(n)$ -factor in $G^* - V_1$ (see Figure 2.4).

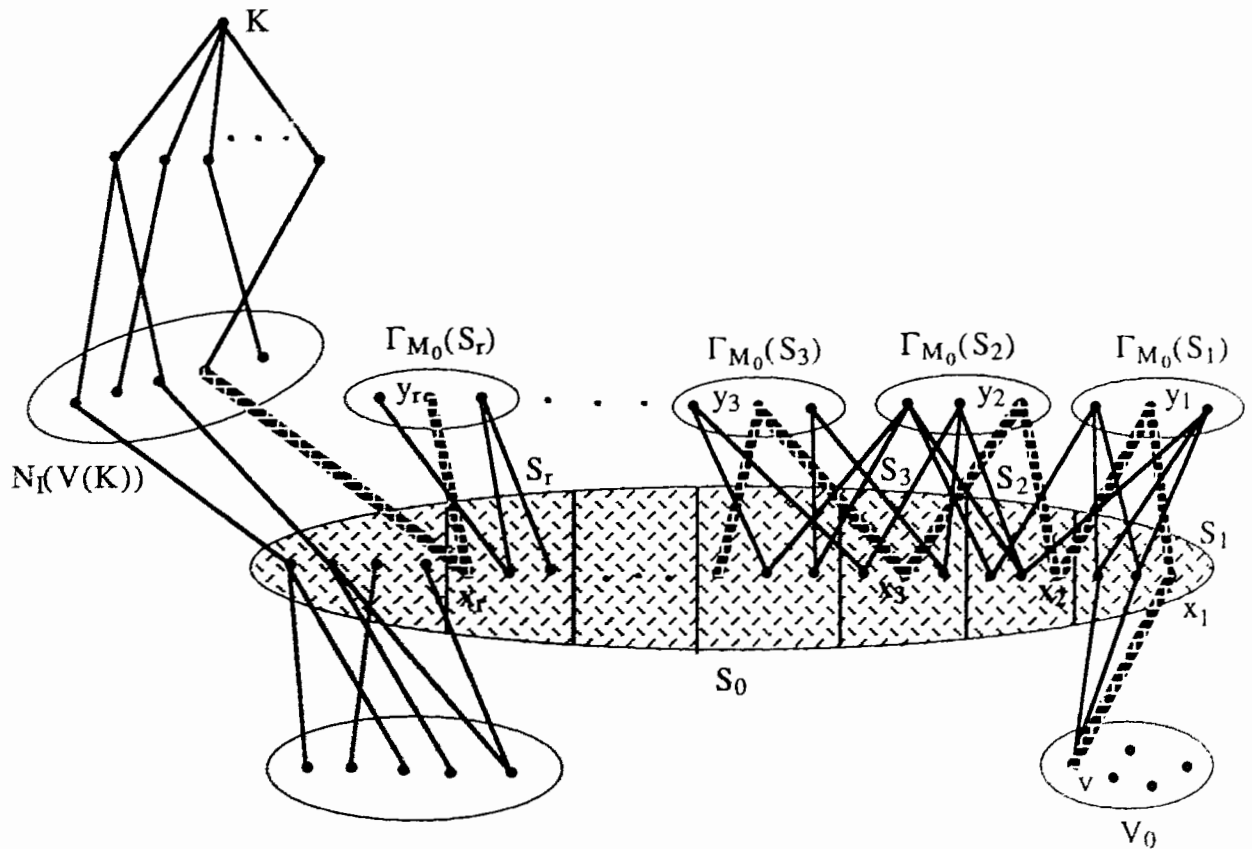


Figure 2.4

If, for every $u \in V_1$, $N_G(\{u\}) \cap V(K) \neq \emptyset$, then the proof is finished by the same argument as applied earlier to V_0 . Otherwise, there is a vertex $u' \in V_1$ with no neighbours in K and we repeat the argument to obtain the independent set V_2 . Continuing in this way we eventually reach a vertex-set V_p ($p \leq |V_0|$) and a $T(n)$ -factor M_p in $G^* - V_p$, so that every vertex in V_p is adjacent to $V(K)$ (that is, $N_G(\{v\}) \cap V(K) \neq \emptyset$ for all $v \in V_p$). An application of the first argument now completes the proof. \square

Corollary 2.2.8 Let G be a graph and $1 \leq k \leq n$. If $i(G-S) \leq n|S| - (n+1)k$ for every $S \subseteq V(G)$, then G is $T(n)$ -factor k -covered ($n \geq 2$).

Proof: Clearly condition (1) of Theorem 2.2.7 is satisfied.

For any given $K_{1,k}$ subgraph K of G , set $G^* = G - V(K)$. For every $S \subseteq V(G^*)$, we have

$$\begin{aligned} i(G^* - S) &= i(G - V(K) - S) \leq n|V(K) \cup S| - (n+1)k \\ &= n|S| + n(k+1) - (n+1)k = n|S| + (n-k). \end{aligned}$$

Thus, condition (2) of Theorem 2.2.7 is also satisfied. So G has a $T(n)$ -factor containing K and G is therefore $T(n)$ -factor k -covered. \square

For a fixed k -claw K of G , from Theorem 2.2.7 we obtain necessary and sufficient conditions for the existence of a $T(n)$ -factor containing K . Our next step is to obtain a characterization of $T(n)$ -factor k -covered graphs. Such a characterization will necessarily be independent of the choice of k -claw.

Theorem 2.2.9 Let G be a graph, and $1 \leq k \leq n$. Then G is $T(n)$ -factor k -covered, $n \geq 2$, if and only if

- (1) $i(G-S) \leq n|S|$ for every $S \subseteq V(G)$ and
- (2) $i(G-S) > n|S| - (n+1)k$ implies that $\Delta(G[S]) < k$.

Proof: Suppose that G is $T(n)$ -factor k -covered and so condition (1) holds. Suppose that there exists a subset of vertices S_0 , $S_0 \subseteq V(G)$, such that $n|S_0| \geq i(G-S_0) > n|S_0|-(n+1)k$ and $\Delta(G[S_0]) \geq k$. Since $\Delta(G[S_0]) \geq k$, then $G[S_0]$ contains a $K_{1,k}$ subgraph K . Set $G^* = G-V(K)$ and $S = S_0-V(K)$. Then

$$\begin{aligned} i(G^*-S) &= i(G-V(K)-(S_0-V(K))) = i(G-S_0) \geq n|S_0|-(n+1)k+1 \\ &= n|S|+(n-k)+1 \end{aligned}$$

and so by Theorem 2.2.7 G has no $T(n)$ -factor containing K , a contradiction.

We next prove the sufficiency of the theorem. Suppose that there were a $K_{1,k}$ subgraph K of G such that G had no $T(n)$ -factor containing K . Set $G^* = G-V(K)$. By Theorem 2.2.7 there exists a set S , $S \subseteq V(G^*)$, such that $i(G^*-S) > n|S|+(n-k)$. Set $S_0 = S \cup V(K)$. Then

$$i(G-S_0) = i(G^*-S) > n|S|+n-k = n|S_0|-k(n+1).$$

But $\Delta(G[S_0]) \geq \Delta(K) = k$, and we have found a set S which does not satisfy (2). \square

Note that from Theorem 2.2.1 and Theorem 2.2.3 it follows that the existence of a star-factor or of a tree-factor in a graph is the same. But, tree-factor covered graphs do not necessarily satisfy the conditions for star-factor covered graphs. For example, the path of length 3, P_4 , is $T(3)$ -factor 1-covered but not $S(3)$ -factor 1-covered. In this sense, the conditions for the existence of $S(n)$ -factor 1-covered graphs are stronger than those required for $T(n)$ -factor 1-covered graphs.

To conclude this chapter, we present the following open problem:

Problem: Characterize star-factor covered graphs.

Chapter 3. On n -extendable graphs.

§3.1. Introduction.

Recall that a graph G is n -**extendable** if it contains a set of n independent edges and every set of n independent edges can be extended to a perfect matching of G . The family of n -extendable graphs is quite large. For example, the cube (see Figure 3.1), the tetrahedron, the dodecahedron and the complete bipartite graph $K_{r,r}$ are 2-extendable. In fact, if the minimum degree $\delta(G)$ is larger than $n+|V(G)|/2$ and $|V(G)| \geq 4$, then G is n -extendable (see Theorem 1.3.4 (3)).

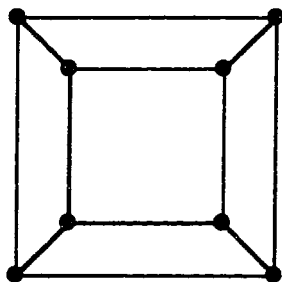


Figure 3.1 The cube is 2-extendable

Plummer [47], [50] studied properties of n -extendable graphs, and the relationship between n -extendability and connectivity. In particular, he investigated the effect on extendability when an edge is deleted from an n -extendable graph, and showed that for any edge $e = xy$ of an n -extendable graph G ($n \geq 1$), $G - e$ is $(n-1)$ -extendable. In the case when an edge is added rather than deleted, Saito [51] made the following conjecture:

Saito's conjecture: If a connected graph G is n -extendable and $G \not\cong K_{r,r}$ or K_{2r} for some r , then there exists an edge $e \in E(\bar{G})$ such that $G \cup e$ is n -extendable.

For $n=1$, Saito noted that the conjecture can be easily proven and such a proof will be presented in the next section. Very recently, Györi (personal communication) proved that $K_r \times K_m$ is a counterexample to Saito's conjecture. In particular, he showed that if both r and m are even then $K_r \times K_m$ is $(\frac{r}{2} + \frac{m}{2} - 1)$ -extendable, but for any edge $e \in E(\overline{K_r \times K_m})$ the graph $(K_r \times K_m) \cup e$ is not $(\frac{r}{2} + \frac{m}{2} - 1)$ -extendable. In section 3.2, we shall show that the conjecture is true if G is bipartite, and that if G is not bipartite and $e \in E(\overline{G})$, then $G \cup e$ is $(n-1)$ -extendable. In light of Györi's counterexample, this result is, in same sense, best possible. However, it would be interesting to determine all n -extendable graphs G in which there exists an edge $e \in E(\overline{G})$ such that $G \cup e$ is n -extendable. In other words, are $K_r \times K_m$ the only counterexamples to Saito's conjecture?

Little, Grant and Holton [33] gave good characterizations of 1-extendable graphs and 1-extendable bipartite graphs. In 1971, Brualdi and Perfect [7] gave a characterization of n -extendable bipartite graphs, but their result is described in terms of matrices and system of distinct representatives. The more results on n -extendable bipartite graphs were obtained by Plummer [48]. In particular, Brualdi and Csima [8] proved that a k -regular bipartite graph of order $2m$ is n -extendable if and only if $k = 1$ or $n \leq 2k - m$. In section 3.3, we shall give criteria for a graph to be n -extendable ($n \geq 1$). Since n -extendable graphs must have a 1-factor, we deal only with graphs of even order. For graphs of odd order, we generalize the idea of n -extendability and introduce $n\frac{1}{2}$ -extendability. A graph G is $n\frac{1}{2}$ -**extendable** if (1) for any vertex v of $V(G)$ there exists a set of n independent edges in G which miss v and (2) for every vertex v and every set of n independent edges $e_1 = x_1y_1, e_2 = x_2y_2, \dots, e_n = x_ny_n$ missing v , there exists a near perfect matching of G which contains e_1, e_2, \dots, e_n and misses v . Analogous to n -extendability, we study the properties of $n\frac{1}{2}$ -extendable

graph and give a characterization of these graphs. The generalizations of factor-critical and bicritical to $n\frac{1}{2}$ -extendability are also discussed.

Several results in this chapter will be based on the following observation.

Observation 3.1.1 A graph G , $|V(G)| \geq 2n+2$, is n -extendable if and only if for any matching M of size i ($1 \leq i \leq n$) the graph $G-V(M)$ is $(n-i)$ -extendable.

Proof: Suppose that G is n -extendable. For any matching M of size i ($1 \leq i \leq n$), let $H = G-V(M)$. Observe that by Theorem 1.3.4 (1) H has a perfect matching. Let M' be a matching of H with $n-i$ edges. Then $M \cup M'$ is an n -matching of G and thus there exists a perfect matching P of G containing $M \cup M'$. Clearly, $P-M$ is a perfect matching of H which contains M' and so H is $(n-i)$ -extendable.

Conversely, for any matching Q of size n in G , let M be a subset of Q with i edges. By assumption $G-V(M)$ is $(n-i)$ -extendable. Thus there exists a perfect matching P of $G-V(M)$ containing $Q-M$ and therefore $P \cup M$ is a perfect matching of G containing Q . \square

§3.2. On Saito's conjecture.

We start this section by stating the characterization of 1-extendable bipartite graphs obtained by Little, Grant and Holton [33].

Theorem 3.2.1 (Little, Grant and Holton [33]) Let $B(X, Y)$ be a bipartite graph with $|X| = |Y|$. Then $B(X, Y)$ is 1-extendable if and only if for every non-empty proper subset S of X we have $|N_Y(S)| > |S|$, where $N_Y(S)$ is the neighbourhood of S in Y .

In order to prove Saito's conjecture for $n = 1$, we introduce the **closure** of a 1-extendable graph G . This is the graph obtained from G by recursively joining pairs of nonadjacent vertices x and y such that $G \cup \{xy\}$ is 1-extendable until no such pair remains. We denote the closure of G by $c(G)$. Notice that $c(G)$ is 1-extendable.

Lemma 3.2.2 $c(G)$ is well defined.

Proof: Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices x and y such that $G \cup \{xy\}$ is 1-extendable until no such pair remains. Denote by e_1, e_2, \dots, e_m and f_1, f_2, \dots, f_n the sequences of edges added to G in obtaining G_1 and G_2 , respectively. We shall show that each e_i is an edge of G_2 and each f_j is an edge of G_1 .

Let $e_{k+1} = uv$ be the first edge in the sequence e_1, e_2, \dots, e_m that is not an edge of G_2 . Set $H = G \cup \{e_1, e_2, \dots, e_k\}$. It follows from the definition of G_1 that $H \cup \{e_{k+1}\}$ is 1-extendable. By the choice of e_{k+1} , H is a spanning subgraph of G_2 . Since $e_{k+1} \in E(G_1)$, there exists a perfect matching F of $H \cup \{e_{k+1}\}$ containing e_{k+1} and F is a perfect matching of $G_2 \cup \{e_{k+1}\}$. That is, $e_{k+1} \in E(G_2)$. This is a contradiction, since u and v are not adjacent in G_2 . Therefore each e_i is an edge of G_2 and, similarly f_j is an edge of G_1 . Hence $G_1 = G_2$ and $c(G)$ is well defined. \square

The next result is the case $n = 1$ of Saito's conjecture. We state it in the form of closure. The proof was first sketched by Saito (private communication).

Theorem 3.2.3 If a connected graph G is 1-extendable, then $c(G)$ is K_{2r} or $K_{r,r}$, where $|V(G)| = 2r$.

Proof: In a connected, 1-extendable graph G , let $F = \{x_1y_1, x_2y_2, \dots, x_ry_r\}$ be a perfect matching of G . Since G is connected, there is an edge adjacent to two edges of F . Suppose this edge is x_1y_2 . Then $x_2y_1 \in c(G)$ as $F - \{x_1y_1, x_2y_2\} \cup \{x_1y_2, x_2y_1\}$ is a

perfect matching containing x_2y_1 . Therefore $\{x_1, y_1, x_2, y_2\}$ induces a complete bipartite graph $K_{2,2}$ in $c(G)$. Suppose we have the subgraph $K_{t,t}$ with vertices $\{x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t\}$ and $t < r$. Since G is connected, there exists an edge of G joining a vertex of $\{x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t\}$ to one of $\{x_{t+1}, x_{t+2}, \dots, x_r, y_{t+1}, y_{t+2}, \dots, y_r\}$, say $x_iy_{t+1} \in E(G)$. Then $x_{t+1}y_i \in c(G)$. Hence $x_{t+1}y_j$ ($j \neq i$) is an edge of $c(G)$ as $F - \{x_iy_i, x_jy_j, x_{t+1}y_{t+1}\} \cup \{x_{t+1}y_j, x_jy_i, x_iy_{t+1}\}$ and, similarly, x_jy_{t+1} ($j \neq i$) lie in $c(G)$. So we have a subgraph $K_{t+1,t+1}$ in $c(G)$. Continuing this argument, we conclude that $c(G)$ contains a subgraph $K_{r,r}$.

If both $\{x_1, x_2, \dots, x_r\}$ and $\{y_1, y_2, \dots, y_r\}$ are independent sets in G , then $c(G) \cong K_{r,r}$ and we are done. Otherwise, without loss of generality, assume that $x_1x_2 \in E(G)$. Then clearly $y_1y_2 \in E(c(G))$. Since $x_1x_2, y_1y_2 \in E(c(G))$, then $x_1x_i \in E(c(G))$ ($3 \leq i \leq r$) as $F - \{x_1y_1, x_2y_2, x_iy_i\} \cup \{x_1x_i, x_2y_i, y_1y_2\}$ (see Figure 3.2) is a perfect matching containing x_1x_i . A similar argument yields $y_1y_i \in E(c(G))$ for $3 \leq i \leq r$. From this we can deduce that for all i and j both x_ix_j and y_iy_j are contained in perfect matchings. Therefore $c(G) \cong K_{2r}$.

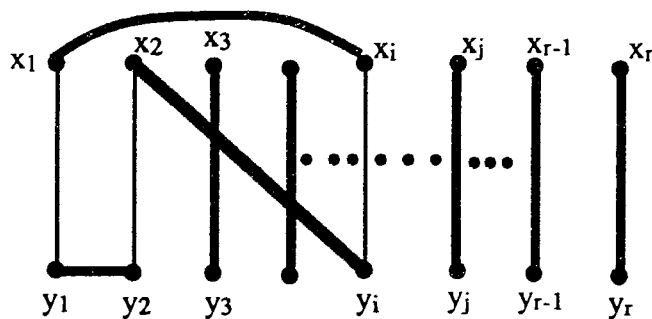


Figure 3.2

□

We now prove that Saito's conjecture holds if G is a bipartite graph.

Theorem 3.2.4 If the connected bipartite graph $G = B(X, Y)$ is n -extendable, then for any edge $e = xy$ of $E(\bar{G})$, with $x \in X$ and $y \in Y$, $G \cup e$ is n -extendable.

Proof: Suppose that the conclusion of the theorem is false. Then there exists an n -extendable bipartite graph $G = B(X, Y)$ so that for some edge $e = xy$, where $x \in X$, $y \in Y$ and $xy \in E(\bar{G})$, $G \cup e$ is not n -extendable. Thus there exist n independent edges of $G \cup e$ which cannot be extended to a perfect matching. Since G is n -extendable, one of these edges is e . Let the others be e_1, e_2, \dots, e_{n-1} , where $e_i = x_i y_i$ ($1 \leq i \leq n-1$), $x_i \in X$ and $y_i \in Y$. Let $X' = X - \{x, x_1, \dots, x_{n-1}\}$ and $Y' = Y - \{y, y_1, \dots, y_{n-1}\}$. Since $B(X', Y')$ has no 1-factor, by Theorem 1.3.6 (Hall's Theorem) there exists a set $S \subseteq X'$ such that $|N_{Y'}(S)| < |S|$. Now $|N_{Y' \cup \{y\}}(S)| \leq |N_{Y'}(S)| + 1$ or $|N_{Y' \cup \{y\}}(S)| \leq |S|$, and thus by Theorem 3.2.1, the bipartite graph $B(X' \cup \{x\}, Y' \cup \{y\}) = G - \{x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}\}$ is not 1-extendable. But this contradicts Observation 3.1.1. \square

Corollary 3.2.5 Saito's conjecture is true for the case of bipartite graphs.

Note that if G is as described in Theorem 3.2.4, then for any edge e with $V(e) \subseteq X$ or Y , $G - V(e)$ is a bipartite graph with a different numbers of vertices in each bipartition. So $G - V(e)$ has no perfect matching. Thus $G \cup e$ is not even 1-extendable. So it was important to choose the edges as described in Theorem 3.2.4. In the case of non-bipartite graphs Györi has provided examples of graphs which are counterexamples to Saito's conjecture. We will show that the graph $K_{2r} \times K_2$ is such a counterexample.

Theorem 3.2.6 (Györi) For any integer $r \geq 1$, $K_{2r} \times K_2$ is r -extendable. But for any edge $e, e \in E(\overline{K_{2r} \times K_2})$, the graph $(K_{2r} \times K_2) \cup e$ is not r -extendable.

Proof: In Theorem 4.2.9 we shall prove that if H is k -extendable, then $H \times K_2$ is $(k+1)$ -extendable. Using this and the fact that K_{2r} is $(r-1)$ -extendable, then $K_{2r} \times K_2$ is r -extendable.

Let $V(K_{2r}) = \{x_1, x_2, \dots, x_{2r}\}$ and $V(K_2) = \{1, 2\}$. Then $E(\overline{K_{2r} \times K_2}) = \{(x_i, 1)(x_j, 2) \mid x_i \neq x_j\}$. To show that $(K_{2r} \times K_2) \cup e$ is not r -extendable for each $e \in E(\overline{K_{2r} \times K_2})$ we need only to consider the edge $e = (x_1, 1)(x_2, 2)$. Let $e_1 = (x_3, 1)(x_4, 1)$, $e_2 = (x_5, 1)(x_6, 1)$, ..., $e_{r-1} = (x_{2r-1}, 1)(x_{2r}, 1)$. Then $e, e_1, e_2, \dots, e_{r-1}$ are r independent edges of $K_{2r} \times K_2$ and $(x_2, 1)$ is an isolated vertex of $(K_{2r} \times K_2) - V(\{e, e_1, e_2, \dots, e_{r-1}\})$. Hence $(K_{2r} \times K_2) \cup e$ is not r -extendable. \square

Actually, Györi proved that if both r and m are even then $K_r \times K_m$ is a counterexample to Saito's conjecture. Although Saito's conjecture is not true in general, we can prove that if G is not bipartite, then for any edge $e, e \in E(\bar{G})$, $G \cup e$ is $(n-1)$ -extendable. This result is rather strong in the sense that it holds for all edges and in view of the falsity of the conjecture this is the best one can expect.

To reach this main result we need the following lemma.

Lemma 3.2.7 If G is a connected n -extendable graph ($n \geq 2$) and M is a matching of size i ($1 \leq i \leq n-1$) in G , then $G - V(M)$ is connected.

Proof: Suppose that $G - V(M)$ is disconnected, where M is a matching of size i ($1 \leq i \leq n-1$). Let e_1, e_2, \dots, e_i be the edges of M and $e_j = x_j y_j$ ($1 \leq j \leq i$). Consider the graphs $G_0 = G$, $G_1 = G_0 - \{x_1, y_1\}$, $G_2 = G_1 - \{x_2, y_2\}$, ..., $G_i = G_{i-1} - \{x_i, y_i\} = G - V(M)$. Since G_0 is connected and n -extendable ($n \geq 2$), by Theorem 1.3.4 (2) it is at least 3-connected. Thus G_1 is connected. Since G_i is disconnected, there exists an h , where $1 \leq h \leq i-1$, such that G_h is connected but G_{h+1} is disconnected. Also, by Observation 3.1.1, G_h is $(n-h)$ -extendable, and so, as $n \geq i+1$ and $h \leq i-1$, G_h is at least 2-

extendable. From Theorem 1.3.4 (2), G_h is 3-connected, and this implies that $G_{h+1} = G_h - \{x_{h+1}, y_{h+1}\}$ is connected, which is a contradiction. \square

We will now prove the main result in this section.

Theorem 3.2.8 Suppose that G is a connected, n -extendable ($n \geq 1$), non-bipartite graph of order at least $2n+2$. For any edge $e = xy \in E(\bar{G})$, $G \cup e$ is $(n-1)$ -extendable.

Proof: Suppose that the connected, non-bipartite graph G is n -extendable. If for any edge $e = xy \in E(\bar{G})$, $G \cup e$ is n -extendable, then by Theorem 1.3.4 (1), we are done. So we assume that for some e , $e = xy \in E(\bar{G})$, $G \cup e$ is not n -extendable.

Suppose also that $G \cup e$ is not $(n-1)$ -extendable, although by Theorem 1.3.4(1) G is. Then there exist $n-1$ independent edges (including e) which cannot be extended to a perfect matching of $G \cup e$. Let these edges be $e = xy$, $e_1 = x_1y_1$, ..., $e_{n-2} = x_{n-2}y_{n-2}$. Thus there is no perfect matching in the subgraph $G' = G - \{x, y, x_1, y_1, \dots, x_{n-2}, y_{n-2}\}$. By Theorem 1.3.1 (Tutte's theorem), there exists a subset S of $V(G')$ such that $o(G'-S) \geq |S|+1$ and a simple parity argument then yields $o(G'-S) \geq |S|+2$. Let $S' = S \cup \{x, y\}$. We will show that S' is an independent set in $G'' = G - \{x_1, y_1, \dots, x_{n-2}, y_{n-2}\}$. Clearly,

$$o(G''-S') = o(G'-S) \geq |S|+2 = |S'|.$$

But since G'' is 2-extendable (by Observation 3.1.1), it follows from Theorem 1.3.4(1) that G'' is 1-extendable. Theorem 1.3.2 then yields, $o(G''-S') \leq |S'|$ and hence $o(G''-S') = |S'|$. Theorem 1.3.2 now implies that S' is independent in G'' .

Next, we shall prove that each odd component of $G''-S'$ is a singleton. If not, let O_1 be an odd component of $G''-S'$ with $|O_1| \geq 3$. By Lemma 3.2.7, G'' is connected. Moreover, from Theorem 1.3.4 (2), G'' is 3-connected (and $|S'| \geq 3$). Consider the following version of Menger's Theorem ([12, p163]):

"A graph G of order $p \geq 2n$ is n -connected if and only if for every two disjoint sets V_1 and V_2 of n vertices each, there exist n vertex-disjoint paths connecting V_1 and V_2 ."

Using this it follows that there exist two independent edges f and g from O_1 to two vertices of S' , say u_1, u_2 . Let z_1 and z_2 be the end-vertices of f and g in O_1 . Letting $S'' = S' - \{u_1, u_2\}$,

$$o((G'' - \{u_1, u_2, z_1, z_2\}) - S'') \geq o(G'' - S') = |S'| > |S''|.$$

That is, $G'' - \{u_1, u_2, z_1, z_2\}$ has no perfect matching, which contradicts the fact that G'' is 2-extendable. So $|O_1| = 1$.

Since G'' is 2-extendable and connected it follows that, as $o(G'' - S') = |S'|$, $G'' - S'$ has no even components. Thus all components of $G'' - S'$ are singletons, and as S' is independent, G'' is a bipartite graph.

Finally, we show that $G[V(G'') \cup \{x_1, y_1\}]$ is bipartite.

From Lemma 3.2.7, we know that $G^* = G[V(G'') \cup \{x_1, y_1\}]$ is connected. By Observation 3.1.1, $G^* = G[V(G'') \cup \{x_1, y_1\}] = G - \{x_2, y_2, \dots, x_{n-2}, y_{n-2}\}$ is 3-extendable, and so is 4-connected. Let the bipartition of G'' be $X'' \cup Y''$ where $|X''| = |Y''|$. Since G^* is 4-connected, each of x_1 and y_1 has degree at least 4. If G^* is not bipartite either $N(\{x_1, y_1\}) \subseteq X''$ (or Y'') or at least one of x_1 and y_1 has a neighbour in each of X'' and Y'' . The first case is eliminated as it implies G^* is not 2-extendable. In the second case, suppose that x_1 has a neighbour in each of X'' and Y'' . But then either $|N(\{x_1, y_1\}) \cap X''| \geq 2$ or $|N(\{x_1, y_1\}) \cap Y''| \geq 2$ and again G^* is not 2-extendable. So G^* is bipartite. Therefore, we can add each of $\{e_1, e_2, \dots, e_{n-2}\}$ to G'' one by one and conclude that $G[V(G'') \cup \{x_1, y_1, \dots, x_{n-2}, y_{n-2}\}] = G$ is bipartite. But this contradicts the assumption. \square

Even though Saito's conjecture is not true for non-bipartite graphs, it does hold for all such graphs with large enough minimum degree. We conclude this section with a statement of that result.

Theorem 3.2.9 Saito's conjecture holds for any n -extendable graph of order p , $p \geq 4$ and minimum degree $\delta(G) \geq \frac{p}{2} + n$.

Proof: Since the minimum degree $\delta(G) \geq \frac{p}{2} + n$, G is a non-bipartite graph. The result follows from Theorem 1.3.4 (3) immediately. \square

§3.3 Some generalizations related to n -extendability.

We begin by giving a characterization of n -extendable graphs which is a generalization of Theorem 1.3.2.

Theorem 3.3.1 A graph G is n -extendable ($n \geq 1$) if and only if for any $S \subseteq V(G)$

(1) $o(G-S) \leq |S|$ and

(2) $o(G-S) = |S| - 2k$ ($0 \leq k \leq n-1$) implies that $F(S) \leq k$, where $F(S)$ is the size of maximum matching in $G[S]$.

Proof: " \Rightarrow " As G has a perfect matching (1) follows from Theorem 1.3.1. Suppose $o(G-S) = |S| - 2k$ ($0 \leq k \leq n-1$) for some vertex-set $S \subseteq V(G)$. We consider first the case that $k = n-1$. In this case, assume $F(S) > n-1$. Let $e_i = x_i y_i$ ($1 \leq i \leq n-1$) be $n-1$ independent edges in $G[S]$. By Observation 3.1.1, $G - \{x_1, y_1, \dots, x_{n-1}, y_{n-1}\}$ is 1-extendable. Let $G' = G - \{x_1, y_1, \dots, x_{n-1}, y_{n-1}\}$ and $S' = S - \{x_1, y_1, \dots, x_{n-1}, y_{n-1}\}$. Then $o(G'-S') = o(G-S) = |S| - 2(n-1) = |S'|$. By Theorem 1.3.2, S' is an independent set. Thus $F(S) \leq F(S') + (n-1) = n-1 = k$, a contradiction. Since k -extendability implies $(k-1)$ -extendability, (2) holds for $0 \leq k \leq n-2$.

" \Leftarrow " The proof will use induction on n .

If $n = 1$, the claim holds from Theorem 1.3.2 as $F(S) = 0$ means that S is independent.

Suppose that the claim holds for $n < r$. Consider $n = r$. By the induction hypothesis, (1) and (2) imply that G is $(r-1)$ -extendable. If G is r -extendable, we are done. Otherwise, there exist $r-1$ independent edges $e_i = x_i y_i$ ($1 \leq i \leq r-1$) so that $G' = G - \{x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$ is not 1-extendable. Since G' has a perfect matching, condition (1) of Theorem 1.3.2 holds. Thus, if G' is not 1-extendable, then there exists a set $S' \subseteq V(G')$ so that $o(G'-S') = |S'|$ and $F(S') \geq 1$. Let $S = S' \cup \{x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$. Then $o(G-S) = o(G'-S') = |S'| = |S| - 2(r-1)$ and $F(S) \geq F(S') + (r-1) \geq r$, which contradicts condition (2). \square

Next we study relationships between n -extendability and $n\frac{1}{2}$ -extendability. It turns out that they are very similar. If a new vertex joins to all vertices of an $n\frac{1}{2}$ -extendable graph G , then the resulting graph is $(n+1)$ -extendable. Thus $(n+1)$ -extendable graphs can be obtained by this manner and in this sense, $n\frac{1}{2}$ -extendability is weaker than $(n+1)$ -extendability. On the other hand, if G is $n\frac{1}{2}$ -extendable, then for any vertex $v \in V(G)$, $G-v$ is n -extendable. Hence $n\frac{1}{2}$ -extendability is "stronger" than n -extendability. However, there exist $(n+1)$ -extendable graphs with the property that on deletion of some vertex the resulting graph is not $n\frac{1}{2}$ -extendable; for example, the cube G of Figure 3.1 is 2-extendable but on deleting any vertex v , $G-v$ is not $1\frac{1}{2}$ -extendable. So it is natural to think of $n\frac{1}{2}$ -extendability as lying between n and $(n+1)$ -extendability. Not surprising then, we can characterize all $n\frac{1}{2}$ -extendable graphs in terms of n -extendable and $(n+1)$ -extendable graphs.

Theorem 3.3.2 A graph G of odd order is $n\frac{1}{2}$ -extendable if and only if $G+K_1$ is $(n+1)$ -extendable.

Proof: Assume that G is $n\frac{1}{2}$ -extendable. Let $H = G+K_1$, where $V(K_1) = \{z\}$ and choose $n+1$ independent edges, $e_i = x_iy_i$ ($i = 1, 2, \dots, n+1$) of $E(H)$.

Case 1. All $n+1$ independent edges lie in $E(G)$. Since G is $n\frac{1}{2}$ -extendable, there exists a near perfect matching M containing e_1, e_2, \dots, e_n and missing x_{n+1} in G . Let w be the vertex adjacent to y_{n+1} in M . Then $M - \{wy_{n+1}\} \cup \{wz, x_{n+1}y_{n+1}\}$ will be a perfect matching of H containing e_1, e_2, \dots, e_{n+1} .

Case 2. Suppose that one of e_1, e_2, \dots, e_{n+1} is not in $E(G)$, say e_{n+1} . Let $e_{n+1} = zw$, where $w \in V(G) - \{x_1, y_1, \dots, x_n, y_n\}$. Then there exists a near perfect matching M of G containing e_1, e_2, \dots, e_n and missing the vertex w . Thus $M \cup \{zw\}$ is a perfect matching of H as required.

Conversely, for any n independent edges e_1, e_2, \dots, e_n of $E(G)$ and vertex v of $V(G)$ not lying on these edges, there exists a perfect matching M of H containing e_1, e_2, \dots, e_n, vz . Then $M' = M - \{z\}$ is a near perfect matching of G which contains e_1, e_2, \dots, e_n and misses v . \square

Remark 3.3.3 Even though when G is $n\frac{1}{2}$ -extendable, $G+K_1$ is $(n+1)$ -extendable, it is not the case that if G is n -extendable, then $G+K_1$ is $n\frac{1}{2}$ -extendable. For example, the cycle C_6 is 1-extendable, but C_6+K_1 is not $1\frac{1}{2}$ -extendable

From the definition of $n\frac{1}{2}$ -extendability, we have the following observation.

Observation 3.3.4 A graph G is $n\frac{1}{2}$ -extendable if and only if $G-v$ is n -extendable for any vertex $v \in V(G)$.

We now give a characterization of $n\frac{1}{2}$ -extendable graphs.

Theorem 3.3.5 A graph G is $1\frac{1}{2}$ -extendable if and only if for any $S \subseteq V(G)$, $S \neq \emptyset$,

(1) $o(G-S) \leq |S|-1$ and

(2) if both $o(G-S) = |S|-1$ and $|S| \geq 3$, then S is independent.

Proof: " \Rightarrow " If G is $1\frac{1}{2}$ -extendable, then G is factor-critical, and by Theorem 1.3.3 condition (1) holds.

Suppose there exists a vertex-set S of $V(G)$ with $|S| \geq 3$ such that $o(G-S) = |S|-1$ but S is not independent. Let $e = xy \in E(G[S])$ and $z \in S - \{x, y\}$. Let $G' = G - \{z\}$ and $S' = S - \{z\}$. Then, as by Observation 3.3.4 G' is 1-extendable, it follows that $o(G'-S') = o(G-S) = |S|-1 = |S'|$. From Theorem 1.3.2 S' must be an independent. But this contradicts the fact that $e \in E(G[S'])$.

" \Leftarrow " Condition (1) guarantees that G has an odd number of vertices (choose $S = \{v\}$, $v \in V(G)$) and then Theorem 1.3.3 implies that G is factor-critical. But we need the stronger result that $G - \{v\}$ is 1-extendable for any $v \in V(G)$. Suppose that for $v \in V(G)$ and $e \in E(G-v)$ there is no perfect matching in $G-v$ containing e . Since $G-v$ has a perfect matching, then by Theorem 1.3.2 and Theorem 1.3.1 we know that there exists a vertex-set $S \subseteq V(G-v)$ so that $o(G-v-S) = |S|$ and S is not independent. Thus $|S| \geq 2$. Let $S'' = S \cup \{v\}$. Then $o(G-S'') = o(G-v-S) = |S| = |S''|-1$ and $|S''| \geq 3$, but S'' is not independent. This contradicts condition (2). \square

Theorem 3.3.6 A graph G is $n\frac{1}{2}$ -extendable if and only if for any $S \subseteq V(G)$, $S \neq \emptyset$,

(1) $o(G-S) \leq |S|-1$ and

(2) if $o(G-S) = |S|-2k-1$ ($0 \leq k \leq n-1$) and $|S| \geq 2k+3$ for some vertex-set $S \subseteq V(G)$, then $F(S) \leq k$, where $F(S)$ is the size of maximum matching in $G[S]$.

Proof: The proof will be by induction on n .

If $n = 1$, we use the claim of Theorem 3.3.5.

Suppose the theorem holds when $n < r$, and consider the case $n = r$.

" \Rightarrow " Assuming that G is $r\frac{1}{2}$ -extendable, it follows that G is factor-critical.

Thus (1) follows from Theorem 1.3.3. If $o(G-S) = |S|-2k-1$ ($0 \leq k \leq r-2$) and $|S| \geq$

$2k+3$, then by the induction hypothesis, $F(S) \leq k$. Suppose then that there exists a set S such that $o(G-S) = |S|-2(r-1)-1$ and $|S| \geq 2r+1$ ($k = r-1$), but $F(S) \geq r$. Let $e_i = x_i y_i$ ($1 \leq i \leq r$) be r independent edges in $G[S]$, $v \in S' = S - \{x_1, y_1, \dots, x_r, y_r\}$ and $G' = G - \{x_1, y_1, \dots, x_r, y_r, v\}$. Then $o(G'-S') = o(G-S) = |S|-2r+1 = |S'|+2 > |S'|$ and by Tutte's theorem, G' has no perfect matching. This contradicts the fact that G is $r \frac{1}{2}$ -extendable.

" \Leftarrow " Suppose that conditions (1) and (2) hold but G is not $r \frac{1}{2}$ -extendable. Then there exists a vertex $v \in V(G)$ such that $G-v$ is not r -extendable. Applying Observation 3.1.1, there exist independent edges $e_i = x_i y_i$ ($1 \leq i \leq r-1$) so that $G' = G - v - \{x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$ is not 1-extendable. However, from the induction hypothesis G is $(r-1) \frac{1}{2}$ -extendable and thus G' has a perfect matching. Then from Theorem 1.3.1 for all $S \subseteq V(G')$, $o(G'-S) \leq |S|$. But now as G' is not 1-extendable, from Theorem 1.3.2, there exists a set $S' \subseteq V(G')$ such that $o(G'-S') = |S'|$ and S' is not independent. Let $S = S' \cup \{v, x_1, y_1, \dots, x_{r-1}, y_{r-1}\}$. Then $o(G-S) = o(G'-S') = |S'| = |S|-2(r-1)-1 = |S|-2r+1$ and so $|S| = |S'|+2(r-1)+1 \geq 2+2(r-1)+1 = 2r+1$. But $F(S) \geq F(S')+(r-1) \geq r$, which contradicts condition (2) when $k = r-1$. \square

Corollary 3.3.7 If G is an $n \frac{1}{2}$ -extendable graph, then G is also $(n-1) \frac{1}{2}$ -extendable.

We now turn to study some of the properties of $n \frac{1}{2}$ -extendable graphs. They are analogous to those of n -extendable graphs.

Theorem 3.3.8 If G is a graph of order $2r+1$, $r \geq n+1 \geq 2$ and $\delta(G) \geq r+n+1$, then G is $n \frac{1}{2}$ -extendable. Moreover, the lower bound on $\delta(G)$ is sharp.

Proof: By Observation 3.3.4, we need only to show that for any $v \in V(G)$ $G-v$ is n -extendable. For any $v \in V(G)$, $\delta(G-v) \geq \delta(G)-1 \geq r+n$. From Theorem 1.3.4 (3), $G-v$ is n -extendable and we are done.

To see that the bound is sharp, consider the graph $G = K_{r+n} + \overline{K_{r-n+1}}$. Since $r \geq n+1$, we take a vertex v and n independent edges $x_1y_1, x_2y_2, \dots, x_ny_n$ from K_{r+n} . There remain $r-n-1$ vertices in K_{r+n} which cannot be matched to the $r-n+1$ vertices in $\overline{K_{r-n+1}}$. Thus $\delta(G) = r+n$ and G is not $n\frac{1}{2}$ -extendable. \square

Theorem 3.3.9 If G is connected and $n\frac{1}{2}$ -extendable ($n \geq 1$), then G is $(n+2)$ -connected and, moreover, there exists an $n\frac{1}{2}$ -extendable graph G of connectivity $n+2$.

Proof: If G is $n\frac{1}{2}$ -extendable, then, by Theorem 3.3.2, $G+K_1$ is $(n+1)$ -extendable. Since $G+K_1$ is connected, by Theorem 1.3.4 (2), $G+K_1$ is $(n+2)$ -connected. Let $K_1 = \{u\}$. Since $n \geq 1$, $G-v = (G+K_1) - \{u,v\}$ is connected for any $v \in V(G)$. By Observation 3.3.4, $G-v$ is n -extendable for any $v \in V(G)$. Thus $G-v$ is $(n+1)$ -connected by applying Theorem 1.3.4 (2).

Suppose that G is not $(n+2)$ -connected. Then there exists a cut-set $S \subseteq V(G)$, $|S| = n+1$. For any $v \in S$, $S - \{v\}$ is a cut-set of $G-v$. Since $|S - \{v\}| = n$, this contradicts the fact that $G-v$ is $(n+1)$ -connected.

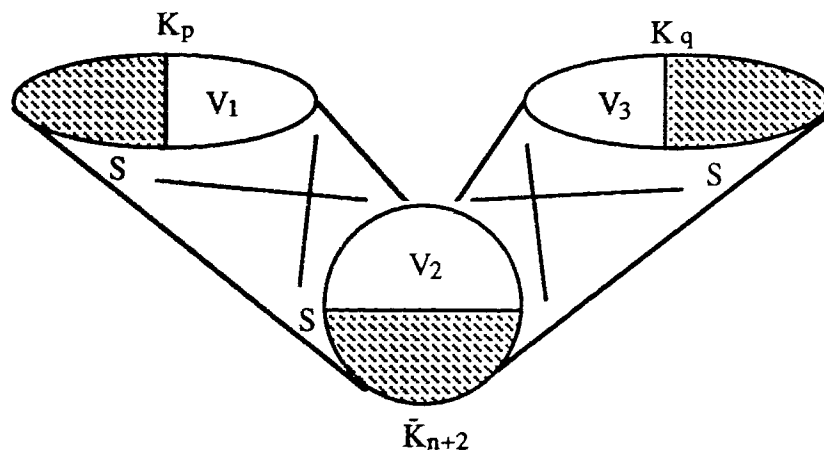


Figure 3.3

To see that an $n\frac{1}{2}$ -extendable graph might not be $(n+3)$ -connected, we consider the graph $G = \overline{K_{n+2}} + (K_p \cup K_q)$ where $n+2+p+q$ is odd and $p \geq q \geq 2n+2$.

Clearly G is not $(n+3)$ -connected as $V(\overline{K_{n+2}})$ is a cut-set of size $n+2$. We next show that G is $n\frac{1}{2}$ -extendable. For any given n independent edges $e_i = x_i y_i$, $1 \leq i \leq n$, and a vertex $v \notin \{x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$, let $S = \{v, x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$, $V_1 = V(K_p) - S$, $V_2 = V(\overline{K_{n+2}}) - S$ and $V_3 = V(K_q) - S$ (see Figure 3.3). We now need only to show that $G - S$ has a perfect matching. Clearly, the existence of a perfect matching in the graph $G - S$ is equivalent to a partition of V_2 into two subsets V_2', V_2'' such that $|V_2'| \leq |V_1|$, $|V_2''| \leq |V_3|$, $|V_2'| \equiv |V_1| \pmod{2}$, and $|V_2''| \equiv |V_3| \pmod{2}$. As $|V(G)|$ is odd and $p, q \geq 2n+2$, we have that $|V_1| + |V_2| + |V_3| = |V(G)| - |S| = p + q + 1 - n$ is even and $|V_1| + |V_3| \geq |V_2| + 2$. Therefore the required partition (V_2', V_2'') can always be achieved. This completes the proof. \square

Remark 3.3.10 Theorem 3.3.10 does not hold for $n = 0$; that is, for factor-critical graphs. The graph below provides an example of a $\frac{1}{2}$ -extendable graph which is not 2-connected.

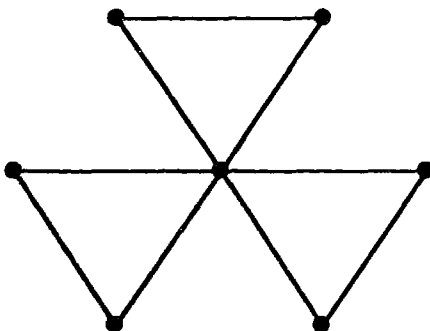


Figure 3.4 This factor-critical graph is not 2-connected.

Corollary 3.3.11 If G is an $n\frac{1}{2}$ -extendable graph of order p , $p \geq 2n+5$, and if u is a vertex of degree $n+2$ in G , then $N_G(u)$ is an independent set.

Proof: Suppose u is a vertex of degree $n+2$ in an $n\frac{1}{2}$ -extendable graph G and let $N_G(u) = \{v_1, v_2, \dots, v_{n+2}\}$. Since $p > 2n+4$, we can choose $n+1$ vertices $w_1, w_2, \dots,$

w_{n+1} in $V(G) - N_G(u) - \{u\}$. As G is $(n+2)$ -connected, by Menger's theorem ([12, p163]) we have $n+2$ vertex-disjoint paths joining $N_G(u)$ and $\{w_1, w_2, \dots, w_{n+1}, u\}$. Hence there are $n+2$ independent edges $e_1 = v_1u, e_2 = v_2w_1', \dots, e_{n+2} = v_{n+2}w_{n+1}'$, where w_i' is the last vertex on the path from w_i to v_{i+1} .

Suppose now that $N_G(u)$ is not independent, say $v_1v_2 \in E(G)$. Then $v_1v_2, e_4, e_5, \dots, e_{n+2}$ are n independent edges. Since u is an isolated vertex of $G - N_G(u)$, there exists no near perfect matching containing $v_1v_2, e_4, e_5, \dots, e_{n+2}$ and missing v_3 . This contradicts the fact that G is $n\frac{1}{2}$ -extendable. \square

A graph G is called **n -critical** if the deletion of any n vertices of $V(G)$ results in a graph with a perfect matching. This concept is a generalization of the notions of factor-critical and bicritical which correspond to the cases when $n = 1$ and $n = 2$, respectively. Hereafter, we will often refer to factor-critical graphs as 1-critical and bicritical graphs as 2-critical. Here we present a characterization of n -critical graphs.

Theorem 3.3.12 A graph G is n -critical if and only if $|V(G)| \equiv n \pmod{2}$ and for any vertex-set $S \subseteq V(G)$ with $|S| \geq n$, $o(G-S) \leq |S| - n$.

Proof: " \Rightarrow " Suppose that G is n -critical. Then it is immediate that $|V(G)| \equiv n \pmod{2}$. Suppose there is a vertex-set $S \subseteq V(G)$ with $|S| \geq n$ and $o(G-S) > |S| - n$. Delete n vertices v_1, v_2, \dots, v_n from S and denote the remaining set by S' . Then $o(G - \{v_1, v_2, \dots, v_n\} - S') = o(G-S) > |S| - n = |S'|$ and by Tutte's theorem, $G - \{v_1, v_2, \dots, v_n\}$ has no perfect matching. But this contradicts the hypothesis.

" \Leftarrow " Suppose that $|V(G)| \equiv n \pmod{2}$ and for any vertex-set $S \subseteq V(G)$ with $|S| \geq n$, $o(G-S) \leq |S| - n$ but G is not n -critical. Then there exist n vertices v_1, v_2, \dots, v_n such that $G - \{v_1, v_2, \dots, v_n\}$ has no perfect matching. Using Tutte's theorem again, there exists a set $S' \subseteq V(G) - \{v_1, v_2, \dots, v_n\}$ so that $o(G - \{v_1, v_2, \dots, v_n\} - S') > |S'|$. Let $S = S' \cup \{v_1, v_2, \dots, v_n\}$. Then $o(G-S) > |S'| = |S| - n$, a contradiction. \square

There are several possible generalizations of n -extendability. One of the generalizations is to consider all graphs G satisfying the property that for any m -matching M and a set of n distinct vertices u_1, u_2, \dots, u_n of G , none of which is incident with any edge of M , there exists any perfect matching M^* of G such that $M \subseteq M^*$ and $u_i u_j \notin M^*$ for $1 \leq i, j \leq n$ and $i \neq j$. Another generalization is to study graphs with the property that for any m independent edges and any n vertices not incident with any one of these m edges, there is a t -matching in G containing the m edges but missing all n vertices. The former is called **(m,n) -extendability** and was studied by Liu and Yu [36]. This concept is stronger than n -extendability and is very helpful for studying the properties of n -extendable graphs.

Chapter 4. Classifications of some families of n -extendable graphs.

§4.1. Introduction.

In this chapter, we discuss the extendability properties of several families of graphs. First, we consider various products of graphs. Two types of products - cartesian product and wreath product - will be studied. The product of certain graphs (for example, $C_m \times C_n$, $C_m \times P_n$) are often the "skeletons" (that is, the spanning subgraphs) of symmetric graphs and thus knowledge of their extendability will be very helpful in understanding the extendability of symmetric graphs. In addition, the question of the extendability of products of graphs is in itself particularly interesting. Products of graphs also provide us with an easy way to construct n -extendable graphs with low degree.

Second, using the results obtained on the extendability of products of graphs we are able to classify 2-extendable Cayley graphs on abelian groups. This is closely related to an earlier result of Chen and Quimpo [15], who proved that every abelian Cayley graph has a Hamiltonian cycle containing a given edge. Their result implies that every abelian Cayley graph is 1-extendable. These results add significantly to our understanding of abelian Cayley graphs.

At the last section of this chapter, we consider generalized Petersen graphs. In [10], Castagna and Prins proved that all generalized Petersen graphs, except for the Petersen graph itself, have a 1-factorization. This result indicates that generalized Petersen graphs are in some sense "1-factor rich" and so we might hope that they are n -extendable for reasonably large n . In [9], Cammack and Schrag conjectured precisely which generalized Petersen graphs are 2-extendable. We shall prove their conjecture and hence classify all 2-extendable generalized Petersen graphs.

Let Γ be an abelian group with operation $+$, and S a generating set of Γ such that the identity element 0 is not in S and $-x \in S$ for each $x \in S$. The **Cayley graph** $G(\Gamma; S)$ on Γ is defined by:

$$V(G(\Gamma; S)) = \Gamma \text{ and } E(G(\Gamma; S)) = \{ xy \mid x, y \in \Gamma, -x+y \in S \}.$$

As an example consider the Cayley graph (Figure 4.1) where $\Gamma = Z_4 \times Z_2 = \{(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (3, 1)\}$ and $S = \{(0, 1), (1, 0), (3, 0), (1, 1), (3, 1)\}$.

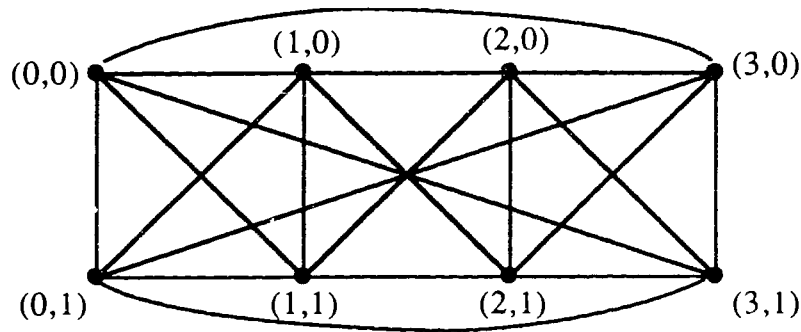


Figure 4.1 The Cayley graph $G(Z_4 \times Z_2; \{(0, 1), (1, 0), (3, 0), (1, 1), (3, 1)\})$

The edge xy in $G(\Gamma; S)$ is said to be **of type a** (or an **a-edge**) if $-x+y \in \{a, -a\}$. For convenience, if $S = \{a_1, a_2, \dots, a_n\}$, we shall often denote $G(\Gamma; S)$ by $G(\Gamma; a_1, a_2, \dots, a_n)$.

For each $a \in \Gamma$, we shall denote by θ_a the mapping from Γ to Γ defined by $\theta_a(x) = a+x$. Clearly, θ_a is an automorphism of $G(\Gamma; S)$, from which it follows that every Cayley graph is **vertex-transitive** (for any two vertices x and y in Γ , $\theta_{y-x}(x) = y$). This also implies that every Cayley graph $G(\Gamma; S)$ is **regular** and in fact the degree of each vertex is $|S|$.

The **generalized Petersen graph** $GP(p, k)$ ($p > k$) has vertex-set $U \cup V$, where $U = \{u_0, u_1, \dots, u_{p-1}\}$ and $V = \{v_0, v_1, \dots, v_{p-1}\}$, and edge-set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+k} \mid i = 0, 1, \dots, p-1\}$, where all subscript arithmetic is performed modulo p . The **generalized Petersen graph** $GP(7, 2)$ is given (Figure 4.2).

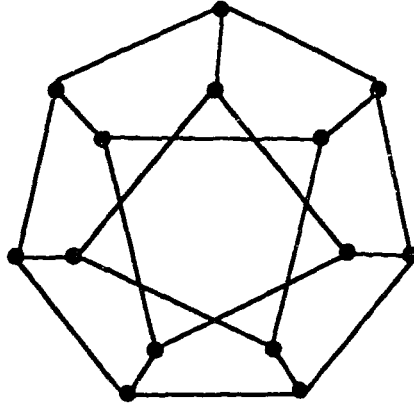


Figure 4.2 GP(7, 2)

§4.2. The extendability of products of graphs.

We start this section with the cartesian product. For convenience, we denote the subgraph induced by $V(G_1) \times \{v_1, v_2, \dots, v_r\}$ (where $\{v_1, v_2, \dots, v_r\} \subseteq V(G_2)$) in $G_1 \times G_2$ by $G_1 \times \{v_1, v_2, \dots, v_r\}$. Thus $V(G_1) \times \{i\}$ is a copy of G_1 . If v_i and v_j are adjacent in G_2 , then $V(G_1) \times \{v_i, v_j\}$ is isomorphic to $G_1 \times P_2$.

Let $e = (x_1, x_2, \dots, x_k)(y_1, y_2, \dots, y_k)$ be an edge of the graph $G_1 \times G_2 \times \dots \times G_k$. By the definition of cartesian product, there exists an integer i so that $x_i y_i \in E(G_i)$ and $x_j = y_j$ for $j = 1, 2, \dots, i-1, i+1, \dots, k$. We denote $x_i y_i$ by e^* and call it the **projection** of e . For an edge $e = (a_1, a_2, \dots, a_{r-1}, x, a_{r+1}, \dots, a_{n-1}, a)(a_1, a_2, \dots, a_{r-1}, y, a_{r+1}, \dots, a_{n-1}, a)$ of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ (notice $xy \in E(G_r)$) the **clone** of e in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$ is defined to be the edge $(a_1, a_2, \dots, a_{r-1}, x, a_{r+1}, \dots, a_{n-1}, b)(a_1, a_2, \dots, a_{r-1}, y, a_{r+1}, \dots, a_{n-1}, b)$. For the set of edges $\{e_1, e_2, \dots, e_r\} \subseteq E(G)$, we denote by $V(\{e_1, e_2, \dots, e_r\})$ the set of all end-vertices of e_1, e_2, \dots, e_r .

Our first object is to study the extendability of $C_m \times P_n$, where $m \geq 3$ and $n \geq 2$. Let $e_1 = (a, b)(c, d)$ and $e_2 = (u, v)(w, x)$ be independent edges in $C_m \times P_n$. We say

that e_1 and e_2 are **perpendicular** if either both $a = c$ and $v = x$, or both $b = d$ and $u = w$. Otherwise, e_1 and e_2 are said to be **parallel**. The following result was obtained by Chen and Quimpo [15] in their study of Hamilton cycles in abelian Cayley graphs.

Lemma 4.2.1 Let m and n be positive integers with mn even, $m \geq 4$ and $n \geq 2$. Then $C_m \times P_n$ is 1-extendable.

Lemma 4.2.2 Let m and n be positive integers with mn even, $m \geq 4$ and $n \geq 2$. Then any two independent and perpendicular edges of $C_m \times P_n$ can be extended to a perfect matching of $C_m \times P_n$.

Proof: Let $C_m = 12\dots m1$ and $P_n = 12\dots n$. Without loss of generality, let $e_1 = (a, b)(a, b+1)$ and $e_2 = (u, v)(u+1, v)$ be two independent perpendicular edges of $C_m \times P_n$.

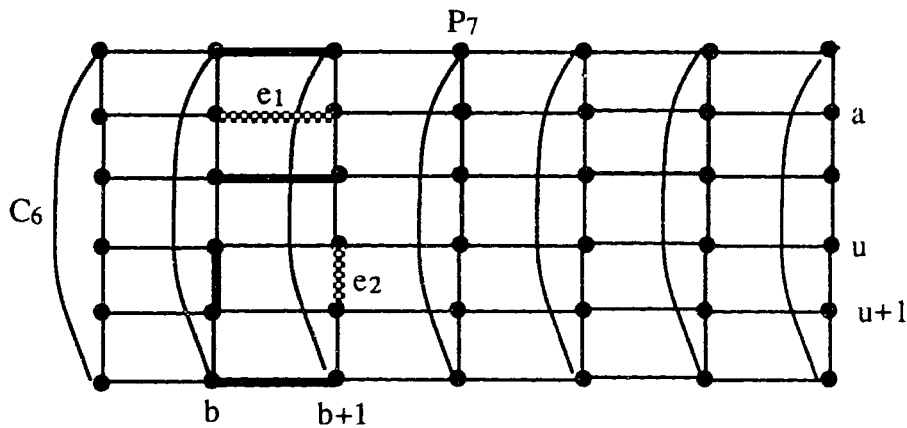


Figure 4.3 m is even, $v = b+1$

We first consider the case when m is even. If $v = b$ or $b+1$, then it is easy to see that $M = \{e_1, (u, b)(u+1, b), (u, b+1)(u+1, b+1)\} \cup \{(g, b)(g, b+1) \mid g \in V(C_m) - \{a, u, u+1\}\}$ is a perfect matching of $C_m \times P_2$, where $P_2 = b(b+1)$ (for example, see Figure 4.3), which contains e_1 and e_2 . M can be extended to a perfect matching of $C_m \times P_n$, since the subgraph of $C_m \times P_n$ induced by the set of vertices not in M can be decomposed into $n-2$ disjoint even cycles of length m ; $C_m \times \{i\}$, $i \in V(P_n) - \{b, b+1\}$. If

$v \neq b$ or $b+1$, let $M = \{(g, b)(g, b+1) \mid g \in V(C_m)\} \cup N$, where N is a perfect matching of the subgraph $C_m \times \{v\}$ containing e_2 (for example, see Figure 4.4). Then M is a set of independent edges containing e_1 and e_2 which can be extended to a perfect matching of $C_m \times P_n$, for the same reason as above.

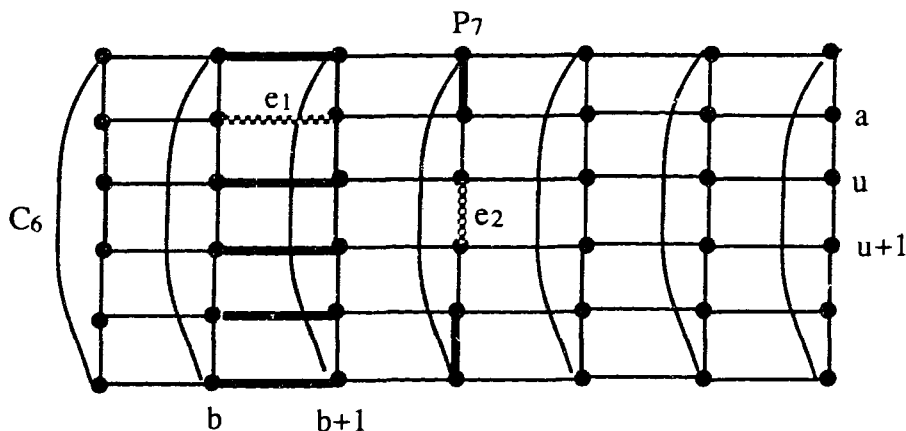


Figure 4.4 m is even, $v \neq b$ or $b+1$

We next consider the case when m is odd, in which case n must be even. Again, assume that $v = b$ or $b+1$ (say $b+1$). If $v = b+1$ is even, then $M = \{e_1, e_2, (u, b)(u+1, b)\} \cup \{(g, b)(g, b+1) \mid g \in V(C_m) - \{a, u, u+1\}\}$ (see Figure 4.5) is a set of independent edges containing e_1 and e_2 . This set can be extended to a perfect matching of $C_m \times P_n$ since the subgraph of $C_m \times P_n$ induced by the set (if this set is not empty) of vertices not in M can be decomposed into the subgraphs $C_m \times \{2x-1, 2x\} \cong C_m \times P_2$, $x = 1, 3, \dots, b-2, b+2, \dots, n-1$, each of which has a perfect matching. On the other hand, when $v = b+1$ is odd, choose y to be any vertex of $V(C_m) - \{a, u, u+1\}$. Let $M = \{e_1, e_2, (u, b)(u+1, b), (y, b-1)(y, b), (y, b+1)(y, b+2)\} \cup \{(g, b)(g, b+1) \mid g \neq a, u, u+1, y\}$. Then M is a set of independent edges containing e_1 and e_2 (an example is given in Figure 4.6), which can be extended to a perfect matching of $C_m \times P_n$, as the subgraph of $C_m \times P_n$ induced by the set of vertices not in M can be decomposed into two even paths $P = (y+1, b-1) (y+2, b-1) \dots (y-1, b-1)$ and $Q = (y+1, b+2)$

$(y+2, b+2)\dots(y-1, b+2)$, and the subgraphs $C_m \times \{x, x+1\} \cong C_m \times P_2$, $x = 1, 3, \dots, b-3, b+3, \dots, n-1$, each of which has a perfect matching.

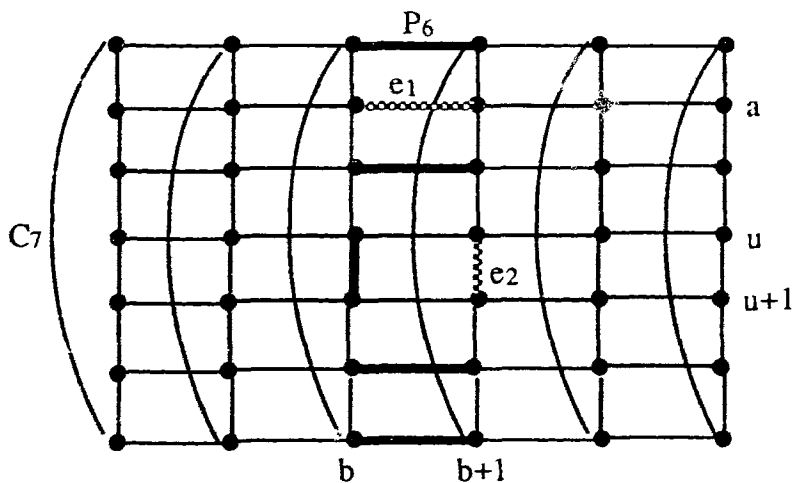


Figure 4.5 m is odd, $v = b+1$ is even

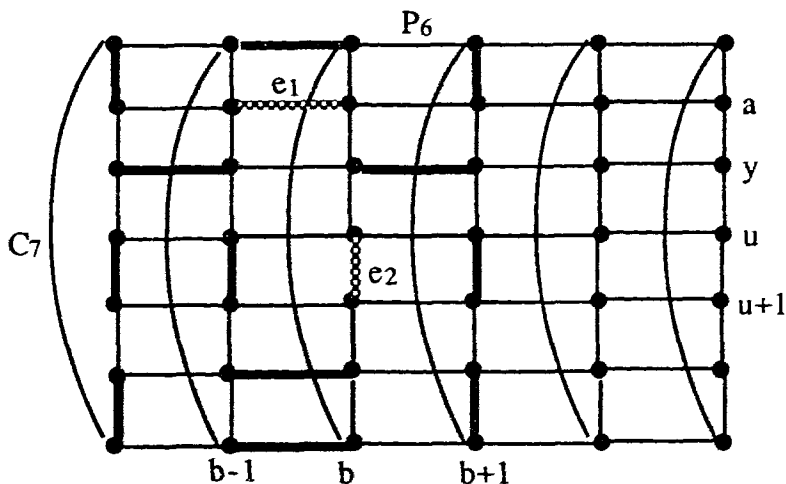


Figure 4.6 m is odd, $v = b+1$ is odd

All that remains is to consider the case when m is odd and $v \neq b$ or $b+1$. Without loss of generality, we may assume that $v > b+1$. If v is odd, let H and K be the graphs induced by the vertex-sets $\{(g, h) \mid h < v\}$ and $\{(g, h) \mid h \geq v\}$, respectively. Since $H \cong C_m \times P_{v-1}$ and $K \cong C_m \times P_{n-v+1}$, v is odd and n is even, then by Lemma 4.2.1, H has a perfect matching M_1 containing e_1 and K has perfect matching

M_2 containing e_2 . Hence $M_1 \cup M_2$ is a perfect matching of $C_m \times P_n$ containing e_1 and e_2 . If v is even and $v > b+2$, let H and K be the graphs induced by the vertex-sets $\{(g, h) \mid h < v-1\}$ and $\{(g, h) \mid h \geq v-1\}$, respectively. Again by Lemma 4.2.1 H has a perfect matching M_1 containing e_1 , K has a perfect matching M_2 containing e_2 and $M_1 \cup M_2$ is a perfect matching of $C_m \times P_n$ containing e_1 and e_2 . The final case is that $v = b+2$ and v is even. Choose a vertex $y \in V(C_m) - \{a, u, u+1\}$ so that $y = u-1$ or $u+2$. Let $M = \{e_1, e_2, (y, b-1)(y, b), (y, b+1)(y, b+2)\} \cup \{(g, b)(g, b+1) \mid g \in V(C_m) - \{a, y\}\}$ (see Figure 4.7). Then M is a set of independent edges containing e_1 and e_2 which can be extended to a perfect matching of $C_m \times P_n$, since the subgraph of $C_m \times P_n$ induced by the set of vertices not in M can be decomposed into two even paths $P = (b-1, y+1)(b-1, y+2) \dots (b-1, y-2)(b-1, y-1)$ and $Q = (b+2, y+1)(b+2, y+2) \dots (b+2, u-1)$ (where addition in the second coordinate is taken by modulo m) and the subgraphs $C_m \times \{x, x+1\} \cong C_m \times P_2$, $x \in \{1, 3, \dots, b-3, b+3, \dots, n-1\}$, each of which has a perfect matching.

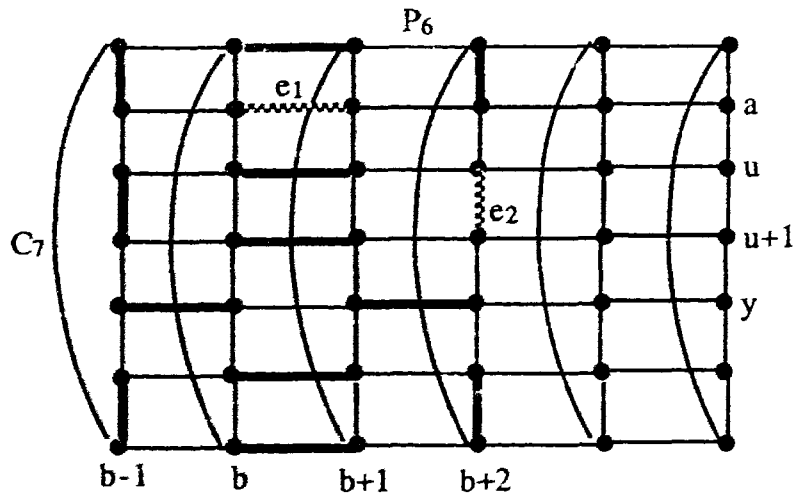


Figure 4.7 m is odd, $v = b+2$ is even.

□

Lemma 4.2.3 If G_1 is a 1-extendable graph with $|V(G_1)| \geq 4$ and G_2 is a connected graph of order at least 2, then $G_1 \times G_2$ is 2-extendable.

Proof: Let $e_1 = (a_1, b_1)(c_1, d_1)$ and $e_2 = (a_2, b_2)(c_2, d_2)$ be two independent edges of $G_1 \times G_2$. We consider the following cases.

Case 1. $b_1 = d_1 \neq b_2 = d_2$.

In this case, $a_1c_1, a_2c_2 \in E(G_1)$. Since G_1 is 1-extendable, there exist perfect matchings F_1 and F_2 in G_1 containing a_1c_1 and a_2c_2 respectively. Thus $\{(g, x)(h, x) \mid gh \in F_1, x \in V(G_2) - \{b_2\}\} \cup \{(g, b_2)(h, b_2) \mid gh \in F_2\}$ is a perfect matching in $G_1 \times G_2$ containing e_1 and e_2 .

Case 2. $b_1 = d_1 = b_2 = d_2$.

Let $b_1 = d_1 = b_2 = d_2 = x$ and suppose that xy is an edge of G_2 . Since G_1 is 1-extendable, there exists a perfect matching F_z of $G_1 \times \{z\}$ for $z \in V(G_2) - \{x, y\}$. Then e_1 and e_2 are contained in the following perfect matching of $G_1 \times G_2$:

$$\{e_1, e_2, (a_1, y)(c_1, y), (a_2, y)(c_2, y)\} \cup \{(g, x)(g, y) \mid g \in V(G_1) - \{a_1, c_1, a_2, c_2\}\} \cup \{F_z \mid z \in V(G_2) - \{x, y\}\}$$

Case 3. $a_1 = c_1 \neq a_2 = c_2$.

Since G_1 is 1-extendable, from Theorem 1.3.4 (2) we have $\delta(G_1) \geq 2$. Choose $x \in N_{G_1}(a_1) - \{a_2\}$. The extendability of G_1 implies that there is a perfect matching F in G_1 which contains xa_1 . Let y be the vertex matched to a_2 in F . We have the following perfect matching of $G_1 \times G_2$ which contains e_1 and e_2 :

$$\{e_1, (x, b_1)(x, d_1)\} \cup \{(a_1, f)(x, f) \mid f \in V(G_2) - \{b_1, d_1\}\} \cup \{e_2, (y, b_2)(y, d_2)\} \cup \{(a_2, f)(y, f) \mid f \in V(G_2) - \{b_2, d_2\}\} \cup \{(g, f)(h, f) \mid gh \in F - \{xa_1, ya_2\}, f \in V(G_2)\}.$$

Case 4. $a_1 = c_1 = a_2 = c_2$.

Let $a_1 = c_1 = a_2 = c_2 = x$. Choose y to be a neighbour of x in G_1 . Then there exists a perfect matching F in G_1 which contains xy . Thus

$$\{e_1, e_2, (y, b_1)(y, d_1), (y, b_2)(y, d_2)\} \cup \{(x, f)(y, f) \mid f \in V(G_2) - \{b_1, d_1, b_2, d_2\}\} \cup \{(g, f)(h, f) \mid gh \in F - \{xy\}, f \in V(G_2)\}$$

is a perfect matching of $G_1 \times G_2$ containing e_1 and e_2 .

Case 5. $b_1 = d_1$ and $a_2 = c_2$.

Let F_1 be a perfect matching of G_1 containing a_1c_1 . If b_1 is b_2 or d_2 , say b_2 , then e_1 and e_2 are contained in the following perfect matching of $G_1 \times G_2$:

$\{e_1, e_2, (a_1, d_2)(c_1, d_2)\} \cup \{(x, b_2)(x, d_2) \mid x \in V(G_1) - \{a_1, c_1, a_2\} \cup \{(g, y)(h, y) \mid gh \in F_1, y \in V(G_2) - \{b_2, d_2\}\}$.

If b_1 is neither b_2 nor d_2 , then the following perfect matching of $G_1 \times G_2$ contains e_1 and e_2 (recall that F_1 contains a_1c_1):

$\{(g, y)(h, y) \mid gh \in F_1, y \in V(G_2) - \{b_2, d_2\}\} \cup \{(x, b_2)(x, d_2) \mid x \in V(G_1)\}$. \square

Corollary 4.2.4 $C_{2m} \times P_n$ is 2-extendable, for $n \geq 2$ and $m \geq 2$.

Proof: Since C_{2m} is 1-extendable, by Lemma 4.2.3 $C_{2m} \times P_n$ is 2-extendable. \square

Corollary 4.2.5 If mn is even, then $C_m \times C_n$ is 2-extendable.

Proof: If mn is even, then one of m and n is even. Thus one of C_m and C_n is 1-extendable. By Lemma 4.2.3 $C_m \times C_n$ is 2-extendable. \square

Corollary 4.2.6 Let G be a 1-extendable graph. Then $G \times P_2$ is 2-extendable.

At this point we know that $C_m \times P_n$ is 2-extendable when $m \geq 4$ is even and $n \geq 2$. When m is odd and n is even we have only partial results (Lemma 4.2.1). We next complete the case when m is odd.

Lemma 4.2.7 The graph $C_{2n+1} \times P_{2r}$ is 2-extendable if and only if $n \geq 2$ and $r \geq 2$.

Proof: Let $C_{2n+1} = 12\dots(2n+1)1$ and $P_{2r} = 12\dots(2r)$. Let e_1 and e_2 be any two independent edges of $C_{2n+1} \times P_{2r}$.

There exists no perfect matching of $C_{2n+1} \times P_2$ containing the edges $e_1 = (1, 1)(2, 1)$ and $e_2 = (2, 2)(3, 2)$ and there is no perfect matching of $C_3 \times P_{2r}$ containing the edges $e_1 = (1, 1)(2, 1)$ and $e_2 = (2, 2)(3, 2)$.

Suppose that $n \geq 2$ and $r \geq 2$. In view of Lemma 4.2.2, we need only consider the case when the independent edges e_1 and e_2 are parallel. Let $e_1 = (a, b)(c, d)$ and $e_2 = (u, v)(w, x)$.

Case 1. $b = d$ (and hence $v = x$).

If $b = v$, choose a vertex y in P_{2r} adjacent to b so that $P_{2r} - \{b, y\}$ is the union of two paths, P_{2m} and P_{2h} . Then $M = \{e_1, e_2, (a, y)(c, y), (u, y)(w, y)\} \cup \{(g, b)(g, y) \mid g \in V(C_{2n+1}) - \{a, c, u, w\}\}$ is a set of independent edges containing e_1 and e_2 . Since the subgraph of $C_{2n+1} \times P_{2r}$ induced by vertices not in M is a union of $C_{2n+1} \times P_{2m}$ and $C_{2n+1} \times P_{2h}$, each of which has a perfect matching, M can be extended to a perfect matching of $C_{2n+1} \times P_{2r}$.

Otherwise, we may assume $b < v$. If $v = b+1$ and b is even or $v > b+1$, then e_1 and e_2 lie in different copies of $C_{2n+1} \times \{x, x+1\} \cong C_{2n+1} \times P_2$, $x = 1, 3, \dots, 2r-1$. Since $C_{2n+1} \times P_2$ is 1-extendable (Lemma 4.2.1), there is a perfect matching containing e_1 and e_2 in $C_{2n+1} \times P_{2r}$. Suppose $v = b+1$ and b is odd. If $|\{a, c\} \cap \{u, w\}| = 0$ or 2 , then $M = \{e_1, e_2, (a, b+1)(c, b+1), (u, b)(w, b)\} \cup \{(g, b)(g, b+1) \mid g \in V(C_{2n+1}) - \{a, c, u, w\}\}$ is a perfect matching of $C_{2n+1} \times \{b, b+1\}$ which (as above) can be extended to a perfect matching of $C_{2n+1} \times P_{2r}$ containing e_1 and e_2 . If $|\{a, c\} \cap \{u, w\}| = 1$, say $c = u$, then choose two vertices y, z in P_{2r} so that either $b(b+1)yz$ or $yzb(b+1)$ is a path in P_{2r} (this is possible as $r \geq 2$ and b is odd). If the path is $b(b+1)yz$, then let $M = \{e_1, e_2, (w, b)(w+1, b), (w+1, b+1)(w+1, y), (w+1, z)(w, z), (w, y)(c, y), (c, z)(a, z), (a, b+1)(a, y)\} \cup \{(g, b)(g, b+1), (g, y)(g, z) \mid g \in V(C_{2n+1}) - \{a, c, w, w+1\}\}$ (see Figure 4.8). By our earlier discussion it is clear that M can be extended to a perfect matching of $C_{2n+1} \times P_{2r}$. (A similar proof applies when the path is $yzb(b+1)$.)

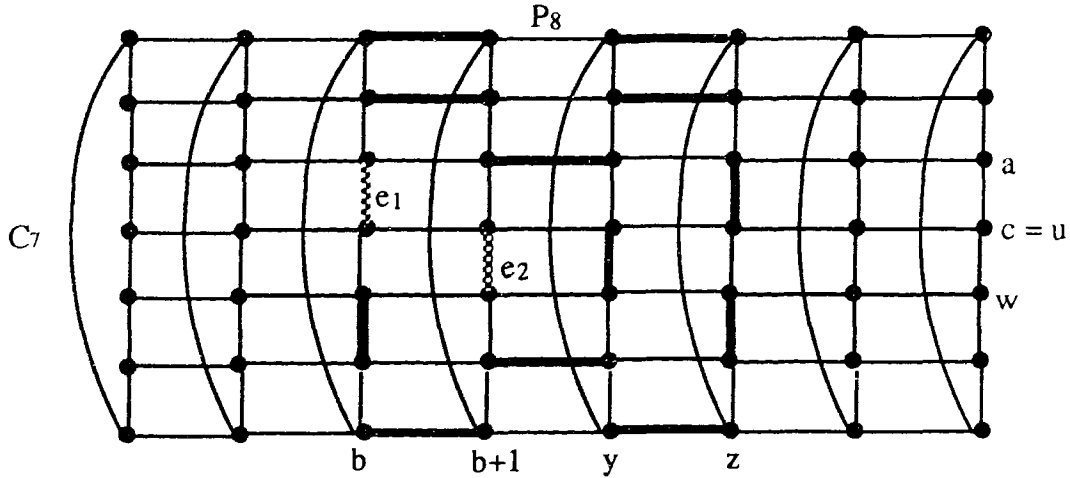


Figure 4.8

Case 2. $a = c$ (and hence $u = w$ and we may assume $d = b+1$ and $x = v+1$).

Case 2.1. $|[b, b+1] \cap [v, v+1]| = 2$, so $b = v$.

If $P_{2r} - \{b, b+1\}$ is the union of even paths, then $M = \{(g, b)(g, b+1) \mid g \in V(C_{2n+1})\}$ is a perfect matching of $C_{2n+1} \times P_2$ containing e_1 and e_2 which can easily be extended to a perfect matching of $C_{2n+1} \times P_{2r}$.

If $P_{2r} - \{b, b+1\}$ is the union of two odd paths, then b is even and $C_{2n+1} \times \{b-1, b, b+1, b+2\} \cong C_{2n+1} \times P_4$. Without loss of generality, we assume $1 = a < u \leq 2n$. Then $M = \{(g, b)(g, b+1) \mid g \in V(C_{2n+1}) - \{2n+1\}\} \cup \{(2n+1, b-1)(2n+1, b), (2n+1, b+1)(2n+1, b+2)\} \cup \{(1, f)(2, f), (3, f)(4, f), \dots, (2n-1, f)(2n, f) \mid f \in \{b-1, b+2\}\}$ (see Figure 4.9) is a perfect matching of $C_{2n+1} \times \{b-1, b, b+1, b+2\}$ and can easily be extended to a perfect matching of $C_{2n+1} \times P_{2r}$.

Case 2.2. $|[b, b+1] \cap [v, v+1]| = 1$, say $b+1 = v$.

Choose a vertex y which is adjacent to b or $b+2$ such that $P_{2r} - \{b, b+1, b+2, y\}$ is the union of even paths. Assume that $y = b+3$. As above we need only find a perfect matching of $C_{2n+1} \times P_4$, $V(P_4) = \{b, b+1, b+2, b+3\}$, containing e_1 and e_2 . Choose a vertex z of C_{2n+1} so that $C_{2n+1} - \{a, u, z\}$ is the union of even paths. Then $(C_{2n+1} \times \{b\}) - \{(a, b)\}$, $(C_{2n+1} \times \{b+1\}) - \{(a, b+1), (u, b+1), (z, b+1)\}$, $(C_{2n+1} \times \{b+2\}) - \{(a, b+2), (u, b+2), (z, b+2)\}$ and $(C_{2n+1} \times \{b+3\}) - \{(a, b+3)\}$ are unions of

even paths (see Figure 4.10). Hence there exists a perfect matching in $C_{2n+1} \times \{b, b+1, b+2, b+3\}$ containing e_1 and e_2 , $(z, b+1)(z, b+2)$ and $(a, b+2)(a, b+3)$.

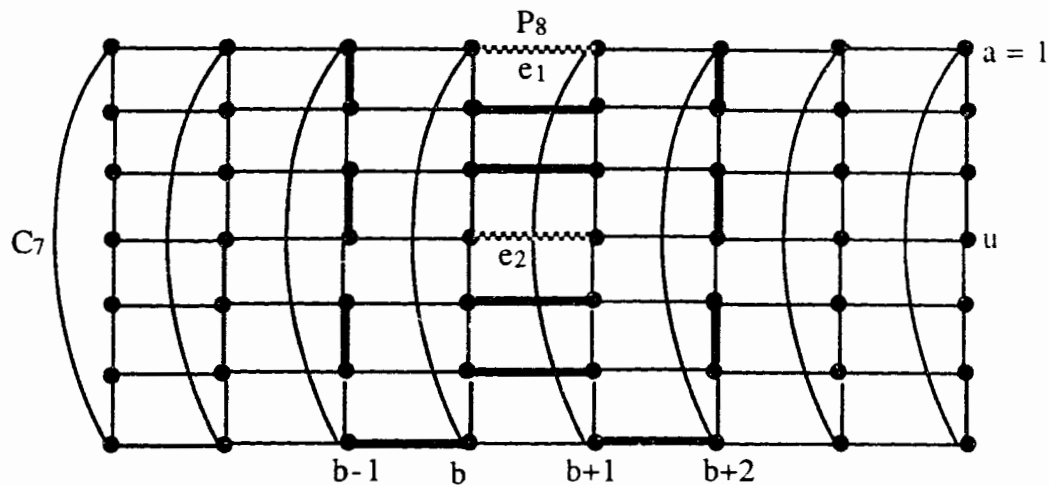


Figure 4.9

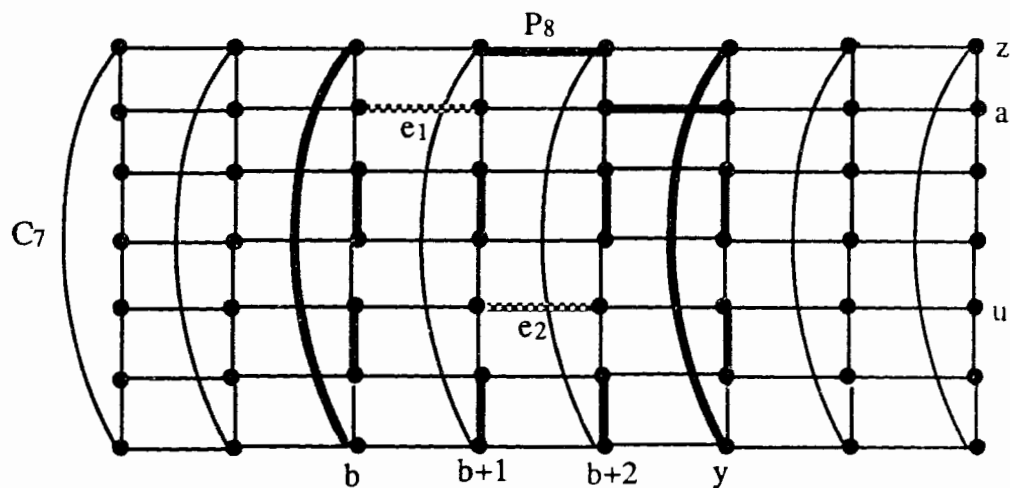


Figure 4.10

Case 2.3. $|\{b, b+1\} \cap \{v, v+1\}| = 0$, and assume $b+1 < v$.

If $v = b+2$ and b is odd or $v > b+2$, then there exists an even integer y so that $b+1 \leq y \leq b+2$, e_1 lies in $C_{2n+1} \times \{1, 2, \dots, y\}$ and e_2 lies in $C_{2n+1} \times \{y+1, \dots, 2r\}$ both of which are 1-extendable by Lemma 4.2.1. Therefore e_1 and e_2 are contained in a perfect matching of $C_{2n+1} \times P_{2r}$. Finally, suppose $v = b+2$ and b is even. If $a = u$, we

may assume $a = 1$. Then $M = \{e_1, e_2, (2n+1, b-1)(2n+1, b), (2n+1, b+1)(2n+1, b+2), (2n+1, b+3)(2n+1, b+4)\} \cup \{(d, g)(d, g+1) \mid d \in V(C_{2n+1}) - \{1, 2n+1\}, g \in \{b, b+2\}\} \cup \{(d, g)(d+1, g) \mid d \in \{1, 3, \dots, 2n-1\}, g \in \{b-1, b+4\}\}$ (see Figure 4.11) is a perfect matching of $C_{2n+1} \times \{b-1, b, b+1, b+2, b+3, b+4\}$ and can easily be extended to a perfect matching of $C_{2n+1} \times P_{2r}$. If $a \neq u$, then there exists a vertex z in C_{2n+1} such that $C_{2n+1} - \{a, u, z\}$ is the union of even paths. As in the case when $a = u$, there is a perfect matching in $C_{2n+1} \times P_6$, $V(P_6) = \{b-1, b, b+1, b+2, b+3, b+4\}$, which contains $(a, b)(a, b+1), (a, b+2)(a, b+3), (u, b)(u, b+1), (u, b+2)(u, b+3), (z, b-1)(z, b), (z, b+1)(z, b+2)$ and $(z, b+3)(z, b+4)$. Hence there is a perfect matching of $C_{2n+1} \times P_{2r}$ which contains e_1 and e_2 .

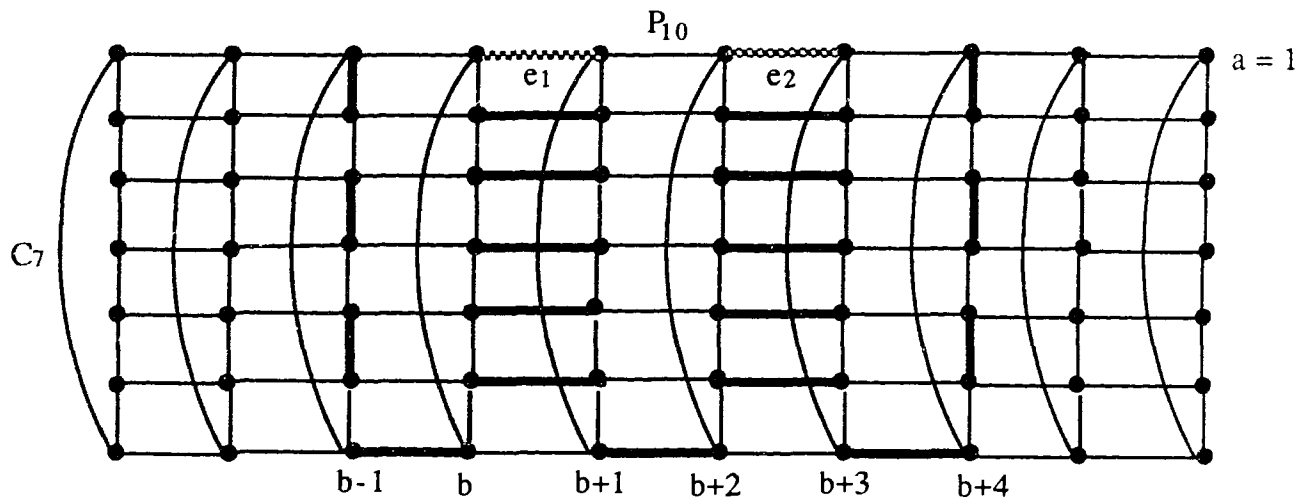


Figure 4.11

□

We have ended a long battle to determine the 2-extendability of $C_m \times P_n$. Basically, we exhaustively considered all possible choices of two independent edges and for each of them we found a perfect matching containing the given edges. It will be much more complex to determine all m and n under which $C_m \times P_n$ is 3-extendable if we attempt to consider all possible sets of three independent edges. Because $C_m \times P_n$

is the "skeleton" of each abelian Cayley graph, it will be even more complicated to classify 3-extendable abelian Cayley graphs. As regards the 2-extendability of $C_m \times P_n$, we summarize the results of Corollary 4.2.4 and Lemma 4.2.7 in the following theorem.

Theorem 4.2.8 Let m and n be positive integers with mn even. Then $C_m \times P_n$ is 2-extendable for all values of m and n except when $m = 3$, and when both $n = 2$ and m is odd. In these cases $C_m \times P_n$ is not 2-extendable.

We next discuss the extendability of product of graphs. This allows us to construct graphs with high extendability from graphs with low extendability by taking cartesian products. First, we need the following lemma.

Lemma 4.2.9 Suppose that G_1 is a connected 1-extendable graph of order at least four and G_2, \dots, G_k are connected graphs of orders at least two. Let e_1, e_2, \dots, e_k be k edges of $G_1 \times G_2 \times \dots \times G_k$. If at most one of $e_1^*, e_2^*, \dots, e_k^*$ (the projections of e_1, e_2, \dots, e_k) belongs to G_1 , then for any vertex $y \in V(G_1 \times \dots \times G_k)$ there exists a vertex x adjacent to y but not adjacent to an end-vertex of any e_i ($1 \leq i \leq k$).

Proof: Let $y = (a_1, a_2, \dots, a_k)$. Since G_1 is 1-extendable and $|V(G_1)| \geq 4$, then by Theorem 1.3.4 (2), $d_{G_1}(a_1) \geq 2$ and there exist two vertices a_1', a_1'' in G_1 so that $a_1'a_1, a_1''a_1 \in E(G_1)$ and $e_1^* \neq a_1'a_1''$. When none of $e_1^*, e_2^*, \dots, e_k^*$ belongs to G_1 , we choose a_1', a_1'' to be any two neighbours of a_1 in G_1 .

As G_i ($2 \leq i \leq k$) are connected graphs of order at least two, $\delta(G_i) \geq 1$. Let a_i' be a neighbour of a_i in G_i , for $2 \leq i \leq k$. Then $(a_1', a_2, \dots, a_k), (a_1'', a_2, \dots, a_k), (a_1, a_2', \dots, a_k), \dots, (a_1, a_2, \dots, a_{k-1}, a_k')$ are $k+1$ neighbours of y in $G_1 \times G_2 \times \dots \times G_k$. By the definition of $G_1 \times G_2 \times \dots \times G_k$, there is no edge among these $k+1$ vertices except possibly the edge $(a_1', a_2, \dots, a_k)(a_1'', a_2, \dots, a_k)$. Since $e_1^* \neq a_1'a_1''$, each edge e_i

covers at most one of these $k+1$ vertices. Therefore, there exists a neighbour of y which is not adjacent to an end-vertex of any e_i ($1 \leq i \leq k$). \square

Györi and Plummer [20] studied extendability of cartesian product of graphs and proved that if G_1 is k -extendable and G_2 is h -extendable, then $G_1 \times G_2$ is $(k+h-1)$ -extendable. We have obtained the following result regarding extendability of products of graphs, in which we only require that one of graphs is 1-extendable; the other can be any graph of order at least two.

Theorem 4.2.10 Let G_1 be a 1-extendable graph with $|V(G_1)| \geq 4$ and G_2, \dots, G_k be connected graphs of order at least two. Then $G_1 \times G_2 \times \dots \times G_k$ is k -extendable.

Proof: We will use induction on k and the fact (Theorem 1.3.4 (1)) that if a graph is n -extendable it is also $(n-1)$ -extendable.

The case $k = 2$ was proven in Theorem 4.2.3.

Suppose that the claim holds for $k \leq n-1$ and consider the case $k = n$. Let e_1, e_2, \dots, e_n be n independent edges of $G_1 \times G_2 \times \dots \times G_n$. By the symmetry of G_2, G_3, \dots, G_n , we need only consider the following cases:

Case 1. The projectors e_i^* satisfy $e_i^* \in E(G_i)$, $1 \leq i \leq n$. Let $e_n^* = ab$.

Case 1.1. $\{e_1, e_2, \dots, e_{n-1}\} \cap E(G_1 \times G_2 \times \dots \times G_{n-1} \times \{a, b\}) = \emptyset$.

By the induction hypothesis, for each $c \in V(G_n) - \{a, b\}$ there exists a perfect matching F_c of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{c\}$ each of which contains clones of the edges e_1, e_2, \dots, e_{n-1} . Then

$$F = \cup \{F_c \mid c \in V(G_n) - \{a, b\}\} \cup \{(a_1, a_2, \dots, a_{n-1}, a)(a_1, a_2, \dots, a_{n-1}, b) \mid (a_1, a_2, \dots, a_{n-1}) \in V(G_1 \times G_2 \times \dots \times G_{n-1})\}$$

is a perfect matching of $G_1 \times G_2 \times \dots \times G_n$ containing the edges e_1, e_2, \dots, e_n .

Case 1.2. $\{e_1, e_2, \dots, e_{n-1}\} \cap E(G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}) \neq \emptyset$ and $\{e_1, e_2, \dots, e_{n-1}\} \cap E(G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}) = \emptyset$.

Assume that e_1, e_2, \dots, e_r are in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$. For each $c \in V(G_n) - \{a, b\}$, let F_c be a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{c\}$ containing the clone of each of the edges of $e_{r+1}, e_{r+2}, \dots, e_{n-1}$. Let e_1', e_2', \dots, e_r' , respectively, be clones of the edges e_1, e_2, \dots, e_r in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$ and let H be the vertex-set of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ excluding the end-vertices of e_1, e_2, \dots, e_r in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$. Then

$$F = \cup\{F_c \mid c \in V(G_n) - \{a, b\}\} \cup \{e_1, e_2, \dots, e_r, e_1', e_2', \dots, e_r'\} \cup \{(z_1, z_2, \dots, z_{n-1}, a)(z_1, z_2, \dots, z_{n-1}, b) \mid (z_1, z_2, \dots, z_{n-1}, a) \in H\}$$

is a perfect matching of $G_1 \times G_2 \times \dots \times G_n$ containing e_1, e_2, \dots, e_n .

Case 1.3. $\{e_1, e_2, \dots, e_{n-1}\} \cap E(G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}) \neq \emptyset$ and $\{e_1, e_2, \dots, e_{n-1}\} \cap E(G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}) \neq \emptyset$.

Without loss of generality, assume that e_1, e_2, \dots, e_r are in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ and $e_{r+1}, e_{r+2}, \dots, e_{r+s}$, $r+s \leq n-1$, are in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$. Let $e_n = xy$, where $x = (x_1, x_2, \dots, x_{n-1}, a)$ and $y = (x_1, x_2, \dots, x_{n-1}, b)$. Let $e_{r+1}', e_{r+2}', \dots, e_{r+s}'$ be the clones of $e_{r+1}, e_{r+2}, \dots, e_{r+s}$, respectively, in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$. By Lemma 4.2.9, there exists a vertex z in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ which is adjacent to x but not adjacent to any of the end-vertices of $e_1, e_2, \dots, e_r, e_{r+1}', e_{r+2}', \dots, e_{r+s}'$. Let $z = (z_1, z_2, \dots, z_{n-1}, a)$ and $w = (z_1, z_2, \dots, z_{n-1}, b)$. By the induction hypothesis, there exists a perfect matching F_1 in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ which contains e_1, e_2, \dots, e_r and xz , and a perfect matching F_2 in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$ which contains $e_{r+1}, e_{r+2}, \dots, e_{r+s}$ and wy . Then the following perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a, b\}$ can easily be extended to a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times G_n$ which contains e_1, e_2, \dots, e_n :

$$F = (F_1 - \{xz\}) \cup (F_2 - \{wy\}) \cup \{xy, wz\}.$$

Case 2. All of e_1, e_2, \dots, e_n lie in the product of exactly $n-1$ of the graphs G_1, G_2, \dots, G_n . We may assume that they all lie in $\cup\{G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\} \mid a \in V(G_n)\}$ or they all lie in $\cup\{\{b\} \times G_2 \times \dots \times G_n \mid b \in V(G_1)\}$.

Case 2.1. Suppose that for some $a \in V(G_n)$ all of e_1, e_2, \dots, e_n are in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ or for some $b \in V(G_1)$, all are in $\{b\} \times G_2 \times \dots \times G_n$.

We begin with the first case. Let ac be an edge of G_n . Let e_1', e_2', \dots, e_n' be the clones of e_1, e_2, \dots, e_n in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{c\}$. Then

$$F = \{e_1, e_2, \dots, e_n, e_1', e_2', \dots, e_n'\} \cup \{(a_1, a_2, \dots, a_{n-1}, a)(a_1, a_2, \dots, a_{n-1}, c) \mid (a_1, a_2, \dots, a_{n-1}, a) \in V(G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}) - V(\{e_1, e_2, \dots, e_n\})\}$$

is a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a, c\}$ which can be extended to a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times G_n$.

In the second case, let F_1 be a perfect matching of G_1 and c be the vertex which is adjacent to b in F_1 . Let e_1', e_2', \dots, e_n' be the clones of e_1, e_2, \dots, e_n in $\{c\} \times G_2 \times \dots \times G_n$. Then

$$F = \{e_1, e_2, \dots, e_n, e_1', e_2', \dots, e_n'\} \cup \{(b, a_2, \dots, a_n), (c, a_2, \dots, a_n) \mid (b, a_2, a_3, \dots, a_n) \in V(\{b\} \times G_2 \times G_3 \times \dots \times G_n) - V(\{e_1, e_2, \dots, e_n\})\}$$

is a perfect matching of $\{b, c\} \times G_2 \times \dots \times G_n$. Hence $F \cup \{(g, a_2, \dots, a_n), (h, a_2, \dots, a_n) \mid (a_2, a_3, \dots, a_n) \in V(G_2 \times G_2 \times \dots \times G_n), gh \in F_1 - \{bc\}\}$ is a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times G_n$.

Case 2.2. Suppose that e_1, e_2, \dots, e_n are contained in different copies of $G_1 \times G_2 \times \dots \times G_{n-1}$; say in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a_i\}$ ($1 \leq i \leq r$, $2 \leq r$). By the induction hypothesis, $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a_i\}$ is $(n-1)$ -extendable. Hence the edges of e_1, e_2, \dots, e_n in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a_i\}$ are contained in a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a_i\}$. Let M be the union of such perfect matchings. Then M can be extended to a perfect matching of $G_1 \times G_2 \times \dots \times G_n$ (as $G_1 \times G_2 \times \dots \times G_{n-1}$ has a perfect matching by the induction hypothesis).

Case 2.3. Suppose that e_1, e_2, \dots, e_n are contained in different copies of $G_2 \times G_3 \times \dots \times G_n$. If there exists a G_r which does not contain any e_i^* ($1 \leq i \leq n$), then e_1, e_2, \dots, e_n are in $\cup\{G_1 \times G_2 \times \dots \times G_{r-1} \times G_{r+1} \times \dots \times G_n \times \{a\} \mid a \in V(G_r)\}$ and the proof follows as in Case 2.2. So we assume that each of G_2, G_3, \dots, G_n contains at least one

of $e_1^*, e_2^*, \dots, e_n^*$. Since $n \geq 3$, one of G_2, G_3, \dots, G_n contains exactly one of $e_1^*, e_2^*, \dots, e_n^*$. Suppose the graph is G_n and the edge is $e_n^* = ab$. Let $e_n = xy$, where $x = (x_1, x_2, \dots, x_{n-1}, a)$ and $y = (x_1, x_2, \dots, x_{n-1}, b)$. By Lemma 4.2.9, there exists a vertex $z = (z_1, z_2, \dots, z_{n-1}, a)$ in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ which is adjacent to x but not adjacent to an end-vertex of any e_1, e_2, \dots, e_{n-1} . Without loss of generality, assume that e_1, e_2, \dots, e_m are contained in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ and $e_{m+1}, e_{m+2}, \dots, e_{m+r}$ are contained in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$. Let $w = (z_1, z_2, \dots, z_{n-1}, b)$. Since $G_1 \times G_2 \times \dots \times G_{n-1}$ is $(n-1)$ -extendable, there exist F_1 and F_2 in $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a\}$ and $G_1 \times G_2 \times \dots \times G_{n-1} \times \{b\}$, respectively, containing e_1, e_2, \dots, e_m, xz and $e_{m+1}, e_{m+2}, \dots, e_{m+r}, yw$ respectively. Then

$$F = (F_1 - \{xz\}) \cup (F_2 - \{yw\}) \cup \{xy, wz\}$$

is a perfect matching of $G_1 \times G_2 \times \dots \times G_{n-1} \times \{a, b\}$. Moreover, F can be extended to a perfect matching of $G_1 \times G_2 \times \dots \times G_n$ containing e_1, e_2, \dots, e_n as $G_1 \times G_2 \times \dots \times G_{n-1} \times (V(G_n) - \{a, b\})$ has a perfect matching which contains $e_{m+r+1}, e_{m+r+2}, \dots, e_{n-1}$ by the induction hypothesis. \square

The **k-cube**, denoted by Q_k , is the graph whose vertices are the ordered k -tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate. Notice that the k -cube has 2^k vertices, $k2^{k-1}$ edges and is isomorphic to $P_2 \times P_2 \times \dots \times P_2$ (k times).

Corollary 4.2.11 the n -cube is $(n-1)$ -extendable.

Proof: Since the n -cube Q_n can be expressed as $Q_n \cong C_4 \times P_2 \times \dots \times P_2$ (where P_2 occurs $n-2$ times in the product), then by Theorem 4.2.10, it is $(n-1)$ -extendable. \square

Remark 4.2.12 Theorem 4.2.10 is best possible. To see this, consider the $(n+1)$ -cube Q_{n+1} . According to Corollary 4.2.11, Q_{n+1} is n -extendable. But since Q_{n+1} is

$(n+1)$ -regular, it cannot be $(n+1)$ -extendable (in view of Theorem 1.3.4(2)). Also, the condition that G_1 is 1-extendable in Theorem 4.2.9 cannot be omitted. For example, let G be any graph with $\delta(G) = 1$. Then $G \times P_2 \times \dots \times P_2$ (where P_2 occurs $n-1$ times in the product) has a vertex of degree n . In light of Theorem 1.3.4 (2), G is not 1-extendable and $G \times P_2 \times \dots \times P_2$ is not n -extendable.

We next consider the extendability of the wreath product of graphs. It seems that determining the extendability of the wreath product of graphs is rather difficult compared with determining the extendability of the cartesian product of graphs. We have not been able to obtain a general result like Theorem 4.2.10 for the wreath product. To better understand the extendability of the wreath product of graphs, we study the wreath products of some special graphs; for example $C_r \otimes \bar{K}_n$, $K_r \otimes \bar{K}_n$. The graph $C_5 \otimes \bar{K}_3$ is shown in Figure 4.12.

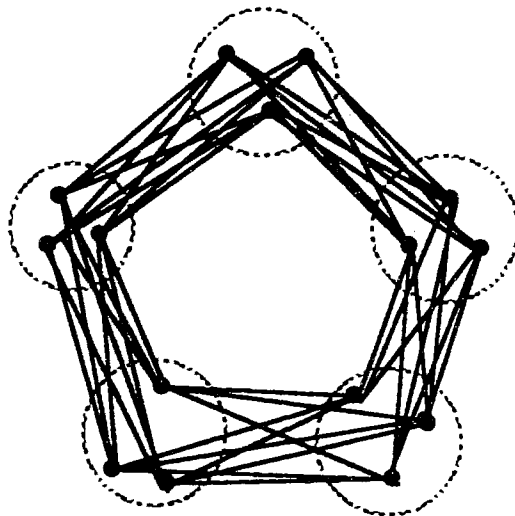


Figure 4.12

Theorem 4.2.13 Let rn be even. If r is even and $r \geq 6$, then $C_r \otimes \bar{K}_n$ is n -extendable and if r is odd, then n is even and $C_r \otimes \bar{K}_n$ is $\frac{n}{2}$ -extendable.

Proof: Let $C_r = 012\dots(r-1)0$ and $\bar{K}_n = \{0, 1, \dots, n-1\}$.

When r is even and $r \geq 6$, we use induction on n .

If $n = 1$, then $C_r \otimes \bar{K}_n \cong C_r$ is 1-extendable.

Suppose that the claim holds for $n < k$ and consider the case $n = k$. For any k independent edges e_1, e_2, \dots, e_k in $C_r \otimes \bar{K}_k$, at most two of the sets $\{i\} \times V(\bar{K}_k)$, $i \in V(C_r)$, have the property that each of their vertices lies in one of the edges e_1, e_2, \dots, e_k . Call such sets **entirely saturated** and call a vertex **saturated by an edge** if it lies on the edge. Suppose that there are exactly b such sets ($b \in \{0, 1, 2\}$) and choose one of e_1, e_2, \dots, e_k , say e_1 , so that exactly b end-vertices of e_1 lie in entirely saturated sets. Without loss of generality, we may assume that $e_1 = (0, 0)(1, 0)$. Since none of $\{\{i\} \times V(\bar{K}_k) \mid i = 2, 3, \dots, r-1\}$ is entirely saturated by e_1, e_2, \dots, e_k , there exists a vertex $v_i \in V(\bar{K}_k)$, $2 \leq i \leq r-1$, such that (i, v_i) is not saturated by e_1, e_2, \dots, e_k in $\{i\} \times V(\bar{K}_k)$. Obviously, $(C_r \otimes \bar{K}_k) - \{(0, 0), (1, 0), (2, v_2), \dots, (r-1, v_{r-1})\} \cong C_r \otimes \bar{K}_{k-1}$ and so by the induction hypothesis there exists a perfect matching F of $C_r \otimes \bar{K}_{k-1}$ containing e_2, e_3, \dots, e_k . Hence $F \cup \{(0, 0)(1, 0), (2, v_2)(3, v_3), \dots, (r-2, v_{r-2})(r-1, v_{r-1})\}$ is a perfect matching of $C_r \otimes \bar{K}_k$ containing the edges e_1, e_2, \dots, e_k .

When r is odd, then n is even and we let $n = 2m$. We use induction on m .

It is easy to show that $C_r \otimes \bar{K}_2$ is 1-extendable.

Suppose that the claim is true for $1 \leq m < k$ and consider the case $m = k$. For any k independent edges e_1, e_2, \dots, e_k in $C_r \otimes \bar{K}_{2k}$, at most two of the sets $\{i\} \times V(\bar{K}_{2k})$, $i \in V(C_r)$, have at least k of their vertices saturated by e_1, e_2, \dots, e_k . Suppose there are exactly b such sets ($b \in \{0, 1, 2\}$) and choose one of e_1, e_2, \dots, e_k , say e_1 , such that exactly b end-vertices of e_1 lie in such sets. Again we may assume $e_1 = (0, 0)(1, 0)$. Since $\{0\} \otimes \bar{K}_{2k}$ and $\{1\} \otimes \bar{K}_{2k}$ are independent sets in $C_r \otimes \bar{K}_{2k}$, there exist vertices $(0, y_0)$ and $(1, y_1)$, $y_0 \neq 0, y_1 \neq 0$, which are not saturated by e_1, e_2, \dots, e_k . Furthermore, since $\{i\} \times V(\bar{K}_{2k})$ ($i = 2, 3, \dots, r-1$) has no more than $k-1$ saturated vertices, there exist two unsaturated vertices (i, y_i) and (i, z_i) in $\{i\} \times V(\bar{K}_{2k})$.

Clearly, $(C_r \otimes \bar{K}_{2k}) - (\{(0, 0), (1, 0), (0, y_0), (1, y_1)\} \cup \{(i, y_i), (i, z_i) \mid i = 2, 3, \dots, r-1\})$
 $\cong C_r \otimes S_{2(k-1)}$. Then by the induction hypothesis, there exists a perfect matching F of
 $C_r \otimes \bar{K}_{2(k-1)}$ containing e_2, e_3, \dots, e_k . Hence $F \cup \{(0, 0)(1, 0), (0, y_0)(1, y_1), (2, y_2)(3,$
 $y_3), (2, z_2)(3, z_3), \dots, (r-2, y_{r-2})(r-1, y_{r-1}), (r-2, z_{r-2})(r-1, z_{r-1})\}$ is a perfect matching
of $C_r \otimes \bar{K}_{2k}$ containing e_1, e_2, \dots, e_k . The induction is complete. \square

Remark 4.2.14 Theorem 4.2.13 is the best possible in the sense that $C_r \otimes \bar{K}_n$ is not
 $(n+1)$ -extendable if r is even ($r \geq 6$), and is not $(\frac{n}{2} + 1)$ -extendable if r is odd. To see
this, when $r = 2m$, let $e_{i+1} = (0, i)(1, i)$, $0 \leq i \leq n-1$, $e_{n+1} = (3, 0)(4, 0)$, and $S = \{(3, j) \mid$
 $1 \leq j \leq n-1\}$. Then there are $n+1$ odd components in $(C_r \otimes \bar{K}_n) - V(\{e_1, e_2, \dots, e_{n+1}\}) -$
 S . But $|S| = n-1$, and by Theorem 1.3.1 there is no perfect matching in $C_r \otimes \bar{K}_n - V(\{e_1,$
 $e_2, \dots, e_{n+1}\})$ and hence $C_r \otimes \bar{K}_n$ has no perfect matching containing e_1, e_2, \dots, e_{n+1}
(see Figure 4.13). When r is odd, then n is even. Let $n = 2m$ and $e_{i+1} = (0, i)(1, i)$, 0
 $\leq i \leq m$. Let $S = \{(0, j), (1, j) \mid j = m+1, \dots, 2m-1\} \cup \{(k, h) \mid k = 3, 5, \dots, r-2; h = 0, 1,$

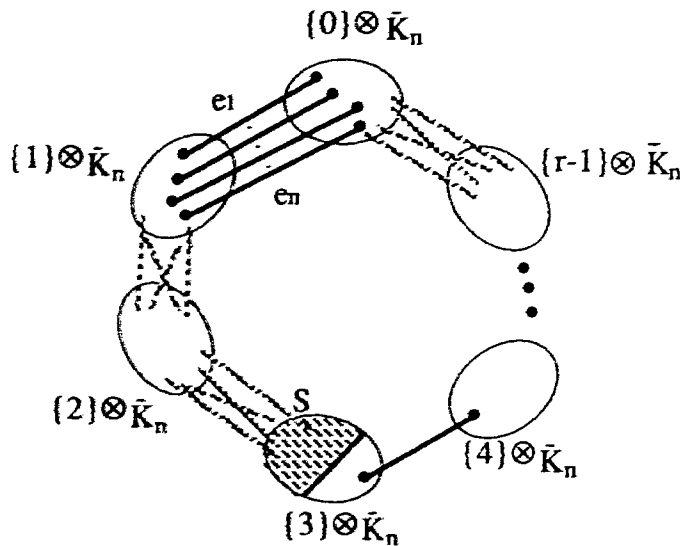


Figure 4.13

..., $2m-1$ }. Then $|S| = 2(m-1)+2m \cdot \frac{r-3}{2} = m(r-1)-2$ and $o((C_r \otimes \bar{K}_{2m}) - V(\{e_1, e_2, \dots, e_{n+1}\}) - S) = m(r-1)$. By Theorem 1.3.1, there is no perfect matching in $C_r \otimes \bar{K}_{2m} - V(\{e_1, e_2, \dots, e_{n+1}\})$. Thus there is no perfect matching of $C_r \otimes \bar{K}_{2m}$ containing e_1, e_2, \dots, e_{n+1} (see Figure 4.14).

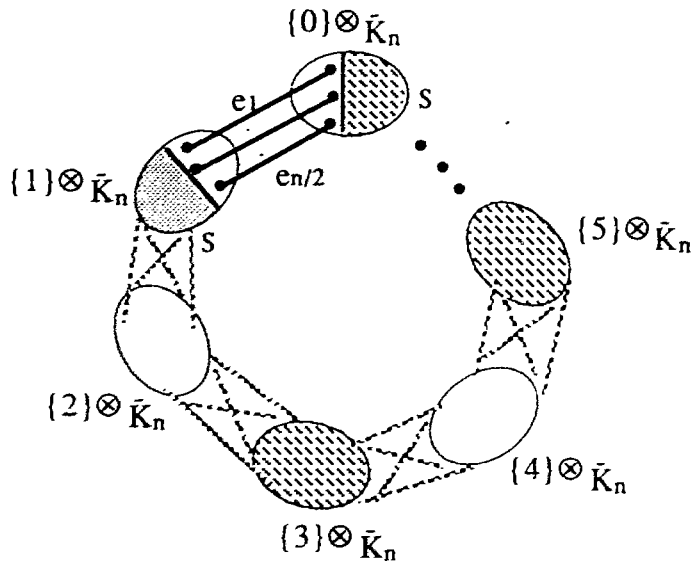


Figure 4.14

The second object of our study of the wreath product is to consider $K_r \otimes \bar{K}_n$, commonly referred to as the complete multipartite graph, which is $(r-1)n$ -regular and has rn vertices.

Theorem 4.2.15 If m is even ($r \geq 3$), then $K_r \otimes \bar{K}_n$ is $\frac{(r-2)n}{2}$ -extendable.

Proof: Let $V(K_r) = \{0, 1, 2, \dots, r-1\}$ and $\bar{K}_n = \{0, 1, \dots, n-1\}$. Let $m = \frac{(r-2)n}{2}$ and e_1, e_2, \dots, e_m be m independent edges of $K_r \otimes \bar{K}_n$.

We claim that for any $2n$ vertices of $K_r \otimes \bar{K}_n$, there exists an n -matching which saturates all these vertices. The proof of this claim uses induction on n . If $n = 1$, then

$K_r \otimes \bar{K}_n \cong K_r$ and the result is immediate. Suppose that the claim is true for $n < k$ and let H be a set of $2k$ vertices chosen from $K_r \otimes \bar{K}_k$.

Since at most two of the sets $\{i\} \times V(\bar{K}_k)$ ($0 \leq i \leq r-1$) are contained in H , we choose r vertices v_0, v_1, \dots, v_{r-1} each from a different $\{i\} \times V(\bar{K}_k)$ such that exactly two of them are in H , say v_0 and v_1 . As v_0 and v_1 belong to different $\{i\} \times V(\bar{K}_k)$, we have $v_0 v_1 \in E(K_r \otimes \bar{K}_k)$. Deleting v_0, v_1, \dots, v_{r-1} from $K_r \otimes \bar{K}_k$, we obtain $K_r \otimes \bar{K}_{k-1}$. By the induction hypothesis, there exists a $(k-1)$ -matching M covering all these $2k-2$ vertices of $H - \{v_0, v_1\}$. Hence $M \cup \{v_0, v_1\}$ is a k -matching as required.

Let e_1, e_2, \dots, e_m be m independent edges of $K_r \otimes \bar{K}_n$. Then $|V(K_r \otimes \bar{K}_n) - V(\{e_1, e_2, \dots, e_m\})| = rn - 2m = 2n$. We know that there exists an n -matching M saturating all these $2n$ vertices. Hence $M \cup \{e_1, e_2, \dots, e_m\}$ is a perfect matching of $K_r \otimes \bar{K}_n$ containing e_1, e_2, \dots, e_m . \square

Remark 4.2.16 Theorem 4.2.15 is also best possible as $K_r \otimes \bar{K}_n$ is not $\frac{(r-2)n+2}{2}$ -extendable. To see this, we consider two cases according the parity of r . Let $V(K_r) = \{0, 1, 2, \dots, r-1\}$ and $V(\bar{K}_n) = \{1, 2, \dots, n\}$.

If r is odd, let $e_1 = (0, 1)(r-1, 1)$, $e_2 = (0, 2)(r-1, 2)$, \dots , $e_n = (0, n)(r-1, n)$, $e_{n+1} = (1, 1)(r-2, 1)$, $e_{n+2} = (1, 2)(r-2, 2)$, \dots , $e_{2n} = (1, n)(r-2, n)$, \dots , $e_{((r-5)n)/2+1} = (\frac{r-5}{2}, 1)(\frac{r+3}{2}, 1)$, $e_{((r-5)n)/2+2} = (\frac{r-5}{2}, 2)(\frac{r+3}{2}, 2)$, \dots , $e_{((r-3)n)/2} = (\frac{r-5}{2}, n)(\frac{r+3}{2}, n)$, $e_{((r-3)n)/2+1} = (\frac{r-3}{2}, 1)(\frac{r+1}{2}, 1)$, \dots , $e_{((r-3)n)/2+(n+2)/2} = (\frac{r-3}{2}, \frac{n+2}{2})(\frac{r+1}{2}, \frac{n+2}{2})$. Then $(K_r \otimes \bar{K}_n) - V(\{e_1, e_2, \dots, e_{((r-2)n)/2+1}\}) \cong (\bar{K}_{n/2-1} + \bar{K}_{n/2-1}) + \bar{K}_n$ which has no perfect matching. Hence there is no perfect matching of $K_r \otimes \bar{K}_n$ containing $e_1, e_2, \dots, e_{((r-2)n)/2+1}$ (see Figure 4.15).

If r is even, let $e_1 = (2, 1)(3, 1)$, $e_2 = (2, 2)(3, 2)$, \dots , $e_n = (2, n)(3, n)$, $e_{n+1} = (4, 1)(5, 1)$, $e_{n+2} = (4, 2)(5, 2)$, \dots , $e_{2n} = (4, n)(5, n)$, \dots , $e_{((r-6)n)/2+1} = (r-4, 1)(r-3, 1)$, $e_{((r-6)n)/2+2} = (r-4, 2)(r-3, 2)$, \dots , $e_{((r-4)n)/2} = (r-4, n)(r-3, n)$, $e_{((r-4)n)/2+1} = (r-1, 1)(0, 1)$, \dots , $e_{((r-4)n)/2+\lceil n/2 \rceil} = (r-1, \lceil \frac{n}{2} \rceil)(0, \lceil \frac{n}{2} \rceil)$, $e_{((r-4)n)/2+\lceil n/2 \rceil+1} = (0, \lceil \frac{n}{2} \rceil+1)(1, \lceil \frac{n}{2} \rceil+1)$, \dots ,

$e_{((r-2)n)/2} = (0, n)(1, n)$, $e_{((r-2)n)/2+1} = (r-1, n)(1, 1)$. Then $K_r \otimes \bar{K}_n - V(\{e_1, e_2, \dots, e_{((r-2)n)/2+1}\}) \cong (\bar{K}_{\lceil n/2 \rceil - 1} + \bar{K}_{\lfloor n/2 \rfloor - 1}) + \bar{K}_n$ and therefore there is no perfect matching of $K_r \otimes \bar{K}_n$ containing $e_1, e_2, \dots, e_{((r-2)n)/2+1}$ (see Figure 4.16).

Figure 4.15

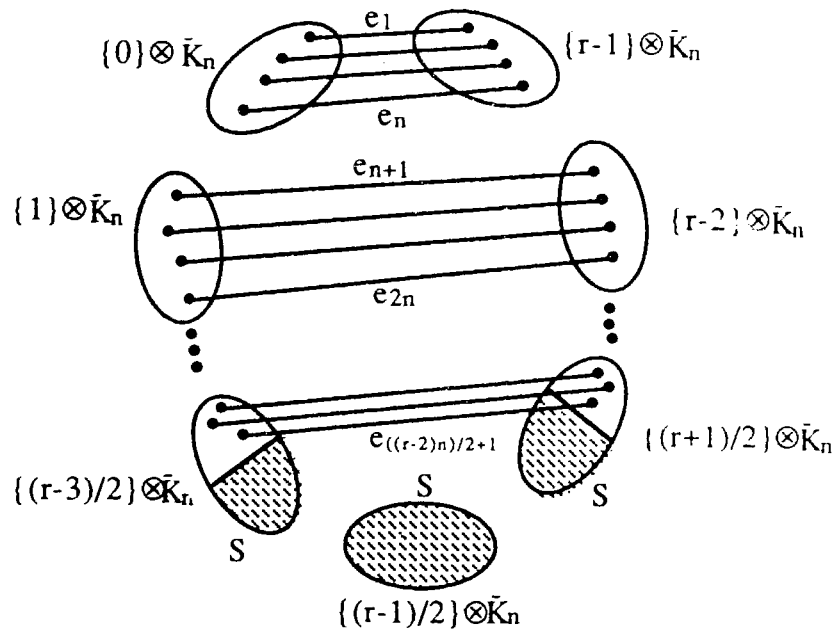
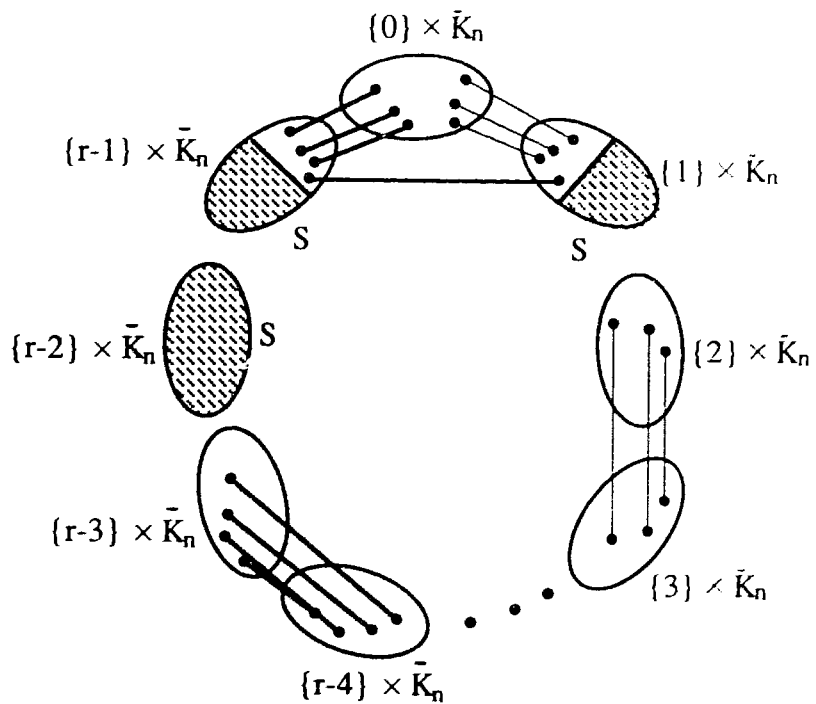


Figure 4.16



Lemma 4.2.17 Let G be a k -extendable graph of order at least $2k+2$. Then for any m independent edges e_1, e_2, \dots, e_m and n vertices v_1, v_2, \dots, v_n of G ($m+n \leq k$) there exist $k+1-(m+n)$ independent edges which are not incident with $V(\{e_1, e_2, \dots, e_m\}) \cup \{v_1, v_2, \dots, v_n\}$.

Proof: Since G is k -extendable, there exists a perfect matching F containing e_1, e_2, \dots, e_m . Let F_1 be the subset of F consisting of all edges in F which have at least one of $\{v_1, v_2, \dots, v_n\}$ as an end-vertex. From Observation 3.1.1 $G_1 = G - V(\{e_1, e_2, \dots, e_m\} \cup F_1)$ is $(k-m-|F_1|)$ -extendable. Since $|V(G_1)| \geq 2(k-m-|F_1|)+2$ and $k-m-|F_1| \geq k-m-n$, by Theorem 1.3.4 (1) G_1 is $(k-m-n)$ -extendable and each perfect matching of G_1 contains at least $k+1-(m+n)$ edges. Therefore, there exist $k+1-(m+n)$ independent edges of G which are not incident with $V(\{e_1, e_2, \dots, e_m\}) \cup \{v_1, v_2, \dots, v_n\}$. \square

Theorem 4.2.18 If G is a k -extendable graph of order at least $2k+4$, then $P_2 \otimes G$ is $(k+2)$ -extendable.

Proof: Let $P_2 = 01$ and e_1, e_2, \dots, e_{k+2} be any $k+2$ independent edges of $P_2 \otimes G$.

Suppose that no edge of e_1, e_2, \dots, e_{k+2} belongs to $\{1\} \times G$. Let e_1, e_2, \dots, e_m be in $\{0\} \times G$ and $e_{m+1}, e_{m+2}, \dots, e_{k+2}$ be in $P_2 \otimes V(G)$. Let $e_j = (0, y_j)(1, z_j)$, $m+1 \leq j \leq k+2$ and let F be a perfect matching of $\{1\} \times G$. Let F_1 be the subset of F consisting of all edges in F which have at least one of $\{z_{m+1}, z_{m+2}, \dots, z_{k+2}\}$ as an end-vertex. Since $|V(G)| \geq 2k+4$, there are at least m edges e_1', e_2', \dots, e_m' of F which are not incident to any of F_1 . Hence $(P_2 \otimes G) - V(\{e_1, e_2, \dots, e_{k+2}, e_1', e_2', \dots, e_m'\})$ has $P_2 \otimes \bar{K}_{|V(G)|-(k+m+2)}$ as a spanning subgraph and $P_2 \otimes G$ then has a perfect matching which contains e_1, e_2, \dots, e_{k+2} .

Otherwise, suppose that m ($m \geq 1$) of e_1, e_2, \dots, e_{k+2} are in $\{0\} \times G$, n ($n \geq 1$) of them are in $\{1\} \times G$ and p ($p \geq 0$) of them lie in $P_2 \otimes \bar{K}_{|V(G)|}$. Thus $m+n+p = k+2$ and

$p \leq k$. Without loss of generality, let $e_i = (0, u_i)(0, v_i)$ ($1 \leq i \leq m$), $e_j = (1, w_j)(1, x_j)$, ($m+1 \leq j \leq m+n$) and $e_h = (0, y_h)(1, z_h)$ ($m+n+1 \leq h \leq m+n+p$) be such edges. Assume that $m \geq n$. Since $\{1\} \times G \cong G$ is k -extendable, by Lemma 4.2.17, for edges $e_{m+1}, e_{m+2}, \dots, e_{m+n}$ and vertices $(1, z_{m+n+1}), \dots, (1, z_{m+n+p})$, there exist $m-n$ edges g_1, g_2, \dots, g_{m-n} (note that $k+1-(n+p) \geq m-n$ as $n \geq 1$) such that $V(\{g_1, g_2, \dots, g_{m-n}\}) \cap (V(\{e_{m+1}, e_{m+2}, \dots, e_{m+n}\}) \cup \{(1, z_{m+n+1}), \dots, (1, z_{m+n+p})\}) = \emptyset$. Since $(P_2 \otimes G) - V(\{e_1, e_2, \dots, e_{m+n}, g_1, g_2, \dots, g_{m-n}\}) - \{(0, y_h), (1, z_h) \mid m+n+1 \leq h \leq m+n+p\}$ contains $P_2 \otimes \bar{K}_{|V(G)|-2m-p}$ as a spanning subgraph and $P_2 \otimes \bar{K}_{|V(G)|-2m-p}$ certainly has a perfect matching, $P_2 \otimes G$ has a perfect matching containing e_1, e_2, \dots, e_{k+2} . \square

There is no example to show that Theorem 4.2.18 is best possible. In fact, we believe that the extendability of $P_2 \otimes G$ should be much larger than $k+2$ if G is a k -extendable graph.

§4.3. On 2-extendable abelian Cayley graphs.

Throughout this section we shall refer to a connected Cayley graph on an abelian group Γ , simply as a abelian Cayley graph. When the abelian group is cyclic, or $\Gamma \cong Z_m$, the Cayley graph $G(\Gamma; S)$ is called a **circulant** and is denoted by $Z_m(S)$.

In this section (which is joint work with O. Chan and C. C. Chen), we shall classify the 2-extendable abelian Cayley graphs. Surprisingly, it turns out that all abelian Cayley graphs which are not 2-extendable are circulants. We state this result formally below; its proof being the main target of this section.

Theorem 4.3.1 Let $G = G(\Gamma; S)$ be a Cayley graph on an even order abelian group Γ . The graph G is 2-extendable if and only if it is not isomorphic to any of the following graphs:

- (I) $Z_{2n}(1, 2n-1)$, $n \geq 3$;
- (II) $Z_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$;
- (III) $Z_{4n}(1, 4n-1, 2n)$, $n \geq 2$;
- (IV) $Z_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$; and
- (V) $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$.

Note that the graph in (I) is just an even cycle of length $2n$, whereas that in (IV) is isomorphic to $C_{2n+1} \times P_2$.

We shall approach the proof by showing that for any two independent edges of $G(\Gamma; S)$ there is a spanning subgraph of G which is the product of a cycle C_m and a path P_n which contains these two edges. We can then apply Theorem 4.2.8 to this subgraph and use the structure of $G(\Gamma; S)$ to classify the 2-extendable abelian Cayley graphs. We begin the proof with the following lemmas.

Lemma 4.3.2 (Chen and Quimpo [15]) Every Cayley graph of even order is 1-extendable.

From Lemma 4.3.2 and Corollary 4.2.6, we have the following.

Corollary 4.3.3 Let G be a Cayley graph. Then $G \times P_2$ is 2-extendable.

Lemma 4.3.4 The cycle C_{2n} is 2-extendable if and only if $n = 2$.

Proof: Clearly, C_4 is 2-extendable. If $n \geq 3$, let $C_{2n} = v_1v_2\dots v_{2n}v_1$. There is no perfect matching in C_{2n} containing the edges v_1v_2 and v_4v_5 . \square

Let Γ be an abelian group and $S \subseteq \Gamma$. We denote by $\langle S \rangle$ the subgroup generated by the elements of S in Γ . We denote by $o(a) = |\langle a \rangle|$ the order of element a in Γ . Let $\langle a \rangle = \{0, a, 2a, \dots, (r-1)a\}$ where $r = o(a)$. We use $b + \langle a \rangle$ to represent the set $\{b, b+a, b+2a, \dots, b+(r-1)a\}$.

At this stage let us look more closely at the structure of abelian Cayley graphs. For any $a \in S \subseteq \Gamma$ the graph $G(\langle a \rangle; \{a, -a\})$ is a cycle of length $o(a)$ and $G(\Gamma; S)$ has a spanning subgraph consisting of the union of $|\Gamma|/o(a)$ disjoint cycles of length $o(a)$. If $b \notin \langle a \rangle$, $r = o(a)$ and $s = o(b)$, then the graph $G(\langle a, b \rangle; \{a, -a, b, -b\})$ has vertex-set $\{ia+jb \mid 0 \leq i \leq r-1, 0 \leq j \leq s-1\}$ and contains s disjoint cycles C_0, C_1, \dots, C_{s-1} of length r (which are all edges of type a in $G(\langle a, b \rangle; \{a, -a, b, -b\})$). Moreover, there is a perfect matching between C_i and C_{i+1} , $0 \leq i \leq s-2$, which is made up of edges of type b in $G(\langle a, b \rangle; \{a, -a, b, -b\})$. By relabelling (if necessary), we see that $G(\langle a, b \rangle; \{a, -a, b, -b\})$ contains a spanning subgraph isomorphic to $C_r \times P_s$. If T is a subset of S with $T = -T$, then $G(\langle T \rangle; T)$ is a subgraph of $G(\Gamma; S)$ and $G(\Gamma; S)$ has a spanning subgraph which is the union of $|\Gamma|/|\langle T \rangle|$ vertex-disjoint copies of $G(\langle T \rangle; T)$.

Lemma 4.3.5 Let $G(\Gamma; S)$ be an abelian Cayley graph and T be a nonempty subset of S with $-T = T$. Then any perfect matching of $G(\langle T \rangle; T)$ can be extended to a perfect matching of $G(\Gamma; S)$.

Proof: This follows immediately from the fact that $G(\Gamma; S)$ can be decomposed into copies of $G(\langle T \rangle; T)$. \square

Recall that the edge xy in $G(\Gamma; S)$ is said to be of **type** a (or an **a -edge**) if $y-x \in \{a, -a\}$. Hence, if xy is of type a , then either $y = x+a$ or $x = y+a$. Also, if $H =$

$G(\langle T \rangle; T) \subseteq G(\Gamma; S)$, then if $c \in \Gamma$ the subgraph $\theta_c(H)$ is a graph with $V(\theta_c(H)) = V(H)+c$ and $(x+c)(y+c) \in E(\theta_c(H))$ if and only if $xy \in E(H)$. Note $\theta_c(H) \cong H$.

Lemma 4.3.6 Let G be an abelian Cayley graph of even order. Then any two independent edges of different types are contained in a perfect matching of G .

Proof: Let $G = G(\Gamma; S)$, where Γ is a finite abelian group of even order. Let $e_1 = ab$ and $e_2 = cd$ be edges of G of types s and t , respectively, where $s, t \in S$ and $s \notin \{t, -t\}$. As G is vertex-transitive, we may assume that $a = 0$ and $b = s$. We shall consider the following cases:

Case 1. s is of even order $2n$ and $t \notin \langle s \rangle$.

Let H be the Cayley graph $G(\langle s, t \rangle; \{s, t, -s, -t\})$. Then H has a spanning subgraph K isomorphic to $C_{2n} \times P_m$, $m \geq 2$, whose edge-set contains e_1 . If e_2 is an edge of H , then we may choose K so that $e_1, e_2 \in E(K)$ and hence by Corollary 4.2.4 there is a perfect matching in H which contains e_1 and e_2 and, by Lemma 4.3.5, can be extended to a perfect matching of G . On the other hand, if e_2 is not an edge of H , then e_2 is in $\theta_c(H)$ for some c . By Lemma 4.3.2 there is a perfect matching M in H containing e_1 and a perfect matching M' in $\theta_c(H)$ containing e_2 . Since $G(\Gamma; S)$ has a spanning subgraph which is the union of $|\Gamma|/|\langle s, t \rangle|$ copies of H , we can extend $M \cup M'$ to a perfect matching of G .

Case 2. s is of even order $2n$ and $t \in \langle s \rangle$.

Let H be the Cayley graph $G(\langle s \rangle; \{s, t, -s, -t\})$, where $t = ks$. If e_2 is not an edge of H , then we can settle this case as in Case 1. Hence we may assume that e_2 is an edge of H with $c = c's$, $d = d's$ and $c' < d'$. Note that $k \in \{c'-d', d'-c'\}$. We then have the following four subcases to consider. For each case, we find a matching M containing e_1 and e_2 such that $V(H)-V(M)$ can be partitioned so that the subgraphs induced by the vertices in each part are of even order and contain a Hamilton path.

Then e_1 and e_2 can be extended to a perfect matching of H . Applying Lemma 4.3.5 again, G has a perfect matching containing e_1 and e_2 .

Case 2.1. If k is odd and c' is even, then d' is odd. Putting $M = \{e_1, e_2\}$, we have the even paths $(2s)(3s)\dots((c'-1)s)$, $((c'+1)s)((c'+2)s)\dots((d'-1)s)$ and $((d'+1)s)\dots((2n-1)s)$.

Case 2.2. If k is odd and c' is odd, then let $M = \{e_1, e_2, ((c'-1)s)((d'-1)s), ((c'+1)s)((d'+1)s)\}$. In this case d' is even and the even paths are $(2s)(3s)\dots((c'-2)s)$, $((c'+2)s)((c'+3)s)\dots((d'-2)s)$ and $((d'+2)s)\dots((2n-1)s)$.

Case 2.3. If k is even and c' is even (hence d' is even), then let $M = \{e_1, e_2, ((c'+1)s)((d'+1)s)\}$. The even paths are $(2s)(3s)\dots((c'-1)s)$, $((c'+2)s)((c'+3)s)\dots((d'-1)s)$ and $((d'+2)s)\dots((2n-1)s)$.

Case 2.4. If k is even and c' is odd, then d' is odd. Let $M = \{e_1, e_2, ((c'-1)s)((d'-1)s)\}$ and the even paths are $(2s)(3s)\dots((c'-2)s)$, $((c'+1)s)((c'+1)s)\dots((d'-2)s)$ and $((d'+1)s)\dots((2n-1)s)$.

Case 3. Both s and t have odd order.

As Γ is of even order, there exists an element $r \in S$ of even order. Hence $r \notin V(H)$, where H is the Cayley graph $G(\langle s, t \rangle; \{s, t, -s, -t\})$. Let K be the Cayley graph $G(\langle r, s, t \rangle; \{r, -r, s, -s, t, -t\})$. If e_2 is not an edge of K , then e_2 is in a subgraph $K' = \theta_c(K) \cong K$ for some $c \notin \langle r, s, t \rangle$. By Lemma 4.3.2 there is a perfect matching M in K containing e_1 and a perfect matching M' in K' containing e_2 . Then $M \cup M'$ can be extended to a perfect matching of G by Lemma 4.3.5. Hence, we assume that $e_2 \in E(K)$. Since $r \notin V(H)$ and r is of even order, K has a spanning subgraph which is isomorphic to $H \times P_{2t}$ (where $2t = o(r)$). Furthermore, K can be partitioned into copies of $L \cong H \times P_2$, where $V(L) = V(H) \cup V(\theta_r(H))$. By Corollary 4.3.3, L is 2-extendable (and hence 1-extendable) and therefore there exists a perfect matching M in K which contains e_1 and e_2 , and can be extended to a perfect matching of G . \square

Lemma 4.3.7 The Cayley graph $Z_{2n}(1, 2n-1, n)$, $n > 2$, is 2-extendable if and only if n is odd.

Proof: Let $G = Z_{2n}(1, 2n-1, n)$. If n is even, we let $e_1 = 01$, $e_2 = (n-1)n$ and $S = \{3, 5, \dots, n-3, n+2, n+4, \dots, 2n-2\}$. Then $|S| = n-3$ and $(G-V(\{e_1, e_2\})) - S$ is a union of $n-1$ isolated vertices. By Tutte's theorem there is no perfect matching in $G-V(\{e_1, e_2\})$ and hence no perfect matching in G containing e_1 and e_2 .

Assume that n is odd. Let $e_1 = ab$ and $e_2 = cd$ be two independent edges of G . In view of Lemma 4.3.6, we may assume that they are of the same type. If they are of type n , then they are contained in the perfect matching consisting of all edges of type n . Thus we need only to consider the case when they are of type 1. Without loss of generality, we may assume that $a = 0$, $b = 1$ and $d = c+1$. If c is even, then e_1 and e_2 are contained in the perfect matching consisting of all edges $x(x+1)$, where $x = 0, 2, 4, \dots, 2(n-1)$. So, let c be odd. Then there exists an even integer $y \in \{2, 3, \dots, c-1\}$, such that e_1 lies on the cycle $C = 012\dots y(y+n)(y+n+1)\dots(2n-1)0$ of length $n+1$ and e_2 lies on the path $P = (y+1)(y+2)\dots(y+n-1)$. (In fact $y = 2$ or $n-1$ depending on c .) As C is an even cycle, it has a perfect matching containing e_1 . Also, as P is of even order and $y+1$ is odd, P has a perfect matching M_2 containing e_2 . Then $M_1 \cup M_2$ is a perfect matching of G containing e_1 and e_2 . \square

Let H be a spanning subgraph of a graph G . We called H an **even path factor** of G if each component of H is a path of even order.

Lemma 4.3.8 The Cayley graph $Z_{4n+2}(1, 4n+1, k, 4n+2-k)$, where $n \geq 1$ and $k < 2n+1$, is 2-extendable if and only if $k \neq 1, 2$ or $2n$.

Proof: Let $G = Z_{4n+2}(1, 4n+1, k, 4n+2-k)$. If $k = 1$, then G is not 2-extendable, by Lemma 4.3.4. If $k = 2$, let $e_1 = 01$ and $e_2 = 34$. Then 2 is an isolated vertex of G -

$V(\{e_1, e_2\})$ and there is no perfect matching in G containing e_1 and e_2 . If $k = 2n$, there is no perfect matching containing $e_1 = 01$ and $e_2 = (2n+1)(2n+2)$ as $(G - V(\{e_1, e_2\})) - \{3, 5, \dots, 2n-1, 2n+4, \dots, 4n\}$ consists of $2n$ isolated vertices (note $| \{3, 5, \dots, 2n-1, 2n+4, \dots, 4n\} | = 2n-2$).

Conversely, assume that $k \neq 1, 2, 2n$. Let $e_1 = ab$ and $e_2 = cd$, $a < b$ and $c < d$, be two independent edges of G . As G is vertex-transitive, we may assume that $a = 0$. By Lemma 4.3.6, we can assume that e_1 and e_2 are of the same type. We consider the following cases:

Case 1. k is odd.

If $e_1 = 01$, then $e_2 = c(c+1)$ and by vertex-transitivity we may assume that $c \leq 2n+1$. Assume first that c is even. Then $M = \{2i(2i+1) \mid i = 0, 1, \dots, 2n\}$ is a perfect matching containing e_1 and e_2 . If c is odd, then e_2 is contained in the even cycle $C = (c-1)c(c+1)\dots(c-1+k)(c-1)$ (which has a perfect matching M_2 containing e_2) and e_1 is contained in the even path $P = (c+k)(c+k+1)\dots(4n+2)012\dots(c-2)$ which has a perfect matching M_1 containing e_1 . Then $M_1 \cup M_2$ is a perfect matching of G containing e_1 and e_2 .

Next, let $e_1 = 0k$. Then $e_2 = c(c+k)$ and we have the following cases to consider.

Case 1.1. c is odd and $c < k$.

Let $M = \{e_1, e_2, (c+1)(c+k+1)\}$. Then $G - V(M)$ has an even path factor $12\dots(c-1), (c+2)(c+3)\dots(k-1), (k+1)(k+2)\dots(c+k-1)$ and $(c+k+2)(c+k+3)\dots(4n+1)$. Thus M can be extended to a perfect matching of G .

Case 1.2. c is odd and $c > k$.

Let $M = \{e_1, e_2, (c-1)(c+k-1), (c+1)(c+k+1)\}$. Then, as $G - V(M)$ has an even path factor $12\dots(k-1), (k+1)(k+2)\dots(c-2), (c+2)(c+3)\dots(c+k-2)$, and $(c+k+2)(c+k+3)\dots(4n+1)$, M can be extended to a perfect matching of G .

Case 1.3. c is even and $c < k$.

Let $M = \{e_1, e_2, (c-1)(c+k-1)\}$. Since $G-V(M)$ has an even path factor $12\dots(c-2), (c+1)(c+2)\dots(k-1), (k+1)(k+2)\dots(c+k-2)$, and $(c+k+1)(c+k+2) \dots(4n+1)$, M can be extended to a perfect matching of G .

Case 1.4. c is even and $c > k$.

Let $M = \{e_1, e_2\}$. Using the same idea as in the previous case, because $G-V(M)$ has an even path factor $12\dots(k-1), (k+1)(k+2)\dots(c-1), (c+1)(c+2)\dots(c+k-1)$, and $(c+k+1)(c+k+2)\dots(4n+1)$, M can be extended to a perfect matching of G .

Thus Case 1 is dealt with. We now suppose that k is even.

Case 2. k is even.

If $e_1 = 01$, then $d = c+1$ and again by vertex-transitivity we may assume that $c \leq 2n+1$. When c is even, $M = \{2i(2i+1) \mid i = 0, 1, \dots, 2n\}$ is a suitable perfect matching. If c is odd, let $M = \{01, (c-1)(c-1+k), c(c+1), (c+2)(c+k+2), (c+k)(c+k+1)\}$. Then M is a set of independent edges containing e_1 and e_2 and $G-V(M)$ has an even path factor $23\dots(c-2), (c+3)(c+4)\dots(c+k-2)$ and $(c+k+3)(c+k+4)\dots(4n+1)$. Thus M can be extended to a perfect matching of G .

Finally, let $e_1 = 0k$. We then have the following subcases to consider. In each subcase, as in the subcases of Case 1, we construct a matching M so that $G-V(M)$ has an even path factor and then M can be extended to a perfect matching of G .

Case 2.1. c is odd and $c < k$.

In this case, let $M = \{e_1, e_2\}$ and it is easy to see that $G-V(M)$ has an even path factor.

Case 2.2. c is odd and $c > k$.

As $k \neq 2n$ we have either $c-k \geq 3$ or $4n-c-k+2 \geq 3$. By vertex-transitivity we may assume the former holds. Let $M = \{e_1, e_2, (c-1)(c+k-1), 1(k+1)\}$. Then M can be extended to a perfect matching of G .

Case 2.3. c is even and $c < k$.

Let $M = \{e_1, e_2, (c-1)(c+k-1), (c+1)(c+k+1)\}$. As in Case 2.1 M can be extended to a perfect matching of G .

Case 2.4. c is even and $c > k$.

Let $M = \{e_1, e_2, (c+1)(c+k+1), 1(k+1)\}$. Again M can be extended to a perfect matching of G . \square

Lemma 4.3.9 The Cayley graph $Z_{4n}(1, 4n-1, k, 4n-k)$, $1 \leq k \leq 2n$, $n \geq 2$, is 2-extendable if and only if $k \neq 1, 2, 2n$.

Proof: Let $G = Z_{4n}(1, 4n-1, k, 4n-k)$. If $k = 1$, then G is not 2-extendable by Lemma 4.3.4. If $k = 2$, then G is not 2-extendable as there is no perfect matching containing 01 and 34 . If $k = 2n$, then G is not 2-extendable by Lemma 4.3.7.

Conversely, assume that k is different from $1, 2$ and $2n$. Let $e_1 = ab$ and $e_2 = cd$, $a < b$ and $c < d$, be two independent edges of G . As G is vertex-transitive, we may let $a = 0$. By Lemma 4.3.6, we may assume that e_1 and e_2 are of the same type t , $t \in \{1, k\}$. We have the following cases to consider.

Case 1. $t = 1$.

By vertex-transitivity, we may assume that $c \leq 2n$. If c is even, then $\{01, 23, \dots, (4n-2)(4n-1)\}$ is a perfect matching containing e_1 and e_2 . Assume that c is odd. If k is odd, let $M = \{e_1, e_2, (c-1)(c+k-1)\}$. Then $G-V(M)$ has an even path factor and so M can be extended to a perfect matching of G . On the other hand, if k is even, let $M = \{e_1, e_2, (c-1)(c+k-1), (c+2)(c+k+2)\}$. Then, for the same reason as in the previous case, M can be extended to a perfect matching of G .

Case 2. $t = k$.

In each subcase we construct a matching M containing $e_1 = 0k$ and $e_2 = c(c+k)$ such that $G-V(M)$ has an even path factor from which it follows that M can be extended to a perfect matching of G . If k is odd, let

$$M = \begin{cases} \{e_1, e_2, (c+1)(c+k+1)\} & \text{if } c \text{ is odd and } c < k; \\ \{e_1, e_2, (c-1)(c+k-1), (c+1)(c+k+1)\} & \text{if } c \text{ is odd and } c > k; \\ \{e_1, e_2, (c-1)(c+k-1)\} & \text{if } c \text{ is even and } c < k; \\ \{e_1, e_2\} & \text{if } c \text{ is even and } c > k. \end{cases}$$

If k is even, let

$$M = \begin{cases} \{e_1, e_2\} & \text{if } c \text{ is odd and } c < k; \\ \{e_1, e_2, (c-1)(c+k-1), 1(k+1)\} & \text{if } c \text{ is odd and } c \geq k+3; \\ \{e_1, e_2, (c+1)(c+k+1), (4n-1)(k-1)\} & \text{if } c \text{ is odd and } c = k+1; \\ \{e_1, e_2, (c-1)(c+k-1), (c+1)(c+k+1)\} & \text{if } c \text{ is even and } c < k; \\ \{e_1, e_2, (c+1)(c+k+1), 1(k+1)\} & \text{if } c \text{ is even and } c > k. \end{cases}$$

It is easy to see that M is as required. The proof is now complete. \square

Lemma 4.3.10 The Cayley graph $Z_{2n}(1, 2n-1, 2, 2n-2, n-1, n+1)$, $n \geq 4$, is 2-extendable.

Proof: Let $G = Z_{2n}(1, 2n-1, 2, 2n-2, n-1, n+1)$ and let $e_1 = ab$, $e_2 = cd$ be two independent edges of G with $a < b$ and $c < d$. As G is vertex-transitive, we may assume that $a = 0$ and by Lemma 4.3.6, we may assume that e_1 and e_2 are of the same type t .

Case 1. $t = 1$.

By the vertex-transitivity of G , we may assume that $b = 1$ and $c \leq n$. If n is even, then by Lemma 4.3.9 the spanning subgraph $Z_{2n}(1, 2n-1, n-1, n+1)$ of G is 2-extendable and so e_1 and e_2 can be extended to a perfect matching of G . Suppose then that n is odd. If c is even, then e_1 and e_2 are contained in the perfect matching $\{01, 23, \dots, (2n-2)(2n-1)\}$ of G . On the other hand, if c is odd, let $M = \{e_1, e_2, (c-1)(c+n-2), (c+n-3)(c+n-1)\}$. Then M can be extended to a perfect matching of G , as $G-V(M)$ has an even path factor: $23\dots(c-2)$, $(c+2)(c+3)\dots(c+n-4)$ and $(c+n)(c+n+1)\dots(2n-1)$.

Case 2. $t = 2$.

Again by vertex-transitivity, we may assume that $b = 2$ and $c \leq n$. If $c = 1$, then the vertices of G which are not on e_1 or e_2 lie on the even path $45\dots(2n-1)$ and so G has perfect matching containing e_1 and e_2 . Let $c > 1$ and recall that $n \geq 4$. According to the parity of c , we construct a matching M so that $G-V(M)$ has an even path factor and hence M can be extended to a perfect matching of G .

$$M = \begin{cases} \{e_1, e_2, 1(2n-1), (c+1)(c+3)\} & \text{if } c \text{ is odd} \\ \{e_1, e_2, 1(2n-1), (c-1)(c+1)\} & \text{if } c \text{ is even} \end{cases}$$

Case 3. $t = n-1$.

We may assume that $b = n-1$ and $c \leq n$ by vertex-transitivity of G . If n is even, then by Lemma 4.3.9, the spanning subgraph $Z_{2n}(1, 2n-1, n-1, n+1)$ of G is 2-extendable and so G has a perfect matching containing e_1 and e_2 . When n is odd, we have several subcases to discuss. For each subcase, as in the proof of Case 2 of Lemma 4.3.9, we construct a matching M containing e_1 and e_2 such that $G-V(M)$ has an even path factor, and hence M can be extended to a perfect matching of G . Assuming $e_1 = 0(n-1)$. The desired matching M is as follows:

$$M = \begin{cases} \{e_1, e_2, (c-1)(c+1), (c+n-2)(c+n)\} & \text{if } c \text{ is even and } c < n-1; \\ \{e_1, e_2\} & \text{if } c \text{ is odd and } c < n-1; \\ \{e_1, e_2, (n-2)(2n-3), (2n-4)(2n-2)\} & \text{if } c \text{ is odd and } c = n. \quad \square \end{cases}$$

Lemma 4.3.11 The Cayley graph $Z_{2n}(1, 2n-1, 2, 2n-2, n)$, $n \geq 3$, is 2-extendable.

Proof: Let $G = Z_{2n}(1, 2n-1, 2, 2n-2, n)$ and let $e_1 = ab$, $e_2 = cd$ be two independent edges of G with $a < b$ and $c < d$. By the vertex-transitivity of G and Lemma 4.3.6, we may assume that $e_1 = 0t$, $e_2 = c(c+t)$ and $c \leq n$.

Case 1. $t = 1$.

If c is even, then e_1 and e_2 are contained in the perfect matching $\{01, 23, \dots, (2n-2)(2n-1)\}$ of G . If both c and n are odd, let $M = \{e_1, e_2, 2(n+2)\}$. Then M can be extended to a perfect matching of G as $G-V(M)$ has an even path factor $34\dots(c-1), (c+2)(c+3)\dots(n+1)$ and $(n+3)(n+4)\dots(2n-1)$. If c is odd and n is even, let

$$M = \begin{cases} \{e_1, e_2, 2(n+2), (n+1)(n+3)\} & \text{if } c = 3 \\ \{e_1, e_2, 24, 3(n+3)\} & \text{if } c \geq 5 \end{cases}$$

Then $G-V(M)$ has an even path factor $56\dots n, (n+4)(n+5)\dots(2n-1)$ (if $c = 3$) or $56\dots (c-1), (c+2)(c+3)\dots(n+2), (n+4)(n+5)\dots(2n-1)$ (if $c \geq 5$).

Case 2. $t = 2$.

If $c = 1$, then $e_1 = 02$ and $e_2 = 13$ are contained in the perfect matching $\{02, 13, 45, \dots, (2n-2)(2n-1)\}$ of G . If $c > 1$, recall that $n \geq 3$. With respect to the parity of c , we construct a matching M so that $G-V(M)$ has an even path factor and hence M can be extended to a perfect matching of G .

$$M = \begin{cases} \{e_1, e_2, 1(2n-1), (c+1)(c+3)\} & \text{if } c \text{ is odd} \\ \{e_1, e_2, 1(2n-1), (c-1)(c+1)\} & \text{if } c \text{ is even} \end{cases}$$

Case 3. $t = n$.

The set of all edges of type n in G form a perfect matching of G which contains e_1 and e_2 . \square

Now we are ready to prove the main theorem.

Proof of Theorem 4.3.1: We first assume that $G = G(\Gamma; S)$ is isomorphic to one of the given graphs.

The graph $Z_{2n}(1, 2n-1)$, $n \geq 3$ is not 2-extendable, by Lemma 4.3.4.

By Lemmas 4.3.8 and 4.3.9 the graph $Z_{2n}(1, 2, 2n-1, 2n-2)$, $n \geq 3$ is not 2-extendable.

The graph $Z_{4n}(1, 4n-1, 2n)$, $n \geq 2$ is not 2-extendable, by Lemma 4.3.7.

If $G = Z_{4n+2}(2, 4n, 2n+1)$, $n \geq 1$, then G is isomorphic to $C_{2n+1} \times P_2$ and so by Lemma 4.2.7 is not 2-extendable.

Finally, the graph $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$, $n \geq 1$ is not 2-extendable, by Lemma 4.3.8.

Conversely, assume that G is not isomorphic to any of the listed graphs. We shall prove that G is 2-extendable.

If G is regular of degree 2, then it must be a 4-cycle and so is 2-extendable.

If G is regular of degree 3, then $|S| = 3$ and we let $S = \{a, b, c\}$. If a, b and c are of order 2, then, as G is connected, G is isomorphic to the complete graph K_4 or the cube $C_4 \times P_2$ and is so 2-extendable. Otherwise, as $S = -S$ we may assume that $a+b = 0$ and $c+c = 0$. If $c \notin \langle a \rangle$, then $G \cong C_m \times P_2$ where $m = o(a)$. By hypothesis ($G \cong Z_{4n+2}(2, 4n, 2n+1)$), m must be even, and so by Corollary 4.2.3, G is 2-extendable. On the other hand, if $c \in \langle a \rangle$, then a must be of even order $2n$ and $c = na$. Hence $G \cong Z_{2n}(1, 2n-1, n)$ and so (by hypothesis) n must be odd. Hence G is 2-extendable, by Lemma 4.3.7.

If G is regular of degree 4, let $S = \{a, b, c, d\}$. Suppose a, b, c and d are of order 2, then G is isomorphic to one of $K_4 \times P_2$, $K_{4,4}$, or $C_4 \times C_4$ and so is 2-extendable.

Suppose a and b are of order 2 and $c+d = 0$. Let e_1 and e_2 be independent edges of G . By Lemma 4.3.6, we may assume that e_1 and e_2 are the same type. If they are of type a or b , then the set of all edges of type a or of type b will be a suitable perfect matching. Thus, we may assume that they are of type c .

If $a \in \langle c \rangle$, then c must be of even order $2n$, $a = nc$ and $b \notin \langle c \rangle$. Hence G has a spanning subgraph $G(\Gamma; \{b, c\})$ which is isomorphic to $C_{2n} \times P_2$ and contains the edges e_1 and e_2 . By Corollary 4.2.4, G has a perfect matching containing e_1 and e_2 . Assume that neither a nor b is in $\langle c \rangle$. If c is of odd order k , then G is isomorphic to $C_k \times P_2 \times P_2$ which, since $C_k \times P_2$ is 1-extendable, is 2-extendable by Corollary 4.2.6.

If c is of even order, either $G \cong C_{2n} \times P_2 \times P_2$ and we are done as in the odd case, or G has a spanning subgraph H , $H \cong C_{2n} \times P_2$, which contains both e_1 and e_2 and again we are done as $C_{2n} \times P_2$ is 2-extendable.

Finally, assume that $a+b = 0$ and $c+d = 0$. At least one of a and c (say a) is of even order $2n$ (since G has even order). If $c \notin \langle a \rangle$, then for any two independent edges e_1 and e_2 of type c in G , there exists a spanning subgraph of G which is isomorphic to $C_{2n} \times P_m$, $m \geq 2$, and contains e_1 and e_2 . By Corollary 4.2.4, G has a perfect matching containing e_1 and e_2 . If $c \in \langle a \rangle$, then $c = ta$, $t \notin \{1, 2, n, n-1\}$ (as $G \cong Z_{2n}(1, 2, 2n-1, 2n-2)$) and $G(\langle a, c \rangle; \{a, b, c, d\}) \cong Z_{2n}(1, 2n-1, t, 2n-t)$. Therefore by Lemmas 4.3.8 and 4.3.9, G is 2-extendable.

Hence, we may assume that G is regular of degree at least 5. Let e_1 and e_2 be any two independent edges of G . As usual, by Lemma 4.3.6, we need only to consider the case when e_1 and e_2 are of the same type (say a). As G is vertex-transitive, we may assume that $e_1 = 0a$. Now, we need only to consider the following two cases.

Case 1. a is of order $2n$.

If $n = 1$, then all the edges of type a in G will be a suitable perfect matching of G . Hence, we may let $n \geq 2$. Then the set of all edges of type a forms a spanning subgraph of G which is the disjoint union of $2n$ -cycles. If e_1 and e_2 lie on two distinct cycles, then clearly they can be extended to a perfect matching of G . Otherwise they are on the same cycle and we let $e_2 = (ta)((t+1)a)$, $1 < t < 2n-1$. If $\langle a \rangle \neq \Gamma$, then there exists $b \in S$ such that $b \notin \langle a \rangle$. Then if the order of b is m , $m \geq 2$, $G(\langle a, b \rangle; \{a, -a, b, -b\}) \subseteq G$ has a spanning subgraph H isomorphic to $C_{2n} \times P_m$ and containing e_1 and e_2 . By Corollary 4.2.4, e_1 and e_2 can be extended to a perfect matching of H , which in turn can be extended to a perfect matching of G by Lemma 4.3.5. Finally, we assume that $\langle a \rangle = \Gamma$ (so $b \in \langle a \rangle$ for all $b \in S$); that is, G is a circulant. As G is of degree at least 5, we have $|S| \geq 5$. If $S = \{a, (2n-1)a, 2a, (2n-2)a, (n-1)a, (n+1)a\}$, then $G \cong Z_{2n}(1, 2n-1, 2, 2n-2, n-1, n+1)$ and by Lemma 4.3.10 G is 2-extendable.

Otherwise, there exists $b \in S - \{a, (2n-1)a, 2a, (2n-2)a, (n-1)a, (n+1)a\}$. If n is odd, then Lemmas 4.3.7 and 4.3.8 imply that the subgraph induced by the set of all edges of types a and b is a 2-extendable spanning subgraph of G . Hence e_1 and e_2 can be extended to a perfect matching of G . On the other hand, if n is even, by Lemma 4.3.11, we may assume $S \neq \{1, 2n-1, 2, 2n-2, n\}$. Thus there exists an element of S other than $a, (2n-1)a, 2a, (2n-2)a$ and na . Then from Lemma 4.3.9, the subgraph induced by the set of all edges of types a and c is a 2-extendable spanning subgraph of G . Hence e_1 and e_2 can be extended to a perfect matching of G .

Case 2. a is of order $2n+1$.

As Γ is of even order, there exists an element $b \in S$ of even order and so $b \notin \langle a \rangle$. Let m be the smallest positive integer such that $mb \in \langle a \rangle$. Then m is even. Let H be the subgraph of G induced by the vertex-set $\{ib + \langle a \rangle \mid i = 0, 1, \dots, m-1\}$, so M is $G(\langle a, b \rangle; \{a, -a, b, -b\})$.

If $m \geq 4$, let H_i be the subgraph of H induced by all the edges of type a and b on the vertex-set $\{ib + \langle a \rangle\} \cup \{(i+1)b + \langle a \rangle\}$, $i = 0, 2, 4, \dots, m-2$. Clearly $H_i \cong C_{2n+1} \times P_2$ and is thus 1-extendable. If e_1 and e_2 are in different H_i , there is clearly (by Lemma 4.3.5) a perfect matching of H containing them, and hence there is such a perfect matching in G . Otherwise, e_2 is an edge in H_0 . If $e_2 = (sa)((s+1)a)$ for some s , $1 < s < 2n$, let $M = \{e_1, e_2, b(b+a), (b+sa)(b+(s+1)a)\} \cup \{(ka)(b+ka) \mid k \neq 0, 1, s, s+1\}$. Then M is a perfect matching of H_0 containing e_1 and e_2 . So we can extend M to a perfect matching of H and then to one of G , by Lemma 4.3.5. Next, let $e_2 = (b+sa)(b+(s+1)a)$ for some $1 \leq s \leq 2n$. If $|\{0, 1\} \cap \{s, s+1\}| = 0$, then $M = \{e_1, e_2, b(b+a), (sa)((s+1)a)\} \cup \{(ka)(b+ka) \mid k \neq 0, 1, s, s+1\}$ is a perfect matching of H_0 containing e_1 and e_2 . We can then extend M to a perfect matching of H and hence to one of G . If $|\{0, 1\} \cap \{s, s+1\}| \neq 0$, then we have either $\{0, 1\} = \{s, s+1\}$ or $|\{0, 1\} \cap \{s, s+1\}| = 1$. In the former case, $M = \{e_1, e_2\} \cup \{(ka)(b+ka) \mid k \neq 0, 1\}$ is a perfect matching of H_0 containing e_1 and e_2 . We can then extend M to a perfect

matching of H and hence to one of G . In the latter case, by vertex-transitivity we may assume $s = 1$ and we consider the subgraph H . Let $(m-1)b+ta$ be the vertex in H_{m-1} such that $mb+ta = 2a$. Let $M = \{e_1, e_2, (2a)((m-1)b+ta), b(2b)\} \cup \{(ka)(b+ka) \mid k \neq 0, 1, 2\}$. Then M can be extended to a perfect matching of H since the subgraphs induced by the vertex-sets $\{2b+\langle a \rangle\} - \{2b\}$ and $\{(m-1)b+\langle a \rangle\} - \{(m-1)b+ta\}$ have even path factors and the subgraph induced by the vertex-set $\{ib+\langle a \rangle \mid i = 3, 4, \dots, m-2\}$ is isomorphic to $C_{2n+1} \times P_{2r}$, where $r = (m-4)/2$, and hence by Lemma 4.3.5 to a perfect matching of G containing e_1 and e_2 .

Finally, we need only to consider the case when $m = 2$. Following the argument in the case $m \geq 4$ we may assume $e_2 = (b+a)(b+2a)$. If $\langle a, b \rangle \neq \Gamma$, then there exists $c \in S-\langle a, b \rangle$ (recall that $\langle a, b \rangle = V(H)$). Let K be the subgraph of $G(\langle a, b, c \rangle; \{a, -a, b, -b, c, -c\})$ induced by the vertices $\langle a, b \rangle \cup (c+\langle a, b \rangle)$. Then K has the following perfect matching which contains e_1 and e_2 :

$$M = \{e_1, e_2, c(c+a), (c+b+a)(c+b+2a), (2a)(2a+c), b(b+c)\} \\ \cup \{(g)(g+c) \mid g \in V(H) - \{0, a, 2a, b, a+b, 2a+b\}\}.$$

And M can be extended to a perfect matching of G , as the subgraph of G induced by the set of all vertices of $V(G) - V(K)$ contains a spanning subgraph which is the disjoint union of copies of H (which has a perfect matching). Therefore it remains to consider the case when $\langle a, b \rangle = \Gamma$. As G is regular of degree at least 5, there exists $c \in S - \{a, -a, b, -b\}$. Since $m = 2$, $V(G) = \langle a \rangle \cup (b+\langle a \rangle)$ and $c = ta$ or $b+ta$ for some $1 < t \leq n$.

If $c = ta$ with $t \geq 3$, then $M = \{e_1, e_2, (2a)((2+t)a), (ta)((t+1)a), (b+(t+1)a)(b+(t+2)a), b(b+ta)\} \cup \{(sa)(b+sa) \mid s \neq 0, 1, 2, t, t+1, t+2\}$ is a perfect matching of G containing e_1 and e_2 . If $c = 2a$, then $M = \{e_1, e_2, (2a)(4a), (3a)(b+3a)\}$ is a set of independent edges containing e_1 and e_2 so that $G - V(M)$ has an even path factor. Thus G has a perfect matching containing e_1 and e_2 . This leaves us with the case $c = b+ta$. As $c \neq b$, then $t \neq 0$ and as $c \neq -b$, then $2b \neq 2tna$. If $t \geq 3$, then $M =$

$\{e_1, e_2, c(a+c), (b+a-c)(b+2a-c)\} \cup \{x(x+c) \mid x \in \langle a \rangle - \{0, a, b+a-c, b+2a-c\}\}$ is a perfect matching of G containing e_1 and e_2 . If $t = 1$, $M = \{e_1, e_2\} \cup \{x(x+c) \mid x \in \langle a \rangle - \{0, a\}\}$ is a perfect matching of G containing e_1 and e_2 . Finally, we assume $t = 2$. Then $2b \neq 4na = (2n-1)a$. If also $2b \neq 0$, then $M = \{e_1, e_2, (2b+a)(2b+2a), (-b)(a-b)\} \cup \{y(y+b) \mid y \in (b+\langle a \rangle) - \{b+a, b+2a, -b, a-b\}\}$ is a perfect matching of G containing e_1 and e_2 . Finally, if $2b = 0$, then $M = \{e_1, e_2, (3a)(4a), (-c)(a-c)\} \cup \{y(y+c) \mid y \in (b+\langle a \rangle) - \{b+a, b+2a, -c, a-c\}\}$ (where $c = b+2a$) is a suitable perfect matching of G containing e_1 and e_2 . \square

To end this section, we would like to raise the following problems.

Problem 4.3.11. Characterize all 3-extendable abelian Cayley graphs and, in general, all k -extendable abelian Cayley graphs.

Lovász and Plummer (see [44]) proved that every vertex-transitive graph of even order is 1-extendable. Since Cayley graphs are vertex-transitive, any Cayley graph of even order is 1-extendable. It is not known which Cayley graphs are 2-extendable.

Problem 4.3.12. Characterize all 2-extendable Cayley graph.

§4.4. Classification of 2-extendable generalized Petersen graphs.

Recall that the **generalized Petersen graph** $GP(p, k)$ ($p > k$) has vertex-set $U \cup V$ where $U = \{u_0, u_1, \dots, u_{p-1}\}$ and $V = \{v_0, v_1, \dots, v_{p-1}\}$, and edge-set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+k} \mid i = 0, 1, \dots, p-1\}$, where all subscript arithmetic is performed modulo p . One

easily sees that $GP(p, k)$ is 1-extendable. The study of 2-extendability of these graphs was begun by Schrag and Cammack [52, 53] who gave necessary and sufficient conditions for the 2-extendability of $GP(p, k)$, when $1 \leq k \leq 7$. In this section, we shall prove the following main result.

Theorem 4.4.1 For $k \geq 3$, the generalized Petersen graph $GP(p, k)$ is 2-extendable if and only if $p \neq 2k$ or $3k$.

For convenience, we call the edge $u_i v_i$ a **spoke** and the edge $v_i v_{i+k}$ a **chord**.

The next result is easily deduced from the definition of $GP(p, k)$.

Theorem 4.4.2 For any positive integers p, k with $p > k \geq 3$:

- (i) $GP(p, k) \cong GP(p, p-k)$; and
- (ii) $GP(p, k)$ has a triangle if and only if $p = 3k$.

Proof: (i) This follows directly from the definition of $GP(p, k)$.

(ii) Suppose $p > k \geq 3$. If $GP(p, k)$ has a triangle, then the triangle must occur on the vertices of V . Thus there exist r, s , and t with $r \leq s \leq t$ so that $v_r v_s$, $v_s v_t$, and $v_t v_r$ are edges of $GP(p, k)$. But $s-r \equiv t-s \equiv r-t \equiv k \pmod{p}$ and thus $p = 3k$. On the other hand, if $p = 3k$, then we have a triangle on the vertex-set $\{v_0, v_k, v_{2k}\}$. \square

As a consequence of Theorem 4.4.2 (i), we will henceforth assume $p \geq 2k$.

Next, we quote three results from Schrag and Cammack [52, 53] which we present as lemmas.

Lemma 4.4.3 (Schrag and Cammack [53]) For all $k \geq 2$, $GP(2k, k)$ and $GP(3k, k)$ are not 2-extendable.

Lemma 4.4.4 (Schrag and Cammack [52]) If $k \geq 4$, then any pair of independent edges of $GP(p, k)$, at least one of which is a spoke, can be extended to a perfect matching.

Lemma 4.4.5 (Schrag and Cammack [52, 53]) If $3 \leq k \leq 7$, $GP(p, k)$ is 2-extendable if and only if $p \neq 2k$ or $3k$.

In order to prove Theorem 4.4.1, we need to show that for any two independent edges e_1, e_2 of $GP(p, k)$ ($p \neq 2k$ or $3k$) there exists a perfect matching containing e_1 and e_2 . Depending on the location of e_1 and e_2 , we consider the following six cases:

- (1) Both e_1 and e_2 are spokes.
- (2) e_1 is a chord and e_2 is a spoke.
- (3) e_1 has both end-vertices in U and e_2 is a spoke.
- (4) Both e_1 and e_2 are chords.
- (5) e_1 has both end-vertices in U and e_2 is a chord.
- (6) Both e_1 and e_2 have their end-vertices in U .

From Lemmas 4.4.4 and 4.4.5, we need only consider $k \geq 8$ and the non-spoke cases (4), (5) and (6). In the proof of Theorem 4.4.1, we will study these cases separately and show that in each case we can always find a perfect matching containing both e_1 and e_2 . For convenience, we introduce a lemma and some notation.

Let $S(p, k)$ denote the set of all spokes of $GP(p, k)$. Given a chord $v_i v_{i+k}$, we call the two chords $v_{i-1} v_{i-1+k}$ and $v_{i+1} v_{i+1+k}$ the **neighbour chords** of $v_i v_{i+k}$. By the definition of $GP(p, k)$, $G[U]$ is an p -cycle and $G[V]$ is a 2-regular graph (we suppose $p \neq 2k$). An even cycle C in $GP(p, k)$ is called **alternating** if the edges of C appear alternately in $\{u_i v_i \mid 0 \leq i \leq p-1\}$ and $\{u_i u_{i+1}, v_i v_{i+k} \mid 0 \leq i \leq p-1\}$.

Lemma 4.4.6 Consider the graph $GP(p, k)$, $p \neq 2k, 3k$, and $k \geq 3$:

(i) If two edges e_1, e_2 of $\{u_i u_{i+1}, v_i v_{i+k} \mid 0 \leq i \leq p-1\}$ are in an alternating cycle, then there exists a perfect matching in $GP(p, k)$ containing e_1 and e_2 .

(ii) Let C_1 and C_2 be vertex-disjoint alternating cycles in $GP(p, k)$. If $e_i \in C_i$ ($i = 1, 2$), then there exists a perfect matching containing e_1 and e_2 .

Proof: (i) Let C be an alternating cycle in $GP(p, k)$. Then $F = C - \{u_i v_i \mid u_i v_i \in C\} \cup \{S(p, k) - E(C)\}$ is a perfect matching containing e_1 and e_2 .

(ii) Let F_1, F_2 be perfect matchings in C_1, C_2 , respectively, which contain e_1, e_2 . Then $F = F_1 \cup F_2 \cup \{S(p, k) - E(C_1) - E(C_2)\}$ is a perfect matching containing e_1 and e_2 . \square

We now turn to prove the main theorem.

Proof of Theorem 4.4.1: By Lemma 4.4.3, we need only to show that if $p \neq 2k$ or $3k$, then for any two independent edges e_1, e_2 in $GP(p, k)$ we can find some perfect matching containing them both. From Lemma 4.4.4 and Lemma 4.4.5 it suffices to consider $k \geq 8$ and the non-spoke cases as follows:

Case 1. Both e_1 and e_2 are chords.

Since e_1 and e_2 are independent chords, we may assume that $e_1 = v_0 v_k$ and $e_2 = v_i v_{i+k}$, where $\{0, k\} \cap \{i, i+k\} = \emptyset$. Without loss of generality, we suppose that e_1 and e_2 satisfy (a) $p > i+k > i > k > 0$ or (b) $p > i+k > k > i > 0$. (Note that if $i > k$ and $i+k > p$, we can relabel so that v_i becomes v_0 and then we are in case(b).)

Case 1.1. Suppose $p > i+k > i > k > 0$.

(1) If $i = k+1$ and $i+k = p-1$, then $p = 2k+2$. In this case, we have an alternating cycle $C = v_0 v_k u_k u_{k+1} v_{k+1} v_{p-1} u_{p-1} u_0 v_0$ and by Lemma 4.4.6 (i), we are done.

(2) If $i = k+1$ and $i+k = p-2$, then $p = 2k+3$. Since $k \geq 8$, we can construct a perfect matching F containing $e_1 = v_0 v_k$ and $e_2 = v_{k+1} v_{2k+1}$ as follows (see Figure 4.17):

$F = \{e_1, e_2, v_{k-3}v_{2k}, v_{k-2}v_{2k-2}, v_{k-1}v_{2k-1}, v_{k+2}v_{2k+2}\} \cup \{u_j u_{j+1} \mid j = k-3, k-1, k+1, 2k-2, 2k, 2k+2\} \cup \{S(p, k) - \{u_j v_j \mid k-3 \leq j \leq k+2 \text{ or } 2k-2 \leq j \leq p\}\}$.

(3) If $i = k+2$ and $i+k = p-1$, then $p = 2k+3$ and by isomorphism, this case is exactly the same as (2).

(4) If $i = k+1$ and $i+k < p-2$, then we obtain two disjoint alternating 8-cycles C_1 and C_2 with $e_i \in C_i$ ($i = 1, 2$), where $C_1 = u_0 v_0 v_k u_k u_{k-1} v_{k-1} v_{p-1} u_{p-1} u_0$ and $C_2 = u_i v_i v_{i+k} u_{i+k} u_{i+k+1} v_{i+k+1} v_{i+1} u_{i+1} u_i$. Applying Lemma 4.4.6 (ii) we are done.

(5) If $i > k+2$ and $i+k = p-1$, then by isomorphism this case is exactly the same as (4).

(6) If $i > k+1$ and $i+k < p-1$, then we can use the spokes and the neighbour chords of e_1 and e_2 to form a perfect matching F which contains e_1 and e_2 . The perfect matching F is as follows (see Figure 4.18):

$F = \{e_1, e_2, v_{p-1}v_{k-1}, v_{i-1}v_{i-1+k}\} \cup \{u_j u_{j+1} \mid j = k-1, i-1, i+k-1, p-1\} \cup \{S(p, k) - \{u_j v_j \mid j = k-1, k, i-1, i, i-1+k, i+k, p-1, 0\}\}$.

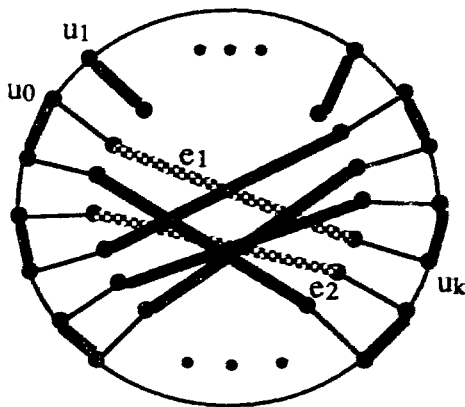


Figure 4.17

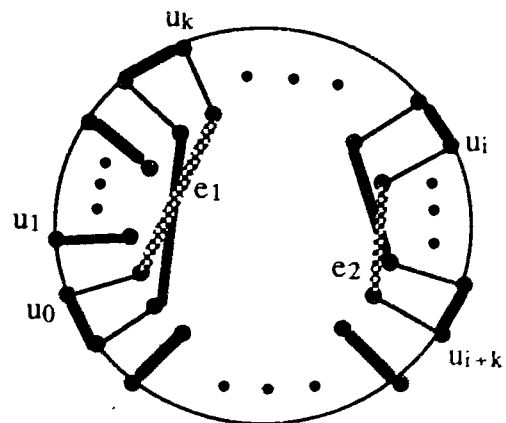


Figure 4.18

Case 1.2. Suppose $p > i+k > k > i > 0$.

(1) If $i = 1$, then $F = \{e_1, e_2\} \cup \{S(p, k) - \{u_j v_j \mid j = 0, 1, k, k+1\}\} \cup \{u_0 u_1, u_k u_{k+1}\}$ is a perfect matching containing $e_1 = v_0 v_k$ and $e_2 = v_1 v_{k+1}$.

(2) If $i = 2$, then as $k \geq 8$, $p \geq 16$ and there is at least one vertex between v_i and v_k . Thus we can use the spokes and the neighbour chords of e_1 and e_2 to form a perfect matching F containing e_1 and e_2 as follows (see Figure 4.19):

$$F = \{v_j v_{j+k} \mid 0 \leq j \leq 3\} \cup \{S(p, k) - \{u_j v_j \mid 0 \leq j \leq 3 \text{ or } k \leq j \leq k+3\}\} \cup \{u_j u_{j+1} \mid j = 0, 2, k, k+2\}.$$

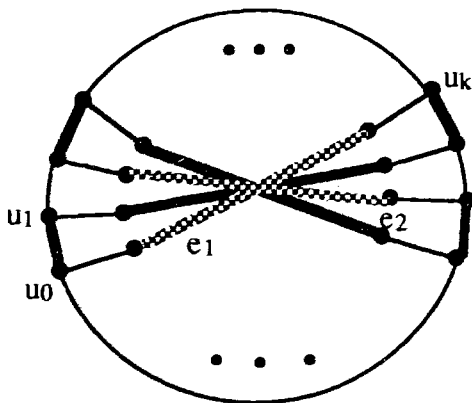


Figure 4.19

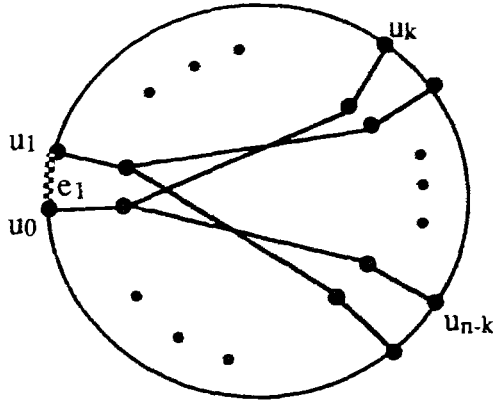
(3) If $i \geq 3$, then there are at least two vertices of V between v_0 and v_i . So we can find neighbour chords $v_1 v_{k+1}$ and $v_{i-1} v_{i+k}$ for e_1 and e_2 , respectively, to obtain two disjoint alternating 8-cycles $C_1 = u_0 u_1 v_1 v_{k+1} u_{k+1} u_k v_k v_0 u_0$ and $C_2 = u_{i-1} u_i v_i v_{i+k} u_{i+k} u_{i-1+k} v_{i-1+k} v_{i-1} u_{i-1}$ so that $e_i \in C_i$ ($i = 1, 2$). Now applying Lemma 4.4.6 (ii) we are done.

Case 2. e_1 lies in $G[U]$ and e_2 is a chord.

As before, we suppose $e_1 = u_0 u_1$ and $e_2 = v_i v_{i+k}$. Let $S_0 = \{v_0, v_1\}$, $S_1 = \{v_k, v_{k+1}\}$, $S_2 = \{v_{p-k}, v_{p-k+1}\}$ and $D = \{v_0 v_k, v_1 v_{k+1}, v_0 v_{p-k}, v_1 v_{p-k+1}\}$. Then $S_1 \cup S_2$ is a set of vertices which are adjacent to S_0 in V , and D is a set of chords incident with S_0 . Since $p \neq 2k$, then either $|S_1 \cap S_2| = 0$ (see Figure 4.20) or $|S_1 \cap S_2| = 1$ (see Figure 4.21). In the second case we must have $p = 2k+1$.

Case 2.1. Suppose e_2 is one of D . If $e_2 = v_0 v_k$ or $e_2 = v_1 v_{k+1}$, then $F = \{u_0 u_1, v_0 v_k, v_1 v_{k+1}, u_k u_{k+1}\} \cup \{S(p, k) - \{u_j v_j \mid j = 0, 1, k, k+1\}\}$ is a perfect matching

containing e_1 and e_2 . A similar perfect matching is constructed if $e_2 = v_1v_{p-k+1}$ or $e_2 = v_0v_{p-k}$.



Figures 4.20

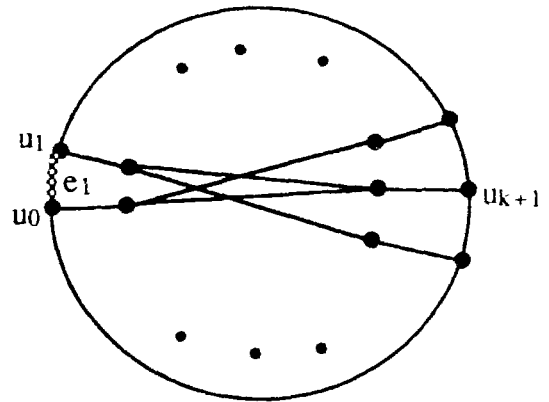


Figure 4.21

Case 2.2. Suppose that the end-vertices of e_2 are disjoint from $S_1 \cup S_2$.

(1) Suppose $|S_1 \cap S_2| = 0$. If $k = 3$ and $i = p-1$, then $C = u_0u_1v_1v_{p-2}u_{p-2}u_{p-1}v_{p-1}v_2u_2u_3v_3v_0u_0$ is an alternating cycle containing $e_1 = u_0u_1$ and $e_2 = v_{p-1}v_2$. Otherwise, there exists a neighbour chord of e_2 which has at most one end-vertex in $S_1 \cup S_2$ and no end-vertex in S_0 . Suppose that $v_{i+1}v_{i+k+1}$ is such a neighbour chord (if it is $v_{i-1}v_{i+k-1}$ essentially the same procedure applies). Since $S_1 \cap S_2 = \emptyset$, one of S_1 and S_2 is disjoint from $\{v_{i+1}, v_{i+k+1}\}$, say S_1 . Then $F = \{e_1 = u_0u_1, u_ku_{k+1}, v_0v_k, v_1v_{k+1}, u_iu_{i+1}, e_2 = v_i v_{i+k}, v_{i+1}v_{i+k+1}, u_{i+k}u_{i+k+1}\} \cup \{S(p, k) - \{u_jv_j \mid j = 0, 1, k, k+1, i, i+1, i+k, i+k+1\}\}$ is a suitable perfect matching.

(2) The case $|S_1 \cap S_2| = 1$ is quite similar to (1). There exists a neighbour chord of e_2 which has at most one end-vertex in $S_1 \cup S_2$. We construct a perfect matching containing e_1 and e_2 by the same argument as (1).

Case 2.3. Suppose the end-vertices of $e_2 = v_i v_{i+k}$ join exactly one of $S_1 \cup S_2$.

(1) When $S_1 \cap S_2 = \emptyset$, we may assume $p > p-k+1 > p-k > k+1 > k > 1$. By the symmetry of $GP(p, k)$, we need only consider the following two cases:

(a) $v_{i+k} = v_{p-k+1}$ (see Figure 4.22) and

(b) $v_{i+k} = v_{p-k}$ (see Figure 4.23).

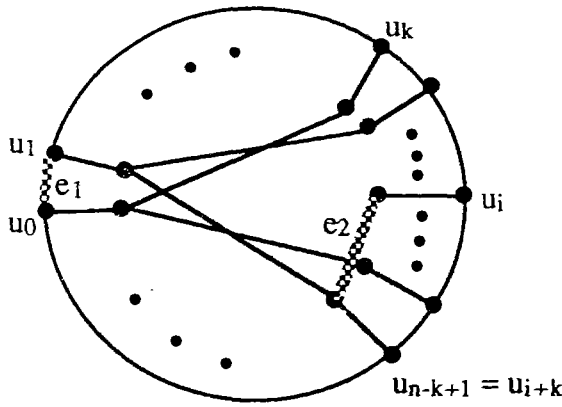


Figure 4.22

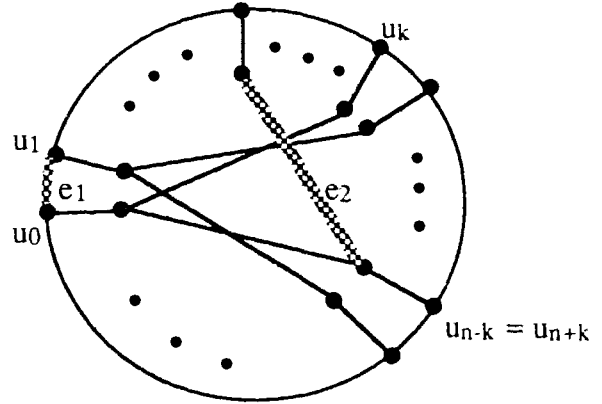


Figure 4.23

In case (a), since $i \notin S_1$, $p \neq 2k+1$ and $k \geq 8$, there exists a neighbour chord d of e_2 such that the end-vertices of d do not intersect $S_1 \cup S_0$ and we have the situation as in case 2.2 (1) and hence a perfect matching containing e_1 and e_2 .

For (b), consider the neighbour chord $v_{i+k+1}v_{i+1}$ of e_2 . If $v_{i+1} \notin S_1$, then we have the situation as above. If $v_{i+1} \in S_1$, then since $v_i \notin S_1$, $v_{i+1} = v_k$. Since $k \geq 8$, the neighbour chord $v_{i+k-1}v_{i-1}$ of e_2 is disjoint from $S_0 \cup S_1$ and we proceed as before.

(2) Suppose now that $|S_1 \cap S_2| = 1$, that is, $p = 2k+1$. Since e_2 has an end-vertex in $S_1 \cup S_2$, either $e_2 = v_2v_{k+2}$ or $e_2 = v_kv_{2k}$. By symmetry we assume $e_2 = v_2v_{k+2}$. From $k \geq 8$, then $d = v_3v_{k+3}$ is a neighbour chord of e_2 which has no end-vertex in $S_0 \cup S_1 \cup S_2$, so as before we are done.

Case 2.4. Suppose that both end-vertices of e_2 are in $S_1 \cup S_2$.

(1) For $|S_1 \cap S_2| = 0$, as $G[V]$ has no triangle, either $e_2 = v_{k+1}v_{p-k}$ or $e_2 = v_kv_{p-k+1}$ (see Figures 4.24 and 4.25). In both cases, we can find an alternating 12-cycle C containing e_1 and e_2 as follows:

$$\text{For } e_2 = v_{k+1}v_{p-k}, C = u_0v_0v_kv_ku_ku_{k+1}v_{k+1}v_{p-k}u_{p-k}u_{p-k+1}v_{p-k+1}v_1u_1u_0.$$

$$\text{For } e_2 = v_kv_{p-k+1}, C = u_0v_0v_{p-k}u_{p-k}u_{p-k+1}v_{p-k+1}v_kv_ku_ku_{k+1}v_{k+1}v_1u_1u_0.$$

We now apply Lemma 4.4.6 (i) to obtain a suitable perfect matching.

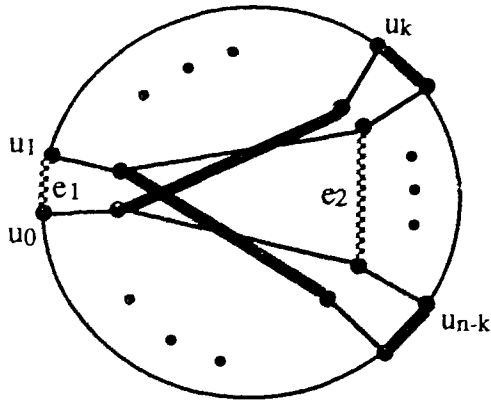


Figure 4.24

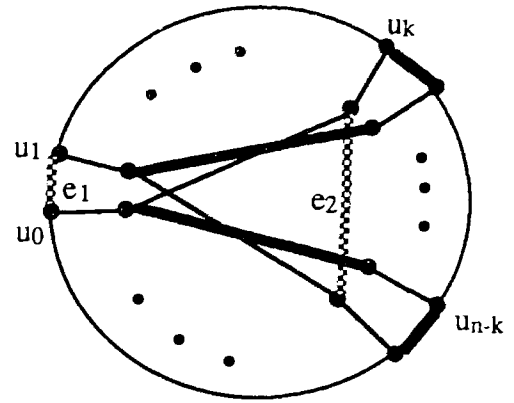


Figure 4.25

(2) If $|S_1 \cap S_2| = 1$, then both end-vertices of e_2 lying in $(S_1 \cup S_2) - (S_1 \cap S_2)$ implies $k = 2$; which is a contradiction.

Case 3. Both e_1 and e_2 lie in $G[U]$.

We may assume $e_1 = u_0u_1$ and $e_2 = u_iu_{i+1}$, where $1 < i \leq p/2$. Let $S_0 = \{v_0, v_1\}$, $S_1 = \{v_k, v_{k+1}\}$, $S_2 = \{v_{p-k}, v_{p-k+1}\}$, $T_0 = \{v_i, v_{i+1}\}$, $T_1 = \{v_{i+k}, v_{i+k+1}\}$, and $x = i-2 = d_{G[U]}(u_i, u_1) - 1$. Notice that $S_0 \cap T_0 = \emptyset$ and $S_1 \cap T_1 = \emptyset$. Except when $i = \lfloor p/2 \rfloor$ and $k = \lceil (p-2)/2 \rceil$ (in which case an alternating 8-cycle containing e_1 and e_2 is easily found), if $i \leq p/2$ and $k < p/2$, then $S_0 \cap T_1 = S_2 \cap T_0 = \emptyset$. Now we consider $S_1 \cap T_0$ as follows.

Case 3.1. If $x \geq k$ or $x \leq k-4$, then $S_1 \cap T_0 = \emptyset$. Thus we obtain two disjoint alternating cycles C_1 and C_2 :

$$C_1 = u_0u_1v_1v_{k+1}u_{k+1}u_kv_kv_0u_0.$$

$$C_2 = u_iu_{i+1}v_{i+1}v_{i+k+1}u_{i+k+1}u_{i+k}v_{i+k}v_iu_i.$$

By Lemma 4.4.6 (ii), there is a perfect matching containing e_1 and e_2 .

Three values of x are unaccounted for and we consider each separately.

Case 3.2. If $k = x+1 = i-1$, then $|S_1 \cap T_0| = 1$ and $e_2 = u_{k+1}u_{k+2}$.

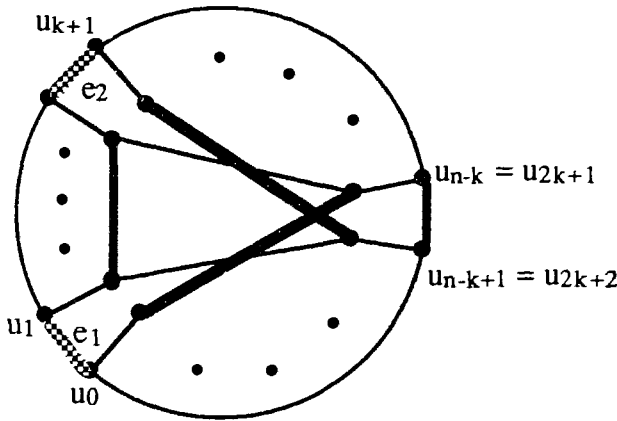
(1) If $|S_2 \cap T_1| = 2$, then $v_{p-k} = v_{2k+1}$ and $v_{p-k+1} = v_{2k+2}$ (see Figure 4.26). We now have an alternating cycle $C = u_0u_1v_1v_{k+1}u_{k+1}u_{k+2}v_{k+2}v_{2k+2}u_{2k+2}u_{2k+1}v_{2k+1}v_0u_0$

containing e_1 and e_2 and hence by Lemma 4.4.6 (i) we have the desired perfect matching.

(2) If $|S_2 \cap T_1| = 1$, then, as $G[V]$ has no triangle, $v_{2k+2} = v_{p-k}$ (see Figure 4.27). Thus $p = 3k+2$. Since $k \geq 8$, there are at least two vertices between v_1 and v_{k+1} , between v_{k+2} and v_{2k+1} , and between v_{2k+3} and v_0 . So we have an alternating cycle C containing e_1 and e_2 :

$$C = u_0 u_1 v_1 v_{k+1} u_{k+1} u_{k+2} v_{k+2} v_{2k+2} u_{2k+2} u_{2k+1} v_{2k+1} v_{3k+1} u_{3k+1} u_{3k} v_{3k} v_{2k} u_{2k} u_{2k-1} v_{2k-1} v_{k-1} u_{k-1} u_k v_k v_0 u_0.$$

By Lemma 4.4.6 (i), we are done.



Figures 4.26

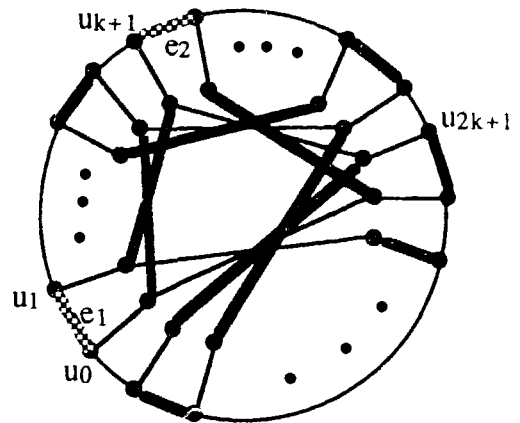


Figure 4.27

(3) If $|S_2 \cap T_1| = 0$, then S_2 and T_1 are disjoint. Since $S_2 \cap T_0 = S_0 \cap T_1 = \emptyset$ (so $v_{2k+2} = v_0$ or $v_{2k+2} = v_1$ is impossible), we easily find two alternating 8-cycles containing e_1 and e_2 , respectively. Applying Lemma 4.4.6 (ii) finishes the case.

Case 3.3. If $k = x+2 = i$, then $S_1 = T_0$ and $e_2 = u_k u_{k+1}$. Now $C = u_0 v_0 v_k u_k u_{k+1} v_{k+1} v_1 u_1 u_0$ is an alternating cycle containing e_1 and e_2 and from Lemma 4.4.6 (i), there is a perfect matching containing e_1 and e_2 .

Case 3.4. If $k = x+3 = i+1$, then $|S_1 \cap T_0| = 1$ and $e_2 = u_{k-1} u_k$.

(1) If $|S_2 \cap T_1| = 2$, then $v_{p-k} = v_{2k-1}$ and $v_{p-k+1} = v_{2k}$ (see Figure 4.28). We obtain an alternating cycle C containing e_1 and e_2 :

$$C = u_0 u_1 v_1 v_2 u_2 u_{2k-1} v_{2k-1} v_{k-1} u_{k-1} u_k v_k v_0 u_0.$$

By Lemma 4.4.6 (1), we are done in this case.

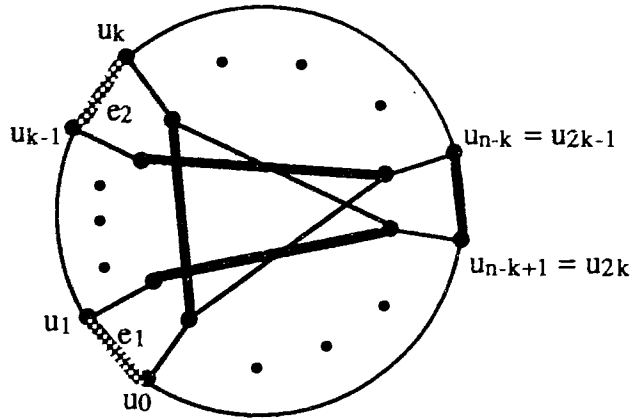


Figure 4.28

(2) If $|S_2 \cap T_1| = 1$, then, as $G[V]$ has no triangle, $v_{2k-1} = v_{p-k+1}$. Thus $p = 3k-2$ and the following perfect matching contains e_1 and e_2 (see Figures 4.29 and 4.30):

If k is even, $F = \{u_{k-2}v_{k-2}, u_0u_1, u_{k-1}u_k, u_{2k-1}u_{2k}, v_1v_{k+1}, v_0v_{2k-2}, v_{k-1}v_{2k-1}, v_kv_{2k}\} \cup \{S(p, k) - \{u_jv_j \mid 0 \leq j \leq 2k\}\} \cup \{v_jv_{j+k} \mid 2 \leq j \leq k-3\} \cup \{u_ju_{j+1} \mid j = 2, 4, \dots, k-4, k+1, k+3, \dots, 2k-3\}$.

If k is odd, $F = \{v_jv_{j+k} \mid 0 \leq j \leq k-1\} \cup \{u_ju_{j+1} \mid j = 0, 2, 4, \dots, 2k-2\} \cup \{u_jv_j \mid 2k \leq j \leq p-1\}$.

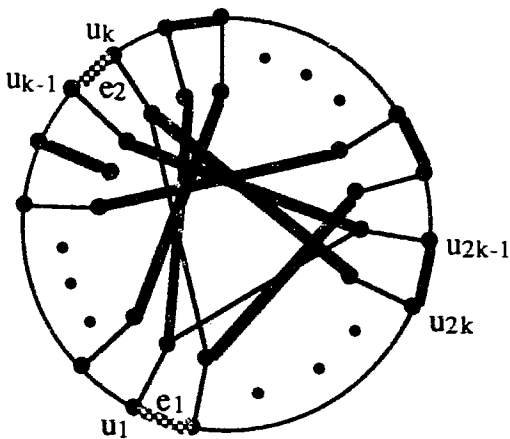


Figure 4.29

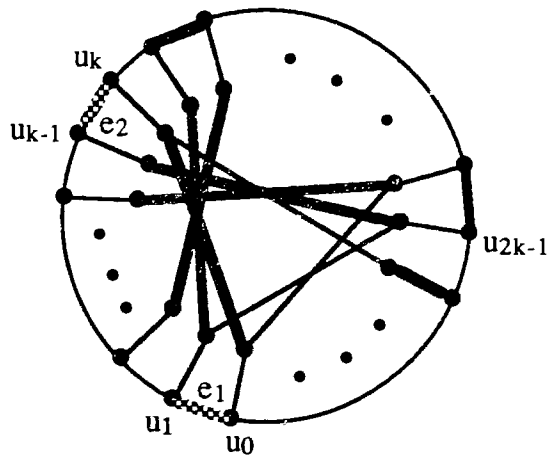


Figure 4.30

(3) The case $|S_2 \cap T_1| = 0$ follows exactly as case 3.2 (3).

The proof is now complete. \square

Chapter 5. Some results about factors.

§5.1. Introduction.

For a fixed integer k , let $S(k) = \{K_{1,i} \mid 1 \leq i \leq k\}$. An $S(k)$ -factor of a graph G is a spanning subgraph of G , each component of which is isomorphic to a member of $S(k)$. (Note that an $S(1)$ -factor is simply a 1-factor.) An $S(k)$ -factor is **proper** if one of its components is isomorphic to $K_{1,k}$. The complete bipartite graph $K_{1,k}$ is called a **k -star** (or simply a **star**). (So we will often call an $S(k)$ -factor a **star-factor**.)

In 1947, Tutte [54] gave a criterion for a graph to have an $S(1)$ -factor (that is, a 1-factor). This criterion was then used by others to study properties of graph with 1-factors. In particular, Lovász [40] showed that a graph with a unique 1-factor cannot have large minimum degree, and Heteyi (see [40]) determined the largest number of edges in a graph with a unique perfect matching. Lovász [40] and Zaks [66] gave a lower bound on the number of 1-factors in an n -connected graph.

We are interested in $S(k)$ -factors when $k \geq 2$. A characterization of $S(k)$ -factor for $k \geq 2$ was given by Las Vergnas [31], Hell and Kirkpatrick [23] and Amahashi and Kano [2] independently. In section 5.2, we study the structure of those graphs with a unique $S(k)$ -factor, $k \geq 2$; obtaining an upper bound on the number of edges such a graph can have, and constructing an extremal graph with a unique $S(k)$ -factor which attains that bound. In section 5.3, we show that any r -regular graph of order n has at least n distinct $S(k)$ -factors ($1 \leq k \leq r$).

The final section, section 5.4, contains a discussion of the extendability of power graphs. The n^{th} **power** G^n of a graph G has the same vertex set as G and two vertices are adjacent if and only if their distance in G is at most n . Chartrand et al. [14] showed that the square G^2 of a connected graph G contains a perfect matching if

and only if G has even order. We consider the extendability of power graphs and show that for any connected graph G , G^2 is elementary and G^3 is 1-extendable.

In 1956, Nordhaus and Gaddum [45] derived two inequalities for the chromatic number of a simple graph G of order n and its complement \bar{G} as follows:

$$\lceil 2\sqrt{n} \rceil \leq \chi(G) + \chi(\bar{G}) \leq n+1 \text{ and}$$

$$n \leq \chi(G) \cdot \chi(\bar{G}) \leq \lfloor (\frac{n+1}{2})^2 \rfloor.$$

This result has had considerable impact, and has been generalized and modified in various directions. Nordhaus-Gaddum type results concerning different graphic parameters, for example edge-colouring number, achromatic number, pseudoachromatic number, covering number, independent number, partition number etc. (see [13]), have been obtained by different researchers. In the last section of this chapter, we obtain a Nordhaus-Gaddum type of result for matchings.

§5.2. Graphs with a unique $S(k)$ -factor.

In order to study the structure of those graphs with a unique $S(k)$ -factor, we need to introduce certain notation.

In the star $K_{1,i}$, $i \geq 2$, we call the vertex of degree i the **centre**, and the vertices of degree 1 the **leaves**. In $K_{1,1}$ we arbitrarily prescribe one vertex to be the centre and the other the leaf.

Let F be an $S(k)$ -factor of G , and suppose that F has m_i components which are isomorphic to $K_{1,i}$, $1 \leq i \leq k$; implying that $\sum m_i(1+i) = |V(G)|$. We denote the centres of these m_i stars by $x(i, 1), x(i, 2), \dots, x(i, m_i)$, $1 \leq i \leq k$, and the leaves of the star with centre $x(i, j)$ by $y_1(i, j), y_2(i, j), \dots, y_i(i, j)$. So the components of F can be described by $S(i, j) = \{ \{x(i, j); y_1(i, j), \dots, y_i(i, j)\} \mid 1 \leq i \leq k, 1 \leq j \leq m_i \}$. For

convenience we write $x(1, j) = x_j$ and $y_1(1, j) = y_j$. Finally, we let S_c denote the set of all centres; $S_c = \{x(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq m_i\}$.

For $k = 1$, an $S(1)$ -factor is simply a 1-factor. Heteyi (see [5]) has determined that the largest number of edges in the graph G of order $2m$ with a unique 1-factor is m^2 . Hence, we may now restrict our attention to the case $k \geq 2$.

Lemma 5.2.1 If G has a unique $S(k)$ -factor F , $k \geq 2$, then there is a set S , $S \subseteq V(G)$, so that $I(G-S) = V(G)-S$, and $|S|$ is the number of components in F .

Proof: We will show that the centres of the stars $K_{1,1}$ in the $S(k)$ -factor can be chosen so that S_c satisfies the requirement. First we choose the centres of the $K_{1,1}$ arbitrarily and let S be the resulting set S_c . Since F is unique, the only possible edges in $G[V-S]$ are those joining leaves of stars $K_{1,1}$. Suppose we have such an edge, say $y_j y_t$. Then x_j and x_t have no other neighbours in $\{y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{t-1}, y_{t+1}, \dots, y_{m_1}\}$ or we get a path of length 5 and hence two $K_{1,2}$ instead of the three $K_{1,1}$'s. Also $x_j y_t \notin E(G)$ and $x_t y_j \notin E(G)$, or the edges $x_j y_j, x_t y_t$ are replaced by a $K_{1,3}$. Finally $x_j x_t \notin E(G)$. Exchange the centre-leaf roles of $x_j y_j$ and $x_t y_t$, and let $S' = (S - \{x_j, x_t\}) \cup \{y_j, y_t\}$. Then $|E(G[V-S'])| < |E(G[V-S])|$. Now we may proceed inductively to complete the proof. \square

From now on, we assume that S_c satisfies $I(G-S_c) = V(G)-S_c$. The following lemma is easily proven.

Lemma 5.2.2 Suppose the graph G has a unique $S(k)$ -factor F . Then the only vertices that the leaves of any component $K_{1,i}$ ($2 \leq i \leq k$) of F can be adjacent to are their own centres and the centres of k -stars.

We next show that for $k \geq 2$, a graph with a proper unique $S(k)$ -factor (that is, the graph has a unique $S(k)$ -factor, and that unique $S(k)$ -factor is proper) has at least k vertices of degree one. Note that this result does not hold when $k = 1$.

Theorem 5.2.3 If G has a unique $S(k)$ -factor F ($k \geq 2$), and F is proper, then G has k vertices of degree 1 which have a common neighbour.

Proof: Consider the k -stars in F . The leaves of $S(k, i)$ cannot be adjacent to any other vertices except centres of k -stars. So if the $S(k)$ -factor has only one k -star, we are done. Otherwise we assume that for each k -star there is an edge from one of its leaves to the centre of another k -star. Construct a digraph H with $V(H) = \{S(k, i) \mid 1 \leq i \leq m_k\}$ and $(S(k, i), S(k, j)) \in A(H)$ if there is an edge from a leaf of $S(k, i)$ to the centre of $S(k, j)$. If H has a directed cycle of length at least two, we exchange edges between the k -stars on this cycle to get another $S(k)$ -factor. So we suppose H is acyclic. Then H has a vertex with outdegree 0 meaning that there are no edges out of the leaves of the corresponding k -star and so the leaves of this k -star are the vertices of degree 1 in G . \square

Corollary 5.2.4 If G has a proper $S(k)$ -factor ($k \geq 2$), and $\delta(G) \geq 2$, then G has at least two $S(k)$ -factors.

Proof: Suppose that G has only one $S(k)$ -factor. By Theorem 5.2.3, we have $\delta(G) = 1$. This contradicts $\delta(G) \geq 2$. Thus G has at least two $S(k)$ -factor. \square

Remark 5.2.5 At this point it is helpful to provide the reader with a description of what we now know of graphs with a unique $S(k)$ -factor F , as shown in Figure 5.1 (the centres are at the top). We describe the other edges which may lie in the graph.

From the leaves of the k -stars we have edges to centres of k -stars so that the digraph H of Theorem 5.2.3 is acyclic (so H is a subdigraph of the transitive tournament

on m_k vertices). Leaves of i -stars, $2 \leq i \leq k$, can only be adjacent to centres of k -stars. Leaves of 1-stars can only be adjacent to their centres. Any two centres can be adjacent but if the centre of a 1-star in F is adjacent to another centre its leaf is not also adjacent to that centre, unless it is the centre of a k - or $(k-1)$ -star. By Lemma 5.2.1 no leaves are adjacent.

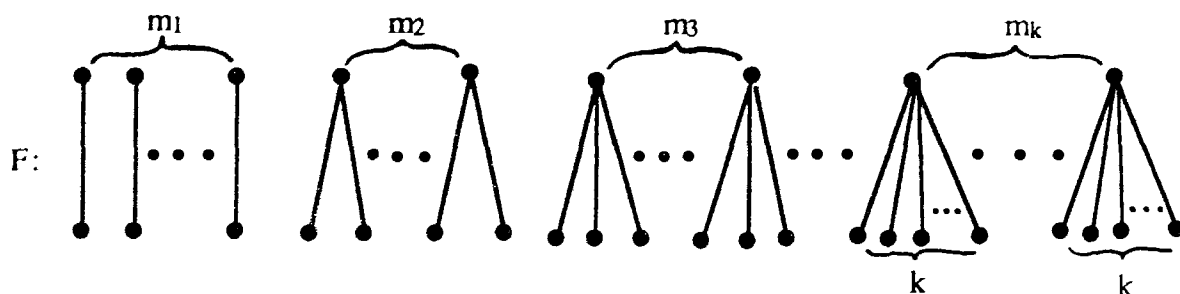


Figure 5.1

In order to provide an upper bound on the number of edges in a graph G with n vertices and a unique $S(k)$ -factor F ($k \geq 2$), we associate with G and F two graphs G_1 and F_1 which we now describe.

Without loss of generality, suppose that $m_k \neq 0$ and let $S(k, 1)$ be the k -star whose leaves are vertices of degree 1 in G (as described in Theorem 5.2.3).

Let $V(G_1) = V(G)$, where the edges of G_1 are those of G except that if both $x(i_1, j)y_s(i_2, r) \in E(G)$ and $x(i_1, j)x(i_2, r) \notin E(G)$, then $x(i_1, j)x(i_2, r) \in E(G_1)$ and $x(i_1, j)y_s(i_2, r) \notin E(G_1)$. So $|E(G)| = |E(G_1)|$.

Define F_1 as follows:

$$V(F_1) = V(F) = V(G)$$

$$E(F_1) = \{x(k, s)x(j, r) \mid x(i, s) \neq x(j, r), 1 \leq i, j \leq k, 1 \leq s \leq m_i, 1 \leq r \leq m_j\} \cup \\ \{x(k, s)y_r(i, h) \mid s = 1, 2, \dots, m_k; \text{ if } i = k, \text{ then } s+1 \leq h \leq m_k, 1 \leq r \leq k; \\ \text{ and if } 1 \leq i < k, \text{ then } 1 \leq h \leq m_i, 1 \leq r \leq i\} \cup E(F).$$

That is, F_1 contains all edges of the $S(k)$ -factor F , a complete subgraph on the vertex-set S_c , and contains edges from leaves to centres of k -stars. It is easy to see that in F_1 , F is the unique $S(k)$ -factor.

Lemma 5.2.6 For a given graph G with a proper unique $S(k)$ -factor F ($k \geq 2$), we define G_1 and F_1 as above. Then $|E(G_1)| \leq |E(F_1)| + \varepsilon$ where if $k = 2$ and $m_1 = 2$ or 3 , or if $m_{k-1} = 1$ and $m_1 \geq 1$, then $\varepsilon = 1$, and in all other cases $\varepsilon = 0$.

Proof: We prove the lemma by constructing a one-to-one mapping f from $E(G_1)$ or $E(G_1) - \{e\}$, $e \in E(G_1)$ (as appropriate), into $E(F_1)$.

The mapping f acts as the identity on (1) the edges of the $S(k)$ -factor $F = \cup S(i, j)$, (2) the edges $x(k, s)y_r(i, h) \in E(G_1)$, and (3) the edges $x(i, s)x(j, r) \in E(G_1)$.

By Lemma 5.2.2 all that remains is to define the action of f on the edges $y_s x(i, j) \in E(G_1)$, $1 \leq i \leq k-1$. If $y_s x(i, j) \in E(G_1)$, then, by the construction of G_1 , $x_s x(i, j) \in E(G_1)$ and so both $y_s x(i, j)$ and $x_s x(i, j)$ are edges of G . If $2 \leq i \leq k-2$ we then obtain another $S(k)$ -factor in G . So $y_s x(i, j) \in E(G_1)$ implies that $i \in \{1, k-1, k\}$. We have already defined $f(y_s x(k, j))$ so only two cases remain. Consider first $y_s x(k-1, j) \in E(G_1)$, $k-1 \neq 1$ so $k \geq 3$.

If $m_{k-1} = 1$, let $I = \{s \mid y_s x(k-1, 1) \in E(G_1)\}$. It is easy to see that if $s, r \in I$, $s \neq r$, then $x_s x_r \notin E(G_1)$. Provided that $|I| \geq 3$, we can extend the one-to-one map f by mapping $\{y_s x(k-1, 1) \mid s \in I\}$ into $\{x_s x_r \mid s, r \in I, s \neq r\}$. If $0 < |I| < 3$ such an extension is only possible from $\{y_s x(k-1, 1) \mid s \in I - \{j\}, j \in I\}$ into $\{x_r x_s \mid r, s \in I, r \neq s\}$ and we have $\varepsilon = 1$. If $m_{k-1} \geq 2$, it is easy to see that if $y_s x(k-1, j) \in E(G_1)$, then $y_s x(k-1, j)$, $x_s x(k-1, j) \in E(G)$ and as the $S(k)$ -factor is unique, then for $t \neq j$ $y_s x(k-1, t)$, $x_s x(k-1, t) \notin E(G)$ and hence are not edges of $E(G_1)$. So we put $f(y_s x(k-1, j)) = x_s x(k-1, t)$.

Finally we consider the edges $y_i x_j \in E(G_1)$, $i \neq j$. Clearly $m_1 \geq 2$ (or there are no such edges). If $k \geq 3$ and $y_i x_j \in E(G_1)$, $i \neq j$, then $x_i x_j \in E(G_1)$ and both $y_i x_j$ and $x_i x_j$ are edges of $E(G)$ and we can construct a 3-star instead of two 1-stars, a contradiction. So $k = 2$. If $m_1 = 2$ or 3 then either there are no edges of type $y_i x_j$, $i \neq j$, or the subgraphs spanned by the 1-stars are isomorphic to one of the four shown in Figure 5.2 (a) (b) (c) (d).

In each case of (a), (b) and (c) the edge y_1x_2 has no image, and in the fourth (Figure 5.2(d)) put $f(y_3x_2) = x_1x_3$ ($x_1x_3 \notin E(G_1)$). (Observe that if $m_{k-1} = 1$ and l is 2 or 3, then $i, j \in I$ $y_ix_j \notin E(G_1)$ and no conflict can arise.)

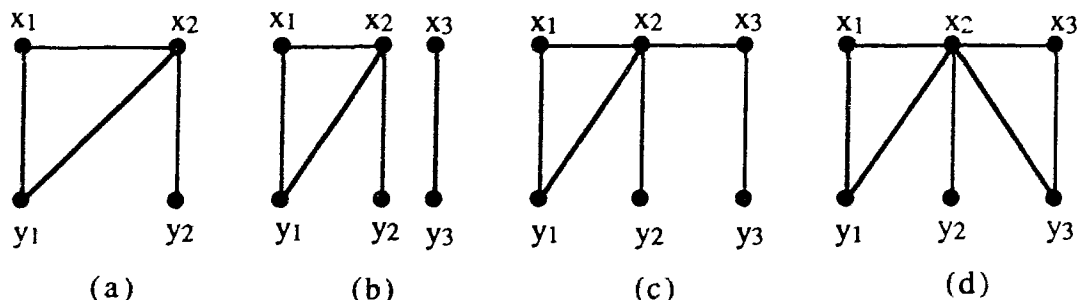


Figure 5.2

What now remains is the case $k = 2$ and $m_1 \geq 4$. In G_1 let $G_1' = G_1[\{x_1, \dots, x_{m_1}, y_1, \dots, y_{m_1}\}]$ and in F_1 $F_1' = F_1[\{x_1, \dots, x_{m_1}, y_1, \dots, y_{m_1}\}]$. If we can show that $|E(G_1')| \leq |E(F_1')|$ we will be able to define f on these remaining edges and so obtain the described one-to-one mapping.

The proof is by induction on m_1 . Calculation shows that the claim is valid when $m_1 = 4$. Suppose now that the claim holds for $m_1 < m$ and consider the case $m_1 = m$. Without loss of generality suppose that $y_1x_2 \in E(G_1)$, implying that $x_1x_2 \in E(G_1)$, $x_1y_i \notin E(G_1)$, $2 \leq i \leq m$, $x_1x_j \notin E(G_1)$ and $y_1x_j \notin E(G_1)$, $3 \leq j \leq m$. Thus $|E(G_1')| = |E(G_1' - \{x_1, y_1\})| + 3$. But $|E(F_1')| = |E(F_1' - \{x_1, y_1\})| + m$ and by the induction hypothesis $|E(G_1' - \{x_1, y_1\})| \leq |E(F_1' - \{x_1, y_1\})|$ so $|E(G_1')| \leq |E(F_1')| + 3 - m \leq |E(F_1')|$ as required.

Thus we have described the required mapping f and the proof is complete. \square

From Remark 5.2.5 and Lemma 5.2.6, we now are able to describe exactly the graphs with maximum number of edges which have F as a proper unique $S(k)$ -factor.

Corollary 5.2.7 If a graph G has the subgraph F as a proper unique $S(k)$ -factor, then $|E(G)| \leq |E(F_1)| + 1$.

We next look at all $S(k)$ -factors F on n vertices and determine that one for which $|E(F_1)|$ is maximized. Given n and $k \geq 2$ we denote by $f(n, k)$ the maximum number of edges in a graph on n vertices which has a proper unique $S(k)$ -factor. Hence for any graph G of order n which has a proper unique $S(k)$ -factor we have $|E(G)| \leq f(n, k)$.

Theorem 5.2.8 If a graph G of order n has proper unique $S(k)$ -factor ($k \geq 2$), then

$$f(n, k) = \begin{cases} \frac{n(n+1)}{6} + 1 & \text{if } k = 2 \text{ and } n \equiv 0, 2 \pmod{3} \\ \frac{(n-1)(n+2)}{6} & \text{if } k = 2 \text{ and } n \equiv 1 \pmod{3} \\ \frac{(n-1)(n+3)}{8} + 1 & \text{if } k = 3 \text{ and } n \equiv 1 \pmod{4} \\ \frac{(n+1)^2}{8} + 1 & \text{if } k = 3 \text{ and } n \equiv 3 \pmod{4} \\ \frac{n(n+2)}{8} & \text{if } k = 3 \text{ and } n \text{ is even} \\ \frac{(n-k)^2 - 9}{8} + n & \text{if } k \geq 4 \text{ and } n \not\equiv k \pmod{2} \\ \frac{(n-k)(n-k-2)}{8} + n & \text{if } k \geq 4 \text{ and } n \equiv k \pmod{2} \end{cases}$$

Proof: As mentioned in the beginning of this section we assume $k \geq 2$. Suppose that G has a proper unique $S(k)$ -factor F which has m_i components isomorphic to $K_{1,i}$. Then $n = |V(G)| = \sum_{i=1}^k m_i(i+1)$. Thus, letting $m = \sum_{i=1}^k m_i$, the number of edges in F_1 is

given by

$$\begin{aligned} |E(F_1)| &= |E(K_m)| + \sum_{i=1}^k m_i + \left(\sum_{i=1}^k i m_i - k \right) + \left(\sum_{i=1}^k i m_i - 2k \right) + \dots + \left(\sum_{i=1}^k i m_i - k m_k \right) \\ &= \frac{1}{2} m(m-1) + (m_k + 1) \sum_{i=1}^k i m_i - (k + 2k + \dots + k m_k) \end{aligned}$$

$$= \frac{1}{2} (m(m-1)+(m_k+1)(2\sum_{i=1}^k im_i-km_k)$$

Let

$$g(F) = 2|E(F_1)| = m(m-1)+(m_k+1)(2n-2m-km_k), \quad \dots (5.2.1)$$

where $m = \sum_{i=1}^k m_i$ and $n = \sum_{i=1}^k m_i(i+1)$. Setting $m^* = \sum_{i=1}^{k-1} m_i$, we obtain

$$\begin{aligned} g(F) &= (m^*+m_k)(m^*-m_k-3)+(m_k+1)(2n-km_k) \\ &= (m^*)^2-3m^*-m_k(m_k+3)+(m_k+1)(2n-km_k). \end{aligned}$$

If m_k is fixed, then $g(F)$ is a quadric function about m^* and reaches its maximum value when m^* is maximized. Hence, in order to maximize $g(F)$ one should take as many components as possible in F . Hence $m_3 = m_4 = \dots = m_{k-1} = 0$ and $m_2 = 0$ or 1. Therefore determining $\frac{1}{2} f(n, k)$ now becomes an integer programming problem as follows:

$$\begin{aligned} \frac{1}{2} f(n, k) &= \max\{(m_1+m_2)^2-3(m_1+m_2)-m_k(m_k+3)+(m_k+1)(2n-km_k)\} \\ \text{s.t.} \quad & 2m_1+3m_2+(k+1)m_k = n \\ \text{(IP)} \quad & m_1, m_2, m_k \geq 0 \quad \text{integers} \end{aligned}$$

We now start to determine the solutions for (IP).

If $n-(k+1)m_k$ is even, then $m_2 = 0$ and $m_1 = \frac{1}{2} (n-(k+1)m_k)$, and if $n-(k+1)m_k$ is odd, then $m_2 = 1$ and $m_1 = \frac{1}{2} (n-3-(k+1)m_k)$. Moreover, in order to obtain the value of m_k which maximizes $g(F)$ we consider following cases.

(1) Suppose that $n-(k+1)m_k$ is even, that is $n = (k+1)m_k+2m_1$ and $m = m_1+m_k$.

Suppose that $k+1$ is even (so $k \geq 3$) and $m_k \geq 2$. Let F' be an $S(k)$ -factor with $|V(F)| = |V(F')|$, $m_k' = m_k-1$, $m_1' = m_1 + \frac{k+1}{2}$ and $m_j' = m_j = 0$, $2 \leq j \leq k-1$. So

$$g(F) = (m_1+m_k)(m_1+m_k-1)+(m_k+1)(km_k+2m_1) \quad \dots (5.2.2)$$

$$\text{and } g(F') = (m_1+m_k + \frac{k-1}{2})(m_1+m_k-1 + \frac{k-1}{2}) + m_k(km_k+2m_1+1).$$

Thus

$$g(F')-g(F) = (k-3)m_1 + \frac{(k-1)(k-3)}{4} \geq 0. \quad \dots (5.2.3)$$

So $g(F') \geq g(F)$ and the maximum number of edges is obtained when we have only one k -star in F .

Suppose that $k+1$ is odd and $m_k \geq 2$, we let F' be an $S(k)$ -factor with $m_k' = m_k - 1$ k -stars and therefore one 2-star and $m_1' = m_1 + \frac{k-2}{2}$ 1-stars. So $m' = m_1 + m_k + \frac{k-2}{2}$

and

$$g(F') = (m_1 + m_k + \frac{k-2}{2})(m_1 + m_k + \frac{k-4}{2}) + m_k(km_k + 2m_1 + 2). \quad \dots (5.2.4)$$

Thus, from (5.2.2) and (5.2.4), we have

$$g(F') - g(F) = (k-4)m_1 + \frac{(k-2)(k-4)}{4} \geq 0 \quad \text{if } k \geq 4. \quad \dots (5.2.5)$$

So for $k = 2$ we expect to have as many 2-stars as possible. This case will later be considered in more detail.

(2) Suppose that $n - (k+1)m_k$ is odd, that is $n = (k+1)m_k + 3 + 2m_1$ and $m = m_1 + m_k + 1$.

Suppose that $k+1$ is even and $m_k \geq 2$, we let F' be an $S(k)$ -factor with $m_k' = m_k - 1$, $m_2' = m_2 = 1$, $m_1' = m_1 + \frac{k+1}{2}$ and $m' = m_1 + m_k + \frac{k+1}{2}$. So

$$g(F) = (m_1 + m_k + 1)(m_1 + m_k) + (m_k + 1)(km_k + 2m_1 + 4) \quad \dots (5.2.6)$$

and
$$g(F') = (m_1 + m_k + \frac{k+1}{2})(m_1 + m_k + \frac{k-1}{2}) + m_k(km_k + 2m_1 + 5).$$

Thus

$$g(F') - g(F) = (k-3)m_1 + \frac{(k+1)(k-1)}{4} > 0 \quad \text{if } k \geq 5. \quad \dots (5.2.7)$$

If $k = 3$, then we expect to have many k -stars. This case will later be considered in detail.

Suppose that $k+1$ is odd and $m_k \geq 2$, we let F' be an $S(k)$ -factor with $m_k' = m_k - 1$, $m_2' = 0$, $m_1' = m_1 + \frac{k+4}{2}$ and $m' = m_1 + m_k + \frac{k+2}{2}$. So

$$g(F') = (m_1 + m_k + \frac{k+2}{2})(m_1 + m_k + \frac{k}{2}) + m_k(km_k + 2m_1 + 4) \quad \dots (5.2.8)$$

Thus, from (2.6) and (2.8), we have

$$g(F') - g(F) = (k-2)m_1 + \frac{k(k+2)}{4} - 4 > 0 \quad \text{if } k \geq 4 \quad \dots (5.2.9)$$

If $k = 2$, then it is better to have more k -stars.

From the above discussion we conclude that, except when (1) $k = 2$ and (2) $k = 3$ and n is odd, if G has a unique proper $S(k)$ -factor F and as many edges as possible, we should choose F to have exactly one k -star, at most one 2-star and all other components 1-stars.

So if $k \geq 4$ we easily obtain

$$|E(F_1)| = \frac{(n-k)^2-9}{8} + n \quad \text{if } n \not\equiv k \pmod{2} \text{ and}$$

$$|E(F_1)| = \frac{(n-k)(n-k-2)}{8} + n \quad \text{if } n \equiv k \pmod{2}.$$

If $k = 3$ and n is even, then $m_3 = 1$, $m_2 = 0$ and $m_1 = \frac{n-4}{2}$. Thus we have $|E(F_1)| = \frac{n(n+2)}{8}$.

We now study the exceptional cases.

When $k = 2$, from (5.2.5) and (5.2.9), we see that $g(F)$ attains its the maximum if m_2 is maximized. So, with fixed n , F_1 has the most edges if the $S(k)$ -factor F has as many 2-stars as possible. Hence, if $n \equiv 0 \pmod{3}$, then $m_1 = 0$, $m_2 = \frac{n}{3}$ and $|E(F_1)| = \frac{n(n+1)}{6}$, if $n \equiv 1 \pmod{3}$, then $m_1 = 2$, $m_2 = \frac{n-4}{3}$ and $|E(F_1)| = \frac{(n+2)(n-1)}{6}$, and if $n \equiv 2 \pmod{3}$, then $m_1 = 1$, $m_2 = \frac{n-2}{3}$ and $|E(F_1)| = \frac{n(n+1)}{6}$.

When $k = 3$ and n is odd, we see from (5.2.7) that $g(F)$ is an increasing function of m_3 . So, with fixed n , F_1 has the most edges if F has as many 3-star as possible. Hence, if $n \equiv 1 \pmod{4}$, then $m_1 = 1$, $m_2 = 1$, $m_3 = \frac{n-5}{4}$ and $|E(F_1)| = \frac{(n-1)(n+3)}{8}$, and if $n \equiv 3 \pmod{4}$, then $m_1 = 0$, $m_2 = 1$, $m_3 = \frac{n-3}{4}$ and $|E(F_1)| = \frac{(n+1)^2}{8}$.

Summarizing the above conclusions, we obtain

$$|E(F_1)| = \begin{cases} \frac{n(n+1)}{6} & \text{if } k = 2 \text{ and } n \equiv 0, 2 \pmod{3} \\ \frac{(n-1)(n+2)}{6} & \text{if } k = 2 \text{ and } n \equiv 1 \pmod{3} \\ \frac{(n-1)(n+3)}{8} & \text{if } k = 3 \text{ and } n \equiv 1 \pmod{4} \\ \frac{(n+1)^2}{8} & \text{if } k = 3 \text{ and } n \equiv 3 \pmod{4} \\ \frac{n(n+2)}{8} & \text{if } k = 3 \text{ and } n \text{ is even} \\ \frac{(n-k)^2-9}{8} + n & \text{if } k \geq 4 \text{ and } n \not\equiv k \pmod{2} \\ \frac{(n-k)(n-k-2)}{8} + n & \text{if } k \geq 4 \text{ and } n \equiv k \pmod{2} \end{cases}$$

But, by Lemma 5.2.6, we have that $f(n, k) = |E(F_1)| + \varepsilon$ where if $k = 2$ and $m_1 = 2$ or 3 , or if $m_{k-1} = 1$, then $\varepsilon = 1$, and otherwise $\varepsilon = 0$. From the calculating, this implies that if $k = 2$ and $n \equiv 0, 1 \pmod{3}$, or if $k = 3$ and $n \equiv 1$ or $3 \pmod{4}$, then $\varepsilon = 1$; otherwise $\varepsilon = 0$. Therefore, we obtain the desired $f(n, k)$. \square

Corollary 5.2.9 If a graph H of order n has an $S(k)$ -factor and $|E(H)| > f(n, k)$, where $f(n, k)$ is as defined in the Theorem 5.2.8, then H has at least two $S(k)$ -factors.

Remark 5.2.10 Hetyei (see [40]) proved that if a graph G of order $2m$ has a 1-factor and $|E(G)| > m^2$, then G has at least two 1-factors. So this corollary is an extension of Hetyei's result.

§5.3. The number of $S(k)$ -factors in an r -regular graph.

For graphs H_1 and H_2 , the **join** of H_1 and H_2 , denote $H_1 + H_2$, is obtained from $H_1 \cup H_2$ by joining all vertices in $V(H_1)$ to those in $V(H_2)$. Let $e_G(S_1, S_2)$ where $S_1 \subseteq V(G)$ and $S_2 \subseteq V(G)$, denote all edges in G which have one end in S_1 and the other in S_2 .

The following result will be used in this section.

Theorem 5.3.1 (Las Vergnas [31]; Hell and Kirkpatrick [23] and Amahashi and Kano [2]) For $k \geq 2$, the graph G has an $S(k)$ -factor if and only if

$$i(G-S) \leq k|S| \quad \text{for all } S \subseteq V(G).$$

Theorem 5.3.2 Let G be a connected r -regular graph ($r \geq 4$) of order n which is not isomorphic to $K_{r,r}$. Then G has at least n star-factors each of which is either a proper $S(r)$ -factor or a proper $S(r-1)$ -factor.

Proof: Let $x \in V(G)$ and the neighbours of x be denoted by $N_G(x) = \{y_1, y_2, \dots, y_r\}$. Let $G_x = G[V(G) - \{x\} - N_G(x)]$ and $I(G_x) = \{z_1, z_2, \dots, z_h\}$ (Recall that $I(G_x)$ denotes the set of isolated vertices in G_x). Obviously, we have $h \leq r-1$. We study the structure of G by considering several cases.

(i) Suppose $|I(G_x)| = 0$. In this case we claim that G_x has an $S(r-1)$ -factor or $G \cong K_{r+1,r+1} - F$, where F is a 1-factor in $K_{r+1,r+1}$. If G_x has no $S(r-1)$ -factor, then by Theorem 5.3.1 there exists a set S in $V(G_x)$ so that $i(G_x - S) > (r-1)|S|$. Since $N_G(I(G_x - S)) \subseteq S \cup N_G(x)$, on counting edges between $S \cup N_G(x)$ and $I(G_x - S)$ we have $r \cdot |I(G_x - S)| \leq r|S| + r(r-1)$ or $|S| + r - 1 \geq i(G_x - S) > (r-1)|S|$. Simplifying we get $|S| = 0$ or 1 as $r \geq 4$. But $I(G_x) = \emptyset$, so $S \neq \emptyset$ and thus $|S| = 1$. Let $S = \{s\}$. Then $i(G_x - \{s\}) \leq r$. But $i(G_x - \{s\}) > (r-1)$, and thus $i(G_x - \{s\}) = r$. Moreover, as $r \cdot i(G_x - \{s\}) = r^2 = e_G(I(G_x - \{s\}), \{s\} \cup N_G(x))$ and G is connected, it follows that $G \cong K_{r+1,r+1} - F$ and the claim is proved.

(ii) If $|I(G_x)| = r-1$, it is easy to see that $G \cong K_{r,r}$. Which has been excluded.

(iii) If $0 < |I(G_x)| < r-1$ and $V(G) = I(G_x) \cup \{x\} \cup N_G(x)$, then $G \cong \{x, z_1, \dots, z_h\} + G[\{y_1, \dots, y_r\}]$.

(iv) Suppose that $0 < |I(G_x)| < r-1$ and $V(G) \neq I(G_x) \cup \{x\} \cup N_G(x)$. Let $G_x' = G[V(G) - \{x, z_1, \dots, z_h, y_1, \dots, y_r\}]$. Then $|V(G_x')| \geq 2$ and $I(G_x') = \emptyset$. We will show that G_x' has an $S(r-2)$ -factor. In fact, if G_x' has no $S(r-2)$ -factor, then by Theorem 5.3.1 there exists a set S' in $V(G_x')$ so that $i(G_x' - S') > (r-2)|S'|$. Moreover, as $I(G_x') = \emptyset$, S' is non-empty. Counting edges we have

$$(r-2)|S'| < i(G_x' - S') \leq |S'| + r - h - 1$$

$$\text{or } 1 \leq |S'| < (r-h-1)/(r-3) = 1 + \frac{2-h}{r-3}.$$

Since $r \geq 4$ and h is a positive integer, $h = 1$. This implies that $|S'| = 1$. Thus we have $i(G_x' - S') = r-1$. Now each vertex of $I(G_x' - S')$ is adjacent to the one vertex of S' and to $r-1$ vertices of $\{y_1, y_2, \dots, y_r\}$. But as x and z_1 are each adjacent to all of $\{y_1, y_2, \dots, y_r\}$, we have at least $2r+(r-1)^2 = r^2+1$ edges incident with $\{y_1, y_2, \dots, y_r\}$ which is impossible.

Thus we conclude that G must be as described in (i),(ii) and (iv) and we now study these graphs.

If $G \cong K_{r+1, r+1} - F$ or $G \cong \{x, z_1, z_2, \dots, z_h\} + G[\{y_1, y_2, \dots, y_r\}]$, it is not hard to find n proper $S(r-1)$ -factors in G . In case (i), each vertex u of G is the centre of an r -star which is easily extended to an $S(r)$ -factor and this $S(r)$ -factor has the only r -star centred at u ; thus giving n distinct proper $S(r)$ -factors in G . In (iv), each vertex is the centre of the only $(r-1)$ -star of the $S(r-1)$ -factor. Thus we obtain n proper $S(r-1)$ -factors and each of these $S(r-1)$ -factors has only one $(r-1)$ -star centred at the different vertices. We have the required factors. \square

§5.4. Miscellaneous results on perfect matchings.

Chartrand et al. [14] have studied perfect matchings in the square of a graph and showed that for any connected graph G the square G^2 has a perfect matching if and only if G has even order. In this section, we study further properties of power graphs with respect to perfect matchings. In particular we look at when the power graphs are elementary and when they are 1-extendable. In order to do so, we start by studying trees and the powers of trees.

Lemma 5.4.1 For any tree T , there exists either a leaf which is adjacent to a vertex of degree two or two leaves with a common neighbour in T .

Proof: Let P be a longest path in T and x an end-vertex of P . Consider the vertex y which is adjacent to x . If y is of degree 2, we are done. If y has degree at least 3, then one of its neighbours is a leaf as otherwise P is not a longest path. \square

Lemma 5.4.2 For any tree T of order $2m$:

(1) there exists a perfect matching F in T^2 which has at least one edge of T ; and

(2) there exists a perfect matching F in T^3 with at least two edges of T , unless T is isomorphic to $K_{1,2m-1}$.

Proof: Use induction on m .

Obviously, both of (1) and (2) are true for $m = 2$.

Suppose that (1) holds for $m < n$. Let $m = n \geq 3$ and T be any tree of order $2n$. If T contains a leaf x which is adjacent to a vertex y of degree 2, then by the induction hypothesis $(T - \{x, y\})^2$ has a perfect matching F' , and $F' \cup \{xy\}$ is a suitable perfect matching of T^2 . Otherwise, by Lemma 5.4.1, there exist two leaves x and y with a common neighbour in T . Thus $xy \in E(T^2)$. Again by the induction hypothesis $(T - \{x, y\})^2$ has a perfect matching F' which contains at least one edge of $T - \{x, y\}$. Then $F' \cup \{xy\}$ is a suitable perfect matching of T^2 .

Next, we suppose that (2) is true for $m < n$. Let $m = n \geq 3$ and let T be a tree of order $2n$, $T \not\cong K_{1,2n-1}$. By Lemma 5.4.1 there are vertices x and y in T so that either x is a leaf adjacent to y which has degree two, or x and y are leaves with a common neighbour. Let $T' = T - \{x, y\}$. If $T' \not\cong K_{1,2n-3}$, then by induction hypothesis, $(T')^3$ has a perfect matching F' with at least two edges of T' and $F' \cup \{xy\}$ is a suitable perfect matching in T^3 . Suppose that $T' \cong K_{1,2n-3}$. Let the centre of $K_{1,2n-3}$ be u and the leaves be $v_1, v_2, \dots, v_{2n-3}$. Since $T \not\cong K_{1,2n-1}$, T must be isomorphic to one of the three trees given in Figure 5.3.

In Figures 5.3(a) and 5.3(b), $F = \{xy, uv_1, v_2v_3, \dots, v_{2n-4}v_{2n-3}\}$ is a perfect matching of T^3 containing two edges of T , and in Figure 5.3(c), $F = \{xv_1, uv_2, yv_3, v_4v_5, \dots, v_{2n-4}v_{2n-3}\}$ is a perfect matching of T^3 containing two edges of T .

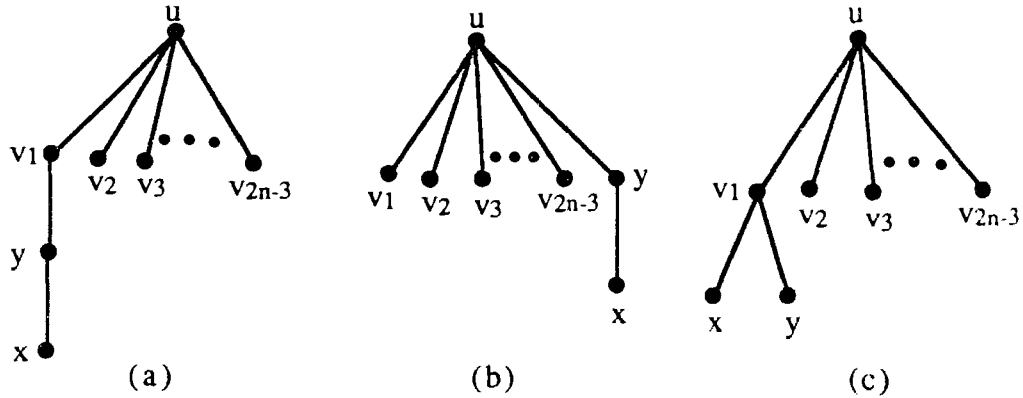


Figure 5.3

This completes the proof. \square

Recall that a graph is **elementary** if the set of all edges which lie in perfect matching of G (allowed edges) form a connected subgraph. Note that the allowed edges form a spanning subgraph. Clearly, for connected graphs the property of being elementary is weaker than that of being 1-extendable. So we begin by studying when powers of trees are elementary.

Theorem 5.4.3 If T is a tree of even order, then T^2 is an elementary graph.

Proof: Let $|V(T)| = 2n$ and $P = v_1v_2\dots v_{m+1}$ be a longest path in the tree T . Then $d_T(v_1) = d_T(v_{m+1}) = 1$.

Claim Let $z = v_3$ (if $d_T(v_2) = 2$) or v_2 (if $d_T(v_2) \geq 3$). Then there exists a perfect matching F of T^2 so that the edge of F which is adjacent to z is in T .

The proof of the claim uses induction on m . When $m = 2$, then $T \cong K_{1,2n-1}$ and the claim follows easily. If $m = 3$, then T is isomorphic to the graph shown in Figure 5.4. If $d(v_2) \equiv d(v_3) \equiv 0 \pmod{2}$ take the edge v_1v_2 and v_3v_4 in F and if $d(v_2) \equiv d(v_3) \equiv 1 \pmod{2}$ put the edge v_2v_3 in F .



Figure 5.4

Suppose that the claim holds when $m < k$. Let $m = k \geq 4$ and choose $e \in E(T^2)$, where

$$e = \begin{cases} v_k v_{k+1} & \text{if } d_T(v_k) = 2 \text{ and} \\ v' v_{k+1} & \text{if } d_T(v_k) \geq 3 \text{ and } v' \in N(v_k) - \{v_{k-1}, v_{k+1}\}, d_T(v') = 1. \end{cases}$$

Set $T' = T - V(e)$. By the induction hypothesis, $(T')^2$ has a perfect matching F' of the required type, and so $F' \cup \{e\}$ is a suitable perfect matching in T^2 . This proves the claim.

We now use induction on $|V(T)|$ to prove the theorem.

It is easy to check that the theorem is true for $n = 2$ and $n = 3$.

Suppose that the theorem holds for $n \leq k$. Let $n = k \geq 4$. If $d(v_2) = 2$, by the claim, there exists a perfect matching F of T^2 so that the edge $e = v_3y$ of F is in $E(T)$. This implies that $v_1v_2 \in F$. Then $F' = F - \{v_1v_2, e\} \cup \{v_1v_3, v_2y\}$ is a perfect matching of T^2 . Since the allowed edges in $(T - \{v_1, v_2\})^2$ form a connected subgraph by the

induction hypothesis (and are also allowed edges in T^2), and v_1v_3 and v_2y are allowed edges, then the allowed edges of T^2 form a connected subgraph.

If $d(v_2) \geq 3$, then by the claim there exists a perfect matching F so that the edge e of F which is adjacent to v_2 is in $E(T)$. If $e = v_2v_3$, then there exists a leaf v' (adjacent to v_2) so that $v'v_1 \in E(F)$. Then $F' = F - \{v'v_1, e\} \cup \{v_1v_2, v'v_3\}$ is a perfect matching of T^2 . By the induction hypothesis, $(T - \{v', v_1\})^2$ is elementary. Since v_1v_2 and $v'v_3$ are allowed edges, T^2 is also an elementary graph. If $e = v_2y$ and $y \neq v_3$ (notice that y must be a leaf), we may assume $y = v_1$ (otherwise choose $yv_2v_3 \dots v_{m+1}$ as a longest path). Let $v' \in N(v_2) - \{v_1, v_3\}$ and $v'z \in E(F)$. Then $F' = F - \{v'z, e\} \cup \{v'v_1, v_2z\}$ is a perfect matching of T^2 . Again using the induction hypothesis, $(T - \{v', v_1\})^2$ is elementary. Since v_1v_2 and $v'v_1$ are allowed edges, T^2 is an elementary graph. \square

Since trees are minimal connected graphs, we can easily generalize the above result to all connected graphs.

Corollary 5.4.4 If G is a connected graph of even order, then G^2 is elementary.

Proof: Let T be a spanning tree of G . By Theorem 5.4.3, T^2 is elementary. Since every spanning supergraph of an elementary graph is elementary, G^2 is also an elementary graph. \square

Although for every tree T of even order T^2 is elementary, it may not be the case that T^2 is 1-extendable. For example, $(P_{2n})^2$ is elementary but not 1-extendable. However, we do have the following result.

Theorem 5.4.5 If T is a tree of even order, then T^3 is 1-extendable.

Proof: Let $|V(T)| = 2n$. We use induction on n to prove the result.

It is easy to see that the claim holds for $n = 1$ and $n = 2$.

Suppose that the claim holds for $n < m$ and let T be a tree with $2m$ vertices. By Lemma 5.4.1 we know that T has vertices v_1 and v_2 so that either $d_T(v_1) = 1$, $d_T(v_2) = 2$ and $v_1v_2 \in E(T)$, or $d_T(v_1) = d_T(v_2) = 1$ and v_1 and v_2 have a common neighbour. Let $T' = T - \{v_1, v_2\}$.

Case 1. Suppose that v_1 is adjacent to v_2 in T and $d_T(v_2) = 2$. Let v_3 be the other neighbour of v_2 . Let $N_1 = N_T(v_3) - v_2$ and $N_2 = N_T(N_1) - v_3$. By the induction hypothesis, $(T')^3$ is 1-extendable. To see that T^3 is 1-extendable, we need only to consider the edges of $\{v_1v_2, v_2v_3, v_1v_3\} \cup \{v_1x \mid x \in N_1\} \cup \{v_2y \mid y \in N_1\} \cup \{v_2z \mid z \in N_2\}$.

For the edge v_1v_2 , let F_1 be a perfect matching of $(T')^3$. Then $F_1 \cup \{v_1v_2\}$ is a perfect matching of T^3 containing v_1v_2 .

For the edges v_1x ($x \in N_1$) and v_2v_3 , let F_2 be a perfect matching of $(T')^3$ containing v_3x , then $F_2 - \{v_3x\} \cup \{v_2v_3, v_1x\}$ is a perfect matching of T^3 containing v_1x and v_2v_3 .

For the edges v_2y ($y \in N_1$) and v_1v_3 , let F_3 be a perfect matching of $(T')^3$ containing yv_3 , then $F_3 - \{yv_3\} \cup \{v_1v_3, v_2y\}$ is a suitable perfect matching.

For the edges v_2z ($z \in N_2$), there exists a vertex w belonging to N_1 so that zw is contained in a perfect matching F_4 of $(T')^3$. Thus $F_4 - \{zw\} \cup \{v_1w, v_2y\}$ is a perfect matching of T^3 as required.

Case 2. Suppose that both v_1 and v_2 are leaves of T with a common neighbour v_3 . By the induction hypothesis, $(T')^3$ is 1-extendable. Let $N_1 = N_T(v_3) - \{v_1, v_2\}$ and $N_2 = N_T(N_1) - v_3$. To see that T is 1-extendable, we shall show that each edge of $\{v_1v_2, v_2v_3, v_1v_3\} \cup \{v_ix \mid i = 1, 2; x \in N_1\} \cup \{v_iy \mid i = 1, 2; y \in N_2\}$ lies in a perfect matching.

For the edge v_1v_2 , let F_1 be a perfect matching of $(T')^3$. Then $F_1 \cup \{v_1v_2\}$ is a perfect matching of T^3 containing v_1v_2 .

For the edges v_1v_3 (respectively v_2v_3) and v_1x (respectively v_2x), $x \in N_1$, let F_2 be a perfect matching of $(T)^3$ containing v_3x . Then $F_2 - \{v_3x\} \cup \{v_1v_3, v_2x\}$ (respectively $F_2 - \{v_3x\} \cup \{v_2v_3, v_1x\}$) is a perfect matching of T^3 as required.

Finally, for the edges v_iy ($i = 1, 2; y \in N_2$), apply the same argument as previously but with F_3 a perfect matching containing v_3y ($y \in N_2$). \square

As before we may strengthen Theorem 5.4.5 so that it applies to all connected graphs.

Corollary 5.4.6 If G is a connected graph of even order, then G^3 is 1-extendable.

Proof: For any edge $e = xy \in E(G^3)$, let $d_G(x, y) = i$ ($1 \leq i \leq 3$). Then there exists an induced path P of G from x to y with length i . Since any subtree of a connected graph can be extended to a spanning tree, there is a spanning tree T of G containing the path P . Hence $e = xy \in E(T^3)$. By Theorem 5.4.5, there exists a perfect matching of T^3 , and hence of G^3 , which contains e . Therefore G^3 is 1-extendable. \square

Even though we have obtained some results on 1-extendability of powers of graphs, there still remain many problems in this area. One is: Given n determine the least integer $m = m(n)$ so that the m^{th} power of a graph is n -extendable. We have shown that $m(1) = 3$. Another is to characterize those graphs G for which G^2 is 1-extendable.

Next, we give a Nordhaus-Gaddum type of result concerning matchings. This result exhibits a relationship between graphs and their complements. More precisely, letting $F(G)$ denote the maximum number of independent edges in G , we shall study upper and lower bounds for $F(G) + F(\bar{G})$ and $F(G) \cdot F(\bar{G})$.

Theorem 5.4.7 Let G be a graph of order $2n$ and $G \not\cong K_{2n}$ or \bar{K}_{2n} . Then

$$n \leq F(G) + F(\bar{G}) \leq 2n, \text{ and}$$

$$n-1 \leq F(G) \cdot F(\bar{G}) \leq n^2.$$

The above bounds are sharp.

Proof: For any graph G of order $2n$ we have $F(G) \leq n$. Hence

$$F(G) + F(\bar{G}) \leq 2n.$$

To see that the bound is sharp, Let G be a perfect matching of K_{2n} . Then $F(G) = n$ and $F(\bar{G}) = F(K_{2n} - nK_2) = n$.

For the lower bound, let M be a perfect matching in K_{2n} and G any graph of order $2n$. Then $M \cap E(G)$ and $M \cap E(\bar{G})$, are independent edges in G and \bar{G} , respectively. Thus $|M \cap E(G)| \leq F(G)$ and $|M \cap E(\bar{G})| \leq F(\bar{G})$. So $F(G) + F(\bar{G}) \geq |M \cap E(G)| + |M \cap E(\bar{G})| = |M| = n$. This bound is seen to be sharp by considering $G = K_{1,2n-1}$.

For any real numbers x and y , $4xy \leq (x+y)^2$. Thus $F(G) \cdot F(\bar{G}) \leq \left(\frac{F(G) + F(\bar{G})}{2}\right)^2 \leq n^2$. This bound is sharp as if $G = nK_2$, $F(G) \cdot F(\bar{G}) = n^2$.

Since neither G nor \bar{G} is empty, then, as $F(G) + F(\bar{G}) \geq n$, we have $F(G) \cdot F(\bar{G}) \geq F(G)(n - F(G))$ and $F(G) \geq 1$. Clearly, if $F(G) = n$, then $F(G) \cdot F(\bar{G}) \geq n$. So we suppose that $1 \leq F(G) \leq n-1$. The function $f(x) = x(n-x)$ is increasing on $[1, n/2]$ and decreasing on $[n/2, n-1]$ and hence

$$F(G) \cdot F(\bar{G}) \geq \min\{f(1), f(n-1)\} = n-1.$$

Therefore for any graph G , $G \not\cong K_{2n}$ or \bar{K}_{2n} , $F(G) \cdot F(\bar{G}) \geq n-1$. Taking $G = K_{1,2n-1}$, the lower bound is achieved. \square

We remark that this theorem can easily be extended to graphs of odd order. Since this is more complicated we will not describe it here.

References.

- [1] J. Akiyama and M. Kano, Factors and factorizations of a graph, *Journal of Graph Theory*, 9(1985) 1-42.
- [2] A. Amahashi and M. Kano, On factors with given components, *Discrete Math.* 42(1982) 1-6.
- [3] N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory 1736-1936*, Oxford University Press, Oxford(1976).
- [4] B. Bollobás and S. E. Eldridge, Maximal matchings in graphs with given minimal and maximal degrees, *Math. Proc. Cambridge Philos. Soc.* 79(1976) 221-234.
- [5] B. Bollobas, *Extremal Graph Theory*, Academic Press (1978).
- [6] J. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan Press Ltd. (1976).
- [7] R. A. Brualdi and H. Perfect, Extension of partial diagonals of matrices I, *Monatsh. Math.* 75(1971) 385-397.
- [8] R. A. Brualdi and J. Csima, Extending subpermutation matrices in regular class of matrices, *Discrete Math.* 62(1986) 99-101.
- [9] L. Cammack and G. Schrag, Research problem 101, *Discrete Math.* 73(1989) 311-312.
- [10] F. Castagna and G. Prins, Every generalized Petersen graph has a Tait colouring, *Pacific J. Math.* 40(1972) 53-58.
- [11] O. Chan, C. C. Chen and Q. L. Yu, On 2-extendable abelian cayley graphs. (Submitted to *Discrete Math.*)
- [12] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Second Edition, Wadsworth, 1986.
- [13] G. Chartrand and J. Mitchem, Graphical theorems of the Nordhaus-Gaddum class, *Recent Trends in Graph Theorey (Proc. 1st New York city Graph Theory*

- Conference, 1970), eds. by M. Capobianco, J. B. Frechen and M. Krolík, Lecture Notes in Mathematics 186, Springer-Verlag, Berlin, 1971, 55-61.
- [14] G. Chantrand, A. D. Polimeni and M. J. Steward, The existence of 1-factors in line graphs, squares, and total graphs, *Indagationes Math.* 35 (Proc. Koninkl. Ned. Akad. Wetensch. Math. Sci. A76) (1973) 228-232.
- [15] C. C. Chen and N. Quimpo, On strongly connected Abelian group graphs, *Combinatorial Math.* VIII, Springer-Verlag Lecture Notes Series, 884 (1981) 23-34.
- [16] C. C. Chen, On edge-Hamiltonian property of Cayley graphs, *Discrete Math.* 72 (1988) 29-33.
- [17] V. Chvátal and D. Hanson, Degrees and matchings, *J. Combin. Theory (B)* 20(1976) 128-138.
- [18] G. Cornuejols, D. Hartvigseen and W. R. Pulleyblank, Packing subgraphs in a graph, *Oper. Res. Letters* 1(1982) 139-143.
- [19] T. Gallai, On factorization of graphs, *Acta. Acad. Sci. Hung.* 1(1950) 133-153.
- [20] E. Györi and M. D. Plummer, Private communication.
- [21] P. Hall, On representatives of subsets, *J. London Math. Soc.*, 10(1935) 26-30.
- [22] G. R. T. Hendry, Extending cycles in graphs, *Discrete Math.* (to appear).
- [23] P. Hell and D. G. Kirkpatrick, Star factors and star packings, TR 82-6, Department of Computing Science, Simon Fraser University (1982).
- [24] P. Hell and D. G. Kirkpatrick, Packings by cliques and finite families of graphs, *Discrete Math.* 49(1984) 118-133.
- [25] P. Hell and D. G. Kirkpatrick, Packings by complete bipartite graphs, *SIAM J. Alg. Disc. Math.* 7(1986) 113-129.
- [26] G. Hetyei, Rectangular configurations which can be covered by 2×1 rectangles, *Pécsi Tan. Föisk. Közl.* 8 (1964) 351-367.

- [27] M. Kano, Graph factors with given properties, Graph Theory Singapore 1983, Springer Lecture Notes Math. 1703, Springer, Berlin (1984) 161-168.
- [28] D. G. Kirkpatrick and P. Hell, On the complexity of general graph factor problem, SIAM J. Computing (1983) 601-609.
- [29] D. G. Kirkpatrick and P. Hell, On the completeness of a generalized matching problem, Proc. Tenth Annual ACM Symposium on Theory of Computing (1978) 240-245.
- [30] D. König, Graphen und Matrizen, Mat. Fiz. Lapok., 38(1931) 116-119.
- [31] M. Las Vergnas, An extension of Tutte's 1-factors of a graph, Discrete Math. 2(1972) 241-255.
- [32] C. H. C. Little, A theorem on connected graphs in which every edge belong to a 1-factor, J. Austral. Math. Sco. 18(1974) 450-452.
- [33] C. H. C. Little, D. D. Grant and D. A. Holton, On defect-d matching in graphs, Discrete Math. 13 (1975) 41-54.
- [34] G. Z. Liu, On $[a,b]$ -covered graphs, JCMCC 5(1989) 14-22.
- [35] G. Z. Liu, On f -factor covered graphs. (preprint)
- [36] J. P. Liu and Q. L. Yu, Matching extensions and products of graphs. (preprint)
- [37] M. Loeb1 and S. Poljak, Subgraph packing - A Survey, Research report No. 89-139, Department of Applied Mathematics, Charles University (1989).
- [38] L. Lovász, A Generalization of König's theorem, Acta. Acad. Sci. Hung. 21(1970) 443-446.
- [39] L. Lovász, Subgraphs with prescribed valencies, J. Combin. Theory, 8(1970) 391-416.
- [40] L. Lovász, On the structure of factorizable graphs, Acta Math. Acad. Sci. Hanger. 23 (1972) 179-195.
- [41] L. Lovász, Matching structure and the matching lattice, J. Combin. Theory, ser.(B) 43 (1987), 187-222.

- [42] L. Lovász and M. D. Plummer, On minimal elementary bipartite graphs, *J. Combin. Theory, Ser. B*, 23(1977) 127-138.
- [43] L. Lovász and M. D. Plummer, On bicritical graphs, *Infinite and Finite Sets, Coll. Math. Soc. J. Bolyai*, No. 10, North-Holland, Amsterdam, 1051-1079.
- [44] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986.
- [45] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* 63(1956) 175-177.
- [46] J. Petersen, Die Theorie der regulären Graph, *Acta Math.* 15(1891) 193-220.
- [47] M. D. Plummer, On n -extendable graphs, *Discrete Math.* 31 (1980) 201-210.
- [48] M. D. Plummer, Matching extension in bipartite graphs, (preprint) 1986.
- [49] M. D. Plummer, Matching extension and the genus of a graph, *J. Combin. Theory, ser.(B)* 44 (1988) 329-337.
- [50] M. D. Plummer, Matching extension and connectivity in graphs, *Congressus Numerantium*, 63(1988), 147-160..
- [51] A. Saito, Research Problem 114, *Discrete Math.*, 79(1989/90),109.
- [52] G. Schrag and L. Cammack, On the 2-extendability of the generalized Petersen graphs, *Discrete Math.* 78 (1989) 169-177.
- [53] G. Schrag and L. Cammack, Some full and fully deterministic generalized Petersen graphs (preprint).
- [54] W.T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* 22(1947) 107-111.
- [55] W.T. Tutte, The factor of graphs, *Canad. J. Math.* 4(1952) 314-328.
- [56] W.T. Tutte, A short proof of the factor theorem for finite graphs, *Canad. J. Math.* 6(1954) 347-352.
- [57] W.T. Tutte, Graph factors, *Combinatorica* 1(1981) 79-97.

- [58] Q. L. Yu, Some theorems on the star-factors, J. of Shandong Uni. (1987) Vol.22, No.4, 24-28.
- [59] Q. L. Yu and C. P. Chen, On tree-factor covered graphs, JCMCC 2(1987), 211-218.
- [60] Q. L. Yu, On barrier sets of star-factors, Graphs and Combinatorics, 6(1990) 71-76.
- [61] Q. L. Yu, On the $\{P_2, P_3\}$ -factor covered graph, J. Of Mathematics, (to appear).
- [62] Q. L. Yu, Classifying two-extendable generalized Petersen graphs. Discrete Math (to appear).
- [63] Q. L. Yu, Counting the number of star-factors in graphs, (submitted to J. Graph Theory).
- [64] Q. L. Yu, t -matching covers and degrees, (to appear Congressus Numerantium).
- [65] Q. L. Yu, A note on n -extendable graphs, (submitted J. Graph Theory).
- [66] J. Zaks, On the 1-factors of n -connected graphs, J. of Combin. Theory (B) 11(1971) 169-180.