

TREE-DECOMPOSABLE THEORIES

by

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Tree-decomposable Theories

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ABSTRACT

We present a notion of tree-decomposition for first-order theories and structures, proving it equivalent to several other conditions. We conclude the discussion with examples of structures and theories satisfying these conditions.

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CONTENTS

Introduction.....	1
1. Background.....	4
2. Main results.....	13
3. Some tree-decomposable structures.....	43
References.....	51

"What had formerly mattered was following the sequences of ideas and the whole intellectual mosaic of a Game with rapid attentiveness, practiced memory, and full understanding. But there now arose the demand for a deeper and more spiritual approach. After each symbol conjured up by the director of a Game, each player was required to perform silent, formal meditation on the content, origin, and meaning of this symbol, to call to mind intensively and organically its full purport. The members of the Order and of the Game associations brought the technique and practice of contemplation with them from their elite schools, where the art of contemplation and meditation was nurtured with the greatest care. In this way the hieroglyphs of the Game were kept from degenerating into mere empty signs."

—Hermann Hesse, *The Glass Bead Game*

(Translated from the German by Richard and Clara Winston.)

"It was not easy. It needed great powers of reasoning and improvisation. The arithmetical problems raised, for instance, by such a statement as "two and two make five" were beyond his intellectual grasp. It needed also a sort of athleticism of mind, an ability at one moment to make the most delicate use of logic and at the next to be unconscious of the crudest logical errors. Stupidity was as necessary as intelligence, and as difficult to attain."

—George Orwell, *Nineteen Eighty-Four*.

Introduction

This thesis is concerned with a branch of model theory known as stability theory, some of whose basic notions are implicit in an early paper of Michael Morley. These notions were made explicit and generalized by Saharon Shelah, who introduced the idea of a *stable* theory. His work was motivated in part by the problem of classifying the models of a given first-order theory according to some organized and simple system of cardinal invariants, such as the dimension of a vector space over a given field, the number and cardinalities of the equivalence classes of an equivalence relation, or the torsion coefficients of an abelian group. One of his principal results, and one that accounts for the great interest in stable theories, is that without the hypothesis of stability, this problem is actually intractable, and no such cardinal invariants can exist. Furthermore, the number of such models in a given infinite cardinality greater than the cardinality of the language, κ say, is actually the maximum possible, namely 2^κ . Thus one can hope to have a reasonable taxonomy for the models of a given theory only if that theory is stable.

There are two definitions of stability common in the literature of the subject. One is that a theory is stable just if no infinite linear ordering of tuples in a model of the theory is definable by a formula; this the definition we use here. The other is that a theory is stable in case the space of types of finite tuples over an infinite subset of a model never exceeds the cardinality of that subset. It can be shown (see, e.g., [P]) that the two definitions are equivalent. Our Chapter 1 is primarily a compendium of basic results about stability, but concludes with two lemmas of

general model theory not especially pertinent to the topics of stability or forking.

Various conditions stronger than stability have been considered in the hope of achieving correspondingly stronger classification results. One such is called *tree-decomposability*, and first appeared in [BS]. This concept is first introduced here in Chapter 2, which is devoted to listing several conditions on a theory and establishing that each is equivalent to tree-decomposability. The main results here are drawn from [BS]. We hope that the treatment given here will be easier to follow than that of Baldwin and Shelah, who were pursuing a much wider investigation. Chapter 3 concludes the work by furnishing a few concrete examples of tree-decomposable theories. Tree-decomposability is such a stringent condition that such examples are fairly difficult to come by.

The notation we have adopted is mostly standard, with a few exceptions which are worth noting here. Where letters $a, A,$ represent objects of some kind, the same letters equipped with overbars, such as $\bar{a}, \bar{A}, \dots,$ are used to represent tuples (finite sequences) of objects of the same kind. The length of a tuple \bar{a} is denoted $\ell(\bar{a})$. Where convenient, the same symbols \bar{a}, \bar{A}, \dots are used for the ranges of these tuples. A symbol for an infinite sequence is obtained by enclosing a symbol for a typical entry in angle brackets, thus: $\langle \bar{a}_i : i < \omega \rangle$. The empty sequence is denoted $\langle \rangle$; if ρ, σ are sequences, then we write $\rho \subset \sigma$ to mean that ρ is a proper initial segment of σ . The sequence obtained by concatenating tuples \bar{a}, \bar{b}, \dots is denoted $\bar{a}\bar{b}\dots$, and this practice is extended even to transfinite concatenation. In certain cases, particularly those which involve the independence symbol \downarrow , we also represent a union of two sets $A \cup B$ or $A \cup \{b\}$ by juxtaposition, thus: AB, Ab . Except where noted, if B is a set and \bar{A} is a

(possibly transfinite) sequence of sets, then $\bar{A} \uparrow B$ is the sequence of sets such that $\bar{A} \uparrow B$ has the same length as \bar{A} and an entry of $\bar{A} \uparrow B$ is obtained from the corresponding entry of \bar{A} by intersecting it with B . If E is an equivalence relation defined on some set including a , then $[a]_E$ is the equivalence class of E to which a belongs; if in addition A is a set, then $E \uparrow A$ is $E \cap (A \times A)$.

Chapter 1

Background

Here we lay the foundation for the material to be presented in Chapter 2. We assume the reader is familiar with the concepts of first-order logic and model theory. A good general reference is the text of Chang and Keisler [CK]. Within model theory we shall be working with the concepts of stability theory, also called classification theory, a theory which has been developed largely by Shelah during the last two decades. In this regard several references are available. We shall rely on Pillay's book [P] which is less intimidating for the beginner than the more comprehensive works of Baldwin [B] and Shelah [S].

First-order theories considered will be assumed complete unless otherwise described. For each theory T , the underlying first-order language L is assumed relational, except that we permit constant symbols. The number of relation and constant symbols in the language L is denoted $|L|$. Models will be denoted by M, N with or without subscripts and occasionally by other letters. The universe of a model M will also be denoted M and its cardinality by $|M|$. The restriction to relational languages should spare the reader any confusion arising from the failure to distinguish notationally between a model and its universe. Subsets of models will be denoted A, B, C, \dots with or without subscripts. We will observe the following convention, first adopted by Shelah. In the context of a particular first-order theory T , all models of T considered are elementary submodels of a fixed *monster model* \mathfrak{C} , referred to in [P] as the *big model*, and therein denoted \bar{M} . This model \mathfrak{C} is κ -saturated for every $\kappa < |\mathfrak{C}|$, and is *homogeneous* in the sense that given two

subsets A and B of \mathfrak{C} of like cardinality $< |\mathfrak{C}|$ and an elementary map f from A to B , there is an automorphism of \mathfrak{C} extending f . The sets C and C' are called *conjugate over* A if there is a automorphism of \mathfrak{C} which fixes A pointwise and maps C onto C' .

The relation of inclusion between elementary submodels of \mathfrak{C} is denoted \leq . If the models of T we wish to consider all have cardinality $< \lambda$, then any model \mathfrak{C} with the following mapping properties will do:

(i) If $M \leq \mathfrak{C}$, N is any elementary extension of M (not necessarily related to \mathfrak{C}), and $|M|, |N| < \lambda$, then there exists an elementary embedding F of N in \mathfrak{C} which fixes M pointwise.

(ii) If $M, N \leq \mathfrak{C}$ have cardinality $< \lambda$ and $F : M \rightarrow N$ is an isomorphism, then F can be extended to an automorphism of \mathfrak{C} .

In the context of T , not only are models assumed to be elementary submodels of \mathfrak{C} , but sets denoted A, B, C with or without subscripts are assumed to be subsets of (the universe of) \mathfrak{C} , and elements and tuples of elements are also assumed to be from \mathfrak{C} . The monster model is valuable because at one stroke it makes accessible all consequences of the amalgamation property for elementary classes; it also makes it possible to shorten and simplify many arguments.

Definition 1.1 The first-order theory T is *unstable* if there are a formula $\phi(\bar{x}, \bar{y})$ of $L(T)$, with $\ell(\bar{x}) = \ell(\bar{y})$, and $\ell(\bar{x})$ -tuples \bar{a}_i ($i < \omega$) such that $\models \phi(\bar{a}_i, \bar{a}_j)$ if and only if $i < j$ for all $i, j < \omega$. T is *stable* if it is not unstable.

This definition is not the one chosen by Pillay, but is shown equivalent in [P, Theorem 2.15]. It is understood that the \bar{a}_i are tuples of elements of the monster model \mathfrak{C} and that \models refers to truth in \mathfrak{C} . Here we have tacitly adopted another model-theoretic convention. When we speak of the formula $\phi(\bar{x}, \bar{y})$ it is understood that a triple $\langle \phi, \bar{x}, \bar{y} \rangle$ is given, consisting of a formula ϕ and disjoint tuples of variables \bar{x}, \bar{y} such that every variable free in ϕ occurs in either \bar{x} or \bar{y} .

When we speak of types, it should be clear from the context whether we are speaking of complete types or arbitrary types. The set of (complete) n -types over A is denoted $S_n(A)$. If $p \in S_n(A)$, then a *defining schema* d for p is a mapping $\phi(\bar{x}, \bar{y}) \mapsto d\phi(\bar{y})$, where \bar{x} is a distinguished n -tuple of variables, $\phi(\bar{x}, \bar{y})$ runs through all formulas without parameters, $d\phi(\bar{y})$ is a formula over A , and

$$\phi(\bar{x}, \bar{b}) \in p \iff \models d\phi(\bar{b})$$

for all $\bar{b} \in A$. One of the pleasant features of stable theories is that complete types are definable (see [P, Proposition 2.19]):

Theorem 1.2 Let T be stable and $n < \omega$. Every $p \in S_n(A)$ has a defining schema.

From now on we are going to assume that the theory we are dealing with is stable.

With the concept of stability in hand, we proceed to introduce the notions of forking and independence. Once again, for the sake of simplicity we use an equivalent condition rather than the definition chosen by Pillay.

Definition 1.3 Let T be a stable first-order theory and $\phi(\bar{x}, \bar{y})$ be a formula of T .

(1) The formula $\phi(\bar{x}, \bar{b})$ is said to *fork over* A if there is a set $\{\bar{b}_i : i < \omega\}$ indiscernible over A such that $\bar{b}_0 = \bar{b}$ and the formulas $\phi(\bar{x}, \bar{b}_i)$ ($i < \omega$) are *almost disjoint* in the sense that for some $n < \omega$,

$$\models \neg \exists \bar{x} \wedge \{\phi(\bar{x}, \bar{b}_i) : i < n\}.$$

(2) A type $p \in S_m(B)$ *forks over* A if there is a formula $\phi(\bar{x}, \bar{b}) \in p$ which forks over A .

(3) B and C are *independent over* A , written $B \downarrow_A C$, if for every $\bar{c} \in C$, $\text{tp}(\bar{c} | A \cup B)$ does not fork over A ; B and C are *independent*, written $B \downarrow C$, if $B \downarrow_{\emptyset} C$.

The property we have chosen to define the forking of a formula over A is called *dividing over* A in $[P]$; see Proposition 6.8. Some useful properties of the independence relation are collected in:

Theorem 1.4 (Properties of the independence relation)

(1) (Finite character)

(i) $A \downarrow_B C$ if and only if $A_0 \downarrow_B C_0$ for all finite A_0, C_0 with $A_0 \subseteq A$ and $C_0 \subseteq C$.

(ii) If $\neg[A \downarrow_B C]$, then there exists finite $B_0 \subseteq B$ such that $\neg[A \downarrow_D C]$ for all D with $B_0 \subseteq D \subseteq B$.

- (2) (Lowering) If $A \downarrow_B C$, then $A \downarrow_D C$ for all D such that $B \subseteq D \subseteq A \cup B \cup C$.
- (3) (Raising) If $A \downarrow_B C$, $D \subseteq B$, and $E \subseteq B$, then $A \cup D \downarrow_B C \cup E$.
- (4) (Transitivity) If $A \downarrow_B C$ and $A \downarrow_{B \cup C} D$, then $A \downarrow_B D$.
- (5) (Symmetry) $A \downarrow_B C$ if and only if $C \downarrow_B A$.
- (6) (Extension) For any A, B, C there exists C' conjugate to C over B such that $A \downarrow_B C'$.
- (7) For all A and B , there exists $C \subseteq B$ such that $|C| \leq |L| + |A| + \aleph_0$ and $A \downarrow_C B$.

Transitivity and extension are Proposition 3.8 (iii) and (iv) of [P], while symmetry is Proposition 3.9 of [P], reformulated in terms of independence. Part (7) is Proposition 3.23 of [P]. In the presence of symmetry, the other properties follow easily from the form of the definition of forking. In applying the above rules for manipulating the independence relation, we shall often use symmetry tacitly.

There is an important connection between the definability of types and independence:

Lemma 1.5 Let M be a model, $\ell(\bar{a}) = \ell(\bar{x})$, and d be a defining schema for $\text{tp}(\bar{a}|M)$. Then $\bar{a} \downarrow_M B$ if and only if for every formula $\phi(\bar{x}, \bar{y})$ and $\bar{b} \in M \cup B$, $\models \phi(\bar{a}, \bar{b}) \leftrightarrow d\phi(\bar{b})$.

We need the notion of *strong type*, which is somewhat awkward to define.

Recall that an *n-type over A* is a consistent set of formulas $\phi(\bar{x})$ over A , where \bar{x} is a designated n -tuple of distinct variables. Further, for an n -tuple \bar{a} , the *type of \bar{a} over A* , denoted $\text{tp}(\bar{a}|A)$, is the set

$$\{\phi(\bar{x}) : \phi(\bar{x}) \text{ is a formula over } A \text{ and } \models \phi(\bar{a})\}.$$

Let $FE^n(A)$ denote the set

$$\{E : E \text{ is an equivalence relation on } \mathfrak{C}^n \text{ which is definable over } A \\ \text{and has only finitely many classes}\},$$

and $CFE^n(A)$ denote $\cup\{\mathfrak{C}^n/E : E \in FE^n(A)\}$. A *strong n-type over A* is a subset p of $CFE^n(A)$ such that $\cap p \neq \emptyset$; p is called *complete* if $p \cap (\mathfrak{C}^n/E) \neq \emptyset$ for all $E \in FE^n(A)$. For an n -tuple \bar{a} the *strong type of \bar{a} over A* , denoted $\text{stp}(\bar{a}|A)$, is $\{X \in CFE^n(A) : \bar{a} \in X\}$. For a strong n -type p over A , the corresponding n -type p^* over A is

$$\{\phi(\bar{x}) : \phi(\bar{x}) \text{ is over } A \text{ and } \exists X \in p \text{ such that the} \\ \text{solution set of } \phi(\bar{x}) \text{ in } \mathfrak{C}^n \text{ is } \bigcup_{\alpha \in \text{Aut}(\mathfrak{C})} \alpha(X)\}.$$

For an n -type p over A the corresponding strong n -type p^s is

$$\{X \in CFE^n(A) : \exists \phi(\bar{x}) \in p \text{ such that } X \text{ is the solution set of} \\ \phi(\bar{x}) \text{ in } \mathfrak{C}\}.$$

Clearly, $p^s = q^s$ if and only if the types p, q are equivalent, and $(p^s)^*$ is equivalent to p for any n -type p over A . If A is the universe of a model, i.e. of an elementary submodel of \mathfrak{C} , and p^* and q^* are equivalent, then p and q are equivalent in the sense that $\cap p = \cap q$. When we restrict to complete types and complete strong types over a model, then the mappings $p \mapsto p^s$ and $p \mapsto p^*$

are bijections, and each is inverse to the other. A basic lemma about strong types is:

Lemma 1.6 (1) For all \bar{a}, A, B there exists \bar{b} such that $\text{stp}(\bar{b}|A) = \text{stp}(\bar{a}|A)$ and $\bar{b} \downarrow_A B$.

(2) For all \bar{a}, \bar{b}, A, B , if $\text{stp}(\bar{a}|A) = \text{stp}(\bar{b}|A)$, $\bar{a} \downarrow_A B$, and $\bar{b} \downarrow_A B$, then $\text{stp}(\bar{a}|A \cup B) = \text{stp}(\bar{b}|A \cup B)$.

Proof. (1) Choose a model $M \supseteq A$ such that $M \downarrow_A B$. This is possible by extension. Again by extension, choose \bar{b} such that $\text{tp}(\bar{b}|M) = \text{tp}(\bar{a}|M)$ and $\bar{b} \downarrow_M B$. By transitivity, $B \downarrow_M \bar{b}$. Further, $\text{stp}(\bar{a}|A) = \text{stp}(\bar{b}|A)$ since \bar{a} and \bar{b} realize the same strong type over M .

(2) This is Proposition 4.34 of [P].

The first part of the lemma is essentially the rule of extension (Theorem 1.2 (6)) for strong types. The analogue for types of the second part of the lemma is not true; this is the reason for introducing strong types.

A sequence $\langle \bar{a}_i : i < \omega \rangle$ is a *Morley sequence over A* if each \bar{a}_i realizes the same strong type over A and $\bar{a}_{i+1} \downarrow_A \bar{a}_0 \dots \bar{a}_i$ for each $i < \omega$.

Lemma 1.7 Let $\langle \bar{a}_i : i < \omega \rangle$ be a Morley sequence over A .

(1) $\{\bar{a}_i : i < \omega\}$ is an indiscernible set over A . Then:

(2) If $\bar{a}_0 \downarrow A$, then $\bar{a}_0 \bar{a}_1 \dots \downarrow A$.

Proof. (1) This is essentially Fact 7.3 of [P].

(2) By induction on i , we show that $\bar{a}_0 \dots \bar{a}_i \downarrow A$ for all $i < \omega$. By finite character, this is enough. The basis is given. Suppose that $\bar{a}_0 \dots \bar{a}_k \downarrow A$. Since $\langle \bar{a}_i : i < \omega \rangle$ is a Morley sequence over A , $\bar{a}_{k+1} \downarrow_A \bar{a}_0 \dots \bar{a}_k$ and $\text{stp}(\bar{a}_k) = \text{stp}(\bar{a}_0 | A)$. From the former by raising A we have $\bar{a}_{k+1} \downarrow_A \bar{a}_0 \dots \bar{a}_k A$, and from the latter, $\bar{a}_{k+1} \downarrow A$ by comparison with \bar{a}_0 . By transitivity, $\bar{a}_{k+1} \downarrow \bar{a}_0 \dots \bar{a}_k A$. By raising, lowering, and finite character, we get

$$\bar{a}_0 \dots \bar{a}_k \bar{a}_{k+1} \downarrow_{\bar{a}_0 \dots \bar{a}_k} A.$$

From this and the induction hypothesis, we have $\bar{a}_0 \dots \bar{a}_k \bar{a}_{k+1} \downarrow A$ by transitivity. This completes the induction hypothesis and the proof.

There are two other standard lemmas we shall need, which are not strongly connected with stability. A formula $\phi(\bar{x})$ over A is called *non-null* if $\models \exists \bar{x} \phi(\bar{x})$.

Lemma 1.8 (Tarski-Vaught Criterion) If every formula $\phi(x)$ over A which is non-null has a solution in A , then A is the universe of a model.

Of course, according to our convention, $A \subseteq \mathfrak{C}$, and the conclusion means that A is the universe of an elementary submodel of \mathfrak{C} .

Lemma 1.9 Let $\phi(\bar{x})$ be a non-null formula over A and B be a finite set such that $\models \forall \bar{x}[\phi(\bar{x}) \rightarrow \bar{x} \cap B \neq \emptyset]$. Then $\text{acl}(A) \cap B \neq \emptyset$.

Proof. If ϕ has only finitely many solutions in M , the proof is easy. If not, let \bar{a}^i ($i < \omega$) be distinct solutions of ϕ in M . Choose $s \subseteq \{1, \dots, \ell(\bar{x})\}$ and b_j ($j \in s$) with $|s|$ as large as possible such that for infinitely many i , $a_j^i = b_j$ for all $j \in s$. Note that $|s| < \ell(\bar{x})$. By thinning we can suppose that $a_j^i = b_j$ for all $i < \omega$ and $j \in s$, and that $a_h^i \neq a_j^k$ for all $i, k < \omega$ and $h, j \in \{1, \dots, \ell(\bar{x})\} - s$. This shows that $s \neq \emptyset$; otherwise the hypothesis about A would be violated. To simplify notation, suppose that s is an initial segment of $\{1, \dots, \ell(\bar{x})\}$, and write $\bar{x} = \bar{y}\bar{z}$, where $\ell(\bar{y}) = |s|$. Let $m = |B|$. Consider the formula

$$\psi(\bar{y}) = \exists \bar{z}_0 \dots \exists \bar{z}_m \left[\bigwedge_{i < k \leq m} \bar{z}_i \cap \bar{z}_k = \emptyset \ \& \ \bigwedge_{i \leq m} \phi(\bar{y}, \bar{z}_i) \right].$$

This formula is non-null since $M \models \psi(\bar{b})$. Also, it is clear that if \bar{b}' is any solution of ψ , then we can choose \bar{c} such that $M \models \phi(\bar{b}'\bar{c})$ and $\bar{c} \cap B = \emptyset$. Thus any solution of ψ meets B . The result follows by induction on $\ell(\bar{x})$.

Chapter 2

Main Results

We recall that a first-order theory T is *unstable* if and only if there exist n , $1 \leq n < \omega$, and a formula $\varphi(\bar{x}, \bar{y})$ of T with $\ell(\bar{x}) = \ell(\bar{y}) = n$, such that in some model M of T , there are n -tuples \bar{a}_i ($i < \omega$) satisfying $M \models \varphi(\bar{a}_i, \bar{a}_j) \iff i < j$ ($i, j < \omega$). This concept can be adapted to monadic logic in various ways. In this chapter, we introduce strong and weak notions of monadic instability for first-order theories, and prove a theorem of Shelah showing that for strong monadic instability, whether the definition is framed in terms of tuples \bar{x}, \bar{y} or singletons x, y makes no difference. Later in the chapter we introduce the notion of tree-decomposability for a first-order theory. We prove that a number of conditions are equivalent to tree-decomposability, among them strong and weak monadic stability.

Definition 2.1 We define a first-order theory T to be *strongly monadically unstable* if there are a first-order formula $\phi(x, y)$ in an extension T' of T by unary predicates and a model M of T' such that $\phi(x, y)$ linearly orders an infinite subset of M . We will say that T is *weakly monadically stable* precisely when it is not strongly monadically unstable. In the same vein, we define T to be *weakly monadically unstable* when there exist an extension T' of T by unary predicates and a monadic formula $\phi(\bar{x}, \bar{y})$ of T' such that $\ell(\bar{x}) = \ell(\bar{y})$, and, for every cardinal λ , there is a model M of T' in which $\phi(\bar{x}, \bar{y})$ linearly orders a

subset of $M^{\ell(\bar{x})}$ of size λ . T is *strongly monadically stable* when it is not weakly monadically unstable.

Theorem 2.2 Let the first-order theory T contain some formula $\phi(\bar{x}, \bar{y})$ such that (1) \bar{x} and \bar{y} are of equal length k , and (2) every linear ordering is realized as the ordering induced by ϕ in a set of k -tuples in some model of T . Then there is a formula $\phi'(x, y, \bar{A})$ with new unary predicates \bar{A} , such that any linear ordering is realized – by interpreting the new unary predicates appropriately – as the ordering of *individuals* defined by ϕ' in some subset of a model of T .

Proof. The following is a modified form of the argument given in Section 8 of [BS]:

Let a theory T and formula $\phi(\bar{x}, \bar{y})$ be given which satisfy the hypotheses of the theorem. Then there exists a model M of T and a collection of k -tuples in M such that ϕ defines among the tuples of the collection a dense linear ordering without endpoints. Invoking Ramsey's theorem and compactness, we lose no generality by assuming the tuples to be order-indiscernible; given order-indiscernibility, we lose no generality by assuming the tuples to be pairwise disjoint. Furthermore, there is clearly nothing to prove unless $k > 1$. Proceeding by induction on k and using compactness again, it is sufficient to show that in some expansion of M by unary predicates some formula linearly orders an infinite collection of $(k-1)$ -tuples or 1-tuples.

Index the tuples as $\langle \bar{a}_s : s \in I \rangle$, where $(I, <)$ is a suitable linear order, and suppose that for s and t in I , $\models \phi(\bar{a}_s, \bar{a}_t)$ if and only if $s < t$. For notational convenience, split each \bar{a}_s , $s \in I$, as $\bar{b}_s c_s$.

Towards a contradiction, assume that no formula in an expansion of M by unary predicates linearly orders an infinite set of $(k-1)$ -tuples or 1-tuples. The key idea of the proof is to show that for $s, t, u, v \in I$,

$$(\#) \quad (u < s < t < v \text{ or } v < s < t < u \rightarrow \models \varphi(\bar{b}_s c_u, \bar{b}_t c_v)).$$

Once this is established we will quickly reach a contradiction. Our main tool is:

Proposition 2.3 Let J be an open interval of I and $\psi(\bar{x})$ and $\chi(y, z)$ be formulas which may contain parameters from $\cup\{\bar{a}_i : i \in I - J\}$ but no others. Let $s, t \in J$ be distinct. Then

- (i) $\models \psi(\bar{b}_s c_t) \leftrightarrow \psi(\bar{b}_t c_s)$
- (ii) $\models \psi(\bar{b}_s c_s) \leftrightarrow \psi(\bar{b}_s c_t)$
- (iii) $\models \chi(c_s, c_t) \leftrightarrow \chi(c_t, c_s)$.

Proof. Without loss of generality, $s < t$. Let $Q = \{c_i : i \in J\}$. Let \bar{x}', \bar{y}' be obtained from \bar{x}, \bar{y} respectively by deleting the last entries.

(i) Suppose the conclusion fails. Without loss of generality, $\models \psi(\bar{b}_s c_t) \ \& \ \neg \psi(\bar{b}_t c_s)$. Then for distinct $u, v \in J$,

$$u < v \leftrightarrow \models (\exists z \in Q)[\psi(\bar{b}_u z) \ \& \ \neg \psi(\bar{b}_v z)].$$

Thus the formula $(\exists z \in Q)[\psi(\bar{x}'z) \ \& \ \neg \psi(\bar{y}'z)]$ linearly orders the set $\{\bar{b}_i : i \in J\}$.

This contradicts our initial assumption.

(ii) Suppose the conclusion fails. Without loss of generality,

$\models \psi(\bar{b}_s c_t) \ \& \ \neg \psi(\bar{b}_s c_s)$. From (i), $\models \psi(\bar{b}_t c_s)$. Thus c_s is the unique solution of $\neg \psi(\bar{b}_s z) \ \& \ Q(z)$, so for distinct $u, v \in J$,

$$u < v \iff \models (\exists w \in Q)(\exists z \in Q)[\neg \psi(\bar{b}_u w) \ \& \ \neg \psi(\bar{b}_v z) \ \& \ \varphi(\bar{b}_u w, \bar{b}_v z)].$$

Again this contradicts our initial assumption.

(iii) Suppose the conclusion fails. Without loss of generality,

$\models \chi(c_s, c_t) \ \& \ \neg \chi(c_t, c_s)$. Then $\chi(x, y)$ linearly orders the set Q , which again contradicts our initial assumption.

Now consider the following sequence of conditions on a 4-tuple (s, t, u, v) of elements of I :

- (1) $s = u < t = v$
- (2) $u < s < t = v$
- (3) $u < s < t < v$
- (4) $s < u < t < v$
- (5) $s < u < v < t$
- (6) $s < v < u < t$
- (7) $v < s < u < t$
- (8) $v < s < t < u$.

We claim that each of these conditions implies $\models \varphi(\bar{b}_s c_u, \bar{b}_t c_v)$. For (1), it is part of our hypothesis about φ . We see that each of (2)–(8) implies $\models \varphi(\bar{b}_s c_u, \bar{b}_t c_v)$ because its predecessor does. For (2), apply (ii) with $\psi(\bar{x}) = \varphi(\bar{x}, \bar{b}_t c_v)$. For (3), apply (ii) with $\psi(\bar{x}) = \varphi(\bar{b}_s c_u, \bar{x})$. The transition from (5) to (6) uses (iii), while all the other transitions are applications of (i).

Notice that since (3) and (8) each imply $\vDash \varphi(\bar{b}_s c_u, \bar{b}_t c_v)$, we have established (#). Reversing the ordering of I and replacing $\varphi(\bar{x}, \bar{y})$ by $\neg \varphi(\bar{y}, \bar{x}) \ \& \ \bar{x} \neq \bar{y}$, we obtain

$$\text{(\#\#)} \quad (v < t < s < u \text{ or } u < t < s < v) \rightarrow \vDash \neg \varphi(\bar{b}_t c_v, \bar{b}_s c_u).$$

From (#) and (\#\#) it follows that if we fix u and v such that $u < v$, then the formula $\varphi(\bar{x}' c_u, \bar{y}' c_v)$ linearly orders the set $\{\bar{b}_i : u < i < v\}$. This final contradiction completes the proof of the theorem.

The reader may have observed that the formula $\varphi'(x, y, \bar{A})$ generated by the last proof contains individual parameters as well as new unary predicates. It is a trivial matter to eliminate these parameters in favour of additional unary predicates.

We now introduce a first-order theory P of the product of two infinite sets. The language of P will be the first-order language determined by three unary predicates U_0, U_1, U_2 , and two binary predicates P_1, P_2 . The intended interpretations of the predicates are as follows: U_0 is the Cartesian product of two infinite sets U_1 and U_2 , and P_1 and P_2 are the projection relations which hold between a pair and its first and second entries. Thus suitable axioms for P are the following: for each positive integer n , axioms stating that there are at least n distinct individuals in each of U_1 and U_2 ; an axiom stating that for each x_1 in U_1 and x_2 in U_2 , there is a unique x_0 in U_0 such that $P_1(x_0, x_1)$ and $P_2(x_0, x_2)$; and an axiom stating that for each x_0 in U_0 there exist unique x_1 in U_1 and x_2 in U_2 such that $P_1(x_0, x_1)$ and $P_2(x_0, x_2)$. For our present purposes, nothing need be said about disjointness, so we leave P incomplete.

The stage is now set for our second theorem.

Theorem 2.4 If P is interpretable in T , then T is strongly monadically unstable.

Proof. If P is interpretable in T , then let $\theta_0(x)$, $\theta_1(x)$, $\theta_2(x)$, $\pi_1(x_0, x_1)$, and $\pi_2(x_0, x_2)$ be the respective interpretations of $U_0(x)$, $U_1(x)$, $U_2(x)$, $P_1(x_0, x_1)$ and $P_2(x_0, x_2)$ in T . In a model M of T , let $\theta_1(M)$, $\theta_2(M)$ be the respective solution sets of the formulas $\theta_1(x)$, $\theta_2(x)$. Select from the infinite sets $\theta_1(M)$, $\theta_2(M)$, equipollent infinite subsets A_1, A_2 , together with a bijection f of A_1 onto A_2 witnessing this equipollence. Let R be a linear ordering of A_1 . Using R and f , we will construct a formula $\phi(x, y, X)$ and a subset S of M such that $\phi(x, y, S)$ linearly orders an infinite subset of M .

Define $S = \{a_0 \in \theta_0(M) : \text{for some } a_1 \in A_1 \text{ and } a_2 \in A_2,$
 $M \models \pi_1(a_0, a_1) \ \& \ \pi_2(a_0, a_2) \text{ and } R(a_1, f^{-1}(a_2))\},$

and

$D = \{a_0 \in \theta_0(M) : \text{for some } a_1 \in A_1, M \models \pi_1(a_0, a_1) \ \& \ \pi_2(a_0, f(a_1))\}.$

Then the required formula $\phi(x, y, X)$ is

$\exists z(X(z) \ \& \ \exists v(\theta_1(v) \ \& \ \pi_1(z, v) \ \& \ \pi_1(x, v)) \ \& \ \exists w(\theta_2(w) \ \& \ \pi_2(z, w) \ \& \ \pi_2(y, w))).$

It is easily verified that D is infinite and that $\phi(x, y, S)$ defines a linear ordering of D .

An important ingredient of our next theorem is the following condition on a theory T :

Definition 2.5 T will be said to satisfy the *triviality condition*, or to be *forking-trivial*, if, for all subsets A, C, D and all elements b of any model of T , $A \downarrow_C D$ implies that either $A \downarrow_{C \cup D} b$ or $C \downarrow_{A \cup D} b$.

Theorem 2.6 If the stable theory T fails to satisfy the triviality condition, then some extension of T by unary predicates interprets P .

Proof. The aim of the argument is to construct some formula which provides a one-to-one pairing function from the Cartesian product of two infinite subsets of a model of T to a third subset of that model. We begin by choosing sets A, C, D , and an element b which constitute a counterexample to forking-triviality. Naming the elements of D , we may suppose that D is empty. From Theorem 1.4 (1)(i), there are tuples $\bar{d}_0 \subseteq A, \bar{e}_0 \subseteq C$ such that neither $\bar{d}_0 \downarrow_C b$ nor $\bar{e}_0 \downarrow_A b$. From Theorem 1.4 (1)(ii), there are tuples $\bar{d}_1 \subseteq A, \bar{e}_1 \subseteq C$ such that neither $\bar{d}_0 \downarrow_{\bar{e}_0 \bar{e}_1} b$ nor $\bar{e}_0 \downarrow_{\bar{d}_0 \bar{d}_1} b$. Applying Theorem 1.4 (1)(i) again, and writing \bar{a} for $\bar{d}_0 \bar{d}_1$, and \bar{c} for $\bar{e}_0 \bar{e}_1$, we have $\bar{a} \downarrow \bar{c}$ but neither $\bar{a} \downarrow_c b$ nor $\bar{c} \downarrow_a b$.

Below, the tuples \bar{a}, \bar{c} will be denoted \bar{a}_0, \bar{c}_0 when convenient. Choose $\bar{a}_1, \bar{a}_2, \dots$ in turn realizing $\text{stp}(\bar{a})$ such that

$$(1) \quad \bar{a}_{i+1} \downarrow \bar{a}_0 \dots \bar{a}_i \bar{c} \quad (i < \omega).$$

Next choose $\bar{c}_1, \bar{c}_2, \dots$ in turn realizing $\text{stp}(\bar{c})$ such that

$$(2) \quad \bar{c}_{i+1} \downarrow \bar{a}_0 \bar{a}_1 \dots b \bar{c}_0 \dots \bar{c}_i \quad (i < \omega).$$

For induction on j suppose

$$(3) \quad \bar{a}_{i+1} \downarrow \bar{a}_0 \dots \bar{a}_i b \bar{c}_0 \dots \bar{c}_j.$$

From (2) by lowering and raising

$$(4) \quad \bar{a}_{i+1} \bar{a}_0 \dots \bar{a}_i b \bar{c}_0 \dots \bar{c}_j \downarrow \bar{a}_0 \dots \bar{a}_i b \bar{c}_0 \dots \bar{c}_{j+1}.$$

By transitivity from (3) and (4)

$$\bar{a}_{i+1} \downarrow \bar{a}_0 \dots \bar{a}_i b \bar{c}_0 \dots \bar{c}_{j+1}.$$

Note that (3) holds for $j = 0$ by (1). Therefore (3) holds for all $i, j < \omega$. By finite character, i.e., Theorem 1.4 (1)(i),

$$(5) \quad \bar{a}_{i+1} \downarrow \bar{a}_0 \dots \bar{a}_i b \bar{c}_0 \bar{c}_1 \dots .$$

From (5) and Lemma 1.6, $\bar{a}_1, \bar{a}_2, \dots$ all realize the same strong type over

$\bar{a} b \bar{c}_0 \bar{c}_1 \dots$. Hence, lowering $\bar{a} b \bar{c}_0 \bar{c}_1 \dots$ in (5) we see that $\langle \bar{a}_i : 1 \leq i < \omega \rangle$ is a

Morley sequence over $\bar{a} b \bar{c}_0 \bar{c}_1 \dots$. Similarly, $\langle \bar{c}_i : 1 \leq i < \omega \rangle$ is a Morley sequence over $\bar{a}_0 \bar{a}_1 \dots b \bar{c}$. Now by Lemma 1.7 (2)

$$(6) \quad \bar{a}_1 \bar{a}_2 \dots \downarrow \bar{a} b \bar{c}_0 \bar{c}_1 \dots, \quad \bar{c}_1 \bar{c}_2 \dots \downarrow \bar{a}_0 \bar{a}_1 \dots b \bar{c}.$$

We also observe from Lemma 1.7 (1) that $\{\bar{a}_i : 1 \leq i < \omega\}$ is indiscernible over $\bar{a} b \bar{c}_0 \bar{c}_1 \dots$ and $\{\bar{c}_i : 1 \leq i < \omega\}$ is indiscernible over $\bar{a}_0 \bar{a}_1 \dots b \bar{c}$.

Since $\bar{a} \downarrow \bar{c}$, from (1) and Lemma 1.6, $\bar{a}_0, \bar{a}_1, \dots$ all realize the same strong type over \bar{c} . It follows from (1) that $\langle \bar{a}_i : i < \omega \rangle$ is a Morley sequence over \bar{c} .

Since $\bar{a} \downarrow \bar{c}$, $\bar{a}_0 \bar{a}_1 \dots \downarrow \bar{c}$ by Lemma 1.7 (2). Now we see from (1) and (2) that

$$\bar{a}_i \downarrow \bar{a}_0 \dots \bar{a}_{i-1} \bar{c} \quad (i < \omega) \text{ and}$$

$$\bar{c}_i \downarrow \bar{a}_0 \bar{a}_1 \dots \bar{c}_0 \dots \bar{c}_{i-1} \quad (i < \omega).$$

Repeating the argument of the last paragraph, we see that $\langle \bar{a}_i : i < \omega \rangle$ is a Morley sequence over $\bar{c}_0 \bar{c}_1 \dots$ and $\langle \bar{c}_i : i < \omega \rangle$ is a Morley sequence over $\bar{a}_0 \bar{a}_1 \dots$. By Lemma 1.7 (1), the sets $\{\bar{a}_i : i < \omega\}$ and $\{\bar{c}_i : i < \omega\}$ are mutually indiscernible. By naming some elements if necessary, we can suppose that $\bar{a}_i \cap \bar{a}_j = \bar{c}_i \cap \bar{c}_j = \emptyset$ for all i, j with $i < j$.

Since neither $\bar{a} \perp_{\bar{c}} b$ nor $\bar{c} \perp_{\bar{a}} b$, there are formulas $\sigma(x, \bar{y}, \bar{z})$ and $\pi(x, \bar{y}, \bar{z})$ such that $\models \sigma(b, \bar{a}, \bar{c}) \ \& \ \pi(b, \bar{a}, \bar{c})$, $\sigma(b, \bar{y}, \bar{c})$ forks over \bar{c} , and $\pi(b, \bar{a}, \bar{z})$ forks over \bar{a} . Let $\chi(x, \bar{y}, \bar{z})$ denote $\sigma(x, \bar{y}, \bar{z}) \ \& \ \pi(x, \bar{y}, \bar{z})$.

Now let $n = \ell(\bar{a})$ and $m = \ell(\bar{c})$. For each integer i between 1 and n , introduce a new unary predicate U_i , and interpret U_i as the collection of i -th members of tuples $\bar{a}_0, \bar{a}_1, \dots$. For each i between 1 and m , introduce a new unary predicate V_i to represent the set of i -th members of tuples $\bar{c}_0, \bar{c}_1, \dots$. Let L^* be the expanded language. Note that types and indiscernibility mentioned below are relative to the original language. Because the tuples $\bar{a}_1, \bar{a}_2, \dots$ are indiscernible over $\bar{a} \bar{b} \bar{c} \bar{c}_1 \bar{c}_2 \dots$ and the tuples $\bar{c}_1, \bar{c}_2, \dots$ are indiscernible over $\bar{a} \bar{b} \bar{c} \bar{a}_1 \bar{a}_2 \dots$, there can be only finitely many types over b among the tuples of $U_1 \times \dots \times U_n \times V_1 \times \dots \times V_m$, so that some single formula distinguishes the type over b of $\bar{a} \bar{c}$ from any other type over b of a tuple in $U_1 \times \dots \times U_n \times V_1 \times \dots \times V_m$. Let $\chi^*(x, \bar{y}, \bar{z})$ be an L^* -formula such that $\models \chi^*(b, \bar{a}', \bar{c}')$ if and only if $\bar{a}' \in U_1 \times \dots \times U_n$, $\bar{c}' \in V_1 \times \dots \times V_m$, and $\text{tp}(\bar{a}' \bar{c}' | b) = \text{tp}(\bar{a} \bar{c} | b)$. Note that $\chi^*(x, \bar{y}, \bar{z})$ implies $\chi(x, \bar{y}, \bar{z})$. We will now show that \bar{a}' must intersect \bar{a} and \bar{c}' must intersect \bar{c} .

If, on the contrary, $\bar{a}' \in U_1 \times \dots \times U_n$ is disjoint from \bar{a} (say) and $\models \chi^*(b, \bar{a}', \bar{c}')$, then $\models \sigma(b, \bar{a}', \bar{c}')$. But $\sigma(b, \bar{y}, \bar{c}')$ forks over \bar{c}' since $\text{tp}(b\bar{c}') = \text{tp}(b\bar{c})$. This contradicts $\bar{a}' \perp_{\bar{c}'} b$ which follows from (6) by lowering \bar{c}' . We conclude that if $\models \chi^*(b, \bar{a}', \bar{c}')$, then \bar{a}' meets \bar{a} and \bar{c}' meets \bar{c} .

Naming b and then applying Lemma 1.9 to the formula $\chi^*(b, \bar{y}, \bar{z})$ we see that $\bar{a} \cap \text{acl}(b)$ is non-empty, where algebraic closure is in the sense of L^* . Thus there is an L^* -formula $\theta(x, y)$ such that $\theta(x, b)$ has only finitely many solutions, of which at least one lies in \bar{a} . Let a_{ki}, c_{ki} denote the i -th entries of \bar{a}_k, \bar{c}_k . By taking the conjunction with $U_i(x)$ for suitable i , we can suppose that the unique solution of $\theta(x, b)$ is a_{0i} . If $\theta(x, b)$ had another solution in U_i , it would have infinitely many, for $\bar{a}_1, \bar{a}_2, \dots$ all realize the same type over b . In the same way, we can construct a formula $\psi(x, y)$ such that for some j , $1 \leq j \leq m$, the unique solution of $\psi(x, b)$ is c_{0j} . Let M be an \aleph_1 -saturated model of T including b, \bar{a}_k, \bar{c}_k ($k < \omega$). For each l and k , choose an element b_{lk} of M such that a_{li} and c_{kj} are, respectively, the unique solutions of $\theta(x, b_{lk})$ in U_i and $\psi(x, b_{lk})$ in V_j . Let M' be obtained from M by adjoining unary predicates which pick out U_i, V_j , and $\{b_{lk} : l, k < \omega\}$. Clearly $T' = \text{Th}(M')$ is an extension of T by unary predicates which interprets P .

Definition 2.7 In a structure M properly containing the set A , define a binary relation E_A on $M - A$ by $a E_A b$ if either $a = b$ or $\neg[a \downarrow_A b]$.

Remarks (a) If $M \leq M'$, then E_A (in the sense of M) is the restriction to M of E_A (in the sense of M'). Thus E_A depends on M only insofar as M restricts the domain.

(b) $a \downarrow_A a$ if and only if $a \in \text{acl}(A)$. Thus when A is algebraically closed, $a E_A b$ is the same as $\neg [a \downarrow_A b]$.

Definition 2.8 Let Γ be a set of L -formulas, M be an L -structure, and A be a proper subset of M . The equivalence relation E on $M - A$ is called a Γ -congruence on M over A if, whenever $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_n$ and $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_n$ are tuples in $M - A$ such that for all i, j , $i < j \leq n$,

- (i) \bar{a}_i is included in a single E -class, \bar{b}_i is included in a single E -class,
- (ii) the E -classes of \bar{a}_i and \bar{a}_j are disjoint, the E -classes of \bar{b}_i and \bar{b}_j are disjoint, and

(iii) $\ell(\bar{a}_i) = \ell(\bar{b}_i)$ and $\text{tp}_\Gamma(\bar{a}_i | A) = \text{tp}_\Gamma(\bar{b}_i | A)$, then $\text{tp}_\Gamma(\bar{a}_0 \bar{a}_1 \dots \bar{a}_n | A) = \text{tp}_\Gamma(\bar{b}_0 \bar{b}_1 \dots \bar{b}_n | A)$.

Bearing the preceding definitions in mind, we can now proceed to the next theorem in the chain.

Theorem 2.9 Let T be a forking-trivial stable theory, and Δ the set of quantifier-free formulas in T . If $N \prec M \models T$, then E_N is a Δ -congruence on M over N . Moreover, if X is any set of equivalence classes of E_N , then $N \cup (\cup X)$ is an elementary submodel of M .

Proof. We first prove that E_N is an equivalence relation, following the method of [BS]. The properties of symmetry and reflexivity are clear. Now suppose that $a E_N b$ and $b E_N c$. We have to show that $a E_N c$. Without loss of generality, suppose that $\neg [a \downarrow_N b]$ and $\neg [b \downarrow_N c]$. Towards a contradiction suppose that $a \downarrow_N c$. Since T is forking-trivial, the following cases are exhaustive:

Case 1. $a \downarrow_{Nc} b$. From $a \downarrow_N c$ we have $a \downarrow_N Nc$. By transitivity we have $a \downarrow_N bc$, whence $a \downarrow_N b$, a contradiction.

Case 2. $c \downarrow_{Na} b$. By the same argument switching a and c , we obtain $b \downarrow_N c$, which is again a contradiction.

This completes the proof that E_N is an equivalence relation.

There is a neat description of forking over N in terms of E_N :

Lemma 2.10 $A \downarrow_N B$ iff $(\forall a \in A)(\forall b \in B)\neg[a E_N b]$.

Proof. It is sufficient to prove that

$$[Ab \downarrow_N C] \iff [A \downarrow_N C \ \& \ b \downarrow_N C].$$

From left to right the implication is trivial. For the other direction suppose $A \downarrow_N C$

and $b \downarrow_N C$. From forking-triviality there are two possibilities:

Case 1. $A \downarrow_{Nc} b$. From $b \downarrow_N C$ we have $b \downarrow_N NC$. From $A \downarrow_{Nc} b$ we have $b \downarrow_{NC} AC$. From $b \downarrow_N NC$ and $b \downarrow_{NC} AC$ by transitivity we have $b \downarrow_N AC$. Hence

$b \downarrow_{NA} C$, and so $C \downarrow_{NA} Ab$. Also, $C \downarrow_N NA$ since $A \downarrow_N C$. From $C \downarrow_N NA$ and $C \downarrow_{NA} Ab$ by transitivity we get $C \downarrow_N Ab$.

Case 2. $C \downarrow_{NA} b$. In this case we get $C \downarrow_{NA} Ab$ immediately. The rest of the argument is the same as in Case 1.

To show that E_N is a Δ -congruence we establish three propositions:

Proposition 2.11 Let $\varphi(\bar{x}, \bar{y})$ be any formula over N , $\ell(\bar{a}_0) = \ell(\bar{a}_1) = \ell(\bar{x})$, $\ell(\bar{b}) = \ell(\bar{y})$, $\varphi\text{-tp}(\bar{a}_0|N) = \varphi\text{-tp}(\bar{a}_1|N)$, and $\bar{a}_i \downarrow_N \bar{b}$ ($i = 0, 1$). Then $\models \varphi(\bar{a}_0, \bar{b}) \leftrightarrow \varphi(\bar{a}_1, \bar{b})$.

Proof. We know that there is a formula $d\varphi(\bar{y})$ over N such that $\varphi(\bar{a}_0, \bar{y})$ and $d\varphi(\bar{y})$ have the same solutions in N . By hypothesis $\varphi(\bar{a}_0, \bar{y})$ and $\varphi(\bar{a}_1, \bar{y})$ have the same solutions in N and so $\varphi(\bar{a}_1, \bar{y})$ and $d\varphi(\bar{y})$ have the same solutions in N . Since $\bar{a}_i \downarrow_N \bar{b}$ ($i = 0, 1$) by Lemma 1.5 we have $\models \varphi(\bar{a}_i, \bar{b})$ if and only if $\models d\varphi(\bar{b})$ ($i = 0, 1$) as required.

Below let $\varphi^-(\bar{x}', \bar{y}')$ be the formula $\varphi(\bar{x}, \bar{y})$ rewritten with $\bar{x}' = \bar{y}$ and $\bar{y}' = \bar{x}$.

Proposition 2.12 Let $\varphi(\bar{x}, \bar{y})$ be a formula over N , $\ell(\bar{a}_i) = \ell(\bar{x})$, $\ell(\bar{b}_i) = \ell(\bar{y})$, and $\bar{a}_i \downarrow_N \bar{b}_i$ ($i = 0, 1$). If $\varphi\text{-tp}(\bar{a}_0|N) = \varphi\text{-tp}(\bar{a}_1|N)$ and $\varphi^-\text{-tp}(\bar{b}_0|N) = \varphi^-\text{-tp}(\bar{b}_1|N)$, then $\models \varphi(\bar{a}_0, \bar{b}_0) \leftrightarrow \varphi(\bar{a}_1, \bar{b}_1)$.

Proof. Choose $\bar{a}_2 \downarrow_N \bar{b}$ such that $\varphi\text{-tp}(\bar{a}_2|N) = \varphi\text{-tp}(\bar{a}_0|N)$. Next choose $\bar{b}_2 \downarrow_N \bar{a}_1\bar{a}_2$ such that $\varphi^-\text{tp}(\bar{b}_2|N) = \varphi^-\text{tp}(\bar{b}_0|N)$. Consider the sentences $\varphi(\bar{a}_0, \bar{b}_0)$, $\varphi(\bar{a}_2, \bar{b}_0)$, $\varphi(\bar{a}_2, \bar{b}_2)$, $\varphi(\bar{a}_1, \bar{b}_2)$, $\varphi(\bar{a}_1, \bar{b}_1)$. After the first one each is equivalent to its predecessor by an application of Proposition 2.11 with respect to either $\varphi(\bar{x}, \bar{y})$ or $\varphi^-(\bar{x}', \bar{y}')$. For example, $\vDash \varphi(\bar{a}_2, \bar{b}_0) \leftrightarrow \varphi(\bar{a}_2, \bar{b}_2)$ since \bar{b}_0, \bar{b}_2 realize the same φ^- -type over N , $\bar{b}_0 \downarrow_N \bar{a}_2$, and $\bar{b}_2 \downarrow_N \bar{a}_2$. Clearly, the desired conclusion follows.

Proposition 2.13 Let $\bar{a}_1, \dots, \bar{a}_n$ and $\bar{b}_1, \dots, \bar{b}_n$ be tuples in $M - N$ such that:

- (i) each of the sets $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n$ is contained in a single E_N -class,
- (ii) for $1 \leq i < j \leq n$, the E_N -classes of \bar{a}_i and \bar{a}_j are disjoint and the E_N -classes of \bar{b}_i and \bar{b}_j are disjoint, and
- (iii) for $1 \leq i \leq n$, $\ell(\bar{a}_i) = \ell(\bar{b}_i)$ and $\text{tp}_\Delta(\bar{a}_i|N) = \text{tp}_\Delta(\bar{b}_i|N)$.

Then

$$\text{tp}_\Delta(\bar{a}_1 \dots \bar{a}_n | N) = \text{tp}_\Delta(\bar{b}_1 \dots \bar{b}_n | N).$$

Proof. We proceed by induction on n . If $n = 1$, the result is clear, so suppose $n = k + 1 > 1$. Define $\bar{c}_0 = \bar{a}_1 \dots \bar{a}_k$, $\bar{c}_1 = \bar{a}_n$, $\bar{d}_0 = \bar{b}_1 \dots \bar{b}_k$, and $\bar{d}_1 = \bar{b}_n$. By the induction hypothesis, $\text{tp}_\Delta(\bar{c}_0|N) = \text{tp}_\Delta(\bar{d}_0|N)$ and $\text{tp}_\Delta(\bar{c}_1|N) = \text{tp}_\Delta(\bar{d}_1|N)$ is given. Further, since no entry of \bar{c}_0 is E_N -related to an entry of \bar{c}_1 , $\bar{c}_0 \downarrow_N \bar{c}_1$. Similarly $\bar{d}_0 \downarrow_N \bar{d}_1$. By Proposition 2.12 applied to all quantifier-free

formulas $\varphi(\bar{x}, \bar{y})$ over N ,

$$\text{tp}_\Delta(\bar{c}_0 \bar{c}_1 | N) = \text{tp}_\Delta(\bar{d}_0 \bar{d}_1 | N),$$

which completes the proof of the proposition.

From Proposition 2.13, it is clear that E_N is a Δ -congruence on M over N .

Let A denote $N \cup (\cup X)$ where X is a set of E_N -classes. We show that A is the universe of an elementary submodel of M by showing that A satisfies the Tarski-Vaught criterion. Let $\bar{a} \in A$ and $\models \exists x \varphi(x, \bar{a})$. We have to show that $\models \varphi(b, \bar{a})$ for some $b \in A$. Choose $b' \in M$ such that $\models \varphi(b', \bar{a})$. If $b' \in A$, we are done. Otherwise, by Lemma 2.10, $b' \downarrow_N \bar{a}$. From Lemma 1.5 it follows that $\models \varphi(b, \bar{a})$ for some $b \in N$. This completes the proof of Theorem 2.9.

To state the next theorem, we have first to introduce one of the principal notions of this work, that of a *tree-decomposable theory*. Here we follow [L2]. The original definition of tree-decomposability was given in [BS]. It will be stated in Chapter 3 when we are in a position to compare some consequences of the two definitions.

Definition 2.14 A *tree* is a set of (ordinal) sequences of elements of some cardinal, closed under initial segments. The elements of the tree are called *nodes* and nodes which have some proper extension within the tree are called *internal*. We

write $\iota(I)$ for the set of internal nodes of the tree I . The natural strict partial ordering of the tree by proper extension is denoted \subset . The *height* of the tree I , denoted $\text{ht}(I)$, is defined to be $\sup\{\ell(\eta) + 1 : \eta \in I\}$. If the length of a node η of I is a successor ordinal $\beta + 1$, then η^- is the initial segment of η of length β .

For the next definition, we suspend the usual convention that all structures considered be elementary submodels of some "monster" model.

Definition 2.15 Let I be a tree, and M a structure for the language L . A *decomposition of M by I* is a collection of L -structures M_η, N_η indexed by the nodes η of I , together with a collection of equivalence relations E_τ indexed by the nodes τ of $\iota(I)$, such that:

- (i) For each η , $|N_\eta| \leq |L| + \aleph_0$.
- (ii) Whenever $\rho \subset \sigma$ in I , then $N_\rho \subset N_\sigma \subseteq M_\sigma \subseteq M_\rho$.
- (iii) For each $\tau \in \iota(I)$, E_τ is a Δ -congruence on M_τ over N_τ , whose equivalence classes are the sets $M_\sigma - N_\tau$, where σ runs through the set of immediate successors of τ in I .

- (iv) If $\eta \in I$ and the domain of η is a limit ordinal, then

$$N_\eta = \cup\{N_\sigma : \sigma \subset \eta\} \text{ and } M_\eta = \cap\{M_\sigma : \sigma \subset \eta\}.$$

- (v) $M_{\langle \rangle} = \cup\{N_\eta : \eta \in I\} = M$.

Remarks Some words of intuitive explanation may be helpful. The definition describes a process whereby a structure is gradually eaten away by small substructures growing inside it, so that any given individual of the structure is eventually consumed by one of these small substructures. (Refer to clause (v) above.) Clauses (ii)–(iv) detail the mechanism by which these small substructures grow: with each internal node η of I there is associated a small substructure N_η and a substructure M_η of M . M_η serves as a pool of individuals from which all elements of N_σ ($\sigma \supset \eta$, $\sigma \in I$) are drawn. If η is not an internal node of I , the structures M_η and N_η are the same.

Definition 2.16 T is *tree-decomposable* if every model of T admits a decomposition by some tree.

Remark A key point is that any tree I by which an L -structure is decomposed has height at most $(|L| + \aleph_0)^+$. This follows from the requirements that N_σ be a proper extension of N_η whenever $\eta \subset \sigma$, and that $|N_\eta| \leq |L| + \aleph_0$ ($\eta, \sigma \in I$).

Theorem 2.17 If T is a forking-trivial stable theory, then T is tree-decomposable.

Proof. Let a model M of T be given, and let L be the language of T .

Without loss of generality $|M| \geq |L|$. Set $M_{\langle \rangle} = M$. By the downward Lowenheim-Skolem theorem, choose a model $N_{\langle \rangle} \subseteq M$ of cardinality $|L| + \aleph_0$. We define $I(0)$ to be $\{\langle \rangle\}$, the singleton of the empty sequence. For $\alpha > 0$, we define $I(\alpha) = \{\eta \in I : \ell(\eta) = \alpha\}$ by transfinite recursion on α , and simultaneously we define M_η and N_η for $\eta \in I(\alpha)$. When $\alpha = \beta + 1$ is a successor ordinal, we also define E_ζ for $\zeta \in I(\beta)$, $M_\zeta \neq N_\zeta$. The definitions are as follows:

Case 1. α is a successor ordinal $\beta + 1$. For each $\zeta \in I(\beta)$ such that $M_\zeta \neq N_\zeta$ we proceed as follows. M_ζ and N_ζ have already been defined so that N_ζ is an elementary substructure of M_ζ . We write E_ζ for the restriction of E_{N_ζ} from the monster model to $M_\zeta - N_\zeta$. Since M_ζ is a substructure of M , E_ζ certainly has no more than $|M|$ equivalence classes. Let κ_ζ be the number of equivalence classes of E_ζ . Labelling the equivalence classes of E_ζ with the ordinals $i < \kappa_\zeta$, we define the next level of the tree by setting $M_{\zeta i}$ equal to the union of N_ζ with the equivalence class of E_ζ labelled i . $N_{\zeta i}$ is then chosen as an elementary submodel of $M_{\zeta i}$ properly extending N_ζ and having the same cardinality. We set

$$I(\alpha) = \{\zeta i : \zeta \in I(\beta), M_\zeta \neq N_\zeta, i < \kappa_\zeta\}.$$

Case 2. α is a limit ordinal. Writing $\eta \upharpoonright i$ for the initial segment of η of length i , we define

$$I(\alpha) = \{\eta : \ell(\eta) = \alpha, (\forall i < \alpha)[\eta \upharpoonright i \in I(i) \ \& \ \cap_{\sigma \subset \eta} M_\sigma \neq \cup_{\sigma \subset \eta} N_\sigma]\}.$$

For $\eta \in I(\alpha)$ we define $M_\eta = \cap_{\sigma \subset \eta} M_\sigma$ and $N_\eta = \cup_{\sigma \subset \eta} N_\sigma$. This completes the construction.

We define $I = \cup\{I(\alpha) : \alpha \text{ an ordinal number}\}$. Let $\eta \in I(\alpha)$; then $\langle N_{\eta \upharpoonright i} : i \leq \alpha \rangle$ is a continuous strictly increasing elementary chain. Therefore $\alpha = \ell(\eta) < |M|^+$. It is clear that I is a tree. Conditions (ii), (iv), and (v) of Definition 2.15 are clear from the construction, while (iii) follows easily from the construction and Theorem 2.9. We establish (i) and complete the proof of the theorem by proving:

Claim. For all $\eta \in I$, $\ell(\eta) < (|L| + \aleph_0)^+$.

Proof of the Claim. Suppose the Claim fails. Denote $(|L| + \aleph_0)^+$ by α . Fix $\eta \in I$ with $\ell(\eta) = \alpha$. From the construction $N_\eta \subset M_\eta$. Also, for $i < \alpha$, $|N_{\eta \upharpoonright i}| \leq |L| + \aleph_0$ by induction on i . Choose $a \in M_\eta - N_\eta$. By Theorem 1.4 (7), there exists $A \subset N_\eta$ such that $|A| \leq |L| + \aleph_0$ and $a \downarrow_A N_\eta$. Since α is a regular cardinal, there exists $\sigma \subset \eta$ such that $A \subset N_\sigma$, whence $a \downarrow_{N_\sigma} N_\eta$. Choose $b \in N_\eta - N_\sigma$. Clearly, a, b are distinct elements of $M_\sigma - N_\sigma$ in the same E_σ -class. This implies that $\neg[a \downarrow_{N_\sigma} b]$, which contradicts $a \downarrow_{N_\sigma} N_\eta$. This completes the proof of the claim and of the theorem.

Our next goal is to prove that for a first-order theory, tree-decomposability implies strong monadic stability. In preparation, we first develop a particular infinitary version of monadic logic; one of the principal features of the logic will be that only set variables will occur free in formulas. Let a structure M be given,

along with a subset A of M . The logic will be tailored to the pair (M,A) . X, Y, Z will be used as variables for subsets of M . Our basic formulas will have the form $\exists \bar{x} \varphi(\bar{x}, \bar{X})$, where $\varphi(\bar{x}, \bar{X})$ is a conjunction of atomic and negated atomic formulas over A . The tuples \bar{x} and \bar{X} are finite, while the conjunction may be infinite. Other formulas are obtained by closing the class of basic formulas under negation, arbitrary conjunctions, and existential quantification of set variables. Conjunctions may have only finitely many free variables. We refer to the class of formulas so obtained as $L_{\infty\omega}(\text{Mon}, L, A)$, where L is the language of M .

Lemma 2.18 Given a (finitary) monadic L -formula over A , there exists a formula of $L_{\infty\omega}(\text{Mon}, L, A)$ equivalent to it in every L -structure M such that $A \subseteq M$.

Proof. By induction on formulas we show that for every monadic L -formula $\varphi(\bar{x}, \bar{Y})$ over A , there is a formula $\varphi'(\bar{X}, \bar{Y})$ in $L_{\infty\omega}(\text{Mon}, L, A)$ such that
$$\forall \bar{X} \forall \bar{Y} \left[\exists x_1 \dots \exists x_k [X_1 = \{x_1\} \ \& \ \dots \ \& \ X_k = \{x_k\} \ \& \ \varphi(\bar{x}, \bar{Y})] \leftrightarrow \varphi'(\bar{X}, \bar{Y}) \right]$$
 is true in every L -structure M such that $A \subseteq M$. The rest is straightforward.

Definition 2.19 If a set $\{\psi_j(\bar{X}) : j \in J\}$ of formulas in $L_{\infty\omega}(\text{Mon}, L, A)$ is such that for every L -structure M with $A \subseteq M$ and every tuple \bar{U} of subsets of M exactly one of the sentences $\psi_j(\bar{U})$ is true in M , then that set is called a *partitioning set*. If M is an L -structure, $A \subseteq M$, and E is a Δ -congruence on M over A , we denote by $\mathfrak{C}(M, E)$ the class of all substructures $N \subseteq M$ such that $A \subseteq N$ and $N - A$ is an E -class.

We remind the reader that Δ denotes the set of quantifier-free formulas of L .

Lemma 2.20 Given a set A and language L , for every $L_{\infty\omega}$ (Mon, L,A)-formula $\varphi(\bar{X})$ over A , there exist a partitioning set $\{\psi_j(\bar{X}) : j \in J\}$ of $L_{\infty\omega}$ (Mon, L,A)-formulas over A and a cardinal λ such that for every L -structure $M \supseteq A$, congruence relation E on M over A , and tuples \bar{U}_0, \bar{U}_1 of subsets of M ,

$$M \models \varphi(\bar{U}_0) \leftrightarrow \varphi(\bar{U}_1)$$

whenever

$|\{N \in \mathfrak{C}(M,E) : N \models \psi_j(\bar{U}_0 \uparrow N)\}| = \kappa \iff |\{N \in \mathfrak{C}(M,E) : N \models \psi_j(\bar{U}_1 \uparrow N)\}| = \kappa$
for all $j \in J$ and $\kappa < \lambda$.

Proof. We proceed by induction on formulas. The reader should observe that the partitioning sets constructed in the proof have the following additional property (*): if $\models \psi_j(\bar{X}) \ \& \ \psi_j(\bar{Y})$, then $\bar{X} \uparrow A = \bar{Y} \uparrow A$.

Consider first the case in which $\varphi(\bar{X})$ is $\exists \bar{x} \Phi(\bar{x}, \bar{X})$, with $\Phi(\bar{x}, \bar{X})$ a possibly infinite conjunction of atomic and negated atomic formulas over A . Let z_1, z_2, \dots be a canonical list of all the individual variables. Let Θ be the set of all atomic and negated atomic formulas over A whose set variables are among those of \bar{X} . Let S be the set of all subsets Ψ of Θ such that there exists n_Ψ such that at most z_1, \dots, z_{n_Ψ} occur in the formulas of Ψ . Let M be a structure, $A \subseteq M$, and \bar{U} a tuple of subsets of M with $\ell(\bar{U}) = \ell(\bar{X})$. We say that (M, \bar{U}) *realizes* $\Psi \in S$ if

there exists $\bar{c} \in M$ such that $\ell(\bar{c}) = n_\Psi$ and $M \models (\wedge \Psi)(\bar{c}, \bar{U})$. Clearly, there is a formula $\chi_\Psi(\bar{X})$ of $L_{\infty\omega}(\text{Mon}, L, A)$ such that (M, \bar{U}) realizes Ψ if and only if $M \models \chi_\Psi(\bar{U})$. Let J denote the power set of S . For each $j \in J$, let $\psi_j(\bar{X})$ be the formula

$$\wedge \{ \chi_\Psi(\bar{X}) : \Psi \in j \} \ \& \ \wedge \{ \neg \chi_\Psi(\bar{X}) : \Psi \in S - j \}.$$

Clearly, for every (M, \bar{U}) there exists a unique $j \in J$ such that $M \models \psi_j(\bar{U})$. Notice that $M_i \models \psi_j(\bar{U}_i)$ ($i = 0, 1$) for some $j \in J$ if and only if the structures (M_0, \bar{U}_0) and (M_1, \bar{U}_1) realize the same quantifier-free finite types over A .

Let M be an L -structure, $A \subseteq M$, E a congruence relation on M over A , and \bar{U}_0, \bar{U}_1 be $\ell(\bar{X})$ -tuples of subsets of M . Suppose that

$$|\{N \in \mathfrak{C}(M, E) : N \models \psi_j(\bar{U}_0 \upharpoonright N)\}| = m \iff |\{N \in \mathfrak{C}(M, E) : N \models \psi_j(\bar{U}_1 \upharpoonright N)\}| = m$$

for all $j \in J$ and $m < \omega$. Suppose that $M \models \varphi(\bar{U}_0)$. To complete the treatment of $\varphi(\bar{X})$, it is sufficient to show that $M \models \varphi(\bar{U}_1)$. There exists \bar{a} such that

$M \models \wedge \Phi(\bar{a}, \bar{U}_0)$. Without loss of generality, $\bar{a} \cap A = \emptyset$ and the entries in \bar{a} are distinct. By writing the variables \bar{x} in a suitable order, we can ensure that \bar{a} has the form $\bar{a}_1 \bar{a}_2 \dots \bar{a}_k$, where $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k$ are the equivalence classes of $E \upharpoonright \bar{a}$. Let

C_i denote the E -class which contains \bar{a}_i , N_i denote $A \cup C_i$ construed as a member of $\mathfrak{C}(M, E)$, and j_i be the unique element of J such that

$N_i \models \psi_{j_i}(\bar{U}_0 \upharpoonright N_i)$. From the equivalence displayed above there are distinct structures

$N'_i \in \mathfrak{C}(M, E)$ such that $N'_i \models \psi_{j_i}(\bar{U}_1 \upharpoonright N'_i)$. Since $\psi_{j_i}(\bar{U})$ specifies which

quantifier-free types over the set A and the language $L \cup \bar{U}$ are realized, there

exists \bar{a}'_i realizing the same quantifier-free type over A in (N'_i, \bar{U}_1) as \bar{a}_i

realizes in (N_i, \bar{U}_0) . From the definition of congruence over A , the quantifier-free

type of $\bar{a}' = \bar{a}'_1 \bar{a}'_2 \dots \bar{a}'_k$ over A in (M, \bar{U}_1) is the same as that of \bar{a} in (M, \bar{U}_0) . Clearly, $M \models \bigwedge \Phi(\bar{a}', \bar{U}_1)$ since each formula in Φ is atomic or negated atomic. Hence $M \models \varphi(\bar{U}_1)$ as required.

Now for the induction steps. The negation case being clear, we pass directly to the case of (arbitrary) conjunction. Consider $\bigwedge \{\varphi_h(\bar{X}) : h \in H\}$. By the induction hypothesis, for each $h \in H$ we have a cardinal λ_h and a partitioning set $\{\psi_{j,h}(\bar{X}) : j \in J_h\}$ which witness the truth of the lemma for $\varphi = \varphi_h$. Define $\lambda = \sup \{\lambda_h : h \in H\}$, $J = \coprod \{J_h : h \in H\}$ and $\psi_j = \bigwedge \{\psi_{h,j(h)} : h \in H\}$ ($j \in J$). Clearly $\{\psi_j(\bar{X}) : j \in J\}$ is a partitioning set which, together with λ , witnesses the truth of the lemma for $\varphi = \bigwedge \{\varphi_h(\bar{X}) : h \in H\}$.

The remaining case is that of existential quantification. Let $\varphi(\bar{X})$ have the form $\exists Y \theta(\bar{X}, Y)$. By the induction hypothesis, there exist a cardinal μ and a partitioning set $\{\chi_k(\bar{X}, Y) : k \in K\}$ which witness the truth of the lemma for $\theta(\bar{X}, Y)$. Let J be the power set of K , and for each $j \in J$, put

$$\psi_j(\bar{X}) = \left(\bigwedge_{k \in j} \exists Y \chi_k(\bar{X}, Y) \right) \& \left(\bigwedge_{k \in K-j} \neg \exists Y \chi_k(\bar{X}, Y) \right).$$

Let λ be the largest of \aleph_0, μ , and $|K|^+$.

Let M be an L -structure, $A \subseteq M$, E a congruence relation on M over A , and \bar{U}_0, \bar{U}_1 be $\ell(\bar{X})$ -tuples of subsets of M . Suppose that

$$|\{N \in \mathfrak{C}(M, E) : N \models \psi_j(\bar{U}_0 \upharpoonright N)\}| = \kappa \iff |\{N \in \mathfrak{C}(M, E) : N \models \psi_j(\bar{U}_1 \upharpoonright N)\}| = \kappa$$

for all $j \in J$ and $\kappa < \lambda$. Suppose that $M \models \varphi(\bar{U}_0)$. To complete the treatment of $\varphi(\bar{X})$, it is sufficient, by the symmetry, to show that $M \models \varphi(\bar{U}_1)$. There exists $V_0 \subseteq M$ such that $M \models \theta(\bar{U}_0, V_0)$.

Define mappings G_0 and G_1 with domain J by:

$$G_0(j) = \{N \in \mathfrak{C}(M,E) : N \vDash \psi_j(\bar{U}_0 \uparrow N)\} \text{ and}$$

$$G_1(j) = \{N \in \mathfrak{C}(M,E) : N \vDash \psi_j(\bar{U}_1 \uparrow N)\}.$$

Observe that the hypothesis about \bar{U}_0 and \bar{U}_1 is equivalent to:

$$(1) |G_0(j)| = |G_1(j)| \text{ or } |G_0(j)|, |G_1(j)| \geq \lambda \text{ for all } j \in J. \text{ Define}$$

$$F_0 : K \rightarrow \mathcal{P}(\mathfrak{C}(M,E)) \text{ by } F_0(k) = \{N \in \mathfrak{C}(M,E) : N \vDash \chi_k((\bar{U}_0 V_0) \uparrow N)\} \text{ (} k \in K\text{)}.$$

Our plan is as follows. We will define $F_1 : K \rightarrow \mathcal{P}(\mathfrak{C}(M,E))$ such that

$\text{rng}(F_1) - \{\emptyset\}$ is a partition of $\mathfrak{C}(M,E)$, and for all $k \in K$

$$(2) |F_1(k)| = |F_0(k)| \text{ or } |F_1(k)|, |F_0(k)| \geq \mu \text{ and}$$

$$(3) F_1(k) \subseteq \cup\{G_1(j) : k \in j \in J\}.$$

Suppose that such F_1 is given. From (3) for each $N \in \mathfrak{C}(M,E)$ we can choose

$$V_1^N \subseteq N \text{ such that } N \in F_1(k) \Rightarrow N \vDash \chi_k((\bar{U}_1 \uparrow N) V_1^N) \text{ (} k \in K\text{)}. \text{ Since}$$

$\{\chi_k(\bar{X}, Y) : k \in K\}$ has the property (*) mentioned at the beginning of the proof,

and since $F_0(k) \neq \emptyset$, $V_1^N \cap A = V_0 \cap A$. Let $V_1 = \cup\{V_1^N : N \in \mathfrak{C}(M,E)\}$. We have

constructed V_1 so that $\{N \in \mathfrak{C}(M,E) : N \vDash \chi_k((\bar{U}_1 V_1) \uparrow N)\} = F_1(k)$, and by

definition $\{N \in \mathfrak{C}(M,E) : N \vDash \chi_k((\bar{U}_0 V_0) \uparrow N)\} = F_0(k)$. Since $|F_0(k)| = |F_1(k)|$

or $|F_0(k)|, |F_1(k)| \geq \mu$ by (2), we have

$$|\{N \in \mathfrak{C}(M,E) : N \vDash \chi_k((\bar{U}_1 V_1) \uparrow N)\}| = \kappa \Leftrightarrow$$

$$|\{N \in \mathfrak{C}(M,E) : N \vDash \chi_k((\bar{U}_0 V_0) \uparrow N)\}| = \kappa$$

for all $\kappa < \mu$. It follows that $M \vDash \theta(\bar{U}_1 V_1)$ and hence that $M \vDash \varphi(\bar{U}_1)$.

It remains to find the function F_1 . Choose a bijection H such that:

$$(4) \text{ dom}(H) = \cup\{G_0(j) : |G_0(j)| < \lambda, j \in J\}$$

$$(5) \text{ rng}(H) = \cup\{G_1(j) : |G_1(j)| < \lambda, j \in J\}$$

$$(6) N \in G_0(j) \Leftrightarrow H(N) \in G_1(j) \text{ (} j \in J, N \in \text{dom}(H)\text{)}.$$

Also, choose a family $\{\mathfrak{C}_k : k \in K\}$ of pairwise disjoint sets such that

$$(7) \mathfrak{C}_k \subseteq \cup\{G_1(j) : k \in j\} - \text{rng}(H)$$

$$(8) F_0(k) \subseteq \text{dom}(H) \Rightarrow \mathfrak{C}_k = \emptyset$$

$$(9) \neg[F_0(k) \subseteq \text{dom}(H)] \Rightarrow |\mathfrak{C}_k| = \min\{|F_0(k)|, \mu\}.$$

To see that such sets can be chosen, consider a fixed k such that

$\neg[F_0(k) \subseteq \text{dom}(H)]$. From (4), there exists j such that $|G_0(j)| \geq \lambda$ and $k \in j$.

From (1), $|G_1(j)| \geq \lambda$. Since $\lambda \geq \mu$ and $G_1(j) \cap \text{rng}(H) = \emptyset$, we can choose $\mathfrak{C}_k \subseteq G_1(j)$. Since $\lambda \geq |K|$, if the same j is chosen for more than one k , there will still be enough elements in $G_1(j)$.

Consider $N \in \mathfrak{C}(M, E)$ such that

$$(10) N \notin \text{rng}(H) \cup \cup\{\mathfrak{C}_k : k \in K\}.$$

Since $N \notin \text{rng}(H)$, there exists unique $j_N \in J$ such that $N \in G_1(j_N)$ and $|G_1(j_N)| \geq \lambda$. By hypothesis, $|G_0(j_N)| \geq \lambda$. For each $P \in G_0(j_N)$, there exists $k_P \in j_N$ such that $P \vDash \chi_{k_P}((\bigcup_0 V_0) \uparrow P)$. Since $\lambda > |K|$ is infinite, by the pigeon-hole principle there exists $k \in j_N$ such that $k_P = k$ for at least λ P 's in $G_0(j_N)$, whence $|F_0(k)| \geq \lambda$. Set $k(N)$ equal to such a value of k for each $N \in \mathfrak{C}(M, E)$ satisfying (10).

Now we can define F_1 by:

$$F_1(k) = \{H(N) : N \in F_0(k)\} \cup \mathfrak{C}_k \cup \{N \in \mathfrak{C}(M, E) : k(N) = k\}.$$

We conclude by showing that F_1 satisfies the requirements laid out above.

Since the sets \mathfrak{C}_k ($k \in K$) are pairwise disjoint and disjoint from $\text{rng}(H)$ by (7) and since $k(N)$ is defined only for N satisfying (10), the sets $F_1(k)$ are pairwise disjoint. Since h is onto and $k(N)$ is defined for every N satisfying (10), $\cup \text{rng}(F_1) = \mathfrak{C}(M, E)$.

Consider k such that $|F_0(k)| < \mu$. Since H is 1-1, $|F_1(k) \cap \text{rng}(H)| = |F_0(k) \cap \text{dom}(H)|$. From (8) and (9), either $F_0(k) \subseteq \text{dom}(H)$ and $\mathfrak{C}_k = \emptyset$, or $|\mathfrak{C}_k| = |F_0(k)|$. Also, for no N do we have $k(N) = k$, so it is clear that $|F_1(k)| = |F_0(k)|$. For k such that $|F_0(k)| \geq \mu$, the same analysis shows that $|F_1(k)| \geq \mu$. Therefore (2) holds.

Consider $N_1 \in F_1(k) \cap \text{rng}(H)$. By definition of F_1 , $N = H^{-1}(N_1) \in F_0(k)$. Let j be the unique member of J such that $N \in G_0(j)$. Then $k \in j$ since $N \models \chi_k((\bigcup_0 V_0) \upharpoonright N)$. By (6), $N_1 \in G_1(j)$. Consider next N_1 such that $k(N_1) = k$. Let j now denote the member of J such that $N_1 \in G_1(j)$. Then $k(N_1) \in j$ by definition of $k(N_1)$. From these remarks and (7), (3) holds. This completes the proof of the lemma.

With the foregoing lemma in hand, we now proceed to the last result of this chapter. This theorem, together with Theorems 2.4, 2.5, 2.9, and 2.17 closes a cycle of implications, and thus proves that certain conditions on a first-order theory T are equivalent. The pertinent conditions are:

- (1) The theory P (see the description just before Theorem 2.4) is not interpretable in any extension of T by unary predicates.
- (2) T is forking-trivial.
- (3) T is strongly monadically stable.
- (4) T is tree-decomposable.
- (5) T is weakly monadically stable.
- (6) T satisfies the conclusions of Theorem 2.9.

This is noteworthy in that it shows both that two equivalent theories are both tree-decomposable or both not, so that tree-decomposability is insensitive to choice of language. It also proves, remarkably, that the two apparently distinct notions of monadic stability actually coincide.

Theorem 2.21 A first-order theory which is tree-decomposable is strongly monadically stable.

Proof. For the sake of a contradiction, let T be tree-decomposable but weakly monadically unstable, with $\varphi(x,y,\bar{X})$ a monadic formula witnessing the instability. Let $M \models T$, and let \bar{U} be a tuple of subsets of M such that $\varphi(x,y,\bar{U})$ orders some subset of M in the order type of an infinite cardinal γ . Let D be the solution set of $\exists y[\varphi(x,y,\bar{U}) \vee \varphi(y,x,\bar{U})]$ in M , so that D includes all the individuals of M representing elements of γ and possibly some others besides.

Let the tree $I \subseteq {}^{\leq \kappa} \lambda$ decompose the model M as described in Definition 2.15. By choosing a large enough γ , we can force the existence of an internal node $\tau \in \iota(I)$ of the tree which has members of D in as many different E_τ -classes as we please. We see this as follows.

Define the subtree I' of I by putting a node σ of I in I' just if M_σ intersects D , so that $\sigma \in I'$ iff $M_\sigma \cap D \neq \emptyset$. Now, for every $d \in D$, there exists $\sigma \in I'$ such that $d \in N_\sigma$. This follows because $M = \bigcup_{\sigma \in I} N_\sigma$, and because $N_\sigma \subseteq M_{\sigma-}$. If now $|D|$ is large, then $|I'|$ is large since $|N_\sigma| \leq |L| + \aleph_0$ for all σ . But $\text{ht}(I') \leq \text{ht}(I) \leq (|L| + \aleph_0)^+$, so there must exist $\tau \in I'$ such that τ has

many immediate successors in I' . More precisely, for any cardinal δ , we can ensure by choice of γ that there exists $\tau \in \iota(I)$ such that D meets at least δ of the equivalence classes of E_τ .

Fix such a τ , and for $\sigma \subset \tau$, let $\sigma+$ be the initial segment of τ of length $\ell(\sigma) + 1$. Augment L by new unary predicate symbols R_i , one for each $i < \ell(\tau)$, writing L^* for the expanded language, and Δ^* for its set of quantifier-free formulas. At the same time, expand M to an L^* -structure M^* by interpreting R_i as $M_{\tau \upharpoonright i} - M_{\tau \upharpoonright (i+1)}$. Let E be the equivalence relation $E_\tau \cup \cup \{E_\sigma \upharpoonright (M_\sigma - M_{\sigma+}) : \sigma \subset \tau\}$.

We now show that E is a Δ^* -congruence on M^* over N_τ . Let $\sigma_1 \subset \sigma_2 \subset \dots \subset \sigma_{n-1} \subset \tau$, and pick $\bar{b}_{ij} \in M_{\sigma_i} - M_{(\sigma_i)+}$ ($1 \leq i < n$, $1 \leq j \leq k_i$) and $\bar{b}_{nj} \in M_\tau - N_\tau$ ($1 \leq j \leq k_n$) such that any two entries of \bar{b}_{ij} are E -related and such that if $j \neq j'$ then no entry of \bar{b}_{ij} is E -related to any entry of $\bar{b}_{ij'}$. Let \bar{c}_{ij} ($1 \leq i \leq n$, $1 \leq j \leq k_i$) be tuples in distinct E -classes such that $\text{tp}_{\Delta^*}(\bar{c}_{ij} | N_\tau) = \text{tp}_{\Delta^*}(\bar{b}_{ij} | N_\tau)$. We need to show that the concatenation of the \bar{c}_{ij} and the concatenation of the \bar{b}_{ij} have the same Δ^* -type over N_τ . (This is what is involved in verifying that the definition of Δ^* -congruence is satisfied.) Let $\bar{a}_i \in N_{\sigma_i} - N_{\sigma_{i-1}}$ ($1 < i < n$) and $\bar{a}_n \in N_\tau - N_{\sigma_{n-1}}$. We are done once we show that the concatenations of the \bar{a}_i and the \bar{c}_{ij} and of the \bar{a}_i and the \bar{b}_{ij} have the same type over N_{σ_1} .

The presence of the new unary predicates R_i in M^* allows us, since $\text{tp}(\bar{c}_{ij} | N_\tau) = \text{tp}(\bar{b}_{ij} | N_\tau)$, to locate \bar{c}_{ij} in $M_{\sigma_i} - M_{(\sigma_i)+}$ ($1 \leq i < n$) and \bar{c}_{nj} in

$M_\tau - N_\tau$. Now, E_τ is a Δ -congruence on M_τ over N_τ . The reader will recall that Δ denotes the set of quantifier-free formulas in the language of M . Thus $\text{tp}_{\Delta^*}(\bar{c}_{n1} \dots \bar{c}_{nk_n} | N_\tau) = \text{tp}_{\Delta^*}(\bar{b}_{n1} \dots \bar{b}_{nk_n} | N_\tau)$. For $1 \leq i \leq n$ write \bar{c}_i for the concatenation of the \bar{c}_{ij} ($1 \leq j \leq k_i$). Then $\text{tp}_{\Delta^*}(\bar{c}_n \bar{a}_n | N_{\sigma_{n-1}}) =$

$\text{tp}_{\Delta^*}(\bar{b}_n \bar{a}_n | N_{\sigma_{n-1}})$. By downward induction on i , we now have

$$\text{tp}_{\Delta^*}(\bar{c}_n \bar{a}_n \bar{c}_{n-1} \bar{a}_{n-1} \dots \bar{a}_{i+1} \bar{c}_i | N_{\sigma_i}) = \text{tp}_{\Delta^*}(\bar{b}_n \bar{a}_n \bar{b}_{n-1} \bar{a}_{n-1} \dots \bar{a}_{i+1} \bar{b}_i | N_{\sigma_i})$$

for $1 \leq i < n - 1$. For the induction, we use that E_{σ_i} is a Δ -congruence on M_{σ_i} over N_{σ_i} , that $\bar{c}_n \bar{a}_n \bar{c}_{n-1} \bar{a}_{n-1} \dots \bar{a}_{i+1}$ and $\bar{b}_n \bar{a}_n \bar{b}_{n-1} \bar{a}_{n-1} \dots \bar{a}_{i+1}$ are in

$M_{(\sigma_i)^+} - N_{\sigma_i}$, which is an E_{σ_i} -class not meeting $\bar{c}_i \bar{b}_i$, and that these tuples have

the same type over N_{σ_i} . Setting $i = 1$ shows that $\bar{c}_n \bar{a}_n \dots \bar{c}_1 \bar{a}_1$ and $\bar{b}_n \bar{a}_n \dots \bar{b}_1 \bar{a}_1$

realise the same type over N_{σ_1} , which completes the proof that E is a

Δ^* -congruence on M^* over N_τ .

Let $\varphi^*(S, U, \bar{X}) \in L_{\omega\omega}(\text{Mon}, L, N_\tau)$ be equivalent to $\exists x \exists y [x \in S \ \& \ y \in U \ \& \ \varphi(x, y, \bar{X})]$. Apply Lemma 2.20 to the formula $\varphi^*(S, U, \bar{X})$, the structure M^* and the congruence E to obtain a partitioning set $\{\psi_j(S, U, \bar{X}) : j \in J\}$ and a cardinal λ fulfilling the conclusion of that Lemma. For the argument to go through, we need δ larger than $|J|$. Note that for this purpose the partitioning set can be determined as a purely syntactical object before the structure M^* is formed. (L^* is formed by adding $|L| + \aleph_0$ new unary predicate symbols to L . M is expanded to M^* by using as many of the new unary predicates as necessary, interpreting those which are left over as \emptyset . In addition, $|L| + \aleph_0$ individual constants are set

aside as names for the elements of N_τ ; if $|N_\tau| < |L| + \aleph_0$, some elements of N_τ are named more than once.) Let $a_i \in D$ ($i < \delta$) lie in distinct E_τ -classes. For $i < \delta$, let M_i be $N_\tau \cup [a_i]_E$ and let j_i be the unique $j \in J$ such that $M_i^* \models \psi_j(\{a_i\}, \emptyset, \mathbb{U} \upharpoonright M_i)$, where M_i^* denotes the structure which results from restricting M^* to (the universe of) M_i . Now $|J| < \delta$, so there are distinct $i, k < \delta$ such that $j_i = j_k$. From the conclusion of Lemma 2.20 we now have $M^* \models \varphi^*(\{a_i\}, \{a_k\}, \mathbb{U}) \leftrightarrow \varphi^*(\{a_k\}, \{a_i\}, \mathbb{U})$, from which $M \models \varphi(a_i, a_k, \mathbb{U}) \leftrightarrow \varphi(a_k, a_i, \mathbb{U})$. This contradicts that $\varphi(x, y, \mathbb{U})$ determines a linear ordering of the a 's, and thus completes the proof.

Chapter 3

Some tree-decomposable structures

Here we present some examples which illustrate the concepts introduced in the previous chapter. A structure M is called *tree-decomposable* if $\text{Th}(M)$ is tree-decomposable. For a first-order theory tree-decomposability is the same as weak monadic stability, so if M is either an expansion by unary predicates or a reduct of a tree-decomposable structure, then M is tree-decomposable. We consider examples which differ from each other by taking reducts, passing to an elementarily equivalent structure, or expanding by unary predicates, to be essentially the same. Let us first describe the examples given in [BS].

Example 3.1 $N = (\mathbb{Z}; S)$, where S is the successor relation. This is essentially the same as Example 3.1.6 of [BS]. An arbitrary model of $\text{Th}(N)$ is a disjoint union of copies of N . For $\text{Th}(N)$, $A \downarrow_B C$ means that if A and C intersect one of the copies of N , then B intersects that same copy of N . Every model M of T is tree-decomposable by a tree of height 2. We can take $N_{\langle \rangle}$ to be the empty structure, each $M_{\langle i \rangle} = N_{\langle i \rangle}$ to be a copy of N , and $E_{\langle \rangle}$ to be the relation true of individuals a, b just when a, b lie in the same copy of N .

Example 3.2 $N = (\omega; E_0, E_1, \dots)$, where $E_i = \{(\sigma, \tau) : \sigma(k) = \tau(k) \text{ for all } k \leq i\}$. Here $A \downarrow_B C$ means that for any $i < \omega$, every E_i -class which meets both A and C also meets B . To see that a theory is tree-decomposable, it is enough to

check that all its reducts to a finite language are tree-decomposable, for any formula witnessing the monadic unstability of the theory would involve at most finitely many symbols of the language. So in the present case it is enough to check that each of the structures $({}^\omega\omega; E_0, \dots, E_n)$ is tree-decomposable. Of course, it is also easy to check tree-decomposability by looking at the independence relation. However, such proofs provide no precise information about the height of a tree decomposing a model of the full theory $\text{Th}(N)$. Employing a variant of the method of Theorem 2.17, we show below that such a model can be decomposed by a tree of height at most $\omega + 2$. The method used here differs from that of Theorem 2.17 in using the equivalence relations supplied with the model, rather than the forking relation, to provide the required Δ -congruences.

Let M be some model of $\text{Th}(N)$. We build a tree $I \subseteq {}^{\leq \omega+1} |M|$ and a decomposition of M by I in stages. At the initial stage, we define $I(0) = \{\langle \rangle\}$, $M_{\langle \rangle} = M$, $N_{\langle \rangle} = \emptyset$, and $E_{\langle \rangle} = E_0$, and write $\kappa_{\langle \rangle}$ for the number of equivalence classes of $E_{\langle \rangle}$. Index these equivalence classes by the ordinals $i < \kappa_{\langle \rangle}$.

The next level $I(1)$ of I is $I(1) = \{\langle i \rangle : i < \kappa_{\langle \rangle}\}$. For each $i < \kappa_{\langle \rangle}$, pick $a_{\langle i \rangle}$ from the i -th equivalence class of $E_{\langle \rangle}$, and define $M_{\langle i \rangle} = [a_{\langle i \rangle}]_{E_{\langle \rangle}}$, $N_{\langle i \rangle} = \{a_{\langle i \rangle}\}$, and $E_{\langle i \rangle} = E_1 \upharpoonright (M_{\langle i \rangle} - N_{\langle i \rangle})$. Subsequent finite levels of the tree are obtained in the same way. If $n < \omega$ and $I(0), I(1), \dots, I(n)$, M_η, N_η, E_η ($\eta \in I(0) \cup \dots \cup I(n)$) are already defined such that $E_\eta = E_{\ell(\eta)} \upharpoonright (M_\eta - N_\eta)$, then $I(n+1)$ and its structures and equivalence relations are obtained as follows. For each $\eta \in I(n)$, let κ_η be the number of equivalence classes of E_η . Note that κ_η is

infinite. Index these equivalence classes by the ordinals $i < \kappa_\eta$, and choose from the i -th equivalence class an element $a_{\eta i}$. By choice of E_η , this $a_{\eta i}$ is not present in N_η . Now define $I(n+1) = \{\eta i : \eta \in I(n), i < \kappa_\eta\}$, and for $\eta \in I(n)$, $i < \kappa_\eta$, put

$$\begin{aligned} N_{\eta i} &= N_\eta \cup \{a_{\eta i}\}, \\ M_{\eta i} &= N_\eta \cup \{a_{\eta i}\} E_\eta, \text{ and} \\ E_{\eta i} &= E_{n+1} \upharpoonright (M_{\eta i} - N_{\eta i}). \end{aligned}$$

This completes the construction at finite levels. For level ω , define

$$I(\omega) = \{\eta \in {}^\omega |M| : \forall i < \omega (\eta \upharpoonright i \in I(i))\},$$

i.e., define the ω -th level of I as the set of ω -sequences from $|M|$

all of whose finite initial segments lie at some finite level. Then for each $\eta \in I(\omega)$, let $M_\eta = \cap \{M_\sigma : \sigma \subset \eta\}$ and $N_\eta = \cup \{N_\sigma : \sigma \subset \eta\}$. For each $\eta \in I(\omega)$ such that $M_\eta - N_\eta$ is nonempty, define $E_\eta = \{(a, a) : a \in M_\eta - N_\eta\}$. For each such η , let $\kappa_\eta = |M_\eta - N_\eta|$ and index the elements of $M_\eta - N_\eta$ as $a_{\eta i}$ for $i < \kappa_\eta$, writing

$$M_{\eta i} = N_{\eta i} = N_\eta \cup \{a_{\eta i}\}.$$

With $I(\omega+1) = \{\eta i : \eta \in I(\omega), N_\eta \neq M_\eta, i < \kappa_\eta\}$ and $I = \cup \{I(\alpha) : \alpha \leq \omega+1\}$,

the construction is complete.

It is evident that with our definitions of I , M_η , N_η , and E_η , parts (i), (ii), (iv), and (v) of Definition 2.15 are satisfied. For (iii), it is also clear that for internal nodes τ of I , the equivalence classes of E_τ are the sets $M_\sigma - N_\tau$ where σ is an immediate successor of τ in I . It remains to show that for such internal nodes τ , E_τ is actually a Δ -congruence on $M_\tau - N_\tau$. For this there are two cases to consider. First suppose that $\ell(\tau) = \omega$. Then any two elements of

$M_\tau - N_\tau$ are E_τ -inequivalent, but E_i -equivalent for all $i < \omega$. Thus any tuples \bar{b}, \bar{c} of distinct elements of $M_\tau - N_\tau$ having the same length also have the same Δ -type over N_τ , which shows that E_τ is a Δ -congruence. Next, suppose that $\ell(\tau) < \omega$, and take $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_n$ and $\bar{c}_0, \bar{c}_1, \dots, \bar{c}_n$ satisfying (i), (ii), and (iii) of Definition 2.8, with $\Gamma = \Delta$, $E = E_\tau$, and \bar{c}_i replacing \bar{a}_i . Let $\bar{b}_0\bar{b}_1\dots\bar{b}_n = b_0b_1\dots b_r$ and $\bar{c}_0\bar{c}_1\dots\bar{c}_n = c_0c_1\dots c_r$. Then $\text{tp}_\Delta(\bar{c}_0\bar{c}_1\dots\bar{c}_n | N_\tau) = \text{tp}_\Delta(\bar{b}_0\bar{b}_1\dots\bar{b}_n | N_\tau)$ just in case for all $k < \omega$, $i, j \leq r$, and $a \in N_\tau$, $E_k(b_i, b_j) \leftrightarrow E_k(c_i, c_j)$ and $E_k(b_i, a) \leftrightarrow E_k(c_i, a)$. Now it was already assumed that $E_k(b_i, a) \leftrightarrow E_k(c_i, a)$ since b_i and c_i are corresponding members of corresponding sub-tuples, and corresponding sub-tuples were supposed to have the same Δ -type over N_τ . Thus we are done if we can show that under these conditions $E_k(b_i, b_j) \leftrightarrow E_k(c_i, c_j)$. Towards a contradiction, suppose without loss of generality that $E_k(b_i, b_j)$ and $\neg E_k(c_i, c_j)$. Then $\neg E_\tau(b_i, b_j)$ and $\neg E_\tau(c_i, c_j)$; otherwise $b_i b_j$ and $c_i c_j$ have the same Δ -type over N_τ . Since $E_k(b_i, b_j)$ and $\neg E_\tau(b_i, b_j)$, we have $k < \ell(\tau)$. Now by construction, $E_k(b_i, a_\tau)$ and $E_k(b_j, a_\tau)$, so that $E_k(c_i, a_\tau)$ and $E_k(c_j, a_\tau)$, whence $E_k(c_i, c_j)$. This is a contradiction. We conclude that E_τ is a Δ -congruence on M_τ over N_τ . Thus our attempted decomposition of M by I was successful.

For this example, there is no canonical decomposition of the kind found in Example 1. This suggests the possibility of recasting the definition of tree-decomposability. In Definition 2.15 (ii) replace the strict inclusion between N_ρ and N_η by \subseteq , while adding "of height $\leq \max(|L|, \aleph_0)^+$ " at the end of Definition 2.16. The theory developed in Chapter 2 requires only trivial changes, and any model M of the theory of Example 3.2 now has a canonical decomposition

by a tree $\underline{c} \leq (\omega+1)_\kappa$ for some κ , constructed in a similar fashion. N_σ is the empty structure when $\ell(\sigma) \leq \omega$. If $1 \leq \ell(\sigma) = i < \omega$, M_σ is an E_{i-1} -class. If $\ell(\sigma) = \omega$, then M_σ is an $(\bigcap_{i < \omega} E_i)$ -class, and if $\ell(\sigma) = \omega + 1$, then $M_\sigma = N_\sigma$ is a singleton.

Example 3.3 This is essentially Example 7.1.1 of [BS]. For $n \geq 1$, let L_n contain unary predicates P_i ($0 \leq i \leq n$) and binary predicates F and G . The axioms of theory T_n are as follows:

- (1) The P_i are pairwise disjoint, and, taken together, exhaust the universe.
- (2) For each x there is a unique y such that $F(x,y)$ and a unique z such that $F(z,x)$. (This can be taken to mean that F is a permutation of the universe.)
- (3) F has no finite cycles. (This actually requires an axiom schema.)
- (4) For each x there is a unique y such that $G(x,y)$. (In other words, G is a function.)
- (5) If $F(x,y_1)$, $G(y_1,z_1)$, $G(x,y_2)$, and $F(y_2,z_2)$, then $z_1 = z_2$. (This means that F and G , taken as functions, commute.)
- (6) If $F(x,y)$, then $P_i(x)$ if and only if $P_i(y)$ ($0 \leq i \leq n$).
- (7) If $G(x,y)$ and $P_i(x)$ ($i > 0$), then $P_{i-1}(y)$.
- (8) If $P_0(x)$ and $G(x,y)$ then $y = x$.
- (9) For each y , $\{x : G(x,y)\}$ is infinite unless $P_n(y)$, in which case $\{x : G(x,y)\}$ is empty. (Once again an axiom schema is required.)

With the axioms given above, T_n is complete and ω -stable. T_n is also $(n+1)$ -tree-decomposable. It then follows from the theory given in Chapter 2 that

T_n is monadically stable. The tree-decomposability is not surprising, for the models of T_n fall apart into components which themselves enjoy a tree structure.

There is another general construction of tree-decomposable theories, first announced in [L3]. To present this construction, we must introduce another definition.

Definition 3.4 A theory T is *coinductive* if it can be axiomatized by the set of all its $\exists\forall$ -theorems, where an $\exists\forall$ -formula is a prenex formula all of whose existential quantifiers precede all its universal quantifiers. A structure is called *coinductive* when its theory is coinductive.

It is a basic result of [L2] that a coinductive theory over a relational language is tree-decomposable. This fact furnishes us with a stock of further examples of tree-decomposable theories.

These new examples are graphs, possibly with some additional unary predicates; in fact, we will consider a graph to be a structure for one of the languages $L_i = \{R, R_1, \dots, R_i\}$, where R is binary and R_j ($1 \leq j \leq i$) are unary, in which $\forall x \forall y ((R(x,y) \rightarrow R(y,x)) \ \& \ \neg R(x,x))$ is true. In graph-theoretic terms, our structures are simple graphs with colours for the vertices, where a vertex may be coloured with more than one colour or with no colour at all. The examples we give are all constructed according to the following plan.

Let T be a universal but not necessarily complete theory of graphs whose class of models is closed under disjoint union. Let $\{M_i : i \in \omega\}$ exhaust the finite models of T in the sense that each M_i is a finite model of T and any finite

model of T is isomorphic to some M_i . Then the *principal model* M_T of T is the disjoint union $\dot{\bigcup}_{i \in \omega} M_i$. (M_T is determined up to isomorphism by the given data, for we can construct an isomorphism between two such principal models by a standard back-and-forth argument.) Now it often happens that if a universal theory T of graphs whose class of models is closed under disjoint union is given, then $\text{Th}(M_T)$ is a (complete) coinductive theory. The conjunction of the following two conditions suffices:

(i) T imposes a uniform finite upper bound on the number of vertices adjacent to each of two distinct vertices, and (ii) in any model of T , no infinite path can be embedded in a substructure of finite diameter. By a result alluded to above, each such theory T yields a tree-decomposable theory $\text{Th}(M_T)$. The following examples are of this kind; all are taken directly from [L3].

Example 3.5 Let T be the L_0 -theory of graphs without cycles.

Example 3.6 Let T be the L_0 -theory of graphs such that all cycles are 3-cycles and any two 3-cycles have at most one vertex in common.

Example 3.7 We begin by defining an L_2 -structure N . The universe of N comes in three parts:

$$R_1 = \cup \{f^n : 1 \leq n \leq \omega\}, \quad R_2 = {}^\omega \omega, \quad \text{with}$$

$$N = R_1 \cup R_2 \cup \{(f, m, n) : f \in {}^\omega \omega, m \leq n < \omega\}.$$

Finally, the adjacency relation R is taken as the least symmetric relation containing all pairs of the forms

$$(f, (f, 0, n)), ((f, m, n), (f, m+1, n)), ((f, n, n), f(0) \dots f(n)),$$

where $f \in {}^\omega \omega$ and $m < n < \omega$. Our theory T is then the universal L_2 -theory

such that M is a finite model of T if and only if M is the disjoint union of a finite number of finite structures embeddable in N .

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