DISTANCE-REGULAR GRAPHS AND EIGENVALUE MULTIPLICITIES

by

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Abstract

This thesis will focus on the consequences of a distance-regular graph possessing an eigenvalue of low multiplicity. We obtain information about such graphs by studying their image configurations under a natural representation in Euclidean space. In particular, the diameter and valency of a distance-regular graph is bounded by a function of the multiplicity of an eigenvalue.

We first discuss spherical 2-distance sets and equiangular lines, and give a new proof for the fact that the size of a spherical 2-distance set can be at most 6 in R^3, 10 in R^4 and 16 in R^5, respectively. Then we classify the distance-regular graphs with an eigenvalue of multiplicity four or five.

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Introduction

A graph G is said to be *distance-regular* if, given any two vertices u and v, the number of vertices in G at distance i from u and distance j from v only depends on the distance between u and v. Since u and v may coincide, a distance-regular graph must be regular. The concept of distance-regular graphs occurred in late 1950's (Bose and Mesner [10], Bose [9]). An important subclass of distance-regular graphs are the distance-transitive graphs. A graph G is said to be *distance-transitive* if, for any vertices u, v, x, y of G satisfying $\partial(u,v) = \partial(x,y)$, there is an automorphism α of G which takes u to x and v to y. (As a common usage in graph theory, $\partial(u,v)$ represents the distance between the two vertices u and v.) Distance-transitivity implies distance-regularity, but the converse is not true.

Starting from very elementary regularity properties, the concept of a distance-regular graph arises naturally as a common setting for regular graphs which are extremal in some sense. The theory of distance-regular graphs has connections to many parts of graph theory, design theory, coding theory, geometry and group theory.

An interesting problem in studying distance-regular graphs and distance-transitive graphs is to classify them. By the great effort of many mathematicians in past years, the distance-transitive graphs have been classified for valency up to thirteen, and the distance-regular graphs have been classified for valency three.

In his recent paper, Godsil [20] studied representations of distance-regular graphs in Euclidean spaces, and derived an upper bound on both diameter and valency of distance-regular graphs in terms of an eigenvalue multiplicity. Representations of graphs provide us with a powerful method for studying distance-regular graphs and their eigenvalue multiplicities. As a byproduct, Godsil's bound on diameter and valency suggests that distance-regular graphs can be classified according to the multiplicity of their eigenvalues, rather than by their valency. This thesis is the beginning of this classification.

In Chapter 1, we introduce the concept and basic results about distance-regular graphs, adjacency matrices and distance-regular line graphs.

In Chapter 2, we study the theory of representations of distance-regular graphs, which is the main machinery throughout this thesis. Roughly speaking, a representation of a graph is a mapping from the vertex set of the graph into a Euclidean space. (In our case, it is usually an eigenspace of the graph.) If the graph is distance-regular, the representation would be "locally injective" and carry considerable information about the original graph. The representation method enables us to study such a graph by investigating the geometric properties of its image configurations under a representation. The idea of representations was first introduced by Godsil [19], and independently by Terwilliger [32].

A set of points in a Euclidean space is said to be a 2-distance set if there exist two numbers a and b such that any two points in the set are at distance either a or b. Under a representation, the image of the neighbourhood of a vertex in a distance-regular graph forms a 2-distance set in an eigenspace. Therefore a bound on the size of a 2-distance set would imply a bound on the valency of a graph. In Chapter 3, we will introduce some known results on 2-distance sets and the so-called "equiangular lines". Based on this, we derive an upper bound on 2-distance sets, which will be useful in the classification work later.

Chapter 4 is basically a reproduction of Godsil's result on bounding diameter and valency of a distance-regular graph by a function of an eigenvalue multiplicity. The bounds as well as various numerical constraints derived in this chapter will be very helpful in the later effort of classifying distance-regular graphs by their eigenvalue multiplicities.

Chapter 5 and Chapter 6 are two working chapters for classifying distance-regular graphs of low multiplicities. It can be easily seen that there is no distance-regular graph with a single eigenvalue other than $\pm k$. Further, the only connected distance-regular graphs with an eigenvalue of multiplicity two are cycles or complete 3-partite regular graphs $\overline{3K_r}$, $r \geq 1$. The first non-trivial case is when eigenvalue multiplicity is three. Without too much difficulty it can be verified that the distance-regular graphs with an eigenvalue of multiplicity three are the five Platonic solids plus all complete 4-partite regular graphs $\overline{4K_r}$. (Surprising!) So our classification work starts with the case when eigenvalue multiplicity is four.

In Chapter 5, we apply our theory and classify the distance-regular graphs with an eigenvalue of multiplicity four. We use representation method as well as elementary arguments to do most of the work by hand. Of course, this would involve us in some lengthy case-by-case discussion. One subcase is worked out using a computer. In the last chapter, we first sharpen an upper bound on the diameter of a distance-regular graph which is derived in Chapter 4. This helps to reduce the possible candidate graphs. Then we use a computer to complete the classification of distance-regular graphs with an eigenvalue of multiplicity five.

Detailed information of distance-regular graphs with an eigenvalue of multiplicity three, four or five is given in the Appendix. There readers can also find the information about distance-regular graphs with valency three, which are classified by Biggs et al. in 1986.

Chapter 1

Distance-Regular Graphs

1.1 Distance-Regular Graphs

We first introduce some standard notation. Let G be a regular graph with valency k and diameter d. Denote by $V(G)$ the vertex set of G and $E(G)$ the edge set of G. For any pair of vertices u, v in G, denote by $\partial(u,v)$ the distance between u and v in G. We write $u \sim v$ if u and v are adjacent (i.e., if $\partial(u,v)$=1). For any vertex $u \in V(G)$ define

$$G_i(u) := \{v \in V(G) \mid \partial(v,u) = i\}, \quad 0 \leq i \leq d.$$

When $i = 1$ we abbreviate $G_1(u)$ to $G(u)$.

A graph G is called *distance-regular* if, given any two vertices u and v, the number of vertices in G at distance i from u and distance j from v only depends on the distance between u and v. Since u and v may coincide, a distance-regular graph is necessarily regular.

Let G be a distance-regular graph with valency k and diameter d. Then for any vertex u in G, the cardinality of $G_i(u)$ depends only on i. (This can be seen by letting v coincide with u and choosing $j = i$ in the definition above.) We denote this cardinality by k_i. Further, there exist integers c_i, b_i and a_i such that for any pair of vertices u and v in G at distance i apart,

$$\begin{aligned} c_i &= |G_{i-1}(u) \cap G_1(v)|, \quad 1 \le i \le d, \\ a_i &= |G_i(u) \cap G_1(v)|, \quad 1 \le i \le d, \\ b_i &= |G_{i+1}(u) \cap G_1(v)|. \quad 1 \le i \le d-1. \end{aligned}$$

The numbers c_i, a_i and b_i are called the *intersection numbers of* G and usually recorded in an array of the form

$$\left\{ \begin{array}{cccccc} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_{d-1} & * \end{array} \right\}.$$

This is called the *intersection array of graph* G. For example, the intersection array of the cube is

$$\left\{ \begin{array}{cccc} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right\}.$$

The intersection array of the complete multipartite regular graph $\overline{sK_r}$ is

$$\left\{ \begin{array}{ccc} * & 1 & (s-1)r \\ 0 & (s-2)r & 0 \\ (s-1)r & r-1 & * \end{array} \right\}.$$

In the following we collect some basic properties of distance-regular graphs. The proofs of (i)–(iv) can be found in Biggs [4] and the proof of (v) in Taylor and Levingston [31].

Lemma 1.1.1 *Let G be a connected distance-regular graph on n vertices with valency k and diameter d. Then we have the following.*

(i) $a_i + b_i + c_i = k, \; 1 \le i \le d-1, \;$ and $a_d + c_d = k$.

(ii) $k_i b_i = k_{i+1} c_{i+1}$, $i \geq 1$, *and* $k_i = (kb_1 \cdots b_{i-1})/(c_2 c_3 \cdots c_i)$, $i \geq 2$. *In particular,* $(kb_1 \cdots b_{i-1})/(c_2 c_3 \cdots c_i)$ *is an integer.*

(iii) $nk \equiv 0 \pmod 2$; $k_i a_i \equiv 0 \pmod 2$, $1 \leq i \leq d$.

(iv) $1 \leq c_2 \leq \cdots \leq c_d$, $k \geq b_1 \geq \cdots \geq b_{d-1}$.

(v) If $i + j \leq d$, *then* $c_i \leq b_j$. □

1.2 The Adjacency Matrix and Spectrum of a Graph

The adjacency matrix of a graph G is the $n \times n$ matrix $A = A(G)$, over the complex field, whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

If follows directly from the definition that A is a real symmetric matrix, and that the trace of A is zero.

Suppose that λ is an eigenvalue of A. Then, since A is real and symmetric, λ is real, and the multiplicity of λ as a root of the characteristic equation $\det(\lambda I - A) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ.

The *spectrum* of a graph G is the set of distinct eigenvalues of $A(G)$, together with their multiplicities. If the distinct eigenvalues of $A(G)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\operatorname{Spec} G = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

For example, the spectrum of a complete graph K_n on n vertices is

$$\operatorname{Spec} K_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

The spectrum of a complete multipartite regular graph $\overline{sK_r}$ is

$$\operatorname{Spec}(\overline{sK_r}) = \begin{pmatrix} r(s-1) & 0 & -r \\ 1 & s(r-1) & s-1 \end{pmatrix}.$$

The spectrum of the cycle C_n on n vertices is given in two forms according as n is even or odd. If n is odd,

$$\operatorname{Spec} C_n = \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \dots & 2\cos\frac{(n-1)\pi}{n} \\ 1 & 2 & \dots & 2 \end{pmatrix};$$

If n is even,

$$\operatorname{Spec} C_n = \begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} & \dots & 2\cos\frac{(n-2)\pi}{n} & -2 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix}.$$

If a graph G is a regular graph with valency k, then its adjacency matrix has the following properties.

(1) The valency k is an eigenvalue of $A = A(G)$.

(2) If G is connected, then the multiplicity of k is one.

(3) For any eigenvalue λ of G, we have $|\lambda| \leq k$.

(4) Let J be the matrix with all entries equal to 1. If G is connected, then $J = p(A)$ for some polynomial $p(x)$. In particular, $AJ = JA$.

Furthermore, if G is distance-regular with diameter d, then $A = A(G)$ has exactly $d+1$ distinct eigenvalues. For brevity, the eigenvalues of the adjacency matrix $A = A(G)$ will also be referred to as the eigenvalues of the graph G.

For more information about adjacency matrices, spectra and characteristic polynomials of graphs, readers are referred to Cvetković, Doob and Sachs [13] and Biggs [4].

1.3 Classification of Distance-Regular Graphs

Distance-transitive graphs and distance-regular graphs are two important classes of highly symmetric graphs. It is no surprise that people would consider classifying these graphs. A natural and traditional idea is to classify these graphs by their valency. Indeed, considerable effort has been made on this since 1971, when Biggs and Smith classified the distance-transitive graphs of valency three. Many mathematicians have contributed to classifying distance-transitive graphs and distance-regular graphs by their valency. The classification has been done for the distance-transitive graphs of valency up to thirteen and for distance-regular graphs of valency three. Details of these classifications can be found in Biggs and Smith [8], Smith [30], Gardiner [18], Ivanov et al. [23], Ivanov et al. [24], Biggs et al. [5], etc. A recent book [11] by Brouwer, Cohen and Neumaier provides a comprehensive account of the information.

It is understandable that the classification work on distance-transitive graphs is more fruitful than on distance-regular graphs, since the former is a proper subclass of the latter. For the distance-regular graphs with valency four or higher, Bannai and Ito have recently shown [2] that there are only finitely many distance-regular graphs of valency four. It is conjectured that for any fixed k, there are only finitely many distance-regular graphs with valency k.

Equivalently, this conjecture can be stated as — There exists a real-valued function $f(k)$, depending only on k, such that any distance-regular graph with valency k is bounded by $f(k)$. Actually, bounding diameter is a common feature for almost all classification of distance-regular graphs and distance-transitive graphs.

In a recent paper, Godsil has shown [20] that the diameter and valency of a distance-regular graph are both bounded by certain functions of its eigenvalue multiplicity. (This is inexact, the precise statement is given in Chapter 4.) This implies that, besides the complete multipartite graphs, there can be only finitely many distance-regular graphs with an eigenvalue of a given multiplicity. This result suggests a different approach for classifying distance-regular graphs, and that is to classify graphs by an eigenvalue multiplicity instead of by the valency.

This thesis is the beginning of this classification. We will study the consequences of distance-regular graphs possessing an eigenvalue of low multiplicity, and then classify the distance-regular graphs with an eigenvalue of multiplicity four or five.

1.4 Distance-Regular Line Graphs

The line graph of a regular graph is again a regular graph. But the line graph of a distance-regular graph is not necessarily distance-regular. It is natural to ask which distance-regular graphs have distance-regular line graphs. This question has been settled in Mohar and Shawe-Taylor [28] and the answer gives a nice relationship between distance-regular line graphs and (k, g)-graphs.

For $k \geq 1$ and $g \geq 3$ we define

$$n_0(k,g) := \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{i-2} + k(k-1)^{i-1}, & \text{if } g \text{ is odd;} \\ 1 + k + k(k-1) + \cdots + k(k-1)^{i-2} + (k-1)^{i-1}, & \text{if } g \text{ is even,} \end{cases}$$

where $i = \lfloor g/2 \rfloor$. A (k, g)-graph is a k-regular graph with girth g and $n_0(k, g)$ vertices. It is well known (see, for example, Proposition 23.1 of Biggs [4]) that, for a given pair of k and g, the number of vertices in a k-regular graph with girth g is at least $n_0(k, g)$. A (k, g)-graph is in some sense a smallest k-regular graph with girth g. But the converse is not true. For (infinitely) many pairs of k and g, a (k, g)-graph does not exist and in these cases every k-regular graph with girth g contains more than $n_0(k, g)$ vertices. The class of (k, g)-graphs is very limited. The classification is almost completed, except that for a few parameter sets the existence question remains open. (For more information, see Chapter 23 of Biggs [4]).

Lemma 1.4.1 (Mohar and Shawe-Taylor) *For a graph G the following conditions are equivalent:*

(i) $L(G)$ is distance-regular and $G \not\cong K_{1,n}$ for $n \geq 2$,

(ii) G and $L(G)$ are both distance-regular,

(iii) G is a (k,g)-graph. $\square$

The *characteristic polynomial* of a graph G is defined to be the characteristic polynomial of the adjacency matrix $A(G)$ of G. For a regular graph G, there is an easy way to calculate the characteristic polynomial $\varphi(L(G);\lambda)$ of the line graph, from the characteristic polynomial of G, $\varphi(G;\lambda)$. This result is stated in the following lemma. A proof of this lemma can be found, for example, in Section 2.4 of Cvetković, Doob and Sachs [13].

Lemma 1.4.2 (Sachs) *Let G be a regular graph of valency k with n vertices and $m = \frac{1}{2}nk$ edges. Then*

$$\varphi(L(G);\lambda) = (\lambda+2)^{m-n}\varphi(G;\lambda+2-k). \qquad \square$$

We rephrase this lemma as follows. If the spectrum of G is

$$\operatorname{Spec} G = \begin{pmatrix} k & \lambda_1 & \dots & \lambda_s \\ 1 & m_1 & \dots & m_s \end{pmatrix},$$

then the spectrum of $L(G)$ is

$$\operatorname{Spec} G = \begin{pmatrix} 2k-2 & k-2+\lambda_1 & \dots & k-2+\lambda_s & -2 \\ 1 & m_1 & \dots & m_s & m-n \end{pmatrix}.$$

(It is worth noting that if G is bipartite, then $k-2+\lambda_s = -2$, and the last two columns get merged into one.) For example, the line graph $L(K_n)$ of the complete graph on n vertices is calculated as

$$\operatorname{Spec} L(K_n) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \frac{1}{2}n(n-3) \end{pmatrix}.$$

Chapter 2

Representations of Distance-Regular Graphs

A representation of a graph is a mapping from the vertex set of the graph to Euclidean space R^m. If this representation mapping has some "good" properties, then we will be able to obtain information about the graph itself by studying its image in R^m. This chapter will discuss representation theory for distance-regular graphs. We will describe how each eigenspace of the adjacency matrix of a graph provides us with a "good" representation and study the basic properties of these representations.

2.1 Representations from Eigenspaces

We describe a very useful way of looking at the eigenvectors of a graph. This is due to Godsil [21] and Terwilliger [32].

Suppose that $A = A(G)$ and that z is an eigenvector of A with eigenvalue θ. As A is a symmetric matrix and all eigenvalues of A are real, we will from now on assume that all our discussions are over the real field. Since the entries of A are either 0 or

1, the equation $Az = \theta z$ is equivalent to the equations

$$\theta z_i = \sum_{j \sim i} z_j, \qquad i = 1, \ldots, n. \tag{2.1}$$

If we view z as a function from $V(G)$ to the real numbers, these equations imply that θ times the value of this function at vertex i is equal to the sum of the values of z at the neighbours of i. Conversely, any function on $V(G)$ which satisfies this condition can be seen to be an eigenvector. We often find it useful to view the values of the function z as "weights" on the vertices of G. If θ is an eigenvalue of A with multiplicity m, we can go further. Let U be an $n \times m$ matrix with its columns forming a basis for the eigenspace corresponding to θ. Then $AU = \theta U$ and so the rows of U give rise to a vector-valued function, u say, on $V(G)$ with the property that $\theta u(i)$ is equal to the sum of the values of u at the neighbours of i. We again find, conversely, that any vector-valued function satisfying this condition determines an eigenspace of A. Any such vector-valued function will be called a *representation* of the graph G.

By way of example, consider the cube in R^3. Identify its vertex set with the eight vectors of the form $(\pm 1, \pm 1, \pm 1)$. Two of these vectors, if not equal, agree in zero, one or two positions. Let Q represent the usual graph of the cube. Then the adjacent vertices in Q correspond to vectors which differ in only one position. We illustrate the situation in Figure 2.1

We now find that if we sum the vectors adjacent to a given vector x, the result is x. Thus, if we consider the vectors adjacent to $(1, 1, 1)$, we have

$$(-1, 1, 1) + (1, -1, 1) + (1, 1, -1) = (1, 1, 1).$$

This shows that the labeled cube in Figure 2.1 is a representation of the usual graph Q in R^3, corresponding to the eigenvalue $\theta = 1$.

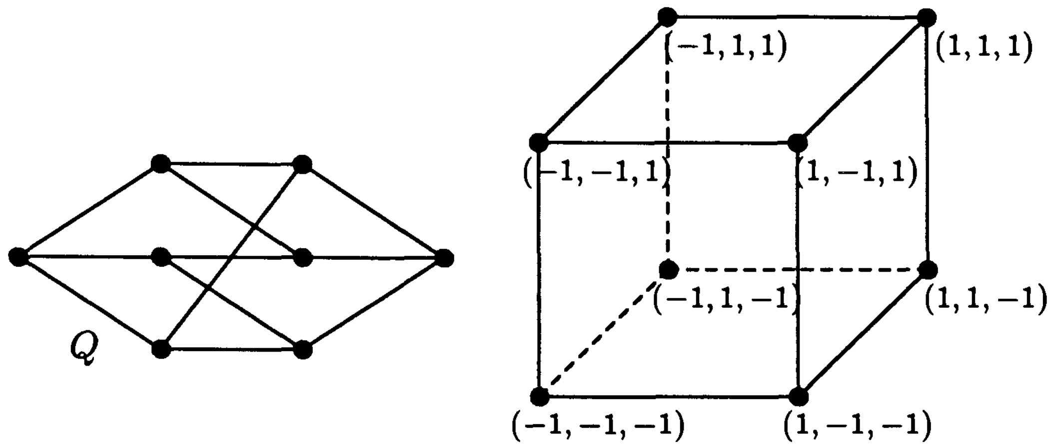

Figure 2.1: A graph and its representation

2.2 Spectral Decompositions

In this section we discuss some matrix theory and prove a useful matrix identity relating to representations of graphs. Suppose A is a real symmetric matrix. Then all of its eigenvalues are real, the eigenvectors of A span R^n and the eigenvectors in different eigenspaces are orthogonal. For each eigenvalue of A, let U_θ be a matrix with its columns forming an orthonormal basis for the eigenspace belonging to θ. Set $Z_\theta = U_\theta {U_\theta}^T$. We will refer to the matrices Z_θ as the *principal idempotents* of A. We denote by $\mathrm{ev}(A)$ the set of distinct eigenvalues of A.

Theorem 2.2.1 (Spectral decomposition of a symmetric matrix) *Let A be a real symmetric matrix, with principal idempotents Z_θ, $\theta \in \mathrm{ev}(A)$. Then the following hold:*

(a) $Z_\theta^2 = Z_\theta$ and $Z_\theta Z_\tau = 0$ *if* $\theta \neq \tau$,

(b) $AZ_\theta = \theta Z_\theta$,

(c) $\sum_{\theta \in \mathrm{ev}(A)} Z_\theta = I$, *(d) For any polynomial p, we have $p(A) = \sum_{\theta \in \mathrm{ev}(A)} p(\theta) Z_\theta$.*

Proof. (a) Since the columns of U_θ are pairwise orthogonal, ${U_\theta}^T U_\theta = I_{m(\theta)}$, where $m(\theta)$ is the multiplicity of θ. Hence

$$Z_\theta Z_\theta = U_\theta {U_\theta}^T U_\theta {U_\theta}^T = U_\theta {U_\theta}^T = Z_\theta$$

(This means that the matrices Z_θ are idempotents). If θ and τ are distinct eigenvalues of A, then, since eigenvectors in different eigenspaces are orthogonal, ${U_\theta}^T U_\tau = 0$ and therefore $Z_\theta Z_\tau = 0$.

(b) Since $AU_\theta = \theta U_\theta$, we find that $AZ_\theta = \theta Z_\theta$.

(c) Let D be the sum of all the matrices Z_θ, as θ ranges over the eigenvalues of A. Then $D^2 = D$ and so all eigenvalues of the symmetric matrix D are 0 or 1. Now

$$\mathrm{tr}(D) = \sum_\theta \mathrm{tr}(Z_\theta) = \sum_\theta m(\theta) = |V(G)|$$

and so the trace of D is equal to its order. This implies that all eigenvalues of D must be equal to 1, and hence that D can be decomposed as $D = L^T I L$ where L is an orthogonal matrix. But this means that D itself is the identity matrix.

(d) If we multiply both sides of (c) by A^r and then note that $A^r Z_\theta = \theta^r Z_\theta$, we see that our claim holds for the polynomials x^r. Hence for any polynomial p,

$$p(A) = \sum_{\theta \in \mathrm{ev}(A)} p(\theta) Z_\theta.$$

We are done. $\square$

As a special case of the above theorem, we have

$$A = \sum_{\theta \in \mathrm{ev}(A)} \theta Z_\theta \tag{2.2}$$

This equation explains the name *spectral decomposition*.

2.3 Basic Properties of Representations

Let G be a graph with vertex set $1, 2, \ldots, n$. In this section we assume A is the adjacency matrix of the graph G. Of course, the assertions in Theorem 2.2.1 hold for A. With the U_θ and Z_θ as defined in last section, it is clear that the matrices U_θ give representations of G associated with θ.

If θ is an eigenvalue of A, with multiplicity m. Then U_θ is an $n \times m$ matrix with columns forming an orthonormal basis for the eigenspace associated with θ. Let $u_\theta(i)$ be the i-th row of U_θ. Then this is a representation of G in R^m. The following result appears in Godsil [19], and Bannai and Ito [1] (Lemma II.8.2).

Lemma 2.3.1 *Let A be the adjacency matrix of the graph G. Then, if i and j are any two vertices in G and r is a non-negative integer,*

$$(A^r)_{ij} = \sum_{\theta \in \mathrm{ev}(A)} \langle u_\theta(i), u_\theta(j) \rangle \theta^r,$$

where $\langle , \rangle$ denotes the usual inner product of two real vectors.

Proof. By Theorem 2.2.1(d), we have

$$A^r = \sum_{\theta \in \mathrm{ev}(A)} \theta^r Z_\theta.$$

It is easily seen that the ij-entry of Z_θ equals the inner product $\langle u_i, u_j \rangle$. The lemma follows immediately. $\square$

Note that $(A^r)_{ij}$ is equal to the number of walks in G from vertex i to vertex j with length r. To apply our theory to distance-regular graphs we need some preliminary information.

Suppose G is a distance-regular graph. We define the *r-th distance matrix* $A_r = A_r(G)$ of G to be the $n \times n$ $(0-1)$-matrix with ij-entry equal to 1 if and only if the distance in G between vertex i and vertex j is r. Thus $A_0 = I$ and A_1 is the adjacency matrix of G. Also $\sum_{i=0}^{d} A_i = J$. The ij-entry of $A_r A_s$ is equal to the number of vertices in G at distance r from i and distance s from j. From the definition of a distance-regular graph it follows that this number only depends on r, s and the distance between i and j. Hence $A_r A_s$ can be written as a linear combination of distance matrices of G. Since the distance matrices are symmetric, it follows that $A_r A_s$ is symmetric too. This implies that $A_r A_s = (A_r A_s)^T = A_s^T A_r^T = A_s A_r$. The next lemma is due to Godsil [20] and Bannai and Ito [1].

Lemma 2.3.2 *Let G be a distance-regular graph of diameter d and let θ be an eigenvalue of G. If i and j are two vertices of G then the inner product $\langle u_\theta(i), u_\theta(j)\rangle$ is determined by the distance between i and j in G, independent of the choice of i and j.*

Proof. By the preceding discussion, for any i and j the product $A_i A_j$ is a linear combination of $A_0, A_1, \ldots, A_d$, and so is $A^2 = A_1 A_1$. It follows by induction that each A^r, $r \geq 1$, can be expressed as linear combination of $A_0, A_1, \ldots, A_d$. In other words, there exist numbers $c_r(t)$ such that

$$(A_1)^t = \sum_{r=0}^{d} c_r(t) A_t.$$

If i and j are at distance s in G, then this implies that the ij-entry of $(A_1)^t$ is equal to $c_s(t)$. Using Lemma 2.3.1 we then deduce that

$$c_s(t) = \sum_{\theta \in \text{ ev }(A)} \langle u_\theta(i), u_\theta(j)\rangle \theta^t.$$

If we fix i and j and take $n = 0, 1, \ldots, d$ this gives us a $(d+1) \times (d+1)$ system of linear equations satisfied by the $d+1$ inner products $\langle u_\theta(i), u_\theta(j)\rangle$. This system is non-singular because the matrix of coefficients is a Vandermonde matrix. Therefore these inner products are determined by the eigenvalues of A and the numbers $c_s(t)$. Since the latter only depend on the distance s between i and j, (independent of the choice of i and j), our lemma is proved. □

One immediate consequence of this lemma is the following.

Corollary 2.3.3 (Inheritance of distance structure) *Suppose that G is a distance-regular graph and θ an eigenvalue of G with multiplicity m. Let u_θ be a representation associated with θ. Then*

(i) The image vectors $u_\theta(v)$, $v \in V(G)$, all have the same length. In other words, u_θ maps $V(G)$ onto a sphere in R^m;

(ii) For any two vertices u and v in G, the distance between $u_\theta(u)$ and $u_\theta(v)$ in R^m is determined by the distance $\partial(u, v)$ in G. □

We now turn to another important property of representations of distance-regular graphs. The following lemma, and its proof, is due to Godsil [20].

Proposition 2.3.4 (Local Injectivity) *Let G be a distance-regular graph with valency k. Let θ be an eigenvalue of G of multiplicity m and θ not equal to $\pm k$. Suppose that u_θ is a representation associated with θ. Assume G is not a complete multipartite graph. Then for any two vertices i and j in G with $\partial(i, j) \leq 2$, $u_\theta(i) \neq u_\theta(j)$.*

Proof. We break the proof into a few steps.

(a) *If u_θ takes equal values on two adjacent vertices of G then $\theta = k$.*
If i and j are two adjacent vertices of G, then by Lemma 2.3.2 we find that

$$\langle u_\theta(x), u_\theta(y)\rangle = \langle u_\theta(i), u_\theta(j)\rangle$$

for any pair of adjacent vertices x and y in G. Since G is connected it follows that if $u_\theta(i)$ and $u_\theta(j)$ are equal, then u_θ is constant on $V(G)$. As we have

$$\theta u_\theta(i) = \sum_{j \sim i} u_\theta(j) \tag{2.3}$$

it follows that $\theta u_\theta(i) = ku_\theta(j) = ku_\theta(i)$. Hence $\theta = k$.

(b) *If i and j are adjacent vertices in G and $u_\theta(i) = -u_\theta(j)$, then $\theta = -k$.* If i and j are adjacent vertices in G and $u_\theta(i) = -u_\theta(j)$, then $u_\theta(x) = -u_\theta(y)$ for any pair of adjacent vertices x and y in G. So Equation 2.3 yields $\theta u_\theta(i) = ku_\theta(j) = -ku_\theta(i)$. Therefore $\theta = -k$.

(c) *If i and j are vertices in G such that $u_\theta(i) = u_\theta(j)$, then $\partial(i,j) > 2$ in G.* Suppose $u_\theta(i) = u_\theta(j)$. It was seen in (a) that i and j cannot be adjacent. Now assume $\partial(i,j) = 2$. Then if x and y are any two vertices in G with $\partial(x,y) = 2$, we must have $u_\theta(x) = u_\theta(y)$.

If G does not have any odd cycle then it is bipartite and u_θ is constant on each of the two color classes. For a pair of adjacent vertices 0 and 1, Equation 2.3 implies that $\theta u_\theta(0) = ku_\theta(1)$ and, interchanging the roles of 0 and 1, also that $\theta u_\theta(1) = ku_\theta(0)$. Consequently $\theta^2 = k^2$ and so $\theta = \pm k$.

Now suppose that G contains odd cycles. If there is an induced odd cycle on at least five vertices, then it follows that u_θ is constant on this cycle and there are pairs of adjacent vertices x and y such that $u_\theta(x) = u_\theta(y)$. Consequently we are back in case (a) above.

The only remaining possibility is that the smallest odd cycle has three vertices. In other words, G contains a triangle, say, vwx. Choose a vertex $y \in G_2(x) \cap G(v)$. If y is not adjacent to w, then $\partial(y,x) = \partial(y,w) = 2$ and thus $u_\theta(x) = u_\theta(y) = u_\theta(w)$. We are again back in case (a) (since x and w are adjacent). This shows that y is adjacent to both of v and w. Assume that the diameter of G is at least three and $z \in G_3(x) \cap G(y)$. Since yvw forms a triangle, vertex z must be adjacent to at least one of v and w. But that means $z \in G_2(x)$, which contradicts the assumption that $z \in G_3(x)$. So the diameter of G is two.

Now choose three vertices z_1, z_2 and z_3 in G such that z_1 is not adjacent to z_2 and z_2 is not adjacent to z_3. As the diameter of G is two, we get $\partial(z_1, z_2) = \partial(z_2, z_3) = 2$, and thus $u_\theta(z_1) = u_\theta(z_2) = u_\theta(z_3)$. It follows from claim (a) that z_1 and z_3 are not adjacent. In other words, the non-adjacency is a transitive relation on the vertex set of G. Therefore G is a complete multipartite graph. □

2.4 The Sequence of Cosines

Suppose G is a distance-regular graph with diameter $d \geq 2$. Let θ be an eigenvalue and u_θ a normalized representation associated with θ. Then the images under u_θ are all unit vectors. Since the inner product of the image vectors of two vertices x and y only depends on the distance $\partial(x, y)$ in G, there are real numbers w_i, $0 \leq i \leq d$, such that for any pair of vertices x and y at distance i,

$$\langle u_\theta(x), u_\theta(y)\rangle = w_i. \tag{2.4}$$

The sequence $w_0, w_1, w_2, \ldots, w_d$ will be called the *sequence of cosines* of G corresponding to the eigenvalue θ. Obviously we have $w_0 = 1$ and $|w_i| \leq 1$, $1 \leq i \leq d$. By the local injectivity of u_θ, it is also clear that $w_1 < 1$ and $w_2 < 1$.

Now fix a pair of vertices u and v in G with $\partial(u, v) = i$. Then the neighbourhood of v contains c_i vertices x_j with $\partial(u, x_j) = i - 1$ and a_i vertices y_j with $\partial(u, y_j) = i$, and b_i vertices z_j with $\partial(u, z_j) = i + 1$. From Equation 2.1 in Section 1, we have

$$\theta u_\theta(v) = \sum u_\theta(x_j) + \sum u_\theta(y_j) + \sum u_\theta(z_j).$$

Taking the inner product with $u_\theta(u)$ on both sides of this equation, we get

$$\theta w_i = c_i w_{i-1} + a_i w_i + b_i w_{i+1}, \qquad 0 \leq i \leq d, \tag{2.5}$$

with the understanding that $w_{-1} = w_{d+1} = 0$. Recall that (in Section 2) representations are obtained from the matrix $Z_\theta = U_\theta {U_\theta}^T$, where U_θ is not uniquely determined by θ. However, from (2.5) we see that the sequence of cosines is uniquely determined by the eigenvalue θ.

We summarize some facts about the sequence of cosines.

Lemma 2.4.1 *Let G be a connected distance-regular graph with diameter d. Let θ be an eigenvalue of G, and w_i, $1 \le i \le d$, be the sequence of cosines of G corresponding to θ. Then we have the following.*

(a) *If $w_1 = 1$, then $\theta = k$.*

(b) *If $w_1 = -1$, then $\theta = -k$.*

(c) *If $w_2 = w_1$, then either $\theta = k$ or $\theta = -1$.*

(d) *If $w_2 = 1$, then G is a complete multipartite graph.*

(e) *If $w_2 = -1$, then either G is $\overline{nK_2}$ for some $n \ge 2$, or G is a cycle of length $4n$ with $n \ge 1$.*

Proof. From Equation 2.5 we have

$$w_1 = \frac{\theta}{k}, \qquad \text{and} \qquad w_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}.$$

Hence claims (a) and (b) are obvious. Claim (c) follows from equation

$$w_1 - w_2 = \frac{(k-\theta)(\theta+1)}{kb_1}.$$

Claim (d) is a rephrasing of Lemma 2.3.4.

For claim (e), we first show that if $w_2 = -1$, then $w_i = \cos\frac{i\pi}{2}$ for $1 \le i \le d$. Let x and y be two vertices of G and u_θ be the representation associated with θ. The equation $w_2 = -1$ implies that if $\partial(x, y) = 2$, then $u_\theta(y) = -u_\theta(x)$, and that if

$\partial(x, y) = 4$, then $u_\theta(y) = u_\theta(x)$. So the equation $w_i = \cos\frac{i\pi}{2}$ will follow by induction if we can prove that $w_1 = 0$. Now let xyz be a path with $\partial(x, z) = 2$ (i.e., a geodetic path). Then $u_\theta(x) = -u_\theta(z)$. Since $\partial(y, x) = \partial(y, z)$, we have the inner products

$$\langle u_\theta(y), u_\theta(x)\rangle = \langle u_\theta(y), u_\theta(z)\rangle = \langle u_\theta(y), -u_\theta(x)\rangle = -\langle u_\theta(y), u_\theta(x)\rangle.$$

Therefore $w_1 = \langle u_\theta(y), u_\theta(x)\rangle = 0$. This in turn implies that $\theta = 0$.

Applying $w_i = \cos\frac{i\pi}{2}$ to Equation 2.5 (the recursive relation), we can calculate that $b_{2s-1} = c_{2s-1}$ and $a_{2s} = 0$ for $s \geq 1$. In particular, $b_1 = c_1 = 1$ and $a_2 = 0$. By Lemma 1.1.1, the equation $b_1 = 1$ implies that all b_i are equal to 1 and all c_i, except c_d, are equal to 1. If the diameter of G is two, then $c_2 = k$ and G has $k + 2$ vertices with valency k being an even integer. So G is a complete multipartite graph $\overline{nK_2}$. If the diameter of G is greater than two, then $b_2 = c_2 = 1$ and $k = c_2 + a_2 + b_2 = 2$. Therefore the intersection array of G has the pattern

$$\left\{\begin{array}{cccccccc} * & 1 & 1 & \ldots\ldots & 1 & 1 & c_d \\ 0 & 0 & 0 & \ldots\ldots & 0 & 0 & a_d \\ 2 & 1 & 1 & \ldots\ldots & 1 & 1 & * \end{array}\right\}.$$

By the equation $b_{2s-1} = c_{2s-1}$, for any odd integer i, the existence of c_i would imply the existence of b_i (and both should be equal to 1). Therefore the diameter d of G must be an even integer, and thus $a_d = 0$ and $c_d = k$. It follows that G is a cycle of length $4n$ with $n \geq 2$. □

Let G be a distance-regular graph with an eigenvalue θ of multiplicity m. In general, the mapping u_θ need not be injective (though it should be locally injective). The following two lemmas will show that in some situations, the representation will be (globally) injective. For a proof, see Godsil [20].

Lemma 2.4.2 *Let θ be an eigenvalue of G. Then θ is the i-th largest eigenvalue of G if and only if the corresponding sequence of cosines $(w_0, w_1, \ldots, w_d)$ has exactly $i - 1$ sign-changes.* □

Lemma 2.4.3 *Let G be a connected distance-regular graph of diameter d, valency k at least three and with an eigenvalue $\theta \neq \pm k$. If the number of eigenvalues of $A(G)$ which are greater than θ is odd, then u_θ is an injective mapping.* □

In other words, if the number of sign-changes of the sequence of cosines is odd, then u_θ is an injective mapping.

Although the representation carries considerable information about the original graph to its image configuration, we cannot expect that the image keeps all the structure information. Even when u_θ is injective on $V(G)$, it does not necessarily follow that the sequence $w_0, \dots, w_d$ is nonincreasing. In particular, the images of the vertices adjacent to a given vertex x need not be the points in $u_\theta(V(G) \setminus x)$ closest to $u_\theta(x)$. There is, however, one important case when this does hold true.

Lemma 2.4.4 (Godsil [20]) *Let G be a distance-regular graph with valency k and diameter d at least two. If θ is the second largest eigenvalue of G and $x \in G$ then the points in $u_\theta(V(G))$ closest to $u_\theta(x)$ are the images of the vertices adjacent to x.*

□

2.5 Gram Matrices

We are going to introduce the basic properties of Gram matrices in this section, which will be useful in the following discussions.

Let A be a real $m \times n$ matrix. Consider the real quadratic form $q(x) = x^T(A^TA)x$. This form is seen at once to be positive semidefinite because $q(x) = (Ax)^TAx$, which is a non-negative scalar since AX is a real vector. So the matrix $G = A^TA$ is a positive semidefinite matrix. On the other hand, letting $A_1, A_2, \dots, A_m$ be the columns of A,

G can be written as

$$G = \begin{pmatrix} A_1{}^T A_1 & \cdots & A_1{}^T A_n \\ \vdots & & \vdots \\ A_n{}^T A_1 & \cdots & A_n{}^T A_n \end{pmatrix}$$

Any real matrix of this form is called a Gram matrix. A Gram matrix must be symmetric. Note that the matrices $A^T A$ and AA^T are both Gram matrices. They have the same rank and the same set of non-zero eigenvalues.

We summarize the basic properties of Gram matrices in the following.

Proposition 2.5.1 *Let $G = A^T A$ be a Gram matrix. Then*

(i) *G is positive semidefinite;*

(ii) *The rank of G equals the rank of A;* and

(iii) *All eigenvalues of G are non-negative.* □

2.6 A Constraint on Eigenvalue Multiplicity

We are going to present a useful theorem of Terwilliger [33], and give a modified proof.

We can write the eigenvalues of G explicitly as

$$\theta_0 \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_{min}.$$

Since G is regular, θ_0 equals the valency k.

Theorem 2.6.1 (Terwilliger) *Let G be a connected distance-regular graph with valency k ($k \geq 2$) and diameter at least 2. Let θ be an eigenvalue of G and $\theta \neq k$. If the multiplicity $m = m(\theta) < k$, then $\theta = \theta_1(G)$* or *$\theta = \theta_{min}(G)$. Furthermore, either*

(i) *θ is an integer such that $1 + \theta$ divides b_1, or*

(ii) *θ and $b_1(1+\theta)^{-1}$ are both quadratic algebraic integers over Q, and θ_1 and θ_{min} are algebraic conjugates over Q with equal multiplicities as eigenvalues of $A(G)$.*

Proof. We first note that if the graph G is a complete multipartite graph, then the diameter of G is 2. Hence G would have only two eigenvalues other than the valency, and both of them are integers. (The spectrum of any complete multipartite regular graph has been presented in Section 1.2.) Therefore nothing needs to be proven. In the rest of the proof we assume that G is not complete multipartite.

Fix a vertex x in G. We use G_1 to represent the subgraph induced by the neighbourhood of x. (The subgraph G_1 itself is a regular graph of valency a_1.) Denote by $A_1 = A(G_1)$ the adjacency matrix for the subgraph G_1.

Now let θ be an eigenvalue of the (original) graph G, and u_θ the representation associated with θ.

Consider the image vectors of the vertices in the neighbourhood G_1 and denote by N the Gram matrix of these vectors. Let $J = J_{k\times k}$ be the all 1's matrix. Then we have

$$\begin{aligned} N &= I + w_1A_1 + w_2\bar{A}_1 \\ &= I + w_1A_1 + w_2(J - I - A_1) \\ &= (1-w_2)I + (w_1-w_2)A_1 + w_2J \\ &= (1-w_2)(I + \frac{w_1-w_2}{1-w_2}A_1 + \frac{w_2}{1-w_2}J), \end{aligned}$$

where w_1 and w_2 are the cosines of a pair of vertices at distance one and two, respectively.

Now let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of the subgraph G_1, in non-increasing order. (Note: we use λ_i for the subgraph G_1, and θ_i for the graph G.) Since we assumed that G would not be a complete multipartite graph, it follows from Lemma 2.4.1 that

$1 - w_2 \neq 0$. As all eigenvectors of $A_1 = A(G_1)$ are also eigenvectors of J, it is easy to see that the eigenvalues of $(\frac{1}{1-w_2})N$ are

$$\mu_1 := 1 + (\frac{w_1 - w_2}{1 - w_2})\lambda_1 + (\frac{w_2}{1 - w_2})k \qquad \text{and}$$

$$\mu_i := 1 + (\frac{w_1 - w_2}{1 - w_2})\lambda_i, \qquad 2 \leq i \leq k.$$

As N is positive semidefinite, all μ_i $(1 \leq i \leq k)$ should be non-negative. Further, since $\operatorname{rank}(N) \leq m < k$ and N is a $k \times k$-matrix, some of these μ_i must equal zero. We claim that there must exist an i, $2 \leq i \leq k$, such that $\mu_i = 0$. (This does not exclude the possibility that μ_1 may equal zero as well.)

Suppose this claim is not true. Then μ_1 would be the only eigenvalue of $(\frac{1}{1-w_2})N$ which equals zero and $\operatorname{rank}(N) = k - 1$. We would have

$$(1 - w_2)\mu_1 = 1 - w_2 + (w_1 - w_2)\lambda_1 + w_2 k = 0.$$

As G_1 is a_1-regular, $\lambda_1 = a_1$. We then derive that

$$\begin{aligned} 0 &= 1 + w_1 a_1 + w_2(k - a_1 - 1) \\ &= w_0 c_1 + w_1 a_1 + w_2 b_1 \\ &= \theta w_1 \\ &= k w_1^2. \end{aligned}$$

So $w_1 = 0$. Recall that w_1 is the inner product of the image vector $u_\theta(x)$ with the image vector of any vertex in G_1. Therefore $w_1 = 0$ implies that $u_\theta(x)$ does not lie in the span of $u_\theta(G_1)$. Since the latter should span a subspace of dimension $k - 1$ as we just assumed, the union $u_\theta(x) \cup u_\theta(G_1)$ would span a subspace of dimension $1 + (k - 1) = k$. But on the other hand, all image vectors under the representation u_θ should lie in a subspace of dimension at most m. This contradiction shows that if

μ_1 is equal to zero, then there must be another μ_i, $2 \le i \le k$, which equals zero too. The claim is proved.

Recall that from Equation 2.5 we have

$$w_1 = \frac{\theta}{k}, \qquad \text{and} \qquad w_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}.$$

From these we derive that

$$\begin{aligned} w_1 - w_2 &= \frac{(k-\theta)(\theta+1)}{kb_1}, \\ 1 - w_2 &= \frac{(k-\theta)(\theta+b_1-1)}{kb_1}, \qquad \text{and} \\ \frac{w_1 - w_2}{1 - w_2} &= \frac{\theta+1}{\theta+1+b_1}. \end{aligned}$$

Now let λ be an eigenvalue of G_1 such that the corresponding $\mu_i = 0$, $2 \le i \le k$. Then

$$1 + \left(\frac{w_1 - w_2}{1 - w_2}\right) = 1 + \left(\frac{\theta+1}{\theta+1+b_1}\right)\lambda = 0. \tag{2.6}$$

It follows from this equation that $\lambda \ne 0$. Notice that N is positive semidefinite for any eigenvalue θ of G with $\theta \ne k$. The sequence

$$\left\{1 + \left(\frac{\theta+1}{\theta+1+b_1}\right)\lambda\right\} \tag{2.7}$$

would remain non-negative for these θ. We further observe that $(\theta+1)/(\theta+1+b_1)$ is a monotone increasing function of the real variable θ. Therefore the sequence 2.7 is monotone as θ runs from second largest eigenvalue to least eigenvalue of G. We conclude that this sequence could reach zero only at the two end points, i.e., θ being either second largest eigenvalue or least eigenvalue of the graph G. This proves the first part of the theorem.

Now since eigenvalues of graphs are algebraic integers, so are θ and λ. It follows from Equation 2.6 that $b_1(1+\theta)^{-1}$ is an algebraic integer, too. Since any irreducible polynomial over Q has no multiple roots over the complex field, all algebraic conjugates of θ over Q are eigenvalues of G with the same multiplicity as θ.

In the first part of the proof we have shown that any eigenvalue of G with multiplicity $m(\theta) < k$ could be either θ_1 or θ_{min}. It follows that θ could have at most one conjugate besides itself. If θ is not in Q, then θ and its unique conjugate comprise the set $\{\theta_1, \theta_{min}\}$. Otherwise, we will have that $b_1(1+\theta)^{-1}$ is in Q, and hence θ and $b_1(1+\theta)^{-1}$ are both integers. $\square$

Chapter 3

Spherical Two-Distance Sets

A set S of points in Euclidean space is said to be a 2-distance set if there are two real numbers a and b such that the distance between any pair of points in S is either a or b. We say S is spherical if it is a subset of a sphere. For distance-regular graphs there is a natural way to represent the vertices of the graph by a set of points on a unit sphere in a Euclidean space. (Refer to Chapter 2.) Under this representation, the image of the neighbourhood of any vertex forms a 2-distance set. In this chapter we will discuss spherical 2-distance sets and related bounds. In Section 2 we discuss equiangular lines, which are closely related to 2-distance sets. Croft proved in 1962 [12] that the exact bound for the cardinality of 2-distance sets in R^3 is 6. In 1979, O. Kristensen proved in an unpublished research report, that the corresponding bound for 2-distance sets in R^4 and R^5 are 10 and 16, respectively. (The author thanks Professor H. Tverberg for kindly providing this information and explaining the outline of the proof, which was originally written in Norwegian.) Both of the above-mentioned proofs are lengthy. For the case of spherical 2-distance sets, we will give a short proof in Section 3.3.

3.1 The Absolute Bound

Similar to the concept of 2-distance set, an r-distance set can be defined for any integer $r \geq 1$. A 1-distance set in R^d is the vertex set of a regular simplex. An interesting problem is to determine the largest possible cardinality of an r-distance set in R^d.

Except for Proposition 3.1.2, most discussion of this section is based on Seidel's work [29].

We are mainly interested in the spherical 2-distance sets. Without loss of generality, we usually consider only those spherical 2-distance sets which lie on the unit sphere Ω_d, where

$$\Omega_d := \{\mathrm{x} \in R^d | \langle \mathrm{x}, \mathrm{x} \rangle = 1\}.$$

In this case the distance between two vectors is determined by their inner product. Hence a spherical set S is a 2-distance set if the vectors in S admit only two distinct inner products, say α and β, and neither α nor β is equal to one.

In R^2, the maximum cardinality of a spherical 2-distance set is five, attained by the vertices of the regular pentagon (Kelly [25]). In R^3, the maximum cardinality of a spherical 2-distance set is six (Croft [12]). In this case, there are many different configurations realizing the bound. For example, the bound is attained by the vertices of the octahedron, and also by any six of the twelve vertices of the icosahedron which do not contain an antipodal pair (Croft [12], Seidel [29]). For general d, the largest (spherical) 2-distance set which has been constructed is the set of $\frac{1}{2}d(d+1)$ mid-points of the edges of a regular simplex. The following theorem yields a universal upper bound for the cardinality of a 2-distance set in terms of d, called the *absolute bound* for spherical 2-distance sets.

Theorem 3.1.1 (Delsarte et al. [14]) *Let S be a spherical 2-distance set in R^d. Then $|S| \leq \frac{1}{2}d(d+3)$.*

Proof. For each vector t in a 2-distance set S of cardinality n with admissible inner products α and β, we define the function

$$F_{\mathbf{t}}(\mathbf{x}) := (\langle \mathbf{t}, \mathbf{x}\rangle - \alpha)(\langle \mathbf{t}, \mathbf{x}\rangle - \beta), \quad \mathbf{x} \in \Omega_d.$$

These are n polynomials of degree at most two in the variables $x_1, \ldots, x_d$ restricted to Ω_d. The linear space of all such polynomials is spanned by the $\frac{1}{2}d(d+3)$ polynomials $\{x_i, 1 \leq i \leq d\} \cup \{x_i{}^2, 1 \leq i \leq d\} \cup \{x_i x_j, 1 \leq i < j \leq d\}$. The polynomials $F_{\mathbf{t}}(\mathbf{x}), \mathbf{x} \in S$, are linearly independent since

$$F_{\mathbf{t}}(\mathbf{x}) = \begin{cases} (1-\alpha)(1-\beta) & \text{if } \mathbf{x} = \mathbf{t}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, their number n cannot exceed $\frac{1}{2}d(d+3)$, the number of the polynomials which spans the linear space. $\square$

So far there are only three known cases in which this bound is reached, namely, when $d = 2$, 6 or 22 (see Larman et al. [26]). On the other hand, we do know that in a few cases the exact bound is smaller than the absolute bound. For example, we are going to show in Section 3.3 that the exact bounds for the cases $d = 3$, 4 and 5 are 6, 10 and 16, respectively, smaller than the corresponding absolute bounds 9, 14 and 20.

General (non-spherical) 2-distance sets have also been studied in depth and some bounds have been obtained (see Delsarte et al. [14] and Blokhuis [7]). For $r = 1$ the problem is trivial and the largest possible number of points of an 1-distance set in R^d is obviously d+1. The *absolute bound* for a (general) 2-distance set in R^d is $\frac{1}{2}(d+1)(d+2)$ (Blokhuis [7]).

In studying regular graphs, it is interesting to consider the (spherical) 2-distance sets satisfying a "regularity" constraint. That is, a spherical 2-distance set S with a fixed integer m such that for any point $\mathbf{x}$ in S there are exactly m points in S having inner product α with $\mathbf{x}$.

Proposition 3.1.2 *Let S be a spherical 2-distance set on Ω_d satisfying the regularity condition. If the sum of all vectors in S is not a zero vector, then $|S| \leq \frac{1}{2}d(d+3)-1$.*

Proof. Let $n = |S|$. Define a function $G(\mathbf{x})$ on the set S.

$$G(\mathbf{x}) := \sum_{\mathbf{y}\in S} \langle \mathbf{y}, \mathbf{x}\rangle = \langle \sum_{\mathbf{y}\in S} \mathbf{y}, \mathbf{x}\rangle, \quad \mathbf{y} \in S$$

Since S satisfies the regularity condition, we have

$$G(\mathbf{y}) = 1 + m\alpha + (n-m-1)\beta.$$

On the other hand,

$$G(\mathbf{x}) = c_1x_1 + c_2x_2 + \cdots + c_dx_d,$$

where $c_1, \ldots, c_d$ are constants determined by the set S. Denote by C the constant $1 + m\alpha + (n-m-1)\beta$. The following equation holds over the set S:

$$c_1x_1 + c_2x_2 + \cdots + c_dx_d = C = C(x_1^2 + \cdots + x_d^2).$$

If the sum of the elements of S is not zero, then $c_1, \ldots, c_d$ cannot all be zero. It follows that the $\frac{1}{2}d(d+3)$ polynomials $\{x_i, 1 \leq i \leq d\} \cup \{x_i{}^2, 1 \leq i \leq d\} \cup \{x_ix_j, 1 \leq i < j \leq d\}$. are linearly dependent. As we have seen in the proof of Theorem 3.1.1, the maximum number of the linearly independent polynomials of degree at most 2 gives a bound for the size of S. It follows that

$$|S| \leq \frac{1}{2}d(d+3) - 1. \qquad \square$$

3.2 Equiangular Lines

A set of lines in Euclidean space is called equiangular, if the angle between each pair of lines is the same. The possibilities for equiangular sets of lines surpass by far those for equiangular sets of vectors. For instance, in Euclidean space R^3 the four diagonals of the cube, and the six diagonals of the icosahedron constitute equiangular sets of lines. Equiangular lines have a close relation to the spherical 2-distance sets. (This section is based on Lemmens and Seidel [27] and Seidel [29].)

Again, we would like to know the maximum size of a set of equiangular lines in R^d. Define $v_\alpha(d)$ to be the maximum number of lines in R^d such that the angle of each pair of lines equals either $\arccos\alpha$ or $\pi - \arccos\alpha$, $\alpha > 0$. Define $v(d)$ to be the maximum number of equiangular lines in R^d, i.e., $v(d) = \max_\alpha(v_\alpha(d))$. So, representing each line by a spanning unit vector, we are interested in the maximum number of unit vectors in R^d whose mutual inner products equal $\pm\alpha$.

There are a few general results on the lower and upper bounds for $v(r)$. First of all, it is shown in Lemmens and Seidel [27] that $v(r) \leq \frac{1}{2}r(r+1)$. As to the lower bounds we have the following lemma.

Lemma 3.2.1 (Lemmens and Seidel [27]) *The following are true.*

(a) $v(q^2+q+1) \geq q(q^2+q+1)$, *if* $q = 2^n$ *for some* n;

(b) $v(q^2-q+1) \geq q^3+1$, *if* $q = p^n$ *for some prime* $p \neq 2$; *and*

(c) $v(r) > 2r$, *for* $r \neq 5$ *or* 14. □

The first two assertions in the above lemma roughly mean that $v(r)$ is at least $r\sqrt{r}$. The next theorem will have an important implication.

Theorem 3.2.2 *Suppose* R^r *contains* n *equiangular lines with the angle* $\arccos\alpha$. *If* $n > 2r$, *then* $1/\alpha$ *is an odd integer.*

Proof. Let G be the Gram matrix of a set of n unit vectors in R^r with mutual inner product $\pm\alpha$. The matrix G is positive semidefinite and has null space of dimension at least $n-r$. Define $A =: (1/\alpha)(G-I)$. Then A has least eigenvalue $-(1/\alpha)$ with multiplicity m, where $m \geq n-r$. Since A is an integer matrix, $-(1/\alpha)$ is an algebraic integer and every algebraic conjugate of $-(1/\alpha)$ is also an eigenvalue of A with the same multiplicity m. If $n > 2r$, then $m > \frac{1}{2}n$ and A, being an $n \times n$ matrix, cannot have more than one eigenvalue of multiplicity m. Therefore $-(1/\alpha)$ is rational, and hence a rational integer.

The eigenspace of A corresponding to the eigenvalue $-(1/\alpha)$ has dimension m and the eigenspace of the all-one matrix J corresponding to the eigenvalue 0 has dimension $v-1$. Since $m > 1$ these subspaces have a nontrivial intersection whose vectors are eigenvectors of the matrix

$$B := \frac{1}{2}(J - I - A),$$

corresponding to the eigenvalue $\lambda = \frac{1}{2}(-1-(1/\alpha))$. Since B is an integer matrix, λ is an algebraic integer. But λ is rational, and therefore λ is a rational integer. Consequently $-(1/\alpha)$ is an odd integer. $\square$

By this theorem and Lemma 3.2.1, in almost all cases a set of equiangular lines with maximum size has a mutual inner product α such that $1/\alpha$ is an odd integer. In other words, almost every set of equiangular lines with maximum size has mutual inner product $\alpha \in \{1/3, 1/5, 1/7, 1/9, \ldots\}$.

Table 3.1 collects the results on the best known lower bounds for $v(r)$ with the corresponding values of $1/\alpha$. For a few values of r, the exact value of $v(r)$ is known.

r	2	3	4	5	6	7	$\cdots$	13	14	15	16
Lower Bound of $v(r)$	3	6	6	10	16	28	$\cdots$	28	28	36	40
Exact Value of $v(r)$	3	6	6	10	16	28	$\cdots$	28		36	
$1/\alpha$	2	$\sqrt{5}$	3	3	3	3	$\cdots$	3	3	5	5

r	17	18	19	20	21	22	23	$\cdots$	42	43
Lower Bound of $v(r)$	48	48	64	80	126	176	276	$\cdots$	276	344
Exact Value of $v(r)$					126	176	276			
$1/\alpha$	5	5	5	5	5	5	5	$\cdots$	5	7

Table 3.1: Lower Bounds on Equiangular Lines

3.3 Constructions of Equiangular Lines and Two-distance Sets

Except for Proposition 3.3.1 and Proposition 3.3.3, the discussion in this section is based on Seidel's work [29].

Any set of n equiangular lines in R^{d+1} gives rise to a spherical 2-distance set of $n-1$ points in R^d. Indeed, for any unit vector u along any of the lines consider the unit vectors at acute angle with u along the $n-1$ remaining lines and project them into a hyperplane perpendicular to u. For instance, the six diagonals of an icosahedron form an equiangular set in R^3. The five neighbours of any vertex of the icosahedron form a regular pentagon, which is a spherical 2-distance set in a plane. Provided with the information in Table 3.1, we can be assured of the existence of spherical 2-distance sets of 9 points in R^4, of 15 points in R^5, of 27 points in R^6, ..., and of 343 points in R^{42}.

Conversely, given any 2-distance set S of n points on the unit sphere in R^d, if the two inner products α and β satisfy $\alpha+\beta \leq 0$, we can construct a set of equiangular lines in R^{d+1} with the same size n.

We imbed R^d into R^{d+1} and view this R^d as a hyperplane in R^{d+1}. Thus there is a unique "axis" through the centre of the unit sphere and perpendicular to the hyperplane. We intend to "lift" the centre of the unit sphere along the axis to a proper position such that the (new) lines from this new centre to the points on the original unit sphere form equiangular lines in R^{d+1}. This can be done as long as there exist a real number $r \geq 1$ and an angle ϕ such that

$$1-\alpha = r^2(1-\cos\phi), \qquad 1-\beta = r^2(1+\cos\phi). \tag{3.1}$$

In this case r would be the distance from any point in S to the new centre, while ϕ would be the common angle for the resulting set of equiangular lines in R^{d+1}.

We verify this for the case of R^2 imbedding into R^3. For the general case, it can be verified in a similar way. Suppose that the required r and ϕ exist. Denote by O the centre of the circle in R^2 on which the points of S lie. After imbedding into R^3 there is an unique axis through O and perpendicular to the original R^2. Since $r \geq 1$, we can always find a point O' on the axis such that the distance from O' to any point in the original circle is r. Now draw lines from O' to the points in S (on the circle). We claim that these (new) lines form a set of equiangular lines in R^3 with mutual angle ϕ.

Assume that $\alpha \geq \beta$. (Then a pair of points with inner product α would be a shorter distance apart than a pair corresponding to β.) Let P and Q be a pair of points in S associated with inner product α. From Figure 3.1(a), it is easy to see (using elementary trigonometry) that

$$\begin{aligned} |PQ|^2 &= 1+1-2\cos\theta && \text{(based on the triangle } OPQ\text{) and} \\ |PQ|^2 &= r^2+r^2-2\cos\phi && \text{(based on the triangle } O'PQ\text{).} \end{aligned}$$

With $\alpha = \cos\theta$, these two equations lead to

$$1-\alpha = r^2(1-\cos\phi).$$

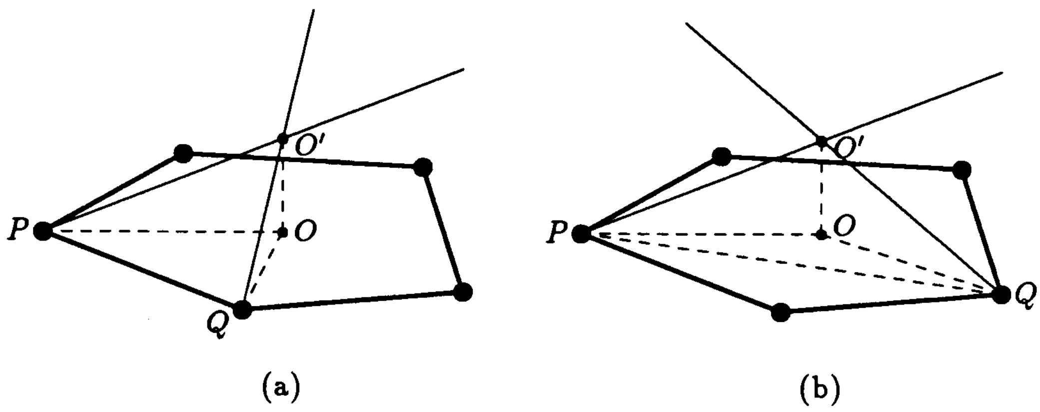

Figure 3.1: Construction of equiangular lines from 2-distance set

Now, let P and Q be a pair of points in S associated with β. From Figure 3.1(b), it follows that

$$\begin{aligned} |PQ|^2 &= 1+1-2\cos r && \text{(based on the triangle } OPQ\text{) and} \\ |PQ|^2 &= r^2+r^2-2\cos(\pi-\phi) && \text{(based on the triangle } O'PQ\text{).} \end{aligned}$$

With $\alpha = \cos\theta$, these two equations lead to

$$1-\beta = r^2(1+\cos\phi).$$

In turn, Equation 3.1 is equivalent to

$$\begin{cases} 2-\alpha-\beta = 2r^2 \\ \beta-\alpha = 2\cos\phi. \end{cases}$$

Since $-1 \leq \alpha, \beta \leq 1$, the required angle ϕ always exists. To ensure the real number r is at least one, we need $\alpha+\beta \leq 0$.

Now we have a way to construct a 2-distance set from a set of equiangular lines, and also a way to construct a set of equiangular lines from a 2-distance set if the associated inner products α and β satisfy $\alpha+\beta \leq 0$. Note that these two constructions are not inverses of each other. With the second construction and the information in Table 3.1, we get the following.

Proposition 3.3.1 *Let S be a 2-distance set on the unit sphere in R^d, with the two corresponding inner products α and β satisfying that $\alpha+\beta \leq 0$. Then $|S| \leq v(d+1)$, where $v(d+1)$ is the maximum cardinality of a set of equiangular lines in R^{d+1}. In particular,*

(i) $|S| \leq 6$, *in* R^3,

(ii) $|S| \leq 10$, *in* R^4, *and*

(iii) $|S| \leq 16$, *in* R^5. □

Let S be a spherical 2-distance set in R^d with associated inner products α and β satisfying $\alpha + \beta \geq 0$. Then Equation 3.1 has a unique non-negative solution $r < 1$ and we cannot construct a set of equiangular lines in the same way. In this situation, we can consider imbedding the Euclidean Space R^d into the Lorentz space $R^{d,1}$.

Lorentz space is defined to be an $(n+1)$-dimensional linear space with the *indefinite inner product* $\langle\ ,\ \rangle_L$. For any two vectors $\mathrm{x} = (x_0, x_1, \ldots, x_d)$ and $\mathrm{x}' = (x'_0, x'_1, \ldots, x'_d)$ in $R^{d,1}$,

$$\langle \mathrm{x}, \mathrm{x}' \rangle_L := -x_0 x'_0 + x_1 x'_1 + \cdots + x_d x'_d.$$

For the 2-distance set S in R^d, we can construct a set Y of vectors in $R^{d,1}$.

$$Y := \{r^{-1}(\sqrt{1-r^2}, \mathrm{x}),\ \ \mathrm{x} \in R^d\},$$

where r is the non-negative solution of Equation 3.1. Since the set S is on the unit sphere of R^d, it is easy to verify that for any vector y in Y,

$$\langle \mathrm{y}, \mathrm{y} \rangle_L = 1.$$

Given Equation 3.1, it follows that for any two distinct vectors y and y′ in Y,

$$\langle \mathrm{y}, \mathrm{y}' \rangle_L = \pm \cos \phi.$$

So, with respect to the indefinite inner product $\langle\ ,\ \rangle_L$, the unit vectors in set Y span a set of equiangular lines in $R^{d,1}$. The following lemma is due to Blokhuis and Seidel [8].

Lemma 3.3.2 *Any set of equiangular lines in Lorentz space $R^{d,1}$ has cardinality*

$$n \leq d(d+1)/2. \qquad \square$$

By this lemma and the construction we just described, we get

Proposition 3.3.3 *Let S be a 2-distance set on the unit sphere in R^d with the two associated inner products α and β satisfying $\alpha+\beta>0$. Then the cardinality of S is at most $d(d+1)/2$. In particular,*

(i) $|S| \leq 6$, *in* R^3;

(ii) $|S| \leq 10$, *in* R^4; *and*

(iii) $|S| \leq 15$, *in* R^5. $\square$

Combining Propositions 3.3.1 and 3.3.3, we conclude that the exact upper bound for the size of a spherical 2-distance set S is 6, 10 and 16 for S in R^3, R^4 and R^5, respectively.

3.4 An Integrality Condition

Lemma 3.4.1 *Let S be a 2-distance set of n points on the unit sphere in R^d with the associated inner products α and β. If $n \geq 2d+3$, then the quotient $(1-\alpha)/(\beta-\alpha)$ is an integer.*

Proof. Let G be the Gram matrix of the 2-distance set S. Then

$$G = I + \alpha A + \beta \bar{A}$$

for a certain symmetric (0,1)-matrix A and $\bar{A} := J - I - A$. Thus,

$$G - \alpha J = (1-\alpha)I + (\beta - \alpha)\bar{A} \tag{3.2}$$

Suppose $\text{rank}\,(G) = m$. Since $\text{rank}\,(J) = 1$, $\text{rank}\,(G - \alpha J) \leq m + 1$. So the null space of

$$(1-\alpha)I + (\beta-\alpha)\bar{A} = (1-\alpha)(I + \frac{\beta-\alpha}{1-\alpha}\bar{A})$$

has dimension at least $n - (m+1)$. Let $\tau = (1-\alpha)/(\alpha-\beta)$. Then the equation above implies that τ is an eigenvalue of the matrix $\bar{A}$ with multiplicity $n - (m+1)$. Since $\bar{A}$ is an integer matrix, τ is an algebraic integer and every algebraic conjugate of τ is also an eigenvalue of $\bar{A}$ with the same multiplicity $n - (m+1)$. It follows that if $n - (m+1) > n/2$, then τ is a rational integer.

Note that $m = \text{rank}\,(G)$ cannot be greater than the dimension of the space R^d, and that

$$n - (m+1) > \frac{n}{2} \quad \text{if and only if} \quad n \geq 2m + 3.$$

Our claim then follows. □

Chapter 4

Bounding the Diameter and Valency of a Distance-Regular Graph

As we discussed in Section 1.3, bounding the diameter is essential for classifying distance-regular graphs. The traditional idea is to find an upper bounds for the diameter of such graphs with a fixed valency. Biggs et al. [5] found upper bound for the diameter of distance-regular graphs with valency three and classified these graphs. Recently Bannai and Ito [2] showed that there are only finitely many distance-regular graphs with valency four because the diameters of such graphs are bounded. It is also conjectured that there exists a function $f(k)$ such that the diameter of any distance-regular graph with valency k is bounded above by $f(k)$. If this were proved, it would guarantee that there are only finitely many such graphs with any fixed valency.

In the course of developing the theory of representations of graphs, it was recently proved (Godsil [20]) that both the diameter and valency of a distance-regular graph are bounded by certain functions $f(m)$ of an eigenvalue multiplicity m. It follows that there are only finitely many non-trivial distance-regular graphs with an eigenvalue of a given multiplicity. This suggests a new approach in classifying distance-regular graphs.

The theory developed in this chapter applies to any eigenvalue θ not equal to $\pm k$. However, since a smaller eigenvalue multiplicity would yield better bounds on diameter and valency, we usually tend to focus on the eigenvalues with a low multiplicity.

A large part of the material in this chapter is based on Godsil's work [20].

4.1 Bounding the Diameter

Theorem 4.1.1 (Godsil [20]) *Let G be a connected distance-regular graph with valency $k \geq 3$. Let θ be an eigenvalue of G with multiplicity m and $\theta \neq \pm k$. Assume G is not a complete multipartite graph. Then the diameter of G is at most $3m-4$.*

We need to prove a few useful lemmas before proving this theorem. Let u_θ denote the representation associated with θ. We call a set S of vertices of G *independent* if its image $u_\theta(S)$ is a linearly independent set of vectors. Since u_θ maps $V(G)$ into R^m, any linearly independent set of vertices of G contains at most m elements. The distance between two vertices x and y will be denoted by $\partial(x, y)$, as usual. A path in G with end-vertices x and y is *geodetic* if its length is equal to $\partial(x, y)$.

Lemma 4.1.2 *If P_1 and P_2 are two geodetic paths in G with the same length, then their images in R^m under u_θ are congruent and there is an orthogonal transformation of R^m mapping $u_\theta(P_1)$ onto $u_\theta(P_2)$.*

Proof. There is an obvious bijection from $V(P_1)$ to $V(P_2)$ which preserves the distances between vertices. Since the distance between any two vertices x and y in G determines the distance between their images $u_\theta(x)$ and $u_\theta(y)$, our assertion follows immediately. □

Let P be a geodetic path in G with length equal to the diameter d. Let x be the initial vertex of P and let Q be the longest path with independent vertex set starting at x and contained in P. Denote the length of Q by q. (Hence $q+1 \leq m$.) It follows from the above lemma that any geodetic path of length smaller than or equal to q is independent and any with length greater than q is dependent.

Lemma 4.1.3 *If P' is a geodetic path containing Q, and with the same initial vertex as Q, then $u_\theta(P')$ is contained in the span of $u_\theta(Q)$.*

Proof. Let x and x_q be the two endpoints of Q. Suppose z is the unique vertex in $P' \setminus Q$ adjacent to x_q. Then $Q \cup \{z\}$ is dependent by our choice of Q and is spanned by the first $q+1$ vertices. By Lemma 4.1.2, the image of any geodetic path with $q+2$ vertices is dependent, being spanned by the first $q+1$ vertices. Now since each subset of $q+2$ consecutive vertices of P' forms a geodetic path, our claim follows by a simple induction argument. $\square$

Lemma 4.1.4 *Let G be a connected distance-regular graph of diameter d, valency k and with an eigenvalue θ of multiplicity m. Assume that any geodetic path in G which is independent with respect to θ has length at most q. Then, if $\theta \neq \pm k$ and $d > q$ we have:*

(i) *$b_i = 1$ for $i \geq q$,*

(ii) *$c_i = 1$ for $i \leq d-q$, and*

(iii) *$a_i + 1 \leq c_{q+i}$ for $i \leq d-q$.*

Proof. Assume, as in the previous lemma, that x and x_q are the two endpoints of a maximal independent path Q. Suppose z and z' are two vertices at distance $q+1$ from x and adjacent to x_q. By Lemma 4.1.3, we have that the images of the two geodetic paths $Q\cup\{z\}$ and $Q\cup\{z'\}$ are both contained in the space spanned by $u_\theta(Q)$. Further, for each vertex y in Q, we have $\partial(y,z) = \partial(y,z')$. This implies that $u_\theta(z) = u_\theta(z')$ because $u_\theta(Q)$ is a basis of this subspace. Since $\partial(z,z') \leq 2$, this contradicts the local injectivity of u_θ. Consequently $b_q = 1$ and so, by the monotonicity of the sequence $\{b_i\}$ (see Lemma 1.1.1), claim (i) is proven.

Recall that the inequality $c_i \leq b_{d-i}$ holds for any distance-regular graph (see Lemma 1.1.1). The claim (ii) then follows immediately.

To prove claim (iii), let P be a geodetic path in G with length equal to d and initial vertex x. Let s_i be the vertex in P at distance $q+i$ from x. Given that $b_{q+i} = 1$ for all $i \geq 0$, a simple induction argument on i shows that s_i is the unique vertex in G at distance i from x_q and at distance $q+i$ from x. It follows that each of the a_i vertices adjacent to s_i and at distance i from x_q are at distance $q+i-1$ from x. This implies that $a_i + 1 \leq c_{q+i}$. □

Proof of Theorem 4.1.1 We are going to show that if $d \geq 3q$, then $k = 2$. Assume that $d \geq 3q$. Then by the first two assertions of Lemma 4.1.4 we know that $c_q = b_q = 1$ and $c_{2q} = 1$. The latter in turn implies that $a_q = 0$ by the third assertion of Lemma 4.1.4. Therefore, we get $k = c_q + a_q + b_q = 1 + 0 + 1 = 2$. In other words, if $k \geq 3$, then $d \leq 3q - 1$. □

4.2 Bounding the Valency

In this section we derive some bounds on the valency of a distance-regular graph with an eigenvalue of multiplicity m.

Lemma 4.2.1 (Godsil [20]) *Let G be a connected distance-regular graph with valency k. Let θ be an eigenvalue of G with multiplicity m and suppose $\theta \neq \pm k$. Assume that G is not a complete multipartite graph. Then*

(i) $k \le \frac{1}{2}(m-1)(m+2)$,

(ii) *if* $a_1 = 0$, *then* $k \le m$, *and*

(iii) *if* $d \ge 2m-1$, *then* $a_1 = 0$ *(and thus* $k \le m$*).*

Proof. Let u_θ be a representation associated with θ. It is clear that for a vertex $u \in V(G)$ any pair of the vertices in $G(u)$ are either at distance 1 or at distance 2 in G. Since the distance between two image vectors under u_θ only depends on the distance of the two vertices in G, it follows that $u_\theta(G(u))$ is a 2-distance set in R^m.

According to Lemma 2.3.3, all image vectors have the same length and the vectors in $u_\theta(G(u))$ lie in a sphere centred at the origin in R^m. In addition, since the vertices in $G(u)$ all have the equal distance (namely 1) to the vertex u, the image vectors in $u_\theta(G(u))$ lie in a second sphere centred at $u_\theta(u)$. Therefore $u_\theta(G(u))$ actually lies in the intersection of two spheres in R^m, which is contained in an $(m-1)$-dimensional affine space. Thus $u_\theta(G(u))$ is a spherical 2-distance set in an $(m-1)$-dimensional affine space. Now apply Lemma 3.1.1 and get

$$|u_\theta(G(u))| \le \frac{1}{2}(m-1)[(m-1)+3] = \frac{1}{2}(m-1)(m+2).$$

By the local injectivity of u_θ, this inequality implies

$$|G(u)| \le \frac{1}{2}(m-1)(m+2).$$

To prove (ii) assume that $a_1 = 0$. This implies that the vertices in $G(u)$ are all distance two apart, so $u_\theta(G(u))$ is a 1-distance set (i.e., a regular simplex) in an $(m-1)$-dimensional affine space. Thus we must have

$$|G(u)| = |u_\theta(G(u))| \le (m-1)+1 = m.$$

Finally, assume that $d \ge 2m-1$. Since the inequality $q+1 \le m$ holds in general, we have $d \ge 2q+1$. It follows from Lemma 4.1.4 that $c_{q+1} = 1$ and this, in turn, implies that $a_1 = 0$. □

Remark. Since the exact upper bounds for spherical 2-distance sets in R^3, R^4 and R^5 have been obtained in Section 3.3, we can substitute these bounds in the above proof and improve Lemma 4.2.1(i) as follows.

(a) If $m = 4$, then $k \leq 6$.

(b) If $m = 5$, then $k \leq 10$.

(c) If $m = 6$, then $k \leq 16$.

A similar geometric argument can be applied to prove the next proposition.

Proposition 4.2.2 *Let G be a connected distance-regular graph with valency k. Let θ be an eigenvalue of G with multiplicity m and suppose $\theta \neq \pm k$. Assume that G is not a complete multipartite graph. Then*

(i) $a_1 \leq \frac{1}{2}(m-2)(m+1)$, $b_1 \leq \frac{1}{2}(m-2)(m+1)$, and

(ii) if $a_1 \neq 0$ and $c_2 = 1$, *then* $k \leq m(a_1+1)/a_1$.

Proof. Recall that $a_1 = |G(u) \cap G(v)|$ for any $u \in V(G)$ and any $v \in G(u)$. Write $a_1 = r$ and $G(u) \cap G(v) = \{w_1, \ldots, w_r\}$. Then we see that

$$\begin{aligned} \partial(w_1,u) = \partial(w_2,u) = \cdots = \partial(w_r,u) = 1, \\ \partial(w_1,v) = \partial(w_2,v) = \cdots = \partial(w_r,v) = 1. \end{aligned}$$

These two equations lead to the corresponding equations for the inner products.

$$\begin{aligned} \langle u_\theta(w_1), u_\theta(u)\rangle = \langle u_\theta(w_2), u_\theta(u)\rangle = \cdots = \langle u_\theta(w_r), u_\theta(u)\rangle, \\ \langle u_\theta(w_1), u_\theta(v)\rangle = \langle u_\theta(w_2), u_\theta(v)\rangle = \cdots = \langle u_\theta(w_r), u_\theta(v)\rangle. \end{aligned}$$

These show that the subspace spanned by the vectors

$$\{u_\theta(w_i) - u_\theta(w_j) | 1 \leq i,j \leq r, \quad i \neq j\}$$

is orthogonal to the subspace spanned by $u_\theta(u)$ and $u_\theta(v)$. The former can have dimension at most $m-2$ because $u_\theta(u)$ and $u_\theta(v)$ are linearly independent. Therefore

$\{u_\theta(w_i) \mid 1 \le i \le r\}$ can be viewed as r points in an $(m-2)$-dimensional affine space. By Lemma 3.1.1, this gives the bound

$$a_1 \le \frac{1}{2}(m-2)(m+1).$$

The bound for b_1 can be proved similarly.

Now we are going to show that if $c_2 = 1$ and $a_1 \ne 0$, then the subgraph induced by $G(u)$ is a union of cliques; after that we shall prove claim (ii). We first show that any vertex x in $G(u) \setminus \{v, w_1, \ldots, w_r\}$ cannot be adjacent to any of the vertices in $\{v, w_1, \ldots, w_r\}$. Assume x is adjacent to one of w_i, say w_1. Since x does not belong to $\{w_1, \ldots, w_r\}$, it is not adjacent to v and thus $\partial(x, v) = 2$. But $G(v) \cap G(x)$ contains two distinct vertices u and w_1. Hence $c_2 \ge 2$, contradicting the assumption that $c_2 = 1$. It then follows that the $a_1 + 1$ $(= r+1)$ vertices $v, w_1, \ldots, w_r$ induce a complete subgraph. Therefore $G(u)$ is partitioned into $k/(a_1+1)$ cliques.

The image of each clique is a regular simplex with $a_1 + 1$ vertices and spans a subspace of dimension $a_1 + 1$ or dimension a_1. Let

$$W := \text{ span}\,\{u_\theta(w_1), \ldots, u_\theta(w_r), u_\theta(w_{r+1})\}$$

be one of these subspaces. Then W itself contains another subspace

$$W_0 := \text{ span}\,\{u_\theta(w_i) - u_\theta(w_j) \quad | \quad 1 \le i, j \le r+1, \quad i \ne j\}.$$

Note that any two vertices in the same clique are at distance 1, whereas any two vertices in different cliques are at distance 2 in G. It follows, as in part (i), that W_0 is orthogonal to each of the subspaces spanned by the cliques other than W. By symmetry we can actually get $k/(a_1+1)$ such subspaces, each of them with dimension exactly a_1 $(= r)$. These subspaces are mutually orthogonal and they are in R^m. So we must have

$$a_1 \frac{k}{a_1+1} \le m,$$

or

$$k \leq \frac{m(a_1+1)}{a_1}.$$

We are done. ◫

Our method can provide considerable information about a distance-regular graph if the diameter is relatively large, compared with the eigenvalue multiplicity m.

A distance-regular graph of diameter d is said to be *antipodal* if its vertex set can be partitioned into classes with the property that any two vertices in the same class are at distance d and any two vertices in different classes are at distance less than d.

Proposition 4.2.3 (Godsil [20]) *Let G be a connected distance-regular graph of diameter d and valency k at least three. Assume that G is not a complete multipartite graph and G has an eigenvalue $\theta \neq \pm k$ of multiplicity m. Let q be the maximal length of a geodetic path in G, independent with respect to θ. If $d = 3q-1$, then G is antipodal, and if $d > 2q$, then $k \leq m + 2q - d + 1$.*

Proof. Assume that $d = 3q-1$. We first verify the following claims:

(a) $b_i = 1$ for $q \leq i \leq 3q-1$;

(b) $c_i = 1$ for $1 \leq i \leq 2q-1$;

(c) $a_i = 0$ for $1 \leq i \leq q-1$;

(d) $b_i = k-1$ for $1 \leq i \leq q-1$;

(e) $b_i = c_i = 1$ for $q \leq i \leq 2q-1$;

(f) $a_i = k-2$ for $q \leq i \leq 2q-1$; and

(g) $c_i = k-1$ for $2q \leq i \leq 3q-1$.

All these claims are consequences of Lemma 4.1.4 and the identity $c_i + a_i + b_i = k$, $1 \leq i \leq k$. Claims (a) and (b) are obvious, and (c) follows from (b). Together (b) and (c) imply (d). Claim (e) yields (f) and thus (g).

With the values of b_i and c_i in hand, one can easily calculate that $k_{3q-1} = 1$ using the relation $k_i b_i = k_{i+1} c_{i+1}$ (see Lemma 1.1.1). So G is antipodal as claimed, and each (antipodal) class consists of exactly two vertices.

Now assume that $d = 3q - r$ for some r, $1 \leq r \leq q - 1$. Since $d \geq 2q + 1$, we have $a_1 = 0$. As easy consequences of Lemma 4.1.4 we deduce that $c_{2q-r} = 1$, $c_{q-r} = 1$ and $a_{q-r} = 0$. Hence $k = c_{q-r} + a_{q-r} + b_{q-r} = 1 + b_{q-r}$. We next show that $b_{q-r} \leq m - (q - r)$.

Let P be a geodetic path of length $q - r$ with two end vertices x and x_{q-r}. Let y and z be two vertices at distance $q - r + 1$ from x and adjacent to x_{q-r}. Since $\partial(u, y)$ and $\partial(u, z)$ are equal for each vertex u in P, it follows that $u_\theta(y) - u_\theta(z)$ is orthogonal to each of the $q - r + 1$ vectors $u_\theta(u)$. Since these $q - r + 1$ vectors are linearly independent, their orthogonal complement in R^m is an $(m - q + r - 1)$-dimensional space. This means that the images of the b_{q-r} vertices which are adjacent to x_{q-r} and at distance $q - r + 1$ from x all lie in an $(m - q + r - 1)$-dimensional affine space.

Finally, $a_1 = 0$ implies that any two of these b_{q-r} vertices are at distance two in G. Therefore their images form a regular simplex in the affine space. Accordingly we must have $b_{q-r} \leq (m - q + r - 1) + 1 = m - q + r$, and $k \leq m - q + r + 1 = m + 2q - d + 1$.
□

The second half of the above proof actually validates the following useful result.

Corollary 4.2.4 *Let G be a distance-regular graph of diameter d and valency k at least three. Assume that G is not a complete multipartite graph and G has an eigenvalue $\theta \neq \pm k$ of multiplicity m. Let q be the maximal length of a geodetic path in G, independent with respect to θ. Then if $a_1 = 0$, $b_i \leq m - i$ for $1 \leq i \leq q$.*
□

Chapter 5

Classifying Distance-Regular Graphs with an Eigenvalue of Multiplicity Four

We are now going to classify the distance-regular graphs with an eigenvalue of multiplicity four. The main result is stated as follows.

Theorem 5.0.1 *The connected distance-regular graphs with an eigenvalue of multiplicity 4 are:*

(1) K_5, $L(K_5)$, $K_{3,3}$, $L(K_{3,3})$, Petersen graph, the line graph of Petersen graph, Pappus' graph, Desargues' graph, 4-cube, dodecahedron, $\overline{4K_2}$, and $K_{5,5}$ minus a 1-factor; and

(2) Complete 5-partite regular graphs $\overline{5K_r}$ with $r \geq 2$. (An infinite family).

The intersection arrays and spectra of these graphs are given in the Appendix.

5.1 The Scheme for Classifying Distance-Regular Graphs with an Eigenvalue Multiplicity Four

The intersection arrays and the spectra for complete multipartite regular graphs have been presented in Section 1.2, from which one can easily see that such a graph can have an eigenvalue with multiplicity four if and only if it is $\overline{5K_n}$ for some n. The rest of this paper will classify the remaining distance-regular graphs.

It is clear from Theorem 4.1.1 and the remark following Lemma 4.2.1 that the diameter of the graphs under investigation is at most eight and the valency is at most six. To complete the classification, we need only to consider the following five cases:

(I) $k = 3$;

(II) $k = 4$ and $a_1 = 0$;

(III) $k = 4$ and $a_1 \neq 0$;

(IV) $k = 5$ and $a_1 \neq 0$; and

(V) $k = 6$ and $a_1 \neq 0$.

As we mentioned in Section 1.3 the distance-regular graphs of valency three have been completely classified. There are exactly thirteen such graphs (Biggs et al. [5]). We list the intersection arrays and spectra of these thirteen graphs in the Appendix. By checking this list, we see that five of them have an eigenvalue of multiplicity four. They are $K_{3,3}$, Petersen graph, Pappus' graph, Desargues graph and the dodecahedron.

To deal with the cases (II)–(V), we apply the representation method as well as some elementary arguments. For one subcase in case (II) we use a computer. We postpone case (II) to the end of this chapter.

5.2 Distance-Regular Graphs with Valency Four and $a_1 \neq 0$

Theorem 5.2.1 *Let G be a connected distance-regular graph with valency $k = 4$. If $a_1 \neq 0$ then either*

(i) $a_1 = 3$ and G is K_5,

(ii) $a_1 = 2$ and G is the octahedron, or

(iii) $a_1 = 1$ and G is the line graph of a (k, g)-graph with $k = 3$ and $g = 4, 5, 6, 8$ or 12.

Proof. The first assertion is obvious. For (ii), suppose $a_1 = 2$. Then for any vertex $u \in V(G)$, the induced subgraph $G(u)$ is a 4-cycle. Thus all induced subgraphs $\{u\} \cup G(u)$ are isomorphic. Now let u be any vertex in $V(G)$ and $G(u) = \{v_1, v_2, v_3, v_4\}$. Since $b_1 = 1$, we may define w to be the unique vertex in $G(v_1)$ such that $\partial(u, w) = 2$. Then $G(v_1) = \{u, v_2, w, v_4\}$. This is a 4-cycle and w is adjacent to v_2 and v_4. Applying the same argument to v_2 will show that the same vertex w is adjacent to v_1 and v_3. Therefore w is actually adjacent to all four vertices in $G(u)$. This shows that G is isomorphic to the octahedron.

To prove (iii), let $a_1 = 1$. Then for any vertex $u \in V(G)$ the induced subgraph $\{u\} \cup G(u)$ is isomorphic to the graph in Figure 5.1. Note that a graph G is a line

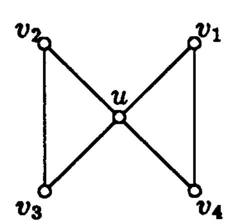

Figure 5.1: Subgraph induced by $\{u\} \cup G(u)$

graph if and only if the edges of G can be partitioned into cliques in such a way that no vertex lies in more than two cliques (see, e.g., Theorem 8.4 of Harary [22]). So $G = L(H)$ is the line graph of another graph H. By Lemma 1.4.1, H is a $(3, g)$-graph for some g. It is known (see, for example, Chapter 23 of Biggs [4]) that $(3, g)$-graph can exist if and only if g = 3, 4, 5, 6, 8 or 12. We note that when g = 4, 5, 6, 8, or 12 the line graph of the $(3, g)$-graph has $a_1 = 1$. This finishes the proof. □

The $(3,4)$-graph is the bipartite graph $K_{3,3}$ and the $(3,5)$-graph is commonly known as Petersen graph. By checking the eigenvalue multiplicities for the graphs listed in Theorem 5.2.1 we get the following

Corollary 5.2.2 *Let G be a connected distance-regular graph with valency $k = 4$ and $a_1 \neq 0$. If G has an eigenvalue of multiplicity 4, then G is either K_5 or the line graph of $K_{3,3}$ or the line graph of Petersen graph.* □

5.3 Distance-Regular Graphs with Valency Five and $a_1 \neq 0$

Theorem 5.3.1 *Let G be a connected distance-regular graph with valency $k = 5$. If $a_1 \neq 0$, then G is either the complete graph K_6 or the icosahedron.*

Proof. Since ka_1 must be even, a_1 can only be 2 or 4. If $a_1 = 4$ we get the complete graph K_6.

Assume $a_1 = 2$. Then for any $u \in V(G)$, the neighbourhood $G(u)$ is a 5-cycle. Let u be a fixed vertex in G and $G(u) = \{v_1, v_2, v_3, v_4, v_5\}$. As $b_2 = 2$, we can write $G(v_1) = \{u, v_2, v_5, x_1, x_2\}$ and these vertices form a 5-cycle. This implies that $a_2 \geq 1$ and $c_2 \geq 2$. By the relation $k_2c_2 = k_1b_1 = 5 \times 2 = 10$ we see that $c_2 = 2$ and $k_2 = 5$. Since $a_2 \leq k - c_2 = 3$ and k_2a_2 is even, it follows that $a_2 = 2$. In summary we now have $a_1 = 2$, $b_1 = 2$, $c_2 = 2$, $a_2 = 2$ and $b_2 = 1$. It is then easy to verify that the subgraph induced by $\{u\} \cup G(u) \cup G_2(u)$ is isomorphic to the graph in Figure 5.2. In particular the induced subgraph $G_2(u)$ is a 5-cycle. Since $b_2 = 1$, for any vertex

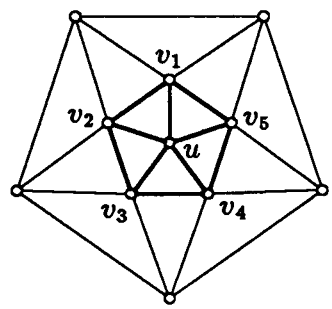

Figure 5.2: Subgraph induced by $\{u\} \cup G(u) \cup G_2(u)$

z in $G_2(u)$, there is a unique vertex x in $G_3(u)$ adjacent to z. Since $G(z)$ is also a 5-cycle, x should be adjacent to the two neighbours of z in $G_2(u)$. Applying this argument to every vertex in $G_2(u)$ we see that the vertex x is the unique vertex in $G_3(u)$, which is adjacent to all the five vertices in $G_2(u)$. So the graph is isomorphic to the icosahedron. $\square$

It is easy to verify that neither K_6 nor the icosahedron has an eigenvalue with multiplicity four.

Remark. The following results are stated (without proof) by Doyen, Hubaut and Reynaert [15]: Suppose G is a connected graph and the subgraph induced by the neighbourhood of any vertex is isomorphic to a fixed graph H.

(a) If H is a 4-cycle, then G is the octahedron;

(b) If H is a 5-cycle, then G is the icosahedron;

(c) If H is a complete multipartite graph $\overline{sK_r}$, then G is $\overline{(s+1)K_r}$.

These results would help to simplify the proofs of Theorem 5.2.1 (when $a_1 = 2$), Theorem 5.3.1 (when $a_1 = 2$) and Lemma 5.4.5. However, since a proof of these results could not be traced after a search in mathematical literature, we choose to produce our proofs independently.

5.4 Distance-Regular Graphs with Valency Six and $a_1 \neq 0$

Theorem 5.4.1 *Suppose G is a connected distance-regular graph of valency $k = 6$, and G is not a complete multipartite graph. If G has an eigenvalue θ with multiplicity $m = 4$, then G is $L(K_5)$.*

We will split Theorem 5.4.1 into four lemmas, in accordance with the four possible values for a_1 ($1 \leq a_1 \leq 4$).

Recall that in Theorem 2.6.1, we have that if a distance-regular graph has an eigenvalue of multiplicity less than the valency, then this eigenvalue is either the second largest one or the least one. Hence the eigenvalue θ in Theorem 5.4.1 is either the second largest or the smallest. In particular, if θ is negative, it must be the smallest eigenvalue of G.

We begin by considering the graphs with $a_1 = 1$.

Lemma 5.4.2 *Suppose G is a connected distance-regular graph of valency $k = 6$, and G is not a complete multipartite graph. If G has an eigenvalue θ with multiplicity $m = 4$, then $a_1 \neq 1$.*

Proof. Suppose that $a_1 = 1$. Then for each vertex $u \in V(G)$, the induced subgraph $\{u\} \cup G(u)$ is isomorphic to the graph in Figure 5.3. Let u_θ be the representation associated with θ. Note that u_θ maps each triangle (K_3) in G to an equilateral triangle in R^4 which is inscribed in a sphere. All these triangles are congruent and span a

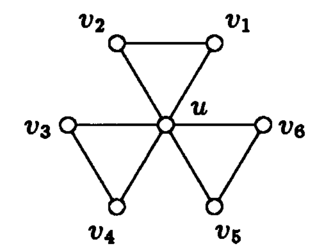

Figure 5.3: Subgraph induced by $\{u\} \cup G(u)$

subspace of the same dimension; either three (non-degenerate) or two (degenerate). We will discuss the two cases separately.

Case 1. Suppose each of these triangles spans a 3-dimensional subspace. In particular the subspace spanned by $\{u_\theta(u), u_\theta(v_1), u_\theta(v_2)\}$ will have dimension three. Write $p = u_\theta(v_3) - u_\theta(v_4)$ and $q = u_\theta(v_5) - u_\theta(v_6)$. Knowing that the inner product of two image vectors is determined by the distance between the two vertices in G (Lemma 2.3.3), one can easily verify the following equations.

$$\langle p, u_\theta(u)\rangle = \langle p, u_\theta(v_1)\rangle = \langle p, u_\theta(v_2)\rangle = 0,$$

$$\langle q, u_\theta(u)\rangle = \langle q, u_\theta(v_1)\rangle = \langle q, u_\theta(v_2)\rangle = 0,$$

$$\langle p, q\rangle = 0.$$

These show that the subspace spanned by $\{p,q\}$ has dimension two and is orthogonal to the subspace spanned by $\{u_\theta(u), u_\theta(v_1), u_\theta(v_2)\}$. It then follows that $u_\theta(\{u\} \cup G(u))$ will span a subspace of dimension at least five, contradicting that $u_\theta(G) \subseteq R^4$.

Case 2. Suppose each triangle formed by the images of K_3's spans only a subspace of dimension two. Since the vertices u, v_1, v_2 form a triangle (refer to Fig. 3), the Gram matrix of the vectors $u_\theta(u)$, $u_\theta(v_1)$ and $u_\theta(v_2)$ is

$$F = \begin{pmatrix} 1 & w_1 & w_1 \\ w_1 & 1 & w_1 \\ w_1 & w_1 & 1 \end{pmatrix} = w_1 J + (1 - w_1) I,$$

where J is the all 1 matrix. As $w_1 < 1$, then F can be singular if and only if $3w_1 + (1 - w_1) = 0$, i.e., $w_1 = -\frac{1}{2}$.

Recall from Equation 2.5 that the sequence of cosines $\{w_i : 0 \le i \le d\}$ satisfies the recurrence

$$c_i w_{i-1} + a_i w_i + b_i w_{i+1} = \theta w_i, \qquad 0 \le i \le d,$$

with the understanding that $w_{-1} = w_{d+1} = 0$. In particular,

$$\begin{aligned} w_1 &= \frac{\theta}{k}, \\ w_2 &= \frac{\theta^2 - \theta a_1 - k}{k b_1}. \end{aligned}$$

We therefore have $\theta = -3$ and $w_2 = \frac{1}{4}$. Knowing that $w_1 = -\frac{1}{2}$ and $w_2 = \frac{1}{4}$, one can derive that

$$d(u_\theta(v), u_\theta(w)) = \begin{cases} \sqrt{3}, & \text{if } \partial(v,w) = 1 \text{ in } G; \\ \sqrt{3/2}, & \text{if } \partial(v,w) = 2 \text{ in } G. \end{cases} \tag{5.1}$$

It is easily seen (by Corollary 2.3.3) that the image of a neighbourhood $G(u)$ forms a 2-distance set in an affine space of dimension $m - 1 = 4 - 1 = 3$. Furthermore, $u_\theta(G(u))$ contains six points, and for each point x in $u_\theta(G(u))$ there is exactly one point in $u_\theta(G(u))$ at distance α from x. (The remaining four points are thus at distance β from x.) This uniquely determines the configuration: it is the regular octahedron.

Summarizing the above calculations, we have that the distance from $u_\theta(u)$ to any of the six points $u_\theta(v_i)$ is $\sqrt{3}$. The six points $u_\theta(v_1), u_\theta(v_2), \ldots, u_\theta(v_6)$ form a regular octahedron with axis length $\sqrt{3}$ and edge length $\sqrt{3/2}$. These properties uniquely determine the configuration $\{u_\theta(u)\} \cup u_\theta(G(u))$ in R^4.

The preceding discussion works for any vertex of G. In particular, the six vertices in $G(u)$ each form a regular octahedron in R^4 in the same manner. This will enable us to determine the image configuration of the second neighbourhood $G_2(u)$ of u. Each v_i, $1 \leq i \leq 6$, has four neighbours in $G_2(u)$. We start with v_1 and its neighbours.

Let z_1, z_2, z_3, and z_4 be the four neighbours of v_1 in $G_2(u)$. Denote by T_1 the regular octahedron formed by the images of $u, v_2, z_1, \ldots, z_4$. We are going to determine the whereabouts of the four points $u_\theta(z_1), \ldots, u_\theta(z_4)$ in R^4. Without loss of generality, we assume the following:

$$\begin{array}{lclcl}
D & := & u_\theta(u) & = & (0,0,0,1); \\
A_1 & := & u_\theta(v_1) & = & (\frac{\sqrt{3}}{2},0,0,-\frac{1}{2}); \\
A_2 & := & u_\theta(v_2) & = & (-\frac{\sqrt{3}}{2},0,0,-\frac{1}{2}); \\
A_3 & := & u_\theta(v_3) & = & (0,\frac{\sqrt{3}}{2},0,-\frac{1}{2}); \\
A_4 & := & u_\theta(v_4) & = & (0,-\frac{\sqrt{3}}{2},0,-\frac{1}{2}); \\
A_5 & := & u_\theta(v_5) & = & (0,0,\frac{\sqrt{3}}{2},-\frac{1}{2}); \\
A_6 & := & u_\theta(v_6) & = & (0,0,-\frac{\sqrt{3}}{2},-\frac{1}{2}).
\end{array}$$

We write $O = (0,0,0,0)$ as the origin of R^4. Figure 5.4 gives a rough demonstration of the configuration. (It is not possible to visualize that $A_1, A_2, \ldots, A_6$ form a regular octahedron in R^4.) Let $X = (x_1, x_2, x_3, x_4)$ represent any of the four points

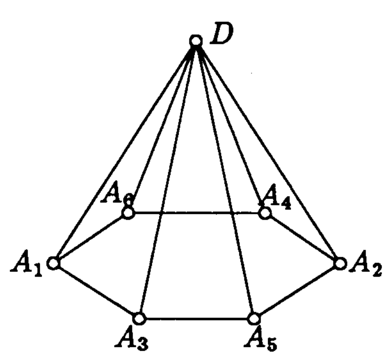

Figure 5.4: Configuration of $u_\theta(\{u\} \cup G(u))$ in R^4

$u_\theta(z_1), \ldots, u_\theta(z_4)$. Since the inner product $\langle X, D\rangle = w_2 = \frac{1}{4}$, we deduce that $x_4 = \frac{1}{4}$. In the same manner $\langle X, A_1\rangle = w_1 = -\frac{1}{2}$ implies that $x_1 = -\frac{\sqrt{3}}{4}$. It then follows from $|X| = 1$ that $x_2^2 + x_3^2 = (\frac{\sqrt{3}}{2})^2$. Thus X satisfies three equations which represent a circle in a plane. In other words, the four vertices $u_\theta(z_1), \ldots, u_\theta(z_4)$ can be nowhere but on that circle.

Applying this argument to the remaining five vertices $v_2, \ldots, v_6$ in $G(u)$, we can easily determine the equations for the corresponding five circles, in which the four "outer" neighbours of each vertex will reside. The equations for all six circles are presented as follows.

$$\text{For} \quad v_1, \quad \begin{cases} x_1 = -\frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_2^2 + x_3^2 = (\frac{\sqrt{3}}{2})^2 \end{cases} \qquad \text{For} \quad v_2, \quad \begin{cases} x_1 = \frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_2^2 + x_3^2 = (\frac{\sqrt{3}}{2})^2 \end{cases}$$

$$\text{For} \quad v_3, \quad \begin{cases} x_2 = -\frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_1^2 + x_3^2 = (\frac{\sqrt{3}}{2})^2 \end{cases} \qquad \text{For} \quad v_4, \quad \begin{cases} x_2 = \frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_1^2 + x_3^2 = (\frac{\sqrt{3}}{2})^2 \end{cases}$$

$$\text{For} \quad v_5, \quad \begin{cases} x_3 = -\frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_1^2 + x_2^2 = (\frac{\sqrt{3}}{2})^2 \end{cases} \qquad \text{For} \quad v_6, \quad \begin{cases} x_3 = \frac{\sqrt{3}}{4} \\ x_4 = \frac{1}{4} \\ x_1^2 + x_2^2 = (\frac{\sqrt{3}}{2})^2 \end{cases}$$

Looking at these equations, one can easily see that

(i) the six planes bearing the six circles lie in the 3-dimensional affine space $x_4 = \frac{1}{4}$, (so they can be treated as lying in R^3),

(ii) these six planes enclose a cube with centre at the origin of R^3, edges parallel to the axes and the length of each edge $\sqrt{3}/2$, and

(iii) the diameter of each circle is twice as long as the edge length of the cube.

Figure 5.5(a) depicts the cube and four of the six planes. From (i), (ii) and (iii), it is clear that each circle has exactly two intersection points with any of the four lateral circles, and has no intersection with the circle on the opposite face of the cube. Hence each circle has eight intersection points. Figure 5.5(c) shows the position of these eight intersection points on the front circle.

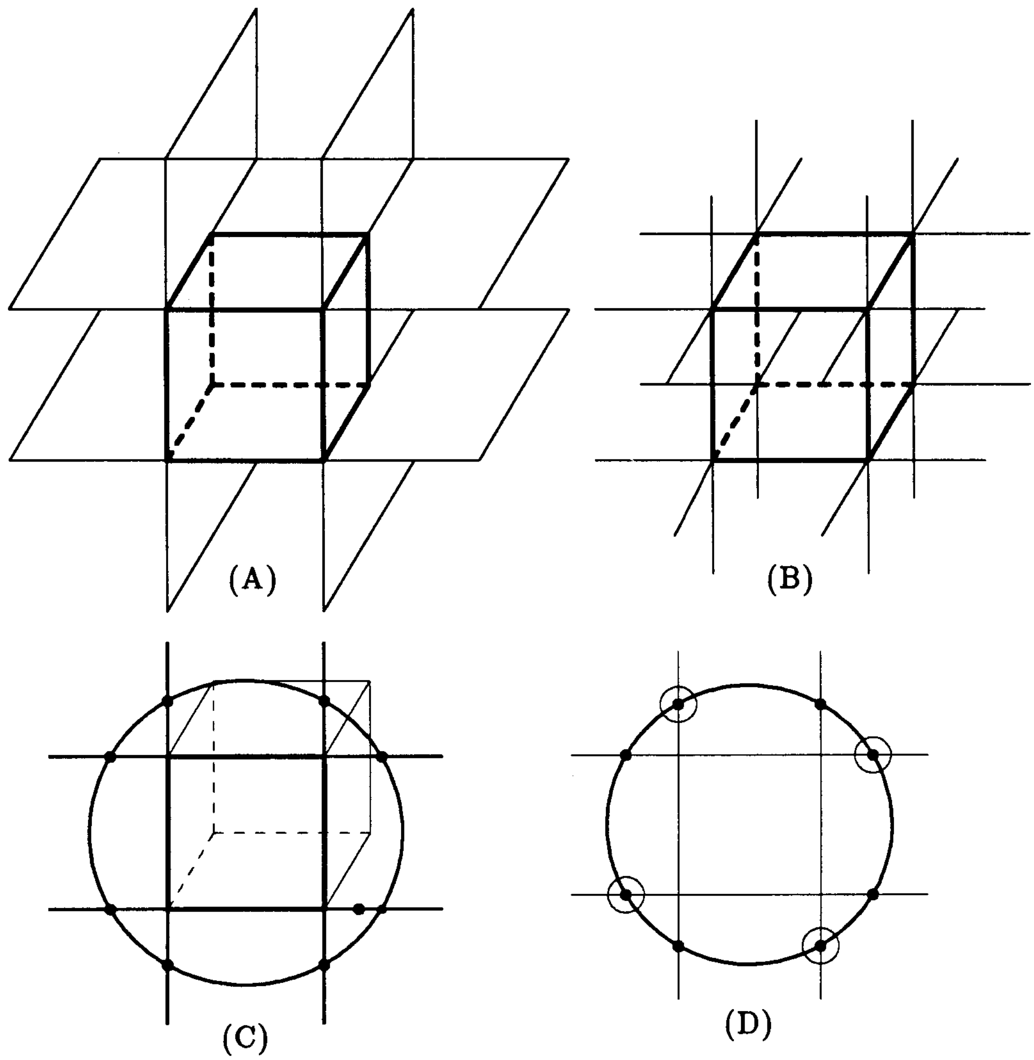

Figure 5.5: Six circles on the six faces of a cube in R^3

So the image vectors of $G_2(u)$ reside in the six circles. We now show that $1 \leq c_2 \leq 2$, and then rule out both possibilities. Since each circle contains four image points (vectors) and there are six circles, the total number of distinct image points is at most 24. However, this number can be smaller than 24 because some circles may share a common image point. From (iii) above we see that no three circles can intersect at a common point. It follows that the possible number of image points would reach its minimum if every image point resided at an intersection of two circles. Then the total number of image points reduces to $24/2 = 12$. We therefore have $12 \leq |u_\theta(G_2(u))| \leq 24$. Noting that $|G_2(u)| = k_1 b_1/c_2 = 24/c_2$, we get $1 \leq c_2 \leq 2$.

Suppose $c_2 = 2$. Then $|G_2(u)| = 12$ and every image point is at an intersection of two circles. Since the four image points in each circle are actually four vertices in a regular octahedron, they should partition the circle into four equal parts. It follows that for each circle there can be only two choices for the position of the four image points on it. The four points of one choice interlace the remaining four (see Figure 5.5(D)). In particular, any two adjacent points among the eight cannot belong to the same choice. Hence for any adjacent pair of points in a circle only one of them could be an image point. Now we observe that around each corner of the cube there are three intersection points (refer to Figure 5.5(B)). Any two of these three are in a circle and they are adjacent points in that circle. It follows that among the three intersection points around each corner of the cube only one can be an image point. A cube has eight corners, so there can be at most eight image points. This contradicts the fact that $|u_\theta(G_2(u))| = 12$.

We are now only left with the subcase $a_1 = 1$ and $c_2 = 1$ in Case 2. Notice that in this case the vertices of $G(v_1)$ have distance either 2 or 3 in G from the vertices of $G(v_2)$. This implies that the four points on one circle can take only two values as distances to any of the four points on the opposite circle. (These two values are determined by w_2 and w_3.) It then follows that the relative position of the eight points on a pair of opposite circles is uniquely determined. The four points on one circle "interlace" the four points on the other. By some elementary calculation (based on the configurations in Figure 5.5) one can easily verify that the shorter one of the two candidate distances should be

$$\sqrt{\frac{33-18\sqrt{2}}{8}} \approx 0.9710919 < 1. \tag{5.2}$$

But on the other hand, we know from Equation 5.1 that

$$d(u_\theta(x), u_\theta(y)) = \sqrt{3/2} > 1$$

for any x, y with $\partial(x, y) = 2$ in G. So the distance in (5.2) could only be realized by a pair of vertices x, y with $\partial(x, y) = 3$ in G.

Since $\theta = -3$, by Lemma 2.6.1, θ must be the smallest eigenvalue of G. We have learned from Lemma 2.4.2 that if θ is the i-th largest eigenvalue of G, then the corresponding sequence of cosines has exactly $(i-1)$ sign changes. Hence the sequence of cosines corresponding to $\theta = -3$ is alternating and w_3 is non-positive. It follows that $d(u_\theta(x), u_\theta(y)) \geq \sqrt{2}$ for any x, $y \in V(G)$ with $\partial(x, y) = 3$ in G. This again contradicts (5.2) above. We are finished. □

Next, we consider distance-regular graphs with valency $k = 6$ and $a_1 = 2$.

Lemma 5.4.3 *Let G be a connected distance-regular graph of valency* 6 *and* $a_1 = 2$. *Then G is the line graph of a* $(4, g)$*-graph with* $g = 4, 6, 8, 12$.

Proof. There are only two non-isomorphic 2-regular graphs on six vertices. Hence, for any vertex u in G, the induced subgraph $\{u\} \cup G(u)$ is isomorphic to one of the two graphs in Figure 5.6. We will prove the lemma in two cases.

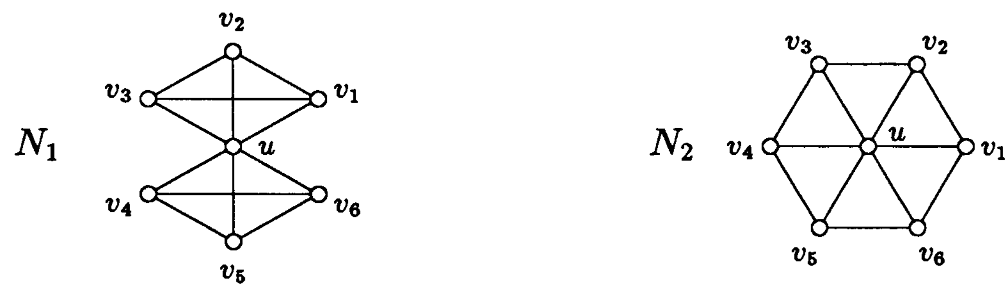

Figure 5.6: Subgraph induced by $\{u\} \cup G(u)$

Case 1. Suppose there exists a vertex u in $V(G)$ such that the induced subgraph $\{u\} \cup G(u)$ is isomorphic to the graph N_1 in Figure 5.6. Then it is easy to see that for any vertex v in $G(u)$ the induced subgraph $\{v\} \cup G(v)$ is also isomorphic to N_1. It follows from the connectedness of G that all the induced subgraphs $\{u\} \cup G(u)$, $u \in V(G)$, are isomorphic to N_1 in Figure 5.6. Therefore the edges of G can be partitioned into cliques such that each vertex lies in exactly two cliques. By Theorem 8.4 in Harary [22], for example, G is the line graph of another graph H, where H must be a $(4, g)$-graph. It is known (see, for example, Chapter 23 in Biggs [4]) that there exist only five $(4, g)$-graphs, namely, the $(4,3)$-graph (K_5), the $(4,4)$-graph ($K_{4,4}$), the $(4,6)$-graph, the $(4,8)$-graph and the $(4,12)$-graph. The line graphs of the last four have $a_1 = 2$, while $L(K_5)$ has $a_1 = 3$.

Case 2. Now suppose that all the induced subgraphs $\{u\} \cup G(u)$, $u \in V(G)$, are isomorphic to the graph N_2 in Figure 5.6. For a fixed $u \in V(G)$, let $G(u) = \{v_1, v_2, \ldots, v_6\}$. The subgraph $G(u)$ is a 6-cycle. Since $b_1 = 3$, write $G_2(u) \cap G(v_1) = \{z_1, z_2, z_3\}$. Then $G(v_1) = \{u, v_6, z_1, z_2, z_3, v_2\}$ must also be a 6-cycle (see Figure 5.7). This clearly implies that $a_2 \geq 2$ and $c_2 \geq 2$. By the relation $k_2 c_2 = k b_1 = 6 \times 3 = 18$, c_2 may be 2, 3 or 6. We deal with these cases in turn.

Suppose $c_2 = 2$. Then $k_2 = |G_2(u)| = 9$. Hence a_2 may be 2 or 4. First assume $a_2 = 2$. Notice that in the subgraph induced by $\{u\} \cup G(u) \cup \{v_1\} \cup G(v_1)$ (see

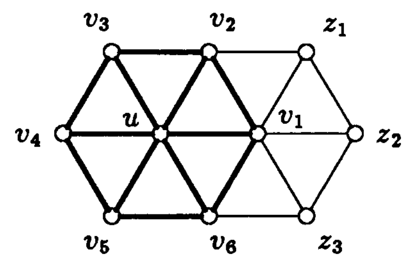

Figure 5.7: Subgraph induced by $\{u\} \cup G(u) \cup (G_2(u) \cap G(v_1))$

Figure 5.7), the vertex z_2 is adjacent to a unique vertex (namely v_1) in $G(u)$ while z_1 and z_3 each have two neighbours in $G(u)$. We call z_2 the *centre-neighbour* of v_1 and call z_1 and z_3 *side-neighbours* of v_1. Thus each vertex v_i, $1 \leq i \leq 6$, in $G(u)$ has a unique centre-neighbour and two side-neighbours.

Now since $c_2 = 2$, z_2 must be adjacent to a second vertex v_j in $G(u)$ and z_2 is the centre-neighbour of that v_j, $(2 \leq j \leq 6)$. It then follows from $a_1 = 2$ that z_1 and z_3 must be the two side-neighbours of that same v_j. This is an impossible situation. So a_2 cannot be 2.

We are left with only $a_2 = 4$ and this gives an array

$$\left\{\begin{matrix} * & 1 & 2 \\ 0 & 2 & 4 \\ 6 & 3 & * \end{matrix}\right\}.$$

This array does not have an eigenvalue with multiplicity four.

We next suppose $c_2 = 3$. Then a_2 may be 2 or 3. If $a_2 = 3$ we get the array

$$\left\{\begin{matrix} * & 1 & 3 \\ 0 & 2 & 3 \\ 6 & 3 & * \end{matrix}\right\}$$

which is not feasible.

Now let $a_2 = 2$. Then $b_1 = 1$ and the graph G has diameter at least three. We know that $k_1 = k_2 = 6$ and $k_3c_3 = k_2b_2 = 6$. Since $c_3 \geq c_2 = 3$, c_3 can only be 3 or 6.

If $c_3 = 3$ then $k_3 = 2$ and $a_3 \geq 2$, which is impossible. So we can have only one array

$$\left\{\begin{matrix} * & 1 & 3 & 6 \\ 0 & 2 & 2 & 0 \\ 6 & 3 & 1 & * \end{matrix}\right\}.$$

This array fails the multiplicity check.

Finally, if $c_2 = 6$, it could give only one candidate array

$$\left\{\begin{matrix} * & 1 & 6 \\ 0 & 2 & 0 \\ 6 & 3 & * \end{matrix}\right\}.$$

If this graph existed, it would be an antipodal graph on 10 vertices and its antipodal quotient would have either five vertices or two vertices. But on the other hand, since a graph and its antipodal cover should have the same valency, the antipodal quotient of G should be a 6-regular graph. This contradiction shows that the graph corresponding to this array does not exist. $\square$

It is easy to check that none of the four line graphs given in Lemma 5.4.3 has an eigenvalue of multiplicity 4. So a distance-regular graph satisfying the assumption of Theorem 5.4.1 cannot have $a_1 = 2$.

We next consider the distance-regular graphs with valency 6 and $a_1 = 3$.

Lemma 5.4.4 *Suppose G is a connected distance-regular graph of valency $k = 6$, and G is not a complete multipartite graph. If G has an eigenvalue θ with multiplicity $m = 4$ and $a_1 = 3$, then G is $L(K_5)$, the line graph of the complete graph K_5.*

Proof. There are only two non-isomorphic 3-regular graphs on six vertices and these are shown as D_1 and D_2 in Figure 5.8. Note that D_1 ($\cong K_{3,3}$) is bipartite and D_2 ($\cong L(K_{2,3})$) is non-bipartite. We divide the proof into two cases.

Case 1. Assume that there exists a vertex u in $V(G)$ such that the induced subgraph $G(u)$ is isomorphic to the graph D_1 in Figure 5.8. Assume further that

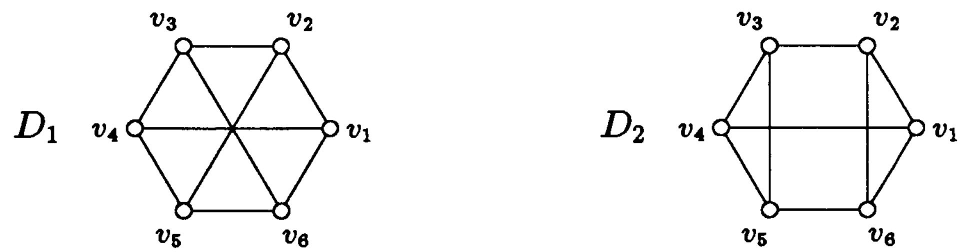

Figure 5.8: Two 3-regular graphs on six vertices

this graph $G(u)$ is actually D_1 (with the same labeling of vertices) and $G(u) = \{v_1, v_2, \ldots, v_6\}$. Let u_θ be the representation associated with the eigenvalue θ, $u_\theta : V(G) \longrightarrow R^4$. Observe that the vertices v_1, v_3 and v_5 are pairwise at distance two in G. Hence the image vectors $u_\theta(v_1)$, $u_\theta(v_3)$ and $u_\theta(v_5)$ form a regular simplex in R^4. Similarly the image vectors $u_\theta(v_2)$,$u_\theta(v_4)$ and $u_\theta(v_6)$ also form a regular simplex. Furthermore, these two simplexes are congruent and lie on the same sphere (centred at the origin) in R^4. So they span two subspaces of the same dimension in R^4. The remainder of the argument is broken into several steps.

(a) *The two subspaces each have dimension three.*

Since $u_\theta(v_1)$, $u_\theta(v_3)$ and $u_\theta(v_5)$ form a spherical regular simplex, they either span a 2-dimensional subspace (degenerate case) or span a 3-dimensional subspace (non-degenerate case). If span $\{u_\theta(v_1), u_\theta(v_3), u_\theta(v_5)\}$ has dimension two then $u_\theta(v_1) + u_\theta(v_3)+u_\theta(v_5)$ is the zero vector in R^4. Similarly $u_\theta(v_2)+u_\theta(v_4)+u_\theta(v_6)=0$. Therefore

$$\sum_{i=1}^{6} u_\theta(v_i) = 0.$$

But on the other hand, we know by Equation 2.1 that

$$\sum_{i=1}^{6} u_\theta(v_i) = \theta u_\theta(u).$$

So $\theta = 0$. This implies by Lemma 2.6.1 that G has either its second largest eigenvalue equal to 0, or its smallest eigenvalue equal to 0. The former implies that G is a complete multipartite graph while the latter implies that G is a single vertex. They both contradict our assumption.

(b) *span* $\{u_\theta(v_1), u_\theta(v_3), u_\theta(v_5)\} \neq$ *span* $\{u_\theta(v_2), u_\theta(v_4), u_\theta(v_6)\}$.

Assume the opposite. If the two subspaces are identical, then by Equation 2.1 all the seven points $\{u_\theta(u), u_\theta(v_1), \ldots, u_\theta(v_6)\}$ lie in a 3-dimensional space. Recall that the image of a graph under a representation is on the unit sphere, and the intersection of a unit sphere with a subspace is again a sphere (though not necessarily a unit sphere). So these seven image vectors would lie on a sphere in the 3-dimensional subspace. This gives a spherical 2-distance set in R^3. But that is impossible since by the discussion in Section 3.3 we know that a 2-distance set in R^3 can have at most six vertices.

(c) *There exists a vertex v_j with $j = 2$, 4 or 6 such that the image vectors in the set $\{u_\theta(v_1), u_\theta(v_3), u_\theta(v_5), u_\theta(v_j)\}$ span the whole space R^4.*

This is an immediate consequence of (b). Without loss of generality we assume $j = 2$.

The proof of Case 1 can now be completed. From the graph $D_1 = G(u)$ we can see that the following are true in G:

$$\begin{aligned} \partial(v_4, v_2) &= \partial(v_6, v_2); \qquad \text{and} \\ \partial(v_4, v_i) &= \partial(v_6, v_i), \quad i = 2, 4, 6. \end{aligned}$$

It follows from Lemma 2.3.3 that the corresponding inner products satisfy

$$\begin{aligned} \langle u_\theta(v_4), u_\theta(v_2)\rangle &= \langle u_\theta(v_6), u_\theta(v_2)\rangle \qquad \text{and} \\ \langle u_\theta(v_4), u_\theta(v_i)\rangle &= \langle u_\theta(v_6), u_\theta(v_i)\rangle, \quad \text{for} \quad i = 2, 4, 6. \end{aligned}$$

Therefore the vector $(u_\theta(v_4) - u_\theta(v_6))$ is perpendicular to each element of a basis of R^4. This forces $u_\theta(v_4) = u_\theta(v_6)$, contradicting the local injectivity of the mapping u_θ.

Case 2. Suppose for any vertex u in G the induced subgraph $G(u)$ is isomorphic to D_2 in Figure 5.8. Fix a vertex $u \in V(G)$. Write $G(u) = \{v_1, v_2, \ldots, v_6\}$ and assume D_2 is the subgraph induced by $G(u)$. Let z_1 and z_2 be the two vertices in $G_2(u)$ adjacent to v_2. Notice that the induced subgraph $G(v_2)$ is also isomorphic to D_2 and that the two triangles in D_2 are disjoint. Since $\{u, v_1, v_3\}$ already forms a triangle in $G(v_2)$, the remaining three vertices $\{v_5, z_1, z_2\}$ form the other triangle. The edges between these two triangles should be a matching. So the induced subgraph $G(u) \cup \{z_1, z_2\}$ contains the graph in Figure 5.9 as a subgraph. Applying the same argument to v_1

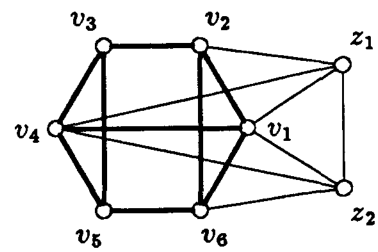

Figure 5.9: A subgraph of the subgraph induced by $G(u) \cup \{z_1, z_2\}$

and noting that z_1 is adjacent to v_1, we see that z_1 should also be adjacent to v_6. It follows that $c_2 \geq 4$ and $a_2 \geq 1$. By the equation $k_2c_2 = kb_1 = 6 \times 2 = 12$ we obtain $c_2 = 4$ and $k_2 = 3$. To ensure that k_2a_2 is even we must have $a_2 = 2$. We are thus led to the array

$$\begin{Bmatrix} * & 1 & 4 \\ 0 & 3 & 2 \\ 6 & 2 & * \end{Bmatrix},$$

which shows that G is a 6-regular graph on ten vertices and has thirty edges. It is easy to verify that $L(K_5)$ is one realization of this array. We are now going to show that $L(K_5)$ is actually the only distance-regular graph with this array.

Since $G(u) \cong D_2$, any maximum clique of the subgraph $G(u)$ is isomorphic to K_3. It follows that any maximum clique of graph G is isomorphic to K_4. As each vertex v_i in $G(u)$ is contained in a maximum clique ($\cong K_3$) of $G(u)$, each edge uv_i is contained in a maximum clique ($\cong K_4$) of G. Furthermore, if an edge uv_i lied in two distinct maximum cliques of G, then the vertex v_i would be contained in two distinct maximum cliques of $G(u)$. But this is impossible because the two triangles in D_2 do not have any vertex in common.

Hence, the edges of graph G can be partitioned into five maximum cliques of G, each of them is isomorphic to K_4. Each vertex of G is contained in exactly two such cliques. It follows (see, for example, Theorem 8.4 of Harary [22]) that G must be the line graph of another graph H and H is on five vertices. As each maximum clique of G consists of four vertices, H should be a 4-regular graph. So we uniquely determine that $H \cong K_5$ and $G \cong L(K_5)$. □

We now consider the distance-regular graphs with valency 6 and $a_1 = 4$, which is the last of the four cases for Theorem 5.4.1.

Lemma 5.4.5 *Suppose G is a connected distance-regular graph of valency $k = 6$. If G has an eigenvalue θ with multiplicity $m = 4$ and $a_1 = 4$, then G is the complete multipartite graph $\overline{4K_2}$.*

Proof. From the equation $k_2c_2 = kb_1 = 6$, we see that c_2 can be 1, 2, 3 or 6. If $c_2 = 6$, then G is the complete multipartite graph $\overline{4K_2}$. If $c_2 = 3$, then $k_2 = 2$ and $a_2 \geq k - c_2 - 1 = 2$, a contradiction. If $c_2 = 2$, then $k_2 = 3$ and $a_2 \geq 3$, a contradiction. If $c_2 = 1$, then the neighbourhood $G(u)$, $u \in V(G)$, is a disjoint union of cliques and by Lemma 4.2.2 we should have $k \leq m(a_1 + 1)/a_1$, or $k \leq 5$, contradicting the assumption that $k = 6$. □

Remark. In this lemma the assumption on eigenvalue multiplicity is not essential. It can be removed with a little effort. In fact, $\overline{4K_2}$ is the only connected graph with the subgraph induced by the neighbourhood of any vertex isomorphic to $\overline{3K_2}$. (See the remark after Theorem 5.3.1.)

Throughout the lengthy discussion in this section, we have examined all possible cases for the distance-regular graphs satisfying the assumption of Theorem 5.4.1. So Theorem 5.4.1 is finally proved.

5.5 Distance-Regular Graphs with Valency $k = 4$ and $a_1 = 0$

Our last lemma will again largely involve geometry.

Lemma 5.5.1 *Let G be a distance-regular graph with valency $k = 4$ and diameter $d \geq 2$. Let θ be an eigenvalue of G with multiplicity $m = 4$ and not equal to $\pm k$. If $a_1 = 0$ and $c_2 \neq 1$, then G is $K_{5,5}$ minus a 1-factor.*

Proof. If $c_2 = 4$, then G is complete bipartite graph $K_{4,4}$ which does not have an eigenvalue of multiplicity four. So we need only consider the cases $c_2 = 2$ and $c_2 = 3$. In what follows we are going to show that $c_2 \neq 2$, and that if $c_2 = 3$, then G is $K_{5,5}$ minus a 1-factor.

We first assume that $c_2 = 2$. In this case we have $k_1 = 4$ and $k_2 = 6$. Let u be a vertex in G and $G(u) = \{v_1, v_2, v_3, v_4\}$. Since $a_1 = 0$, the four vertices in $G(u)$ are pairwise at distance two in G. It follows from $c_2 = 2$ that $(G(v_i) \cap G(v_j)) \setminus \{u\}$ consists of a single vertex for any $1 \leq i < j \leq 4$. We denote this vertex by the unordered pair (ij). Therefore

$$G_2(u) = \{(12), (13), (14), (23), (24), (34)\}.$$

We see that the vertex (ij) is a neighbour to both v_i and v_j in $G(u)$, and the neighbourhood of v_i consists of u and the vertices corresponding to the three unordered pairs containing i.

Let u_θ be the representation associated with θ and denote by w_i, $1 \leq i \leq d$, the sequence of cosines corresponding to θ. We are going to consider the image configuration of $\{u\} \cup G_2(u)$ in R^4 under the mapping u_θ.

We know from Lemma 2.4.1 that if $w_2 = -1$, then G is $\overline{nK_2}$ for some n or a cycle of length $4n$. In our case G would be $\overline{3K_2}$ if $w_2 = -1$. But $\overline{3K_2}$ does not have an eigenvalue with multiplicity four. Hence we have $w_2 \neq -1$.

We will next prove that the representation u_θ is injective on $\{u\} \cup G_2(u)$. By the local injectivity of u_θ, the image of u cannot coincide with any vertex in $G_2(u)$. Let (ij) and (kl) be two vertices in $G_2(u)$. If the two pairs have one number in common, say $j = l$, then they are neighbours of v_j and are at distance two in G. So their images cannot coincide. Now suppose that (ij) and (kl), as unordered pairs, are disjoint and $u_\theta((ij))$ and $u_\theta((kl))$ coincide in R^4. Denote by P the common image point $u_\theta((ij)) = u_\theta((kl))$. Then P will have the same distance to each of $u_\theta(v_1)$, $u_\theta(v_2)$, $u_\theta(v_3)$ and $u_\theta(v_4)$. Note that these four points form a regular simplex and the point P itself should be on the unit sphere in R^4. There is only one choice left for point P, and that is at the opposite pole on the unit sphere from $u_\theta(u)$. This would imply that $w_2 = -1$, a contradiction.

We next show that the images of the five vertices $\{u, (12), (34), (13), (24)\}$ lie on a circle in R^4. It is easily proved by linear algebra that, given three points in R^4 which are not collinear, any three spheres respectively centred at these three points intersect in a circle. (This circle may be degenerate and become a single point or the empty set.) Now observe that $\{u, (12), (34), (13), (24)\}$ is a subset of $G_2((14)) \cap G_2((23))$. By Lemma 2.3.3 the images of the five vertices $\{u, (12), (34), (13), (24)\}$ should lie on two spheres centred at $u_\theta((14))$ and $u_\theta((23))$, respectively. Since every image vector falls on a sphere centred at the origin O, so do these five image points. It follows from $w_2 \neq -1$ that the centres of these three spheres are distinct and not collinear.

This yields the claim.

Since $\{(12), (34), (13), (24)\} \subseteq G_2(u)$, it follows that $u_\theta(u)$ has equal distance to the images of $\{(12), (34), (13), (24)\}$ in R^4. But this is not possible when all these five points lie on a circle. Hence $c_2 \neq 2$.

Now we consider the case $c_2 = 3$ and determine the unique distance-regular graph. Since $a_1 = 0$, the graph G contains no triangle. It follows from $c_2 = 3$ that a_2 can be either 1 or 0. First assume that $a_1 = 1$. Fix a vertex u in G, and let v_1 and v_2 be a pair of adjacent vertices in $G_2(u)$. Since $c_2 = 3$ and $k_1 = 4$, we see that

$$|G(u) \cap G(v_1) \cap G(v_2)| \geq 2.$$

Hence G would contain a triangle, a contradiction.

Now assume that $a_2 = 0$. Then $b_2 = 1$. Since $c_3 \geq c_2 = 3$, it follows that c_3 could be either 3 or 4. However, as

$$k_3 = |G_3(u)| = \frac{kb_1b_2}{c_2c_3} = \frac{12}{3c_3}$$

should be an integer, c_2 must be 4. So G has diameter three and its intersection array is

$$\left\{\begin{matrix} * & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & * \end{matrix}\right\}.$$

This graph does not have any odd cycles because all the a_i are zero. Hence G is a bipartite regular graph on ten vertices with valency four. Therefore, G is $K_{5,5}$ minus a 1-factor. □

If $c_2 = 1$, we use a computer to search for feasible intersection arrays. Our program consists of three subroutines. First, we generate candidate arrays which fall in the range $d \leq 8$, $k = 4$, $a_1 = 0$ and $c_2 = 1$, and satisfy the conditions in

Lemma 1.1.1. Secondly, for each candidate array we check the numerical constraints derived in Lemma 4.1.4 and Lemma 4.2.1. Finally, we compute the eigenvalues and their multiplicities for each array which passed the first two examinations. This process produced one qualified array, which represents the 4-cube.

Combining all the discussions in Chapter 5, we have completed the classification for distance-regular graphs with an eigenvalue of multiplicity four, which is summarized in Theorem 5.0.1.

Chapter 6

Classifying Distance-Regular Graphs with an Eigenvalue of Multiplicity Five

In this chapter we classify distance-regular graphs with an eigenvalue of multiplicity five. We will first improve the bounds derived earlier, then use a computer to carry out detailed calculations. The main result of this chapter is

Theorem 6.0.1 *Let G be a connected distance-regular graph with an eigenvalue of multiplicity five. Then G is one of the following graphs.*

(a) *K_6, $L(K_6)$, complement of $L(K_6)$, icosahedron, complement of $L(K_{2,6})$, halved Foster graph, 5-cube, folded graph of 5-cube, complement of folded graph of 5-cube, the unique graph with array $\{5,4,1,1;1,1,4,5\}$, of the folded 5-cube), Petersen graph, line graph of Petersen graph, $L(K_5)$, dodecahedron, Desargues graph, Johnson graph $J(6,3)$, $\overline{5K_2}$.*

(b) *All complete multipartite graphs $\overline{6K_r}$ with $r \geq 2$.*

All of these graphs are uniquely determined by their intersection arrays.

The intersection arrays and spectra of these graphs are listed in the Appendix.

In this chapter, Proposition 6.1.4 and 6.1.5 are joint work with W.J. Martin.

6.1 On the Graphs with Large Diameter

Recall that a distance-regular graph G of diameter d is said to be antipodal if the vertex set of G can be partitioned in such a way that any two vertices in the same cells are at distance d, and any two vertices in different cell are at distance less than d. We list some properties of antipodal graphs which will be used in later discussion.

Lemma 6.1.1 (Taylor and Levingston [31]) *Let G be a distance-regular graph with diameter d. Then G is antipodal if and only if $b_i = c_{d-i}$ for $0 \le i \le d$ and $i \ne \lfloor \frac{d}{2} \rfloor$.* □

Lemma 6.1.2 (Yoshizawa [34]) *Let G be a distance-regular graph satisfying either $k_d \le 2$, or $k_d < k$ and girth $g \ge 5$. Then G is antipodal.* □

The following lemma is due to Terwilliger [32].

Lemma 6.1.3 *Let G be a distance-regular graph with valency k and girth g ($g \ge 4$). Let θ be an eigenvalue of G with multiplicity m, and $\theta \ne \pm k$. Then*

$$\begin{array}{rcll} m & \ge & k(k-1)^{r-1}, & \text{if } g = 4r, \text{ or } 4r+1; \\ m & \ge & 2(k-1)^{r}, & \text{if } g = 4r+2, \text{ or } 4r+3. \end{array}$$

We write the above formula explicitly for a few initial cases:

- If $g \ge 4$, then $k \le m$;
- If $g \ge 6$, then $2(k-1) \le m$;
- If $g \ge 8$, then $k(k-1) \le m$;
- If $g \ge 10$, then $2(k-1)^2 \le m$; and
- If $g \ge 12$, then $k(k-1)^2 \le m$.

By Theorem 4.1.1 and the remark following Lemma 4.2.1, we see that for a distance-regular graph with an eigenvalue of multiplicity five, the diameter could be at most 11 and valency at most 10. In what follows, we will further reduce these bounds.

It has been shown (Godsil [20]) that if $d = 3m - 4$, then G is the dodecahedron. We are now going to show that in general there is no distance-regular graph with $d = 3m - 5$, and for the case $m = 5$ there is no such graph with $d = 3m - 6$.

Proposition 6.1.4 *Let G be a distance-regular graph with valency k and diameter $d \geq 2$. If G has an eigenvalue $\theta \neq \pm k$ with multiplicity m, then $d \neq 3m - 5$.*

Proof. Distance-regular graphs with $m = 3$ or $m = 4$, or with valency three have already been classified. By checking the list of these graphs (in the Appendix) we find that the claim is true for all of these graphs. Therefore we assume that $m \geq 5$ and $k \geq 4$.

As in the proof of Theorem 4.1.1 we denote by q the maximum length of a geodetic path in G which is independent with respect to θ. (Then $q + 1 \leq m$.) There it is proved that $d \leq 3q - 1$. Suppose that G has diameter $d = 3m - 5$. Then we must have $q = m - 1$, and thus $d = 3q - 2$. Applying Lemma 4.1.4 we find that $c_i = 1$ for $1 \leq i \leq 2m - 4$, and $a_1, a_2, \ldots, a_{m-3}$ are all zero. It follows that the girth g is at least $2(m-3) + 2 = 2(m-2)$.

If $m = 5$, then $g \geq 6$. It follows from Lemma 6.1.3 that $2(k-1) \leq m$, and so $k \leq 3$. Hence, in the rest of the proof we can further assume that $m \geq 6$.

In general, Lemma 6.1.3 implies that

$$m \geq (k-1)^{\lfloor g/4 \rfloor}.$$

So

$$m \geq (k-1)^{\lfloor (m-2)/2 \rfloor}.$$

It is easy to verify that this inequality cannot hold for any integers $k \geq 4$ and $m \geq 6$. This contradiction shows that there does not exist distance-regular graphs with $d = 3m - 5$. ◫

Proposition 6.1.5 *Let G be a distance-regular graph with valency k and diameter d. Let $\theta \neq \pm k$ be an eigenvalue of G with multiplicity $m = 5$. Then $d \neq 3m - 6$.*

Proof. Suppose that G has diameter $d = 3m - 5 = 10$. By the inequality $d \leq 3q - 1$, it is easy to see that $q = m - 1 = 4$ and $d = 3q - 3$. Applying Lemma 4.1.4 we get:

(a) $b_i = 1$ for $4 \leq i \leq 8$;

(b) $c_i = 1$ for $1 \leq i \leq 5$;

(c) $a_4 = a_5 = k - 2$;

(d) $c_8 = k - 1$; (since $a_i + 1 \leq c_{q+i}$)

(e) $a_1 = 0$; (since $a_i + 1 \leq c_{q+i}$) and

(f) $b_1 = k - 1$.

Therefore the intersection array of the graph should have the pattern

$$\left\{ \begin{array}{cccccccccc} * & 1 & 1 & 1 & 1 & 1 & c_6 & c_7 & k-1 & c_9 \\ 0 & 0 & a_2 & a_3 & k-2 & k-2 & a_6 & a_7 & 0 & a_9 \\ k & k-1 & b_2 & b_3 & 1 & 1 & 1 & 1 & 1 & * \end{array} \right\}$$

Since $a_1 = 0$, it follows from Lemma 4.2.1 that $k \leq m = 5$. Recall that by Corollary 4.2.4 we have $b_i \leq m - i$ for $1 \leq i \leq q$. So $b_2 \leq m - 2 = 3$ and $b_3 \leq 2$. Therefore $c_6 \leq b_3 \leq 2$ and $c_7 \leq b_2 \leq 3$.

If $a_2 = 0$, then $g \geq 7$ and that would force $k \leq 3$. So we assume that $a_2 \geq 1$. It follows that $c_6 \geq a_2 + 1 \geq 2$. Therefore $c_6 = 2$ and $b_3 = 2$. It is easy to calculate that $k_d = kb_2/c_7c_9$. As $b_2 \leq 3$ and $c_9 \geq c_8 = k - 1$, we get

$$k_d = \frac{kb_2}{c_7c_9} \leq \frac{3k}{(k-1)c_7} \leq \frac{3k}{2(k-1)} < k.$$

By Lemma 6.1.2 we conclude that G is antipodal. It follows from Lemma 6.1.1 that $c_9 = b_0 = k$ and $b_2 = c_7$.

If $k = 4$, then $a_2 \geq 1$ and $b_2 \geq b_3 = 2$ which imply that $a_2 = 1$ and $b_2 = 2$. We get one candidate array. This array is not feasible because the calculation for eigenvalue multiplicity does not give integers. Now assume $k = 5$. As $2 \leq c_7 \leq b_2 \leq 3$, we have two candidate arrays corresponding to $b_2 = c_7 = 2$ and $b_2 = c_7 = 3$, respectively. Both of them failed the integrality check for eigenvalue multiplicities. □

An immediate consequence of the above discussion is

Corollary 6.1.6 *Let G be a distance-regular graph with valency k and diameter d. Let $\theta \neq \pm k$ be an eigenvalue of G with multiplicity $m = 5$. Then $d \leq 3m - 7$.* □

The following lemma is basically a rephrasing of some known results which will be helpful in classifying graphs.

Lemma 6.1.7 *Let G be a distance-regular graph with valency k and diameter d. Let $\theta \neq \pm k$ be an eigenvalue of G with multiplicity $m = 5$. Then the following is true.*

(a) *If $a_1 = 0$, then $k \leq m = 5$.*

(b) *If $c_2 > 1$, then $d \leq m = 5$.*

(c) *If $a_1 = a_2 = 0$ and $c_2 = 1$, then $k \leq 3$.*

(d) *If $a_1 \neq 0$, then $a_i \neq 0$ for $2 \leq i \leq d - 1$.*

Proof. Claim (a) is from Lemma 4.1.1. By Lemma 4.1.4 we have $c_{d-q} = 1$. If $c_2 > 1$, then $c_2 > c_{d-q}$ and $d - q < 2$. Since $q \leq m - 1$ we get $d < 2 + q \leq 2 + 4 = 6$. By the condition of claim (c), it follows that girth $g \geq 6$. Applying Lemma 6.1.3 we get $2(k - 1) \leq m$, or $k \leq 3$.

To prove the last claim, let $x_0, x_1, \ldots, x_i, x_{i+1}$ be a geodetic path from x_0 to x_{i+1}. (Then $\partial(x_0, x_{i+1}) = i+1$.) It can be seen that the set $G(x_i) \cap G(x_{i+1})$ of size a_1 is covered by the two sets $G_i(x_0) \cap G(x_i)$ and $G_i(x_1) \cap G(x_{i+1})$, each of size a_i. Therefore $2a_i \geq a_1$. In particular, if $a_1 \neq 0$, then $a_i \neq 0$. □

6.2 Computer-Aided Search

All complete multipartite graphs $\overline{6K_r}$, $r \geq 2$ are distance-regular and have an eigenvalue of multiplicity five, as we have seen in Section 1.2. By the discussion in Section 1, it is also clear that, besides the complete 6-partite graphs, the intersection arrays of the distance-regular graphs with an eigenvalue of multiplicity five fall in one of the following five cases.

Index	a_1	c_2	a_2	Diameter	Valency
Case 1:	$a_1 = 0$	$c_2 > 1$	a_2 free	$d \leq 5$	$k \leq 5$
Case 2:	$a_1 = 0$	$c_2 = 1$	$a_2 = 0$	$d \leq 8$	$k \leq 3$
Case 3:	$a_1 = 0$	$c_2 = 1$	$a_2 > 0$	$d \leq 8$	$k \leq 5$
Case 4:	$a_1 > 0$	$c_2 > 1$	$a_2 > 0$	$d \leq 5$	$k \leq 10$
Case 5:	$a_1 > 0$	$c_2 = 1$	$a_2 > 0$	$d \leq 8$	$k \leq 10$

For these five cases, we use a computer to search for feasible intersection arrays and then identify the resulting arrays.

Our program consists of two phases. In the first phase, the program does the following.

(a) Generate arrays with diameter and valency in the range specified in the above table.

(b) Check the usual feasibility condition stated in Lemma 1.1.1.

(c) Check various numerical constraints derived in Chapter 4. (In the actual running, these constraints are quite restrictive and effectively eliminate infeasible arrays.)

The second phase of the computing mainly consists of a program calculating the eigenvalue multiplicities for distance-regular graphs. We use this program to check the integrality of the eigenvalue multiplicities for those arrays which passed all the examinations in first phase. (This program uses the so-called QR-method.)

The final output of the computing gives out 18 arrays. One of them with an eigenvalue of multiplicity three is easily eliminated because there is no such graph in the known list (see the Appendix) for distance-regular graphs with an eigenvalue of multiplicity three. The rest of the arrays are identified easily and it turns out that all of these distance-regular graphs are known ones. The complete classification is summarized in Theorem 6.0.1. The intersection arrays and spectra of these graphs are listed in the Appendix.

Appendix

DISTANCE-REGULAR GRAPHS WITH

AN EIGENVALUE OF MULTIPLICITY THREE

Intersection Array	Spectrum	n	Name
$\left\{\begin{matrix} * & 1 \\ 0 & 2 \\ 3 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$	4	K_4 (tetrahedron)
$\left\{\begin{matrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}$	6	octahedron ($L(K_4) = \overline{3K_2}$)
$\left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$	8	cube ($K_{4,4}$ minus 1-factor)
$\left\{\begin{matrix} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & \sqrt{5} & -1 & -\sqrt{5} \\ 1 & 3 & 5 & 3 \end{pmatrix}$	12	icosahedron
$\left\{\begin{matrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{5} & 1 & 0 & -2 & -\sqrt{5} \\ 1 & 3 & 5 & 4 & 4 & 3 \end{pmatrix}$	20	dodecahedron
$\left\{\begin{matrix} * & 1 & 3r \\ 0 & 2r & 0 \\ 3r & r-1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3r & 0 & -r \\ 1 & 4r-4 & 3 \end{pmatrix}$	$4r$	$\overline{4K_r}$, $r \geq 2$

DISTANCE-REGULAR GRAPHS WITH AN EIGENVALUE OF MULTIPLICITY FOUR

Intersection Array	Spectrum	n	Name
$\begin{Bmatrix} * & 1 \\ 0 & 3 \\ 4 & * \end{Bmatrix}$	$\begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}$	5	K_5
$\begin{Bmatrix} * & 1 & 3 \\ 0 & 0 & 0 \\ 3 & 2 & * \end{Bmatrix}$	$\begin{pmatrix} 3 & 0 & -3 \\ 1 & 4 & 1 \end{pmatrix}$	6	$K_{3,3}$
$\begin{Bmatrix} * & 1 & 1 \\ 0 & 0 & 2 \\ 3 & 2 & * \end{Bmatrix}$	$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$	10	Petersen graph ($\overline{L(K_5)}$)
$\begin{Bmatrix} * & 1 & 2 \\ 0 & 1 & 2 \\ 4 & 2 & * \end{Bmatrix}$	$\begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & 4 \end{pmatrix}$	9	$L(K_{3,3})$
$\begin{Bmatrix} * & 1 & 6 \\ 0 & 4 & 0 \\ 6 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 6 & 0 & -2 \\ 1 & 4 & 3 \end{pmatrix}$	8	$\overline{4K_2}$
$\begin{Bmatrix} * & 1 & 4 \\ 0 & 3 & 2 \\ 6 & 2 & * \end{Bmatrix}$	$\begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 5 \end{pmatrix}$	10	$L(K_5)$
$\begin{Bmatrix} * & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 4 & 1 & -1 & -4 \\ 1 & 4 & 4 & 1 \end{pmatrix}$	10	$K_{5,5}$ minus a 1-factor
$\begin{Bmatrix} * & 1 & 1 & 4 \\ 0 & 1 & 2 & 0 \\ 4 & 2 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 4 & 2 & -1 & -2 \\ 1 & 5 & 4 & 5 \end{pmatrix}$	15	line graph of Petersen graph
$\begin{Bmatrix} * & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 3 & \sqrt{3} & 0 & -\sqrt{3} & -3 \\ 1 & 6 & 4 & 6 & 1 \end{pmatrix}$	18	Pappus graph
$\begin{Bmatrix} * & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 4 & 2 & 0 & -2 & -4 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$	16	4-cube
$\begin{Bmatrix} * & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 3 & 2 & 1 & -1 & -2 & -3 \\ 1 & 4 & 5 & 5 & 4 & 1 \end{pmatrix}$	20	Desargues graph
$\begin{Bmatrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{Bmatrix}$	$\begin{pmatrix} 3 & \sqrt{5} & 1 & 0 & -2 & -\sqrt{5} \\ 1 & 3 & 5 & 4 & 4 & 3 \end{pmatrix}$	20	dodecahedron
$\begin{Bmatrix} * & 1 & 4r \\ 0 & 3r & 0 \\ 4r & r-1 & * \end{Bmatrix}$	$\begin{pmatrix} 4r & 0 & -r \\ 1 & 5r-5 & 4 \end{pmatrix}$	$5r$	$\overline{5K_r}$, $r \geq 2$

DISTANCE-REGULAR GRAPHS WITH AN EIGENVALUE OF MULTIPLICITY FIVE

Intersection Array	Spectrum	n	Name
$\left\{\begin{matrix} * & 1 \\ 0 & 4 \\ 5 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & -1 \\ 1 & 5 \end{pmatrix}$	6	K_6
$\left\{\begin{matrix} * & 1 & 1 \\ 0 & 0 & 2 \\ 3 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$	10	Petersen graph ($\overline{L(K_5)}$)
$\left\{\begin{matrix} * & 1 & 2 \\ 0 & 0 & 3 \\ 5 & 4 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & 1 & -3 \\ 1 & 10 & 5 \end{pmatrix}$	16	antipodal quotient of 5-cube
$\left\{\begin{matrix} * & 1 & 4 \\ 0 & 3 & 2 \\ 6 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 6 & 1 & -2 \\ 1 & 4 & 5 \end{pmatrix}$	10	$L(K_5)$
$\left\{\begin{matrix} * & 1 & 3 \\ 0 & 1 & 3 \\ 6 & 4 & * \end{matrix}\right\}$	$\begin{pmatrix} 6 & 1 & -3 \\ 1 & 9 & 5 \end{pmatrix}$	15	$\overline{L(K_6)}$
$\left\{\begin{matrix} * & 1 & 8 \\ 0 & 6 & 0 \\ 8 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 8 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$	10	$\overline{5K_2}$
$\left\{\begin{matrix} * & 1 & 4 \\ 0 & 4 & 4 \\ 8 & 3 & * \end{matrix}\right\}$	$\begin{pmatrix} 8 & 2 & -2 \\ 1 & 5 & 9 \end{pmatrix}$	15	$L(K_6)$
$\left\{\begin{matrix} * & 1 & 6 \\ 0 & 6 & 4 \\ 10 & 3 & * \end{matrix}\right\}$	$\begin{pmatrix} 10 & 2 & -2 \\ 1 & 5 & 10 \end{pmatrix}$	16	complement of quotient of 5-cube
$\left\{\begin{matrix} * & 1 & 1 & 4 \\ 0 & 1 & 2 & 0 \\ 4 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 4 & 2 & -1 & -2 \\ 1 & 5 & 4 & 5 \end{pmatrix}$	15	line graph of Petersen graph
$\left\{\begin{matrix} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & \sqrt{5} & -1 & -\sqrt{5} \\ 1 & 3 & 5 & 3 \end{pmatrix}$	12	icosahedron
$\left\{\begin{matrix} * & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & 1 & -1 & -5 \\ 1 & 5 & 5 & 1 \end{pmatrix}$	12	$K_{6,6}$ minus 1-factor ($\overline{L(K_{2,6})}$)

(cont'd)

DISTANCE-REGULAR GRAPHS WITH AN EIGENVALUE OF MULTIPLICITY FIVE (cont'd)

Intersection Array	Spectrum	n	Name
$\left\{\begin{matrix} * & 1 & 4 & 9 \\ 0 & 4 & 4 & 0 \\ 9 & 4 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 9 & 3 & -1 & -3 \\ 1 & 5 & 9 & 5 \end{pmatrix}$	20	Johnson graph $J(6,3)$
$\left\{\begin{matrix} * & 1 & 1 & 4 & 5 \\ 0 & 0 & 3 & 0 & 0 \\ 5 & 4 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & \sqrt{5} & 1 & -\sqrt{5} & -3 \\ 1 & 8 & 10 & 8 & 5 \end{pmatrix}$	32	double cover of quotient of 5-cube
$\left\{\begin{matrix} * & 1 & 1 & 4 & 6 \\ 0 & 1 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 6 & 3 & 1 & -2 & -3 \\ 1 & 12 & 9 & 18 & 5 \end{pmatrix}$	45	halved Foster graph
$\left\{\begin{matrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{5} & 1 & 0 & -2 & -\sqrt{5} \\ 1 & 3 & 5 & 4 & 4 & 3 \end{pmatrix}$	20	dodecahedron
$\left\{\begin{matrix} * & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 2 & 1 & -1 & -2 & -3 \\ 1 & 4 & 5 & 5 & 4 & 1 \end{pmatrix}$	20	Desargues graph
$\left\{\begin{matrix} * & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5 & 3 & 1 & -1 & -3 & -5 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}$	32	5-cube
$\left\{\begin{matrix} * & 1 & 5r \\ 0 & 4r & 0 \\ 5r & r-1 & * \end{matrix}\right\}$	$\begin{pmatrix} 5r & 0 & -r \\ 1 & 6r-6 & 5 \end{pmatrix}$	6r	$\overline{6K_r}$, $r \geq 2$

DISTANCE-REGULAR GRAPHS WITH VALENCY THREE

Intersection Array	Spectrum	n	Name
$\left\{\begin{matrix} * & 1 \\ 0 & 2 \\ 3 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$	4	K_4
$\left\{\begin{matrix} * & 1 & 3 \\ 0 & 0 & 0 \\ 3 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 0 & -3 \\ 1 & 4 & 1 \end{pmatrix}$	6	$K_{3,3}$
$\left\{\begin{matrix} * & 1 & 1 \\ 0 & 0 & 2 \\ 3 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$	10	Petersen graph
$\left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$	8	cube
$\left\{\begin{matrix} * & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{2} & -\sqrt{2} & -3 \\ 1 & 6 & 6 & 1 \end{pmatrix}$	14	Heawood graph
$\left\{\begin{matrix} * & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{3} & 0 & -\sqrt{3} & -3 \\ 1 & 6 & 4 & 6 & 1 \end{pmatrix}$	18	Pappus graph
$\left\{\begin{matrix} * & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 3 & 2 & 2 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 2 & \sqrt{2}-1 & -1 & -\sqrt{2}-1 \\ 1 & 8 & 6 & 7 & 6 \end{pmatrix}$	28	Coxeter graph
$\left\{\begin{matrix} * & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 2 & 0 & -2 & -3 \\ 1 & 9 & 10 & 9 & 1 \end{pmatrix}$	30	8-cage
$\left\{\begin{matrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{5} & 1 & 0 & -2 & -\sqrt{5} \\ 1 & 3 & 5 & 4 & 4 & 3 \end{pmatrix}$	20	dodecahedron
$\left\{\begin{matrix} * & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 2 & 1 & -1 & -2 & -3 \\ 1 & 4 & 5 & 5 & 4 & 1 \end{pmatrix}$	20	Desargues graph
$\left\{\begin{matrix} * & 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 2 & 2 & 2 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{6} & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{6} & -3 \\ 1 & 21 & 27 & 28 & 27 & 21 & 1 \end{pmatrix}$	126	12-cage
$\left\{\begin{matrix} * & 1 & 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 3 & 2 & 2 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & 2.56 & 2 & 0.88 & 0 & -1.35 & -1.56 & -2.53 \\ 1 & 9 & 18 & 16 & 17 & 16 & 9 & 16 \end{pmatrix}$	102	
$\left\{\begin{matrix} * & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}$	$\begin{pmatrix} 3 & \sqrt{5} & 2 & 1 & 0 & -1 & -2 & -\sqrt{5} & -3 \\ 1 & 12 & 9 & 18 & 10 & 18 & 9 & 12 & 1 \end{pmatrix}$	90	Foster graph (triple cover of 8-cage)

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