FRACTIONAL SPIN IN THE GAUGED O(3) NON LINEAR SIGMA MODEL

by

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FRACTIONAL SPIN IN THE GAUGED O(3) NONLINEAR SIGMA MODEL.

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ABSTRACT

It is known that the (2+1)-dimensional non-linear sigma model has topological solitonic solutions and that these solitons acquire fractional spin $\dfrac{\theta}{2\pi}$ through a dynamical term called Hopf Invariant.

In this thesis the soliton of the non-linear sigma model coupled to a topologically massive(m) abelian gauge field through a topological current is quantized semi classically. The spin of the solitons with unit topological charge is evaluated and found to be $\dfrac{\theta}{2\pi}$ ($\theta = \dfrac{e^2}{2m}$, where e is gauge coupling constant) independent of mass(m) of the gauge field to all orders in m (m $\neq$ 0) and it is zero for the mass zero. Thus gauging the sigma model does not affect the fractional spin of the soliton and hints that there is a phase transition at the mass zero(Taejin Lee, Chekuri N. Rao, K. S. Viswanathan. Phys. Rev. D. **39**(1989)2350).

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DEDICATION

To My teacher Dr. V. Srinivasan

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CHAPTER 1

INTRODUCTION.

It is by now well known that topological properties give rise to "unusual spins" for systems which admit topological solitons and where the fundamental fields are bosonic. We call the spin "unusual" because it arises from a specific dynamical term rather than from kinematics. Finkelstein and Rubinstein showed in their remarkable papers (Ref. 3, 4) that in (3+1) dimensions the existence of the kinks in the field configuration is necessary for a bosonic theory to exhibit Fermionic character. As we shall see below the "unusual spin "arises because the field configuration space is non simply connected. This fact in turn determines the non single valuedness of the wave functional defined on the field configuration space.

There are two procedures to find the "unusual spin" :

(1) Canonical method (Ref. 12, 35). (We provide a detailed exposition of this method in subsequent chapters).

(2) Path Integral method (Ref. 9, 33, 35, 36, 37, 38)

The path integral method makes the connection between the "unusual spin" and connectedness of the field configuration space explicit. We therefore briefly explain this method in this chapter .

Let us consider two Vacuum –to–Vacuum process. In one process (i) we create a soliton - anti - soliton pair at $t = - \infty$ (ii) we separate the pair by a large distance (iii)we rotate the soliton adiabatically through an angle 2π and finally (iv) we annihilate the pair at $t = + \infty$ In the other process we perform the same steps as before except step (iii) i. e. we do not rotate the soliton. Comparing the amplitudes of these two processes, we find that they differ by a phase factor e^{is} where s is the action corresponding to the adiabatic rotation. Feynman proved that this phase factor is equal to $e^{\frac{i}{\hbar} 2\pi J}$ where J is spin part of the angular momentum of the soliton. $J = \frac{s}{2\pi} \hbar$.

In the case of simply connected configuration spaces each path can continuously be deformed into one another and various paths contribute to the total amplitude (each path contributes a factor $e^{\frac{i}{\hbar}s}$, where s is the action of the corresponding path). If the configuration space (Q) is not simply connected there exists some paths which cannot continuously be deformed into one another. Thus the paths in the non simply connected space(Q) fall into different equivalence classes determined by the first homotopy group $\Pi_1(Q)$ (which characterizes the connectivity of the field space). Paths belonging to different equivalence classes need not give the same contribution to the total amplitude. Each path contributes an extra multiplicative factor $\chi(\alpha_i)$ which will be called its weight hereafter to the total amplitude in addition to $e^{\frac{i}{\hbar}s}$, where α_i is the topological index (or connectivity index) that characterizes the path. Thus the contribution from each path is $\chi(\alpha_i)\, e^{\frac{i}{\hbar}s}$.

In simply connected spaces the amplitude of a path from a time interval t_i to t' followed by its development over the time interval t' to t_f is taken equal to the amplitude of the path developed over the time interval t_i to t_f In non simply connected spaces for a path decomposed into two paths belonging to two different equivalence classes characterized by α_i and α_j is assigned a weight factor $\chi(\alpha_{i+j})$ $(=\chi(\alpha_i)\,\chi(\alpha_j)$ because of the additive property of $\Pi_1(Q))$.

The propagator is the sum over all paths connecting the intial state $\phi_i(x)$ at t_i to the final $\phi_f(x)$ at t_f.

$$K(\phi_f(x),t_f\,;\phi_i(x),t_i) = \sum_{\alpha_i \in \Pi_1(Q).} \chi(\alpha_i) \int_{\phi(x)\in\alpha_i} \mathcal{D}\phi\, e^{\frac{i}{\hbar}S_0[\phi(x)]}$$

where $\phi(x,t_i) = \phi_i(x), \phi(x,t_f) = \phi_f(x)$ and S_0 is the action . Probability conservation requires $|\chi(\alpha_i)| = 1$. The complex weight is given by $\chi(\alpha_i) = e^{i\,S_{topological}}$ Where $S_{topological}$ is the topological action that arises because of the connectivity of the configuration space[35,36].

To find the spin of the soliton we compare the Vacuum-to- Vacuum amplitudes of the two processes. In one we create a soliton , anti-soliton pair and separate them by a large distance and then annihilate them.

The amplitude is given by [37]

$$\langle S(T)|S(0)\rangle = e^{-\frac{i}{\hbar}TM}$$

where M is the soliton mass and T is time . In the other process the soliton is infinitely slowly rotated through 2π. The rotation corresponds to a path in the field configuration space with a weight factor $\chi(\alpha_i)$ that is different from the weight factor of the unrotated soliton path [9]. The amplitude is given by

$$\langle S(T)|S(0)\rangle_{2\pi} = \chi(\alpha_i)\, e^{-\frac{i}{\hbar}TM}$$

We see that the amplitude differ by a weight factor $\chi(\alpha_i) = e^{i\,S_{topological}}$ which is equal to $e^{\frac{i}{\hbar}2\pi J}$

$$J = \frac{S_{topological}}{2\pi}\hbar.$$

Thus we see that the non trivial topology of the field configuration space contributesto the unusual spin.

An example of a theory which exhibits unusual spin in (3+1) dimensions is the Skyrme [37,38] model. The fields are mappings from $S^{(3)} \to SU(2)$ and the third homotopy group $\Pi_3(SU(2)) \approx \Pi_3(S^{(3)}) = Z$. (Z is an additive group of integers) implies that the field configuration space has infinitely many homotopy classes. The topological solitons are field configurations with the topological charge $Q \neq 0$. These solitons have been identified as baryons and the topological charge has been associated with the baryon number [38,39]. The connectivity of the field configuration space is determined by the first homotopy group

$\Pi_1(SU(2)) \approx \Pi_4(SU(2)) = Z_2$. where Z_2 is a group of integers modulo. 2. This shows that the field configuration space is doubly connected and hints that the Skyrme solitons may be quantized as fermions on addition of an extra topological term to the original action. Witten[37] showed that these solitons acquire half odd integral spin by adding a term called Wess-Zumino term $(n\Gamma_{w-z} = S_{topological})$ (He proved $\Gamma_{w-z} = \pi$ and n takes odd values :

$$J = \frac{S_{topological}}{2\pi}\hbar. = \frac{n\Gamma_{w-z}}{2\pi}\hbar = \frac{1}{2}\hbar \text{ for } n = 1).$$

The (2+1) dimensional O(3) non linear sigma model also exhibits solitons which possess unusual spins. The field manifold $\sum_{a=1}^{3} n^a(x,t)n^a(x,t) = 1$ is $S^{(2)}$(field) where n^a are three real scalar fields. The field configuration space is described by space of continuous maps from $S^{(2)}$ (space) $\rightarrow S^{(2)}$(field) and $\Pi_2(S^{(2)}(\text{field})) = Z$, where Z is an additive group of integers. Belavin and Polyakov [6] first proved that the model admits solitons. The solitons are the field configurations characterized by the topological charge $Q \neq 0$. The connectivity of the field configuration space is determined by $\Pi_1(Q) \approx \Pi_3(S^{(2)}(\text{field})) = Z$. i. e. infinitely connected. This leads to the interesting possibility of fractional spin . Wilczek and Zee [8] showed that when one adds an additional term known as the Hopf term($\theta H = S_{topological}$) to the action , the solitons acquire fractional spin. They showed that $H = 1$ and the coefficient θ can take any arbitrary value .

$$J = \frac{S_{topological}}{2\pi}\hbar. = \frac{\theta H}{2\pi}\hbar = \frac{\theta}{2\pi}\hbar.$$

These result were reestablished by Bowick,Karabali and Wijewardhana [12]. They provide a general quantization scheme which we adapt to our problem. Karabali and Murthy [23] considered the possibility of coupling the O(3) non linear sigma model to an abelian gauge field to achieve fractional spin. They found that the solitons of the model acquire fractional spin by coupling the

model to a massive abelian gauge field without explicitly adding the Hopf term. But they were able to show that the induced fractional spin is $\frac{\theta}{2\pi}$ ($\theta = \frac{e^2}{2m}$)upto terms of order $(1/ml)^5$; where m is the topological mass of the gauge field, l is the length scale determining the size of the soliton and e is interaction coupling constant. They considered the following action

$$\mathcal{L} = \frac{1}{2f}\partial_\mu n^a \ \partial^\mu n^a - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m}{4}\varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + eA_\mu j^\mu,$$

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon^{abc} \ n^a \ \partial_\nu n^b \partial_\lambda n^c$$

where j^μ is the topological current of the sigma model coupling the gauge field $F_{\mu\nu}$ and A_μ is the gauge potential. They integrated out A_μ in the path integral and there by obtained a non local effective action. They expanded the non local part in local operators in powers of (1/m) and used the collective coordinate formalism to quantize the theory . The resulting theory contained infinitely higher time derivatives in the collective coordinate. This leads to a system with the second class constraints. The net outcome is that they can only take finite number of terms to calculate the spin. They therefore conjectured that

(a). The induced fractional spin is independent of m{i. e. $\frac{\theta}{2\pi}$ } for large m and it is known that the fractional spin is zero for m = 0.

(b). There might be a phase transition typically for (ml) $\approx$ O(1). i. e. for ml < 1 the spin is zero and it is $\frac{\theta}{2\pi}$ for ml >1

In this thesis we basically consider the same model that Karabali and Murthy considered. We expand the field variables n^a, $A_\mu(x, t)$ around the classical soliton solutions and thus quantize the fields semiclassically by canonical method. We show that the gauged O(3) sigma model exhibits solitons with

(a). The fractional spin is strictly $\dfrac{\theta}{2\pi}$ to all orders in m for m ≠ 0 and it is zero for m = 0 at least

for Q = 1sector .

(b). The phase transition occurs at m = 0 i. e. the fractional spin is zero for m = 0 and it is $\dfrac{\theta}{2\pi}$ for

m ≠ 0.

One can also prove the same result in general for any sector[35].

Motivation

(i) If one ignores the kinetic term for the gauge fields in the above lagrangian and then integration over A_μ yields the Hopf term in the action. So the origin of the Hopf term lies in the gauge fields.

(ii). It is not unreasonable to expect the sigma model coupled to an abelian gauge field as a possible model for physical applications. Belavin and Polyakov established that the continuum limit of the Heisenberg anti-ferromagnet may be approximated by O(3) sigma model [6]. More recently, Polyakov, Wiegman and Dzyaloshinskii proposed a possible mechanism for high T_c superconductors based on the above equivalence. The solitons of O(3) sigma model become neutral fermions when θ = π which binds the electrons (or holes) and become charged bosons.

In Chapter . 2. we discuss a (2+1) dimensional non linear sigma model, canonical procedure to find angular momentum and thereby spin in (2+1) dimensions . An attempt is made to explain in simple terms topological concepts such as the winding number , the connectivity of field configuration space, etc. which are important for this work. We will also discuss in the same chapter the Hopf term and show how the dynamical (Hopf) term gives rise to fractional spin by using canonical method.

In Chapter. 3. we discuss a (2+1) dimensional massive abelian gauge field and briefly mention the origin of the mass term for the gauge field. We couple the massive gauge field with nonlinear sigma model and establish the possibility of solitons acquiring fractional spin though we do not explicitly add the Hopf term.

In Chapter. 4. and Appendices A, B, we show the procedure we followed to obtain the fractional spin $\dfrac{\theta}{2\pi}$ for $m \neq 0$ to all orders of m and it is zero for $m = 0$, which made us to predict the phase transition at $m = 0$.

CHAPTER 2

NON-LINEAR SIGMA MODEL AND FRACTIONAL SPIN OF THE SOLITONS

In this chapter we discuss the O(3) non-linear sigma model, its soliton solutions, the spin and other properties of these solitons. We illustrate through canonical method how a dynamical term i. e. Hopf term gives rise to fractional spin. This model was used in the past to describe nucleons as solitons in a theory of mesons(in 3+1 dimensions)[1] and to describe the metastable states of Heisenberg ferromagnets (in 2+1 dimensions) [6].

2. 1 O(3) non-linear sigma model

The (2+1) dimensional O(3) non-linear sigma model we consider is one in which the fields $(n^a(x,t)$ a = 1,2,3,) are real and scalars with respect to the Lorentz transformations. These fields are constrained to have unit magnitude at each space time point

$$\sum_{a=1}^{3} n^a(\mathbf{x},t)n^a(\mathbf{x},t) \equiv \mathbf{n} \cdot \mathbf{n} = 1 \qquad (2.1.1)$$

where $\mathbf{x}$ and t represent two space $x^1, x^2,$ and time coordinates respectively. n^1, n^2, n^3 are components of a vector $\mathbf{n}$ in "internal space" i. e. three dimensional field space. The above equation defines the field space as 2-sphere (denoted as $S^{(2)}_{fld}$)

The action of the model has only kinetic term

$$S[\mathbf{n}] = \int dt \, L \; , \qquad (2.1.2)$$

where the lagrangian L is

$$L = \int d^2x \mathcal{L} \qquad (2.1.3)$$

and the lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2f} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} \quad . \tag{2.1.4}$$

μ is space time index taking values 0,1, 2. The above action however does not describe a free field theory because of the constraint Eq. (2. 1. 1) and the fields are clearly dimensionless[L^0]. The coupling constant f has dimensions [L^1]. Both the lagrangian density eq. (2. 1. 4.), and the constraints Eq. (2. 1. 1) are invariant under global orthogonal (O(3)) rotations in three dimensional internal space. i. e. $\mathbf{n} \rightarrow \mathbf{n}' = e^{i\mathbf{R}.\theta}\mathbf{n}$.where $\mathbf{R}$ are the generators of the O(3) group and θ are rotational parameters, independent of space time.

The equations of motion subjected to the constraint $\mathbf{n}.\mathbf{n} = 1$ read

$$\partial_\mu \partial^\mu \mathbf{n} - (\mathbf{n}.\partial_\nu \partial^\nu \mathbf{n})\mathbf{n} = 0 . \tag{2.1.5}$$

In this chapter we are interested in the static configurations of the fields. Their time evolution can be obtained by Lorentz boosting. The equations of motion for such fields reduce to

$$\partial_i \partial^i \mathbf{n} - (\mathbf{n}.\partial_j \partial^j \mathbf{n})\mathbf{n} = 0 . \tag{2.1.6}$$

Where i, j take values 1, 2. Among various solutions of the above equation, there exist a class of solutions called topological solutions, which we describe below. These topological solutions are finite energy ($0 \leq E < \infty$)field configurations.

The energy momentum tensor defined by $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathbf{n})} \partial^\nu \mathbf{n} - g^{\mu\nu}\mathcal{L}$, yields for the above lagrangian density

$$T^{\mu\nu} = \frac{1}{f} \partial^\mu \mathbf{n} \cdot \partial^\nu \mathbf{n} - \frac{1}{2f} g^{\mu\nu} \partial_\lambda \mathbf{n} \cdot \partial^\lambda \mathbf{n} \tag{2.1.7}$$

The energy is

$$E \equiv \int T^{00} d^2 x = \frac{1}{2f} \int \partial_i \mathbf{n} \cdot \partial^i \mathbf{n} \ d^2 x \tag{2.1.8}$$

Finiteness of the energy yields the following boundary condition from the above equation

$$r \sqrt{(\partial_i n \partial_i n)} \to 0 \quad \text{as } r \to \infty \qquad (2.1.9)$$

or

$$\lim_{r \to \infty} \mathbf{n} = \mathbf{n}^0 \qquad (2.1.10)$$

where r is the radial coordinate, $\mathbf{n}^0$ is a constant unit vector in the internal space and $\mathbf{n}$ should take same $\mathbf{n}^0$ value in all directions . Otherwise $\mathbf{n}$ will depend on the angular coordinate θ, even at $r = \infty$ and $\frac{1}{r} \frac{\partial \mathbf{n}}{\partial \theta}$ component of $\partial_i \mathbf{n}$ will not satisfy Eq. (2. 1. 9).

Eq. (2. 1. 10) implies that all points at $r = \infty$ should be identified i. e. mapped onto same point in the field space. Thus the physical coordinate plane $R^{(2)}$ is compactified into a unit 2-sphere (denoted as. $S_{phy}^{(2)}$). The circle with $r = \infty$ is mapped into north pole of the sphere $S_{phy}^{(2)}$. We have seen before that the fields n^a take values on $S_{fld}^{(2)}$ (due to Eq. (2. 1. 1)) . Obviously the fields n^a map the physical space, $S_{phy}^{(2)}$ into the field space, $S_{fld}^{(2)}$. Each point in $S_{fld}^{(2)}$. correspond to a value of the field $\mathbf{n}$. Let us consider three examples. (1). mapping of entire surface of $S_{phy}^{(2)}$ to a point in $S_{fld}^{(2)}$. (2) mapping of entire surface of $S_{phy}^{(2)}$ to an open surface on $S_{fld}^{(2)}$. This open surface can continuously be deformed to a point. (3). the surface of $S_{phy}^{(2)}$ can be wrapped over $S_{fld}^{(2)}$, ones, twice, thrice, . . .Each of these mappings can neither be deformed to a point nor can one into another. The number of times $S_{phy}^{(2)}$ wraps $S_{fld}^{(2)}$ is called the winding number Q. Examples (1),(2) have winding number Q = 0. These are trivial mappings. Example (3) has winding number Q = 1, 2, 3, . . . depending on the number of wrappings. These are non-trivial mappings. Clearly these winding numbers form a group under addition. The corresponding mappings, being isomorphic to winding numbers, form a group called the second homotopy group[23] of $S_{fld}^{(2)}$ represented by $\Pi_2(S_{fld}^{(2)}) = Z$. where Z is the additive group of integers.

The winding number Q is clearly equal to $\dfrac{\int dS_{fld}^{(2)}}{\int dS_{phy}^{(2)}}$ and $\int dS_{phy}^{(2)} = 4\pi$.

Therefore

$$Q = \frac{1}{4\pi} \int dS_{fld}^{(2)} = \frac{1}{4\pi} \int dS_{fld}^{(2)^a} n^a \tag{2.1.11}$$

where $dS_{fld}^{(2)^a}$ is surface area element vector on $S_{fld}^{(2)}$ in n^a direction, given as

$$dS_{fld}^{(2)^a} = dn^b \wedge dn^c = \mathcal{E}^{abc} dn^b dn^c = (\tfrac{1}{2}\mathcal{E}^{abc} \frac{\partial n^b}{\partial x^i} \frac{\partial n^c}{\partial x^j} dx^i dx^j \mathcal{E}^{ij}) \tag{2.1.12}$$

where " $\wedge$ " represents wedge product between two one forms dn^b and dn^c

Thus Q becomes

$$Q = \frac{1}{8\pi} \int d^2x (\mathcal{E}^{ij} \mathcal{E}^{abc} n^a \frac{\partial n^b}{\partial x^i} \frac{\partial n^c}{\partial x^j}) \tag{2.1.13}$$

We can identify the integrand as charge density j^0, of some conserved current j^μ, given as

$$j^\mu = \frac{1}{8\pi} \mathcal{E}^{\mu\nu\lambda} \mathcal{E}^{abc} n^a \frac{\partial n^b}{\partial x^\nu} \frac{\partial n^c}{\partial x^\lambda} = \frac{1}{8\pi} \mathcal{E}^{\mu\nu\lambda} \mathcal{E}^{abc} n^a \partial_\nu n^b \partial_\lambda n^c \tag{2.1.14a}$$

We can easily see that the current j^μ is conserved. i. e $\partial_\mu j^\mu = 0$, by using the anti-symmetry property of $\mathcal{E}^{\mu\nu\lambda}$ and remembering that the vector triple product of three vectors, all lying in the plane perpendicular to **n** is zero (since there are only two independent directions in this plane). Notice that we did not use the equations of motion of the fields to prove the conservation of the current j^μ. A current which is conserved without invoking the dynamics of the field is called "topological current". The existence of the conserved current j^μ defines a gauge potential $B_\mu(x,t)$ through the curl equation

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$$j^{\mu} = \mathcal{E}^{\mu\nu\lambda} \partial_{\nu} B_{\lambda} \; . \tag{2.1.14b}$$

Let us start with an inequality to see the physics that Q gives.

$$\int d^2x \Big[\big\{ \partial_i n \pm \mathcal{E}^{ij}(n \times \partial_j n) \big\} \cdot \big\{ \partial_i n \pm \mathcal{E}^{ik}(n \times \partial_k n) \big\} \Big] \geq 0. \tag{2.1.15}$$

The above inequality follows from the fact that the integrand is positive definite. The dot represents the scalar product. We can simplify the inequality as below by using the facts $n.n = 1$ and the derivative of a vector is always perpendicular to it $(n \cdot \partial_j n) = 0$

$$\int d^2x (\partial_i n) \cdot (\partial_i n) \geq \pm \int d^2x \, \mathcal{E}^{ij} n \cdot (\partial_i n \times \partial_j n) \tag{2.1.16}$$

We can easily identify the above equation with(because of (2. 1. 8) and (2. 1. 13))

$$E \geq \frac{4\pi}{f} |Q|. \tag{2.1.17}$$

Thus we see that the energy E is bounded by the topological charge (Q) through equation 2. 1. 17 The condition of minimum energy ($E = \frac{4\pi}{f} |Q|$) implies

$$\partial_i n = \pm \, \mathcal{E}^{ij}(n \times \partial_j n). \tag{2.1.18}$$

Differentiating Eq. (2. 1. 18) with ∂^i ,and using the anti symmetry property of $\mathcal{E}^{ij}$, $(n \cdot \partial_j n) = 0$ and Eq. (2.1.18).we obtain

$$- \partial_i \partial^i n = \pm \, n(\partial^k n . \partial_k n) \tag{2.1.19}$$

let us take $(n \cdot \partial_j n) = 0$ and differentiate it with ∂^j

$$\partial^j \mathbf{n} \cdot \partial_j \mathbf{n} = - (\mathbf{n} \cdot \partial_i \partial^i \mathbf{n}) \qquad\qquad (2.1.20)$$

using Eq. (2. 1. 20) in Eq. (2. 1. 19) we get back the static equations of motion

$$\partial_i \partial^i \mathbf{n} - (\mathbf{n} \cdot \partial_j \partial^j \mathbf{n}) \, \mathbf{n} = 0. \qquad\qquad (2.1.6)$$

Therefore the fields satisfying minimum energy condition are solutions of static equations of motion Eq. (2. 1. 6) but converse may not be true. Since our interest is to find the topological solutions, we solve Eq. (2. 1. 18) by stereographically projecting $S^{(2)}_{fld}$ onto a plane parametrised by (w^1, w^2). The stereographic projection takes the surface of three dimensional sphere $(S^{(2)}_{fld})$ onto a two dimensional plane (w^1, w^2).

The stereographic projection is given by

$$w^1 = \frac{(2n^1)}{(1 - n^3)} \, , \qquad\qquad (2.1.21a)$$

$$w^2 = \frac{(2n^2)}{(1 - n^3)} \, , \qquad\qquad (2.1.21b) \qquad\qquad (2.1.21)$$

where $w = w^1 + i\, w^2 = \dfrac{2n}{(1 - n^3)} \, ,$

and $n = n^1 + i\, n^2$ $\qquad\qquad (2.1.21c)$

and now the field space is a two dimensional complex plane. We can write Eq. (2. 1. 18) in terms of n as

$$- \partial_i n = \pm \, \varepsilon_{ij} \, i \, (n \overset{\leftrightarrow}{\partial_j} n^3) \, , \qquad\qquad (2.1.22)$$

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where $(n \overset{\leftrightarrow}{\partial_j} n^3) = n(\partial_j n^3) - (\partial_j n) n^3$.

We will find $\partial_1 w$ and $\partial_2 w$ by using e. qs(2. 1. 21,a, b,) and (2. 1. 22) as Cauchy-Riemann condition

$$- \partial_1 w = \pm i \, \partial_2 w. \qquad (2.1.23)$$

Any analytic function of z is a solution of this equation , where $z = x^1 + ix^2$.

Consider a function of the form

$$w = [(z - z_0)]^q = \rho^q e^{iq\phi} \qquad (2.1.24)$$

where q is a positive number ρ is the radius vector at z_0 and ϕ is polar angle of w-plane. Since w is analytic in z it is clearly a solution of Eq. (2. 1. 23). We can easily see that q is equal to the winding number Q

Let us write Eq. (2. 1. 13) in complex plane by using (2. 1. 21), $(z - z_0) = \rho \, e^{i\phi}$ and $d^2x = \rho \, d\rho \, d\phi$

$$Q = \frac{1}{4\pi} \int \rho \, d\rho \, d\phi \frac{n^2 \, \rho^{2(q-1)}}{\left(1 + \frac{1}{4} \rho^{2q}\right)^2} \quad . \qquad (2.1.25)$$

Upon integration we get $Q = q$. Thus q is the winding number . The solutions of Eq. (2. 1. 22) represent the static topological solitons , characterized by the winding number q, with static energy

$E = \frac{4\pi}{f} |q|$. and z_0 is the location of the soliton. Having obtained topological solitons, we now turn our attention to their spins in the next section.

2. 2 The Soliton spin

In this section we define angular momentum generator in (2+1) dimensions and calculate the spin of the solitons of the non-linear sigma model.

In (2+1) dimensions there is only one angular momentum generator corresponding to the rotation in the two space dimensions. The angular momentum generator is ambiguous upto the addition of an arbitrary constant. Since there is only one angular momentum generator, we have no non trivial Lie algebra to resolve this ambiguity. We use the Poincare algebra to uniquely define the angular momentum generator (M)[12,25].

The angular momentum generator (M) is defined as

$$M = \frac{1}{2} i \, \varepsilon_{ij} \left[M^{0i}, M^{0j} \right] \qquad (2.2.1)$$

where i, j take values 1, 2 and the boosts M^{0i} s are defined by the Poincare generators

$$M^{\mu\nu} \equiv \int d^2x \left[x^\mu T^{0\nu} - x^\nu T^{0\mu} \right] \qquad (2.2.2a)$$

$$P^\mu \equiv \int d^2x \, T^{0\mu} \, . \qquad (2.2.2b)$$

$T^{\mu\nu}$ is the energy momentum tensor and the Poincare algebra satisfied by these generators is[12,25]

$$\left[P^\mu, P^\nu \right] = 0 \qquad (2.2.3a)$$

$$i \left[P^\mu, M^{\nu\lambda} \right] = g^{\mu\lambda} P^\nu - g^{\mu\nu} P^\lambda \qquad (2.2.3b)$$

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$$i\left[M^{\mu\nu},M^{\kappa\lambda}\right] = g^{\nu\lambda} M^{\mu\kappa} - g^{\mu\lambda} M^{\nu\kappa} + g^{\mu\kappa} M^{\nu\lambda} - g^{\nu\kappa} M^{\mu\lambda} \qquad (2.2.3c)$$

where μ, ν, λ, κ take values 0, 1, 2 and $g^{\mu\nu}$ is the metric tensor. We use (2. 2. 3c) in (2. 2. 1) to get expression for the angular momentum as

$$M = \tfrac{1}{2}\, i\, \varepsilon_{ij}\left[M^{0i},M^{0j}\right] = \tfrac{1}{2}\, \varepsilon_{ij} M^{ij} = \varepsilon_{ij} \int d^2x\, x^i\, T^{oj}$$

$$M = \varepsilon_{ij} \int d^2x\, x^i\, T^{oj} \qquad (2.2.4)$$

Thus the angular momentum generator(M) is uniquely defined in (2+1) dimensions and is clearly a pseudo scalar.

We now proceed to calculate the spin of the soliton of the non-linear sigma model. The canonical momentum (Π^a)conjugate to the field, n^a from Eq. (2. 1. 3)is

$$\Pi^a \equiv \frac{\delta L}{\delta(\partial_0 n^a)} = \frac{1}{f}\, \partial_0 n^a \qquad (2.2.5)$$

The expression for angular momentum (M_0) of the soliton can be written by using equations (2. 1. 7),(2. 2. 4) and (2. 2. 5) as

$$M_0 = \varepsilon_{ij} \int d^2x\, x^i\, \Pi^a\, (\partial^j n^a)\,. \qquad (2.2.6)$$

M_0 is a combination of both orbital angular momentum and spin. The spin can be extracted from M_0 by going over to the rest frame of the soliton . From Eq. (2. 2. 6), we see that the spin of the soliton is zero.

Fractional spin

Solitons of the model acquire fractional spin if we add a term, called Hopf invariant to the action Eq. (2. 1. 2). Before discussing fractional spin, let us understand a topological concept, connectivity of the configuration space (Q). This is most easily done by considering a simple example of a two component real scalar fields(f) in (1+1) dimensions with $f^2 = 1$.

Configuration space and its connectivity : Configuration space is a function space whose points are functions f representing all possible fields .

Let Q be set of all mappings f that map $S^{(1)}_{phy}$ to $S^{(1)}_{fld}$. Q is divided into a set of homotopy classes $Q_0, Q_1, Q_2 \ldots \ldots Q_i \ldots \ldots$ .Q_i is the set of all mappings with winding number "i".This set of homotopy classes denoted by $\Pi_1(S^{(1)}_{fld})$ is isomorphic to Z, and therefore has infinite number of homotopy classes.The collection of mappings f_i with winding number i belong to Q_i. To determine the connectivity of Q, take any two arbitrary points in Q_i and see if there is any one parameter function which interpolates between these two points this is clearly given by $\Pi_1(Q_i)$. This implies that Q_i is further divided into different equivalence classes Q_{io}, Q_{i1}, Q_{i2}, · $\ldots \ldots Q_{i\alpha}$. $\Pi_1(Q_i)$ is isomorphic to $\Pi_1(Q_j)$ for all i, j . The mapping $f_{i\alpha}$ with winding number i and with α number of kinks (see fig. 1)belong to $Q_{i\alpha}$. Any two mappings $f_{i\alpha}$, $f_{i\beta}$ cannot be deformed into each other if $\alpha \neq \beta$.If $\Pi_1(Q_i) = z_1$, then only one class of mappings exist in each Q_i and the configuration space is said to be simply connected. If $\Pi_1(Q_i) = z_2$, then two different equivalent classes of mappings exist in Q_i and the configuration space Q is said to be doubly connected . If $\Pi_1(Q_i) = Z$, then the number of distinct classes of mappings in each Q_i is infinite and the configuration space Q is multiply connected.
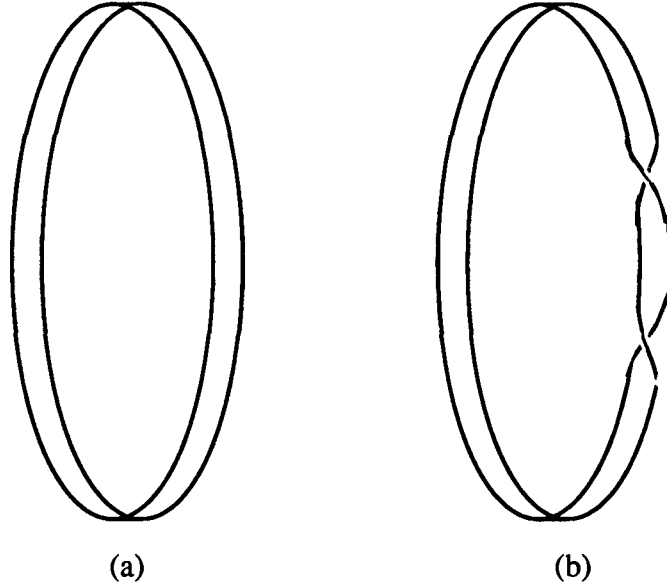
Fig 1. A simple example of a kink is a 2π twist in a rubber band. (a) A rubber band without twist. (b) A rubber band with a 2π twist, which cannot be undone of by any continuous process.

We determine $\Pi_1(Q_i)$ in the following way. We define a functional on Q_i and evaluate it along a closed path in Q_i and observe the change in the functional at the end of the operation. A closed path in Q_i is defined by a function $\phi(t):[o,2\pi] = S^{(1)} \rightarrow Q_i$ where $o \leq t \leq 2\pi$ such that $\phi(0) = f_{i\alpha} = \phi(2\pi)$, clearly the set of all these mappings is isomorphic to $\Pi_1(Q_i)$ The function $f \equiv f(x,t)$ where $x \in S^{(1)}_{phy}$ and $t \in [o,2\pi] = S^{(1)}$. Therefore $f(x): S^{(1)}_{phy} \rightarrow S^{(1)}_{fld}$ becomes $f(x,t): S^{(2)}_{phy} \rightarrow S^{(1)}_{fld}$. The homotopy group for these mappings is $\Pi_2(S^{(1)})$. It has been proved[5] that $\Pi_1(Q_i)$ is isomorphic to $\Pi_{1+1}(S^{(1)})$ and $\Pi_2(S^{(1)}) = 1$. Hence in our example the configuration space is simply connected. If, in general $\Pi_n(S^{(m)})$ is the homotopy of the functions that map $S^{(n)}$ onto $S^{(m)}_{fld}$. then the connectivity of the field configuration space $S^{(m)}_{fld}$ ($\equiv Q$)is determined by $\Pi_{(n+1)}(S^{(m)}_{fld})$. The material we discussed here is found in references 3, 4, and 5.

We now turn our attention to the connectivity of the $S_{fld}^{(2)}$. We know from the previous section that $n(x)$ map the physical space $S_{phy}^{(2)}$ onto $S_{fld}^{(2)}$

$$n(x)\big|_{t=constant} : S_{phy}^{(2)} \rightarrow S_{fld}^{(2)} \tag{2.2.7}$$

which gives the static field configuration of $S_{fld}^{(2)}$ at any instant of time and due to the above argument, we have

$$n(x,t): S_{phy}^{(3)} \rightarrow S_{fld}^{(2)} \tag{2.2.8}$$

where $x \in S_{phy}^{(2)}$ and $t \in [0,2\pi] = S^{(1)}$. These mappings $n(x,t)$ yield both static configuration and its evolution with time. These mappings are called Hopf invariant mappings in mathematics, and form infinitely different homotopy classes $\Pi_3(S^{(2)}) = Z$. Therefore $\Pi_1(Q_i) \approx \Pi_3(S_{fld}^{(2)}) = Z$, the $S_{fld}^{(2)}$ is multiply connected . The connectivity of the $S_{fld}^{(2)}$ hints that we can add a term that describes Hopf invariant mapping to our model.

The action (Eq. 2. 1. 2) with Hopf invariant term is

$$S[n] = \frac{1}{2f}\int d^3x \partial_\mu n \cdot \partial^\mu n \quad + \theta H \tag{2.2.9}$$

where θ is an arbitrary real parameter and H is Hopf invariant of the mappings $n(x,t)$. This is given by

$$H = -\int_{S_{phy}^{(3)}} A \wedge dA. \tag{2.2.10}$$

Here "A" is a 1-form and "dA" is a closed exact 2-form, both belong to $S_{phy}^{(3)}$. "dA" is a particular 2-form equal to the induced mapping $(n^*(x,t))$[24] of any 2-form belong to $S_{fld}^{(2)}$.

$$A = A_\mu dx^\mu \text{ and } dA = \frac{\partial A_\lambda}{\partial x_\nu} dx^\nu \wedge dx^\lambda$$

$$A \wedge dA = A_\mu dx^\mu \wedge \partial_\nu A_\lambda \; dx^\nu \wedge dx^\lambda = A_\mu \partial_\nu A_\lambda \; dx^\mu \wedge dx^\nu \wedge dx^\lambda$$

$$A \wedge dA = A_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} d^3x = \frac{1}{2} A_\mu j^\mu d^3 x \qquad (2.2.11)$$

Substituting in Eq. (2. 2. 10) we get

$$H = - \int_{S^{(3)}_{phy}} A_\mu \partial_\nu A_\lambda \varepsilon^{\mu\nu\lambda} d^3x = \frac{1}{2} \int_{S^{(3)}_{phy}} A_\mu j^\mu d^3 x \; . \qquad (2.2.12)$$

A_μ is gauge potential defined through the equation (2. 1. 14b) and depends on **n** fields. A_μ is determined in terms of **n** fields by equations (2. 1. 14a) and (2. 1. 14b). The Hopf invariant (H) is related to a topological quantity, called the linking number. For any mapping $\mathbf{n}(x,t)$: $S^{(3)}_{phy} \rightarrow S^{(2)}_{fld}$ consider the induced mapping $\mathbf{n}^*(x,t)$ of any two points of $S^{(2)}_{fld}$ into $S^{(3)}_{phy}$.

These two points will be two curves in $S^{(3)}_{phy}$. The Hopf invariant is the number of times these two curves link . The value of H is equal to one [2] for any $\mathbf{n}(x,t)$ and therefore these two curves link ones. We can determine the spin of the soliton using the linking number. Let us consider two vacuum - to - vacuum processes. In one we create a soliton and anti soliton pair at some time and pull them apart. We allow to annihilate after suitable period of time. In other we do the same except that we rotate the soliton by 2π. Both amplitudes differ by a phase factor $e^{i\theta}$. This gives the soliton spin as $\dfrac{\theta}{2\pi}$. Were the soliton not rotated the mapping would be homotopically trivial, we do not find any phase factor. If we observe the trajectories of the soliton and anti-soliton, in the first process, we find that both the trajectories do not link. In the second process the trajectories of the soliton link the trajectory of the anti-soliton by ones due to 2π rotation of the soliton.

The Hopf term plays the role of an effective long range interaction among the **n** fields [11]. This term is a total divergence term [9] and therefore does not contribute to both the equations of motion

(Eq. 2. 1. 5) and the energy momentum tensor $T^{\mu\nu}$ (Eq. 2. 1. 7).

We explicitly find out the angular momentum operator and identify the fractional spin operator from the obtained expression. We follow the discussion given in reference 12. The action given by Eq. (2. 2. 9) is invariant under the gauge transformations $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$, where $\Lambda(x,t)$ is a scalar potential. We are thus free to choose the radiation gauge $\partial_i A_i = 0$.

Therefore A_i can be written as

$$A_i(x) = \varepsilon_{ij} \partial_j \psi(x) \tag{2.2.13}$$

where ψ is a scalar potential. We substitute (2. 2. 13) in (2. 1. 14b) to get

$$j^0 = -\nabla^2 \psi \tag{2.2.14}$$

and solving for Greens function

$$\nabla^2 G(x) = -\delta(x) \tag{2.2.15}$$

we get

$$G(x) = -\frac{1}{4\pi} \ln(x^2) + \text{constant} \tag{2.2.16}$$

Therefore

$$\psi(x) = \int d^2x' \, G(x - x') j_0(x') \tag{2.2.17}$$

and

$$A_i(x) = \varepsilon_{ij} \partial_j \int d^2x' \, G(x - x') j_0(x') \tag{2.2.18}$$

We find the canonical momentum (Π^a) conjugate to the fields n^a from Eq. (2. 2. 9).

$$\Pi^a \equiv \frac{\delta L}{\delta(\partial_0 n^a)} = \frac{1}{f} \partial_0 n^a + \frac{\theta}{2\pi} \varepsilon_{ij} \varepsilon^{abc} A_i \, n^b \, \partial_j n^c \tag{2.2.19}$$

We evaluate the angular momentum generator (M) by using equations (2. 1. 7),(2. 2. 4) and (2. 2. 19) as

$$M_{Soliton} = \mathcal{E}_{ij} \int d^2x \, x^i \, \Pi^a \, (\partial^j n^a) - \frac{\theta}{2\pi} \int d^2x \mathcal{E}_{ij} \mathcal{E}_{kl} \mathcal{E}^{abc} x_i (A_k n^b \partial_l n^c) \partial_j n^a \qquad (2.2.20)$$

We simplify the θ term by using equations (2. 1. 1.), (2. 1. 14a) and (2. 2. 18) as

$$I = \frac{\theta}{\pi} \int d^2x \, d^2x \,' \frac{x_i(x - x')^i}{|x - x'|^2} \qquad j^0 (x) \quad j^0 (x') \; . \qquad (2.2.21)$$

Symmetrizing and using equation(2. 1. 13)

$$I = \frac{\theta}{2\pi} Q^2$$

and

$$M = \mathcal{E}_{ij} \int d^2x \, x^i \, \Pi^a \, (\partial^j n^a) + \frac{\theta}{2\pi} Q^2. \qquad (2.2.22)$$

We go to the soliton rest frame to observe its spin. In this frame the first term in (2. 2. 22) vanishes because $\Pi^a = 0$ and hence the spin of the soliton is $\frac{\theta}{2\pi}$ for $Q = 1$. Thus the soliton acquires fractional spin due to the Hopf term.

CHAPTER 3

FRACTIONAL SPIN OF THE SOLITONS COUPLED TO AN ABELIAN GAUGE FIELD

In the preceding chapter, we discussed how the solitons acquire fractional spin when we add the Hopf term to the non linear sigma model. In nature real systems are the interacting ones. We in this chapter, consider a physical model in which the solitons of the sigma model interact with a gauge field. The gauge field we consider is a (2+1) dimensional massive one and it attains mass at classical level through the term $\mathcal{E}^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$, which is quadratic in gauge potential A_μ [14,15,26.]

This term is called the Chern-Simons density. The C. S. term is local U(1) gauge invariant upto a total divergence term as opposed to the quadratic term $A_\mu A^\mu$ which is not gauge invariant. The addition of this term is not unusual. It has been shown that the above quadratic term can be derived from a local U(1) gauge invariant action [27].

$$S[A,\psi] = \int d^3x \left[\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + i\,\overline{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi \right]$$

where ψ are fermion fields, $F_{\mu\nu}$ is the usual field strength tensor given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_ν is gauge potential and γ^μ are Dirac matrices. The gauge field effective action S_{eff} for the above action S is obtained in path integral formulation by integrating out the ψ fields. $S_{eff} = i \ln$. det. { $\gamma^\mu(\partial_\mu + A_\mu)$ }. The ultraviolet divergences are regulated in a gauge invariant way by introducing a massive, Pauli-Villars regulator. The regulated effective action S_{eff}^R contains Chern-Simons term.

$$S_{eff}^R = S_{eff} \text{ (finite,m=0)} \pm \frac{1}{8\pi} \int d^3x\, \mathcal{E}^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \qquad (3.1.1.)$$

The second term is a topological quantity, since it is invariant under local variations of the potentials A_μ. The integrand of the second term is the so called Chern-Simons term. In the first

section of this chapter, we discuss a (2+1) dimensional massive abelian gauge field and in the second section we couple the non linear sigma model to the gauge field and then recover the Hopf term that is responsible for the fractional spin.

3. 1. (2+1) dimensional massive abelian gauge field.

We generate mass to an abelian gauge field $F_{\mu\nu}$ by adding the topological term (Eq. 3. 1. 1), which is local U(1) gauge invariant upto a total divergence, to the kinetic action of the gauge field [14,15]. The energy momentum tensor is obtained by varying action with respect to the metric tensor $g^{\mu\nu}(x)$. The Chern-Simons term, being independent of the metric tensor $g^{\mu\nu}$, does not contribute to the energy momentum tensor but it does contribute to the equation of motion. This is another way of seeing that the C. S. term is topological. It is independent of the metric which describes the local properties of the manifold M.

The action of an abelian gauge field with a mass term is given by

$$S_g = -\frac{1}{4}\int d^3x \, F^{\mu\nu}F_{\mu\nu} - \frac{m}{2}\int d^3x \, \mathcal{E}^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \qquad (3.1.2)$$

where the first term is the kinetic term and $F^{\mu\nu}$ is the usual field strength tensor given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The gauge potential A^ν has dimensions $[L^{-1/2}]$ as can be seen from the kinetic term in (3. 1. 2). The coefficient m in the second term has dimensions $[L^{-1}]$. The second term is a topological term we discussed in equation(3. 1. 1.).

Let us now study the transformation properties of the action S_g under the U(1) local gauge transformations $A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\Lambda$, where $\Lambda(x)$ is a scalar potential. We know that the field strength $F^{\mu\nu}$ is invariant under these transformations i. e. $F'^{\mu\nu} = F^{\mu\nu}$ and hence the kinetic term is invariant. We can write the integrand of the second term $\mathcal{E}^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$ as $\frac{1}{2}\mathcal{E}^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$ by using the identity $\frac{1}{2}\mathcal{E}^{\mu\nu\lambda}F_{\nu\lambda} = \mathcal{E}^{\mu\nu\lambda}\partial_\nu A_\lambda$.

$$S'_g = -\frac{1}{4}\int d^3x \, F'^{\mu\nu}F'_{\mu\nu} - \frac{m}{4}\int d^3x \, \mathcal{E}^{\mu\nu\lambda}A'_\mu \, F'_{\nu\lambda}$$

$$= -\frac{1}{4}\int d^3x \, F^{\mu\nu}F_{\mu\nu} - \frac{m}{4}\int d^3x \, \mathcal{E}^{\mu\nu\lambda}\{A_\mu + \frac{1}{e}(\partial_\mu\Lambda)\}F_{\nu\lambda}$$

$$S'_g = S_g - \frac{m}{4e}\int d^3x \, \mathcal{E}^{\mu\nu\lambda}(\partial_\mu\Lambda)F_{\nu\lambda} \qquad (3.1.3)$$

Using the relation .

$$\mathcal{E}^{\mu\nu\lambda}\partial_\mu\{\Lambda(x)F_{\nu\lambda}\} = \mathcal{E}^{\mu\nu\lambda}(\partial_\mu\Lambda)F_{\nu\lambda} + \mathcal{E}^{\mu\nu\lambda}\Lambda(\partial_\mu F_{\nu\lambda}) \ . \qquad (3.1.4)$$

We multiply the Bianchi identity $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ by $\mathcal{E}^{\mu\nu\lambda}$ to get $\mathcal{E}^{\mu\nu\lambda}(\partial_\mu F_{\nu\lambda}) = 0$. We write equation (3. 1. 4)by using this result as

$$\mathcal{E}^{\mu\nu\lambda}\partial_\mu\{\Lambda(x)F_{\nu\lambda}\} = \mathcal{E}^{\mu\nu\lambda}(\partial_\mu\Lambda)F_{\nu\lambda} \qquad (3.1.5)$$

We write equation (3. 1. 3) by using equation(3. 1. 5) and denoting $\frac{m}{4e}\mathcal{E}^{\mu\nu\lambda}\Lambda F_{\nu\lambda} = \tilde{j}^\mu$ as

$$S'_g = S_g + \int d^3x(\partial_\mu \tilde{j}^\mu) \ . \qquad (3.1.6)$$

Thus the action changes by a total divergence term under local U(1) gauge transformations. The contribution of the total divergence term depends on the behaviour of $\tilde{j}^\mu$ at $|x| = \infty$. Since it vanishes as $|x| \to \infty$, the total divergence term is not topological and hence it does not contribute to the action. Thus the action (Eq. 3. 1. 2.) is invariant under local U(1) gauge transformations and m can take arbitrary values. We show below that m is indeed mass of the gauge field $F^{\mu\nu}$[14].

Variation of equation (3. 1. 2) with respect to A^μ yields the equation of motion as

$$\partial_\mu F^{\mu\nu} - \frac{m}{2} \mathcal{E}^{\nu\mu\lambda} F_{\mu\lambda} = 0 \,. \tag{3.1.7}$$

We can easily see that the equation of motion is also local gauge invariant. The identity $\mathcal{E}^{\mu\nu\lambda}(\partial_\mu F_{\nu\lambda}) = 0$ defines the conserved current j^μ, where

$$j^\mu = \frac{1}{2} \mathcal{E}^{\nu\mu\lambda} F_{\mu\lambda} = \mathcal{E}^{\mu\nu\lambda} \partial_\nu A_\lambda \,. \tag{3.1.8}$$

We rewrite the equation of motion (3. 1. 7) by using equation (3. 1. 8) as

$$\mathcal{E}^{\nu\mu\alpha} \partial_\mu j_\alpha + m\, j^\nu = 0. \tag{3.1.9}$$

Multiplying equation (3. 1. 9) by $(- m\, g_{\sigma\nu} + \mathcal{E}_{\sigma\rho\nu}\, \partial^\rho)$ and then making use of equations (3. 1. 8) and the fact $\partial_\mu j^\mu = 0$, we find

$$(\partial^\mu \partial_\mu + m^2)F^{\alpha\beta} = 0 \,. \tag{3.1.10}$$

This is the equation of motion for a gauge field with mass m. We find the energy momentum tensor $T_g^{\mu\nu}$ by varying the action given by Eq. (3. 1. 2) with respect to $g^{\mu\nu}(x)$ as

$$T_g^{\mu\nu} = - F^{\mu\nu} F^\nu_{.\lambda} + g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \tag{3.1.11.}$$

We note that the mass term , being independent of $g^{\mu\nu}(x)$, does not contribute to the energy - momentum tensor . Hence the name topological mass.
We find the angular momentum (M_g) by using equations (2. 2. 4.) and (3. 1. 11.) (for future use) as

$$M_g = \varepsilon_{ij} \int d^2x \; x^i \; T_g^{0j} = -\int d^2x \; x^i \; E_i \; B \qquad (3.1.12)$$

Where $E_i = F_{i0}$ and $B = \varepsilon^{ij}\partial_i A_j$.

In this section we proved the following . The topological term produces mass to the gauge field $F^{\mu\nu}$. It could be used as an alternative to the Higgs mechanism to produce mass. Thus mass for the gauge field is produced without symmetry breaking, where as in the Higgs mechanism there is spontaneous breakdown of symmetry. The action S_g is still local U(1) gauge invariant and the parameter m takes arbitrary values. The topological term contributes to the equations of motion but does not contribute to the energy momentum tensor. These conclusions remain the same even upon the quantization[15] . In the case of non abelian gauge theories the parameter m is quantized. The Chern-Simons term under the gauge transformations (U) changes by the winding number $w = m \int d^3x \varepsilon^{\alpha\beta\gamma}$ trace $\left[U^{-1}(\partial_\alpha U) \; U^{-1}(\partial_\beta U) \; U^{-1}(\partial_\gamma U) \right]$ of the gauge transformations (This term is zero in the abelian case.) in addition to the $\Lambda(x)$ dependent surface term [15]. Thus in non abelian case the action is not gauge invariant and the parameter m is restricted to take integral values in order to have the state functional single valued under the gauge transformations . Thus m in non abelian case is quantized. In the next section we discuss the fractional spin of the solitons of the non linear sigma model in the presence of the abelian gauge field.

3. 2 Recovery of the Hopf term.

The action of the non linear sigma model coupled to the massive abelian gauge field is given by

$$S = S_g + S_\sigma + S_I \qquad (3.2.1)$$

Where S_g is the action of the massive abelian gauge field (we discussed in the section .3.1).

$$S_g = \int d^3x \mathcal{L}_g = \int d^3x \left[-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{m}{4} \varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} \right] \qquad (3.1.1)$$

S_σ is the action of the non linear sigma model (we discussed in section. 2. 1.).

$$S_\sigma = \int d^3x \mathcal{L}_\sigma = \frac{1}{2f}\int d^3x \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} \quad . \tag{2.1.4}$$

and S_I is the interaction action for the fields $\mathbf{n}$ and A_μ

$$S_I = e\int d^3x \mathcal{L}_I = e\int d^3x \, A_\mu \, j^\mu, \tag{3.2.2}$$

e is a parameter that measures the strength of the interaction. The dimensions of e is $[L^{-1/2}]$. j^μ is the topological current given (we discussed in section. 2. 1) .

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon^{abc} \, n^a \, \partial_\nu n^b \partial_\lambda n^c \, . \tag{2.1.14a}$$

The interaction action S_I is invariant under the local U(1) gauge transformations as we can see from below .

$$S'_I = e\int d^3x \, A'_\mu \, j^\mu \quad = e\int d^3x \{A_\mu + \frac{1}{e}(\partial_\mu \Lambda)\} \, j^\mu \quad = S_I + \int d^3x \, (\partial_\mu \Lambda) \, j^\mu$$

$$= S_I - \int d^3x \, \Lambda \, (\partial_\mu j^\mu) \tag{3.2.2}$$

We know that the topological current is conserved. i. e. $\partial_\mu j^\mu = 0$.

Therefore

$$S'_I = S_I \, . \tag{3.2.3}$$

The S_I is different from the interaction action which couples A_μ to the Noether's current. It does not require the equations of motion to prove its invariance under the gauge transformations as we saw in Eq. (3. 2. 3.). Thus the parameter e is not to be thought of as electric charge. It has been proved [35] that the coupling of A_μ to the n fields is analogous to that of charged particle to an external gauge field. Recent work on anyonic super conductors [40] makes use of such abelian gauge field.

We now turn our attention to recover the Hopf term. To recover the Hopf term we integrate out the gauge fields A_μ, which are quadratic, in the path integral formalism and get the effective action S_{eff} as [23].

$$S_{eff} = \int d^3x \left[\frac{1}{2f} (\partial_\mu n^a)^2 - \theta \, j^\mu \frac{1}{\left(\frac{1}{m} D_1 + D_2 \right)_{\mu\nu}} j^\nu \right] \qquad (3.2.4)$$

where $\theta = \frac{e^2}{2m}$, $D_1^{\mu\nu} = \left(g^{\mu\nu} \partial^\alpha \partial_\alpha - \partial^\mu \partial^\nu \right)$ which arise from $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

and $D_2^{\mu\nu} = \varepsilon^{\mu\nu\lambda} \partial_\lambda$ which arise from $\frac{m}{4} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}$. If the kinetic term $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is absent in the action S (Eq. 3. 2. 1), the parameters e and m can be taken to be dimensionless. The θ term of the effective action Eq. (3. 2. 4), will not contain $D_1^{\mu\nu}$. We can write the θ term as

$$- \theta \int d^3x \, j^\mu \frac{1}{\left(D_2 \right)_{\mu\nu}} j^\nu \qquad (3.2.5)$$

We find after simple algebra and by using (2. 1. 14b)

$$- \theta \int d^3x \, j^\mu \frac{1}{\left(D_2 \right)_{\mu\nu}} j^\nu = - \theta \int d^3x \, \varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu B_\lambda \, . \qquad (3.2.6)$$

If the kinetic term is present both e and m pickup dimensions $[L^{-1/2}]$ and $[L^{-1}]$. θ is still dimensionless. We can recover the Hopf term when $m = \infty$. Thus our action (Eq. 3. 2. 1) has Hopf term in disguise. We can expect the fractional spin to the soliton of the model described by Eq. (3. 2. 1.).

If the mass term is absent

$$S_{eff} = \int d^3x \left[\frac{1}{2f} (\partial_\mu n^a)^2 - \frac{e^2}{2} \int d^3x \, j^\mu \frac{1}{(D_1)_{\mu\nu}} j^\nu \right] \qquad (3.2.7)$$

The second term cannot be written as Hopf term. Hence the fractional spin of the soliton is zero if $m = 0$.

We now briefly discuss the procedure followed by Karabali and Murthy[23]. They expanded the second term in Eq. (3. 2. 4) in local operators in powers of $\frac{1}{m}$ upto an order $1/m^5$ and used $Q = 1$ collective coordinate ansatz with the collective coordinate $\alpha(t)$ to obtain the effective action in higher order time derivatives of $\alpha(t)$ with complicated coefficients. The coefficients being integrals of functions of m, θ, f, g, g', g". With the result the Lagrangian of the system contain higher time derivatives in $\alpha(t)$ and is constrained to four second class constraints. Thus forcing them to use the Dirac's method of canonical quantization . They found by using these tools that the fractional spin of the soliton is independent of the mass parameter m for large m (ml > 1 where l is the soliton size) and since the 1/m expansion breaks down for small m (ml < 1) they failed to find if the fractional spin depends on m for small m. They ran into this problem because they integrated out the gauge field.

In the next chapter we will show by using the back ground field method and classical equations of motion that the fractional spin is independent of the gauge mass m to all orders in m i. e. $\frac{\theta}{2\pi}$ for $m \neq 0$ and it is zero for $m = 0$.

CHAPTER. 4

MASS INDEPENDENCE OF THE FRACTIONAL SPIN

We discussed in the last chapter that the action (Eq. 3. 2. 1) of the gauged O(3) non-linear sigma model has Hopf term in disguise and therefore we can expect the solitons of the model to have fractional spin. We also expect the spin of the solitons is zero if m=0(since no Hopf term arises from the action). In this chapter we discuss the details of the model in sect. 4. 1, quantization in 4. 2. In sect. 4. 3. we show that the spin of the soliton of the model is zero if m=0 and it is fractional and independent of m if $m \neq 0$. The Hopf coefficient θ that is induced is given by $\theta = \dfrac{e^2}{2m}$.

4. 1 **The Model**

The action of the gauged O(3) non- linear sigma model is given by (We discussed in Sect. 3. 2)

$$S = S_g + S_\sigma + S_I \qquad (3.\ 2.\ 1)$$

Where

$$S_g = \int d^3x \mathcal{L}_g = \int d^3x [\ -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{m}{4}\varepsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}\ ] \qquad (3.1.2)$$

$$S_\sigma = \int d^3x \mathcal{L}_\sigma = \frac{1}{2f}\int d^3x \partial_\mu n^a \cdot \partial^\mu n^a \qquad (2.1.4)$$

$$S_I = e\int d^3x \mathcal{L}_I = e\int d^3x\ A_\mu\ j^\mu \qquad (3.2.2)$$

$$j^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon^{abc}\ n^a\ \partial_\nu n^b\ \partial_\lambda n^c \qquad (2.1.14a)$$

where $F_{\mu\nu}$ is the abelian gauge field with mass m, and A_{μ} is the gauge potential. e is the parameter that measures the strength of interaction and has dimension [$L^{-1/2}$], f is the coupling constant with dimensions [L^{+1}].

In order to quantize the model we first find the equations for the soliton.

Variation of the action (Eq. 3. 2. 1) with respect to n and A_{μ} give the equations of motion as

$$\partial_{\mu} F^{\mu\nu} - m \, \varepsilon^{\nu\mu\lambda} \, \partial_{\mu} A_{\lambda} + e \, j^{\nu} = 0 \qquad (4.1.1a)$$

$$-\frac{1}{f}(\partial_{\mu} \, \partial^{\mu} n) + \frac{1}{f} n \, (\, n \, .(\partial_{\mu} \, \partial^{\mu} n) \,) + \frac{e}{4\pi} \, \varepsilon^{\mu\nu\lambda}(\, n \times \partial_{\lambda} n) \, \partial_{\nu} A_{\mu} = 0 \qquad (4.1.1b)$$

We look for radially symmetric static soliton solution. Since the soliton is characterized by the same topological charge

$$Q = \int d^2x \, j^0 = \frac{1}{8\pi} \int d^2x \, \varepsilon^{0ij}\varepsilon^{abc} \, n^a \, \partial_i n^b \, \partial_j n^c \qquad (2.1.13)$$

as in the usual O(3) sigma model, we take the following ansatz for n^a depending on a single function g(r) of radial coordinate only

$$
\begin{aligned}
n^1 &= \cos\phi \; \sin g(r) \\
n^2 &= \sin\phi \; \sin g(r) \\
n^3 &= \cos g(r)
\end{aligned}
\qquad (4.1.2)
$$

where (r, ϕ) are the polar coordinates and the boundary conditions on the g(r) are

$$g(\infty) = \pi \text{ and } g(0) = 0 \, . \qquad (4.1.3)$$

The solution given by Eq. (4. 1. 2) with the boundary conditions (Eq. 4. 1. 3) describe a soliton known as Baby Skyrmion in the literature[36]. We represent the vector **n** by an arrow ($\rightarrow$) at each space point. As we move from r = 0 to r = ∞ the vector **n** takes values from (0 0 +1) to (0 0 -1). A typical map of baby skyrmion is shown below.

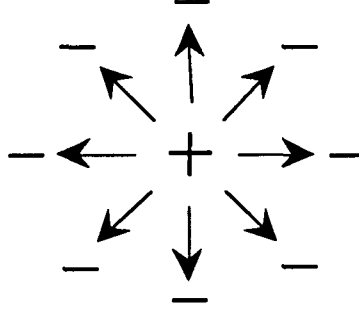Fig 2. Baby skyrmion. $+$ ((0 0 1)) indicates **n** out of the page, $-$ ((0 0 -1)) indicates **n** into the page.

We substitute (4. 1. 2) in (2. 1. 14a) to get

$$j^0 = -\frac{1}{4\pi r}\,(\cos g)' \qquad\qquad (4.\,1.\,4a)$$

$$j^i = 0 \qquad\qquad (4.1.4b) \qquad (4.1.4)$$

where prime denotes differentiation with respect to r. Substituting (4. 1. 4a) in (2. 1. 13) we find

$$Q = \int d^2x\, j^0 = -\int r\, dr\, d\phi\, \frac{1}{4\pi r}\,(\cos g)' = -\frac{1}{2}\,[\cos g(\infty) - \cos g(_0)] = 1.$$

Thus equation (4. 1. 2) with the boundary condition (4. 1. 3) describe mapping $S^{(2)}_{phy}$ onto $S^{(2)}_{fld}$.

with winding number one and hence describes a soliton of Q = 1. It is convenient to choose the coulomb gauge for the static solution $\partial_i A^i = 0$. Then we can write A^i as

$$A^i = \epsilon^{ij}\,\partial_j \psi. \qquad\qquad (4.1.5)$$

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To respect spherical symmetry for **n**, we take A^0 and ψ to be function of r only.

$$A^0 = A^0(r) \qquad (4.1.6a)$$

$$\psi = \psi(r) \qquad (4.1.6b) \qquad (4.1.6)$$

The equations of motion reduce to

$$\partial_i F^{i0} - m\,\varepsilon^{ij}\,\partial_i A_j + e\,j^0 = 0 \qquad (4.1.7a)$$

$$\partial_i F^{ij} - m\,\varepsilon^{ji}\,\partial_i A_0 = 0 \qquad (4.1.7b)$$

and

$$-\frac{1}{f}(\partial_i \partial^i \mathbf{n}) + \frac{1}{f}\,\mathbf{n}\,(\,\mathbf{n}.(\partial_j \partial^j \mathbf{n})\,) - \frac{e}{4\pi}\,\varepsilon^{ij}(\,\mathbf{n} \times \partial_j \mathbf{n})\,\partial_i A_0 = 0. \qquad (4.1.7c)$$

The above equations can be rewritten by using equations (4.1.4), (4.1.5) and (4.1.6) as

$$\partial_i(\partial_j \partial^j \psi - mA_0) = 0 \qquad (4.1.8a)$$

$$\partial_j \partial^j(A_0 - m\psi) = e\,j_0 \qquad (4.1.8b) \qquad (4.1.8)$$

and

$$\frac{1}{f}\{(\,g'' + \frac{1}{r}g'\,) - \frac{1}{2r^2}\sin 2g\} - \frac{e}{4\pi r}\,\sin g\;A'_0 = 0. \qquad (4.1.8c)$$

Choosing the integration constant to be zero in equation (4.1.8a) and substituting it for $(\partial_j \partial^j \psi)$ in (4.1.8b), we get the equation for A_0 as

$$(\partial_j \partial^j - m^2)A_0 = e\,j_0(x) \qquad (4.1.9)$$

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The solution of this can be written as

$$A_0(x) = e \int d^2x' \, G_{A_0}(x - x') \, j_0(x') \tag{4.1.10}$$

where the Greens function $G_{A_0}(x - x')$ satisfies

$$(\partial_j \partial^j - m^2) G_{A_0}(x - x') = \delta^2(x - x'), \tag{4.1.11}$$

where $\delta^2(x - x')$ is a two dimensional Dirac delta function. Fourier transforming (4. 1. 11) we find

$$G_{A_0}(x - x') = \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2 + m^2)} e^{-i\mathbf{k}.(x - x')} \tag{4.1.12a}$$

In polar coordinates

$$G_{A_0}(|x - x'|) = \int \frac{kdk}{(2\pi)} \frac{1}{(k^2 + m^2)} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i|\mathbf{k}|.|x - x'|\cos\theta} \tag{4.1.12b}$$

We use the integral representation of the Bessel function of the first kind

$$\mathcal{J}_0(|\mathbf{k}||x - x'|) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i|\mathbf{k}|.|x - x'|\cos\theta} \tag{4.1.12c}$$

in the above equation to get

$$G_{A_0}(|x - x'|) = \int_0^\infty \frac{k\,dk}{2\pi} \frac{1}{(k^2 + m^2)} \mathcal{J}_0(|k||x - x'|) \qquad (4.1.12d)$$

$$= \frac{1}{2\pi} \mathcal{K}_0(m|x - x'|) \qquad (4.1.13)$$

where $\mathcal{K}_0$ is the modified Bessel function $(m > 0)$. Therefore A_0 can be written as

$$A_0(x) = \frac{e}{2\pi} \int d^2x' \, \mathcal{K}_0(m|x - x'|) \, j_0(x') \qquad (4.1.14a)$$

We can write A^i by using (B. 2),(B. 6) (Appendix. B) and (4. 1. 5) as

$$A^i(x) = -\frac{e}{2\pi m} \varepsilon^{ij} \partial_j \int d^2x' \left[\ln(m|x - x'|) + \mathcal{K}_0(m|x - x'|) \right] j_0(x') \qquad (4.1.14b)$$

(By using 4.1.14a)

$$A^i(x) = -\frac{e}{2\pi m} \varepsilon^{ij} \partial_j \int d^2x' \, \ln(m|x - x'|) \, j_0(x') - \frac{1}{m} \varepsilon^{ij} \partial_j A_0(x). \qquad (4.1.14c)$$

The above equations (4. 1. 14a,b) describe a soliton characterized by the topological charge $Q = 1$ sitting at x' producing a gauge potential $A_\mu(x)$ at x . Eq. (4. 1. 14c) shows the connection between the vector potential $A(x)$, scalar potential A_0 and the topological source $j_0(x')$. Equation (4. 1. 8c) with (4. 1. 14) describe static soliton of the model. We note that the soliton with $Q = 1$ of the usual non-linear sigma model described by the ansatz (4. 1. 2) with

$$g(r) = \cos^{-1}\left(\frac{r^2 - \lambda^2}{r^2 + \lambda^2}\right) \tag{4.1.15}$$

satisfies

$$g'' + \frac{1}{r} g' - \frac{1}{2r^2} \sin 2g = 0. \tag{4.1.16}$$

λ in Eq. (4. 1. 15) is an arbitrary scale parameter of the soliton. We now turn to the quantization of the soliton sector with $Q = 1$.

4. 2   Quantization

We follow the semi classical method of quantization of the solitons by using the collective coordinates[7,28]. Basically the method involves expanding the fields around the classical solutions by using one parameter functions called collective coordinates.

We expand the field variables $\mathbf{n}$ and A_μ around the soliton solution and apply the canonical Hamiltonian method[7, 28]. Since our primary concern is the induced spin of the soliton, we consider a U(1) family of configurations characterized by a single collective coordinate $\alpha(t)$ corresponding to the zero mode of rotation[12]

$$n^1 = \cos(\phi + \alpha(t)) \sin g(r)$$

$$n^2 = \sin(\phi + \alpha(t)) \sin g(r) \tag{4.2.1}$$

$$n^3 = \cos g(r).$$

Evidently the $\mathbf{n}$ given by Eq. (4. 2. 1) describe the mappings from $S^{(3)}_{phy}$ onto $S^{(2)}_{fld}$ with $Q = 1$.

We know from Chapter 2. that the field configuration space is multiply connected $(\Pi_1(Q_i) \approx \Pi_3(S^{(2)}_{fld}) = Z)$. This means that there exist infinitely many field functions($\mathbf{n}$'s) in each sector. We can easily see from Eq. (4. 2. 1) that there can be many one parameter functions

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots \text{etc.})$ satisfying $Q = 1$. We will, however, take into account the full degrees of freedom for the gauge fields. We write

$$A_\mu(x,t) = A_\mu^{cl}(x) + A_\mu^q(x,t) \qquad (4.2.2)$$

where A_μ^{cl} is the gauge part of the classical soliton solution. We write the topological current j^μ in terms of the collective coordinate $\alpha(t)$ and $g(r)$ by using the equations (4. 2. 1) and (2. 1. 14a) as

$$j^0 = -\frac{1}{4\pi r} (\cos g)'$$

$$j^1 = -\frac{1}{4\pi r} \sin\phi \, (\cos g)' \, \dot\alpha \qquad (4.2.3)$$

$$j^2 = +\frac{1}{4\pi r} \cos\phi \, (\cos g)' \, \dot\alpha$$

Where dot represents differentiation with respect to t. Substituting equations (4. 2. 1),
(4. 2. 2), (4. 2. 3) in the action (3. 2. 1) and writing $A_\mu^{cl}(x)$ in terms of ψ by using (4. 1. 5),

(4. 1. 8a) integrating by parts where ever necessary and noting ψ is a function of r only, we get.

$$S = \int d^3x \left[-\frac{1}{4} F_{\mu\nu}^q F^{\mu\nu\, q} - \frac{m}{2} \mathcal{E}^{\mu\nu\lambda} A_\mu^q (\partial_\nu A_\lambda^q) + \frac{1}{2f} \sin^2 g \, \dot\alpha^2 - \frac{e}{4\pi r} (\cos g)' \, \mathcal{E}^{ij} A_i^q x_j \, \dot\alpha + \right.$$

$$\left. -\frac{e}{4\pi} (\cos g)' \psi' \dot\alpha - (\partial_i \partial^i A_0^{cl}) \, A_0^q + m^2 A_0^{cl} A_0^q + e j_0 A_0^q \right] - M_s \qquad (4.2.4)$$

where M_s is the soliton mass

$$M_s = \frac{1}{2} \int d^2x \left[\frac{1}{f} \left((g')^2 + \frac{1}{r^2} \sin 2g \right) + \frac{e}{4\pi r} (\cos g)' A_0^{cl} \right]. \qquad (4.2.5)$$

Let us take the following terms from equation (4. 2. 4) and use (4. 1. 9).

$$- (\partial_i \partial^i A_0^{cl}) \, A_0^q + m^2 A_0^{cl} A_0^q + ej_0 A_0^q = - \{(\partial_i \partial^i A_0^{cl}) - m^2 A_0^{cl} - ej_0\} A_0^q = 0 \qquad (4.2.6)$$

and denote

$$\kappa = \frac{1}{f} \int d^2x \, \sin^2 g \qquad (4.2.7a)$$

$$(4.2.7)$$

$$\rho = -\frac{e}{4\pi} \int d^2x (\cos g)' \psi'. \qquad (4.2.7b)$$

We write Eq. (4. 2. 4) by using (4. 2. 6) and (4. 2. 7) and dropping the indices q to express the action in terms of the quantum fields $\alpha(t)$ and A_μ^q

$$S = \int dt \, L \quad - \quad M_s \qquad (4.2.8)$$

where

$$L = L_g + L_\alpha + L_{int} \qquad (4.2.9)$$

$$L_g = \int d^2x \, \mathcal{L}_g = \int d^2x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m}{4} \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \Big] \qquad (4.2.10a)$$

$$L_{int} = \int d^2x \, \mathcal{L}_{int} = \int d^2x \Big[\frac{e}{4\pi r} (\cos g)' \epsilon^{ij} x_i A_j \dot{\alpha} \Big] \qquad (4.2.10b)$$

$$L_\alpha = \int d^2x \, \mathcal{L}_\alpha = \frac{\kappa}{2} \dot{\alpha}^2 + \rho \dot{\alpha} \qquad (4.2.10c)$$

We have dropped the superscript q on A^{μ} and $F^{\mu\nu}$. Now we construct the Hamiltonian for the system described by (4. 2. 8). Let us start the procedure by defining momenta conjugate to A_{μ} and $\alpha(t)$

$$\Pi_{\mu} = \frac{\delta L}{\delta \dot{A}^{\mu}} \qquad (4.2.11a)$$

(4.2.11)

$$\beta = \frac{\partial L}{\partial \dot{\alpha}} . \qquad (4.2.11b)$$

We find from (4. 2. 8) by using (4. 2. 11) that the momentum (Π_0) conjugate to A_0 is zero i.e.

$$\Pi_0 = 0. \qquad (4.2.12a)$$

and momenta conjugate to A^i and α are given by

$$\Pi_i = E_i + \frac{m}{2} \epsilon_{ij} A^j \qquad (4.2.12b)$$

$$\beta = \kappa \dot{\alpha} + \rho + \gamma(t) \qquad (4.2.12c)$$

where

(4.2.12)

$$\gamma(t) = \int d^2x \, \frac{e}{4\pi r} (\cos g)' \, \epsilon^{ij} \, x_i \, A_j \qquad (4.2.12d)$$

$$E_i = F_{i0}. \qquad (4.2.12e)$$

Equation (4. 2. 12a) implies that the system has constraints. We follow Dirac's procedure to find out the full set of constraints. We show the details in Appendix A. We see from the appendix A. that the system has two first class constraints .

$$. \chi_0 \equiv \Pi_0 = 0 \qquad (4.2.12a)$$

$$\chi_1 \equiv (\partial_i \Pi^i + \frac{m}{2} B) = 0 \qquad (A.7)$$

These two constraints are those of the abelian gauge field with Chern-Simon's term. A Legendre transform yields the Hamiltonian

$$H = \int d^2x \left(\Pi_i \dot{A}^i + \beta \dot{\alpha} - \mathcal{L} \right) \qquad (4.2.13a)$$

$$H = H_\alpha + H_g \qquad (4.2.13b) \qquad\qquad (4.2.13)$$

where

$$H_\alpha = \frac{1}{2\kappa}\left(\beta - \rho - \gamma(t)\right)^2 \qquad (4.2.13c)$$

$$H_g = \int d^2x \,\mathcal{H}_g = \frac{1}{2}\int d^2x \left[\left(\Pi_i + \frac{m}{2}\varepsilon_{ij}A^j\right)^2 + B^2 \right] \qquad (4.2.13d)$$

$$B = \varepsilon_{ij}\partial^i A^j . \qquad (4.2.13e)$$

We have upon quantization

$$[\Pi_\mu(x,t), A_\nu(x',t)] = -i\,\eta_{\mu\nu}\delta(x-x') \qquad (4.2.14a)$$

$$\qquad\qquad (4.2.14)$$

$$[\beta(t), \alpha(t)] = -i . \qquad (4.2.14b)$$

We thus have reduced the system of soliton field coupled to A_μ into a system with one degree of freedom (particle) described by $\alpha(t)$ interacting with a gauge field. The constraints are those of abelian gauge field. There are no constraints on the soliton. It has been shown[7] the commutator given by Eq. (4. 2. 14b) is consistent with the equal time canonical commutator.

$$[\Pi^a(x,t), n^b(x',t)] = -i\,\delta^{ab}\delta(x-x')$$

where $\Pi^a(x,t)$ is the canonical momentum to $n^b(x',t)$.

4. 3 Mass independence of the fractional spin.

We know the expression for the angular momentum as

$$M = \varepsilon_{ij} \int d^2x \, x^i \, T^{0j} \tag{2.2.4}$$

where $T^{\mu\nu}$ is the energy momentum tensor. Variation of the action (3. 2. 1) with respect to the metric $g^{\mu\nu}$ yields the energy momentum tensor ($T^{\mu\nu}$)

$$T_{\mu\nu} = - F_\mu{}^\lambda F_{\nu\lambda} + \frac{1}{2} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} + \frac{1}{f} \partial_\mu \mathbf{n} \cdot \partial_\nu \mathbf{n} - \frac{1}{2f} g_{\mu\nu} \partial_\lambda \mathbf{n} \cdot \partial^\lambda \mathbf{n} \tag{4.3.1}$$

$$T^{0j} = - F^{0\lambda} F^j{}_\lambda + \frac{1}{f} \partial^0 \mathbf{n} \cdot \partial^j \mathbf{n} \tag{4.3.2a}$$

(By using Eq. (3.1.11))

$$= T_g^{0j} + \frac{1}{f} \partial^0 \mathbf{n} \cdot \partial^j \mathbf{n} \tag{4.3.2b}$$

Substituting Eq. (4. 3. 2b) in Eq. (2. 2. 4) we get the expression for the angular momentum(M)

$$M = \varepsilon_{ij} \int d^2x \, x^i \left[- F^{0\lambda} F^j{}_\lambda + \frac{1}{f} \partial^0 \mathbf{n} \cdot \partial^j \mathbf{n} \right] \tag{4.3.3a}$$

$$\tag{4.3.3}$$

$$= \varepsilon_{ij} \int d^2x \, x^i \, T_g^{0j} + \frac{1}{f} \, \varepsilon_{ij} \int d^2x \, x^i \partial^0 \mathbf{n} \cdot \partial^j \mathbf{n}$$

$$= -\int d^2x \, x^i E_i B + \frac{1}{f} \, \varepsilon_{ij} \int d^2x \, x^i \partial^0 \mathbf{n} \cdot \partial^j \mathbf{n} . \tag{4.3.3b}$$

We know from Eq. (4. 2. 12b) that the momenta (Π^μ) conjugate to A^μ do not couple to $\mathbf{n}$ fields and hence the first term of Eq. (4. 3. 3b) is purely electromagnetic angular momentum (M_g) (Eq. 3. 1. 12). We now express the angular momentum operator M in terms of canonical momentum operator (Π^a) similar to the expression $\mathbf{L} = \mathbf{r} \times \mathbf{P}$.

We find canonical momentum (Π^a) conjugate to n^a fields Eq. (3. 2. 1) as

$$\Pi^a \equiv \frac{\delta L}{\delta(\partial_o n^a)} = \frac{1}{f}\partial_o n^a + \frac{e}{2\pi}\varepsilon_{ij}\varepsilon^{abc}A_i n^b \partial_j n^c \qquad (4.3.4)$$

Substituting (4. 3. 4) in (4. 3. 3b) and writing M_g for the first term we get

$$M = M_g + \varepsilon_{ij}\int d^2x \; x^i \Pi^a (\partial^j n^a) - \frac{e}{2\pi}\int d^2x \varepsilon_{ij}\varepsilon_{kl}\varepsilon^{abc} x_i (A_k n^b \partial_l n^c)\partial_j n^a \qquad (4.3.5)$$

where A_k is given by Eq. (4. 1. 14b). We can easily notice that the last two terms are same as Eq. (2. 2. 20)

$$M_{Soliton} = \varepsilon_{ij}\int d^2x \; x^i \Pi^a (\partial^j n^a) - \frac{\theta}{2\pi}\int d^2x \varepsilon_{ij}\varepsilon_{k'l}\varepsilon^{abc} x_i (A_{k'} n^b \partial_l n^c)\partial_j n^a \qquad (2.2.20)$$

except that 'e' takes the place of 'θ' and the Greens functions for A_k (Eq. 4. 1. 14b) and $A_{k'}$ (Eq. 2. 2. 18) are different. Thus we expect that the angular momentum (M) split into purely electromagnetic part (M_g), orbital part of the soliton and fractional part (with $\frac{e^2}{2m} = \theta$).

We replace the A^μ in Eq. (4. 3. 3b) by $A^{cl}_\mu(x) + A^q_\mu(x,t)$, write $A^{cl}_\mu(x)$ in terms of ψ by using (4. 1. 5), (4. 1. 8) and $\mathbf{n}$, $A^q_\mu(x,t)$ in terms of canonical variables by using equations (4. 2. 1), (4. 2. 12). We obtain the following expression for M

$$M = -\int d^2x \; x^k\left[(\nabla^2\psi + B)E_k - \frac{1}{m}(\partial_k\nabla^2\psi)B\right] + \beta - \gamma(t) - \rho + \int d^2x \; x^k(\partial_k A^{cl}_0)\nabla^2\psi \qquad (4.3.6)$$

where β is canonical momentum conjugate to α and

$$\gamma(t) = e \int d^2x \, j_0(x) \, \epsilon^{ij} \, x_i \, A_j(x,t) \qquad (4.2.12d)$$

$$\rho = -\frac{e}{4\pi} \int d^2x (\cos g)' \psi'. \qquad (4.2.7b)$$

We notice from Eq. (4. 3. 6), or(4. 3. 5) that explicit mass dependence appears with terms linear in operator $A_k(x,t)$. To eliminate the explicit mass dependence in Eq. (4. 3. 6), we consider the linear terms in $A_k(x,t)$ and show that they vanish by using Gauss' law constraints , classical equations of motion and the fact that ψ is radially symmetric i. e. $\psi = \psi(r)$.

The linear terms are

$$-\int d^2x \, x^k \left[(\nabla^2 \psi) E_k - \frac{1}{m} \left(\partial_k \nabla^2 \psi \right) B + e j_0(x) \, \epsilon^{kl} \, A_l(x,t) \right]. \qquad (4.3.7)$$

The Gauss' law constraints in terms of E and B and the classical equations of motion we need are

$$\partial_i E^i - mB = 0 \qquad (A.7)$$

$$\nabla^2 \psi - m A_0^{cl} = 0 \qquad (4.1.8a)$$

$$(\nabla^2 - m^2) A_0^{cl} - e j_0 = 0 \qquad (4.1.9)$$

since $\psi = \psi(r)$

$$\nabla^2 \psi = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \psi \right) \qquad (4.3.8a)$$

and
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4.3.8)

$$\frac{\partial}{\partial x^i} = \frac{x_i}{r} \frac{\partial}{\partial r} \qquad (4.3.8b)$$

Let take the first term of Eq. (4. 3. 7) and use equations (A. 7) and (4. 3. 8)

$$-\int d^2x \; x^k \left[(\nabla^2\psi)E_k\right] = -\int d^2x \; x^k \left\{ \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}\psi)\right\} E_k = -\int d^2x \left\{\partial_k (r\frac{d}{dr}\psi)\right\}E^k$$

Integrating by parts the R. H. S. and using (A. 7)

$$= -m\int d^2x \; (r\frac{d}{dr}\psi)B = -m\int d^2x \; (r\frac{d}{dr}\psi)\epsilon^{ij}\partial_i A_j.$$

Integrating by parts the R. H. S. and using (4. 3. 8), (4. 1. 8a)

$$-\int d^2x \; x^k\left[(\nabla^2\psi)E_k\right] = m\int d^2x \; \epsilon^{ij}x_i A_j (\nabla^2\psi) = m^2\int d^2x \; \epsilon^{ij}x_i A_j A_0^{cl} \; . \qquad (4.3.9a)$$

Let us take the second term of Eq. (4. 3. 7)

$$\frac{1}{m}\int d^2x \; x^k\left(\partial_k\nabla^2\psi\right)B = \frac{1}{m}\int d^2x \; x^k\left\{\frac{x_k}{r}\frac{d}{dr}(\nabla^2\psi)\right\}\epsilon^{ij}\partial_i A_j.$$

Integrating by parts (R. H. S.) and using (4. 3. 8),(4. 1. 8a)

$$= -\frac{1}{m}\int d^2x \; x_i\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\nabla^2\psi\right)\epsilon^{ij}A_j = -\int d^2x \; \epsilon^{ij}x_i A_j\left(\nabla^2\left(\frac{\nabla^2\psi}{m}\right)\right)$$

$$\frac{1}{m}\int d^2x \; x^k\left(\partial_k\nabla^2\psi\right)B = -\int d^2x \; \epsilon^{ij}x_i A_j\left(\nabla^2 A_0^{cl}\right). \qquad (4.3.9b)$$

Substituting (4. 3. 9a,b) in (4. 3. 7) we find the coefficient of $\epsilon^{ij}x_i A_j$ is zero by using

Eq. (4. 1. 9).

$$-\int d^2x \; x^k\left[(\nabla^2\psi)E_k - \frac{1}{m}\left(\partial_k\nabla^2\psi\right)B + ej_0(x)\;\epsilon^{kl}A_l(x,t)\right] =$$

$$= -\int d^2x \left[(\nabla^2 - m^2)A_0^{cl} - ej_0\right]\epsilon^{ij}x_i A_j$$

$$= 0 \; . \qquad (4.3.10)$$

Thus the terms linear in the quantum operator A_μ vanish by using Gauss' law constraint and classical equations of motion.

We use (4. 3. 10) in (4. 3. 6) to get

$$M = -\int d^2x \left(\mathbf{x} \cdot \mathbf{E} \right) B + \beta - \rho + \int d^2x \, x^k \left(\partial_k A_0^{cl} \right) \nabla^2 \psi. \qquad (4.3.11)$$

Clearly we can identify the first term of the above equation as the angular momentum operator (M_g) of the gauge field (Eq. 3. 1. 12)

$$M = M_g + \beta - \rho + \int d^2x \, x^k \left(\partial_k A_0^{cl} \right) \nabla^2 \psi. \qquad (4.3.12)$$

Thus as expected in Eq. (4. 3. 5) the angular momentum operator (M) separates into three terms, the electromagnetic contribution, the soliton canonical angular momentum β and the $\dfrac{e}{2\pi}$ term.

The last two terms in the equation (4. 3. 12) can be simplified by using the equations of motion E. qs (4. 1. 8) written in the form (The detailed calculations are shown in Appendix. B).

$$\psi = em \int d^2x' \, G_\psi(\mathbf{x}\text{-}\mathbf{x}') j^0(\mathbf{x}') \,, \qquad (4.3.13)$$

where $j^0 = -\dfrac{1}{4\pi r} \left\{ \cos g(r) \right\}'$. The Green's function $G_\psi(\mathbf{x}\text{-}\mathbf{x}')$ is given by

$$G_\psi(\mathbf{x}\text{-}\mathbf{x}') = \int \frac{d^2k}{(2\pi)^2} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}\text{-}\mathbf{x}')}}{k^2(k^2+m^2)}$$

$$= -\frac{1}{2\pi m^2}\Big[\ln(m|\mathbf{x}\text{-}\mathbf{x}'|) + \mathcal{K}_0(m\,|\mathbf{x}\text{-}\mathbf{x}'|)\Big]. \qquad (4.3.14)$$

In Eq. (4. 3. 14), $\mathcal{K}_\nu$ is the modified Bessel function. Using Eqs. (4. 3. 13) and (4. 314), we can rewrite Eq. (4. 312) as

$$M = M_g + \beta + \frac{\theta}{2\pi}\int d^2x \; d^2x' j^0(x) j^0(x') - \frac{\theta}{2\pi}\int d^2x \; d^2x' j^0(x) \Big[(m|x-x'|) \mathcal{K}_1(m|x-x'|) \Big] j^0(x')$$

(4.3.15)

where $\theta = \frac{e^2}{2m}$, we now show by using equation of motion (Eq. 4. 1. 8c) that the last term in

Eq. (4. 3. 15) is zero. We show the detailed calculations in Appendix. B. We perform the angular

integration of the last term by utilizing the fact that the topological current j_0 is a function of only

radial coordinate

$$\int d^2x \; d^2x' j^0(x) \Big[(m|x-x'|) \mathcal{K}_1(m|x-x'|) \Big] j^0(x') =$$

$$= - 8\pi^2 \Big[\int_0^\infty rdr \; j^0(r) \; mr \; \mathcal{K}_1(mr) \int_0^r r'dr' \; j^0(r') \; I_0(mr') +$$

$$- \int_0^\infty rdr \; j^0(r) \; mr \; I_1(mr) \int_r^\infty r'dr' \; j^0(r') \; \mathcal{K}_0(mr') \Big].$$

(4.3.16)

We generate the same terms from Equation(4. 1. 8c) and show that it is a total derivative . The

equation of motion is .

$$\frac{1}{f}\Big((g'' + \frac{1}{r} g') - \frac{1}{2r^2} \sin 2g \Big) - \frac{e}{4\pi r} (\sin g) A'_0 = 0$$

(4.1.8c)

We know that

$$A_0 (x) = \frac{e}{2\pi} \int d^2x' \; \mathcal{K}_0(m |x - x'|) \; j_0(x') \;.$$

(4.1.14a)

Performing the angular integration (Appendix. B) we get

$$A_0 = e \Big[\int_0^r r'dr' \; j^0(r') \; \mathcal{K}_0(mr) I_0(mr') + \int_r^\infty r'dr' \; j^0(r') \; I_0(mr) \mathcal{K}_0(mr') \Big].$$

(4.3.17)

Differentiating with respect to r

$$A_0^{'}(r) = e\left[-\int_0^r r'dr' \, j^0(r') \, I_0(mr') \, m\mathcal{K}_1(mr) + \int_r^\infty r'dr' \, j^0(r') \, mI_1(mr) \, \mathcal{K}_0(mr')\right] \qquad (4.3.18)$$

We substitute Eq. (4. 3. 18) in (4. 1. 8c), multiply through out by " $g' \, r^2$ " and use (4. 12. 4a) integrate with respect to r.

$$\int_0^\infty dr \frac{1}{f}\left[\left(r^2 g' g'' + r\,(g')^2\right) - \frac{g'}{2}\sin 2g\right] =$$

$$= -e^2\left[\int_0^\infty rdr\, j^0(r)\, mr\, \mathcal{K}_1(mr)\int_0^r r'dr'\, j^0(r')\, I_0(mr') +\right.$$

$$\left. -\int_0^\infty rdr\, j^0(r)\, mr\, I_1(mr)\int_r^\infty r'dr'\, j^0(r')\, \mathcal{K}_0(mr')\right] . \qquad (4.3.19)$$

Let us consider

$$\frac{1}{2}\frac{\partial}{\partial r}(rg')^2 = r\, g'^2 + r^2\, g'\, g'' \qquad (4.3.20a)$$

$$(4.3.20)$$

$$-\frac{1}{2}\frac{\partial}{\partial r}(\cos 2g) = g'\sin 2g . \qquad (4.3.20b)$$

By using (4.3. 20),the left hand side of (4. 3. 19) can be written as

$$\int_0^\infty dr \frac{1}{f}\left[\left(r^2 g'g'' + r\,(g')^2\right) - \frac{g'}{2}\sin 2g\right] = \frac{1}{2f}\int dr\frac{\partial}{\partial r}\left\{(rg')^2 - \frac{1}{2}\cos 2g\right\}$$

$$= \frac{1}{2f}\left[(rg')^2 \Big|_0^\infty - 0\right] \qquad \text{(by using Eq. (4. 1. 3))}$$

We see from Eq. (4. 1. 15) $g'(\infty) = 0$ and $g'(0) =$ finite,(r g') is zero for r = 0 and r= ∞.

There fore

$$\int_0^\infty dr \frac{1}{f}\left[\left(r^2 g'g'' + r\,(g')^2\right) - \frac{g'}{2}\sin 2g\right] = 0. \qquad (4.3.21)$$

By using Eq. (4. 3. 21) in Eq. (4. 3. 19) we get

$$\int_0^\infty rdr\, j^0(r)\, mr\, \mathcal{K}_1(mr) \int_0^r r'dr'\, j^0(r')\, I_0(mr') +$$

$$-\int_0^\infty rdr\, j^0(r)\, mr\, I_1(mr) \int_r^\infty r'dr'\, j^0(r')\, \mathcal{K}_0(mr') \Big] = 0 \quad (4.3.22)$$

Therefore by using (4. 3. 22) the last term of Eq. (4. 3. 15) is zero

$$\int d^2x\, d^2x'\, j^0(x) \Big[(m|x-x'|)\, \mathcal{K}_1(m|x-x'|) \Big] j^0(x') = 0. \quad (4.3.23)$$

Substituting $Q = \int d^2x\, j_0$ and using Eq. (4. 3. 23) in Eq. (4. 3. 15) we get

$$M = M_g + \beta + \frac{\theta}{2\pi} Q^2 \quad (4.3.24)$$

$$M = M_g + M_{\text{Soliton}} \quad (4.3.25)$$

Where we write M_{Soliton} as the soliton total angular momentum operator

$$M_{\text{Soliton}} = \beta + \frac{\theta}{2\pi} Q^2 \quad (4.3.26)$$

This is the expression for the angular momentum operator for the soliton . It contains a quantum operator β and a C- number term $\frac{\theta}{2\pi} Q^2$. β is canonical momentum conjugate to the angular coordinate α and hence describes the angular momentum operator. The C- number term is proportional to Q^2. In the case studied here $Q = 1$. The fractional part of the spin is induced by the

topological mass term of the gauge field. The mass enters only through $\theta = \dfrac{e^2}{2m}$. It is clear that this is a non-perturbative phenomenon as m appears in the denominator. Thus coupling the σ-model to a dynamical gauge field through topological current does not change the physics contained in the Hopf term and demonstrates that the Hopf invariant is induced by the gauge field which has dynamical degree of freedom. Such 'fictitious' U(1) gauge field has been considered recently[40] in the theory of Anyonic super conductors. We expect our result will thus have important application in this area. We can now see why Karabali and Murthy were able to show that the fractional spin is independent of m upto an order $1/m^5$. They considered the soliton solutions corresponding to $Q = 1$ given by our ansatz (Eq. 4. 2. 1.) but they ignored the equations of motion modified by coupling to gauge field. As we saw in our calculations both the Gauss' law constraint and the classical equations of motion for the coupled fields play a crucial role in demonstrating that the fractional spin is $\theta/2\pi$ to all orders in m. We discuss the eigen values of the angular momentum of the soliton sector $Q = 1$.

$$M_{Soliton} = \beta + \frac{\theta}{2\pi} \qquad (4.3.27)$$

The eigen values of β are integers, since β is $\dfrac{1}{i}\dfrac{\partial}{\partial\alpha}$ in the coordinate representation, whose eigen functions are $e^{in\alpha}$ $n \in Z$.

$$\left(\beta + \frac{\theta}{2\pi}\right) e^{in\alpha} = \left(\frac{1}{i}\frac{\partial}{\partial\alpha} + \frac{\theta}{2\pi}\right) e^{in\alpha} = \left(n + \frac{\theta}{2\pi}\right) e^{in\alpha} . \qquad (4.3.28)$$

Thus the soliton of the non-linear sigma model coupled to U(1) gauge field with the topological mass term in (2+1) dimensions has a fractional, whose value is strictly $\dfrac{\theta}{2\pi}$.

We now show that the fractional spin is zero if $m = 0$. We rewrite equation (4. 3. 12) by using Eq. (B. 7) as

$$M = M_g + \beta - e^2 m \int d^2x \, j^0(x) \, x^i \int d^2x' \left\{ \partial_i^x G_\psi(x - x') \right\} j^0(x') +$$

$$+ e^2 m \int d^2x \, x^i \int d^2x'' \left\{ \partial_i^x \nabla_x^2 G_\psi(x - x'') \right\} j^0(x'') \int d^2x' \left\{ \nabla_x^2 G_\psi(x - x') \right\} j^0(x') \quad (4.3.29)$$

where $\partial_i^x = \dfrac{\partial}{\partial x^i}$, $\nabla_x^2 = \partial_i^x \partial^{xi}$. Clearly for $m = 0$ Eq. (4. 3. 29) becomes

$$M = M_g + \beta . \qquad (4.3.30)$$

Thus the spin of the soliton is zero for $m = 0$ and the spin of the soliton is fractional and independent of m for $m \neq 0$.

4. 4 Conclusions

In this thesis we have discussed the (2+1) dimensional O(3) non-linear sigma model coupled to an abelian gauge field with a topological mass (m) and have evaluated the fractional spin of the soliton with the unit topological charge. We have carried out a semi classical quantization of the model treating the non-vanishing classical gauge fields in the soliton sector as back ground gauge fields. We have shown that though the model is not identical to sigma model with the Hopf invariant term, the fractional spin of the solitons is strictly $\dfrac{\theta}{2\pi}$ for $m \neq 0$. and that the spin has arisen from the dynamical term $\dfrac{e}{2\pi} \int d^2x \, \varepsilon_{ij} \varepsilon_{kl} \varepsilon^{abc} \, x_i \, (A_k \, n^b \, \partial_l n^c) \partial_j n^a$. We have found that the fractional spin vanishes for $m = 0$. This fact has lead us to predict a phase transition at $m = 0$. We hope that that this route for realizing fractional spins may be helpful in looking for other (2+1) dimensional models. Finally the mechanism for super conductivity proposed by Polyakov using $\theta = \pi$ is still applicable in the presence of the electro magnetic field.

Appendix. A

We calculate the constraints of the system described by the Lagrangian (4. 2. 9) by Dirac's method. (Ref. 15, 29, 30, 31, 32.). We have obtained in Sect. 4. 2. the constraint equation $\chi_0 \equiv \Pi_0 = 0$ (Eq. 4. 2. 12a)(called the primary constraint). Primary constraints are those that are direct consequence of the definition of the momenta (Π_μ) (independent of equations of motion). The fact $\Pi_0 = 0$ tells us that among the variables A_0, A_1, A_2, Π_0, Π_1, Π_2 only A_1, A_2 and Π_1, Π_2 are truly independent. If $\chi_0 \equiv \Pi_0$ is zero at some time , it has to be zero through out all the time for consistency and therefore $\dot{\chi}_0 \equiv \dot{\Pi}_0 = 0$. This condition generates further constraints χ_1, χ_2, $\chi_3 \cdot \ldots\ldots\ldots \chi_k$.

$$0 = \dot{\chi}_0 = [\chi_0, \ H]_{PB} \equiv \chi_1 \tag{A.1}$$

where the Poisson bracket $[,]_{PB}$ is defined by

$$[C,D]_{PB} = \frac{\partial C}{\partial \alpha} \frac{\partial D}{\partial \beta} - \frac{\partial D}{\partial \alpha} \frac{\partial C}{\partial \beta} + \int d^2 x \left(\frac{\delta C}{\delta A^\mu} \frac{\delta D}{\delta \Pi_\mu} - \frac{\delta D}{\delta A^\mu} \frac{\delta C}{\delta \Pi_\mu} \right) . \tag{A.2}$$

Further demanding $\dot{\chi}_1 = 0$ produces χ_2 and so on.Thus we find the constraints $\chi_1, \chi_2,$ $\chi_3 \ldots\ldots\ldots \chi_{k-1}$.until we get the value of the Poisson bracket $[\chi_{k-1}, H]$ is zero . (While evaluating the Poisson bracket, we should not use the fact $\chi_i = 0$.).

$$0 = \dot{\chi}_{k-1} = [\chi_{k-1}, \ H]_{PB} \equiv \chi_\kappa \tag{A.3a}$$

$$0 = 0 \equiv \chi_2 \tag{A.3b}$$

χ_1, χ_2, χ_3.........χ_{k-1}.are called secondary constraints.These arise only after the Equations of motion are used at least once. We evaluate $[\Pi_0, H]_{PB}$ to find out χ_1. H is the Hamiltonian obtained by using (4. 2. 13a) and (4. 2. 9) and is given by

$$H = H_\alpha + H_G \tag{A. 4}$$

$$H_\alpha = \frac{1}{2\kappa}(\beta - \rho - \gamma(t))^2 \tag{A.5a}$$

$$H_G = \int d^2x \, \mathcal{H}_G = \int d^2x \left[\frac{1}{2}(\Pi_i + \frac{m}{2}\epsilon_{ij}A^j)^2 + \frac{1}{2}B^2 - A_o(\partial_i\Pi^i + \frac{m}{2}B) \right] \tag{A.5b}$$

$$[\Pi_0(x,t), \, H(t)] = [\Pi_0(x,t), H_G(t)] = \left[\Pi_0(x,t), \int d^2x' \, \mathcal{H}_G(x',t)\right]$$

$$= \int d^2x' \, d^2x'' \left(\frac{\delta\Pi_0(x,t)}{\delta A_\mu(x'',t)} \frac{\delta\mathcal{H}_G(x',t)}{\delta\Pi^\mu(x'',t)} - \frac{\delta\mathcal{H}_G(x',t)}{\delta A_\mu(x'',t)} \frac{\delta\Pi_0(x,t)}{\delta\Pi^\mu(x'',t)} \right)$$

$$= -\int d^2x' \, d^2x'' \left(\frac{\delta\mathcal{H}_G(x',t)}{\delta A_0(x'',t)} \frac{\delta\Pi_0(x,t)}{\delta\Pi^0(x'',t)} \right) .$$

$$= -\int d^2x' \, d^2x'' \left[\partial_i'\Pi^i(x',t) + \frac{m}{2}\epsilon^{ij}\partial_i'A_j(x,t)\right]\delta(x'-x'')\delta(x-x'')$$

Where $\partial_i' = \frac{\partial}{\partial x'^i}$ and $\delta(x-x'')$ is a two dimensional delta function. By using the property of the delta functions we find .

$$[\Pi_0(x,t), \, H(t)] = \partial_i\Pi^i(xt) + \frac{m}{2}B(x,t) \equiv \chi_1 \tag{A.6}$$

$$\chi_1 \equiv \partial_i\Pi^i(xt) + \frac{m}{2}B(x,t) = 0 \tag{A.7}$$

We evaluate

$$[\chi_{1'}(x,t),\ H(t)] = \int d^2x'\, d^2x'' \left(\frac{\delta\chi_1(x,t)}{\delta A_k(x'',t)} \frac{\delta\mathcal{H}_G(x',t)}{\delta\Pi^k(x'',t)} - \frac{\delta\mathcal{H}_G(x',t)}{\delta A_k(x'',t)} \frac{\delta\chi_1(x,t)}{\delta\Pi^k(x'',t)} \right).\quad (A.8)$$

Let us evaluate the first term by using Eq. (A. 7) and Eq. (A. 5b)

$$\int d^2x'\, d^2x'' \left(\frac{\delta\chi_1(x,t)}{\delta A_k(x'',t)} \frac{\delta\mathcal{H}_G(x',t)}{\delta\Pi^k(x'',t)} \right) = \int d^2x'\, d^2x''\, \frac{m}{2}\varepsilon^{ik}\left\{ \partial_i'' \delta(x-x'') \right\} \times$$

$$\times \left\{ \Pi^k(x'',t)\delta(x'-x'') - \frac{m}{2}\varepsilon^{kj}A_j(x'',t)\delta(x'-x'') + \left(\partial_k'' A_0(x'',t)\right)\delta(x'-x'') \right.$$

$$\left. + A_0(x'',t)\left(\partial_k''\delta(x'-x'')\right) \right\}$$

$$(A.9)$$

We further simplify the right hand side (R. H. S) of Eq. (A. 9) by using the property of ε^{ij} and integration by parts as

$$\text{R.H.S} = \int d^2x'\, d^2x'' \left[\frac{m}{2}\varepsilon^{ik}\left(\partial_i \delta(x-x'')\right)\Pi^k(x',t)\delta(x'-x'') + \right.$$

$$\left. + \frac{m^2}{4}\left(\partial_i\delta(x-x'')\right) A_i(x',t)\delta(x'-x'') \right]$$

By using the properties of the delta functions

$$= +\frac{m}{2}\varepsilon^{ik}\partial_i\Pi_k(x',t) - \frac{m^2}{4}\ \partial_iA_i(x,t) \qquad (A.10)$$

Let us evaluate the second term of Eq. (A. 8)

$$+\int d^2x' \, d^2x'' \left(\frac{\delta \mathcal{H}_G(x',t)}{\delta A_k(x'',t)} \frac{\delta \chi_1(x,t)}{\delta \Pi^k(x'',t)} \right) = + \int d^2x' \, d^2x'' \left[\left\{ +\frac{m}{2} \varepsilon^{ik} \Pi_i(x'',t)\delta(x'-x'') + \right.\right.$$

$$-\frac{m^2}{4} A^k(x'',t)\delta(x'-x'') + \frac{1}{2}\varepsilon^{ik}\varepsilon^{mn}(\partial_i'' \partial_m'' A_n(x'',t))\delta(x'-x'') +$$

$$+\frac{1}{2}\varepsilon^{ik}\varepsilon^{mn}(\partial_m'' A_n(x'',t))(\partial_i'' \delta(x'-x'')) + \frac{1}{2}\varepsilon^{ij}\varepsilon^{mk}(\partial_m'' \partial_i'' A_j(x'',t))\delta(x'-x'') +$$

$$+ \frac{1}{2}\varepsilon^{ij}\varepsilon^{mk}(\partial_i'' A_j(x'',t))(\partial_m'' \delta(x'-x'')) + \frac{m}{2} \varepsilon^{ik}(\partial_i'' A_0(x'',t))\delta(x'-x'') +$$

$$\left.\left.+\frac{m}{2} \varepsilon^{ik} A_0(x'',t)\left(\partial_i'' \delta(x'-x'')\right) \right\} \left(\partial_k'' \delta(x'-x'')\right)\right] \qquad (A.11)$$

We simplify (A. 11) further by using the property of ε^{ij}, integration by parts, and the properties of the delta functions . We get the R. H. S as

$$\text{R.H.S} = +\frac{m}{2} \varepsilon^{ik}\partial_i \Pi_k(x',t) - \frac{m^2}{4} \partial_i A_i(x,t) \qquad (A.12)$$

By observing (A. 10) and (A. 12), we notice that the first term and the second term of the R. H. S of Eq. (A. 8) are same except they differ by a minus sign .

Therefore

$$[\chi_1,(x,t), \, H(t)] = 0 \qquad (A.13)$$

and hence

$$\chi_2 \equiv 0 \qquad (A.14)$$

Thus we see that the system has two constraints

$$\chi_0 \equiv \Pi_0 = 0 \qquad\qquad (4.2.12a)$$

$$\chi_1 \equiv (\partial_i \Pi^i + \frac{m}{2} B) = 0 \qquad\qquad (A.7)$$

and since $[\chi_0, \chi_1]$ is clearly zero these are first class constraints.

Appendix. B

In this appendix we show that the last two terms of Eq. (4. 3. 12)(θ term) is equal to the last two terms of Eq. (4. 3. 15). i. e.

$$-\rho + \int d^2x \, x^k \left(\partial_k A_0^{cl}\right) \nabla^2 \psi = \frac{\theta}{2\pi} \int d^2x \, d^2x' \, j^0(x) \left\{ 1 - (m|x-x'|) \mathcal{K}_1(m|x-x'|) \right\} j^0(x'). \quad \text{(B.1a)}$$

and

$$\int d^2x \, d^2x' j^0(x) \Big[(m|x-x'|) \mathcal{K}_1(m|x-x'|) \Big] j^0(x')$$

$$= -8\pi^2 \Big[\int_0^\infty r dr \, j^0(r) \, mr \, \mathcal{K}_1(mr) \int_0^r r' dr' \, j^0(r') \, I_0(mr') +$$

$$- \int_0^\infty r dr \, j^0(r) \, mr \, I_1(mr) \int_r^\infty r' dr' \, j^0(r') \, \mathcal{K}_0(mr') \Big]. \quad \text{(4.3.16)}$$

$$A_0 = e \left[\int_0^r r' dr' \, j^0(r') \, \mathcal{K}_0(mr) I_0(mr') + \int_r^\infty r' dr' \, j^0(r') \, I_0(mr) \mathcal{K}_0(mr') \right] \quad \text{(4.3.17)}$$

For this purpose we need the Green's function G_ψ of ψ. The equation for ψ by using (4. 1. 9) and (4. 1. 8a) can be written as

$$(\nabla^4 - m^2 \nabla^2)\psi = emj_0(x) \quad \text{(B.1b)}$$

We can write ψ in terms of $j_0(\mathbf{x})$ as

$$\psi = em \int d^2x' \; G_\psi(\mathbf{x-x'})j^0(\mathbf{x'}) \;, \tag{B.2}$$

where $G_\psi(\mathbf{x-x'})$ is the Green's function for ψ and can be evaluated as below

$$(\nabla^4 - m^2\nabla^2)G_\psi(\mathbf{x-x'}) = \delta^2(\mathbf{x-x'}) \tag{B.3}$$

$\delta^2(\mathbf{x-x'})$ is a two dimensional delta function. Fourier transforming (B. 3) we find .

$$G_\psi(\mathbf{x-x'}) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x-x'})}}{k^2(k^2+m^2)} \tag{B.4}$$

By using Eq. (4. 1. 12c)

$$G_\psi(\mathbf{x-x'}) = \int\limits_0^\infty \frac{k\,dk}{2\pi} \frac{1}{k^2(k^2+m^2)} \mathcal{J}_0(|\mathbf{k}||\mathbf{x-x'}|) \tag{B. 5}$$

$$= -\frac{1}{2\pi m^2}\Big[\ln(m|\mathbf{x-x'}|) + \mathcal{K}_0(m\,|\mathbf{x-x'}|)\Big]. \quad \text{(Upon integration)} \tag{B.6}$$

Let us take the left hand side(L. H. S.) of Eq. (B. 1a) and use Eq. (B. 2) ,Eq. (4. 3. 8b) and (4. 2. 7b)

$$\frac{\partial}{\partial x^i} = \frac{x_i}{r} \frac{\partial}{\partial r} \tag{4.3.8b}$$

$$\rho = -\frac{e}{4\pi}\int d^2x(\cos g)'\psi' \tag{4.2.7b}$$

To get the L. H. S of Eq. (B. 1a) as

$$-\rho + \int d^2x \, x^k \left(\partial_k A_0^{cl}\right) \nabla^2 \psi = - e^2 \, m \int d^2x \, j^0(x) \, x^i \int d^2x' \left\{ \partial_i^x G_\psi(x - x') \right\} j^0(x') +$$

$$+ e^2 \, m \int d^2x \, x^i \int d^2x'' \left\{ \partial_i^x \nabla_x^2 G_\psi(x - x'') \right\} j^0(x'') \int d^2x' \left\{ \nabla_x^2 G_\psi(x - x') \right\} j^0(x') \quad \text{(B. 7)}$$

where $\partial_i^x = \dfrac{\partial}{\partial x^i}$, and $\nabla_x^2 = \partial_i^x \partial^{xi}$. We simplify the second term on the R. H. S. of (B. 7). Since

x,x',x" are dummy variables, we change $x \rightarrow x''$, $x'' \rightarrow x$ and then integration by parts yields

$$\int d^2x \, d^2x' \, d^2x'' \, x''^i \left\{ \partial_i^{x''} \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x') =$$

$$= -2 \int d^2x \, d^2x' \, d^2x'' \left\{ \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x') -$$

$$- \int d^2x \, d^2x' \, d^2x'' \, x''^i \left\{ \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \partial_i^{x''} \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x'). \quad \text{(B.8)}$$

The last term of R. H. S of Eq. (B. 8) is same as its L. H. S. with opposite sign if
$x \rightarrow x'$,
$x' \rightarrow x$. Hence (B. 8) can be written as

$$\int d^2x \, d^2x' \, d^2x'' \, x''^i \left\{ \partial_i^{x''} \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x') =$$

$$= - \int d^2x \, d^2x' \, d^2x'' \left\{ \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x')$$

Integrating by parts twice the R. H. S. we get

$$\int d^2x \ d^2x' \ d^2x'' \ x''^i \left\{ \partial_i^{x''} \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x') =$$

$$= -\int d^2x \ d^2x' \ d^2x'' \left\{ \nabla_{x''}^4 G_\psi(x'' - x) \right\} j^0(x) \ G_\psi(x'' - x') \ j^0(x') \qquad (B.9)$$

We use Eq. (B. 3) and the property $\int d^2x'' \ \delta(x'' - x) f(x'') = f(x)$ in Eq. (B. 9) to get

$$\int d^2x \ d^2x' \ d^2x'' \ x''^i \left\{ \partial_i^{x''} \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \left\{ \nabla_{x''}^2 G_\psi(x'' - x') \right\} j^0(x') =$$

$$= -m^2 \int d^2x \ d^2x' \ d^2x'' \left\{ \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \ G_\psi(x'' - x') \ j^0(x') +$$

$$- \int d^2x \ d^2x' \ j^0(x) \ G_\psi(x - x') \ j^0(x') \qquad (B.10)$$

We substitute (B. 10) in (B. 7) and rearrange the terms to yield

$$-\rho + \int d^2x \ x^k \left(\partial_k A_0^{cl} \right) \nabla^2 \psi = -e^2 m \int d^2x \ d^2x' j^0(x) \left\{ x^i \left(\partial_i^x G_\psi(x - x') \right) + \right.$$

$$\left. + G_\psi(x - x') \right\} j^0(x') +$$

$$- e^2 m^3 \int d^2x \ d^2x' \ d^2x'' \left\{ \nabla_{x''}^2 G_\psi(x'' - x) \right\} j^0(x) \ G_\psi(x'' - x') \ j^0(x') \qquad (B.11)$$

We simplify the right hand side of (B. 11) term by term by using (B. 4)

Let us take

$$\int d^2x \ d^2x'j^0(x) \ x^i \ (\partial_i^x G_\psi(x - x')) = \int d^2x \ d^2x'j^0(x) \ x^i \ \int \frac{d^2k}{(2\pi)^2} \ (-ik_i) \frac{e^{-ik_i(x-x')^i}}{k^2(k^2+m^2)} j^0(x')$$

We symmetrize the expression by changing $x \rightarrow x'$, using $x^i \dfrac{\partial}{\partial x^i} = x'^i \dfrac{\partial}{\partial x'^i}$ and $k_i \rightarrow -k_i$. We get

$$\int d^2x \ d^2x'j^0(x) \ x^i \ (\partial_i^x G_\psi(x - x')) =$$

$$= -\frac{1}{2}\int d^2x \ d^2x'j^0(x) \ \int \frac{d^2k}{(2\pi)^2} \ ik_i(x-x')^i \frac{e^{-ik_i(x-x')^i}}{k^2(k^2+m^2)} j^0(x') \qquad (B.12)$$

$$= \frac{1}{2}\int d^2x \ d^2x'j^0(x) \ \int \frac{kdk \ d\theta}{(2\pi)^2} \ (-i|k||x-x'| \cos \theta) \frac{e^{-i|k||x-x'| \cos \theta}}{k^2(k^2+m^2)} j^0(x')$$

Using Eq. (4. 1. 12c)

$$= \frac{1}{2}\int d^2x \ d^2x'j^0(x) \ \int \frac{k^2dk \ d\theta}{(2\pi)} \ \frac{(-i)|x-x'| \cos \theta}{k^2(k^2+m^2)} \ \mathcal{J}_0(|k||x - x'|) \ j^0(x') \quad (B.13)$$

We write

$$(-i)|x-x'| \cos \theta \ \mathcal{J}_0(|k||x - x'|) = \frac{\partial \mathcal{J}_0(|k||x - x'|)}{\partial k} = - (|x-x'|) \ \mathcal{J}_1(|k||x - x'|) \qquad (B.14)$$

Substituting (B. 14) in (B. 13) and performing k- integration to get

$$\int d^2x \, d^2x' j^0(x) \, x^i \, (\partial_i^x G_\psi(x - x')) = - \frac{1}{4\pi m^2} \int d^2x \, d^2x' j^0(x) \left[1 - m|x\text{-}x'| \mathcal{K}_1(m|x\text{-}x'|) \right] j^0(x').$$

(B.15)

Let us take the following term from (B. 11) and use (B. 4)

$$\int d^2x \, d^2x' \, d^2x'' \left\{ \nabla^2_{x''} G_\psi(x'' - x) \right\} j^0(x) \, G_\psi(x'' - x') \, j^0(x') \quad =$$

$$= \int d^2x \, d^2x' \, d^2x'' \, j^0(x) \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} (-k^2) \frac{e^{-ik_i (x''-x')^i}}{k^2(k^2+m^2)} \frac{e^{-ik'_i (x''-x')^i}}{k'^2(k'^2+m^2)} j^0(x')$$

(B.16)

By using

$$\int \frac{d^2x''}{(2\pi)^2} \, e^{-i(k+k')_i \, x''^i} = \delta(k+k')$$

$$\int \frac{d^2k'}{(2\pi)^2} \delta(k+k') = 1 \qquad\qquad (k = -k')$$

We can write (B. 16) as

$$\int d^2x \, d^2x' \, d^2x'' \left\{ \nabla^2_{x''} G_\psi(x'' - x) \right\} j^0(x) \, G_\psi(x'' - x') \, j^0(x') \quad =$$

$$= - \int d^2x \, d^2x' \, j^0(x) \int \frac{d^2k}{(2\pi)^2} \frac{e^{-ik_i (x'-x)^i}}{k^2(k^2+m^2)^2} j^0(x') \qquad = \text{R.H.S}$$

(B.17)

We write

$$\frac{1}{k^2(k^2+m^2)^2} = - \frac{\partial}{\partial m^2} \left\{ \frac{1}{k^2(k^2+m^2)} \right\}$$

and use Eq. (4. 1. 12c) to get R. H. S of (B. 17) as

$$\text{R. H. S.} = \int d^2x \, d^2x' j^0(x) \frac{\partial}{\partial m^2} \int \frac{k \, dk}{(2\pi)} \frac{1}{k^2(k^2+m^2)} \mathcal{J}_0(\|k\| \|x - x'\|) \, j^0(x')$$

Integrating with respect to k and performing $\frac{\partial}{\partial m^2}$ on the result ,we get (B.17) as

$$\int d^2x \, d^2x' \, d^2x'' \left\{ \nabla^2_{x''} G_\psi(x'' - x) \right\} j^0(x) \, G_\psi(x'' - x') \, j^0(x') \quad =$$

$$= \frac{-1}{2\pi m^4}\Big[\int d^2x \, d^2x' \, j^0(x) \big\{ - \ln(m|x-x'|) - \mathcal{K}_0(m \, |x - x'|)+$$

$$+\frac{1}{2} - \frac{1}{2}(m|x-x'|)\mathcal{K}_1(m|x-x'|) \big\} j^0(x'). \Big] \qquad (B.18)$$

Substituting (B.15),(B. 6),(B. 18) in (B. 11) and writing $\theta = \frac{e^2}{2m}$,we get

$$-\rho + \int d^2x \, x^k \big(\partial_k A_0^{cl} \big) \nabla^{\,2}\psi = \int d^2x \, d^2x' \, j^0(x) \Big[\frac{\theta}{2\pi}\big\{ + \frac{1}{2} - \frac{1}{2}(m|x-x'|)\mathcal{K}_1(m|x-x'|) \big\} +$$

$$+ \frac{e^2}{2\pi m}\big\{ \ln(m|x-x'|) + \mathcal{K}_0(m \, |x - x'|) - \frac{e^2}{2\pi m}\big\{ \ln(m|x-x'|) + \mathcal{K}_0(m \, |x - x'|) +$$

$$+ \frac{\theta}{2\pi}\big\{ + \frac{1}{2} - \frac{1}{2}(m|x-x'|)\mathcal{K}_1(m|x-x'|) \big\} \Big] j^0(x').$$

$$-\rho + \int d^2x \, x^k \big(\partial_k A_0^{cl} \big) \nabla^{\,2}\psi = \frac{\theta}{2\pi} \int d^2x \, d^2x' \, j^0(x) \big\{ 1 - (m|x-x'|)\mathcal{K}_1(m|x-x'|) \big\} j^0(x'). \quad (B.19)$$

We prove (4. 3. 16). Let us take the left hand side (L. H. S.) and perform the angular integration by using the fact that j^0 is function of radial coordinate r and write $\frac{\partial}{\partial x}\mathcal{K}_0(x) = -\mathcal{K}_1(x)$.

$$\text{L.H.S.} = -\int d^2x\, d^2x'\, j^0(r) \left\{ m\frac{\partial}{\partial m}\mathcal{K}_0(m|x-x'|)\right\} j^0(r')$$

$$= \int r\, dr\, d\phi\, r'\, dr'\, d\phi'\, j^0(r)\left[m\frac{\partial}{\partial m}\mathcal{K}_0\left(m\sqrt{(r^2 + r'^2 - 2\,r\,r'\cos\theta)}\right)\right] j^0(r')$$

(B.20)

Where $\theta = (\phi - \phi')$. We treat ϕ' as constant, $d\theta = d\phi$ and $\phi = \phi'$ if $\theta = 0$; $\phi = (2\pi + \phi')$ if $\theta = 2\pi$ and shift the integration region $\phi' - (2\pi + \phi')$ to $0 - 2\pi$.

$$\text{L.H.S.} = \int_0^\infty r\, dr\, r'\, dr'\, d\phi'\, j^0(r)\left[m\frac{\partial}{\partial m}\int_0^{2\pi}d\theta\, \mathcal{K}_0\left(m\sqrt{(r^2 + r'^2 - 2\,r\,r'\cos\theta)}\right)\right] j^0(r')$$

(B.21)

Using

$$\mathcal{K}_0(x) = i\frac{\pi}{2}\left[\mathcal{J}_0(ix) + i\mathcal{N}_0(ix)\right]$$

(B.22)

where $\mathcal{J}_0$ is Bessel function and $\mathcal{N}_0$ is Neuman function .

$$\text{L.H.S.} = \int_0^\infty r\, dr\, r'\, dr'\, d\phi'\, j^0(r)\,(i\pi)\times$$

$$\times\, m\frac{\partial}{\partial m}\int_0^\pi d\theta\left[\mathcal{J}_0\left(\sqrt{(-(mr)^2 - (mr')^2 + 2\,m^2 r\,r'\cos\theta)}\right)\right. +$$

$$\left. +\, i\,\mathcal{N}_0\left(\sqrt{(-(mr)^2 - (mr')^2 + 2\,m^2 r\,r'\cos\theta)}\right)\right]\, j^0(r')$$

(B.23)

We use the integrals

$$\int_0^\pi d\theta \; \mathscr{J}_0\left(\sqrt{(-(mr)^2 - (mr')^2 + 2\,m^2 r\,r'\cos\theta)}\right) =$$

$$= \pi \; \mathscr{J}_0\,(imr)\;\mathscr{J}_0\,(imr') \qquad\qquad (B.24)$$

$$\int_0^\pi d\theta \; \mathscr{N}_0\left(\sqrt{(-(mr)^2 - (mr')^2 + 2\,m^2 r\,r'\cos\theta)}\right) =$$

$$= \pi \; \mathscr{J}_0\,(imr)\;\mathscr{N}_0\,(imr') \qquad\qquad \text{(For } r' > r\text{)}$$

$$= \pi \; \mathscr{J}_0\,(imr')\;\mathscr{N}_0\,(imr) \qquad\qquad \text{(For } r > r'\text{)}$$

$$(B.25)$$

We write Eq.(B.23) by using (B.24),(B. 25),(B. 22) and $\mathscr{J}_0\,(ix) = I_0\,(x)$,

$$\frac{\partial}{\partial m}\mathscr{K}_0(mr) = -\,r\;\mathscr{K}_1(mr),\; \frac{\partial}{\partial m}I_0(mr) = r\;I_1(mr).$$

$$\text{L.H.S.} = (2\pi)^2 \left[-\int_0^\infty dr' \int_0^{r'} dr \; r'\,j^0(r')\;r\,j^0(r)\;mr'\;\mathscr{K}_1(mr')I_0(mr) + \right.$$

$$+ \int_0^\infty dr' \int_0^{r'} dr \; r'\,j^0(r')\;r\,j^0(r)\;mr\;\mathscr{K}_0(mr')I_1(mr) -$$

$$- \int_0^\infty dr' \int_{r'}^\infty dr \; r'\,j^0(r')\;r\,j^0(r)\;mr\;\mathscr{K}_1(mr)I_0(mr') +$$

$$\left. + \int_0^\infty dr' \int_{r'}^{\infty'} dr \; r'\,j^0(r')\;r\,j^0(r)\;mr'\;\mathscr{K}_0(mr)I_1(mr') \right] \quad (B.26)$$

Inverting the order of integration and writing $r \to r'$ and $r' \to r$, we get

$$\text{L.H.S.} = -(8\pi)^2 \left[\int_0^\infty dr \int_0^r dr' \, r \, j^0(r) \, r' \, j^0(r') \, mr \, \mathcal{K}_1(mr) I_0(mr') + \right.$$

$$\left. + \int_0^\infty dr \int_r^\infty dr' \, r' \, j^0(r') \, r \, j^0(r) \, mr \, \mathcal{K}_0(mr') I_1(mr) \right] \qquad \text{(B.27)}$$

Therefore we can write

$$\int d^2x \, d^2x' j^0(x) \left[(m|x-x'|) \mathcal{K}_1(m|x-x'|) \right] j^0(x')$$

$$= -8\pi^2 \left[\int_0^\infty rdr \, j^0(r) \, mr \, \mathcal{K}_1(mr) \int_0^r r'dr' \, j^0(r') \, I_0(mr') + \right.$$

$$\left. - \int_0^\infty rdr \, j^0(r) \, mr \, I_1(mr) \int_r^\infty r'dr' \, j^0(r') \, \mathcal{K}_0(mr') \right]. \qquad \text{(4.3.16)}$$

We now prove Eq. (4. 3. 17). Let us take Eq. (4. 1. 14a)

$$A_0(x) = \frac{e}{2\pi} \int d^2x' \, \mathcal{K}_0(m|x-x'|) j_0(x') \qquad \text{(4.1.14a)}$$

$$= \frac{e}{2\pi} \int_0^\infty r' \, dr' \int_0^{2\pi} d\theta \, \mathcal{K}_0 \left(m\sqrt{(r^2 + r'^2 - 2 \, r \, r' \cos\theta)} \right) j^0(r')$$

$$\text{(B.28)}$$

We can write (B. 28) by using (B. 22),(B. 24),(B. 25) and $\mathcal{J}_0(ix) = I_0(x)$, as Eq. (4. 3. 17)

$$A_0 = e \left[\int_0^r r'dr' \, j^0(r') \, \mathcal{K}_0(mr) I_0(mr') + \int_r^\infty r'dr' \, j^0(r') \, I_0(mr) \mathcal{K}_0(mr') \right] \qquad \text{(4.3.17)}$$

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