RESULTS ON THE COVERING OF 2-PATHS BY CYCLES

by

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ABSTRACT

The main problem which is presented in this thesis is that of finding collections of cycles of length four such that each 2-path of K_n occurs on exactly λ of these cycles. In Chapter 2 it is shown that the necessary conditions for the existence of such a collection of cycles is also sufficient. Block designs are used in solving many of the cases and, in addition, some new methods of creating block designs are given.

In Chapters 3 and 4, respectively, we present the covering and packing variants of the above problem. That is, we look for maximal (minimal) collections of 4-cycles containing each 2-path of K_n at least (at most) λ times. In Chapter 5 we look at finding resolvable collections of such 4-cycles.

The cases where the 2-paths are covered by cycles of length n and cycles of length five are discussed in Chapters 1 and 6, respectively.

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CHAPTER 1

INTRODUCTION

In most graph decomposition problems one is concerned with partitioning the edges of a graph into subgraphs, each of which is isomorphic to a graph in a fixed family of graphs, or has a particular property. In this thesis we consider the problem in which the graph is the complete graph, the items we are interested in partitioning are the paths of length two and the subgraphs formed by the edges in each partition are cycles of a fixed length. The history of one of these particular problems is presented below and also appears in [N].

1.1 Dickson's Problem

Originally the problem presented below was most probably thought of as some sort of intriguing puzzle, as were many other graph theoretical problems. Perhaps for this reason, the work which was done on this problem was, apparently, 'lost' for quite some time. I would like to thank A. Rosa for providing me with the reference [D1] to work done by L.E. Dickson. This provided the necessary clue to finding the earlier papers and Dickson's 1905 paper.

1.1.1 Introduction

In 1899, C. H. Judson [J1] posed the following problem.

"Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated."

Exactly one year later, Judson [J2] gave seating arrangements for the same problem with six people and with eight people. The problem of seating seven people now began to look quite difficult, and the editor of the journal offered one year's free subscription to the first person to provide a solution. This prize no doubt went to F. H. Safford [S1] who gave a seating arrangement for seven people (which he believed was unique) and also provided another solution to the "six-problem" which he claimed was non-isomorphic to the solution given by Judson. In [S2], Safford showed that there are at most two non-isomorphic solutions to the "six-problem" and L. E. Dickson [D2] proved that Judson's solution and Safford's solution were, in fact, distinct.

In 1905, L. E. Dickson [D3] generalized the original problem to seating n people at a round table on (n-1)(n-2)/2 consecutive days. In this paper he gave necessary and sufficient conditions for a group solution and used these conditions to obtain solutions for n = 4, 5, 6, 8, 10 and 12. Also in 1905, H. E. Dudeney (see [D4]), apparently unaware of the earlier work, asked the readers of the Daily Mail to solve the problem of seating six people on ten occasions.

In The Canterbury Puzzles ([D4], originally published in 1907), H. E. Dudeney asked the same question that Judson had asked in 1899 [J1]. In the solution section of the same book, Dudeney provided a solution to this question. He also wrote that Ernest Bergholt had found an easy method for solving all cases where n=p+1, p prime, and E. D. Bewley had found a method for solving all cases where n is even. Since these statements were not accompanied by any proofs it is doubtful that either Bergholt or Bewley had solved what was claimed. In [D5] Dudeney provided solutions to the above problem for $4 \le n \le 12$ and claimed that he also had solutions for $13 \le n \le 25$ and n=33. The reasons for the omission of the proofs to these results was quite possibly due to the the recreational (and not mathematical) nature of the questions.

Graph theoretically, the problem becomes that of finding (n-1)(n-2)/2 Hamilton cycles in K_n, the complete graph on n vertices, so that every path of length two occurs on exactly one of the cycles. We call such a set of Hamilton cycles a <u>complete set of pairwise orthogonal Hamilton cycles in K_n</u> (abbreviated to: complete set of POHC(n)). C. Huang and A. Rosa [HR] were also unaware of Dickson's work when they wrote their paper which contained a construction for a complete set of POHC(n) where n=p+1 and p is a prime. In the same paper they mention that they have found several distinct cyclic solutions for n=13 and 15.

More recently, David Wagner (personal communication) observed that if there exists a perfect 1-factorization of the complete graph on n points, K_n, then the set of Hamilton cycles formed by the union of every two 1-factors in the 1-factorization forms a complete set of POHC(n). It is known that perfect 1-factorizations of K_n exist for n=p+1, n=2p (where p is an odd prime), all even n where n≤38, n=50, n=244 and n=344 ([MR], [SS1] and [SS2]). Notwithstanding the claims made by Ernest Bergholt and E. D. Bewley, this provides us with many new solutions to our problem.

1.1.2 Dickson's Method

Let $X=\{0, 1, a_3, a_4, \ldots, a_{n-1}\}$. We want to find a set of (n-1)(n-2)/2 Hamilton cycles in K_n, where $X \cup \{\infty\}$ is the vertex set of K_n, so that every path of length two occurs on exactly one cycle. Let the 'first' Hamilton cycle be denoted by

$$\infty \quad 0 \quad 1 \quad a_3 \quad a_4 \quad \cdots \quad a_{n-1} \tag{1}$$

(∞ will always appear first). We will also call Hamilton cycles <u>arrangements</u> since they are an arrangement of the vertices of K_n. The arrangement (1) is called the <u>initial arrangement</u>. We wish to obtain the remaining Hamilton cycles from (1) by suitable group operations on 0, 1, ..., a_{n-1}. Let G be a group of permutations acting on the set X. Then if we consider, for now, the arrangement (1) to be different from

$$\infty \quad a_{n-1} \cdots a_4 \quad a_3 \quad 1 \quad 0 \tag{2}$$

then we see that the permutation group G=(G,X) by which the initial arrangement is permuted into the remaining arrangements must be sharply 2-transitive, since ∞ must be between each pair of group elements exactly once. Thus G, which has degree n-1, is of order (n-1)(n-2). The following theorem concerning sharply 2-transitive groups appears in Permutation Groups and Combinatorial Structures by N. L. Biggs and A. T. White ([BW], p. 127). We present it here in a shorter form.

1.1 THEOREM: If (G,X) is a sharply 2-transitive group, then $|X|$ is a prime power, p^e.

That is, $n=p^e+1$ and we can consider X to be the elements of $GF(p^e)$, where p is any prime number. Since the group of affine transformations of a finite field is a group that acts sharply 2-transitively on the elements of the field, Dickson claimed that we may take this group as G. That is, G is the group of affine transformations of the finite field $GF(p^e)$ and the a_i are elements of $GF(p^e)$, $a_1=0$ and $a_2=1$.

In 1905, all the 2-transitive groups of degree p^e and order $p^e(p^e-1)$ for all p when e = 1, 2, or 3, and for $p^e=2^4$ were known. It is interesting to note that after stating this, Dickson wrote "if further exceptions occur, they arise for $p^e \geq 81$, so that their use in the present problem would be impracticable".

Now, given an arrangement (1) the remaining arrangements will be obtained by applying the affine transformations $S_{r,s} \in G$ to the elements of $GF(p^e)$, where r and s belong to $GF(p^e)$ and $r \neq 0$. $S_{r,s}$ replaces (1) by

$$\infty \quad s \quad r+s \quad a_3r+s \quad a_4r+s \quad \ldots \quad a_{n-1}r+s \qquad (3).$$

Since a complete set of POHC(n) consists of (n-1)(n-2)/2 Hamilton cycles, only half of the arrangements (2) can be taken. In order to be able to choose exactly half of the cycles and still cover every 2-path exactly once, the a_i must be chosen in such a way that there exists a $T \in G$ that replaces (1) by (2).

That is, $T(a_i)=a_{n-i}$ and it is not difficult to see that T is defined by $T(x) = -x+a_{n-1}$. If now we define $q = \lfloor (p^e-1)/2 \rfloor$, then since we require

$a_{n-i} = -a_i + a_{n-1}$ we have the following conditions on the a_i:

$$a_{n-i} - a_{n-1-i} = a_{i+1} - a_i \qquad 1 \le i \le q \qquad (4).$$

The transformations $S_{r,s}$ and $TS_{r,s}$ thus give the same arrangements, but in reverse order, and therefore we keep only one of these two transformations. This results in $|G|/2=(n-1)(n-2)/2$ arrangements as required.

We must now make sure that these $(n-1)(n-2)/2$ Hamilton cycles give a complete set of POHC(n). The equations which arise from the conditions that no two Hamilton cycles have a 2-path in common can be written as:

$$(a_{i+2}-a_i)/(a_{i+1}-a_i) \ne (a_{j+2}-a_j)/(a_{j+1}-a_j) \qquad 1 \le i < j \le n-3 \qquad (5).$$

Thus Dickson arrived at the following theorem.

1.2 THEOREM: If $n=p^e+1$ and there exists an arrangement

$$0,\ 1,\ a_3,\ \ldots,\ a_{n-1}$$

of the elements of $GF(p^e)$ satisfying conditions (4) and (5) then there exists a complete set of pairwise orthogonal Hamilton cycles in K_n.

PROOF: Since the a_i satisfy conditions (4), the Hamilton cycles obtained by applying the transformations in G to (1) occur twice each. By choosing only one of the two transformations $S_{r,s}$ and $TS_{r,s}$ in G we get $(n-1)(n-2)/2$ Hamilton cycles as required.

The a_i also satisfy conditions (5) and so no 2-path occurs on more than one Hamilton cycle. Thus, since there are $(n-1)(n-2)/2$ Hamilton cycles, every 2-path occurs on exactly one Hamilton cycle. ■

Using the $(n-3)(n-4)/2 + q$ conditions (4) and (5), Dickson gave solutions for $p^e = 3, 4, 5, 7, 9$ and 11. We give below the (corrected) example in Dickson's paper [D3] for $p^e=5$. We list the transformations $S_{r,s}$ and $TS_{r,s}$ before the Hamilton cycle obtained from them.

$S_{1,0}=S_{4,3}$: ∞ 0 1 4 2 3 $\qquad$ $S_{2,0}=S_{3,1}$: ∞ 0 2 3 4 1

$S_{1,1}=S_{4,4}$: ∞ 1 2 0 3 4

$S_{1,2}=S_{4,0}$: ∞ 0 4 1 3 2

$S_{1,3}=S_{4,1}$: ∞ 1 0 2 4 3

$S_{1,4}=S_{4,2}$: ∞ 4 0 3 1 2

$S_{2,1}=S_{3,2}$: ∞ 1 3 4 0 2

$S_{2,2}=S_{3,3}$: ∞ 2 4 0 1 3

$S_{2,3}=S_{3,4}$: ∞ 4 2 1 0 3

$S_{2,4}=S_{3,0}$: ∞ 0 3 2 1 4

If we define $f_i=(a_{i+2}-a_i)/(a_{i+1}-a_i)$ then condition (5) tells us that the f_i must be distinct ($1\le i\le n-3$) and, since the a_i's are also distinct, different from 0 and 1. Now let $g_i=f_i-1$. Then clearly the g_i form a permutation of the elements of $GF(p^e)-\{0,-1\}$. If we also let $\pi_i=g_1g_2\ldots g_i$ then, since $a_3=a_2+\pi_1$, it can be shown by induction that

$$a_i=a_{i-1}+\pi_{i-2} \qquad 3\le i\le n-1 \qquad (6).$$

We now see that conditions (4) and equation (6) give $\pi_{n-3}=1$ and $\pi_{n-2-i}=\pi_{i-1}$, $2\le i\le q$.

The following theorem, also due to Dickson, offers an alternate way of finding an initial arrangement.

__1.3__ __THEOREM:__ Let $n=p^e+1$ and let g_1, g_2, ..., g_{n-3} be a permutation of the elements of $GF(p^e)-\{0,-1\}$. If the a_i defined by (6) are all distinct and satisfy conditions (4) then there exists a complete set of pairwise orthogonal Hamilton cycles of K_n.

__PROOF:__ Since the a_i are all distinct and satisfy conditions (4) it remains to be shown that the a_i satisfy conditions (5). This, however, follows from the definition of the f_i and g_i. The rest of the proof follows from Theorem 1.2. ■

__1.4__ __THEOREM:__ There exists a complete set of pairwise orthogonal Hamilton cycles in K_{17}.

__PROOF:__ Consider the polynomial $a^4=a+1$ in the field $GF(2^4)$. The arrangement

$$\infty\ 0\ 1\ a^4\ a^8\ a^9\ a^7\ a^{12}\ a^2\ a^{14}\ a\ a^5\ a^{10}\ a^3\ a^{11}\ a^6\ a^{13}$$

satisfies conditions (4) and (5) and hence by Theorem 1.2 there is a complete set of pairwise orthogonal Hamilton cycles in K_{17}. ■

The above arrangement was found in the following way. Using the method presented in [D3] (and also presented above), a computer was used to find Hamilton cycles in K_{17} which could be extended to complete sets of POHC(17). First, the addition and multiplication tables of elements in $GF(2^4)$ were defined using the polynomial $a^4=a+1$, which is irreducible in Z_2. There are $2^7 7!$ orderings of the elements in $GF(2^4)-\{0,1\}$ satisfying (7). For each of these orderings $g_1, g_2, \ldots, g_{14}$, the products π_i were calculated. Next, the a_i were defined recursively by

$$a_i := a_{i-1} + \pi_{i-2} \qquad 3 \leq i \leq 16.$$

If the a_i satisfied the conditions of Theorem 1.3 then

$$\infty \quad 0 \quad 1 \quad a_3 \quad a_4 \quad \ldots \quad a_{16}$$

was known to be an initial arrangement which could be extended to a complete set of POHC(17).

In fact, using the above algorithm, 3316 initial Hamilton cycles were found. By Theorem 1.2 these can all be extended to complete sets of POHC(17). One of these complete sets of POHC(17) is given in Appendix 1.

Dickson claimed that the number of arrangements (1) which satisfy the conditions in Theorem 1.2 increases rapidly with n. Unfortunately, he did not supply a proof of this claim. The large number of initial arrangements found for n=17 does, however, lend support to his claim.

The first unsolved case of Dickson's problem for which the above method can be used is $n=2^5+1=33$. However, due to the size of this problem (there are 450 conditions on the a_i), a solution was not attempted. Due to the number of solutions found for $p^e+1=17$ the author feels safe in presenting the conjecture that many solutions also exist for $p^e+1=33$.

The first value of n for which it is not known whether or not a complete set of POHC(n) exists is now n=19.

1.2 A More General Problem

More generally, one can ask for a family of k-cycles (cycles of length k) in K_n so that every 2-path lies on exactly λ cycles. Such a family of cycles will be called an exact covering of the 2-paths of K_n by k-cycles.

1.5 DEFINITION: A $C(n,k,\lambda)$ design is a family of k-cycles in K_n in which each 2-path of K_n occurs exactly λ times.

The case k=3 is, of course, trivial since one would simply take as the 3-cycles λ copies of all of the subsets of size 3 of the n-set representing the vertices of K_n.

The case k=n was presented in Section 1.1. That is, we have already discussed all that is known about $C(n,n,1)$ designs. When k=5 the problem is also interesting although not much work has been done in this area. In Chapter 6 this problem is discussed briefly and some results are presented.

The case k=4 is particularly interesting because of its close connection with Steiner quadruple systems. A Steiner quadruple system SQS(n) is an ordered pair (X,β) where β is a family of 4-subsets (blocks) chosen from an n-set, X, so that every 3-subset occurs in exactly one of the blocks. In terms of the graph K_n, this is a covering of the triangles of K_n by K_4 subgraphs so that every triangle is in exactly one of the quadruples. Steiner quadruple systems of order n are also known as $3-(n,4,1)$ designs and it is well known (Hanani, [H1]) that a $3-(n,4,1)$ design exists if and only if $n \equiv 2,4 \pmod 6$. Since a $C(4,4,1)$ design is easily constructed one sees immediately that Hanani's result implies the existence of a $C(n,4,1)$ design whenever $n \equiv 2,4 \pmod 6$.

In general, a $3-(n,4,\lambda)$ design is a family of subsets of size four taken from an n-set so that each 3-subset occurs exactly λ times. For such designs with $\lambda \geq 1$ we have the following result of Hanani [H2].

1.6 THEOREM: Necessary and sufficient conditions for the existence of a 3-$(n,4,\lambda)$ design are

1. $\lambda n \equiv 0 \pmod 2$
2. $\lambda(n-1)(n-2) \equiv 0 \pmod 3$ and
3. $\lambda n(n-1)(n-2) \equiv 0 \pmod 8$.

Again, this gives rise to $C(n,4,\lambda)$ designs. A more general statement is possible, however, by making use of 3-(n,K,λ) designs. A 3-(n,K,λ) design is a family of blocks with elements chosen from an n-set with the property that every 3-subset occurs in exactly λ of them and the size of each is a member of the set K.

The following shows how one can recursively construct $C(n,4,1)$ designs given that certain 3-designs exist. Although Lemma 1.7 can obviously be extended to cycles of length r, $r \geq 3$, it is presented here for $r=4$ as this case is the one studied for the most part.

1.7 LEMMA: If there exists a 3-(n,K,λ) design, and if for every $k \in K$ there exists a covering S of the edges of K_k by 4-cycles so that every 2-path occurs on exactly μ 4-cycles, then the edges of K_n can be covered by 4-cycles so that each 2-path occurs on exactly $\mu \cdot \lambda$ 4-cycles.

PROOF: Replace each block $B_i = \{v_{i1}, \ldots, v_{ik}\}$ of the 3-(n,K,λ) design by the covering S of K_k based on the set B_i. Call this covering S_i. Thus every 2-path having vertices from the elements of the block B_i will occur on exactly μ 4-cycles. Since every 3-element subset of the n-element set occurs in exactly λ blocks we see that every 2-path will occur $\mu \cdot \lambda$ times in the union of the S_i. ∎

The following theorem due to Hanani [H3] can be useful in finding $C(n,k,\lambda)$ designs. In light of Theorem 1.9 which follows, this result does not contribute anything further to the problem of finding $C(n,4,\lambda)$ designs.

1.8 THEOREM: If q is a power of a prime and d is a positive integer then there exists a $3-(q^d+1,q+1,1)$ design.

If one point is deleted from this design we get a $3-(q^d,\{q+1,q\},1)$ design and if two points are deleted we get a $3-(q^d-1,\{q+1,q,q-1\},1)$ design. If r points are deleted from the same block ($3\le r\le q-2$) we get a $3-(q^d+1-r,\{q+1,q,q-1,q+1-r\},1)$ design.

For example, given a $3-(q^d+1-r,\{q+1,q,q-1,q+1-r\},1)$ design and a $C(n,k,\lambda)$ design for each $n\in\{q+1,q,q-1,q+1-r\}$, then by Lemma 1.7 there exists a $C(q^d+1-r,k,\lambda)$ design.

Before continuing, we need to establish some notation and define some terms. We denote by (u,v) the edge incident with vertices u and v and by (u,v,w,x) the cycle of length four containing the four edges (u,v), (v,w), (w,x) and (x,u). By $[x,y,z]$ we mean the path of length two containing the two edges (x,y) and (y,z). For all other graph theory notation the reader is referred to Bondy and Murty [BM].

Consider a graph with n vertices labelled 1, ..., n. The <u>distance</u> between two vertices i and j in this graph is $\min\{i-j,\ j-i\}$, where arithmetic operations are carried out modulo n on the residue class 1, ..., n.

Theorem 1.6 and Lemma 1.7, along with several direct constructions, will enable us to prove the main result of this thesis.

<u>1.9</u> <u>THEOREM:</u> There exists a $C(n,4,\lambda)$ design if and only if one of the following hold.

1. n is even
2. $n\equiv 1 \pmod 4$ and $\lambda\equiv 0 \pmod 2$
3. $n\equiv 3 \pmod 4$ and $\lambda\equiv 0 \pmod 4$.

To see that these conditions are necessary one simply observes that there are $n(n-1)(n-2)/2$ paths of length two in K_n, each 2-path occurs λ times and each 4-cycle contains four 2-paths. Thus $\lambda n(n-1)(n-2)\equiv 0 \pmod 8$

is required. Let the vertices of K_n be 1, 2, ..., n and consider the 4-cycles (1,x,2,y). Since each 2-path [1,z,2] must occur λ times we require $\lambda n \equiv 0 \pmod 2$. Together these two conditions give the three situations as stated in the theorem.

Chapter 2 deals with the proof of Theorem 1.9 and is divided into three sections, each dealing with one of the conditions of the theorem. When there does not exist a $C(n,4,\lambda)$ design we consider the problems of finding minimal covers and maximal packings of the 2-paths in K_n by 4-cycles. These problems are discussed in Chapters 3 and 4, respectively. When n is a multiple of four a $C(n,4,1)$ design always exists and a natural question is to ask if the 4-cycles of the design can be partitioned into parallel classes; Chapter 5 deals with this question. Finally, in Chapter 6, we look quickly at the problem of finding $C(n,5,1)$ designs.

CHAPTER 2

EXACT COVERINGS

Let $L(K_n)$ denote the line graph of K_n and let $\lambda L(K_n)$ be the multigraph in which there are λ edges between each pair of vertices of $L(K_n)$. One can ask the following graph decomposition problem, which is equivalent to asking when a $C(n,4,\lambda)$ design exists. When can the edges of $\lambda L(K_n)$ be decomposed into cycles of length four, each of which satisfies the additional property that the cycle in the line graph corresponds to a cycle in the original complete graph?

The vertices of $L(K_n)$ are labelled by the edges that they represent, and so the vertex (a,b) is the same as the vertex (b,a). In this graph the edge ((a,b),(b,c)) corresponds to the 2-path [a,b,c] in K_n. Thus if a set of 4-cycles of K_n can be found so that each 2-path of K_n occurs on exactly λ 4-cycles then a set of 4-cycles of $L(K_n)$ can be found so that each edge of $L(K_n)$ occurs on exactly λ 4-cycles.

There are basically three types of 4-cycles in the line graph of K_n. Let a, b, c, d and e be five distinct vertices of K_n. In $L(K_n)$, cycles of the form ((a,b), (a,c), (a,d), (a,e)) and ((a,b), (a,c), (a,d), (b,d)) do not correspond to 4-cycles in K_n. However, cycles of the form ((a,b), (b,c), (c,d), (d,a)) do, in fact, define 4-cycles in K_n. It is this type of 4-cycle in $L(K_n)$, which corresponds to the 4-cycle (a,b,c,d) in K_n, that we are interested in.

In this chapter we will prove Theorem 1.9, which is restated below. The proof of the theorem, which is divided into three parts, also appears in [HN]. When n is even the theorem will be proved by considering the graph decomposition problem described above.

1.9 THEOREM: There exists a $C(n,4,\lambda)$ design if and only if one of the following hold.

1. n is even
2. $n \equiv 1 \pmod 4$ and $\lambda \equiv 0 \pmod 2$
3. $n \equiv 3 \pmod 4$ and $\lambda \equiv 0 \pmod 4$.

2.1 <u>Case:</u> n is even

We now show that when n is even, $n=2m$, the complete graph on n vertices can be covered by 4-cycles so that every 2-path occurs exactly once on some 4-cycle. In Theorem 2.1 such a covering set is produced. These coverings can also be constructed recursively from 3-designs.

<u>2.1 THEOREM:</u> The edges of K_{2m} can be covered by 4-cycles so that each 2-path occurs exactly once on some 4-cycle.

<u>PROOF:</u> In this proof addition is modulo $2m-1$ on the residue class 1, 2, ..., $2m-1$.

Label the vertices of K_{2m} with the symbols ∞, 1, 2, ..., $2m-1$ and consider the line graph of K_{2m}, $L(K_{2m})$. We wish to cover the edges of $L(K_{2m})$ by 4-cycles so that we satisfy the conditions

1. each edge of $L(K_{2m})$ is in exactly one 4-cycle and
2. these 4-cycles correspond to 4-cycles in K_{2m}.

Arrange the vertices of $L(K_{2m})$ into a $(2m-1)\times m$ array $A=(a_{i,j})$, where the vertex in cell $a_{i,1}$ is (∞,i) and in cell $a_{i,j}$ is $(i,i+j-1)$, $1\le i\le 2m-1$ and $2\le j\le m$. Now one easily sees that the following 4-cycles of $L(K_{2m})$ satisfy condition 2 as given above:

$$C_{1,i}(0) := ((\infty,1),\ (1,i),\ (i,2i-1),\ (\infty,2i-1)) \qquad 2\le i\le m$$

and

$$C_{i,j}(0) := ((1,i),\ (1,j),\ (j,i+j-1),\ (i,i+j-1)) \qquad 2\le i<j\le m.$$

Notice that the cycle $C_{i,j}$ contains only vertices from columns i and j. For $0\le k\le 2m-2$, develop these $m(m-1)/2$ cycles as follows (see Figures 1 and 2):

$$C_{1,i}(k) := ((\infty,1+k),\ (1+k,i+k),\ (i+k,2i+k-1),\ (\infty,2i+k-1,))$$

and

$$C_{i,j}(k) := ((1+k,i+k),\ (1+k,j+k),\ (j+k,i+j+k-1),\ (i+k,i+j+k-1)).$$

A total of $(2m-1)(m)(m-1)/2$ cycles of length four have just been defined and these contain a total of $2m(2m-1)(m-1)$ edges. This is precisely the number of 2-paths in K_{2m}. Thus if each edge of $L(K_{2m})$ appears in some 4-cycle then it appears in exactly one 4-cycle. We now show that this is indeed the case.

The edges of the line graph can be divided into four categories:

1. Edges of the form $((\infty,i),\ (\infty,j))$.
2. Edges of the form
   a. $((\infty,i),\ (i,i-j+1))$
   b. $((\infty,i),\ (i,i+j-1))$

   which are each incident with one vertex from column 1 and one vertex from column j $(2 \le j \le m)$.
3. Edges of the form $((i-j+1,i),\ (i,i+j-1))$ which are incident with vertices from column j only $(2 \le j \le m)$.
4. Edges of the form
   a. $((i+j-1,i),\ (i,i+k-1))$
   b. $((i+j-1,i),\ (i,i-k+1))$
   c. $((i-j+1,i),\ (i,i+k-1))$
   d. $((i-j+1,i),\ (i,i-k+1))$

   which are each incident with one vertex from column j and one vertex from column k $(2 \le j < k \le m)$. These edges are found by considering a fixed vertex $(i,\ i+j-1)$ in column j. This vertex is adjacent to the vertices $(i,\ i+k-1)$, $(i,\ i-k+1)$, $(i+j-1,\ i+j-k)$ and $(i+j-1,\ i+j+k-2)$ in column k. If we then add $1-j$ to the vertices in the latter two edges just defined we obtain the four edges given above.

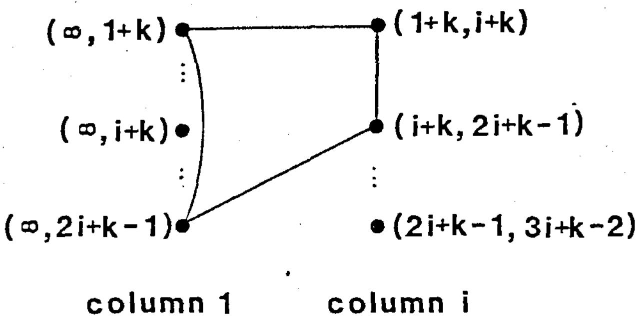


Figure 1

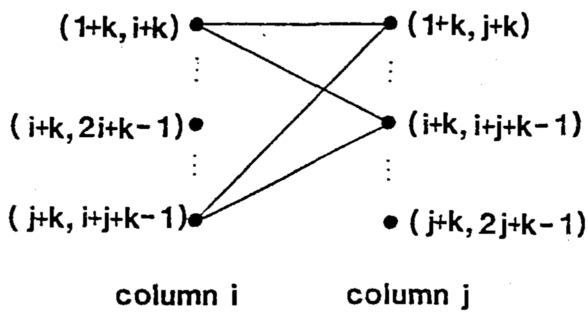


Figure 2

We now check to see that each edge from each of these categories appears in one of the previously defined cycles of $L(K_{2m})$.

1. Suppose without loss of generality that $j>i$. We now consider separately the cases $j-i$ even and $j-i$ odd.
   a. If $j-i$ is even let $r=(j-i+2)/2$. The edge $((\infty,i), (\infty,j))$ belongs to $C_{1,r}(i-1)$.
   b. If $j-i$ is odd let $r=(2m-1+i-j+2)/2$. The edge $((\infty,i), (\infty,j))$ belongs to $C_{1,r}(j-1)$.
2. a. The edge $((\infty,i), (i,i+j-1))$ belongs to $C_{1,j}(i-1)$.
   b. The edge $((\infty,i), (i,i-j+1))$ belongs to $C_{1,j}(i-2j+1)$.
3. The edge $((i-j+1,i), (i,i+j-1))$ belongs to $C_{1,j}(i-j)$.
4. a. The edge $((i+j-1,i), (i,i+k-1)$ belongs to $C_{j,k}(i-1)$.
   b. The edge $((i+j-1,i), (i,i-k+1)$ belongs to $C_{j,k}(i-k)$.
   c. The edge $((i-j+1,i), (i,i+k-1)$ belongs to $C_{j,k}(i-j)$.
   d. The edge $((i-j+1,i), (i,i-k+1)$ belongs to $C_{j,k}(i-j-k+1)$.

So each edge of $L(K_{2m})$ belongs to some 4-cycle and therefore each 2-path of K_{2m} belongs to a 4-cycle. ∎

Hanani [H2] has shown that for every even n, $n\geq4$, there exists a $3-(n,\{4,6\},1)$ design. Applying Lemma 1.7 and using the trivial $C(4,4,1)$ design and the easily constructed $C(6,4,1)$ design will also yield the desired coverings.

2.2 Case: $n \equiv 1 \pmod 4$

When $n \equiv 1 \pmod 4$ it is not possible to cover the edges of K_n with 4-cycles so that each 2-path occurs exactly once. However, it is possible to cover the edges of K_{4m+1} with 4-cycles so that each 2-path occurs exactly λ times, where $\lambda \equiv 0 \pmod 2$. Since a covering of the edges of K_{4m+1} by 4-cycles so that each 2-path occurs exactly λ times can be obtained by taking $\lambda/2$ copies of a covering of K_{4m+1} by 4-cycles so that each 2-path occurs exactly twice, we only consider the case $\lambda=2$.

2.2 THEOREM: When $n \equiv 1,5 \pmod{12}$ the edges of K_n can be covered by 4-cycles so that every 2-path occurs exactly twice.

PROOF: This follows from Theorem 1.6 and Lemma 1.7. ■

2.3 LEMMA: The edges of K_9 can be covered by 4-cycles so that each 2-path occurs on exactly two 4-cycles.

PROOF: Take two copies of the C(8,4,1) design that is obtained by replacing each K_4 in Figure 3 by the C(4,4,1) design on those points. Remove the following fourteen 4-cycles (in bold in the figure) once only:

(1,2,3,4), (1,2,5,8), (1,6,2,7), (1,3,7,5), (1,3,6,8), (1,4,8,7),
(2,3,5,6), (2,7,3,8), (2,4,7,5), (2,4,6,8), (3,5,4,8), (3,4,7,6),
(5,6,7,8), (1,5,4,6).

Now define the diagonals of a 4-cycle (a,b,c,d) to be the two unordered pairs of vertices {a,c} and {b,d}. Each unordered pair of vertices of K_8 occurs once as a diagonal and twice as an edge in the above fourteen 4-cycles. If (a,b,c,d) is one of these fourteen 4-cycles, then replace it with the four 4-cycles (9,b,c,d), (a,9,c,d), (a,b,9,d) and (a,b,c,9).

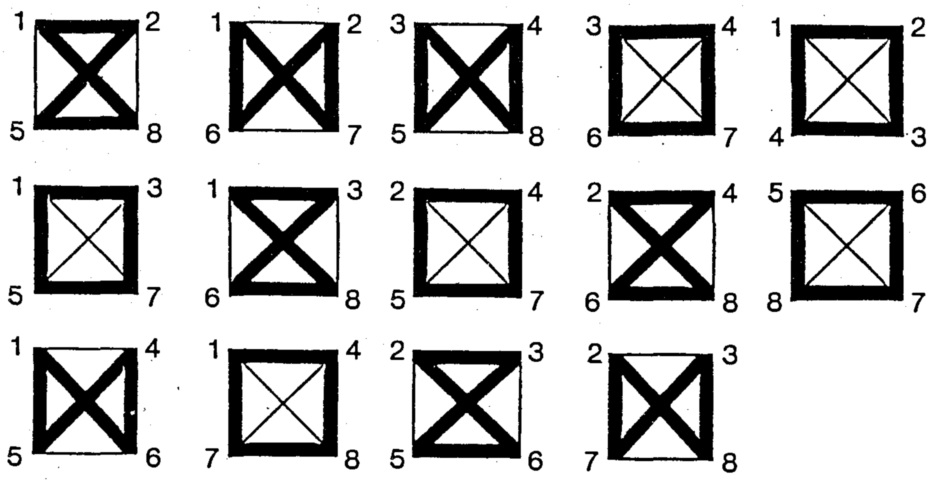


Figure 3

Thus all 2-paths [x,9,y] and [9,x,y] occur twice and all 2-paths not containing vertex 9 still occur twice. ∎

We now present three theorems which can be used to construct $C(4m+1,4,2)$ designs recursively for all remaining values of n, where $n=4m+1$. First, we present a definition and construction which will be used in the proofs of these theorems.

2.4 DEFINITION: An orthogonal array $OA(n,k)$ of strength t ($1\le t\le k<n$) and index 1 is a k by n^t array of elements from an n-set such that for any fixed set of t rows in the array, each ordered t-tuple from the n-set occurs exactly once as a column.

2.5 CONSTRUCTION: It is known ([T], [W1] and [W2]) that whenever $n\ge4$ and $n\ne6,10$ there exist three mutually orthogonal Latin squares of order n. If A, B and C are three mutually orthogonal Latin squares of order n based on the set $\{1, 2, \ldots, n\}$, then the set of n^3 column vectors $(i,j,A(x,y),B(x,y))^T$ (where $1\le i,j\le n$ and (x,y) runs through all ordered pairs such that $C(x,y)=C(i,j)$) is an $OA(n,4)$ of strength 3. This orthogonal array contains an $OA(n,4)$ of strength 2 as can be seen by taking only those n^2 column vectors $(i,j,A(i,j),B(i,j))^T$. Call these n^2 columns Type I and call the remaining n^3-n^2 columns Type II.

2.6 THEOREM: If $m \geq 4$, $m \neq 6$ or 10, λ is even and there exists a $C(m+1,4,\lambda)$ design then there exists a $C(4m+1,4,\lambda)$ design.

PROOF: Let $S_i=\{S_i(1), \ldots, S_i(m)\}$, $1 \leq i \leq 4$, be four disjoint sets of size m and let ∞ be an element which does not belong to any of these sets. Now let $S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{\infty\}$ be the vertex set of K_{4m+1}. We want to partition the edges of this graph into 4-cycles so that every 2-path occurs exactly λ times. We do this in three steps (see Figure 4).

Step 1. Take four copies of a $C(m+1,4,\lambda)$ design in which the vertex set is, in turn, $S_1 \cup \{\infty\}$, $S_2 \cup \{\infty\}$, $S_3 \cup \{\infty\}$ and $S_4 \cup \{\infty\}$. Every 2-path containing only vertices from S_i occurs on exactly λ 4-cycles as does every 2-path that contains ∞ and two vertices from S_i, $1 \leq i \leq 4$.

Step 2. We now form 4-cycles so that every 2-path containing two vertices from S_i and one vertex from S_j appears λ times for all pairs i,j with $i \neq j$.

If m is even we take a 1-factorization $F_1, F_2, \ldots, F_{m-1}$ of K_m with vertex set S_i and a 1-factorization $G_1, G_2, \ldots, G_{m-1}$ of K_m with vertex set S_j. For each edge (u,v) of F_k and each edge (x,y) of G_k construct a K_4 based on the vertex set {u,v,x,y}. This gives a total of $(m-1)(m^2/4)$ K_4's. Note that every pair of vertices from S_i appears with every vertex from S_j exactly once so that all possible triples with two vertices from S_i and one vertex from S_j appears exactly once in some K_4. For each of these we take a $C(4,4,\lambda)$ design.

If m is odd we take a Hamilton decomposition $F_1, F_2, \ldots, F_{(m-1)/2}$ of K_m with vertex set S_i and a Hamilton decomposition $G_1, G_2, \ldots, G_{(m-1)/2}$ of K_m with vertex set S_j. For each edge (u,v) of F_k and each edge (x,y) of G_k construct a K_4 based on the vertex set {u,v,x,y}. This will give a total of $(m-1)(m^2/2)$ K_4's. Note that every pair of vertices from S_i appears

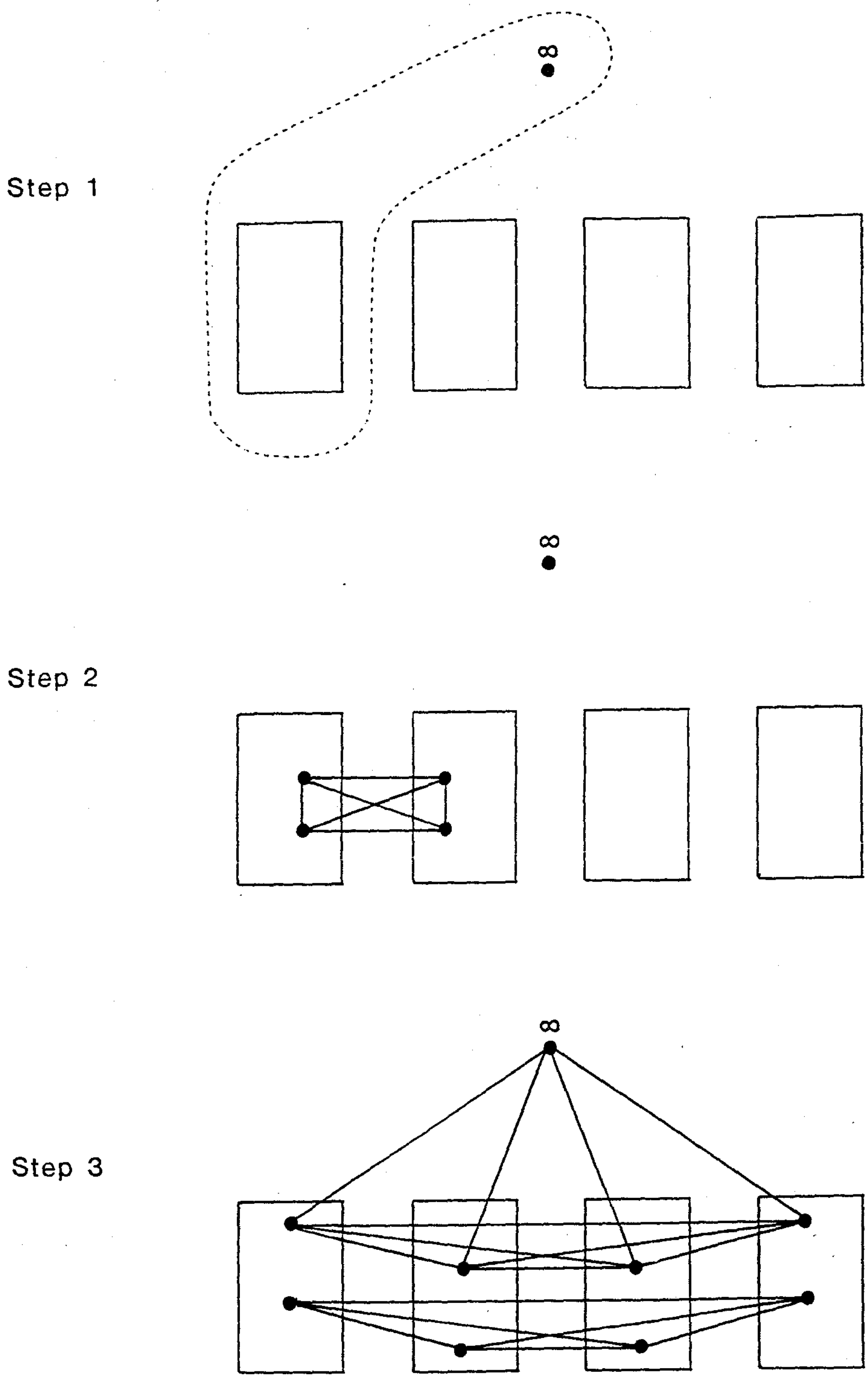


Figure 4

with every vertex from S_j exactly twice since in F_k and G_k each vertex has degree two. Thus all possible triples with two vertices from S_i and one vertex from S_j appear in exactly two different K_4's, and for each of these we take a $C(4,4,\lambda/2)$.

Step 3. Let A, B and C be three mutually orthogonal Latin squares of order m. These exist since $m \geq 4$ and $m \neq 6,10$. Define an OA(m,4) of strength 3 as in Construction 2.5.

We now define the 4-cycles which cover all the 2-paths remaining; that is, all the 2-paths which contain at most one point from each of the sets S_1, S_2, S_3, S_4 and $\{\infty\}$.

For each Type I column $(i,j,a,b)^T$ in the OA(m,4), we view the set $\{S_1(i), S_2(j), S_3(a), S_4(b), \infty\}$ as the vertex set of a K_5. Since the Type I columns give an OA(m,4) of strength 2 this means that ∞ appears with each pair of vertices x and y, where $x \in S_u$ and $y \in S_v$, $1 \leq u < v \leq 4$. Now take a $C(5,4,\lambda)$ design on each of these vertex sets. Such a design exists since λ is even.

For each Type II column $(i,j,a,b)^T$ in the OA(m,4) we take a $C(4,4,\lambda)$ design with vertex set $\{S_1(i), S_2(j), S_3(a), S_4(b)\}$. Since the Type I and Type II columns together give an OA(m,4) of strength 3 we see that among these coverings and the coverings of size five defined in the previous paragraph we get all 2-paths $[s_i,s_j,s_k]$, where $s_i \in S_i$, $s_j \in S_j$ and $s_k \in S_k$ and i,j and k are distinct.

Combining the 4-cycles from each of the three steps gives a set of 4-cycles which cover all the 2-paths of K_{4m+1} exactly λ times each. ■

2.7 **THEOREM:** If

1. $t \equiv 2,4 \pmod 6$ and $t \geq 8$,
2. λ is even,
3. there exist t-2 mutually orthogonal Latin squares of order m, and

4. there exists both a $C(m+1,4,\lambda)$ design and a $C(t+1,4,\lambda)$ design, then there exists a $C(tm+1,4,\lambda)$ design.

PROOF: Let $S_i:=\{S_i(1), \ldots, S_i(m)\}$, $1\le i\le t$, be t mutually disjoint sets of size m and let ∞ be a new element not in the union of the S_i's. We want to construct a $C(tm+1,4,\lambda)$ design on the vertex set $S_1 \cup S_2 \cup \ldots \cup S_t \cup \{\infty\}$. The first two steps are to be carried out as in Theorem 2.6, but with $1\le i\le t$. Note that since t is even, a $C(t+1,4,\lambda)$ design cannot exist when λ is odd (see the remarks at the beginning of the section).

Step 3. Let $L_3, L_4, \ldots, L_t$ be t-2 mutually orthogonal Latin squares of order m and define an OA(m,t) of strength 2 with columns $(i, j, L_3(i,j), \ldots, L_t(i,j))^T$, $1\le i,j\le m$.

For each column $(i, j, a_3, \ldots, a_t)^T$ in the orthogonal array let $\{S_1(i), S_2(j), S_3(a_3), \ldots, S_t(a_t), \infty\}$ be the vertex set of a K_{t+1} and take a $C(t+1,4,\lambda)$ design based on this set.

Hanani proved that whenever $t \equiv 2,4 \pmod 6$ then there exists a 3-(t,4,1) design. Now suppose we have such a 3-design based on the set $\{1,2,\ldots,t\}$. For each block $\{w,x,y,z\}$ in the design consider only rows w,x,y and z of the orthogonal array. These give an OA(m,4) of strength 2.

This orthogonal array can easily be extended to an OA(m,5) of strength 2 by adding one more row of the OA(m,t). Consider the elements of the first two rows of this array to define rows and columns, respectively, and the elements of the third and fourth rows of this array to be the entries of Latin squares A and B. Then it is easy to see that A and B are orthogonal. Moreover, if we consider any fixed element in the fifth row then the columns in which it appears define a common transversal in A and in B. That is, the entries in the fifth row can be taken as entries of a Latin square C which is also orthogonal to A and B. Returning to Construction 2.5 we see that the OA(m,4) of strength 2 can be imbedded in an OA(m,4) of strength 3. The m^3-m^2 columns added to the OA(m,4) of strength 2 to obtain the OA(m,4) of strength 3 are called (as before) Type II columns.

For each Type II column $(a,b,c,d)^T$ take a $C(4,4,\lambda)$ design based on the set $\{S_w(a),\ S_x(b),\ S_y(c),\ S_z(d)\}$.

The union of these 4-cycles gives the required set of 4-cycles which cover each 2-path exactly λ times. ■

2.8 THEOREM: If

1. $m\neq6$ and $m\neq10$,
2. $m\lambda$ is even, and
3. there exists a $C(m,4,\lambda)$ design and a $C(t,4,\lambda)$ design,

then there exists a $C(mt,4,\lambda)$ design.

PROOF: Let $S_i:=\{S_i(1),\ \ldots,\ S_i(m)\}$, $1\le i\le t$, be t mutually disjoint sets of size m and let $S_1 \cup S_2 \cup \ldots \cup S_t$ be the vertex set of K_{mt}. We want to construct a $C(mt,4,\lambda)$ design on this vertex set. As before, we do this in three steps.

Step 1. For $1\le i\le t$ take a $C(m,4,\lambda)$ design in which the vertex set is S_i.

Step 2. This is the same as Step 2 of Theorem 2.6. If m is even we use a $C(4,4,\lambda)$ design and if m is odd we use a $C(4,4,\lambda/2)$ design.

Step 3. Let A, B and C be three mutually orthogonal Latin squares of order m based on $\{1,\ \ldots,\ m\}$ and define an OA(m,4) of strength 3 using the method in Construction 2.5.

Replace each 4-cycle (w,x,y,z) in the $C(t,4,\lambda)$ design by the m^3 4-cycles $(S_w(a),\ S_x(b),\ S_y(c),\ S_z(d))$, where $(a,b,c,d)^T$ is a column in the OA(m,4).

Combining these three steps gives the required result. ■

2.9 COROLLARY: The edges of K_n, $n\equiv 1 \pmod 4$, can always be covered by 4-cycles so that each 2-path occurs exactly twice.

PROOF: When $n\equiv 1,5 \pmod{12}$ we use Theorem 2.2. This leaves the case $n\equiv 9 \pmod{12}$.

For $n=9$ use Lemma 2.3. Theorem 2.6 with $m=5$ and $\lambda=2$ yields $n=21$ and with $m=8$ and $\lambda=2$ yields $n=33$. If $n=48k+9$ and $n\neq 441$ use Theorem 2.7 with $t=8$, $m=6k+1$ and $\lambda=2$. Using [B] and [W3] we see that there exist six mutually orthogonal Latin squares of order $6k+1$, where $6k+1\neq 55$. The conditions of Theorem 2.7 are thus satisfied for all values of n other than $n=441$. In this case we use Theorem 2.8 with $m=9$, $t=49$ and $\lambda=2$. For $n=48k+21$, $48k+33$ or $48k+45$ use Theorem 2.6 with $m=12k+5$, $12k+8$ or $12k+11$, respectively, and $\lambda=2$. We know that there exists a $C(12k+6,4,2)$ design and a $C(12k+12,4,2)$ design from Theorem 2.1, and there exists a $C(12k+9,4,2)$ design since there exists a $C(12k^*+9,4,2)$ design for all $k^*<4k$. ■

2.3 Case: $n\equiv 3 \pmod 4$

When $n\equiv 3 \pmod 4$ a simple counting argument shows that it is not possible to cover the edges of K_n with 4-cycles so that each 2-path occurs exactly λ times where $\lambda\equiv 1,2,3 \pmod 4$. In this section we show that it is, however, possible to cover the edges of K_{4m+3} with 4-cycles so that each 2-path occurs exactly λ times, where $\lambda\equiv 0 \pmod 4$. Since a $C(4m+3,4,\lambda)$ design can be obtained by taking $\lambda/4$ copies of a $C(4m+3,4,4)$ design we only consider the case $\lambda=4$.

2.10 THEOREM: When $n\equiv 7,11 \pmod{12}$ there exists a $C(n,4,4)$ design.

PROOF: This follows from Lemma 1.7 and Theorem 1.6. ■

2.11 THEOREM: There exists a $C(15,4,4)$ design.

PROOF: Take four copies of the $C(14,4,1)$ design as given by Theorem 2.1.

For $0 \le i \le 12$, remove the following 4-cycles <u>once only</u>:

$(1+i,\infty,9+i,5+i)$, $(1+i,\infty,13+i,7+i)$, $(1+i,2+i,4+i,3+i)$, $(1+i,2+i,7+i,6+i)$, $(1+i,3+i,7+i,5+i)$, $(1+i,4+i,9+i,6+i)$, $(1+i,4+i,10+i,7+i)$.

Each unordered pair of vertices from the 14 vertices occurs twice as a diagonal and four times as an edge in these ninety-one 4-cycles. Replace each 4-cycle (w,x,y,z) in the above set by the four 4-cycles $(15,x,y,z)$, $(w,15,y,z)$, $(w,x,15,z)$ and $(w,x,y,15)$. The 2-path $[a,15,b]$ occurs four times since it occurs twice for each time the diagonal a-b occurs. The 2-path $[a,b,15]$ occurs once for every time the edge (a,b) occurs and so this 2-path occurs four times. All 2-paths not containing 15 occur four times each as before. So when $\lambda=4$ we can find a $C(15,4,\lambda)$ design. ∎

<u>2.12</u> <u>LEMMA:</u> There exists a $C(27,4,4)$ design.

<u>PROOF:</u> There exists a 3-$(27,6,4)$ design [HHK] and we use this, together with Lemma 1.7 and Theorem 2.1, to prove the lemma. ∎

<u>2.13</u> <u>THEOREM:</u> If $n \equiv 3 \pmod 4$ then the edges of K_n can be covered by 4-cycles so that every 2-path occurs on exactly four 4-cycles.

<u>PROOF:</u> Let $K=\{4,5,6,7,9,11,13,15,19,23,27,29,31\}$. Then for every $k \in K$ we have a $C(k,4,4)$ design. Hanani [H3] has shown that for all $n \ge 4$ there exists a 3-$(n,K,1)$ design. The result now follows using Lemma 1.7. ∎

By combining Theorems 2.1 and 2.13 and Corollary 2.9 we see that there exists a $C(n,4,4)$ design for every $n \ge 4$.

2.4 New Construction Techniques for 3-designs

The construction techniques of section 2.2 can also be used to obtain new 3-(n,K,λ) designs. For instance, using Theorem 2.6 we see that for every $m\geq 2$ $(2m\neq 6,10)$ there exists a 3-$(8m+1,\{2m+1,5,4\},1)$ design and for every $m\geq 2$ there exists a 3-$(8m+5,\{2m+2,5,4\},2)$ design.

To get the first design, we take one copy of each of the four blocks of size $2m+1$ based on the sets $S_i \cup \{\infty\}$, $1\leq i\leq 4$, as in Step 1 of the proof. Next, we take one copy of each block of size four given in the first half of Step 2. Finally, we take one copy each of all blocks of size five and four as defined in Step 3. It is easy to see that we get all triples from the set of size $8m+1$.

To get the second design, we take two copies of each of the four blocks of size $2m+2$ based on the sets $S_i \cup \{\infty\}$, as in Step 1. We next take one copy of each block of size four as given in the second half of Step 2. Finally we take two copies of each block of size five and four defined in Step 3.

CHAPTER 3
MINIMAL COVERS

By a <u>covering</u> of the 2-paths of K_n by 4-cycles we mean a collection of cycles of length four which contain all of the $n(n-1)(n-2)/2$ paths on three vertices (2-paths) at least once each. Coverings are also defined for larger values of λ. That is, we can also ask for the minimum number of 4-cycles so that each 2-path of K_n is on at least λ 4-cycles.

A. Hartman et al. [HMM] have looked at the related problem of covering all of the triples of an n-set by quadruples. When there exists an SQS(n) the triples can, of course, each be covered exactly once by quadruples. J. Schoenheim (see [HMM]) showed that at least $\lceil n/4 \lceil (n-1)/3 \lceil (n-2)/2 \rceil\rceil\rceil$ quadruples are needed to cover the triples of a set of size n. In [HMM], it was shown, using an existing construction and some recursive techniques, that for all $n \geq 52423$ one could cover all the triples of an n-set by exactly $\lceil n/4 \lceil (n-1)/3 \lceil (n-2)/2 \rceil\rceil\rceil$ quadruples. They also showed that if a certain group divisible design on 54 points exists then this bound can be lowered considerably.

Since the number of 2-paths in K_{2m+1} is not a multiple of four we see that it is not possible to cover the 2-paths of K_{2m+1} by 4-cycles so that each 2-path occurs exactly once. The following theorem gives a lower bound for the minimum number of 4-cycles needed.

<u>**3.1 LEMMA:**</u> The minimum number of 4-cycles needed to cover all the 2-paths in K_{2m+1} at least once is $m^3 + \lceil m^2/2 \rceil$.

<u>**PROOF:**</u> For any two vertices a and b of K_{2m+1} we have $2m-1$ different 2-paths of the form [a,x,b]. Any 4-cycle containing such a 2-path must contain another 2-path of the form [a,y,b]. Thus the number of 4-cycles containing a 2-path of the form [a,x,b] must be at least $\lceil (2m-1)/2 \rceil = m$. There are $m(2m+1)$ ways in which a and b can be chosen, but the 4-cycle

containing the 2-paths [a,x,b] and [a,y,b] also contains the 2-paths [x,a,y] and [x,b,y]. Thus we need at least $\lceil (m/2)m(2m+1) \rceil = m^3 + \lceil m^2/2 \rceil$ 4-cycles to cover all the 2-paths in K_{2m+1}. ■

3.2 DEFINITION: $L_i(n)$ is the least integer such that there exist $L_i(n)$ 4-cycles which cover all the 2-paths of K_n at least i times each.

In Chapter 2 we saw by Theorem 2.1 that $L_1(4m) = 8m^3 - 6m^2 + m$ and that $L_1(4m+2) = 8m^3 + 6m^2 + m$. Using the above theorem we thus see that $L_1(4m+1) \geq 8m^3 + 2m^2$. In fact, we will now demonstrate in Theorem 3.3 that $L_1(4m+1) = 8m^3 + 2m^2$ and so the lower bound calculated in Lemma 3.1 is attained.

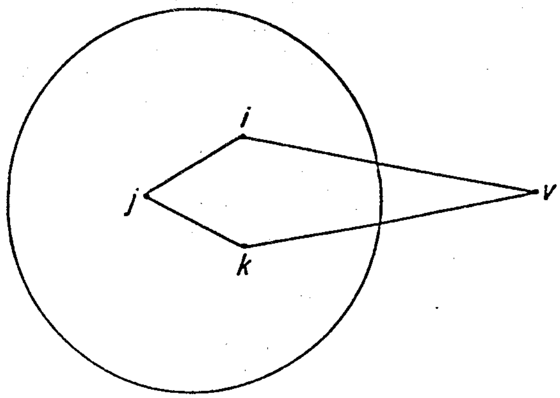
3.3 THEOREM: The edges of K_{4m+1} can be covered by $8m^3 + 2m^2$ 4-cycles so that each 2-path occurs at least once on some 4-cycle.

PROOF: In this proof addition is modulo 4m on the residue class 1, ..., 4m.

Label the vertices of K_{4m+1} with the elements of $\{1, \ldots, 4m, \nu\}$ and label the vertices of K_{4m} with the elements of $\{1, \ldots, 4m\}$. Thus we can write $K_{4m} = K_{4m+1} - \{\nu\}$.

We now sketch the proof of this theorem; complete details follow later. We will find a set of $8m^3 + 2m^2$ cycles of length four which contain each 2-path of K_{4m+1} at least once by first taking the set of $8m^3 - 6m^2 + m$ cycles of length four which cover all the 2-paths of K_{4m} exactly once each. This is done according to the proof of Theorem 2.1.

Next, we add to these $L_1(4m)$ 4-cycles a further $8m^2$ cycles of length four (see Figure 5). Note that each 4-cycle in Figure 5 contains a 2-path that has already been covered. These new 4-cycles will be chosen so that they contain all 2-paths of the form $[\nu,*,*]$ and $[*,\nu,*]$ at least once each.



K_{4m}

Figure 5

That is, if we consider the 4-cycle (ν,i,j,k) then for a fixed j we want vertices i and k to take on all values r, $1 \le r \ne j \le 4m$. In this way we get all 2-paths $[\nu,r,j]$. As j varies, $1 \le j \le 4m$, we see that we must also require the vertices i and k to take on all values of r and s, $1 \le r < s \le 4m$. In this way we get all 2-paths of the form $[r,\nu,s]$.

However, $L_1(4m) + 8m^2 = 8m^3 + 2m^2 + m$ and so it will be necessary to delete m 4-cycles so that in the set of 4-cycles which remain each 2-path still occurs on at least one 4-cycle. These m 4-cycles, which will be chosen so that they are vertex disjoint in order to facilitate vertex relabeling, will be removed from the covering of K_{4m}.

Based on these m disjoint 4-cycles we will define 4m subgraphs G_r, $1 \le r \le 4m$, of K_{4m} as follows.

1. For each of the m 4-cycles (h,i,j,k) that will be removed from the covering of K_{4m} we define the four subgraphs G_h, G_i, G_j and G_k so that $(i,k) \in E(G_h)$, $(h,j) \in E(G_i)$, $(i,k) \in E(G_j)$ and $(h,j) \in E(G_k)$.
2. In G_r every vertex other than r has positive degree and vertex r has degree 0.
3. The union of these 4m subgraphs is a multigraph $K_{4m} \cup F$, where F is a 1-factor in K_{4m}.

For each r, $1 \le r \le 4m$, and for each $(x,y) \in G_r$ we define the 4-cycle (x,r,y,ν). By (1) above, we see that we will have defined the following four 4-cycles

(i,h,k,ν), (h,i,j,ν), (i,j,k,ν), (h,k,j,ν)

for each 4-cycle (h,i,j,k) that will be removed. Thus even after the removal of such a 4-cycle every 2-path on $\{1, \ldots, 4m\}$ still occurs at least once.

By (2) above, it is clear that we get all 2-paths $[r,x,\nu]$ for $1 \le r,x \le 4m$, $r \ne x$.

Since (by (3)) the union of these 4m subgraphs contains K_{4m} we see that we get all 2-paths $[x,\nu,y]$ for every edge $(x,y) \in E(K_{4m})$.

Thus by defining these $8m^2$ 4-cycles based on the subgraphs G_r we see that

1. all 2-paths on $\{1, \ldots, 4m, \nu\}$ have been covered and
2. all 2-paths in the m disjoint 4-cycles have been covered at least twice and so these m cycles can be removed.

Thus we have covered all the 2-paths of K_{4m+1} by

$$(8m^3 - 6m^2 + m) + 8m^2 - m = 8m^3 + 2m^2$$

cycles of length four so that each 2-path occurs at least once.

The ideas for the proof that we have just discussed are now presented in more detail.

By the proof of Theorem 2.1, a set of 4-cycles covering all the 2-paths of K_{4m} contains the following m 4-cycles:

$(\infty, 1, 2, 3)$	(from $C_{1,2}(0)$)	
$(4a, 4a+1, 4a+3, 4a+2)$	(from $C_{2,3}(4a-1)$)	$1 \le a \le m-1$.

Since these 4-cycles are disjoint, then by a suitable relabelling of the vertices, which changes $\{\infty, 1, \ldots, 4m-1\}$ to $\{1, 2, \ldots, 4m\}$, they can be written:

$(i, m+i, 2m+i, 3m+i)$ $\quad 1 \le i \le m$.

The desired covering of all the 2-paths of K_{4m+1} by 4-cycles will contain all the 4-cycles of the $C(4m,4,1)$ design as described in Theorem 2.1, with vertices relabelled as above, except for the aforementioned m vertex disjoint 4-cycles. This gives $8m^3 - 6m^2$ 4-cycles. The remaining 4-cycles to be added will now be defined.

Consider the graph G_1 (see Figure 6) containing the following edges:

$$(2, 4m), (3, 4m-1), \ldots, (m+1, 3m+1), (m+1, 3m),$$
$$(m+2, 3m-1), \ldots, (2m, 2m+1).$$

Note that vertex 1 has degree 0. All other graphs G_r, $1 \le r \le 4m$, are obtained from this by rotating G_1 so that vertex r has degree 0 (and vertex m+r has degree 2).

In each subgraph G_r all distances appear exactly once: the distances 2, 4, ..., 2m occur in the first m edges defined and the distances 1, 3, ..., 2m-1 occur in the last m edges defined. Thus in the union of these 4m graphs each edge of K_{4m} appears once except for edges of distance 2m which appear twice.

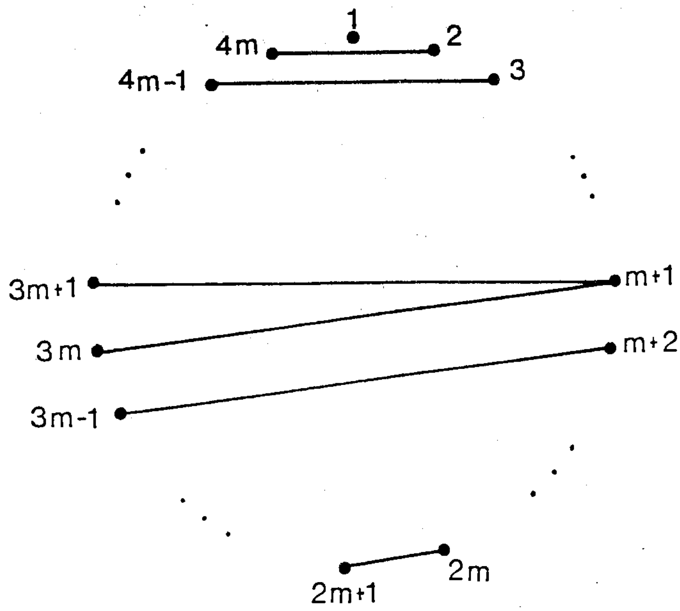
As was mentioned before, if the edge (x,y) belongs to G_r then we include the 4-cycle (x,r,y,ν) in our covering of the 2-paths of K_{4m+1}. The number of 4-cycles which were added was $4m \cdot 2m = 8m^2$, where 4m is the number of graphs G_r and 2m is the number of edges in G_r. We must now check that these $8m^3 + 2m^2$ cycles of length 4 do, in fact, cover all the 2-paths of K_{4m+1} exactly once each.

It suffices to show that

1*. all 2-paths on the m vertex disjoint 4-cycles that were removed occur in some 4-cycle of the covering and

2*. all 2-paths containing ν occur on some 4-cycle.

The edge $(m+r,3m+r)$ belongs to G_r, $1 \le r \le 4m$, so that we get the 2-path $[m+r,r,3m+r]$. Since all 2-paths on the 4-cycles that were removed were of this form we see that (1*) is satisfied. Each edge (x,y) of K_{4m} belongs to



G_1

Figure 6

some graph G_r so that we get the 2-path $[x,\nu,y]$. For each r and for each $x \neq r$ there is an edge incident with vertex x in G_r so that we get the 2-path $[r,x,\nu]$. Thus (2*) is satisfied.

Thus we have a solution to the covering problem for K_{4m+1} using

$$L_1(4m) + 8m^2 - m = (8m^3 - 6m^2 + m) + 8m^2 - m = 8m^3 + 2m^2$$

4-cycles. Thus $L_1(4m+1) = 8m^3 + 2m^2$. ■

Using Lemma 3.1 we find that $L_1(4m+3) \geq 8m^3 + 14m^2 + 8m + 2$. We show here that equality is also obtained in this case.

3.4 THEOREM: The edges of K_{4m+3} can be covered by $8m^3 + 14m^2 + 8m + 2$ 4-cycles so that each 2-path occurs at least once on some 4-cycle.

PROOF: This proof is similar to the proof of Theorem 3.3. The main difference lies in the way the subgraphs G_i are defined.

Addition is modulo 4m+2 on the residue class 1, ..., 4m+2. Label the vertices of K_{4m+3} with the elements $\{1, \ldots, 4m+2, \nu\}$ and label the vertices of K_{4m+2} with the elements $\{1, \ldots, 4m+2\}$. Thus we can write $K_{4m+2} = K_{4m+3} - \{\nu\}$.

By the proof of Theorem 2.1, the set of 4-cycles covering all the 2-paths of K_{4m+2} contains the following m disjoint 4-cycles:

$(\infty, 1, 2, 3)$ (from $C_{1,2}(0)$)

$(4a, 4a+1, 4a+3, 4a+2)$ (from $C_{2,3}(4a-1)$) $1 \le a \le m-1$.

Since these 4-cycles are disjoint, then by a suitable relabelling of the vertices, which changes $\{\infty, 1, \ldots, 4m+1\}$ to $\{1, 2, .., 4m+2\}$, they can be written:

$(i, m+1+i, 2m+1+i, 3m+2+i)$ $1 \le i \le m$.

Note that vertices m+1 and 3m+2 do not occur on any of the m disjoint cycles.

As in Theorem 3.3, the covering of all the 2-paths of K_{4m+3} by 4-cycles will contain all the 4-cycles of the C(4m+2,4,1) design as described in Theorem 2.1 (with vertices relabelled as above), with the exception of the above m vertex disjoint 4-cycles. This gives $8m^3 + 6m^2$ 4-cycles. The remaining 4-cycles to be added will now be defined.

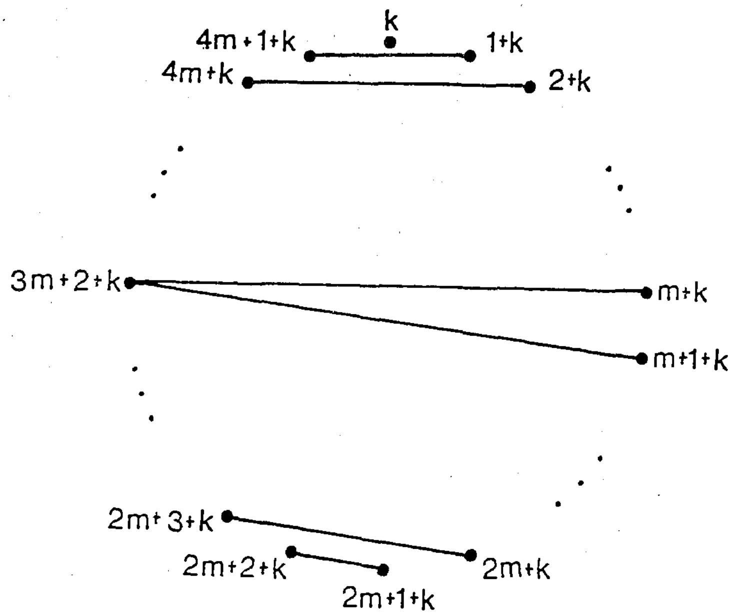
Let the multigraph G be given by $G = K_{4m+2} \cup \{(i,j) : |i-j|=2m+1\}$. We will now define a family of graphs whose union is G.

For $1 \le k \le m$ and $2m+2 \le k \le 3m+1$, let H_k be the graph containing the following edges (see Figure 7):

$$(1+k, 4m+1+k), (2+k, 4m+k), \ldots, (m+k, 3m+2+k),$$
$$(m+1+k, 3m+2+k), \ldots, (2m+k, 2m+3+k), (2m+1+k, 2m+2+k).$$

For $m+2 \le k \le 2m+1$ and $3m+3 \le k \le 4m+2$, let H_k be the graph containing the following edges (see Figure 8):

$$(1+k, 4m+1+k), (2+k, 4m+k), \ldots, (m+k, 3m+2+k),$$
$$(m+k, 3m+1+k), \ldots, (2m-1+k, 2m+2+k), (2m+k, 2m+1+k).$$



The subgraphs H_k, $1 \leq k \leq m$ and $2m+2 \leq k \leq 3m+1$

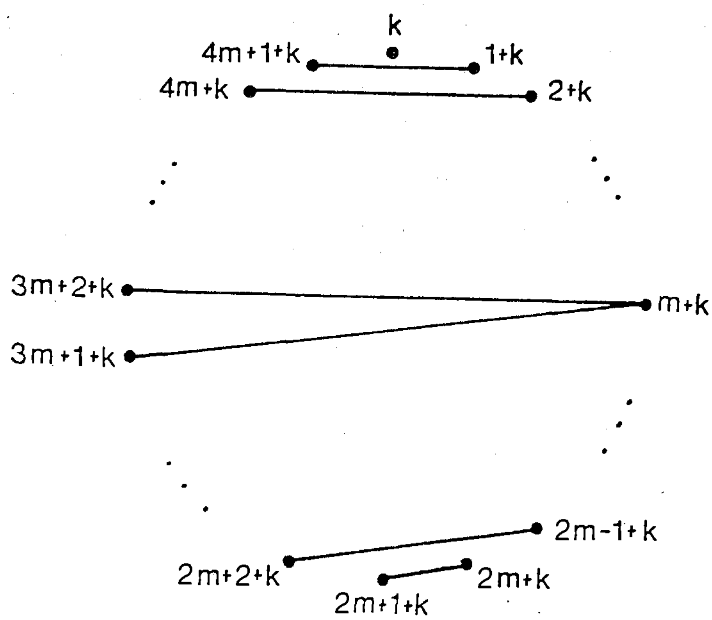
Figure 7

The subgraphs H_{m+1} and H_{3m+2} have not yet been given, and are described below. Define H_{m+1} to be the graph containing the following edges: (1, 4m+2), (2, 4m+1), ..., (m+1, 3m+2), ..., (2m+1, 2m+2) and define H_{3m+2} to be the graph containing: (m, m+2), (m-1, m+3), ..., (3m+1, 3m+3), (m+1, 3m+2). It is not difficult to check that the union of these 4m+2 subgraphs H_i is G.

We now define

$$G_{m+1} = H_{m+1} \cup (m+2,\ 3m+2) - (m+1,\ 3m+2)$$
$$G_{3m+2} = H_{3m+2} \cup (m+1,\ 3m+3) - (m+1,\ 3m+2)$$
$$G_1 = H_1 \cup (m+1,\ 3m+2) - (m+1,\ 3m+3)$$
$$G_{2m+2} = H_{2m+2} \cup (m+1,\ 3m+2) - (m+2,\ 3m+2).$$

For all other values of i we define $G_i = H_i$. Since the union of the 4m+2 subgraphs H_i is G it is trivial to check that the union of the 4m+2 subgraphs G_i is also G. Note that in G_k only vertex k has degree 0.



The subgraphs H_k, $m+2 \le k \le 2m+1$ and $3m+3 \le k \le 4m+2$

Figure 8

As in Theorem 3.3, if the edge (x,y) belongs to G_k, $1 \le k \le 4m+2$, then we will include the 4-cycle (x,k,y,ν) in the covering of the 2-paths of K_{4m+3} by 4-cycles. By doing so we add a further $8m^2 + 8m + 2$ 4-cycles.

The procedure to check for the occurence of all 2-paths on $\{1, \ldots, 4m+2, \nu\}$ is similar to that for Theorem 3.3 and is thus omitted.

The 2-paths of K_{4m+3} can be covered by

$$L_1(4m+2) - m + 8m^2 + 8m + 2 = (8m^3 + 6m^2 + m) - m + 8m^2 + 8m + 2 = 8m^3 + 14m^2 + 8m + 2$$

4-cycles. Thus $L_1(4m+3) = 8m^3 + 14m^2 + 8m + 2$. ∎

3.5 LEMMA: When $n=4m+1$ and $\lambda=2k+1$ at least

$$16m^3k - mk + 8m^3 + 2m^2$$

4-cycles are needed to cover the 2-paths of K_n so that each 2-path occurs at least λ times.

PROOF: This number is calculated in a similar way as the bound in Lemma 3.1. For any two vertices a and b we need $\lceil (2k+1)(4m-1)/2 \rceil = 4mk + 2m - k$ 4-cycles of the form (a,x,b,y) so that each 2-path with end vertices a and b occurs λ times. There are $2m(4m+1)$ ways that a and b can be chosen, but the 4-cycle (a,x,b,y) also contains the 2-paths [x,a,y] and [x,b,y]. Thus we need at least $(4mk+2m-k)(2m)(4m+1)/2$ 4-cycles. ∎

With the following Theorem 3.6 we complete the covering problem for every $n \equiv 1 \pmod 4$. In general then, for every λ we now know the minimum number of 4-cycles needed to cover all the 2-paths of K_{4m+1} at least λ times each.

3.6 THEOREM: When $n=4m+1$ and $\lambda=2k+1$ we can find a set of

$$16m^3k - mk + 8m^3 + 2m^2$$

4-cycles in K_n so that each 2-path occurs on at least λ 4-cycles. That is,

$$L_{2k+1}(4m+1) = 16m^3k - mk + 8m^3 + 2m^2.$$

PROOF: A simple way to achieve the lower bound given in Lemma 3.5 is to take k copies of the covering with $\lambda=2$ as given in Section 2.2 and also take the covering as given in Theorem 3.3. We thus have a total of $k(4m+1)(m)(4m-1) + 8m^3 + 2m^2$ cycles of length four in which each 2-path of K_{4m+1} occurs at least $2k+1$ times. ∎

3.7 LEMMA: When $n=4m+3$ and $\lambda=4k+1$ at least

$$32m^3k + 48m^2k + 22mk + 3k + 8m^3 + 14m^2 + 8m + 2$$

4-cycles are needed to cover the 2-paths of K_n so that each 2-path occurs at least λ times.

PROOF: Following the proof of Lemma 3.5 we see that for any two vertices a and b we need $\lceil (4k+1)(4m+1)/2 \rceil = 8mk + 2m + 2k + 1$ 4-cycles of the form (a,x,b,y) so that each 2-path with end vertices a and b occurs $4k+1$ times.

There are $(4m+3)(2m+1)$ ways that a and b can be chosen, but the 4-cycle (a,x,b,y) also contains the 2-paths $[x,a,y]$ and $[x,b,y]$. Therefore the total number of 4-cycles that are required is $\lceil (8mk+2m+2k+1)(4m+3)(2m+1)/2 \rceil$.

■

We now show that the lower bound calculated in Lemma 3.7 can be achieved. This proof follows that of Theorem 3.6 very closely.

3.8 THEOREM: The 2-paths of K_{4m+3} can be covered by

$$32m^3k + 48m^2k + 22mk + 3k + 8m^3 + 14m^2 + 8m + 2$$

4-cycles so that each 2-path occurs on at least 4k+1 4-cycles. That is,

$$L_{4k+1}(4m+3) = 32m^3k + 48m^2k + 22mk + 3k + 8m^3 + 14m^2 + 8m + 2.$$

PROOF: Take k copies of the covering with $\lambda = 4$ as given in Section 2.3 and also take the covering as given in Theorem 3.4. Thus we have a total of $k(4m+3)(2m+1)(4m+1) + 8m^3 + 14m^2 + 8m + 2$ cycles of length four which contain each 2-path of K_{4m+3} at least 4k+1 times each. ■

When $n=4m+1$ and $\lambda=2k+1$ the minimum number of 4-cycles required to cover all 2-paths so that each 2-path occurs at least λ times has been determined exactly. Similarly, when $n=4m+3$ and $\lambda=4k+1$ the minimum number of 4-cycles required to cover all 2-paths so that each 2-path occurs at least λ times has also been determined exactly.

Using calculations similar to those in Lemma 3.1 for $n=4m+3$ and $\lambda=2$ one finds that one needs at least

$$16m^3 + 24m^2 + 11m + 2$$

4-cycles to cover each 2-path in K_{4m+3} at least twice. In the next theorem we show that this lower bound on $L_2(4m+3)$ can never be attained; that is, $L_2(4m+3) > 16m^3 + 24m^2 + 11m + 2$.

3.9 LEMMA: When $n=4m+3$ and $\lambda=2$ the least number of 4-cycles that are required to cover each 2-path of K_{4m+3} at least twice is

$$16m^3 + 24m^2 + 11m + 3.$$

PROOF: We will count the number of times each vertex occurs in a 4-cycle supposing that each 2-path occurs at least twice. Each vertex x occurs on at least $2(4m+2)(4m+1)$ 2-paths of the form [x,*,*] and on at least $(4m+2)(4m+1)$ 2-paths of the form [*,x,*]. If a 4-cycle contains the vertex x it contains three 2-paths that contain x and thus the vertex x occurs on at least $(2(4m+2)(4m+1) + (4m+2)(4m+1))/3 = 16m^2 + 12m + 2$ cycles of length four.

Clearly, if the 2-paths of K_{4m+3} are covered by 4-cycles so that each 2-path occurs at least twice then there is some 2-path that occurs at least three times. Suppose this is the 2-path [1,2,3]. The vertices 1, 2 and 3 thus occur on at least $16m^2 + 12m + 3$ cycles of length four.

The sum over all vertices of the number of 4-cycles on which each vertex occurs is

$$4m(16m^2 + 12m + 2) + 3(16m^2 + 12m + 3).$$

Each 4-cycle contains four vertices and so the minimum number of 4-cycles needed to cover all of the 2-paths of K_{4m+3} so that each 2-path occurs at least twice is thus

$$16m^3 + 24m^2 + 11m + 3. \blacksquare$$

Theorem 3.4 shows that one can find a set of $8m^3 + 14m^2 + 8m + 2$ 4-cycles in K_{4m+3} in which each 2-path occurs at least once. By taking two copies of each 4-cycle in this set we see that

$$L_2(4m+3) \le 16m^3 + 28m^2 + 16m + 4.$$

Using this observation and Lemma 3.9 we have thus proven Theorem 3.10.

3.10 THEOREM: As a bound on the minimum number of 4-cycles needed to cover each of the 2-paths of K_{4m+3} at least twice we have

$$16m^3 + 24m^2 + 11m + 3 \le L_2(4m+3) \le 16m^3 + 28m^2 + 16m + 4.$$

A similar result can be obtained for $L_{4k+2}(4m+3)$ using the ideas from Theorem 3.8. That is, to the upper and lower bounds on $L_2(4m+3)$ we add $L_{4k}(4m+3) = k(4m+3)(2m+1)(4m+1)$.

Although we have not been able to find a construction which produces $16m^3 + 24m^2 + 11m + 3$ cycles of length four which cover all of the 2-paths of K_{4m+3} at least twice each, we have a construction which yields $L_2(4m+3) \le 16m^3 + 24m^2 + 13m + 2$ for some values of m. In such cases the upper bound of Theorem 3.10 would therefore be reduced.

Label the vertices of K_{4m+2} with the elements $\{1, \ldots, 4m+1, \infty\}$ and label the vertices of K_{4m+3} with the elements $\{1, \ldots, 4m+1, \infty, \nu\}$. Now, take a C(4m+2,4,2) design and suppose that in this design one could find m(4m+1) cycles satisfying the following two conditions:

1. each pair of vertices of K_{4m+2} occurs at most once as non-adjacent vertices of a cycle and
2. each pair of vertices of K_{4m+2} occurs at most twice as the edge of a cycle.

Then we have the following lemma.

<u>3.11</u> <u>LEMMA:</u> If there exist m(4m+1) 4-cycles in a C(4m+2,4,2) design satisfying conditions 1 and 2 above, and one can find a certain set of subgraphs of $2K_{4m+2}$, then one can find $16m^3 + 24m^2 + 13m + 2$ 4-cycles which contain each 2-path of K_{4m+3} at least twice each.

<u>PROOF:</u> For each $x \in \{1, \ldots, 4m+1, \infty\}$ we define a subgraph G_x of $2K_{4m+2}$ (every edge of K_{4m+2} occurs twice) so that every vertex in G_x has degree 2, except for vertex x which has degree 0. For each 4-cycle (a,x,b,c) in the distinguished set of m(4m+1) 4-cycles we put the edge (a,b) in G_x. This subgraph will also contain other edges which are chosen arbitrarily, however we require the union of these 4m+2 subgraphs to be $2K_{4m+2}$. Now remove the set of m(4m+1) 4-cycles from the C(4m+2,4,2) design and for each $x \in \{1, \ldots, 4m+1, \infty\}$ and for each edge (a,b) in G_x add the 4-cycles

(a,x,b,ν).

It is easy to see that all 2-paths on $\{1, \ldots, 4m+1, \infty\}$ have been recovered. Also, since every edge (a,b) occurs twice, we see that we get all 2-paths $[a,\nu,b]$ twice and since every vertex in G_x has degree 2 we see that we get all 2-paths $[x,a,\nu]$ twice each. We have used a total of

$$(16m^3+12m^2+2m)-m(4m+1)+(4m+2)(4m+1) = 16m^3+24m^2+13m+2$$

4-cycles to cover all of the 2-paths of K_{4m+3} by 4-cycles at least twice each. ■

3.12 EXAMPLE: Using the above lemma we can find a collection of 55 4-cycles in K_7 so that each 2-path occurs at least twice. This number is one more than the minimum of 54 given in Theorem 3.10. First we get a $C(6,4,2)$ design by taking two copies of the $C(6,4,1)$ design obtained using Theorem 2.1. Remove the following five 4-cycles:

$$(\infty,1,2,3),\ (\infty,2,3,4),\ (\infty,3,4,5),\ (\infty,4,5,1) \text{ and } (\infty,5,1,2).$$

The subgraphs G_x are shown in Figure 9; add the thirty 4-cycles defined by them to obtain $2(15) - 5 + 30 = 55$ 4-cycles which cover all the 2-paths of K_7 at least twice each. ■

3.13 EXAMPLE: When $m=3$ the conditions of Lemma 3.11 can be satisfied and thus there exists a set of 689 4-cycles in K_{15} containing each 2-path of K_{4m+3} at least twice each. This number is in contrast to the minimum of 684 calculated in Theorem 3.10.

The following thirty-nine 4-cycles (which belong to the $C(14,4,2)$ design obtained using Theorem 2.1) satisfy the two conditions which precede Lemma 3.11:

$$(2+i,1+i,4+i,5+i),\ (3+i,1+i,6+i,8+i),\ (\infty,1+i,5+i,9+i) \qquad 1\le i\le 13.$$

Addition is modulo 13 on the residue class $1, \ldots, 13$.

Each pair of non-adjacent vertices in the above 4-cycles defines an edge in some subgraph G_x. In fact, the distinguished 4-cycles above define

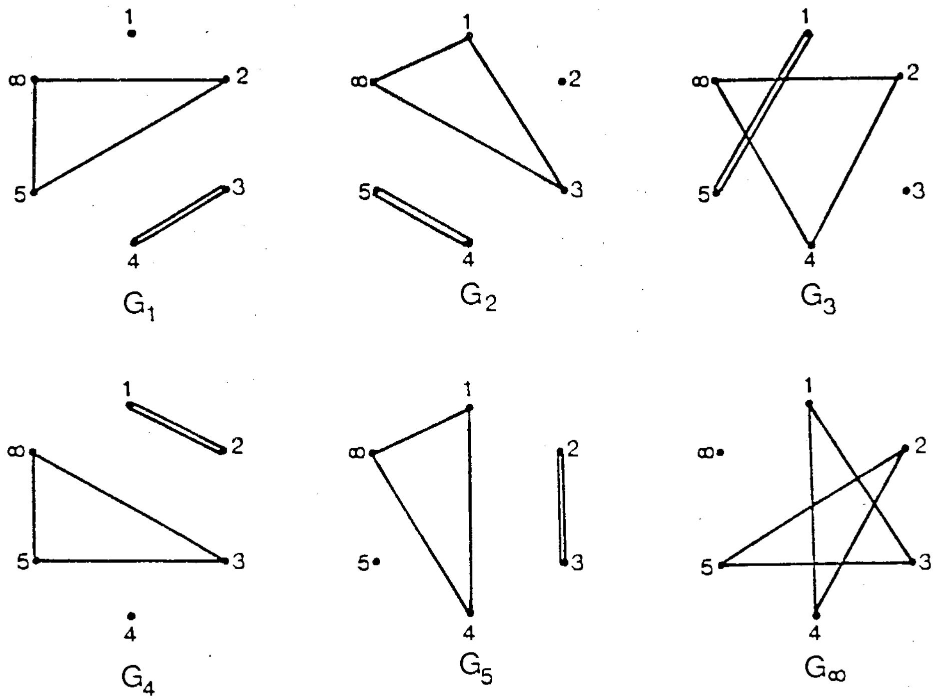


Figure 9

two vertex disjoint 4-cycles and a single edge in each subgraph G_x $(1\leq x\leq 13)$ of $2K_{14}$. Now 'double' this single edge so that the vertices incident with it have degree two. Thus every vertex in G_x other than x, x+6, x+7 and ∞ have degree two. If we take a triangle on these latter three vertices then every vertex in G_x (other than x) has degree two. Finally, we take as G_∞ the graph which consists of the edges (x, x+1) for $1\leq x\leq 13$.

It is not difficult to check that the union of these 14 subgraphs is $2K_{14}$. This checking is made easier by drawing 13 vertices on a circle (labelled 1 to 13) and putting the vertex ∞ in the centre. Now draw G_1, say, and note that the edges in this graph have distances:

$$1,\ 2,\ 2,\ 3,\ 3,\ 4,\ 4,\ 5,\ 5,\ 6,\ 6,\ \infty,\ \infty.$$

Thus in the union of the subgraphs G_i, $1\leq i\leq 13$, every edge of $2K_{14}$ appears exactly twice, except for the edges of distance 1 which appear exactly once. The subgraph G_∞ contains each edge of distance once exactly once each and therefore the union of these fourteen subgraphs is $2K_{14}$. ∎

3.14 EXAMPLE: When n=4 we can again satisfy the conditions of Lemma 3.11. Thus in K_{19} we can find a collection of 1462 cycles of length four which cover all the 2-paths at least twice each, whereas the lower bound in Theorem 3.10 is 1455. The solution to this problem is similar to that of Example 3.13, and sixty-eight 4-cycles which satisfy the two conditions preceding Lemma 3.11 are:

$(17+i,1+i,7+i,6+i)$, $(4+i,1+i,6+i,9+i)$,

$(5+i,1+i,8+i,12+i)$, $(\infty,1+i,16+i,14+i)$ $\quad 1\le i\le 17$.

Addition is modulo 17 on the residue class 1, ..., 17.

These four cycles come from the C(18,4,2) design obtained by taking two copies of each of the 4-cycles as given by Theorem 2.1. If we define the subgraphs G_x, $1\le x\le 17$, as in Example 3.13, then every vertex in G_x other than x, x+8, x+9 and ∞ has degree two (these vertices have degree 0). Take a triangle on the latter three vertices so that now every vertex in G_x other than x has degree two. The subgraph G_∞ consists of the edges (x, x+1) for $1\le x\le 17$.

It is not difficult to check that the union of these subgraphs is $2K_{18}$. ■

3.15 LEMMA: When n=4m+3 and λ=3 at least

$$24m^3 + 36m^2 + \lceil 16.5m \rceil + 3$$

4-cycles are required in order for each 2-path of K_n to appear on at least λ 4-cycles.

PROOF: Assume that each 2-path occurs at least three times and count the number of 4-cycles on which each vertex appears. Each vertex x appears on at least $3(4m+2)(4m+1)$ 2-paths of the form $[x,*,*]$ and on at least $3(2m+1)(4m+1)$ 2-paths of the form $[*,x,*]$. If a 4-cycle contains the vertex x then it contains three 2-paths that contain x. Thus the vertex x occurs on at least $(3(4m+2)(4m+1) + 3(2m+1)(4m+1))/3 = 24m^2 + 18m + 3$ cycles of length four.

However, not all 2-paths occur exactly three times and so there is at least one 2-path, say [1,2,3], that occurs at least four times. The vertices 1, 2 and 3 thus occur on at least $24m^2 + 18m + 4$ cycles of length four.

The minimum number of 4-cycles in K_{4m+3} needed to cover each 2-path at least three times each is thus

$$(4m(24m^2 + 18m + 3) + 3(24m^2 + 18m + 4))/4 = 24m^3 + 36m^2 + 33m/2 + 3. \quad \blacksquare$$

3.16 THEOREM: The minimum number of 4-cycles needed to cover each of the 2-paths of K_{4m+3} at least three times is bounded by:

$$24m^3 + 36m^2 + \lceil 16.5m \rceil + 3 \le L_3(4m+3) \le 24m^3 + 42m^2 + 24m + 6.$$

PROOF: The lower bound in the above inequality comes from Lemma 3.15 and the upper bound is obtained by taking three copies of the design on 4m+3 vertices in which each 2-path occurs on at least one 4-cycle (see Theorem 3.4). $\blacksquare$

If the conditions of Lemma 3.11 are satisfied, then we can reduce the upper bound in the above theorem to $24m^3 + 38m^2 + 21m + 4$. The bounds on the number of 4-cycles of K_{4m+3} needed in order for each 2-path to occur on at least 4k+3 cycles of length four can easily be obtained by adding $L_{4k}(4m+3) = k(4m+3)(2m+1)(4m+1)$ to both the upper and lower bounds on $L_3(4m+3)$ as given in Theorem 3.16. Clearly, we can reduce the upper bound on $L_{4k+3}(4m+3)$ if the conditions of Lemma 3.11 are satisfied.

To summarize this chapter we have the following table. Here we describe, for $\lambda \in \{4k, 4k+1, 4k+2, 4k+3\}$ and $n \in \{4m, 4m+1, 4m+2, 4m+3\}$, the general results known about $L_\lambda(n)$ and give the theorem or corollary used to justify the statements.

(n,λ)	Results on $L_\lambda(n)$	Refer to:
$(4m,4k)$	$L_\lambda(n) = 4mk(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+1)$	$L_\lambda(n) = (4mk+m)(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+2)$	$L_\lambda(n) = (4mk+2m)(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+3)$	$L_\lambda(n) = (4mk+3m)(4m-1)(2m-1)$	Theorem 2.1
$(4m+1,4k)$	$L_\lambda(n) = 2mk(4m-1)(4m+1)$	Corollary 2.9
$(4m+1,4k+1)$	$L_\lambda(n) = (16m^3-m)(2k)+8m^3+2m^2$	Theorem 3.6
$(4m+1,4k+2)$	$L_\lambda(n) = (2mk+m)(4m-1)(4m+1)$	Corollary 2.9
$(4m+1,4k+3)$	$L_\lambda(n) = (16m^3-m)(2k+1)+8m^3+2m^2$	Theorem 3.6
$(4m+2,4k)$	$L_\lambda(n) = 4mk(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+1)$	$L_\lambda(n) = (4mk+m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+2)$	$L_\lambda(n) = (4mk+2m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+3)$	$L_\lambda(n) = (4mk+3m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+3,4k)$	$L_\lambda(n) = (2mk+k)(4m+3)(4m+1)$	Theorem 2.13
$(4m+3,4k+1)$	$L_\lambda(n) = k(32m^3+48m^2+22m+3)+8m^3+14m^2+8m+2$	Theorem 3.8
$(4m+3,4k+2)$	$k(4m+3)(2m+1)(4m+1)+16m^3+24m^2+11m+3$	Theorem 3.10
	$\le L_\lambda(n) \le k(4m+3)(2m+1)(4m+1)+16m^3+28m^2+16m+4$	
$(4m+3,4k+3)$	$k(4m+3)(2m+1)(4m+1)+24m^3+36m^2+\lceil 33m/2 \rceil+3$	Theorem 3.16
	$\le L_\lambda(n) \le k(4m+3)(2m+1)(4m+1)+24m^3+42m^2+24m+6$	

CHAPTER 4

MAXIMAL PACKINGS

Whenever one asks questions about the covering problem, it is a natural assumption that analagous questions will then be asked for the packing problem. In this case the packing problem is to find the largest collection of 4-cycles in K_n so that each 2-path occurs on at most λ 4-cycles.

<u>4.1</u> <u>DEFINITION:</u> Let $U_i(n)$ be the largest integer such that the 2-paths of K_n can be packed into $U_i(n)$ cycles of length four, so that no 2-path occurs on more than i 4-cycles.

We have the trivial results (see Theorem 2.1) that

$U_1(4m) = L_1(4m) = 8m^3 - 6m^2 + m$ and

$U_1(4m+2) = L_1(4m+2) = 8m^3 + 6m^2 + m.$

<u>4.2</u> <u>LEMMA:</u> The maximum number of 4-cycles of K_{2m+1} that contain each of the 2-paths of K_{2m+1} at most once is no more than $m^3 - m(m+1)/2$.

<u>PROOF:</u> For any two vertices a and b of K_{2m+1} we have 2m-1 different 2-paths of the form [a,x,b]. Any 4-cycle containing such a 2-path must contain another 2-path of the form [a,y,b]. Thus the number of 4-cycles containing a 2-path of the form [a,x,b] must be at most $\lfloor (2m-1)/2 \rfloor = m-1$. There are m(2m+1) ways in which a and b can be chosen, but the 4-cycle containing the 2-paths [a,x,b] and [a,y,b] also contains [x,a,y] and [x,b,y]. Thus we can have at most $(m-1)(m)(2m+1)/2 = m^3 - m(m+1)/2$ 4-cycles if each 2-path of K_{2m+1} is to occur at most once. ■

Thus we see that

$U_1(4m+1) \le 8m^3 - 2m^2 - m$ and

$U_1(4m+3) \le 8m^3 + 10m^2 + 3m.$

In fact it is easy to see that in both cases we have equality. This is shown in the following theorems.

4.3 THEOREM: In K_{4m+1} one can find

$$8m^3 - 2m^2 - m$$

4-cycles so that each 2-path occurs on at most one 4-cycle. In other words,

$$U_1(4m+1) = 8m^3 - 2m^2 - m.$$

PROOF: Consider the set of 4-cycles of K_{4m+2} (as given by Theorem 2.1) in which each 2-path occurs exactly once. There are $8m^3 + 6m^2 + m$ cycles of length four in this set and each vertex appears on $8m^2 + 2m$ cycles of length four. If the vertex ∞, say, is removed (as are all cycles containing it) then one is left with $8m^3 - 2m^2 - m$ cycles of length four and each 2-path on the remaining 4m+1 vertices appears at most once. ∎

4.4 THEOREM: In K_{4m+3} there exists a set of

$$8m^3 + 10m^2 + 3m$$

4-cycles so that each 2-path occurs on at most one 4-cycle. In other words,

$$U_1(4m+3) = 8m^3 + 10m^2 + 3m.$$

PROOF: Theorem 2.1 tells us that one can find $8m^3 + 18m^2 + 13m + 3$ cycles of length four which contain each 2-path of K_{4m+4} exactly once each. Each vertex of K_{4m+4} appears on $8m^2 + 10m + 3$ cycles of length four and thus if one of these vertices, along with all 4-cycles containing it, is removed then we are left with $8m^3 + 10m^2 + 3m$ cycles of length four. These 4-cycles contain each 2-path on the remaining 4m+3 vertices at most once. ∎

4.5 LEMMA: When $n=4m+1$ and $\lambda=2k+1$ at most

$$16m^3k - mk + 8m^3 - 2m^2 - m$$

4-cycles can pack the 2-paths of K_{4m+1} so that each 2-path occurs on at most λ 4-cycles.

PROOF: Simply count 2-paths as in Lemmas 3.5 and 4.2. ■

4.6 THEOREM: In K_{4m+1} we can find

$$16m^3k - mk + 8m^3 - 2m^2 - m$$

4-cycles which contain each 2-path at most 2k+1 times.

PROOF: Take k copies of the collection of 4-cycles which contain each 2-path of K_{4m+1} exactly twice (as given in Section 2.2). Add to these 4-cycles the $8m^3 - 2m^2 - m$ cycles of length four which contain each 2-path at most once (see Theorem 4.3). This gives the required number of 4-cycles. ■

With the above result we have finished the packing problem for all $n \equiv 1 \pmod 4$ and for all λ.

4.7 LEMMA: When $n=4m+3$ and $\lambda=4k+1$ at most

$$32m^3k + 48m^2k + 22mk + 3k + 8m^3 + 10m^2 + 3m$$

4-cycles can be taken if each 2-path of K_{4m+3} is to occur on at most λ 4-cycles.

PROOF: Again, count as in Lemmas 3.7 and 4.2. ■

4.8. THEOREM: $U_{4k+1}(4m+3) = 32m^3k + 48m^2k + 22mk + 3k + 8m^3 + 10m^2 + 3m.$

PROOF: From Section 2.3 we know that one can cover the 2-paths of K_{4m+3} by 4-cycles so that each 2-path occurs on exactly four 4-cycles. Take k copies of each of these 4-cycles as well as the $8m^3 + 10m^2 + 3m$ 4-cycles (see Theorem 4.4) which contain each 2-path of K_{4m+3} at most once. ■

Using the argument in the proof to Lemma 4.2, one obtains $U_2(4m+3) \leq 16m^3 + 24m^2 + 11m + 1$. However, this bound can be improved.

4.9 LEMMA: The number of 4-cycles of K_{4m+3} which contain each 2-path at most twice is at most

$$16m^3 + 24m^2 + 11m.$$

PROOF: The vertex x occurs on at most $16m^2 + 12m + 2$ cycles of length four and, because we know that there is some 2-path that does not occur twice, we can assume that the vertex x occurs on at most $16m^2 + 12m + 1$ cycles of length four. These $16m^2 + 12m + 1$ cycles contain $16m^2 + 12m + 1$ 2-paths which do not contain x.

There are at most $(4m+2)(4m+1)(4m)$ 2-paths not containing x and we have used at most $16m^2 + 12m + 1$ of them. This leaves $64m^3 + 32m^2 - 4m - 1$ 2-paths to be packed into 4-cycles. Thus at most $16m^3 + 8m^2 - m - 1$ cycles of length four can be added. The total number of 4-cycles is thus at most $16m^3 + 24m^2 + 11m$. ∎

Theorem 4.4 shows that $U_1(4m+3) = 8m^3 + 10m^2 + 3m$ and thus

$$U_2(4m+3) \geq 16m^3 + 20m^2 + 6m.$$

This result and Lemma 4.9 allow us to put both upper and lower bounds on $U_2(4m+3)$ and we have thus proven Theorem 4.10.

4.10 THEOREM: The maximum number of 4-cycles of K_{4m+3} which cover each 2-path at most twice is bounded by

$$16m^3 + 20m^2 + 6m \leq U_2(4m+3) \leq 16m^3 + 24m^2 + 11m.$$

By adding $U_{4k}(4m+3) = k(4m+3)(2m+1)(4m+1)$ to both sides in the above inequality we obtain upper and lower bounds on $U_{4k+2}(4m+3)$.

Although we have not been able to attain the bound given in Lemma 4.9, we have a construction which, for some values of m, shows that $U_2(4m+3) \geq 16m^3 + 24m^2 + 5m$. Consider the following. Let the vertices of K_{4m+2} be labelled by the elements $\{1, \ldots, 4m+1, \infty\}$ and let the vertices of K_{4m+3} be labelled by the elements of $\{1, \ldots, 4m+1, \infty, \nu\}$. First, take a

C(4m+2,4,2) design. Next, suppose that one could find $4m^2 + m$ cycles of length four in this C(4m+2,4,1) design satisfying

1. each pair of elements from $\{1, \ldots, 4m+1, \infty\}$ occurs as a pair of non-adjacent vertices of at most one 4-cycle and
2. each pair of elements from $\{1, \ldots, 4m+1, \infty\}$ occurs as the edge of at most two 4-cycles.

Then it would be possible to replace each of these $4m^2 + m$ 4-cycles (a,b,c,d) by the four 4-cycles (a,b,c,ν), (b,c,d,ν), (c,d,a,ν) and (d,a,b,ν) to get $16m^3 + 12m^2 + 2m - (4m^2 + m) + 4(4m^2 + m) = 16m^3 + 24m^2 + 5m$ 4-cycles of K_{4m+3} that contain each 2-path at most twice. Thus we have proven the following lemma.

4.11 LEMMA: If there exist $4m^2 + m$ cycles in a C(4m+2,4,2) design satisfying conditions 1 and 2 above, then one can find $16m^3 + 24m^2 + 5m$ 4-cycles which contain each 2-path of K_{4m+3} at most twice.

Consider the line graph of K_{4m+2}, $L(K_{4m+2})$, and arrange its vertices in a $(4m+1) \times (2m+1)$ array as in Theorem 2.1. The 4-cycles in K_{4m+2} containing only vertices from column 1 and column i can be written as $(\infty,1+k,i+k,2i+k-1)$ for some k and the 4-cycles in K_{4m+2} containing vertices from column i and column j can be written as $(i+k,1+k,j+k,i+j+k-1)$ for some k. The two distances in K_{4m+2} between the pairs of non-adjacent vertices in any of these 4-cycles can thus be given as

∞, $\min\{2i-2, 2-2i\}$ between columns 1 and i

and

$\min\{i-j, j-i\}$, $\min\{2-i-j, i+j-2\}$ between columns i and j.

In the above calculations addition is modulo 4m+1 on the residue class 1, ..., 4m+1.

If one could find m disjoint pairs of columns in this array so that the distances given above are all distinct, then the $4m^2 + m$ cycles of length four lying between these m pairs of columns satisfy conditions 1 and 2.

Condition 2 is easily seen to be satisfied since each vertex of the line graph (an edge in K_{4m+2}) appears on at most two 4-cycles. The differences between non-adjacent vertices in the 4-cycles between the chosen pairs of columns are all distinct and thus when these 4-cycles are all developed modulo 4m+1 the distances between non-adjacent vertices on any of the chosen 4-cycles is also distinct.

4.12 EXAMPLE: Using Lemma 4.9 with m=1 we see that the maximum number of 4-cycles which contain each 2-path of K_7 at most twice is 51. Lemma 4.11 tells us that if we can find a particular set of five 4-cycles in a C(6,4,2) design then we will be able to find a collection of 45 4-cycles which between them contain the 2-paths of K_7 at most twice each. The following set of five 4-cycles

$(\infty,1,2,3)$, $(\infty,2,3,4)$, $(\infty,3,4,5)$, $(\infty,4,5,1)$ and $(\infty,5,1,2)$.

satisfies conditions 1 and 2 and thus the required set of 45 4-cycles exists. ∎

4.13 LEMMA: When n=4m+3 and λ=3 at most

$$24m^3 + 36m^2 + \lfloor 16.5m + 1.5 \rfloor$$

4-cycles can be found so that each 2-path of K_{4m+3} occurs at most three times.

PROOF: Consider a vertex x; it can occur on at most 3(2m+1)(4m+1) 2-paths of the form [*,x,*] and at most 3(4m+2)(4m+1) 2-paths of the form [x,*,*]. Since each 4-cycle that contains x also contains three 2-paths that contain x we see that each vertex can occur on at most $24m^2 + 18m + 3$ cycles of length four.

We know that there is at least one 2-path, say [x,y,z] that does not occur three times and so the vertex occurs on at most $24m^2 + 18m + 2$ 4-cycles. These 4-cycles which contain x contain $24m^2 + 18m + 2$ 2-paths which do not contain x.

There are $6m(4m+2)(4m+1)$ 2-paths not containing x and we have accounted for $24m^2 + 18m + 2$ of them. This leaves $96m^3 + 48m^2 - 6m - 2$ 2-paths to be packed into 4-cycles. Thus at most $24m^3 + 12m^2 - \lceil (3m+1)/2 \rceil$ cycles of length four can be added. The number of 4-cycles of K_{4m+3} containing each 2-path at most three times each is thus at most

$$24m^3 + 36m^2 + \lfloor 16.5m + 1.5 \rfloor . \quad \blacksquare$$

4.14 THEOREM: The maximum number of 4-cycles of K_{4m+3} in which each 2-path occurs at most three times is bounded by:

$$24m^3 + 30m^2 + 9m \leq U_3(4m+3) \leq 24m^3 + 36m^2 + \lfloor 16.5m + 1.5 \rfloor .$$

PROOF: This result follows from Theorem 4.4 and Lemma 4.13. $\blacksquare$

The bounds on the number of 4-cycles of K_{4m+3} containing each 2-path at most $4k+3$ times each can be obtained by adding $U_{4k}=k(4m+3)(2m+1)(4m+1)$ to each side of the above inequality.

The following table summarizes the results obtained in this chapter.

(n,λ)	Results on $U_\lambda(n)$	Refer to:
$(4m,4k)$	$U_\lambda(n) = 4mk(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+1)$	$U_\lambda(n) = (4mk+m)(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+2)$	$U_\lambda(n) = (4mk+2m)(4m-1)(2m-1)$	Theorem 2.1
$(4m,4k+3)$	$U_\lambda(n) = (4mk+3m)(4m-1)(2m-1)$	Theorem 2.1
$(4m+1,4k)$	$U_\lambda(n) = 2mk(4m-1)(4m+1)$	Corollary 2.9
$(4m+1,4k+1)$	$U_\lambda(n) = (16m^3-m)(2k)+8m^3-2m^2-m$	Theorem 4.6
$(4m+1,4k+2)$	$U_\lambda(n) = (2mk+m)(4m-1)(4m+1)$	Corollary 2.9
$(4m+1,4k+3)$	$U_\lambda(n) = (16m^3-m)(2k+1)+8m^3-2m^2-m$	Theorem 3.6
$(4m+2,4k)$	$U_\lambda(n) = 4mk(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+1)$	$U_\lambda(n) = (4mk+m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+2)$	$U_\lambda(n) = (4mk+2m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+2,4k+3)$	$U_\lambda(n) = (4mk+3m)(2m+1)(4m+1)$	Theorem 2.1
$(4m+3,4k)$	$U_\lambda(n) = (2mk+k)(4m+3)(4m+1)$	Theorem 2.13
$(4m+3,4k+1)$	$U_\lambda(n) = k(32m^3+48m^2+22m+3)+8m^3+10m^2+3m$	Theorem 4.8
$(4m+3,4k+2)$	$k(4m+3)(2m+1)(4m+1)+16m^3+20m^2+6m \le U_\lambda(n) \le$ $k(4m+3)(2m+1)(4m+1)+16m^3+24m^2+11m$	Theorem 4.10
$(4m+3,4k+3)$	$k(4m+3)(2m+1)(4m+1)+24m^3+30m^2+9m \le U_\lambda(n) \le$ $k(4m+3)(2m+1)(4m+1)+24m^3+36m^2+\lfloor (33m+1)/2 \rfloor$	Theorem 4.14

CHAPTER 5

RESOLVABLE DESIGNS

In this Chapter we look at the problem of finding resolvable C(n,4,1) designs. A t-(n,k,λ) design (X,β) is resolvable if the blocks in β can be partitioned into classes so that in each class every element of X appears exactly once. D. Jungnickel and S. Vanstone [JV] and A. Hartman [H8] have recently published results concerning the resolvability of certain types of 3-designs.

In [JV] Jungnickel and Vanstone show that the necessary conditions for a 3-(n,4,3) design to be resolvable are also sufficient. The construction of a family of 3-(n,4,3) designs and the proof that they are resolvable (see [JV]) are presented below.

It is easy to see that the only necessary condition for such a design to be resolvable is that n be a multiple of 4. Now let $\mathcal{F}$ be any 1-factorization of K_n. For each 1-factor F of $\mathcal{F}$ and for each pair of edges (a,b) and (c,d) in F forms the block {a,b,c,d}.

Consider any subset {a,b,c} of the n-set. Since any pair of these three vertices is an edge of exactly one 1-factor it is clear that this subset occurs in three blocks of size four. Thus the collection of these (n-1)(n/2)(n/2-1)/2 blocks form the blocks of a 3-(n,4,3) design.

Since the blocks of this design were obtained by taking the (n/2)((n/2)-1)/2 pairs of edges from each 1-factor, then for every 1-factor F we can construct the complete graph on n/2 vertices whose vertices are the edges of F. This complete graph has an even number of vertices and hence it also has a 1-factorization, say $\mathcal{G}$(F). Every edge ((a,b), (c,d)) in a 1-factor $G \in \mathcal{G}(F)$ defines a block {a,b,c,d} in the 3-design and thus the collection of these n/4 edges gives a parallel class in the design. If we repeat this for all $F \in \mathcal{F}$ then we will have partitioned the blocks of the 3-design into resolution classes.

Since an SQS(n) exists if and only if $n \equiv 2$ or 4 (mod 6) and may be resolvable only if n is a multiple of four, the necessary condition for the existence of a resolvable SQS(n) is that $n \equiv 4$ or 8 (mod 12).

One infinite class of resolvable Steiner quadruple systems was known to T. P. Kirkman [K] as early as 1847. These were the quadruple systems of order 2^m, with points being the elements of $GF(2^m)$, obtained by taking the planes of the affine space of dimension m over GF(2) to be the blocks of the quadruple system. Each parallel class is then defined to be a subspace of dimension two and all of its translates.

For example, if m=3 then we would take as the eight points of the quadruple system the following:

000 001 010 100 011 101 110 111.

The subspaces of dimension two are:

S_1=(000,001,010,011) S_2=(000,001,100,101)

S_3=(000,001,110,111) S_4=(000,010,100,110)

S_5=(000,010,101,111) S_6=(000,100,011,111)

S_7=(000,011,101,110).

For $v \in GF(2^m)$ we define by S_j+v the set of points s+v such that $s \in S_j$. The parallel classes of this quadruple system are S_j+w, $1 \le j \le 7$, where w is any element of $GF(2^3)$ that does not belong to S_j.

More recently, Alan Hartman [H8] has shown that the necessary conditions for the existence of a resolvable Steiner quadruple system are sufficient, except in at most twenty-three cases. We thus have the following theorem.

5.1 THEOREM: Whenever $n \equiv 4$ or 8 (mod 12) there exists a resolvable Steiner quadruple system of order n, except possibly for $n \in S$, where

$S = \{220, 236, 292, 364, 460, 596, 676, 724,$
$1076, 1100, 1252, 1316, 1820, 2236, 2308,$
$2324, 2380, 2540, 2740, 2812, 3620, 3820, 6356\}$.

5.2 DEFINITION: Let (X,β) be 3-$(n,4,1)$ design. By a partition of β into nonempty subsets $P_1, \ldots, P_t$ we mean a disjoint union $\beta = \cup P_i$, where $1 \le i \le t$. We also use the notation $\beta = P_1|P_2| \ldots |P_t$ to identify the partition. If each $x \in X$ is contained in exactly one 4-subset of each P_i then this partition is called a resolvable partition. In this case we also call each P_i a parallel class.

The following theorem shows that the doubling construction presented in [H1] preserves resolvability.

5.3 THEOREM: If there exists a resolvable Steiner quadruple system of order n then there exists a resolvable Steiner quadruple system of order 2n.

PROOF: Addition is modulo n/2 on the residue class 1, ..., n/2.

Let (X_1, β_1) and (X_2, β_2) be two resolvable Steiner quadruple systems of order n, where X_1 and X_2 are disjoint sets of size n. Now define two 1-factorizations of K_n, $F = \{F_1, \ldots, F_{n-1}\}$ and $G = \{G_1, \ldots, G_{n-1}\}$, with vertex set X_1 and X_2, respectively, and arbitrarily order the edges of each 1-factor.

We now define a resolvable SQS(2n), $(X_1 \cup X_2, \beta_1 \cup \beta_2 \cup \beta)$, where β is a set of blocks which will be defined in Step 2. This Steiner quadruple system will have $(2n-1)(2n-2)/6$ parallel classes.

Step 1. If $P_1|P_2|\ldots|P_t$ and $Q_1|Q_2|\ldots|Q_t$ (where $t=(n-1)(n-2)/6$) are resolvable partitions of β_1 and β_2, respectively, then $P_1 \cup Q_1$, $P_2 \cup Q_2$, ... $P_t \cup Q_t$ are t parallel classes in the SQS(2n).

Step 2. For each i, $1 \le i \le n-1$, and for each k, $1 \le k \le n/2$, a parallel class is defined as follows. For $1 \le j \le n/2$, take the j^{th} edge (a,b) of F_i together with the $(j+k)^{th}$ edge (c,d) of G_i to form the block $\{a,b,c,d\}$. These n/2 blocks form a parallel class and the collection of blocks in all such parallel classes forms β.

It is a simple matter to check that these $n(n-1)/2$ parallel classes, together with the $(n-1)(n-2)/6$ parallel classes defined in Step 1, form the $(2n-1)(2n-2)/6$ parallel classes in a resolvable SQS(2n). ∎

We now turn our attention to the problem of finding resolvable $C(n,4,1)$ designs. The following definition is analogous to Definition 5.2.

5.4 DEFINITION: A $C(n,4,1)$ design is resolvable if its 4-cycles can be partitioned into classes so that every vertex appears exactly once in each class. Each such class is called a parallel class (of the design).

It is easy to see that a necessary condition for the existence of a resolvable $C(n,4,1)$ design is that n be a multiple of four.

In the same way that Steiner quadruple systems of order n can be used to obtain $C(n,4,1)$ designs, resolvable Steiner quadruple systems of order n give rise to resolvable $C(n,4,1)$ designs. We have the following theorem.

5.5 THEOREM: If there exists a resolvable SQS(n) then there exists a resolvable $C(n,4,1)$ design.

PROOF: Let (X, β) be a resolvable Steiner quadruple system of order n with resolvable partition $P_1|P_2|\ldots|P_t$ (where $t=(n-1)(n-2)/6$). For each parallel class P_i in the SQS(n) we define three parallel classes P_i^1, P_i^2 and P_i^3 in the $C(n,4,1)$ design. This is done as follows. For each block $\{u,v,w,x\}\in P_i$ (with $u<v<w<x$ and $1\le i\le t$) we place the three 4-cycles it defines in the parallel classes according to: $(u,v,w,x)\in P_i^1$, $(u,w,v,x)\in P_i^2$ and $(u,v,x,w)\in P_i^3$.

These 3t sets form the parallel classes of a resolvable $C(n,4,1)$ design. ∎

The following theorem for $C(n,4,1)$ designs is analogous to Theorem 5.3, which gives a corresponding result for Steiner quadruple systems.

<u>5.6</u> <u>**THEOREM:**</u> If there is a resolvable $C(n,4,1)$ design then there is a resolvable $C(2n,4,1)$ design.

<u>**PROOF:**</u> This proof closely follows that of Theorem 5.3. In this proof addition is modulo $n/2$ on the residue class $1, \ldots, n/2$.

Take two resolvable $C(n,4,1)$ designs, one based on the vertex set X_1 and the other based on the vertex set X_2, where X_1 and X_2 are disjoint sets of size n. The resolvable $C(2n,4,1)$ design we define is based on the vertex set $X_1 \cup X_2$. Let $F = \{F_1, \ldots, F_{n-1}\}$ and $G = \{G_1, \ldots, G_{n-1}\}$ be two 1-factorizations of K_n with vertex set X_1 and X_2, respectively, and arbitrarily order the edges in each 1-factor. The resolvable $C(2n,4,1)$ design will now be defined.

<u>Step</u> <u>1.</u> Let $P_1|P_2|\ldots|P_t$ and $Q_1|Q_2|\ldots|Q_t$ (where $t=(n-1)(n-2)/2$) be resolvable partitions of the two $C(n,4,1)$ designs. Then $P_i \cup Q_i$, $1 \le i \le t$, are parallel classes in the $C(2n,4,1)$ design.

<u>Step</u> <u>2.</u> This is the same as Step 2 of Theorem 5.3, except that each parallel class of blocks is replaced by three parallel classes of 4-cycles using Theorem 5.5. ■

<u>5.7</u> <u>**DEFINITION:**</u> Given a $C(4m+2,4,1)$ design, a <u>near-parallel</u> <u>class</u> $P_{i,j}$, $1 \le i < j \le 4m+2$, is a set of m vertex disjoint 4-cycles based on the set $\{1,2,\ldots,4m+2\}-\{i,j\}$.

<u>5.8</u> <u>**DEFINITION:**</u> For $m \ge 0$, a $C(4m+2,4,1)$ design is <u>near-resolvable</u> if the 4-cycles in the design can be partitioned into $(4m+2)(4m+1)/2$ near-parallel classes $P_{i,j}$ $(1 \le i < j \le 4m+2)$.

Ideas similar to those contained in the above two definitions have been used before (see [H4]) to double Steiner quadruple systems to obtain resolvable Steiner quadruple systems. Definitions 5.7 and 5.8 are less general than the corresponding definitions in [H4], simply because the minimum number of classes into which the 4-cycles can be partitioned is equal to the number of edges in K_{4m+2}. Clearly the maximum number of distinct vertices in each class of the partition is 4m, and we can consider (for now) this class as containing m 4-cycles and an edge. Thus in each class there must be <u>exactly</u> one edge of K_{4m+2}.

The following four lemmas are useful in constructing resolvable C(n,4,1) designs for those values of n for which there does not exist a resolvable SQS(n); that is, when $n \equiv 0 \pmod{12}$. The C(2,4,1) design is trivially near-resolvable. Lemma 5.12 was obtained by computer using a simple algorithm and there is no reason to believe that near-resolvable designs do not always exist. The main reason the computer search was stopped after a near-resolvable C(18,4,1) design was found was the length of time it took for the program to run.

<u>5.9</u> <u>LEMMA:</u> There is a near-resolvable C(6,4,1) design.

<u>PROOF:</u> Consider the fifteen 4-cycles in a C(6,4,1) design. Each cycle in this design is disjoint from exactly two vertices, say i and j, and this cycle gives a near-parallel class $P_{i,j}$. ■

<u>5.10</u> <u>LEMMA:</u> There is a near-resolvable C(10,4,1) design.

<u>PROOF:</u> Consider the cycles in the C(10,4,1) design obtained by using the 'line-graph' solution as described in the proof of Theorem 2.1. We want to partition the 4-cycles into 45 near-parallel classes $P_{i,j}$, where either $1 \le i < j \le 9$ or $i = \infty$ and $1 \le j \le 9$. This is done by choosing one cycle $C_{i,j}(k)$ from each of the ten pairs of columns in the array (see Theorem 2.1). These ten cycles are then taken in pairs so that

1. in each of the five pairs the cycles are vertex disjoint, and
2. the edges formed by the vertices missed in each of the pairs of cycles have distance ∞, 1, 2, 3 and 4, respectively.

By applying the permutation (1,2,3,4,5,6,7,8,9) to the vertices of the cycles (∞ is a fixed point) we obtain the required 45 near-parallel classes. That is, we can think of the five pairs of cycles as 'starter' cycles.

We give below five pairs of 4-cycles which satisfy conditions 1 and 2. The near-parallel class $P_{i,j}$ that is defined by them is also listed.

<u>The two 4-cycles used</u>		<u>Near-parallel class</u>
(1,4,8,5)	(2,3,7,6)	$P_{\infty,9}$
(∞,1,4,7)	(2,3,6,5)	$P_{8,9}$
(∞,1,3,5)	(7,6,8,9)	$P_{2,4}$
(∞,1,2,3)	(6,4,7,9)	$P_{5,8}$
(∞,1,5,9)	(4,2,6,8)	$P_{3,7}$

If the vertices in these pairs of 4-cycles are permuted using the permutation given above we see that all edges of the line graph $L(K_{10})$ are used (i.e. all 2-paths of K_{10}) and each pair of distinct vertices i and j of K_{10} defines exactly one near-parallel class $P_{i,j}$. Thus we have a near-resolvable C(10,4,1) design. ∎

<u>5.11</u> <u>LEMMA:</u> There exists a near-resolvable C(14,4,1) design.

<u>PROOF:</u> As in the proof of Lemma 5.10, we use the C(14,4,1) design obtained from the line graph solution described in Theorem 2.1. We want to partition the 4-cycles into 91 near-parallel classes $P_{i,j}$, where either $1 \le i < j \le 13$ or $i = \infty$ and $1 \le j \le 13$. This is done by choosing one cycle $C_{i,j}(k)$ from each of the twenty one pairs of columns in the 13×7 array (see Theorem 2.1). These twenty one cycles are then partitioned into seven groups of three so that

1. in each of the seven groups of three 4-cycles the cycles are vertex

disjoint and

2. the edges formed by the missing vertices in each of the pairs of cycles have distance ∞, 1, 2, 3 ,4, 5 and 6 respectively.

These seven 3-tuples of cycles are then developed modulo 13 on the residue class 1, ..., 13 (∞ is a fixed point) to produce the required ninety-one near-parallel classes.

Seven groups of 4-cycles which satisfy the above two conditions are given below. The near parallel class $P_{i,j}$ that each set of cycles belongs to is also given.

The three 4-cycles used			Near Parallel class
(3,1,5,7)	(8,6,11,13)	(9,4,10,12)	$P_{\infty,2}$
(2,1,3,4)	(7,6,11,12)	(∞,8,13,5)	$P_{9,10}$
(2,1,7,8)	(3,13,6,9)	(∞,5,11,4)	$P_{10,12}$
(2,1,4,5)	(11,8,13,3)	(∞,6,9,12)	$P_{7,10}$
(4,1,5,8)	(3,2,6,7)	(∞,10,11,12)	$P_{9,13}$
(3,1,7,9)	(10,6,11,2)	(∞,4,8,12)	$P_{5,13}$
(3,1,4,6)	(9,5,11,2)	(∞,8,10,12)	$P_{7,13}$

If the three four-cycles in each group are developed modulo 13 then this gives us the required ninety-one near-parallel classes. ■

5.12 LEMMA: There is a near-resolvable C(18,4,1) design.

PROOF: As the proof is similar to that of Lemma 5.10, we present here only the nine groups of four 4-cycles which satisfy the following two conditions:

1. in each of the nine groups the four 4-cycles are vertex disjoint and
2. the edges formed by the points missed in each of the groups have distance ∞, 1, 2, ..., 8 respectively.

The four 4-cycles used				Near Parallel class
(1,5,10,6)	(7,11,17,13)	(8,12,2,15)	(9,14,4,16)	$P_{\infty,3}$
(4,5,7,6)	(8,9,12,11)	(10,13,17,14)	(∞,1,2,3)	$P_{15,16}$
(6,7,11,10)	(8,9,17,16)	(13,15,4,2)	(∞,1,3,5)	$P_{12,14}$
(2,4,9,7)	(8,10,17,15)	(12,14,5,3)	(∞,1,6,11)	$P_{13,16}$
(2,5,11,8)	(13,16,6,3)	(4,7,15,12)	(∞,1,9,17)	$P_{10,14}$
(5,6,11,10)	(13,14,3,2)	(8,9,16,15)	(∞,1,4,7)	$P_{12,17}$
(6,12,2,13)	(14,3,11,5)	(9,16,7,17)	(∞,1,8,15)	$P_{4,10}$
(3,6,11,8)	(10,15,4,16)	(9,14,5,17)	(∞,1,7,13)	$P_{2,12}$
(8,10,13,11)	(15,17,4,2)	(12,16,7,3)	(∞,1,5,9)	$P_{6,14}$

If these groups of four 4-cycles are each developed modulo 17 (the residue class is 1, ..., 17 and ∞ is a fixed point) then all the edges of $L(K_{18})$ will have been used. That is, all 2-paths of K_{18} have been used and we have a near-resolvable C(18,4,1) design. ∎

The following theorem is a special case of Theorem 5.15, with s=2. It is presented here, however, because its proof is simple and it allows us to simplify the proof of Theorem 5.15.

5.13 THEOREM: If there exists a near-resolvable C(t,4,1) design then there exists a resolvable C(2t,4,1) design.

PROOF: Suppose we have a near-resolvable C(t,4,1) design with vertex set T. Label the vertices of K_t with the elements of $T:=\{1,...,t\}$ and label the vertices of K_{2t} with the elements of $T\times\{1,2\}$, which will also be the vertex set of the resolvable C(2t,4,1) design. We will write i_j instead of (i,j). Let K_t^i be the restriction of K_{2t} to the vertex set $T\times\{i\}$, i=1,2.

Let $\mathcal{F}=\{F_1, \ldots, F_{t-1}\}$ be a 1-factorization of K_t and arbitrarily order the edges in each F_i. If (a,b) is the j^{th} edge of F_i in K_t then we define

(a_1,b_1) to be the j^{th} edge of F_i in K_t^1 and (a_2,b_2) to be the j^{th} edge of F_i in K_t^2. The proof is presented in three steps, which are illustrated in Figure 10.

<u>Step 1.</u> Each near-parallel class $P_{x,y}$ in the C(t,4,1) design determines a parallel class $R_{x,y}$ in the C(2t,4,1) design as follows. For each 4-cycle $(a,b,c,d)\epsilon P_{x,y}$ we have $(a_1,b_1,c_1,d_1)\epsilon R_{x,y}$ and $(a_2,b_2,c_2,d_2)\epsilon R_{x,y}$. These $(t-2)/2$ 4-cycles, together with the 4-cycle (x_1,y_1,x_2,y_2), form the parallel class $R_{x,y}$ in the C(2t,4,1) design. In this way we get $t(t-1)/2$ parallel classes, one for each near-parallel class in the near-resolution of the C(t,4,1) design.

The remaining parallel classes that are to be defined will contain those 4-cycles with two vertices from $T\times\{1\}$ and two vertices from $T\times\{2\}$.

<u>Step 2.</u> For $1\le j\le t-1$, consider the 1-factor F_j. This 1-factor will be used to define two parallel classes, S_j^1 and S_j^2, as follows. For each $(x,y)\epsilon F_j$ we have $(x_1,x_2,y_1,y_2)\epsilon S_j^1$ and $(x_1,x_2,y_2,y_1)\epsilon S_j^2$. This gives $2(t-1)$ parallel classes. (Note that the 4-cycle (x_1,y_1,x_2,y_2) already belongs to the parallel class $R_{x,y}$ defined in Step 1.)

<u>Step 3.</u> So far, each pair of vertices of $T\times\{1\}$ has appeared on exactly three 4-cycles with only one pair of vertices from $T\times\{2\}$. In this step of the proof we define the $3(t-1)(t-2)/2$ parallel classes which remain. Addition is modulo $t/2$ on the residue class 1, 2, ..., $t/2$.

For each i, $1\le i\le t-1$, and for each k, $1\le k\le (t-2)/2$, take the j^{th} edge, $1\le j\le t/2$, (a_1,b_1) of F_i in K_t^1 together with the $(j+k)^{th}$ edge (c_2,d_2) of F_i in K_t^2. Take a C(4,4,1) design on the vertex set $\{a_1,b_1,c_2,d_2\}$. For each i and k we get $3(t/2)$ 4-cycles which can easily be partitioned into three parallel classes. This gives $3(t-1)(t-2)/2$ parallel classes.

Now take the $3(t-1)(t-2)/2$ parallel classes just defined, together with the parallel classes S_j^1 and S_j^2 $(1\le j\le t-1)$ and $R_{x,y}$ $(1\le x<y\le t)$. These $3(t-1)(t-2)/2 + 2(t-1) + t(t-1)/2 = (2t-1)(t-1)$ parallel classes contain all

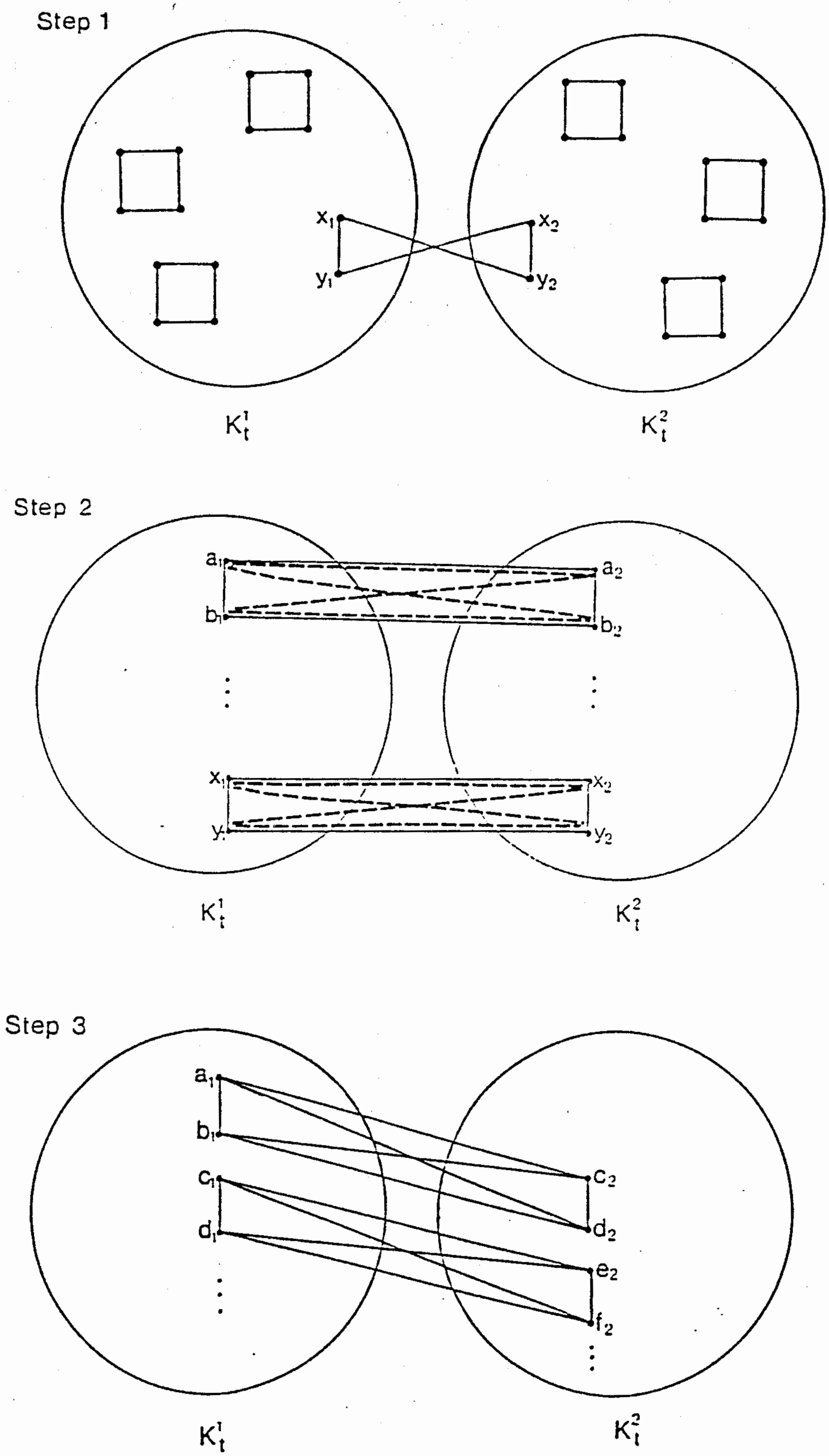

Step 1
x_1
y_1
x_2
y_2
K_t^1
K_t^2
Step 2
a_1
b_1
a_2
b_2
x_1
y_1
x_2
y_2
K_t^1
K_t^2
Step 3
a_1
b_1
c_1
d_1
c_2
d_2
e_2
f_2
K_t^1
K_t^2

Figure 10

the 2-paths in K_{2t} exactly once each. ∎

The following construction allows us to take an orthogonal array of strength three and partition it into n^2 orthogonal arrays of strength one. In a later construction of resolvable C(n,4,1) designs we will find that this property will be essential in maintaining the resolvability of the designs.

<u>5.14</u> <u>**CONSTRUCTION:**</u> In this construction we will define an OA(n,4) of strength three from a Latin square of order n. The columns of this orthogonal array can be partitioned into n^2 sets, each of which is an OA(n,4) of strength one.

Let A=A(i,j) be a Latin square of order n with rows, columns and entries taken from the set {1, ..., n}. Define n permutations ρ_i, $1 \le i \le n$, of the elements of A by $\rho_i A(1,c)=A(i,c)$, $1 \le c \le n$.

It is not difficult to verify that for $1 \le i,j \le n$ and $0 \le d \le n-1$ the n^3 columns

$$(i,j,i+d,\rho_{i+d}(A(i,j)))^T$$

form an OA(n,4) of strength three. Here addition is modulo n on the residue class 1, 2, ..., n.

Fix k, $1 \le k \le n$, and consider all ordered pairs (i,j_i) such that $A(i,j_i)=k$. Then for a fixed d, $0 \le d \le n-1$, the n columns

$$(i,j_i,i+d,\rho_{i+d}(k))^T$$

form an OA(n,4) of strength one. These orthogonal arrays will be denoted OA(n,4;d,k).

<u>5.15</u> <u>**THEOREM:**</u> If there exists a near-resolvable C(s,4,1) design and a near-resolvable C(t,4,1) design then there exists a resolvable C(st,4,1) design.

<u>**PROOF:**</u> Taking s=2 we see that we have proved this result in Theorem 5.13.

Thus in what follows we assume without loss of generality that both $s \geq 6$ and $t \geq 6$.

Suppose we have a near-resolvable $C(s,4,1)$ design and a near-resolvable $C(t,4,1)$ design with vertex sets $S:=\{1, \ldots, s\}$ and $T:=\{1, \ldots, t\}$, respectively. Label the vertices of K_s and K_t with the elements of S and T, respectively, and label the vertices of K_{st} with the elements of $T \times S$, which will also be the vertex set of the resolvable $C(st,4,1)$ design. We write i_j instead of (i,j). Let K_t^i be K_{st} restricted to the vertex set $T \times \{i\}$, $i \in S$.

Let $\mathcal{F}=\{F_1, \ldots, F_{s-1}\}$ and $\mathcal{G}=\{G_1, \ldots, G_{t-1}\}$ be 1-factorizations of K_s and K_t, respectively. Now, arbitrarily order the edges in each 1-factor and if (a,b) is the j^{th} edge of F_i (G_i) in K_s (K_t) then we define (a_k,b_k) to be the j^{th} edge of F_i (G_i) in K_s^k (K_t^k). The proof is presented in three steps, which are illustrated in Figures 11a, 11b and 11c. For this illustration we use $s=6$.

<u>Step</u> <u>1.</u> In Figure 11a we show this step pictorially and have used $F_1=\{(1,2),(3,4),(5,6)\}$.

Each near-parallel class $P_{x,y}$ in the $C(t,4,1)$ design determines a parallel class in the $C(st,4,1)$ design as follows. For each 4-cycle $(a,b,c,d) \in P_{x,y}$ we create the 4-cycles (a_i,b_i,c_i,d_i), $1 \leq i \leq s$. These $s(t-2)/4$ cycles of length four, together with the $s/2$ 4-cycles (x_i,y_i,x_j,y_j), where $(i,j) \in F_1$, form a parallel class in the $C(st,4,1)$ design. In this way we get $t(t-1)/2$ parallel classes, one for each near-parallel class in the near-resolution of the $C(t,4,1)$ design.

<u>Step</u> <u>2.</u> In this step we do <u>not</u> define any parallel classes of the $C(st,4,1)$ design. Instead, for each edge $(u,v) \in K_s$ we define partial parallel classes of 4-cycles on $T \times \{u,v\}$. These partial parallel classes of 4-cycles contain all 2-paths on the vertices of $T \times \{u,v\}$ which have not yet been used in some parallel class of Step 1.

Step 1

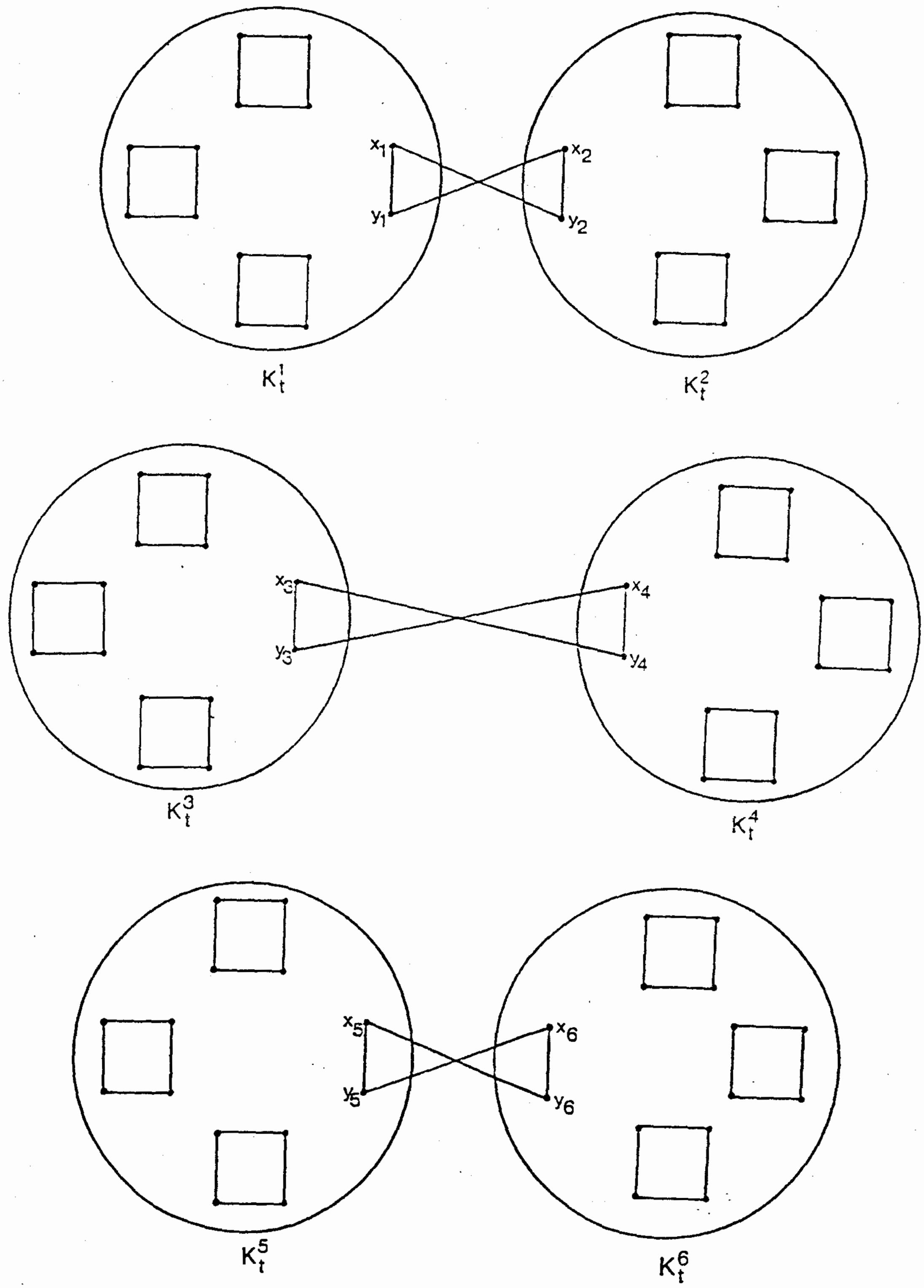


Figure 11a

a) If $(u,v) \in F_1$ there will be $2(t-1) + 3(t-1)(t-2)/2$ partial parallel classes of 4-cycles defined as follows.

For each 1-factor G_j in K_t, $1 \le j \le t-1$, we define two partial parallel classes of 4-cycles, S_j^1 and S_j^2. For each $(a,b) \in G_j$ we have $(a_u,a_v,b_u,b_v) \in S_j^1$ and $(a_u,a_v,b_v,b_u) \in S_j^2$. Thus we get $2(t-1)$ partial parallel classes of 4-cycles on $T \times \{u,v\}$. (Note that the 4-cycle (a_u,b_u,a_v,b_v) was already used in Step 1.)

For each h and k, $1 \le h \le t-1$ and $1 \le k \le (t-2)/2$, we define three partial parallel classes of 4-cycles on $T \times \{u,v\}$. For $1 \le j \le t/2$ take the j^{th} edge (a_u,b_u) of G_h in K_t^u together with the $(j+k)^{th}$ edge (c_v,e_v) of G_h in K_t^v to form the block $\{a_u,b_u,c_v,e_v\}$. Take the C(4,4,1) design on these vertices and distribute the three 4-cycles in this design among three different sets. Thus for each h and k we have 3(t/2) cycles of length four which can easily be partitioned into three partial parallel classes, each containing t/2 vertex disjoint 4-cycles on $T \times \{u,v\}$.

b) If $(u,v) \in F_i$, $2 \le i \le s-1$, there will be $3(t-1)(t/2)$ partial parallel classes on $T \times \{u,v\}$ defined. For each h and k, $1 \le h \le t-1$ and $1 \le k \le t/2$, a partial parallel class of blocks of size four is formed by taking, for $1 \le j \le t/2$, the j^{th} edge (a_u,b_u) of G_h in K_t^u together with the $(j+k)^{th}$ edge (c_v,e_v) of G_h in K_t^v to form the block $\{a_u,b_u,c_v,e_v\}$. This partial parallel class of blocks is now replaced, using Theorem 5.5, by three partial parallel classes of 4-cycles.

Step 3. In Figure 11b we illustrate this step under the assumption that (1,2,4,3) is a cycle in the near-parallel class $P_{5,6}$ of the near-resolvable C(6,4,1) design.

For each near-parallel class $P_{x,y}$ in the C(s,4,1) design we will define t^2 parallel classes in the C(st,4,1) design.

Step 3

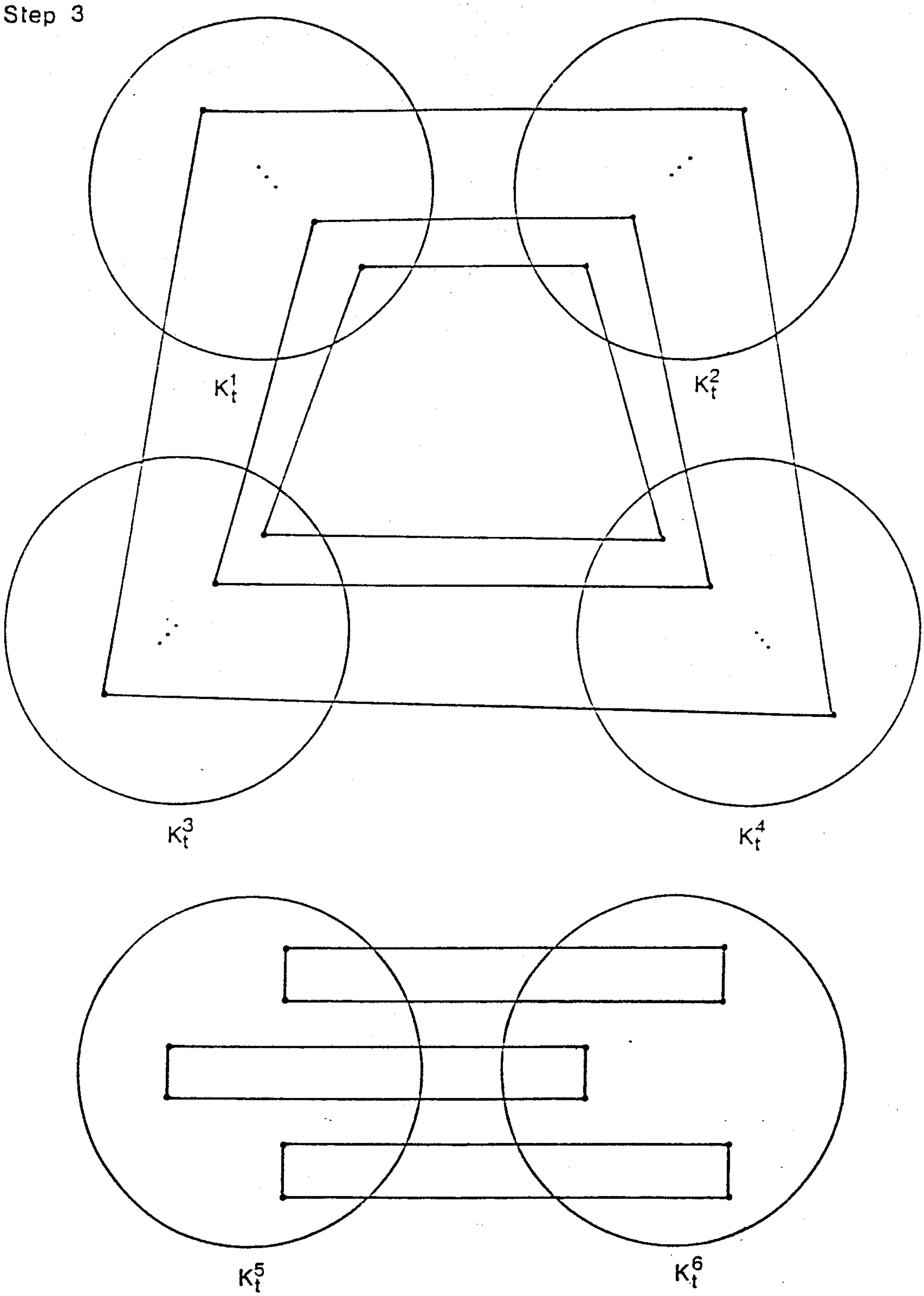


Figure 11b

For $0 \le d \le t-1$ and $1 \le k \le t$ consider the $OA(t,4;d,k)$ as defined in Construction 5.14. We define a parallel class $P(x,y;d,k)$ of 4-cycles based on this orthogonal array of strength one by:

For each cycle $(a,b,c,e) \in P_{x,y}$ and for each column $(u,v,w,z)^T$ of the $OA(t,4;d,k)$ we have

$$(u_a, v_b, w_c, z_e) \in P(x,y;d,k).$$

In addition, we add to $P(x,y;d,k)$ 4-cycles on the vertex set $T \times \{x,y\}$ as follows:

1. If $(x,y) \in F_1$ then any partial parallel class of 4-cycles defined in Step 2a is added to $P(x,y;d,k)$. This partial parallel class is then deleted from the sets of 4-cycles constructed in Step 2a.
2. If $(x,y) \in F_i$, $2 \le i \le s-1$, then any partial parallel class of 4-cycles defined in Step 2b is added to $P(x,y;d,k)$. This partial parallel class is then deleted from the sets of 4-cycles constructed in Step 2b.

It is easy to see by a counting argument that the number of partial parallel classes of 4-cycles which were defined in Steps 2a and 2b is at least as great as the number of partial parallel classes of 4-cycles which are to be deleted from these steps.

<u>Step 4.</u> This final step is shown in Figure 11c. We assume in the illustration that $F_2 = \{(1,6),(2,4),(3,5)\}$. The cycles shown are thus taken from Step 2b.

For each edge $(x,y) \in F_1$ the number of partial parallel classes of 4-cycles on $T \times \{x,y\}$ remaining from Step 2a is $3(t/2-1)(t-1) + 2(t-1) - t^2 = (t^2-5t+2)/2$. For each edge $(x,y) \in F_i$, $2 \le i \le s-1$, the number of partial parallel classes of 4-cycles on $T \times \{x,y\}$ remaining from Step 2b is $3(t/2)(t-1) - t^2 = (t^2-3t)/2$.

For each edge in F_1 take a partial parallel class of 4-cycles from Step 2a. The collection of these s/2 partial parallel classes forms a

Step 4

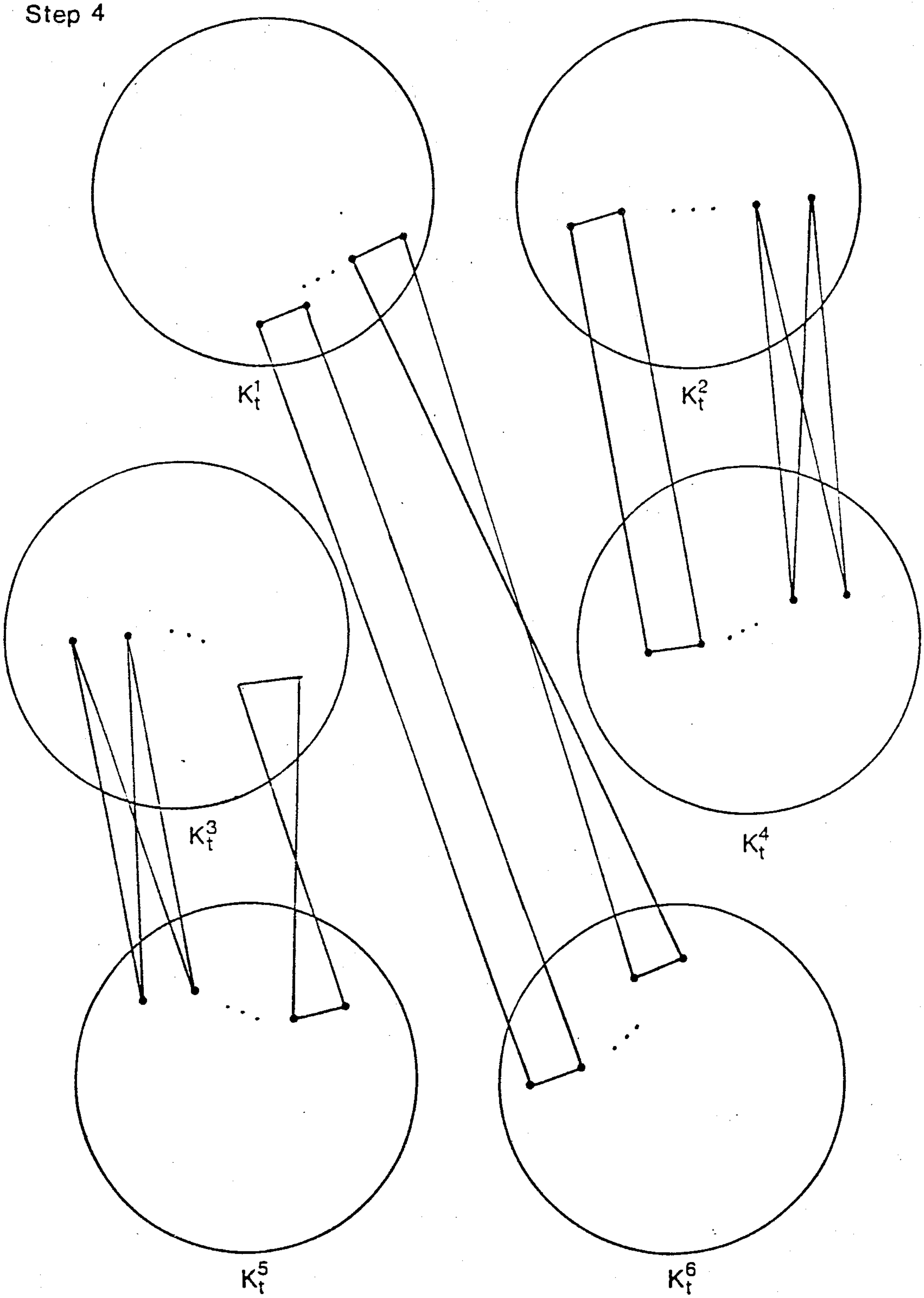


Figure 11c

parallel class in the $C(st,4,1)$ design. Delete these partial parallel classes from Step 2a. Repeat this step $(t^2-5t+2)/2$ times until all such partial parallel classes are accounted for.

For $2 \le i \le s-1$ take, for each edge in F_i, a partial parallel class of 4-cycles from Step 2b. These $s/2$ partial parallel classes form a parallel class in the $C(st,4,1)$ design. Delete these partial parallel classes from Step 2b. Repeat this step $(t^2-3t)/2$ times so that no partial parallel classes remain in Step 2b.

We now check to see that we have the correct number of parallel classes. The total number of parallel classes created at each step of the proof is:

Step 1	$t(t-1)/2$
Step 3	$t^2 s(s-1)/2$
Step 4	$(t^2-5t+2)/2 + (t^2-3t)(s-2)/2$.

The sum of these numbers is $(st-1)(st-2)/2$ as required and thus we have constructed a resolvable $C(st,4,1)$ design (of course one also needs to check that every 2-path is accounted for). ∎

<u>**5.16 THEOREM:**</u> If there exists a resolvable $C(s,4,1)$ design and a near-resolvable $C(t,4,1)$ design, then there exists a resolvable $C(st,4,1)$ design.

<u>**PROOF:**</u> The proof of this theorem is very similar to the proof of Theorem 5.15.

Suppose we have a resolvable $C(s,4,1)$ design with vertex set $S:=\{1, \ldots, s\}$ and a near-resolvable $C(t,4,1)$ design with vertex set $T:=\{1, \ldots, t\}$. Label the vertices of K_s and K_t with the elements of S and T, respectively, and label the vertices of K_{st} with the elements of $T \times S$. This set will also be the vertex set of the $C(st,4,1)$ design and we write

i_j instead of (i,j). Finally, we define K_t^i to be K_{st} restricted to the vertex set $T\times\{i\}$, $i\in S$.

Let $F=\{F_1, \ldots, F_{s-1}\}$ and $G=\{G_1, \ldots, G_{t-1}\}$ be 1-factorizations of K_s and K_t, respectively. Now arbitrarily order the edges in each 1-factor, and if (a,b) is the j^{th} edge of F_i (G_i) in K_s (K_t) then we define (a_k,b_k) to be the j^{th} edge of F_i (G_i) in K_s^k (K_t^k).

Step 1. This is the same as Step 1 of Theorem 5.15. We get $t(t-1)/2$ parallel classes, one for each partial parallel class in the near-resolution of the $C(t,4,1)$ design.

Step 2. This is the same as Step 2 of Theorem 5.15.

Step 3. This step differs from the corresponding step of Theorem 5.15 because the $C(s,4,1)$ design is actually resolvable, not near-resolvable. For each parallel class P in the $C(s,4,1)$ design we define t^2 parallel classes in the $C(st,4,1)$ design.

For $0\le d\le t-1$ and $1\le k\le t$ consider the $OA(t,4;d,k)$ as defined in Construction 5.14. We define a parallel class $P(d,k)$ of 4-cycles based on this orthogonal array of strength one by:

For each cycle $(a,b,c,e)\in P$ and for each column $(u,v,w,z)^T$ of the $OA(t,4;d,k)$ we have

$$(u_a,v_b,w_c,z_e)\in P(d,k).$$

Step 4. This is similar to Step 4 of Theorem 5.15, except for the fact that no partial parallel classes of 4-cycles have yet been used from Step 2. For each edge $(x,y)\in F_1$ the number of partial parallel classes of 4-cycles on $T\times\{x,y\}$ to be used from Step 2a is $(3t-2)(t-1)/2$. For each edge $(x,y)\in F_i$, $2\le i\le s-1$, the number of partial parallel classes of 4-cycles on $T\times\{x,y\}$ to be used from Step 2b is $3t(t-1)/2$.

The rest of the proof closely follows Step 4 of Theorem 5.15.

We now count the total number of parallel classes created at each step of the proof.

Step 1	$t(t-1)/2$
Step 3	$t^2(s-1)(s-2)/2$
Step 4	$(3t-2)(t-1)/2 + 3t(t-1)(s-2)/2$

The sum of these numbers is $(st-1)(st-2)/2$ as required and thus we have constructed a resolvable $C(st,4,1)$ design. ■

5.17 THEOREM: If there exists a resolvable $C(s,4,1)$ design and a resolvable $C(t,4,1)$ design then there exists a resolvable $C(st,4,1)$ design.

PROOF: The proof of this theorem is very similar to the proof of Theorem 5.15.

Suppose we have resolvable $C(s,4,1)$ and $C(t,4,1)$ designs with vertex sets $S:=\{1, \ldots, s\}$ and $T:=\{1, \ldots, t\}$, respectively. Label the vertices of K_s and K_t with the elements of S and T, respectively, and label the vertices of K_{st} with the elements of $T \times S$. This set will also be the vertex set of the $C(st,4,1)$ design and we write i_j instead of (i,j). Finally, we define K_t^i to be K_{st} restricted to the vertex set $T \times \{i\}$, $i \in S$.

Let $\mathcal{F}=\{F_1, \ldots, F_{s-1}\}$ and $\mathcal{G}=\{G_1, \ldots, G_{t-1}\}$ be 1-factorizations of K_s and K_t, respectively. Now, arbitrarily order the edges in each 1-factor and if (a,b) is the j^{th} edge of F_i (G_i) in K_s (K_t) then we define (a_k,b_k) to be the j^{th} edge of F_i (G_i) in K_s^k (K_t^k).

Step 1. For each parallel class P in the $C(t,4,1)$ design we get a parallel class in the $C(st,4,1)$ design as follows. For each 4-cycle $(a,b,c,d) \in P$ we create the 4-cycles (a_i,b_i,c_i,d_i), $1 \leq i \leq s$. These $st/4$ cycles of length four form a parallel class in the $C(st,4,1)$ design. We thus get $(t-1)(t-2)/2$ parallel classes, one for each parallel class in the resolution of the $C(t,4,1)$ design.

Step 2. In this step we do not define any parallel classes of the $C(st,4,1)$ design. Instead, for each edge $(x,y)\epsilon K_s$ we define partial parallel classes of 4-cycles on $T\times\{x,y\}$. These partial parallel classes of 4-cycles contain all 2-paths on the vertices of $T\times\{x,y\}$ which have not yet been used in some parallel class of Step 1.

For each $(x,y)\epsilon F_i$, where $1\le i\le s-1$, $3(t-1)(t/2)$ partial parallel classes on $T\times\{x,y\}$ will be defined. For each $1\le h\le t-1$ and $1\le k\le t/2$ a partial parallel class of blocks of size four is formed by taking, for $1\le j\le t/2$, the j^{th} edge (a_x,b_x) of G_h in K_t^x together with the $(j+k)^{th}$ edge (c_y,e_y) of G_h in K_t^y to form the block $\{a_x,b_x,c_y,e_y\}$. This partial parallel class of blocks is now replaced, using Theorem 5.5, by three partial parallel classes of 4-cycles.

Step 3. This step is the same as Step 3 of Theorem 5.16.

Step 4. Again, this is similar to Step 4 of Theorem 5.15, except for the fact that no classes of 4-cycles have yet been used from Step 2. For each edge $(x,y)\epsilon F_i$, $1\le i\le s-1$, the number of classes of 4-cycles on $T\times\{x,y\}$ to be used from Step 2 is $3t(t-1)/2$.

The rest of the proof closely follows Step 4 of Theorem 5.15.

We now count the total number of parallel classes created at each step of the proof.

Step 1	$(t-1)(t-2)/2$
Step 3	$t^2(s-1)(s-2)/2$
Step 4	$3t(t-1)(s-1)/2$

The sum of these numbers is $(st-1)(st-2)/2$ as required and the $C(st,4,1)$ design is thus resolvable. ■

Using the theorems and lemmas of this chapter we are able to find resolvable $C(n,4,1)$ designs for 87 of the 100 admissable values of n up to 400.

5.18 THEOREM: Let $X = \{132, 156, 204, 220, 228, 236, 276, 292, 300, 348, 364, 372, 396\}$. If $4 \le n \le 400$ and $n \equiv 0 \pmod 4$ then there exists a resolvable $C(n,4,1)$ design, except possibly for $n \in X$.

PROOF: The proof is presented in Appendix 2. ■

CHAPTER 6

EXACT COVERINGS USING 5-CYCLES

In this chapter we look at the problem of finding $C(n,5,\lambda)$ designs. There are $(n)(n-1)(n-2)/2$ 2-paths in K_n and every 5-cycle contains five 2-paths. Thus if there exists a $C(n,5,\lambda)$ design then $\lambda n(n-1)(n-2) \equiv 0 \pmod{10}$. If λ is not a multiple of five then the necessary conditions for the existence of a $C(n,5,\lambda)$ design are that $n \equiv 0$, 1 or 2 (mod 5). If λ is a multiple of five then there are no conditions on n for the existence of a $C(n,5,\lambda)$ design.

The following work was done with Dr. R. Mathon of the University of Toronto. We construct $C(n,5,1)$ designs for n=5,6,7 and 10. These are presented below.

6.1 THEOREM: There exists a $C(5,5,1)$ design.

PROOF: The six 5-cycles in the design are listed below (addition is modulo 5 on the residue class 1, ..., 5). It is a simple matter to verify the existence of all paths of length two.

$(1,2,3,4,5)$ $\quad$ $(1+i,5+i,2+i,4+i,3+i)$ $\quad$ $0 \le i \le 4$. ■

6.2 THEOREM: There exists a $C(6,5,1)$ design.

PROOF: Using the point set $\{\infty, 1, ..., 5\}$, the twelve cycles of length five in a $C(6,5,1)$ design are listed below. The point ∞ is a fixed point, addition is modulo 5 on the residue class 1, ..., 5, and $0 \le i \le 4$.

$(1,2,3,4,5)$ $\quad$ $(\infty,1+i,3+i,2+i,4+i)$

$(1,3,5,2,4)$ $\quad$ $(\infty,1+i,2+i,4+i,5+i)$. ■

6.3 THEOREM: There exists a $C(7,5,1)$ design.

PROOF: The following twenty one cycles of length five (with addition modulo 7 on the residue class 1, ..., 7) cover every 2-path exactly once each. Here we take $0 \le i \le 6$.

(1+i,4+i,6+i,3+i,2+i) (3+i,4+i,6+i,7+i,5+i)

(1+i,3+i,6+i,7+i,4+i). ■

6.4 LEMMA: There exists a C(10,5,1) design.

PROOF: The point set that we use is $\{\infty, 1, \ldots, 9\}$, where ∞ is a fixed point. Addition is modulo 9 on the residue class 1, ..., 9, and $0 \le i \le 8$. The following seventy-two cycles of length five contain each 2-path once.

(∞,1+i,3+i,2+i,4+i) (∞,2+i,3+i,5+i,6+i)

(∞,7+i,4+i,2+i,8+i) (∞,7+i,3+i,4+i,9+i)

(2+i,3+i,4+i,8+i,5+i) (3+i,4+i,1+i,6+i,8+i)

(5+i,7+i,9+i,6+i,1+i) (1+i,3+i,8+i,5+i,4+i). ■

The C(11,5,1) design and the C(12,5,1) design given below were found very recently by R. Mathon. I would like to thank him for allowing me to reproduce them here.

6.5 LEMMA: There exists a C(11,5,1) design.

PROOF: The following design is based on the set {0, ..., 10}. Addition is modulo 11 and $0 \le i \le 10$. The ninety-nine 5-cycles listed contain each 2-path of K_{11} once.

(0+i,1+i,2+i,10+i,3+i) (0+i,1+i,5+i,10+i,8+i)

(0+i,3+i,6+i,8+i,9+i) (0+i,10+i,6+i,1+i,3+i)

(0+i,9+i,7+i,2+i,5+i) (0+i,1+i,10+i,3+i,6+i)

(0+i,5+i,10+i,6+i,4+i) (0+i,10+i,1+i,8+i,5+i)

(0+i,4+i,8+i,7+i,1+i).

When i=0 we note that each of the 5-cycles in the first column can be obtained by multiplying each entry in the preceding 5-cycle by 3. Also note that in the second column the first and second 5-cycles and the third and fourth 5-cycles are 'additive inverses' of each other. ■

6.6 LEMMA: There exists a C(12,5,1) design.

PROOF: The following design is based on the set $\{\infty, 0, \ldots, 10\}$. Addition is modulo 11 and $0 \le i \le 10$. The 132 cycles of length 5 that are listed below contain each 2-path of K_{12} once.

$(\infty+i,0+i,10+i,7+i,2+i)$	$(0+i,1+i,2+i,10+i,3+i)$
$(\infty+i,0+i,8+i,10+i,6+i)$	$(0+i,3+i,6+i,8+i,9+i)$
$(\infty+i,0+i,2+i,8+i,7+i)$	$(0+i,9+i,7+i,2+i,5+i)$
$(\infty+i,0+i,6+i,2+i,10+i)$	$(0+i,5+i,10+i,6+i,4+i)$
$(\infty+i,0+i,7+i,6+i,8+i)$	$(0+i,4+i,8+i,7+i,1+i)$
$(0+i,1+i,10+i,3+i,6+i)$	$(0+i,10+i,6+i,1+i,3+i)$.

Note that when $i=0$ the second through fifth 5-cycles in each column can be obtained from the one preceding it by multiplying each number (other that ∞) by 3. ■

As in the case of $C(n,4,\lambda)$ designs we are able to use 3-designs and existing $C(n,5,\lambda)$ designs to construct more $C(n,5,\lambda)$ designs. The following lemma is an obvious extension to Lemma 1.7, as is its proof.

6.7 LEMMA: Let $K=\{n_1, \ldots, n_r\}$. If for each n_i, $1 \le i \le r$, there exists a $C(n_i,5,\lambda)$ design and if there exists a $3\text{-}(n,K,\mu)$ design then there exists a $C(n,5,\lambda\cdot\mu)$ design.

For example, using this lemma and various 3-designs (see [HHK]) we see that there are also $C(n,5,1)$ designs for $n \in \{17,21,22,25,26\}$.

Whether or not $C(n,5,1)$ designs exist for all permissible values of n is an interesting problem to pursue. However, even for the small cases, the problem is large and could take a very long time to solve on the computer. It therefore appears that using the computer to solve this problem is inappropriate, but an attempt to use design theory or group theory could possibly result in further C(n,5,1) designs.

APPENDIX 1

In the Introduction, we discussed the problem of finding a set of $(n-1)(n-2)/2$ Hamilton cycles in K_n so that every 2-path lies on exactly one Hamilton cycle. Using a method described by L. E. Dickson [D2], we found 3316 distinct solutions to the problem when $n=17$. The irreducible polynomial in $GF(2^4)$ that was used was $a^4=a+1$. We now present the 120 cycles for one of these solutions, except that we write i instead of a^{i-1} when $2 \le i \le 15$.

∞	0	1	5	9	10	8	13	3	15	2	6	11	4	12	7	14
∞	1	0	2	3	8	10	12	9	4	5	11	6	15	13	14	7
∞	2	5	1	11	4	15	14	6	8	0	3	9	10	7	12	13
∞	3	9	11	1	12	13	8	0	14	6	2	5	7	10	4	15
∞	4	15	8	14	2	5	11	7	1	10	12	13	0	6	3	9
∞	5	2	0	6	15	4	7	11	10	1	9	3	8	14	13	12
∞	6	11	9	5	7	14	15	2	13	3	0	1	12	4	10	8
∞	10	8	15	13	0	1	9	12	5	4	7	14	2	3	6	11
∞	0	2	6	10	11	9	14	4	1	3	7	12	5	13	8	15
∞	1	5	11	8	6	3	7	15	0	9	14	13	2	12	10	4
∞	2	0	3	4	9	11	13	10	5	6	12	7	1	14	15	8
∞	3	6	2	12	5	1	15	7	9	0	4	10	11	8	13	14
∞	5	1	9	15	3	6	12	8	2	11	13	14	0	7	4	10
∞	6	3	0	7	1	5	8	12	11	2	10	4	9	15	14	13
∞	7	12	10	6	8	15	1	3	14	4	0	2	13	5	11	9
∞	11	9	1	14	0	2	10	13	6	5	8	15	3	4	7	12
∞	0	3	7	11	12	10	15	5	1	4	8	13	6	14	9	1
∞	2	6	12	9	7	4	8	1	5	10	15	14	3	13	11	5
∞	3	0	4	5	10	12	14	11	9	7	13	8	2	15	1	9
∞	4	7	3	13	6	2	1	8	15	0	5	11	12	9	14	15
∞	6	2	10	1	4	7	13	9	11	12	14	15	0	8	5	11
∞	7	4	0	8	2	6	9	13	14	3	11	5	10	1	15	14
∞	8	13	11	7	9	1	2	4	10	5	0	3	14	6	12	10
∞	12	10	2	15	0	3	11	14	13	6	9	1	4	5	8	13
∞	0	4	8	12	13	11	1	6	2	5	9	14	7	15	10	1
∞	2	10	15	7	14	9	5	3	0	1	11	13	12	8	4	5
∞	3	7	13	10	8	5	9	2	6	11	1	15	4	14	12	9
∞	4	0	5	6	11	13	15	12	10	8	14	9	3	1	2	15
∞	6	12	14	4	15	1	11	0	3	9	5	8	10	13	7	11
∞	7	3	11	2	5	8	14	10	12	13	15	1	0	9	6	14
∞	8	5	0	9	3	7	10	14	15	4	12	6	11	2	1	10
∞	12	6	9	0	1	15	13	4	7	14	8	5	2	11	3	13
∞	0	5	9	13	14	12	1	7	3	6	10	15	8	1	11	2
∞	1	2	3	12	7	13	0	14	9	11	8	4	10	0	6	5
∞	3	11	1	8	15	10	9	4	0	2	12	14	13	9	5	6
∞	4	8	14	11	9	6	15	3	7	12	2	1	5	15	13	10
∞	7	13	15	5	1	2	14	0	4	10	6	9	11	14	8	12
∞	8	4	12	3	6	9	10	11	13	14	1	2	0	10	7	15
∞	9	6	0	10	4	8	3	15	1	5	13	7	12	3	2	11
∞	13	7	10	0	2	1	12	5	8	15	9	6	3	12	4	14
∞	0	6	10	14	15	13	2	8	4	7	11	1	9	1	12	3
∞	1	11	8	7	4	12	5	10	15	14	6	0	3	0	13	9
∞	2	3	4	13	8	14	0	15	10	12	9	5	11	5	7	6

∞	4	12	2	9	1	11	10	5	0	3	13	15	14	15	6	7
∞	5	9	15	12	10	7	1	4	8	13	3	2	6	2	14	11
∞	8	14	1	6	2	3	15	0	5	11	7	10	12	10	9	13
∞	10	7	0	11	5	9	4	1	2	6	14	8	13	8	3	12
∞	14	8	11	0	3	2	13	6	9	1	10	7	4	7	5	15
∞	0	7	11	15	1	14	3	9	5	8	12	1	10	2	13	4
∞	1	14	6	4	0	7	9	3	2	10	13	0	8	5	12	15
∞	2	12	9	8	5	13	6	11	1	15	7	5	4	0	14	10
∞	3	4	5	14	9	15	0	1	11	13	10	9	12	6	8	7
∞	5	13	3	10	2	12	11	6	0	4	14	2	15	1	7	8
∞	6	10	1	13	11	8	2	5	9	14	4	11	7	3	15	12
∞	9	15	2	7	3	4	1	0	6	12	8	3	13	11	10	14
∞	11	8	0	12	6	10	5	2	3	7	15	6	14	9	4	13
∞	0	8	12	1	1	15	4	10	6	9	13	2	11	3	14	5
∞	1	10	13	0	0	4	15	8	11	3	12	5	6	9	7	2
∞	3	13	10	9	9	14	7	12	2	1	8	6	5	0	15	11
∞	4	5	6	15	15	1	0	2	12	14	11	10	13	7	9	8
∞	6	14	4	11	11	13	12	7	0	5	15	3	1	2	8	9
∞	7	11	2	14	14	9	3	6	10	15	5	12	8	4	1	13
∞	10	1	3	8	8	5	2	0	7	13	9	4	14	12	11	15
∞	12	9	0	13	13	11	6	3	4	8	1	7	15	10	5	14
∞	0	9	13	1	2	1	5	11	7	10	14	3	12	4	15	6
∞	1	3	12	0	5	0	2	6	14	8	7	9	13	15	4	11
∞	2	11	14	5	0	5	1	9	12	4	13	6	7	10	8	3
∞	4	14	11	15	10	15	8	13	3	2	9	7	6	0	1	12
∞	5	6	7	2	1	2	0	3	13	15	12	11	14	8	10	9
∞	7	15	5	14	12	14	13	8	0	6	1	4	2	3	9	10
∞	8	12	3	10	15	10	4	7	11	1	6	13	9	5	2	14
∞	13	10	0	12	14	12	7	4	5	9	2	8	1	11	6	15
∞	0	10	14	2	3	1	6	12	8	11	15	4	13	5	1	7
∞	1	8	7	5	9	0	11	13	10	6	4	15	12	2	0	14
∞	2	4	13	0	6	5	3	7	15	9	8	10	14	1	5	12
∞	3	12	15	6	0	9	2	10	13	5	14	7	8	11	9	4
∞	5	15	12	1	11	2	9	14	4	3	10	8	7	0	2	13
∞	6	7	8	3	2	11	0	4	14	1	13	12	15	9	11	10
∞	8	1	6	15	13	10	14	9	0	7	2	5	3	4	10	11
∞	9	13	4	11	1	3	5	8	12	2	7	14	10	6	3	15
∞	0	11	15	3	4	2	7	13	9	12	1	5	14	6	1	8
∞	1	6	4	9	15	5	14	12	3	13	0	2	7	11	0	10
∞	2	9	8	6	10	0	12	14	11	7	5	1	13	3	5	15
∞	3	5	14	0	7	6	4	8	1	10	9	11	15	2	9	13
∞	4	13	1	7	0	10	3	11	14	6	15	8	9	12	15	5
∞	6	1	13	2	12	3	10	15	5	4	11	9	8	0	11	14
∞	7	8	9	4	3	12	0	5	15	2	14	13	1	10	14	11
∞	9	2	7	1	14	11	15	10	0	8	3	6	4	5	3	12
∞	0	12	1	4	5	3	8	14	10	13	1	6	15	7	2	9
∞	1	13	0	15	2	9	10	7	8	12	0	11	4	14	5	3
∞	2	7	5	10	1	6	15	13	4	14	5	3	8	12	0	11
∞	4	6	15	0	8	7	5	9	2	11	15	12	1	3	10	14
∞	5	14	2	8	0	11	4	12	15	7	2	9	10	13	1	6
∞	7	2	14	3	13	4	11	1	6	5	14	10	9	0	12	15
∞	8	9	10	5	4	13	0	6	1	3	10	14	2	11	15	12
∞	10	3	8	2	15	12	1	11	0	9	8	7	5	6	4	13
∞	0	13	1	5	6	4	9	15	11	14	2	7	1	8	3	10
∞	1	12	0	2	11	15	3	4	6	7	5	14	0	10	9	8
∞	2	14	5	1	3	10	11	8	9	13	0	12	5	15	6	4
∞	3	8	9	11	2	7	1	14	5	15	6	4	9	13	0	12
∞	5	7	2	0	9	8	6	10	3	12	1	13	2	4	11	15
∞	6	15	11	9	0	12	5	13	1	8	3	10	11	14	2	7
∞	9	10	3	6	5	14	0	7	2	4	11	15	3	12	1	13
∞	11	4	6	3	1	13	2	12	0	10	9	8	6	7	5	14
∞	0	14	2	6	7	5	10	1	12	15	3	8	1	9	4	11
∞	1	7	5	11	14	2	8	0	13	4	9	10	0	3	15	6

∞	2	13	0	3	12	1	4	5	7	8	6	15	5	11	10	9
∞	3	15	6	2	4	11	12	9	10	14	0	13	9	1	7	5
∞	4	9	10	12	3	8	2	15	6	1	7	5	15	14	0	13
∞	7	1	12	10	0	13	6	14	2	9	4	11	14	15	3	8
∞	10	11	4	7	6	15	0	8	3	5	12	1	8	13	2	14
∞	12	5	7	4	2	14	3	13	0	11	10	9	13	8	6	15
∞	0	15	3	7	8	6	11	1	13	1	4	9	2	10	5	12
∞	1	4	9	14	10	11	6	0	12	0	15	3	5	8	2	13
∞	2	8	6	12	15	3	9	5	14	5	10	11	0	4	1	7
∞	3	14	0	4	13	2	5	9	8	9	7	1	6	12	11	10
∞	4	1	7	3	5	12	13	15	11	15	0	14	10	2	8	6
∞	5	10	11	13	4	9	3	2	7	2	8	6	1	15	0	14
∞	8	2	13	11	0	14	7	10	3	10	5	12	15	1	4	9
∞	11	12	5	8	7	1	0	6	4	6	13	2	9	14	3	15

APPENDIX 2

We present here the proof of Theorem 5.18. The values of n ($4 \le n \le 400$) for which a resolvable C(n,4,1) design is known to exist are listed in tabular form, along with the results used to justify their resolvability. For those values of n for which there exists a resolvable SQS(n), we write * to mean Theorems 5.1 and 5.5 are used in the proof. All unsolved values are listed with a dash (-) beside them; their resolvability could be proved by finding near-resolvable designs of an appropriate order.

n	Theorems & Lemmas used	n	Theorems & Lemmas used
4	*	84	5.9, 5.11 and 5.15
8	*	88	*
12	5.9 and 5.13	92	*
16	*	96	5.17
20	*	100	*
24	5.6	104	*
28	*	108	5.9, 5.12 and 5.15
32	*	112	*
36	5.12 and 5.13	116	*
40	*	120	5.10 and 5.16
44	*	124	*
48	5.17	128	*
52	*	132	-
56	*	136	*
60	5.9, 5.10 and 5.15	140	*
64	*	144	5.17
68	*	148	*
72	5.12 and 5.16	152	*
76	*	156	-
80	*	160	*

n	Theorems and Lemmas used	n	Theorems and Lemmas used
164	*	284	*
168	5.11 and 5.16	288	5.17
172	*	292	-
176	*	296	*
180	5.10, 5.12 and 5.15	300	-
184	*	304	*
188	*	308	*
192	5.17	312	5.9 and 5.16
196	*	316	*
200	*	320	*
204	-	324	5.12 and 5.15
208	*	328	*
212	*	332	*
216	5.12 and 5.16	336	5.17
220	-	340	*
224	*	344	*
228	-	348	-
232	*	352	*
236	-	356	*
240	5.17	360	5.9 and 5.16
244	*	364	-
248	*	368	*
252	5.11, 5.12 and 5.15	372	-
256	*	376	*
260	*	380	*
264	5.9 and 5.16	384	5.17
268	*	388	*
272	*	392	*
276	-	396	-
280	*	400	*

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