CYCLE DECOMPOSITIONS OF COMPLETE GRAPHS

by

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CYCLE DECOMPOSITIONS OF ~~K~~ COMPLETE GRAPHS

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Abstract

The problem of finding a decomposition of the complete graph, directed or undirected, into cycles of a fixed length is one on which there has been much research, and for which there are still many cases left unsolved. We investigate in particular the decomposition of $K_{2n}-I$, the complete graph on $2n$ vertices with a one-factor removed, into cycles of fixed even length.

We begin with a brief exposition of known results in the area. We then construct a decomposition of the graph $K_{2n}-I$ into cycles of even length $2m$, for cases when n is even and $3m/2 \leq n < 2m$.

Table of contents

List of figures

Chapter 1

§1.1 Definitions and Notation.

1.1.1 K_n and K_n^* will be used to denote the complete undirected graph and the complete directed graph, respectively. $K_{n,s}$ and $K_{n,s}^*$ will be used to denote the complete undirected and directed bipartite graphs, respectively. The graph $K_{2n}-I$ is the complete undirected graph with a one-factor removed.

1.1.2 A path is a sequence $P=(x_1, ..., x_{n+1})$ of vertices together with the edges x_ix_{i+1}, $i=1, ..., n$, where $x_i \neq x_j$ if $i \neq j$. The vertices x_1 and x_{n+1} are the endvertices of P. The length of P, denoted $l(P)$, is the number of edges in P. A path of length m is also called an m-path. A Hamilton path in a graph G is a path which meets every vertex of G.

A cycle is a sequence $C=(x_1, ..., x_{n+1})$ of vertices together with the edges x_ix_{i+1}, $i=1, ..., n$, where $x_1=x_{n+1}$, and $x_i \neq x_j$ if $i \neq j$ and $\{x_i, x_j\} \neq \{x_1, x_{n+1}\}$. The length $l(C)$ of C is the number of edges in C. An m-cycle is a cycle of length m, and a Hamilton cycle in a graph G is one which meets every vertex of G.

A directed cycle or dicycle is a cycle in a directed graph, where x_ix_{i+1} is the arc directed from x_i to x_{i+1} for $i=1, ..., n$.

1.1.3 If $A=(x_1, ..., x_n)$ and $B=(x_n, ..., x_{n+k})$, then $A+B$ is the concatenation of A and B, that is, $A+B=(x_1, ..., x_n, x_{n+1}, ..., x_{n+k})$.

1.1.4 A one-factor of a graph G (also called a perfect matching) is a spanning subgraph of G in which every vertex has degree 1. A 2-factor of G is a spanning subgraph of G in which every vertex has degree 2.

1.1.5 We write $G = H_1 \oplus H_2$ if G is the edge-disjoint union of the subgraphs H_1 and H_2.

1.1.6 If G is a graph and n is a natural number, then nG denotes the graph consisting of n vertex-disjoint copies of G.

1.1.7 If $G = H_1 \oplus H_2 \oplus ... \oplus H_k$, where $H_1 \cong H_2 \cong ... \cong H_k \cong H$, then we say that G has an H-decomposition, or that G may be decomposed into subgraphs isomorphic to H.

1.1.8 Let G and H be graphs. The wreath product of G and H, denoted $G\text{wr}H$, is formed by replacing each vertex of G with a copy of H, and joining vertices in different copies of H by an edge if and only if the corresponding vertices of G are adjacent.

We state here for convenience some basic results which are used repeatedly throughout the thesis.

1.1.9 Lemma ([Lu]) The graph K_{2n+1} can be decomposed into Hamilton cycles for every natural number n.

1.1.10 Corollary The graph K_{2n} can be decomposed into Hamilton paths for every natural number n.

1.1.11 Lemma ([Lu]) The graph $K_{2n}-I$ can be decomposed into Hamilton cycles for every natural number n.

1.1.12 Lemma The graph $K_{2n}-I$ is isomorphic to the graph $K_n \text{wr} \bar{K}_2$.

Proof The graph $K_n \text{wr} \bar{K}_2$ is formed by replacing each vertex v of K_n with a pair of independent vertices v_1 and v_2, and adding the edges u_1v_1, u_1v_2, u_2v_1 and u_2v_2 for every edge uv of K_n. Thus each vertex v_1 of $K_n \text{wr} \bar{K}_2$ is adjacent to every other vertex of $K_n \text{wr} \bar{K}_2$ except its 'partner' v_2. ■

§ 1.2 Introduction

In this thesis we consider the decomposition of complete graphs (the complete directed graph, the complete undirected graph, and the graph $K_{2n}-I$) into cycles of fixed length. The problem has interested many authors and has proven to be quite challenging. Hamilton decompositions of the complete undirected graph and the graph $K_{2n}-I$ (stated here as Lemmas 1.1.9 and 1.1.11) appear in Lucas' Récréations Mathématiques, where they are attributed to Walecki. More recently, the question of finding a decomposition of the complete graph into 2-factors whose components are all 3-cycles appeared in [R1] in 1963, as a reformulation of Kirkman's Schoolgirl problem. (Ray-Chaudhuri and Wilson's solution to the Schoolgirl problem was later used by Bermond in finding a decomposition of the complete directed graph into 3-dicycles.) Since then the problem has been studied in general, and has been solved for various cases. In

Chapter 2 we give an outline of the known solutions, primarily for decompositions into even-length cycles.

In Chapter 3 we construct a decomposition of the graph $K_{2n}-I$ into cycles of length $2m$, where n is even and $3m/2 \leq n < 2m$. The construction uses the following method of composition, due to Häggkvist: If a graph G may be decomposed into paths and cycles of length m, then $G\text{wr}\bar{K}_2$ may be decomposed into cycles of length $2m$. Now if G is the complete graph K_n, then $G\text{wr}\bar{K}_2 \cong K_{2n}-I$ (see Lemma 1.1.11). Häggkvist's result therefore allows us to use methods of decomposing the graph K_n, in solving the problem for the graph $K_{2n}-I$.

Chapter 2

In this chapter we give a brief survey of known results in the area of decompositions of the complete graph (or the complete directed graph) into cycles (or dicycles) of fixed even length. We consider three different classes of decompositions: the decomposition of the complete directed graph into even-length dicycles (§2.1); the decomposition of the complete undirected graph into even-length cycles (§2.2); and the decomposition of the graph $K_{2n}-I$ into even-length cycles (§2.3).

§ 2.1 Directed cycles

We first look at the directed case - that is, decompositions of the complete directed graph K_n^* into dicycles of fixed, even length m. The necessary conditions (NC1) for such a decomposition are:

(1) $n(n-1) \equiv 0 \pmod{m}$ (the number of arcs of K_n^* is a multiple of m);

and (2) $n \geq m$.

It has been conjectured (see for example [B/F]) that these necessary conditions are also sufficient, for m even or odd, except for the cases $(n,m) = (6,3)$, (4,4) and (6,6) when the decomposition is known not to exist. The conjecture has been shown to be true if either $n \equiv 0 \pmod{m}$ or $(n-1) \equiv 0 \pmod{m}$, but apart from this relatively few cases have been solved.

We begin with some lemmas which are useful in recursive constructions. The first is from [S] and will be used repeatedly in later sections.

2.1.1 Lemma ([S])

(i) The complete directed bipartite graph $K^*_{n,s}$ can be decomposed into $2m$-dicycles if and only if $n,s \geq m$ and $ns \equiv 0 \pmod{m}$.

(ii) The complete (undirected) bipartite graph $K_{n,s}$ can be decomposed into $2m$-cycles if and only if $n,s \geq m$, $ns \equiv 0 \pmod{2m}$ and n and s are even.

We refer the reader to the proof given in [S], which is constructive and too long to include here.

The next two lemmas are straightforward.

2.1.2 Lemma If K^*_n, K^*_s, and $K^*_{n,s}$ can be decomposed into m-dicycles, then K^*_{n+s} can be decomposed into m-dicycles.

Proof We write $K^*_{n+s} = K^*_n \cup K^*_{n,s} \cup K^*_s$. The result is obvious. ■

2.1.3 Lemma ([B/F]) If K^*_{n+1}, K^*_{s+1} and $K^*_{n,s}$ can be decomposed into m-dicycles, then K^*_{n+s+1} can be decomposed into m-dicycles.

Proof Let $V(K^*_{n+s+1}) = X \cup Y \cup \{x_0\}$, where $|X| = n$ and $|Y| = s$. Then we may partition the arcs of K^*_{n+s+1} into those of K^*_{n+1} on vertex-set $X \cup \{x_0\}$, those of K^*_{s+1} on vertex-set $Y \cup \{x_0\}$, and those of $K^*_{n,s}$ on vertex-set $X \cup Y$. Thus a decomposition of K^*_{n+1}, K^*_{s+1} and $K^*_{n,s}$ into m-dicycles gives us a decomposition of K^*_{n+s+1} into m-dicycles. ■

The following three lemmas give solutions for the cases $n = m$ and $n = m+1$. These will be used in the solutions for the cases with $n \equiv 0,1 \pmod{m}$.

2.1.4 Lemma ([B/F]) For every integer m, even or odd, K^*_{m+1} can be decomposed into m-dicycles.

2.1.5 Lemma ([Tl]) For $2m \geq 8$, K^*_{2m} can be decomposed into $2m$-dicycles.

2.1.6 Lemma ([B/F]) There is no decomposition of K^*_{2m} into $2m$-dicycles for $2m = 4,6$.

We now give solutions to the problem for the cases when $n \equiv 0,1 \pmod{m}$. They are due to Bermond and Faber and appear in [B/F]. We first consider $n \equiv 0 \pmod{m}$.

2.1.7 Theorem ([B/F]) If m is even and K^*_m can be decomposed into m-dicycles, then K^*_{qm} can be decomposed into m-dicycles for all $q \geq 1$.

Proof The proof is by induction on q. When $q = 1$, the theorem is true by hypothesis. Let $q \geq 1$ and assume that K^*_{qm} can be decomposed into m-dicycles. Now $K^*_{(q+1)m} = K^*_{qm} \cup K^*_{qm,m} \cup K^*_m$. By Lemma 2.1.1 $K^*_{qm,m}$ is decomposable into m-dicycles, and by hypothesis K^*_m is also; therefore $K^*_{(q+1)m}$ is decomposable into m-dicycles. The theorem follows by induction. ■

Combining Theorem 2.1.7 with Lemma 2.1.5 we have the following corollary.

2.1.8 Corollary For all even $m \geq 8$ and for all $q \geq 1$, K^*_{qm} can be decomposed into m-dicycles.

Secondly, we consider $n \equiv 1 \pmod{m}$.

2.1.9 Theorem ([B/F]) If m is even, then K^*_{qm+1} can be decomposed into m-dicycles for all $q \geq 1$.

Proof Again the proof is by induction on q. If $q = 1$, the theorem follows from Lemma 2.1.4. Let $q \geq 1$ and assume K^*_{qm+1} can be decomposed into m-dicycles. Then by the induction base and Lemma 2.1.3, $K^*_{qm+m+1} = K^*_{(q+1)m+1}$ can be decomposed into m-dicycles. By induction the theorem holds for all $q \geq 1$. ■

Thus if $n \equiv 1 \pmod m$, or $n \equiv 0 \pmod m$ and $m > 6$, where m is even, there is a decomposition of K^*_n into m-dicycles if and only if n and m satisfy the necessary conditions (NC1). Complete solutions for $m \leq 6$ are given below. For the cases when $n \not\equiv 0,1 \pmod m$, the problem is not yet completely solved. The known results are the following.

2.1.10 Theorem (Hartnell and Milgram, [H/M]) If $n = p^e$ for some prime p and integer $e \geq 1$, then K^*_n can be decomposed into m-dicycles if and only if n and m satisfy the necessary conditions (NC1) (m may be even or odd).

2.1.11 Theorem (Hartnell, [Hl]) If x and y are odd integers and $y \geq 5$, then K^*_{xy} can be decomposed into $2x$-dicycles.

Finally, the problem has been completely solved for several small values of m, that is, for even m satisfying $4 \leq m \leq 16$. These results can all be found in [B/H/S].

2.1.12 Theorem ([B/H/S])

(i) K^*_n can be decomposed into 4-dicycles if and only if $n \equiv 0,1 \pmod 4$ and $n > 4$.

(ii) K_n^* can be decomposed into 6-dicycles if and only if $n \equiv 0,1 \pmod 3$ and $n > 6$.

(iii) K_n^* can be decomposed into 8-dicycles if and only if $n \equiv 0,1 \pmod 8$.

(iv) K_n^* can be decomposed into 10-dicycles if and only if $n \equiv 0,1 \pmod 5$.

(v) K_n^* can be decomposed into 12-dicycles if and only if $n \equiv 0,1,4$ or $9 \pmod{12}$.

(vi) K_n^* can be decomposed into 14-dicycles if and only if $n \equiv 0,1 \pmod 7$.

(vii) K_n^* can be decomposed into 16-dicycles if and only if $n \equiv 0,1 \pmod{16}$.

In [B/H/S] the authors show, using composition methods, that the problem of finding decompositions of K_n^* into m-dicycles (for even m) can be reduced to checking a finite number of cases for each m. More specifically, they derive from the necessary conditions (NC1) the following necessary conditions, which we will call (NC1*):

(1) $n \geq m$,

and (2) there exist positive integers y and z such that $yz = m$ and $n \equiv ly \pmod{yz}$, where $0 \leq l < z$ and $ly \equiv 1 \pmod z$ if $l > 0$.

They then show that, for a given m, there is an integer $n_0(m)$ such that if the conditions (NC1*) are sufficient for the existence of a decomposition of K_n^* into m-dicycles when $n_0(m) \leq n < n_0(m) + m$, then these conditions are sufficient for the existence of such a decomposition for all n.

§2.2 Undirected cycles

We now consider decompositions of the complete undirected graph K_n into m-cycles, for even m. Notice that if C is an m-cycle in K_n, then every vertex of K_n has degree either 2 or 0 in C. Thus if there is a decomposition of K_n into m-cycles,

every vertex of K_n must have even degree. Therefore, n must be odd. The necessary conditions (NC2) are then:

(1) $n(n\text{-}1) \equiv 0 \pmod{2m}$ (the number of edges of K_n is a multiple of m);

(2) n is odd;

and (3) $n \geq m$.

The first two results in this area are from 1965 and 1966, respectively.

2.2.1 Theorem (Kotzig, [K]) If n is odd, $(n-1)/2 \equiv 0 \pmod{2m}$, and $m \equiv 0 \pmod 4$, then K_n can be decomposed into m-cycles.

2.2.2 Theorem (Rosa, [R]) If n is odd, $(n-1)/2 \equiv 0 \pmod{2m}$, and $m \equiv 2 \pmod 4$, then K_n can be decomposed into m-cycles.

Thus the problem has been solved for the case $n \equiv 1 \pmod m$. Notice that, since m is even and n must be odd, $n \equiv 0 \pmod m$ is impossible. For the cases when $n \not\equiv 1 \pmod m$, there are few known results. The following two theorems give solutions for the cases when $4 \leq m \leq 16$, and when $m = 2p^e$ for some prime p and integer $e \geq 1$. In the first case the results appear in various papers but can all be found in [B/H/S].

2.2.3 Theorem ([B/H/S])

(i) K_n can be decomposed into 4-cycles if and only if $n \equiv 1 \pmod 8$.

(ii) K_n can be decomposed into 6-cycles if and only if $n \equiv 1,9 \pmod{12}$.

(iii) K_n can be decomposed into 8-cycles if and only if $n \equiv 1 \pmod{16}$.

(iv) K_n can be decomposed into 10-cycles if and only if $n \equiv 1,5 \pmod{20}$.

(v) K_n can be decomposed into 12-cycles if and only if $n \equiv 1,9 \pmod{24}$.

(vi) K_n can be decomposed into 14-cycles if and only if $n \equiv 1,21 \pmod{28}$.

(vii) K_n can be decomposed into 16-cycles if and only if $n \equiv 1 \pmod{32}$.

2.2.4 Theorem (Alspach/Varma, [A/V]) If $m = 2p^e$ for some prime p and integer $e \geq 1$, then K_n can be decomposed into m-cycles if and only if m and n satisfy the necessary conditions (NC2).

§2.3 The graph $K_{2n}-I$

In this section we look at decompositions of the graph $K_{2n}-I$, the complete graph on $2n$ vertices with a one-factor removed, into cycles of fixed, even length $2m$. The necessary conditions (NC3) for such a decomposition are the following:

(i) $n(n-1) \equiv 0 \pmod{m}$ (the number of edges of $K_{2n}-I$ is a multiple of $2m$);

and (ii) $n \geq m$.

Some results for this problem are given in [H/K/R], although this paper is primarily concerned with finding those 2-factors which decompose $K_{2n}-I$. Their results are the following.

2.3.1 Proposition ([H/K/R]) For all $m \geq 2$, $K_{6m}-I$ can be decomposed into $2m$-cycles.

2.3.2 Proposition ([H/K/R]) For all $m \geq 2$, $K_{4m}-I$ can be decomposed into $2m$-cycles.

2.3.3 Proposition ([H/K/R]) If n is even, the graph $K_{2n}-I$ can be decomposed into 4-cycles.

The remaining results in this section rely on the following lemma which provides us with a particular method for recursively constructing cycle decompositions. The construction in Chapter 3 will also use this method of composition.

2.3.4 Lemma (Häggkvist, [H]) Let G be a path or a cycle with m edges, and let H be a 2-regular graph on $2m$ vertices with all components even. Then $G\text{wr}\bar{K}_2$ is the edge-disjoint union of G' and G'', where $G' \cong G'' \cong H$.

Proof Let H consist of k disjoint cycles with lengths $2m_1, 2m_2, ..., 2m_k$, where $\sum_1^k m_i = m$. Let $s_j = \sum_1^j m_i$ and $s_0 = 0$.

Let G be the path or cycle $(u_0, u_1, ..., u_m)$, where $u_0 = u_m$ if G is a cycle. Let G_i be the segment $(u_{s_{i-1}}, u_{s_{i-1}+1}, ..., u_{s_i})$ of G, of length m_i, $i = 1, 2, ..., k$.

Let $G\text{wr}\bar{K}_2$ be obtained from G by replacing each vertex u_i, $0 \le i \le m$, by the independent vertices $x(u_i)$ and $y(u_i)$.

Let G_i' have vertex-set $V(G\text{wr}\bar{K}_2)\backslash\{y(u_{s_{i-1}}), x(u_{s_i})\}$, and edges $(x(u_{s_{i-1}}), x(u_{s_{i-1}+1}))$, $(x(u_{s_{i-1}}), (y(u_{s_{i-1}+1}))$, together with any pair of independent edges between $\{x(u_j), y(u_j)\}$ and $\{x(u_{j+1}), y(u_{j+1})\}$ for $j = s_{i-1}, s_{i-1}+1, ..., s_i-2$, and finally the edges $(x(u_{s_i-1}), y(u_{s_i}))$, $(y(u_{s_i-1}), y(u_{s_i}))$ [see Fig. 2.1]

Clearly G_i' is a cycle of length $2m_i$, whose edge-induced complement in $G_i\text{wr}\bar{K}_2$ is another $2m_i$-cycle, G_i'' [see Fig. 2.1].

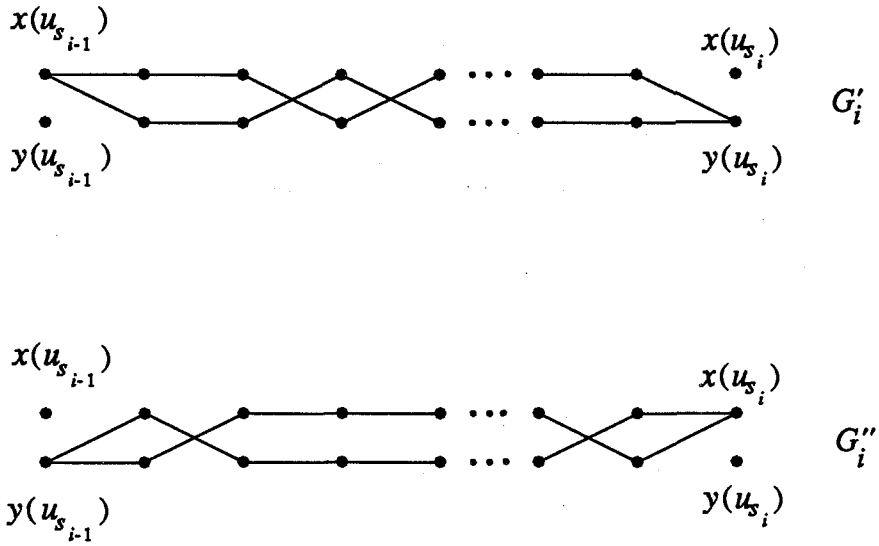


Fig. 2.1

Also, the graphs G_i', $i = 1, 2, ..., k$, are pairwise disjoint, and each of $G' = \bigcup_1^k G_i'$ and $G'' = \bigcup_1^k G_i''$ is isomorphic to H. Moreover $G\text{wr}\bar{K}_2 = G' \oplus G''$, and the lemma follows. ■

2.3.5 Corollary Let G be a path or a cycle with m edges. Then $G\text{wr}\bar{K}_2$ is the edge-disjoint union of two $2m$-cycles.

Proof Let $k = 1$ in Lemma 2.3.4. ■

One consequence of this is the following. Suppose that a graph G can be decomposed into m-paths and m-cycles, say $P_1, ..., P_k, C_1, ..., C_l$. Then $G\text{wr}\bar{K}_2$ is the edge-disjoint union of the graphs $P_1\text{wr}\bar{K}_2, ..., P_k\text{wr}\bar{K}_2, C_1\text{wr}\bar{K}_2, ..., C_l\text{wr}\bar{K}_2$. Applying Lemma 2.3.4 to each P_i and C_j gives us a decomposition of $G\text{wr}\bar{K}_2$ into

$2m$-cycles. The following lemma allows us to use this method to decompose $K_{2n}-I$ into $2m$-cycles when $n(n-1) \equiv 0 \pmod{2m}$.

2.3.6 Lemma (Tarsi, [T]) Necessary and sufficient conditions for a decomposition of K_n into m-paths are:

(i) $n(n-1) \equiv 0 \pmod{2m}$;

and (ii) $n > m$.

2.3.7 Corollary If m,n satisfy (NC3) and in addition $n(n-1) \equiv 0 \pmod{2m}$, then $K_{2n}-I$ can be decomposed into $2m$-cycles.

Remark This result is also proven in [H], for odd n and $m \neq n-1$, using the same method.

Now if G has a Hamilton decomposition, then by applying Lemma 2.3.4 to each cycle of the decomposition we get the following result.

2.3.8 Proposition ([H]) If G has a Hamilton decomposition, then $G\text{wr}\bar{K}_2$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times.

In addition we have the following proposition, due to Laskar ([L]).

2.3.9 Proposition ([L]) If G has a Hamilton decomposition, then the graph $G\text{wr}\bar{K}_m$ also has a Hamilton decomposition.

2.3.10 Corollary ([A/H]) If G has a Hamilton decomposition, then $G\text{wr}\bar{K}_{2m}$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times.

We now show how this method of composition may be used to find a decomposition of $K_{2n}-I$ into $2m$-cycles for the cases when $n \equiv 0 \pmod{m}$. The following result was proven in [A/H] in 1985.

2.3.11 Theorem ([A/H]) For any $m \geq 2$ and any natural number n, the graph $K_{2nm}-I$ can be decomposed into $2m$-cycles.

Proof The proof is divided into two cases:

Case 1. Let n be odd. Notice that $K_{2nm}-I \cong K_{nm}\text{wr}\bar{K}_2$, and that $K_{nm}\text{wr}\bar{K}_2 = H_1 \oplus H_2$, where $H_1 \cong nK_m\text{wr}\bar{K}_2$ and $H_2 \cong K_n\text{wr}\bar{K}_{2m}$. Since n is odd, K_n has a Hamilton decomposition. Therefore by Corollary 2.3.10, $K_n\text{wr}\bar{K}_{2m}$ can be decomposed into any collection of bipartite 2-factors in which each 2-factor appears an even number of times. In particular, $K_n\text{wr}\bar{K}_{2m}$ can be decomposed into $m(n-1)$ copies of nC_{2m}, since $n-1$ is even. Therefore H_2 can be decomposed into $2m$-cycles. Moreover, $K_m\text{wr}\bar{K}_2 = K_{2m}-I$ can be decomposed into $2m$-cycles by Lemma 1.1.12, so H_1 can also be decomposed into $2m$-cycles. Therefore, $K_{2nm}-I$ can be decomposed into $2m$-cycles.

Case 2. Let n be even. In this case $K_{nm}\text{wr}\bar{K}_2 = H_1 \oplus H_2$, where $H_1 \cong (n/2)K_{2m}\text{wr}\bar{K}_2$ and $H_2 \cong K_{n/2}\text{wr}\bar{K}_{4m}$. If $n/2$ is odd, then $K_{n/2}\text{wr}\bar{K}_{4m}$ can be decomposed into $m(n-2)$ copies of nC_{2m} as above ($n-2$ is even in this case). If $n/2$ is even, then $K_{n/2}$ has a one-factorisation. Let $K_{n/2} = F_1 \oplus \ldots \oplus F_{(n-2)/2}$,

where each F_i is a one-factor. For each i, $F_i \text{wr} \bar{K}_{4m} \cong (n/4)K_{4m,4m}$. Now $K_{2m,2m} = C_4 \text{wr} \bar{K}_m$ has a Hamilton decomposition, since C_4 has. Therefore $K_{4m,4m} \cong K_{2m,2m} \text{wr} \bar{K}_2$ can be decomposed into $2m$ 2-factors each isomorphic to $4C_{2m}$, and so $K_{n/2} \text{wr} \bar{K}_{4m}$ can be also. Thus H_2 can be decomposed into $2m$-cycles. Finally, applying Proposition 2.3.2 to each component of $(n/2)K_{2m} \text{wr} \bar{K}_2 \cong (n/2)(K_{4m} - I)$ yields a decomposition of H_1 into $2m$-cycles. Therefore, $K_{2nm} - I$ can be decomposed into $2m$-cycles. ■

Chapter 3

As stated in the introduction, the new results we have found concern the decomposition of the graph $K_{2n}-I$ into edge-disjoint $2m$-cycles. We discussed in the previous chapter how this problem has been solved in some cases; in summary, there is known to be a decomposition of $K_{2n}-I$ into edge-disjoint 2m-cycles if

(1) $n \equiv 0 \pmod{m}$

(2) $n(n-1) \equiv 0 \pmod{2m}$

or (3) $m = 2$.

In this chapter we give partial results for the case when m does not divide n, and the quotient $n(n-1)/m$ is odd. In Section 3.1 we give an outline of the methods we use in the constructions; Section 3.2 contains the constructions themselves.

§ 3.1 Outline of the construction

Our constructions are based on the method of Lemma 2.3.4. In the case where $n(n-1)/m$ is even, we are able to decompose K_n into edge-disjoint m-paths, and using Lemma 2.3.4, this decomposition of K_n yields a decomposition of $K_{2n}-I \cong K_n \mathrm{wr} \bar{K}_2$ into $2m$-cycles. When $n(n-1)/m$ is odd, however, there is no decomposition of K_n into m-paths, since $|E(K_n)| \not\equiv 0 \pmod{m}$. Thus we need to modify the construction somewhat.

Suppose that $n(n-1)/m$ is odd. Then in fact $|E(K_n)| \equiv m/2 \pmod{m}$. Our method will be to find a subgraph D of K_n with the following properties:

(1) $|E(D)| \equiv m/2 \pmod{m}$;

(2) $K_n - D$ can be decomposed into m-paths and m-cycles;

and (3) $D \text{wr} \bar{K}_2$ can be decomposed into $2m$-cycles.

Notice that since $|E(K_n)| \equiv m/2 \pmod{m}$, property (1) implies that $|E(K_n - D)| \equiv 0 \pmod{m}$.

We will apply Lemma 2.3.4 to the decomposition of $K_n - D$ into m-paths and m-cycles, which will give us a decomposition of $(K_n - D)\text{wr}\bar{K}_2$ into $2m$-cycles. Since $D\text{wr}\bar{K}_2$ may be decomposed into $2m$-cycles, we will then have a decomposition of $[(K_n - D)\text{wr}\bar{K}_2] \oplus [D\text{wr}\bar{K}_2] = K_n\text{wr}\bar{K}_2 \cong K_{2n} - I$ into $2m$-cycles.

We begin with a lemma which reduces the problem to 'small' values of n, that is, to values of n satisfying $m < n < 2m$.

3.1.1 Lemma To find decompositions of $K_{2n} - I$ into $2m$-cycles for all m,n satisfying (NC3), it is sufficient to find decompositions of $K_{2n} - I$ into $2m$-cycles for all m,n satisfying both (NC3) and $m < n < 2m$.

Proof Suppose we have decompositions of $K_{2n} - I$ into $2m$-cycles for all m,n satisfying both (NC3) and $m < n < 2m$.

Let n and m satisfy (NC3) but not $m < n < 2m$. By (NC3), $n \geq m$, so either $m = n$ or $n \geq 2m$. By Lemma 1.1.11, $K_{2n} - I$ can be decomposed into Hamilton cycles, so we assume $n \geq 2m$. Thus we may write $n = km + (m+r)$, where $k \geq 1$

and $0 \le r < m$. Therefore $2n = k(2m) + (2m + 2r)$, and $K_{2n} - I$ is the edge-disjoint union of the following:

k copies of $K_{2m} - I$;

one copy of $K_{2m+2r} - I$;

$\binom{k}{2}$ copies of $K_{2m,2m}$;

and k copies of $K_{2m,\,2m+2r}$.

Again by Lemma 1.1.11, $K_{2m} - I$ can be decomposed into Hamilton cycles. By Lemma 2.1.1, each of $K_{2m,2m}$ and $K_{2m,2m+2r}$ can be decomposed into $2m$-cycles. If $r = 0$ this gives us the desired decomposition of $K_{2n} - I$ into $2m$-cycles. Suppose $r > 0$. By (NC3), $|E(K_{2n} - I)| \equiv 0 \pmod{2m}$, and so $|E(K_{2m+2r} - I)| \equiv 0 \pmod{2m}$. Since in addition $m < m + r < 2m$, by hypothesis we have a decomposition of $K_{2m+2r} - I$ into $2m$-cycles. Thus $K_{2n} - I$ may be decomposed into $2m$-cycles, and the lemma follows. ■

In the light of Lemma 3.1.1, for the remainder of the chapter we restrict ourselves to cases where $m < n < 2m$. Our aim is to find a subgraph D of K_n with the three properties described above. We first define a subgraph D with properties (1) and (3); and for $n \ge 3m/2$, where n is even, we construct a decomposition of $K_n - D$ into m-paths and m-cycles (this is property (2) required of D). Thus we have solved the problem for m and n satisfying (NC3) and for which $3m/2 \le n < 2m$, with n even.

In the construction we first write $K_n = K_m \cup K_{m,r} \cup K_r$, where $r = n - m$. Now as $n(n-1)/m$ is odd, then necessarily m is even (since n is even). Thus we can decompose K_m into $m/2-1$ m-cycles and a one-factor F. Notice that $|E(F)| =$

$m/2$. We will show that there is a cycle C_1 in the decomposition of K_m such that $C_1 \cup F$ has the properties (1) and (3) required for the subgraph D. From this point we will proceed to find a decomposition of $K_n - (C_1 \cup F)$ into m-paths and m-cycles.

Now since n and m are even, r is also even. If in addition $r \geq m/2$ (or equivalently $n \geq 3m/2$), then by Lemma 2.1.1 $K_{m,r}$ can be decomposed into m-cycles. However $n < 2m$ implies $r < m$, so there is no decomposition of K_r into m-paths and m-cycles. Instead, we will decompose K_r into r paths of length at most $r-1$. Using the edges of two m-cycles from the decomposition of $K_{m,r}$ as a 'bridge' between K_r and K_m, we will extend each of these paths to an m-path with segments of one or more of the m-cycles from the decomposition of K_m. This will yield r m-paths which together cover all the edges of K_r, the edges of two m-cycles from $K_{m,r}$, and the edges of some (or all) of the m-cycles comprising $K_m - (C_1 \cup F)$. In addition the construction will be such that the remainder (if any) of K_m will be a collection of m-cycles, as will be the remainder of $K_{m,r}$. Thus we will have a decomposition of $K_n - (C_1 \cup F)$ into m-paths and m-cycles.

We now give the constructions.

§ 3.2 The construction

Throughout this section we assume that:

(*) n and m satisfy (NC3), n is even, $m < n < 2m$ and the quotient $n(n-1)/m$ is odd.

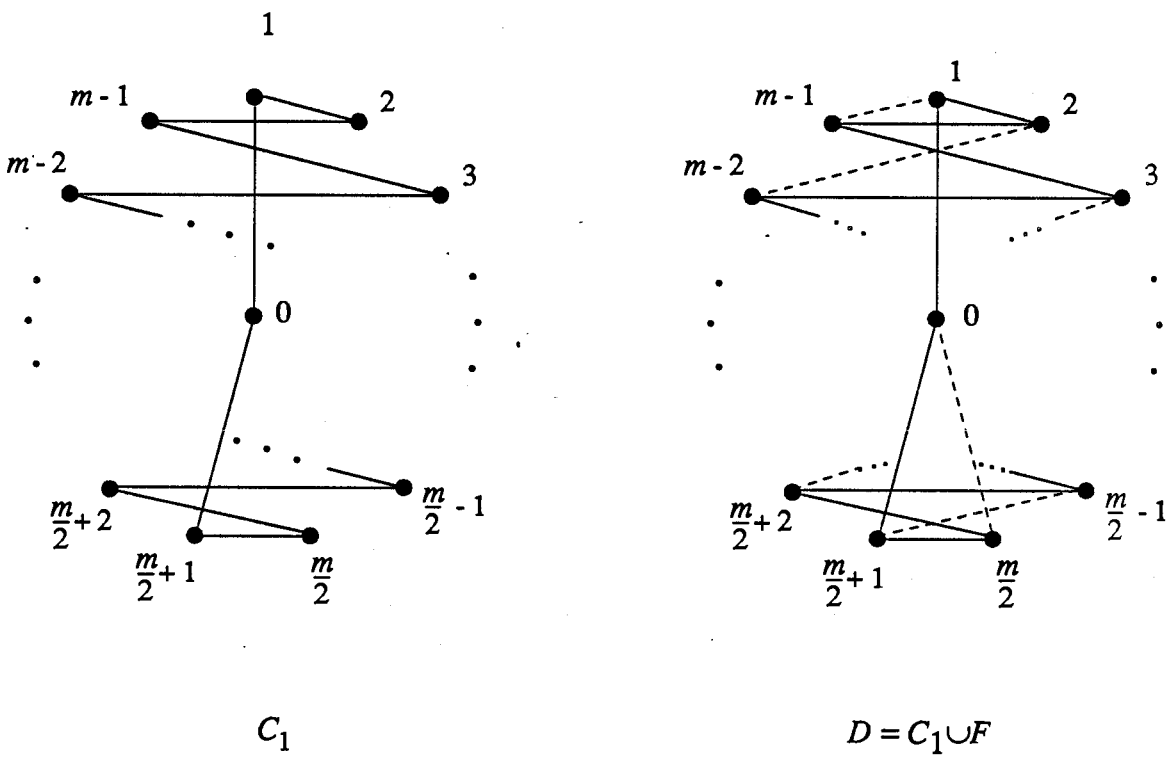
We begin with the construction of the subgraph D. We write $K_n = K_m \cup K_{m,r} \cup K_r$, where $V(K_m) = \{0, 1, ..., m-1\}$ and $V(K_r) = \{z_1, z_2, ..., z_r\}$. We decompose K_m

(remember that m is even) into $m/2-1$ m-cycles $C_1, \ldots, C_{m/2-1}$ and a one-factor F, defined as follows. Let $\sigma = (0)(1\ 2\ \ldots\ m-1)$, and $C_1 = (0, 1, 2, m-1, 3, m-2, \ldots, m/2-1, m/2+1, 0)$.

For $i = 2, \ldots, m/2-1$, let $C_i = \sigma^{i-1}(C_1)$. Finally, let

$F = \{(m-1,1), (m-2,2), \ldots, (m/2+1,m/2-1), (0,m/2)\}$.

Thus C_1 is the m-cycle of Fig. 3.1, and for $i \geq 2$, C_i is obtained from C_1 by a clockwise rotation of i places, with the vertex 0 fixed.



C_1 $\qquad\qquad$ $D = C_1 \cup F$

Fig. 3.1

Letting $D = C_1 \cup F$ (see Fig. 3.1) we have

3.2.1 Lemma The graph $D \text{wr} \bar{K}_2$ is the edge-disjoint union of three $2m$-cycles.

Proof We show that D is the edge-disjoint union of three perfect matchings with the property that the union of any two perfect matchings is a Hamilton cycle. We then use the three resulting Hamilton cycles to construct three edge-disjoint $2m$-cycles in $D\text{wr}\overline{K}_2$.

It is easy to see that the perfect matchings M_1, M_2 and M_3, where M_1 and M_2 consist of alternate edges of the m-cycle C_1 and $M_3 = F$, partition the edges of D. By construction, the union of M_1 and M_2 is the m-cycle C_1.

Let $M_1 = \{(0,1), (2,m-1), (3,m-2), ..., (m/2,m/2+1)\}$,

and $M_2 = \{(1,2), (m-1,3), (m-2,4), ..., (m/2+2,m/2), (m/2+1,0)\}$.

We have $M_3 = F = \{(m-1,1), (m-2,2), ..., (m/2+1,m/2-1), (m/2,0)\}$.

To show that $M_1 \cup M_3$ is an m-cycle, define a permutation π of $V(K_m)$ by $\pi(0) = 0$, $\pi(1) = 1$, and $\pi(i) = m-i+1$, $2 \le i \le m-1$. The effect of π is to interchange the endvertices of each edge of M_1, except for (0,1) which is unchanged. Thus $\pi(M_1) = M_1$.

Now $M_3 = \{(m/2,0), (m-1,1)\} \cup \{(m-i,i) : 2 \le i \le m/2-1\}$.

Thus $\pi(M_3) = \{m/2+1,0), (2,1)\} \cup \{(i+1,m-i+1) : 2 \le i \le m/2-1\}$

$= \{(m/2+1,0), (1,2)\} \cup \{(j+2,m-j) : 1 \le j \le m/2-2\}$

$= M_2$.

Similarly $\pi(M_2) = M_3$. So $\pi : V(D) \to V(D)$ is an automorphism, and $\pi(M_1 \cup M_3) = M_1 \cup M_2$. Thus, $M_1 \cup M_3$ is also an m-cycle.

We now show that $M_2 \cup M_3$ is an m-cycle. We have

$$\begin{aligned} M_2 \cup M_3 &= \{(1,2), (m/2+1,0)\} \cup \{(j+2, m-j) : 1 \le j \le m/2-2\} \cup \\ &\quad \{(m-1,1), (m/2,0)\} \cup \{(m-i,i) : 2 \le i \le m/2-1\} \\ &= \{(1,2), (1,m-1)\} \cup \{(m-j,2+j), (1+j, m-j-1) : 1 \le j \le m/2-2\} \cup \\ &\quad \{(m/2,0), (m/2+1),0)\}. \end{aligned}$$

Now $M_2 \cup M_3$ clearly contains the 2-path $(m-1,1,2)$. To this we add the pair of edges $(m-1,3)$ and $(2,m-2)$, which gives us a 4-path with endvertices $m-2$ and 3. We continue adding edges in pairs of the form $(m-j,2+j)$, $(1+j,m-j-1)$, for $j = 1, 2, ..., m/2-2$, until a cycle is formed; after each addition we obtain a path or cycle of length two more than the previous path, and with endvertices $m-j-1$ and $j+2$ (where $2 \le j \le m/2-2$). A cycle will be formed only if $m-i-1 = i+2$ for some i; that is, if $m-3 = 2i$. But m is even, so $m-3$ is odd, and this is impossible. Therefore after the addition of the last pair of edges $(m/2+2,m/2)$ and $(m/2-1,m/2+1)$ we have a path in $M_2 \cup M_3$ of length $m-2$ with endvertices $m/2+1$ and $m/2$. The addition of the two remaining edges $(m/2+1,0)$ and $(m/2,0)$ completes this path to an m-cycle; thus $M_2 \cup M_3$ is an m-cycle.

Let $M_1 \cup M_2 = C_{12}$ ($C_{12} = C_1$),

$M_2 \cup M_3 = C_{23}$

and $M_3 \cup M_1 = C_{31}$.

We now use these three m-cycles to partition the edges of $D\text{wr}\bar{K}_2$ into three $2m$-cycles. We first observe that every edge of D lies on exactly one of the perfect matchings M_i, and so on exactly two of the m-cycles C_{ij}.

Let $V(D\text{wr}\bar{K}_2) = \{x(i),\ y(i) :\ i \in V(D)\}$, so that

$E(D\text{wr}\bar{K}_2) = \{(x(i),y(j)),\ (x(i),x(j)),\ (y(i),y(j)),\ (y(i),x(j)) :\ ij \in E(D)\}$.

For each $ij \in E(D)$, let

$p(ij) = \{(x(i),x(j)),\ (y(i),y(j))\}$ (the corresponding 'parallel' edges of $D\text{wr}\bar{K}_2$),

and $c(ij) = \{(x(i),y(j)),\ (y(i),x(j))\}$ (the corresponding 'crossing' edges of $D\text{wr}\bar{K}_2$).

We define C'_{12} from C_{12} as follows. We let C'_{12} consist of the edges $(x(0),x(1))$, $(x(0),y(1))$, together with either pair of independent edges $p(ij)$ or $c(ij)$ for each subsequent edge ij, $j \neq 0$, of C_{12}, and finally the edges $(x(m/2+1),y(0))$, $(y(m/2+1),y(0))$ (see Fig. 3.2).

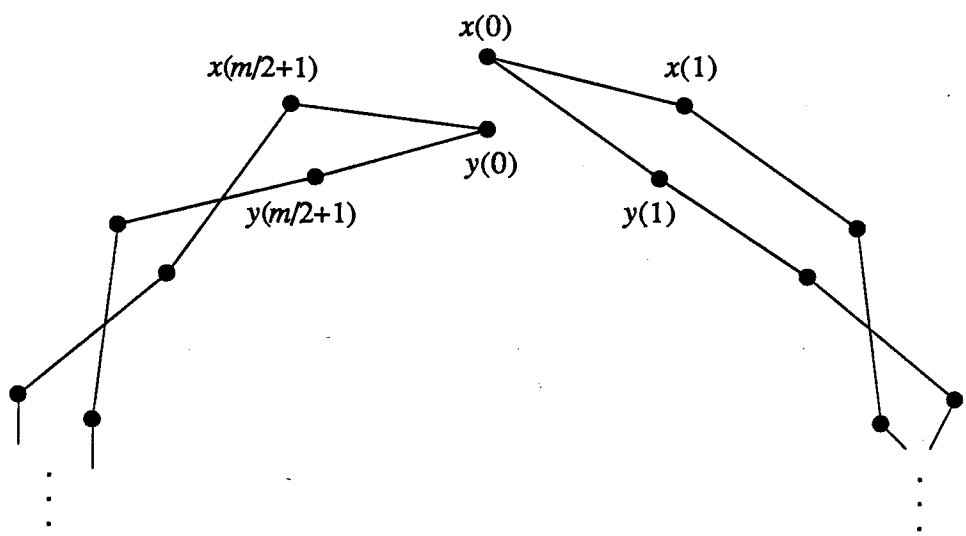


Fig. 3.2

We let C'_{23} consist of the edges $(x(0),x(m/2+1))$, $(x(0),y(m/2+1))$, together with either pair of independent edges $p(ij)$ or $c(ij)$ for each subsequent edge ij, $j\neq 0$, of $C_{23}\backslash C_{12}$, and whichever pair of independent edges $p(ij)$, $c(ij)$ does not lie on C'_{12} for each subsequent edge of $C_{23} \cap C_{12}$, and finally the two edges $(x(m/2),y(0))$, $(y(m/2),y(0))$.

Finally we let C'_{31} consist of the edges $(x(0),x(m/2))$, $(x(0),y(m/2))$, together with whichever pair of independent edges $p(ij)$, $c(ij)$ lies on neither C'_{12} nor C'_{23}, for each subsequent edge ij, $j\neq 0$, of C_{31}, and finally the edges $(x(1),y(0))$ and $(y(1),y(0))$.

It is clear that each of C'_{12}, C'_{23} and C'_{31} is a $2m$-cycle in $D\text{wr}\overline{K}_2$. In addition, if C'_{pq} and C'_{kl} are not edge-disjoint, then they share either a pair $p(ij)$ or $c(ij)$ for some $ij \in E(D)$, $i,j\neq 0$, or an edge incident with either $x(0)$ or $y(0)$. The first case cannot occur since in defining each C'_{pq} we choose a pair $p(ij)$ or $c(ij)$ only if it does not lie on a previously defined cycle C'_{kl}. It is easy to check in the construction that the second case cannot occur. Thus the three $2m$-cycles C'_{12}, C'_{23} and C'_{31} partition the edges of $D\text{wr}\overline{K}_2$. ■

Therefore D has property (3) described earlier, and clearly $|E(D)| = 3m/2$ so D also has property (2). We now show that D has property (1).

3.2.1 Lemma The graph $K_n - D$ can be decomposed into m-paths and m-cycles.

Proof We begin with an important observation about the cycles $C_1, ..., C_{m/2-1}$ of the decomposition of $K_n - D$. Let W be the Eulerian walk of $K_m - D$ defined by

$W = C_2 + C_3 + ... + C_{m/2-1}$ (where the cycles C_i are oriented so that $(0,i)$ is the first directed edge of C_i)

$= (0, 2, 3, ..., m/2+1, m/2+2, 0, 3, 4, ..., m/2+2, m/2+3, 0, 4, 5, ..., 0, m/2-1, m/2, ..., m-2, m-1, 0)$.

3.2.2 Claim The shortest cycle in W has length $m-2$.

Proof First, as we have already pointed out, each cycle C_i, $i \geq 3$, is simply a rotation of C_2. Also, any segment of W which lies entirely within some C_i must be either a path or an m-cycle. Thus it is sufficient to show that any cycle in W which begins in C_2 and ends in C_3 has length at least $m-2$. To do this we find the length of the cycle which begins at the occurrence of a vertex v in C_2 and ends at the occurrence of the same vertex v in C_3, for each $v \neq 0$ of K_m (clearly the cycle which begins and ends with 0 has length m).

Let v be the k^{th} vertex of C_2 (where we orient C_i as in W so that 0 is the first vertex, i the second, and so on) (see Fig. 3.3).

Now C_3 is obtained from C_2 by a clockwise rotation through one place, with 0 fixed.

Thus (i) if $4 \le v \le m/2+1$, then v is the $(k-2)^{\text{nd}}$ vertex of C_3;

(ii) if $m/2+3 \le v \le m-1$ or $v = 1$ or 2, then v is the $(k+2)^{\text{nd}}$ vertex of C_3;

(iii) if $v = 3$, then v is the third vertex of C_2 and the second of C_3;

and (iv) if $v = m/2+2$, then v is the m^{th} vertex of C_2 and the $(m-1)^{\text{st}}$ of C_3.

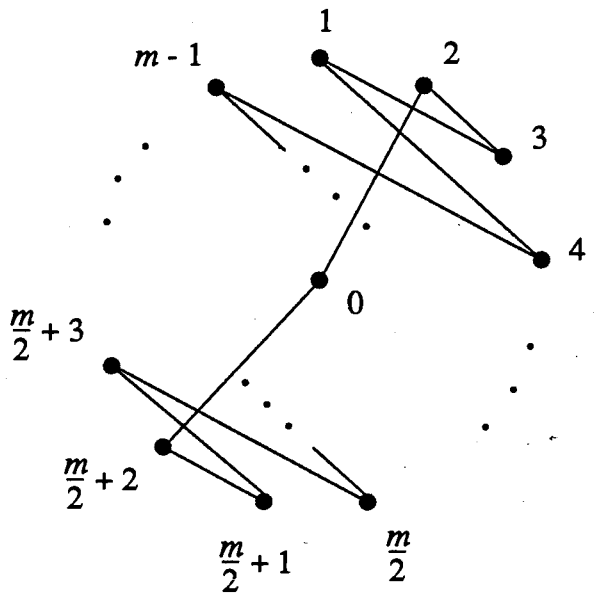


Fig. 3.3

Therefore the closed walk of $C_2 + C_3$ which begins and ends at vertex $v \neq 0$ of K_m has length $m-2$ (if $4 \le v \le m/2+1$), $m+2$ (if $m/2+3 \le v \le m-1$ or $v = 1$ or 2) or $m-1$ (if $v = 3$ or $v = m/2+2$). So the shortest cycle in W has length $m-2$. ∎

Thus W is a trail in $K_m - D$ with the property that any segment of length at most $m-3$ is a path, and such that $E(K_m - D) = E(W)$. Notice also that if S is an initial segment of W whose length is a multiple of m, then $W \setminus S$ consists of a

collection of entire m-cycles $C_j, ..., C_{m/2-1}$ for some j. These properties of W are crucial for the constructions.

The remainder of the proof of Lemma 3.2.1 is divided into three cases. Case 1 contains the basic construction which is valid for all n and m satisfying (*) with $r > m/2+1$ and $r > 8$. Cases 2 and 3 contain modifications of this construction for the cases $r = m/2+1$ and $r \leq 8$, respectively. Observe that if $r = m/2$, then $n(n-1) = (3m/2)(n-1)$, which is not divisible by m when $n-1$ is odd. Thus we may assume that $r > m/2$.

Case 1 Let $r > m/2+1$ and $r > 8$.

Recall that since n and m are even, then r is also even. In addition, since both $n(n-1)/m$ and $n-1$ are odd, then for any integer e, 2^e divides m if and only if 2^e divides n. In particular, $m \equiv n \pmod 4$, and so $r \equiv 0 \pmod 4$.

Now since r is even, we may decompose K_r into $r/2-1$ r-cycles and a one-factor. If we add one edge of the one-factor to each r-cycle, we obtain a decomposition of K_r into $r/2-1$ subgraphs G_i and a single edge zz', where each G_i is an r-cycle with a chord. These subgraphs have the following useful property.

3.2.3 Claim Given $2 \leq y_i \leq r-1$, there is a vertex x_i of G_i such that G_i may be divided into two paths P_i and P_i', of lengths y_i and $r+1-y_i$ respectively, with a common endvertex x_i. Moreover, we may choose the vertices x_i in such a way that distinct values of y_i will determine distinct vertices x_i.

Proof Each G_i is an r-cycle with a chord. Let p and q be the endvertices of the chord. Thus p and q have degree 3 in G_i while all other vertices of G_i have degree 2.

Now p and q divide the r-cycle of G_i into two segments S_1 and S_2, where we assume $l(S_1) \leq l(S_2)$ (Fig. 3.4).

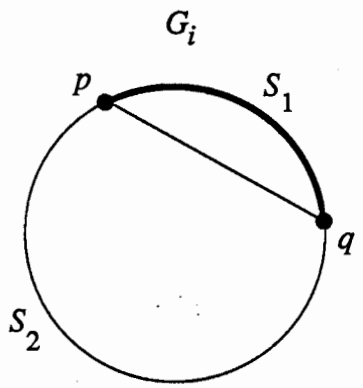


Fig. 3.4

If $y_i \leq l(S_1)$, we let P_i be the path which consists of the edge qp and the first (beginning with p) $y_i - 1$ edges of S_1. We let x_i be the terminal vertex of P_i. Since $2 \leq y_i \leq l(S_1)$, x_i lies on S_1, $x_i \neq p,q$, and for distinct values of y_i with $2 \leq y_i \leq l(S_1)$, the corresponding vertices x_i are distinct. Since $x_i \neq p$, the remainder P_i' of G_i, which consists of a segment of S_1 and all of S_2, is also a path (see Fig. 3.5)

If $y_i > l(S_1)$, we let P_i be the path of length y_i which begins at p, follows S_1 to q, and continues along S_2; and we let x_i be the terminal vertex of P_i. In this case,

since $l(S_1) < y_i \leq r-1$, x_i lies on S_2 and $x_i \neq p,q$. Thus the remainder P_i' of G_i, which consists of a segment of S_2 and the edge pq, is a path. For distinct values of y_i with $l(S_1) < y_i \leq r-1$, the corresponding vertices x_i are distinct (see Fig. 3.5).

Notice that each vertex of degree 2 in G_i will be the vertex x_i corresponding to exactly one value of y_i, $2 \leq y_i \leq r-1$, and that no value of y_i gives $x_i = p$ or q. Notice also that we cannot divide G_i into paths P_i and P_i' for which $l(P_i) < 2$ or

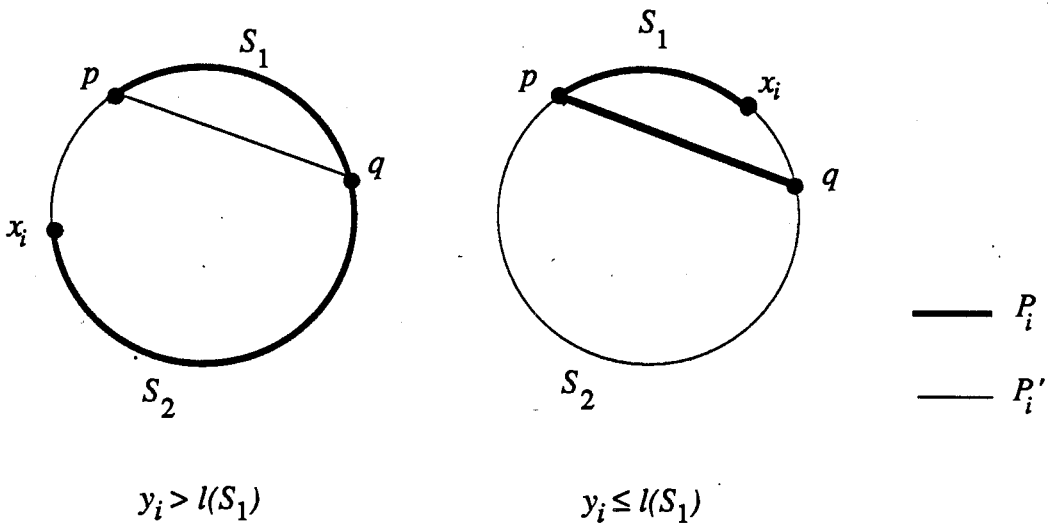


Fig. 3.5

$l(P_i') < 2$. Finally, $2 \leq y_i \leq r-1$ implies $2 \leq r+1-y_i \leq r-1$, so that for each i, $2 \leq l(P_i), l(P_i') \leq r-1$. ■

In the construction, we divide each subgraph G_i into two paths, P_i and P_i', of lengths y_i and $r+1-y_i$, respectively, as in the claim. We use the edges of one m-cycle C from $K_{m,r}$, together with an initial segment of W, to complete each of the paths P_i, P_i', $1 \leq i \leq r/2-1$, to an m-path in $K_n - D$.

To do this, we choose $r-2$ edge-disjoint paths $Q_1, Q_1', Q_2, Q_2', ..., Q_{r/2-1}, Q_{r/2-1}'$, each a segment of W, so that Q_i and Q_i' will complete P_i and P_i', respectively, to a path of length m, using one or more edges of C as a bridge between K_r and K_m. We will use all the edges of C to do this, so that the remainder of $K_{m,r}$ will consist of entire m-cycles. The union of the paths $Q_1, Q_1', Q_2, Q_2', ..., Q_{r/2-1}, Q_{r/2-1}'$ will be an initial segment of W.

We will then use a second m-cycle C' from $K_{m,r}$, together with the single edge zz' from K_r and the first $m-1$ edges of $W \setminus \cup\{Q_i \cup Q_i' : 1 \le i \le r/2-1\}$, to construct two more m-paths in $K_n - D$.

Thus we will use the edges of K_r, some or possibly all of those of $K_m - D$, and those of two m-cycles from $K_{m,r}$, to construct r m-paths in $K_n - D$. We must check that there are enough edges in $E(K_r) \cup E(K_m - D)$ to construct these paths. We need

$$|E(K_r)| + |E(K_m - D)| + 2m \ge rm.$$

Now $|E(K_r)| = r(r-1)/2$, and $|E(K_m - D)| = m(m/2 - 2) = m^2/2 - 2m$.

So we need

$$r(r-1)/2 + m^2/2 - 2m + 2m \ge rm,$$

or equivalently,

$$m^2 - 2rm + r^2 - r \ge 0. \qquad (1)$$

We are assuming that $m > r$, and that both m and r are even. So we may let $r = m - 2k$, for some positive integer k. Then (1) becomes

$$m^2 - 2(m-2k)m + (m-2k)(m-2k-1) \ge 0,$$

or $4k^2 + 2k \ge m$.

Now by (NC3), $m|(n(n-1))$. Therefore $m|(2m-2k)(2m-2k-1)$. Equivalently, $m|(4k^2+2k)$. But this implies that $4k^2+2k \geq m$. Therefore we have

$$|E(K_r)| + |E(K_m - D)| + 2m \geq rm,$$

as required. So there are enough edges in $K_r \cup (K_m - D)$ for the construction of our r m-paths.

To construct these m-paths we will need to use two particular m-cycles C and C' from $K_{m,r}$ (for example, if v is the endvertex of Q_i and x the endvertex of P_i, then in order to use edges of C to cross from K_r to K_m, and so join P_i to Q_i, we require that x and v lie on C).

By Lemma 2.1.1 we know that there is a decomposition of $K_{m,r}$ into m-cycles. For our construction we want the decomposition to contain two particular cycles C and C'. So we choose two cycles from the given decomposition and relabel their vertices with those of the required cycles C and C'. This of course induces a relabelling of the entire decomposition of $K_{m,r}$. The cycles C and C' are related to some extent, so we must choose the original cycles from the given decomposition of $K_{m,r}$ carefully.

In particular, we want to use zz', C', and the first $m-1$ edges of $W \setminus \cup\{Q_i \cup Q_i' : 1 \leq i \leq r/2-1\}$ to construct two m-paths. We do not know at this point the length of the initial segment $\cup\{Q_i \cup Q_i' : 1 \leq i \leq r/2-1\}$ of W. However the r m-paths which we are constructing in $K_n - D$ together cover rm edges of $K_n - D$. Since $E(K_n - D) = E(K_m - D) \cup E(K_{m,r}) \cup E(K_r)$, and we know that m divides each of $|E(K_n - D)|$, $|E(K_m - D)|$ and $|E(K_{m,r})|$, then m also divides $|E(K_r)|$. In addition, m divides $|E(C \cup C')|$. Therefore, since we are using $E(K_r)$, $E(C \cup C')$ and a segment of W to construct these r m-paths, the total number of

edges we use from W is also a multiple of m. Since W begins at 0 in the cycle C_2, the segment of W which we use (including the first $m-1$ edges of $W \setminus \cup\{Q_i \cup Q_i' : 1 \le i \le r/2-1\}$) consists of a collection $C_2, C_3, ..., C_j$ of m-cycles from K_m, where $j \le m/2-1$. Therefore the $m-1$ edges of $W \setminus \cup\{Q_i \cup Q_i' : 1 \le i \le r/2-1\}$ which we use will be the last $m-1$ edges of the cycle C_j. Since $C_j = (0, j, j+1, j-1, ..., j+m/2, 0)$, these $m-1$ edges will be the segment $S = (j, j+1, j-1, ..., j+m/2, 0)$ of C_j.

We use S, C' and (z, z') to construct two m-paths as follows. We label one edge of C' $(j+1, z)$. The two paths are

$R_{r/2} = [S \setminus (j, j+1)] + (j+1, z) + (z, z')$,

and $R'_{r/2} = (j, j+1) + [C' \setminus (j+1, z)]$ (Fig. 3.6).

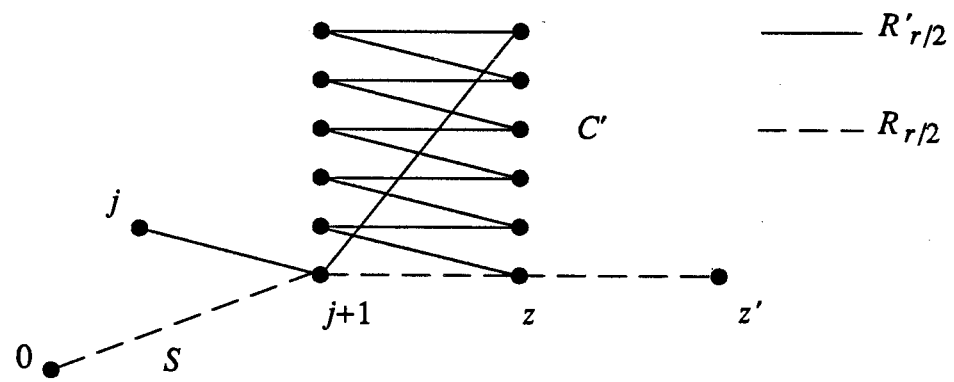


Fig. 3.6

In order for $R'_{r/2}$ to be a path, we must ensure that j does not lie on C'. This is the only restriction on the labelling of the m remaining vertices of C'. However, requiring that $(j+1, z) \in E(C')$ and $j \notin V(C')$ will put some restriction on how we may label the other m-cycle, C. For this reason we choose C and C' as follows.

3.2.4 Claim In any decomposition of $K_{m,r}$ into m-cycles we can find two vertices a and b of K_m, and a vertex w of K_r, such that the cycle containing the edge aw does not contain b, and the cycle containing the edge bw does not contain a.

Proof There are $\binom{m}{2}$ possible pairs $\{a,b\} \subseteq V(K_m)$. For each cycle C^* of $K_{m,r}$, $V(C^*)$ contains $\binom{m/2}{2}$ pairs $\{a,b\}$. There are r cycles in the decomposition, and so $r\binom{m/2}{2}$ pairs in total. But

$$\begin{aligned} r\binom{m/2}{2} &= (r/2)(m/2)(m/2-1) \\ &= (r/4)(m)(\mathrm{m}/2-1) \\ &< (r/4)(m)(m/2-1/2) \\ &= (r/4)(m(m-1))/2 \\ &= r/4\binom{m}{2}. \end{aligned}$$

Therefore at least one pair $\{a,b\} \subseteq V(K_m)$ occurs on fewer than $r/4$ cycles. Let $\{a,b\}$ be such a pair. Each cycle C^* containing both a and b contains four edges of the form aw_0 or bw_0, for $w_0 \in V(K_r)$. Thus the set $\mathbf{C}$ of all cycles containing both a and b covers at most $4(r/4 - 1) = r - 4$ edges of the form aw_0 or bw_0, for $w_0 \in V(K_r)$. So there is a vertex w_0 of K_r for which no cycle of $\mathbf{C}$ contains either aw_0 or bw_0 (in fact there are at least four such vertices). The three vertices a, b, and w_0 satisfy the claim. ■

Therefore we may choose a, b, and w_0 as in the claim, relabel a with j+1, b with j, and w_0 with z, and let C' be the cycle containing $(j+1, z)$ but not j, and C be the cycle containing (j,z) but not j+1.

We now proceed to label C.

We arrange the paths Q_i', Q_i, $1 \le i \le r/2-1$, along W so that

$$W = Q_1' + Q_2 + Q_3 + Q_2' + Q_4 + Q_3' + \ldots + Q_i + Q_{i-1}' + \ldots + Q_{r/2-2}$$
$$+ Q_{r/2-3}' + Q_{r/2-1} + Q_{r/2-2}' + Q_{r/2-1}' + Q_1 + S + R,$$

where R is the remaining segment (if any) of W which will not lie on any of the r m-paths.

Recall that W begins with the cycle C_2, so that $Q_1' = (0, 2, 3, 1, \ldots)$, and that S consists of the last $m-1$ edges of C_j, so that $S = (j, j+1, j-1, \ldots, j+m/2, 0)$. Thus R consists of the $(m/2 - 1 - j)$ m-cycles $C_{j+1}, C_{j+2}, \ldots, C_{m/2-1}$.

Let v_1 be the terminal vertex of Q_1' (and so also the initial vertex of Q_2). For $2 \le i \le r/2 - 2$, let v_i be the terminal vertex of Q_{i+1} (and the initial vertex of Q_i'). Let $v_{r/2-1}$ be the terminal vertex of $Q_{r/2-1}'$ (and the initial vertex of Q_1).

We will label $r-2$ of the vertices of C with $v_1, \ldots, v_{r/2-1}$ and $x_1, \ldots, x_{r/2-1}$ so that the paths P_i, Q_i and P_i', Q_i' match up as in Fig. 3.7. Thus we need the vertices $v_1, \ldots, v_{r/2-1}$ to be distinct, and the corresponding vertices $x_1, \ldots, x_{r/2-1}$ to be distinct. Now from Claim 3.2.3, we may choose $l(P_i), l(P_i') \in \{2, 3, \ldots, r-1\}$, where we require of course that $l(P_i) + l(P_i') = r+1$, and distinct values of $l(P_i)$ will give us distinct vertices x_i to label on C. Similarly the vertices $v_1, \ldots, v_{r/2-1}$ will be determined by our choices of $l(Q_1), l(Q_1'), \ldots, l(Q_{r/2-1}), l(Q_{r/2-1}')$. For $i \ne 2$, we will use precisely one edge of C to join P_i and P_i' to Q_i and Q_i', respectively. Thus for $i \ne 2$

$$l(Q_i) = m - 1 - l(P_i),$$

and $$l(Q_i') = m - 1 - l(P_i'),$$

so $$m - r \le l(Q_i), l(Q_i') \le m - 3.$$

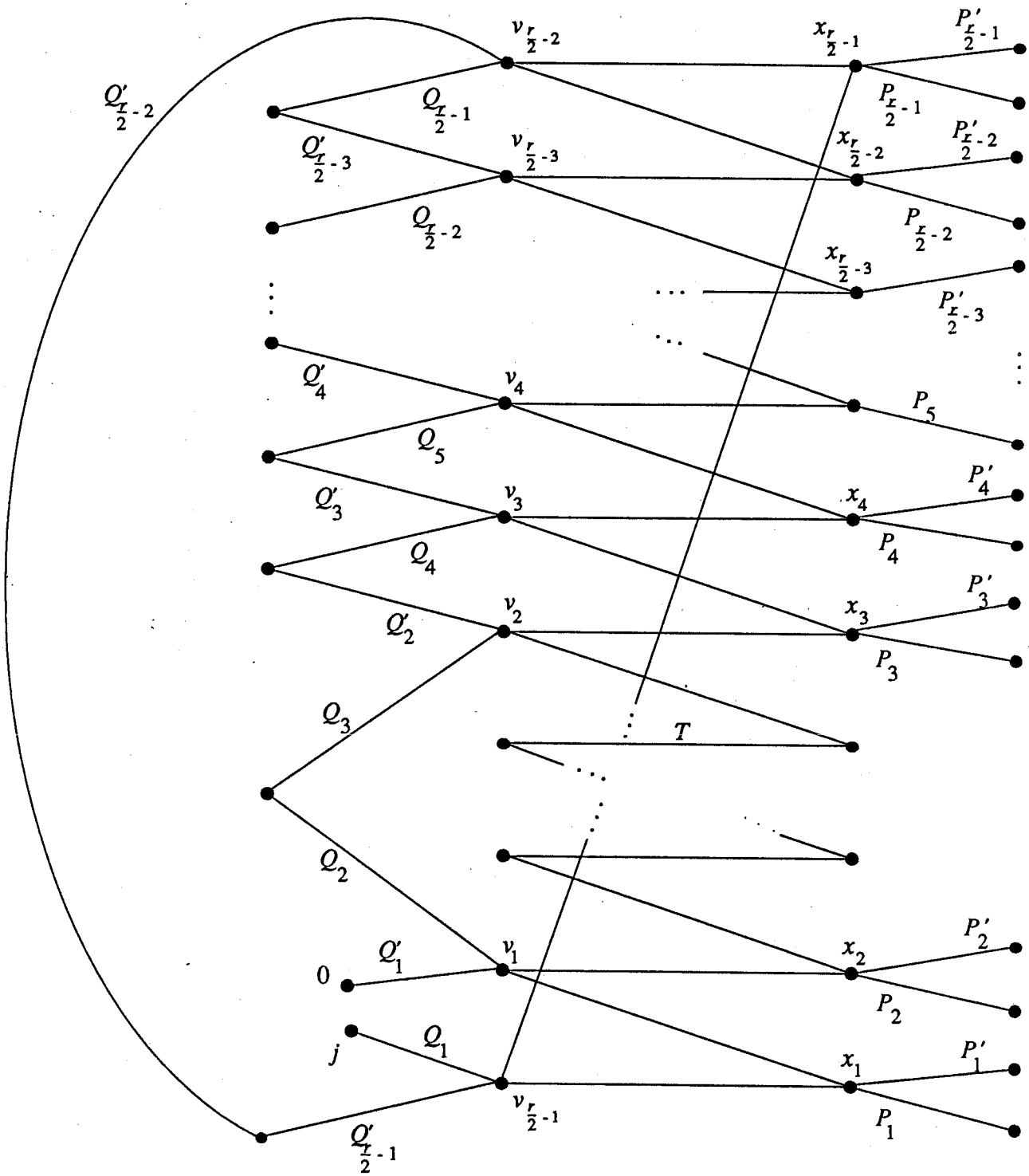

$v_{\frac{r}{2}-2}$
$x_{\frac{r}{2}-1}$
$P'_{\frac{r}{2}-1}$
$Q'_{\frac{r}{2}-2}$
$Q_{\frac{r}{2}-1}$
$P_{\frac{r}{2}-1}$
$Q'_{\frac{r}{2}-3}$
$v_{\frac{r}{2}-3}$
$x_{\frac{r}{2}-2}$
$P'_{\frac{r}{2}-2}$
$Q_{\frac{r}{2}-2}$
$P_{\frac{r}{2}-2}$
$x_{\frac{r}{2}-3}$
$P'_{\frac{r}{2}-3}$
Q'_4
v_4
P_5
Q_5
Q'_3
v_3
x_4
P'_4
P_4
Q_4
Q'_2
v_2
x_3
P'_3
P_3
Q_3
T
Q_2
Q'_1
v_1
x_2
P'_2
0
P_2
j
Q_1
x_1
P'_1
$v_{\frac{r}{2}-1}$
P_1
$Q'_{\frac{r}{2}-1}$

Fig. 3.7

Therefore by Claim 3.2.2, for any choice of $l(P_i)$ from $\{2, 3, ..., r-1\}$, Q_i and Q_i' will be segments in W and hence paths in K_m. Our procedure is to choose the lengths $l(P_i)$, $1 \le i \le r/2-1$, so that the resulting vertices $v_1, ..., v_{r/2-1}$ and $x_1, ..., x_{r/2-1}$ are distinct. In addition, recall that for the construction of the two m-paths $R_{r/2}$ and $R'_{r/2}$ we require $(j,z) \in E(C)$, and $j+1 \notin V(C)$, as in Claim 3.2.4. First let us look at what happens when $i = 2$.

When $i = 2$ we have a segment T of C, of length $m - r + 3$, joining x_2 to v_2 (and consequently joining P_2' to Q_2'). In order for $P_2' + T + Q_2'$ to be a path we must ensure that the internal vertices of T lie on neither P_2' nor Q_2'. So having labelled $x_1, ..., x_{r/2-1}$ and $v_1, ..., v_{r/2-1}$, we will need to be able to label the remaining $m - r + 2$ vertices of C (which are exactly the internal vertices of T) with vertices which do not lie on P_2' or Q_2'. To do this we will need at least $(m - r + 2)/2$ vertices of K_m which do not belong to $\{v_1, ..., v_{r/2-1}\} \cup V(Q_2')$, and at least $(m - r + 2)/2$ vertices of K_r which do not belong to $\{x_1, ..., x_{r/2-1}\} \cup V(P_2')$. Therefore we need

$m - |\{v_1, ..., v_{r/2-1}\} \cup V(Q_2')| \ge (m - r + 2)/2$, and

$r - |\{x_1, ..., x_{r/2-1}\} \cup V(P_2')| \ge (m - r + 2)/2$.

Since $v_2 \in V(Q_2')$ and $x_2 \in V(P_2')$ and $r > 4$, these conditions reduce to $l(Q_2') \le m/2$ and $l(P_2') \le r - m/2$. We cannot satisfy these unless $r - m/2 \ge 2$, since from Claim 3.2.3 we need $l(P_2') \ge 2$. It is for this reason that we made this assumption at the outset.

Since $r \ge m/2+2$, we may set $l(P_2') = r - m/2$ and consequently $l(Q_2') = m/2 - 3$ (clearly $m/2 - 3 \le m - 3$, so Q_2' will indeed be a path). This will give us enough

freedom in the labelling of the last $m-r+2$ vertices of C to ensure that $P_2' + T + Q_2'$ will be a path.

We now begin the labelling of C. We first choose v_1. Notice that we have arranged the paths Q_i, Q_i', $1 \le i \le r/2-1$, along W in such a way that the length of the segment between v_1 and $v_{r/2-1}$ is precisely

$[l(Q_2) + l(Q_2')] + [l(Q_3) + l(Q_3')] + \ldots + [l(Q_{r/2-1}) + l(Q_{r/2-1}')]$.

Since $l(Q_i) + l(Q_i') = \begin{cases} 2m-r-3, & i \neq 2 \\ m-5, & i = 2, \end{cases}$

the length of this segment depends only upon m and r. Therefore the choice of v_1 will uniquely determine $v_{r/2-1}$. In addition, the length of the segment of $C_{j-1} \cup C_j$ between $v_{r/2-1}$ and j (travelling in the direction specified for W) is precisely $l(Q_1)$. Therefore $v_{r/2-1}$ will in turn determine $l(Q_1)$.

Since we must ensure that (j,z) is an edge of C, we will show how to choose $l(Q_1')$ so that $v_1 = j$, and $v_{r/2-1} \neq j, j+1$ (this is because $j+1$ may not lie on C, and v_1 must be different from $v_{r/2-1}$). We will then choose G_1 so that $x_1 = z$.

Now since C_j is the last cycle from W used in constructing these m-paths, we use in total $j-1$ cycles C_i from W (recall that C_1 is contained in D). We construct r m-paths in total, and so

$$|E(K_r)| + |E(C \cup C')| + (j-1)m = rm.$$

Thus $r(r-1)/2 + 2m + (j-1)m = rm$,

so $j = r - 1 - r(r-1)/2m.$ (2)

We want to set $v_1 = j$, so that $j \in V(C)$. Since v_1 determines $v_{r/2-1}$, we must make sure that setting $v_1 = j$ does not force $v_{r/2-1} = j$ or $j+1$. Consider C_2, the

first cycle from W we use in constructing the m-paths. Clearly, Q_1' will be a segment of C_2, beginning (0, 2, 3, 1, ...) (see Fig. 3.8)

The (forward) path from 0 to j in C_2 has length $2(j-2)$, provided that $2 < j < m/2 + 2$. Now j is the index of one of the cycles in the decomposition of K_m, and so $j \le m/2 - 1 < m/2 + 2$. Secondly, by (2) $j > 2$ if and only if $r - 1 - r(r-1)/2m > 2$, or equivalently, $m > r(r-1)/(2r-6)$. Now since we are assuming $r > 8$, we have $r - 1 < 2r - 6$ and so $(r-1)/(2r-6) < 1$. Therefore $r[(r-1)/(2r-6)] < r < m$, and so $j > 2$ as required. So setting $l(Q_1') = 2(j-2)$ we will have $v_1 = j$.

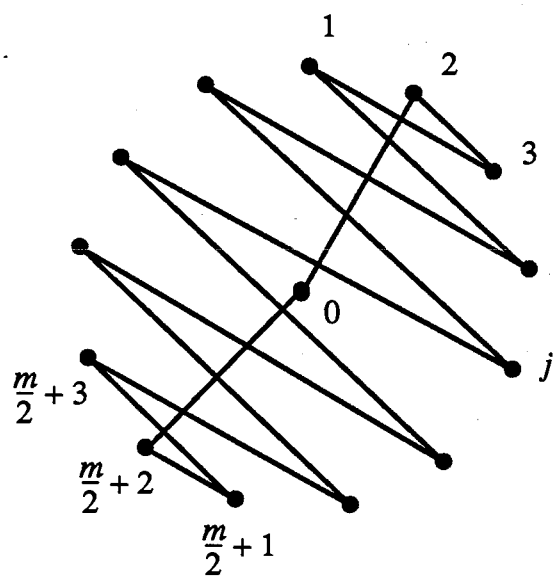


Fig. 3.8

First, we must check that this will not force $v_{r/2-1} = j,\ j+1$. Now the segment $(C_1 + ... + C_j) \setminus S$ of W ends at the vertex j (more precisely, at the edge $(0, j)$) of C_j. Therefore, if $l(Q_1) \le m - 3$ we will have $v_{r/2-1} \ne j$ (by Claim 3.2.2). In addition, $j+1$ is the second vertex of S (recall that S is the segment

$(j, j+1, j-1, ..., 0)$ of C_j). So again by Claim 3.2.2, if $l(Q_1) \le m-4$ we will have $v_{r/2-1} \neq j+1$.

We also need $m - r \le l(Q_1') \le m-3$. Combining this with the above requirement that $l(Q_1) \le m-4$ and the fact that $l(Q_1) + l(Q_1') = 2m - r - 3$, we need $m - r < l(Q_1') \le m-3$. First, since $j \le m/2 - 1$, then $l(Q_1') = 2(j-2) \le m-6 < m-3$. Second, we need $m - r < 2(j-2) = 2j - 4$, or equivalently, $j > (m - r + 4)/2$. So by (2) we need $r - 1 - r(r-1)/2m > (m - r + 4)/2$, or equivalently, $-m^2 + (3r-6)m - r(r-1) > 0$.

Let $F(m) = -m^2 + (3r-6)m - r(r-1)$. The roots of F are

$$r_1 = (3r-6)/2 - \sqrt{5r^2/4 - 8r + 9}$$

and $r_2 = (3r-6)/2 + \sqrt{5r^2/4 - 8r + 9}$,

and $F(m) > 0$ whenever $r_1 < m < r_2$. We are assuming that $r \ge m/2 + 1$ and $r \le m-2$. Therefore $r + 2 \le m \le 2r - 2$. It is straightforward to check that, if $r > 8$, then $r_1 < r + 2 < 2r - 2 < r_2$. Therefore, since $r > 8$, we have $F(m) > 0$ whenever $r + 2 \le m \le 2r - 2$. This gives $2(j-2) > m - r$ as required, and so we may set $l(Q_1') = 2(j-2)$. We have $m - r \le l(Q_1') \le m-3$, and $v_{r/2-1} \neq j, j+1$.

Finally we want to set $x_1 = z$. We have a decomposition of K_r into r-cycles and a one-factor, and we want to add one edge of the one-factor to each r-cycle to obtain a decomposition of K_r into the $r/2-1$ subgraphs G_i and a single edge zz'. First, choose any r-cycle and any edge of the one-factor, and let G_1 be the union of the chosen r-cycle with the chosen edge. Since we have set $l(Q_1') = 2(j-2)$, we know $l(P_1)$ and $l(P_1')$. Moreover, $m - r < l(Q_1') < m-3$ and

$l(Q_1)+l(Q_1')=2m-r-3$ imply $2<l(P_1), l(P_1')<r-1$. As in Claim 3.2.3, the value of $l(P_1)$ will uniquely determine the vertex x_1. Also by Claim 3.2.3, the vertex x_1 has degree 2 in G_1, that is, x_1 is not an endvertex of the chord of G_1. So the edge e of the one-factor F which contains x_1 is not the chord of G_1. Thus we may set $z=x_1$, and let z' be the other endvertex of e. In other words, we choose e to be the single edge of K_r not contained in any G_i.

We may now construct the subgraphs $G_2, ..., G_{r/2-1}$, adding each remaining edge of the one-factor to one of the r-cycles. All we require is that each G_i be an r-cycle with a chord, where the chord is not e (and of course that $E(G_1), ...,$ $E(G_{r/2-1})$ and e partition $E(K_r)$).

We have now labelled the vertices v_1, $v_{r/2-1}$ and x_1 on C. For the labelling of v_2 and x_2, recall that we have set $l(P_2')=r-m/2$ and $l(Q_2')=m/2-3$, which gives $l(P_2)=m/2+1$ and $l(Q_2)=m/2-2$. Now $l(P_2)$ will determine the vertex x_2 of G_2. Suppose $x_2=x_1=z$. Since $l(P_2)+l(P_2')=r+1$ is odd, $l(P_2)\neq l(P_2')$. So we may interchange P_2 and P_2' so that $l(P_2')=m/2+1$ and $l(P_2)=r-m/2$. This will give us a different vertex x_2 so that $x_2\neq x_1$. The two m-paths we construct will now be $P_2'+x_2v_1+Q_2$ and P_2+T+Q_2'. Thus we may assume $x_2\neq x_1$.

Notice that the vertex v_2 will depend on $l(Q_3)$ (see Fig. 3.7). Since $m-r\leq l(Q_3)\leq m-3$, there are $r-2$ choices for $l(Q_3)$. Each of these $r-2$ values of $l(Q_3)$ determines a vertex v_2, and these vertices are consecutive vertices of W. Since $r-2<m-2$, then by Claim 3.2.2 they are all different. Thus we have $r-2$ different choices for v_2. Now we must choose $l(Q_3)$ so that $v_2\neq v_1, v_{r/2-1}, j+1$. In addition, each choice of $l(Q_3)$ will determine a

corresponding vertex x_3 of G_3. We must choose $l(Q_3)$ so that $x_3 \neq x_2, x_1$. Therefore in total we might have to exclude five of the possible values of $l(Q_3)$. But we are assuming $r > 8$, so $r - 2 > 6$, and so we may certainly choose $l(Q_3)$ so that both $v_2 \neq v_1, v_{r/2-1}, j+1$ and $x_3 \neq x_2, x_1$.

Notice that for $2 \leq i \leq r/2 - 2$, once we have chosen $v_1, ..., v_{i-1}, v_{r/2-1}$, the vertices $x_1, ..., x_i$ are all fixed, and the choice of v_i (equivalently the choice of $l(Q_{i+1})$) will determine x_{i+1}.

Assume that we have chosen distinct vertices $v_1, ..., v_{i-1}, v_{r/2-1}$ and that the resulting vertices $x_1, ..., x_i$ are all different, where $3 \leq i \leq r/2 - 2$. We now choose v_i. The value of $l(Q'_{i-1})$ is fixed. Thus the vertex v_i depends on $l(Q_{i+1})$. Again $m - r \leq l(Q_{i+1}) \leq m - 3$, giving us $r - 2$ choices of $l(Q_{i+1})$, and consequently $r - 2$ distinct choices of v_i. Of these, we will have to exclude at most $i + 1$ to ensure that $v_i \notin \{v_1, ..., v_{i-1}, v_{r/2-1}, j+1\}$, and at most i others might result in $x_{i+1} \in \{x_1, ..., x_i\}$. We therefore have at least $r - 2 - (i + 1) - i = r - 2i - 3$ valid choices for v_i. But $i \leq r/2 - 2$, so $r - 2i - 3 \geq 1$. Thus there is at least one valid choice of v_i and hence x_{i+1}.

Once $v_{r/2-2}$ has been chosen, we have labelled $v_1, ..., v_{r/2-1}$ and $x_1, ..., x_{r/2-1}$. We must now label the remaining $m - r + 2$ vertices of C (recall that these are the internal vertices of the segment T of C).

We have $(m - r + 2)/2$ vertices of C to label in K_r. None of these may be labelled with vertices from $\{x_1, ..., x_{r/2-1}\} \cup V(P'_2)$. Since x_2 lies on P'_2, this leaves us with at least $r - (r/2 - 1 + r - m/2) = (m - r + 2)/2$ available labels, which is just enough. Similarly we have $(m - r + 2)/2$ vertices of C to label in K_m.

In this case we may not use vertices from $\{v_1, ..., v_{r/2-1}, j+1\} \cup V(Q_2')$. We have $v_2 \in V(Q_2')$, and so there are at least

$m - (r/2 - 1 + 1 + m/2 - 3) = (m - r + 6)/2 > (m - r + 2)/2$ available labels.

We have now labelled the cycle C. Since we also labelled z and $j+1$ on C', we have in total labelled $m/2 + 1$ vertices of K_m and $m/2$ vertices of K_r (in the decomposition of $K_{m,r}$). We may label the remaining vertices of $K_{m,r}$ arbitrarily.

The construction is now complete. We have constructed the following r m-paths in $K_n - D$:

$$R_1 = P_1 + (x_1, v_{r/2-1}) + Q_1,$$
$$R_2 = P_2 + (x_2, v_1) + Q_2,$$
$$\vdots$$
$$R_{r/2-1} = P_{r/2-1} + (x_{r/2-1}, v_{r/2-2}) + Q_{r/2-1},$$
$$R_1' = P_1' + (x_1, v_1) + Q_1',$$
$$R_2' = P_2' + T + Q_2',$$
$$R_3' = P_3' + (x_3, v_3) + Q_3',$$
$$\vdots$$
$$R_{r/2-1}' = P_{r/2-1}' + (x_{r/2-1}, v_{r/2-1}) + Q_{r/2-1}',$$
$$R_{r/2} = (S \setminus (j, j+1)) + (j+1, z) + (z, z'),$$

and $R_{r/2}' = (j, j+1) + (C' \setminus (j+1, z))$.

These paths cover all the edges of K_r, $2m$ edges of $K_{m,r}$, and an initial segment (or possibly all) of W. Now the remaining edges of $K_{m,r}$ are partitioned into m-cycles. In addition, the 'unused' portion R of W consists of the $(m/2 - 1 - j)$ m-cycles $C_{j+1}, ..., C_{m/2-1}$. Thus we have a decomposition of $K_n - D$ into m-paths and m-cycles. ∎

Case 2 Let n be even, and let $r = m/2+1$, where $r > 8$. As in Case 1, m and r are even, and $r \equiv 0 \pmod 4$.

The problem with the previous construction, when $r = m/2+1$, was that we might not be able to label the internal vertices of T so that $R_2' = P_2' + T + Q_2'$ would be a path. In this case we set $l(P_2') = r - m/2 + 1 = 2$ and we choose G_2 so that x_1 is one endvertex of its chord (so of course we can no longer set $x_1 = z$). By Claim 3.2.3, the vertex x_2 will be neither endvertex of this chord (in particular, we will have $x_2 \neq x_1$). Moreover, each endvertex of the chord lies on both P_2 and P_2', so this will force $x_1 \in V(P_2')$. This will leave us, when we come to label the internal vertices of T, with

$$r - |\{x_1, \ldots, x_{r/2-1}\} \cup V(P_2')| \geq r - [(r/2 - 1) + (r - m/2 + 1) - 1]$$
$$= (m - r + 2)/2$$

available vertices to use as labels.

In order to let x_1 be an endvertex of the chord in G_2, we must first ensure that $x_1 \neq z$, since the edge (z, z') of the one-factor lies on no G_i. In addition, x_1 must not be an endvertex of the chord in G_1. (But this is guaranteed by Claim 3.2.3.)

Since we now require $x_1 \neq z$ (whereas before we set $x_1 = z$), we cannot label the edge v_1x_1 of C as jz. However, since $r = m/2+1$, and since each vertex of K_r lies on $m/2$ cycles in the decomposition, then each vertex of K_r in fact lies on $(r - 1)$ of the r m-cycles in the decomposition. Now if we choose the vertices a, b and z of K_n as in Claim 3.2.4, we have at most $(r/4 - 1)$ cycles containing both a and b. Since each vertex of K_m lies in total on $r/2$ of the cycles in the m-cycle decomposition of $K_{m,r}$, there are then at least $(r/4 + 1)$ cycles containing b and

not containing a. Since at most one of these cycles does not contain z, we have, in addition to the cycle containing the edge bz but not a, at least $(r/4 + 1) - 2 \geq 1$ cycles containing both b and z (but not a) on which b and z are non-adjacent. Consequently it does not matter whether or not we label j and z as adjacent on C.

The modified construction is as follows. As before we let $l(Q_1) = 2(j-2)$, so that $v_1 = j$ while $v_{r/2-1} \neq j, j+1$. This is still valid since we are again assuming that $r > 8$. For G_1 we choose any r-cycle and any edge of the one-factor from the decomposition of K_r, and we determine the resulting vertex x_1.

For G_2 we choose any remaining r-cycle together with the edge of the one-factor containing x_1. We set $l(P_2') = r - m/2 + 1 = 2$, which determines the vertex x_2. For z, we then choose any vertex of K_r different from x_1 and x_2 and the endvertices of the chords in G_1 and G_2 (recall that one of these is x_1). Notice that this will guarantee $z \notin V(P_2')$, since, letting (x^*, x_1) be the chord in G_2, we have $V(P_2') = \{x_2, x_1, x^*\}$ [see Fig. 3.9]. Thus there are $r - 5$ choices for z, and since $r > 8$ we have $r - 5 > 3$. Our choice of z determines the edge (z, z') of the one-factor which will lie on no subgraph G_i.

We now proceed to construct $G_3, ..., G_{r/2-1}$ arbitrarily (of course not using (z, z') on any G_i) and to label the vertices $\{v_2, ..., v_{r/2-2}\}$ and $\{x_3, ..., x_{r/2-1}\}$ exactly as before.

It now remains to label the internal vertices of T. For those which lie in K_m, since $l(Q_2') = m - l(T) - l(P_2') = m/2 - 4$, we have at least

$$m - |\{v_1, ..., v_{r/2-1}\} \cup \{j+1\} \cup V(Q_2')| \geq m - (r/2 + m/2 - 4)$$
$$= (m - r + 8)/2$$

available vertices to use as labels. For those which lie in K_r, as in the previous argument we have at least $(m-r+2)/2$ available vertices to use as labels. If

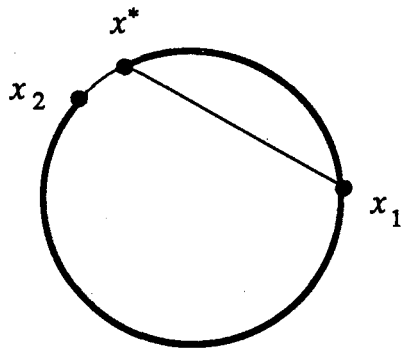


Fig. 3.9

$z \notin \{x_1, ..., x_{r/2-1}\}$ then we label one vertex of T with z, and the others arbitrarily (notice that $(m-r+2)/2 > 0$, so there is a vertex of T which we may label z, and recall that $z \notin V(P_2')$). If $z \in \{x_1, ..., x_{r/2-1}\}$, then we may label all $(m-r+2)/2$ of these vertices arbitrarily, as before.

Finally, we construct the paths $R_{r/2}$ and $R'_{r/2}$ exactly as before, using the cycle C' containing the edge $(j+1, z)$ (and not containing j), the edge (z, z'), and the last $m-1$ edges of C_j.

This gives us, as before, a decomposition of $K_n - D$ into the r m-paths R_i, R_i', $1 \le i \le r/2$, the remaining $(r-2)$ m-cycles of $K_{m,r}$ and, if $j < m/2-1$, the $(m/2-1-j)$ cycles $C_{j+1}, ..., C_{m/2-1}$ from K_m. ■

Case 3 Let n be even and let $r \le 8$. We have either $r = 4$ or $r = 8$, since $r \equiv 0 \pmod 4$.

(i) Let $r = 4$. Then $n = m + 4$; so (NC3) imply $m \mid (m + 4)(m + 3)$, or equivalently, $m \mid (m^2 + 7m + 12)$. Since in addition we assume $n(n-1)/m$ is odd, we have $m \mid 12$, where $12/m$ is even and m is even. This implies $m = 6$, so $n = 10$.

We give a decomposition of $K_{10} - D$ into 6-paths and 6-cycles. We let $K_{10} = K_6 \cup K_{6,4} \cup K_4$ where $V(K_6) = \{u_0, u_1, ..., u_5\}$ and $V(K_4) = \{w_0, w_1, w_2, w_3\}$. We have

$D = C_1 \cup F = (u_0, u_1, u_2, u_5, u_3, u_4, u_0) \cup \{(u_5, u_1), (u_4, u_2), (u_3, u_0)\}$ and

$W = C_2 = (u_0, u_2, u_3, u_1, u_4, u_5, u_0)$.

We decompose $K_{6,4}$ into the following 6-cycles:

$A = (u_0, w_0, u_2, w_2, u_1, w_1, u_0)$,

$B = (u_1, w_0, u_4, w_1, u_2, w_3, u_1)$,

$C = (u_3, w_0, u_5, w_2, u_4, w_3, u_3)$,

and $E = (u_0, w_2, u_3, w_1, u_5, w_3, u_0)$.

We decompose K_4 into the 3-paths (w_3, w_0, w_2, w_1) and (w_0, w_1, w_3, w_2). We use these 3-paths together with the 6-cycles A, B, and C_2 to form the following 6-paths [see Fig. 3.10]:

$R_1 = (w_3, w_0, u_0, w_1, u_1, w_2, u_2)$,

$R_2 = (w_1, w_2, w_0, u_2, u_3, u_1, u_4)$,

$R_3 = (w_2, w_3, u_1, w_0, u_4, w_1, u_2)$,

and $R_4 = (w_0, w_1, w_3, u_2, u_0, u_5, u_4)$.

We have decomposed $K_{10} - D$ into the 6-paths R_1, R_2, R_3, and R_4, and the 6-cycles C and E.

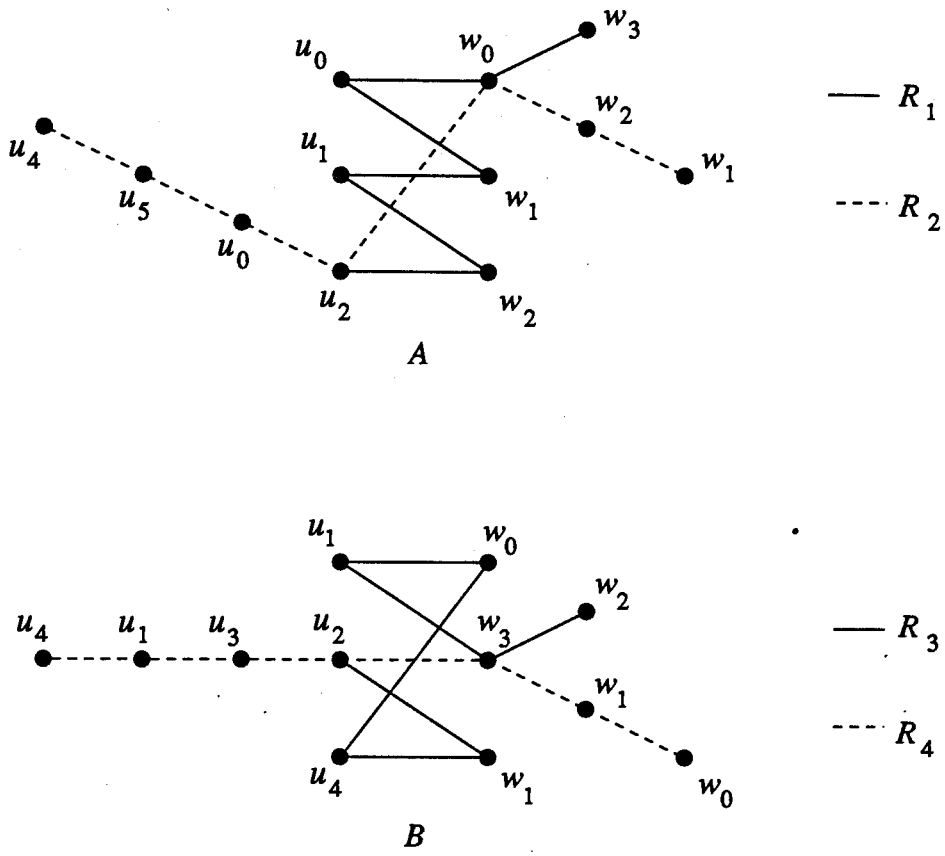


Fig. 3.10

(ii) Let $r = 8$. Then $n = m + 8$, so (NC3) imply that $m|(m + 8)(m + 7)$, or equivalently, $m|(m^2 + 15m + 56)$. Since we assume $n(n - 1)/m$ is odd, we have $m|56$, where $56/m$ is even and m is even. Finally, $r < m$ implies $m > 8$, and $r \geq m/2 + 1$ implies $m \leq 14$. Thus the only case to consider is $m = 14$ (and so $n = 22$).

We give a decomposition of $K_{22} - D$ into 14-paths and 14-cycles. We let $K_{22} = K_{14} \cup K_{14,8} \cup K_8$, where $V(K_{14}) = \{u_0, ..., u_{13}\}$ and $V(K_8) = \{w_0, ..., w_7\}$. We have $K_{14} = F \cup \cup\{C_i : 1 \le i \le 6\}$, so

$D = C_1 \cup F$

$= (u_0, u_1, u_2, u_{13}, ..., u_7, u_8, u_0) \cup \{u_{13}u_1, u_{12}u_2, ..., u_8u_6, u_0u_7\}$, and W

$= C_2 + C_3 + ... + C_6$.

We decompose K_8 into the following four 7-paths:

$G_1 = (w_0, w_1, w_7, w_2, w_6, w_3, w_5, w_4)$,

$G_2 = (w_1, w_2, w_0, w_3, w_7, w_4, w_6, w_5)$,

$G_3 = (w_2, w_3, w_1, w_4, w_0, w_5, w_7, w_6)$,

and $G_4 = (w_3, w_4, w_2, w_5, w_1, w_6, w_0, w_7)$.

By Lemma 2.1.1, there is a decomposition of $K_{14,8}$ into 14-cycles. Let C be any cycle from this decomposition. As in the construction in Case 1, we will relabel the vertices of C (and so the entire decomposition of $K_{14,8}$) so that we can use the edges of C as a bridge between K_{14} and K_8. We will divide each of the 7-paths G_i into two paths P_i and P_i'. If $i \neq 2$, we will set $l(P_i) = 3$ and $l(P_i') = 4$, and we will extend each of P_i and P_i' to a 14-path in $K_{22} - D$ using one edge of C and a segment of W. We will set $l(P_2) = 6$ and $l(P_2') = 1$. To extend P_2 to a 14-path, we use one edge of C and a segment of W (of length 7). To extend P_2' to a 14-path, we use the remaining 7 edges of C (which we again call T) and a segment of W (of length 6). As before, we let Q_i and Q_i' be the segments of W which we use to extend P_i and P_i', respectively, and we arrange these segments along W so that $W = Q_1' + Q_2 + Q_3 + Q_2' + Q_4 + Q_3' + Q_4' + Q_1$. We define $v_1, ..., v_4$ and $x_1, ..., x_4$ as before.

Here, we want to construct eight 14-paths from the edges of K_8, $K_{14} - D$, and one 14-cycle. We have $|E(K_{14} - D)| + 14 + |E(K_8)| = 14(5) + 14 + 28$

$= 14(8)$.

So our construction will use all the edges of W.

We relabel C so that

$v_1, v_2, v_3, v_4 = u_{11}, u_9, u_0, u_8,$

$x_1, x_2, x_3, x_4 = w_2, w_6, w_4, w_5,$

and $T = (w_6, u_1, w_0, u_2, w_1, u_7, w_3, u_9)$.

The remaining vertices of $K_{14,8}$ may be relabelled arbitrarily.

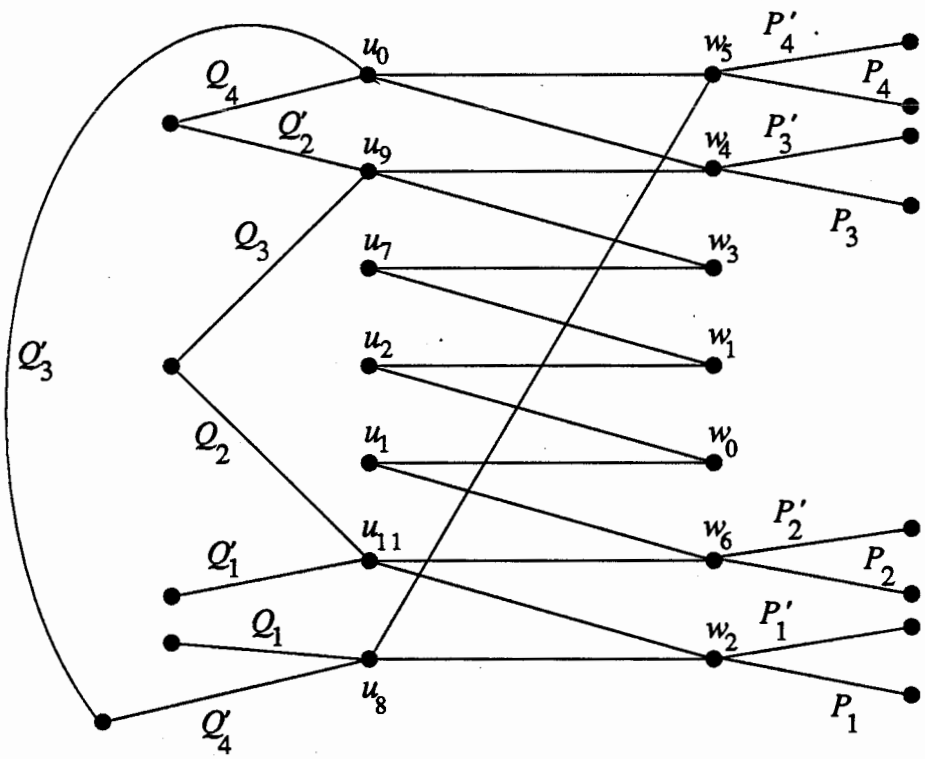


Fig. 3.11

The eight 14-paths we construct are the following (see Fig. 3.11):

$$R_1 = P_1 + x_1 v_4 + Q_1$$
$$= (w_0, w_1, w_7, w_2, u_8, u_4, u_9, u_3, u_{10}, u_2, u_{11}, u_1, u_{12}, u_{13}, u_0),$$
$$R_1' = P_1' + x_1 v_1 + Q_1'$$
$$= (w_4, w_5, w_3, w_6, w_2, u_{11}, u_6, u_{12}, u_5, u_{13}, u_4, u_1, u_3, u_2, u_0),$$
$$R_2 = P_2 + x_2 v_1 + Q_2$$
$$= (w_1, w_2, w_0, w_3, w_7, w_4, w_6, u_{11}, u_7, u_{10}, u_8, u_9, u_0, u_3, u_4),$$
$$R_2' = P_2' + T + Q_2'$$
$$= (w_5, w_6, u_1, w_0, u_2, w_1, u_7, w_3, u_9, u_{10}, u_0, u_4, u_5, u_3, u_6),$$
$$R_3 = P_3 + x_3 v_2 + Q_3$$
$$= (w_2, w_3, w_1, w_4, u_9, u_{11}, u_8, u_{12}, u_7, u_{13}, u_6, u_1, u_5, u_2, u_4),$$
$$R_3' = P_3' + x_3 v_3 + Q_3'$$
$$= (w_6, w_7, w_5, w_0, w_4, u_0, u_5, u_6, u_4, u_7, u_3, u_8, u_2, u_9, u_1),$$
$$R_4 = P_4 + x_4 v_3 + Q_4$$
$$= (w_3, w_4, w_2, w_5, u_0, u_{11}, u_{10}, u_{12}, u_9, u_{13}, u_8, u_1, u_7, u_2, u_6)$$

and $R_4' = P_4' + x_4 v_4 + Q_4'$

$$= (w_7, w_0, w_6, w_1, w_5, u_8, u_5, u_7, u_6, u_0, u_{12}, u_{11}, u_{13}, u_{10}, u_1).$$

These eight 14-paths together cover all the edges of $K_{14} - D$, all the edges of K_8, and the edges of one 14-cycle from the decomposition of $K_{14,8}$. Since the remainder of $K_{14,8}$ consists of 14-cycles (now relabelled), we have a decomposition of $K_{22} - D$ into 14-paths and 14-cycles. By Corollary 2.3.5 and Lemma 3.2.1, this yields a decomposition of $K_{44} - I$ into 28-cycles.

This completes the proof of Lemma 3.2.2. ■

Thus we have a decomposition of $K_{2n} - I$ into $2m$-cycles if n and m satisfy the conditions (*) on page 20. The remaining cases are those for which the quotient $n(n-1)/m$ is odd, $m < n < 2m$, and either n is odd or $n < 3m/2$. We hope that a construction similar to the above will give results in some or all of these cases, particularly the case when n is odd and at least $3m/2$.

List of References

[A/H] B. Alspach, R. Häggkvist, Some observations on the Oberwolfach Problem. Journal of Graph Theory **9** (1985) 177-187.

[A/V] B. Alspach, B. Varma, Decomposing complete graphs into cycles of length $2p^e$. Annals of Discrete Mathematics **9** (1980) 155-162.

[B/F] J. C. Bermond, V. Faber, Decomposition of the complete directed graph into k-circuits. Journal of Combinatorial Theory **21** (B) (1976) 146-155.

[B/H/S] J. C. Bermond, C. Huang, D. Sotteau, Balanced cycle and circuit designs: even cases. Ars Combinatoria **5** (1978) 293-318.

[H] R. Häggkvist, A lemma on cycle decompositions. Annals of Discrete Mathematics **27** (1985) 227-232.

[Hl] B. Hartnell, Decomposition of K^*_{xy}, x and y odd, into $2x$-circuits. Proc. Fourth Manitoba Conference on Numerical Math. (1974) 265-271.

[H/K/R] C. Huang, A. Kotzig, A. Rosa, On a variation of the Oberwolfach Problem. Discrete Mathematics **27** (1979) 261-277.

[H/M] B. Hartnell, M. Milgram, Decomposition of K_p, for p a prime into k-circuits. Proc. Journées franco-belges sur les graphes et hypergraphes, Paris, Mai 1974.

[K] A. Kotzig, On the decomposition of complete graphs into $4k$-gons. Mat.-Fyz. Časopis Sloven. Akad. Vied. **15** (1965) 229-233 (in Russian).

[L] R. Laskar, Decomposition of some composite graphs into Hamilton cycles. Colloq. Math. Soc. János Bolyai **18** (1976) 705-716.

[Lu] E. Lucas, *Récréations Mathématiques,* Vol. 2, Gauthier-Villars, Paris (1884).

[R] A. Rosa, On cyclic decompositions of the complete graph into $(4m+2)$-gons. Mat.-Fyz. Časopis Sloven. Akad. Vied. **16** (1966) 349-353.

[R1] A. Rosa, The use of graphs for the formulation of Kirkman's problem. Mat.-Fyz. Časopis Sloven. Akad. Vied. **13** (1963) 105-113.

[S] D. Sotteau, Decomposition of $K_{m,n}$ ($K^*_{m,n}$) into cycles (circuits) of length $2k$. Journal of Combinatorial Theory (B) **30** (1981) 75-81.

[T] M. Tarsi, Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs. Journal of Combinatorial Theory (A) **34** (1983) 60-70.

[Tl] T. Tillson, A Hamilton decomposition of K^*_{2m}, $2m \geq 8$. Journal of Combinatorial Theory (B) **29** (1980) 68-74.