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# RESOLVABLE AND NEAR-RESOLVABLE ORIENTED 3- AND 4-CYCLE DECOMPOSITIONS OF THE COMPLETE SYMMETRIC DIGRAPH

by

Susan Hamm

B.Sc., Simon Fraser University, 1987

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE (MATHEMATICS)

in the Faculty of

Mathematics and Statistics

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SIMON FRASER UNIVERSITY

December 1989

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ISBN 0-315-59347-4

Canada

# Approval

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DEGREE: Master of Science (Mathematics)

TITLE OF THESIS: Resolvable and Near-Resolvable Oriented 3- and 4-cycle Decompositions of the Complete Symmetric Digraph

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Resolvable and Non-Resolvable Oriented 3- and 4-cycle Decompositions of the Complete Symmetric Digraph

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# ABSTRACT

In this thesis we study resolvable and near-resolvable decompositions of the complete symmetric digraph on $v$ vertices, $DK_v$, into each of the two oriented 3-cycles, $CT_3$ and $TT_3$, and into each of the four oriented 4-cycles, A, B, C and D, (where A, B, and C, are the oriented 4-cycles with longest path lengths one, two and three respectively, and D is the directed 4-cycle). In Chapter One we present a brief history of the problem, together with some preliminary results. This is followed, in Chapter Two, by a discussion of known results for oriented 3-cycle decompositions. In Chapters Three and Four we study necessary and sufficient conditions for the existence of resolvable and near-resolvable decompositions of $DK_v$ into each of A, B, C and D. We show that $DK_v$ admits resolvable decompositions into B if and only if $v \equiv 0 \pmod 4$, $v \neq 4$ (with possible exceptions $v = 20$ and $v = 52$); into C if and only if $v \equiv 0 \pmod 4$; and into D if and only if $v \equiv 0 \pmod 4$, $v \neq 4$. $DK_v$ cannot be resolvably decomposed into A. Near-resolvable decompositions of $DK_v$ into A and into D exist if and only if $v \equiv 1 \pmod 4$, and into B and into C if and only if $v \equiv 1 \pmod 4$, $v \neq 5$ (with the possible exception of decompositions of $DK_{21}$ into near B-factors).

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# Chapter 1 - Introduction

Let G be a finite multigraph with no loops. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a collection of connected graphs on the vertex set of G whose edge-disjoint union is isomorphic to G. Then we say that $\mathcal{H}$ is a decomposition of the graph G. In particular, if $H_i$ is a cycle for $i = 1,2,\ldots,n$, then $\mathcal{H}$ is a cycle decomposition of G. If $H_i \cong H$ for all $i$ then we say that H divides G, denoted H|G. The question of when a given graph G has a certain type of cycle decomposition has been of considerable interest over the past several years. For a general survey we refer the reader to [4] and [14]. In particular, there has been much work done when G is the complete graph on $v$ vertices with edge multiplicity $\lambda$, $\lambda K_v$, and all of the cycles in the decomposition have the same length. The problem formally stated is to determine the values of $v$ for which $\lambda K_v$ has a cycle decomposition into cycles of length $k$. Clearly, it is necessary that $v \geq k$, that $k$ divide the number of edges in $\lambda K_v$, and that the degree, $\lambda(v-1)$, of each vertex be even. In this thesis we concentrate on the cases where $k = 3$ and $k = 4$.

A Steiner Triple System on $v$ points, (an STS($v$)) is a collection of 3-subsets of a $v$-set such that each pair of elements in the $v$-set occurs exactly once in some 3-subset. If we let the vertices of $K_v$ be the elements of the $v$-set, each 3-subset gives a 3-cycle in $K_v$ and each edge in $K_v$ occurs in exactly one 3-cycle. Hence $K_v$ can be decomposed into cycles of length 3 exactly when an STS($v$) exists; that is, when $v \equiv 1$ or 3(mod 6) [30]. Much work has been done on triple systems. For a bibliographic sketch and for constructions of both STS($v$) and of triple systems with various $\lambda$, we refer the reader to [30].

In 1965, Kotzig [24] investigated decompositions of $K_v$ into $4t$-cycles.

**1.1. Theorem:** (Kotzig, [24]) If $v \equiv 1 \pmod{8t}$, then there is a decomposition of $K_v$ into $4t$-cycles; the condition being also necessary if $t$ is a power of two.

In particular, if $t=1$, we have that $K_v$ can be decomposed into 4-cycles if and only if $v \equiv 1 \pmod 8$.

In this paper we restrict ourselves to the study of decompositions into 3- and 4-cycles. However, many other results for different cycle lengths are known and we refer the interested reader to [27] and [28].

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a decomposition of a graph G with $|V(G)| = v$. If we can partition the graphs $H_i$ into classes, such that the $H_i$ in a given class are vertex-disjoint, and their union is a spanning subgraph of G, then we say $\mathcal{H}$ is a resolvable decomposition of G and call each class a parallel class. If each of the $H_i \in \mathcal{H}$ is a cycle of length $k$, then we say that $\mathcal{H}$ is a resolvable $k$-cycle decomposition. If in addition, $H_i \cong H$ for all $i$, we may also say H divides G resolvably, denoted $H|_R G$. In this case the parallel classes are called H-factors and we say G has an H-factorization. Observe that for a resolvable $k$-cycle decomposition to exist we must have $v \equiv 0 \pmod k$, $v \geq k$ and $\lambda\left(\frac{v(v-1)}{2}\right) \equiv 0 \pmod k$.

If we can partition the graphs $H_i \in \mathcal{H}$ into classes such that the $H_i$ in each class are vertex-disjoint and their union is a spanning subgraph of $G-\{x\}$, the graph G with one vertex removed, we say that $\mathcal{H}$ is a near-resolvable decomposition of G and again call the classes parallel classes. If all $H_i \in \mathcal{H}$ are $k$-cycles we say that $\mathcal{H}$ is a near-resolvable $k$-cycle decomposition of G. If in addition, $H_i \cong H$ we may also say that H divides G near-resolvably, denoted $H|_{NR} G$. In this case the parallel classes are called near H-factors, and we say G has a near H-factorization. For a near-resolvable $k$-cycle decomposition of G we must have $v \equiv 1 (\mathrm{mod}\ k)$, $v \geq k$, and $\lambda\left(\frac{v(v-1)}{2}\right) \equiv 0 (\mathrm{mod}\ k)$.

We define a 1-factor of a graph G to be a set of vertex-disjoint edges which span G. A near 1-factor of a graph G is a set of vertex-disjoint edges which span $G-\{x\}$, the graph G with a vertex removed.

The question of when $K_v$ can be resolvably decomposed into cycles dates back to the famed Oberwolfach problem, first formulated by Ringel and first mentioned in print in [16]. The specific case of finding resolvable decompositions of $K_v$ into 3-cycles is better known as Kirkman's schoolgirl problem and was solved by Ray-Chaudhuri and Wilson [26]. Such decompositions are called Kirkman triple systems, KTS($v$).

**1.2. Theorem:** (Ray-Chaudhuri and Wilson, [26]) There is a resolvable decomposition of $K_v$ into 3-cycles (a KTS($v$)) if and only if $v \equiv 3 (\mathrm{mod}\ 6)$.

A proof of this theorem can also be found in [30, pp. 254-260].

We observe that there can be no resolvable decomposition of $K_v$ into 4-cycles since this would require that $v \equiv 0 (\mathrm{mod}\ 4)$ and that $v$ be odd, which is impossible.

After many years of research and papers by various mathematicians, the general problem for resolvable $k$-cycle decompositions of $K_v$ was solved. The interested reader can find the culmination of the results in three papers, one by Alspach, Schellenberg, Stinson and Wagner [2], the second by Alspach and Häggkvist [1], and a later paper by Hoffman and Schellenberg [22].

In [18], Hanani settled the question of resolvable and near-resolvable decompositions of $2K_v$ into 3-cycles.

**1.3. Theorem:** (Hanani, [18]) Resolvable decompositions of $2K_v$ into 3-cycles exist if and only if $v \equiv 0 \pmod 3$, $v \neq 6$, and near-resolvable decompostions of $2K_v$ into 3-cycles exist if and only if $v \equiv 1 \pmod 3$.

The existence of near-resolvable $k$-cycle decompositions of $2K_v$ was completely resolved in [21] and [12]. (We note that no near-resolvable $k$-cycle decomposition of $K_v$ exists. Recall that the degree of each vertex must be even in order for the graph to admit a $k$-cycle decomposition. Hence $v$ must be odd. Also, each parallel class uses $v$-1 edges, hence $|E(K_v)| = \frac{v(v-1)}{2}$ must be divisible by $v$-1. But this is not possible if $v$ is odd.)

In [21], Heinrich, Lindner and Rodger show that the necessary condition that $v \equiv 1 \pmod k$ is sufficient for the existence of a near-resolvable $k$-cycle decomposition of $2K_v$ for $k$ odd, $k \geq 3$, and in [12], Burling and Heinrich show that it is also sufficient for $k$ even. In particular we have near-resolvable 4-cycle decompositions.

**1.4. Theorem:** (Burling and Heinrich, [12]) Near-resolvable 4-cycle decompositions of $2K_v$ exist if and only if $v \equiv 1 \pmod 4$.

Analogous questions have also been asked concerning decompositions of directed graphs. If G is a graph, then let DG be the directed graph obtained by replacing each edge $ab \in E(G)$ with the two arcs $(a,b)$ and $(b,a)$. In particular we have the complete symmetric digraph, $DK_v$. Decompositions of digraphs are particularly interesting since different orientations of the arcs are possible. For example, if we wish to decompose $DK_v$ into oriented 3-cycles we can consider the two possible orientations given in Figure 1:

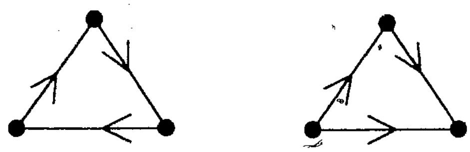



Figure 1

The first we call a *cyclic triple*, denoted $CT_3$, and the second we call a *transitive triple*, denoted $TT_3$.

Mendelsohn was the first to study decompositions of $DK_v$ into cyclic triples. In [25] he presents the idea of decomposing $DK_v$ into cyclic triples as a generalization of Steiner triple systems and gives necessary and sufficient conditions for the existence of such decompositions.

**1.5. Theorem:** (Mendelsohn, [25]) $DK_v$ can be decomposed into cyclic triples if and only if $v \equiv 0$ or $1 (\text{mod } 3)$, $v \neq 6$.

Later, Hung and Mendelsohn [23] established the analagous result for transitive triples.

**1.6. Theorem:** (Hung and Mendelsohn, [23]) $DK_v$ can be decomposed into transitive triples if and only if $v \equiv 0$ or $1 (\text{mod } 3)$, $v \neq 1$.

Thus whenever the necessary conditions are satisfied, $DK_v$ can be decomposed into either the cyclic or the transitive triple unless $v = 6$.

The case of decomposing $DK_v$ into oriented 4-cycles is again more complex as there are four possible orientations as shown in Figure 2.

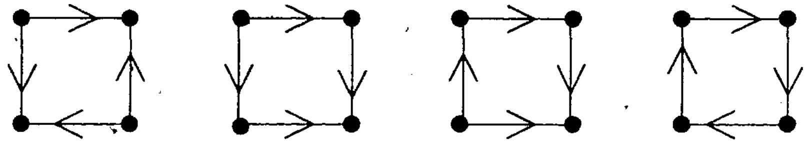



Figure 2

We adopt the notation of [19] in naming these four graphs, denoting them A, B, C, and D respectively, where the later the letter, the longer the longest directed path. A is known as the alternator and D is often called the 4-circuit.

Schönheim [29] and Bermond and Faber [6] independently worked on the problem of decomposing $DK_v$ into D. Schönheim refers to Mendelsohn's generalization of triple systems [25] as his motivation for studying oriented 4-cycle decompositions and in [29] gives necessary and sufficient conditions for such decompositions. Bermond worked on the more general problem of determining the values of $v$ for which $DK_v$ can be decomposed into $k$-circuits, directed $k$-cycles where the longest directed path is of length $k$. In [3] he conjectured that the necessary condition $v(v\text{-}1) \equiv 0(\text{mod } k)$ is also sufficient except for $v$=6, $k$=3; $v$=4=$k$; and $v$=6=$k$. In a joint paper with Faber [6] he developed many results for $k$ even. In particular they resolve the case $k$=4.

**1.7. Theorem:** (Schönheim [29], Bermond and Faber [6]) $DK_v$ can be decomposed into D if and only if $v$>4 and $v \equiv 0$ or 1(mod4).

Necessity is clear since the number of edges must be divisible by 4. If $v$=4 it can be shown by exhaustion that the decomposition does not exist.

Harary, Wallis and Heinrich [19] were the first to discuss the other possible orientations of the 4-cycle.

**1.8. Theorem:** (Harary, Wallis and Heinrich[19])

(a) $A|DK_v$ if and only if $v\equiv 1 \pmod 4$;

(b) $B|DK_v$ if and only if $v\equiv 0$ or $1 \pmod 4$, $v\neq 4$ or 5;

(c) $C|DK_v$ if and only if $v\equiv 0$ or $1 \pmod 4$, $v\neq 5$.

In what follows we focus on resolvable and near-resolvable decompostions of $DK_v$, restricting ourselves to the study of oriented 3- and 4-cycle decompostions. In Chapter 2 we give an overview of work done on resolvable and near-resolvable decompositions of $DK_v$ into the two oriented 3-cycles. In Chapter 3 we discuss resolvable decompositions into the four oriented 4-cycles and establish the following theorem.

**1.9. Theorem:** **a)** $A\nmid_R DK_v$.

**b)** $B|_R DK_v$ if and only if $v\equiv 0 \pmod 4$, $v\neq 4$, with the possible exceptions $v$=20 and $v$=52.

**c)** $C|_R DK_v$ if and only if $v\equiv 0 \pmod 4$.

**d)** $D|_R DK_v$ if and only if $v\equiv 0 \pmod 4$, with the possible exception of $v$=12.

In Chapter 4 we discuss near resolvable decompositions into the four oriented 4-cycles and prove that :

**1.10. Theorem:** a) A and D divide $DK_v$ near resolvably if and only if $v \equiv 1 \pmod 4$.

b) B and C divide $DK_v$ near resolvably if and only if $v \equiv 1 \pmod 4$, $v \neq 5$, with the possible exception that B may not divide $DK_{21}$ near resolvably.

# Chapter - 2 Resolvable and Near-Resolvable Oriented 3-Cycle Decompositions

## §2.1. Definitions and Notation

In addition to the definitions and notation introduced in Chapter 1, the following terms and conventions are used.

Let $C_k$ denote the non-oriented $k$-cycle. In particular, $C_3$ is the non-oriented 3-cycle. The cyclic triple $CT_3$ with vertex-set $\{a, b, c\}$, has arcs $(a,b)$,$(b,c)$ and $(c,a)$; while the transitive triple, $TT_3$, on the same vertex set has arcs $(a,b)$, $(b,c)$ and $(a,c)$. In each case the triple is denoted $(a, b, c)$. In the discussion that follows we use the symbol $X_3$ to denote an oriented 3-cycle.

Given an oriented $k$-cycle C, the oriented cycle obtained by reversing the direction of each arc in C is called the *converse* of C. If C is isomorphic to its converse then we say that C is *self-converse*. In particular, we note that $CT_3$ and $TT_3$ are both self-converse.

Let $K_A$ denote the complete graph with vertex set A and $C_A$ denote a cycle with vertex set A. Let $K(n,m)$ denote the complete multipartite graph with vertex set consisting of $n$ parts of $m$ vertices each, and let $C(n,m)$ be the graph with vertex set consisting of $n$ parts of $m$ vertices each, $X_1, X_2,...X_n$, with $E(C(m,n))=\{xy : x\in X_i$ and $y\in X_{(i+1)(\text{mod } n)}\}$.

## §2.2. Resolvable 3-cycle decompositions of $DK_v$

In 1979, Bermond, Germa and Sotteau [5] established necessary and sufficient conditions for resolvable decompositions of $DK_v$ into $CT_3$ and into $TT_3$.

**2.1. Theorem:** (Bermond, Germa, Sotteau [5]) $DK_v$ admits resolvable decompositions into $TT_3$ and into $CT_3$ if and only if $v \equiv 0 \pmod 3$, $v \neq 6$.

It is clear that for such decomposition to exist we require $v \equiv 0 \pmod 3$ as the number of vertices of $DK_v$ must be a multiple of 3. To see that $v \neq 6$, suppose that either $CT_3|_R DK_6$ or $TT_3|_R DK_6$. Then on deleting the orientations of the arcs we have a resolvable decomposition of $2K_6$ into $C_3$ which contradicts Theorem 1.3.

We will prove the sufficiency of the theorem via a series of Lemmas.

**2.2. Lemma:** (Bermond, Germa, and Sotteau, [5]) When $v \equiv 3 \pmod 6$, $X_3|_R DK_v$.

**Proof:** From Theorem 1.3., we have $C_3|_R K_v$ if $v \equiv 3 \pmod 6$. To each $C_3$, associate an oriented 3-cycle (either $CT_3$ or $TT_3$) and its converse. Thus for each $C_3$-factor of $K_v$,we obtain two $CT_3$-(or $TT_3$-) factors of $DK_v$, giving resolvable decompostions of $DK_v$ as required. ■

We require several lemmas and another Theorem in order to provide resolvable decompositions when $v \equiv 0 \pmod 6$.

**2.3. Lemma:** $X_3|DK_4$.

**Proof:** Let the vertices of $DK_v$ be the four elements of GF(4): $0, 1, x, x^2$ with $x^2 = x+1$. In each case the triples of a decomposition are $\{(\alpha+1, \alpha+x, \alpha+x^2): \alpha \in GF(4)\}$. ■

**2.4. Lemma:** If $X_3|_R DK_v$, then $X_3|_R DK_{4v}$.

**Proof:** Partition the vertices of $DK_{4v}$ into $v$ sets $A_1, A_2, \ldots, A_v$ with $|A_i| = 4$. Denote the vertices of $A_i$ by $\{a_i^\alpha : \alpha \in GF(4)\}$. Let $C_1, C_2, \ldots, C_{v-1}$ be the $X_3$-factors of an $X_3$-factorization of $DK_v$. From $C_1$ we construct seven edge disjoint $X_3$-factors of

$DK_{4v}$ by associating with each $(i,j,k) \in C_1$ the seven sets of triples, each triple isomorphic to $X_3$: $\{(a_i^\alpha, a_j^\alpha, a_k^\alpha), (a_i^{\alpha+1}, a_i^{\alpha+x}, a_i^{\alpha+x^2}), (a_j^{\alpha+1}, a_j^{\alpha+x}, a_j^{\alpha+x^2}),$ $(a_k^{\alpha+1}, a_k^{\alpha+x}, a_k^{\alpha+x^2})\}$, where $\alpha \in GF(4)$; and $\{(a_i^{x^p}, a_j^{x^{p+1}}, a_k^{x^{p+2}}),$ $(a_i^{x^p+1}, a_j^{x^{p+1}+1}, a_k^{x^{p+2}+1}), (a_i^{x^p+x}, a_j^{x^{p+1}+x}, a_k^{x^{p+2}+x}), (a_i^{x^p+x^2}, a_j^{x^{p+1}+x^2}, a_k^{x^{p+2}+x^2})\}$ for $p \in \{1,2,3\}$.

From each $C_l$, $2 \le l \le v-1$ we construct four edge disjoint $X_3$-factors $DK_{4v}$ by associating with each $(i,j,k) \in C_l$, the four sets of triples, each triple isomorphic to $X_3$: $\{(a_i^\alpha, a_j^\alpha, a_k^\alpha), (a_i^{\alpha+1}, a_j^{\alpha+x}, a_k^{\alpha+x^2}), (a_i^{\alpha+x}, a_j^{\alpha+x^2}, a_k^{\alpha+1}), (a_i^{\alpha+x^2}, a_j^{\alpha+1}, a_j^{\alpha+x})\}$, $\alpha \in GF(4)$.

This yields $7+4(v-2) = 4v-1$ $X_3$-factors of $DK_{4v}$ and hence $X_3|_R DK_{4v}$. ■

We state the following three lemmas without proof.

**2.5. Lemma:** (Bermond, Germa, and Sotteau, [5]) $X_3|_R DK_{18}$, $X_3|_R DK_{24}$, $X_3|_R DK_{30}$, and $X_3|_R DK_{42}$.

**2.6. Lemma:** (Bermond, Germa, and Sotteau, [5]) $DK_{A \cup B} - DK_B$, where $|A| = 12$ and $|B| = 6$, can be decomposed into seventeen subgraphs, twelve of which are $X_3$-factors of $DK_{A \cup B}$, and five of which are $X_3$-factors of $DK_A$.

**2.7. Lemma:** (Brouwer, Hanani, and Schrijver, [10]) For $r \geq 4$, $K_4 | K(r,12)$.

We are now in a position to show:

**2.8. Lemma:** (Bermond, Germa, and Sotteau, [5]) When $v \equiv 6 \pmod{12}$, $X_3|_R DK_v$.

**Proof:** Let $v = 12u + 6$. When $u \le 3$, the claim follows from Lemma 2.5. Let $u \geq 4$, and partition the set X of vertices of $DK_v$ as follows: $X = \bigcup_{i=1}^{u} A_i \cup B$, where $A_i = \{a_j^i: 1 \le j \le 12\}$ and $|B| = 6$. By Lemma 2.5, $DK_{A_1 \cup B} \cong DK_{18}$ can be decomposed into $X_3$-factors $C_j^1, 1 \le j \le 17$. By Lemma 2.6, for $i = 2,3,\ldots,u$, $DK_{A_i \cup B} - DK_B$ can be decomposed into precisely twelve $X_3$-factors, $\mathcal{D}_1^i, \mathcal{D}_2^i, \ldots, \mathcal{D}_{12}^i$, of $DK_{A_i \cup B}$, and five $X_3$-factors, $\mathcal{E}_{13}^i, \mathcal{E}_{14}^i, \ldots, \mathcal{E}_{17}^i$, of $DK_{A_i}$. From Lemma 2.7, the graph $DK(u,12)$ with vertex set $\bigcup_{i=1}^{u} A_i$ where the $A_i$ are the independent sets, has a $DK_4$-decomposition. Let $\mathcal{S}$ be the set of all $DK_4$ in such a decomposition.

Let $a_j^i \in A_i$ and let $\mathcal{P}_j^i = \{DK_4 : DK_4 \in \mathcal{S}$ and $a_j^i \in V(DK_4)\}$. By Lemma 2.3, each of these $DK_4$ has an $X_3$-decomposition. Then let $\mathcal{F}_j^i = \{X_3 : DK_4 \in \mathcal{P}_j^i$, $X_3$ is an oriented 3-cycle in the decomposition of $DK_4$, and $a_j^i \notin X_3\}$. Clearly $\mathcal{F}_j^i$ is an $X_3$-factor of $DK(u\text{-}1, 12)$ with vertex set $V(DK(u,12)) - A_i$.

We obtain an $X_3$-factorization of $DK_{12u+6}$ with the following $12u+5$ parallel classes: $\mathcal{C}_j^1 \cup \mathcal{F}_j^1$ for $j=1,2,\ldots,12$; $\mathcal{C}_j^1 \cup \bigcup_{i=2}^{u} \mathcal{E}_j^i$ for $j = 13,\ldots,17$; and $\mathcal{D}_j^i \cup \mathcal{F}_j^i$ for $j=1,2,\ldots,12$, $2 \leq i \leq u$. Hence $X_3|_R DK_v$ when $v \equiv 6 \pmod{12}$. ■

**2.9. Lemma:** (Bermond, Germa, and Sotteau, [5]) If $v \equiv 0 \pmod{12}$, then $X_3|_R DK_v$.

**Proof:** Let $v=4^\alpha q$ where $q \equiv 0 \pmod 3$ but $q \not\equiv 0 \pmod{12}$. Since $X_3|_R DK_q$, for $q \neq 6$, (Lemmas 2.2 and 2.8) by repeatedly applying Lemma 2.4, we see that $X_3|_R DK_v$, except when $q=6$. When $q=6$, let $v=4^\alpha(6)=4^{\alpha-1}(24)$ and since $X_3|_R DK_{24}$, by Lemma 2.5, again on repeatedly applying Lemma 2.4, we find $X_3|_R DK_v$, which completes the proof. ■

The techniques used in the proof of this theorem are very useful in the following chapter on resolvable oriented 4-cycle decompositions.

## §2.3 Near-Resolvable Oriented 3-Cycle Decompositions

In 1981, Bennett and Sotteau [8] addressed the question of near-resolvable decompositions of $DK_v$ into the oriented 3-cycles.

**2.10. Theorem:** (Bennett and Sotteau [8]) $DK_v$ admits a near-resolvable decomposition into $X_3$ if and only if $v \equiv 1 \pmod 3$.

Clearly $v \equiv 1 \pmod 3$ is necessary since each near $X_3$-factor consists of oriented triples and an isolated vertex of $DK_v$. Recall from Lemma 2.3 that $X_3|_{NR}DK_4$.

In order to establish sufficiency we require a series of lemmas. Before we continue, we remind the reader of the definition of pairwise balanced designs.

A pairwise balanced design PBD($v$,I,$\lambda$) is a collection of $i$-subsets, $i \in$ I, called blocks, of a $v$-set such that each pair of elements in the $v$-set occurs in exactly $\lambda$ blocks. In particular, we observe that if $K_v$ has a decomposition into H-factors where H is the edge-disjoint union of complete graphs, with orders in I, then there exists a PBD($v$,I,$\lambda$) and conversely.

**2.11. Lemma:** $X_3|_{NR}DK_7$.

**Proof:** Let the vertices of $DK_7$ be labelled by the elements of $Z_7$ (the additive group of residues modulo 7). The seven parallel classes of a near-resolvable decomposition of $DK_7$ into $X_3$ are $\{i, (i+1, i+2, i+4), (i+6, i+5, i+3)\}$, $i \in Z_7$. ■

**2.12. Lemma:** $X_3|_{NR}DK_{10}$.

**Proof:** Let the vertices of $DK_{10}$ be labelled by the elements of $Z_{10}$. The ten parallel classes of a near-resolvable decomposition of $DK_{10}$ into $CT_3$ are:
{0,(1,2,3),(4,7,8), (5,9,6)}, {1,(2,6,0), (3,8,7), (4,9,5)}, {2, (1,9,7), (3,5,8), (4,0,6)}, {3,(1,5,6), (2,0,7), (4,8,9)}, {4,(1,7,5), (2,8,6), (3,9,0)}, {5, (1,6,8), (2,7,9), (3,0,4)},

{6,(1,3,4), (2,9,8), (5,7,0)}, {7, (1,8,0), (2,4,5), (3,6,9)}, {8, (1,0,9), (2,5,3), (4,6,7)}, {9,(1,4,2), (3,7,6), (5,0,8)}. The ten parallel classes of a near-resolvable decompositon of $DK_{10}$ into $TT_3$ are:

{0,(1,2,3),(8,7,4), (9,6,5)}, {1,(0,6,2), (7,8,3), (5,9,4)}, {2, (9,7,1), (3,5,8), (6,4,0)}, {3,(5,1,6), (7,2,0), (4,9,8)}, {4,(1,7,5), (2,8,6), (0,3,9)}, {5, (6,1,8), (2,7,9), (3,0,4)}, {6, (4,3,1), (8,9,2), (5,0,7)}, {7, (0,8,1), (4,2,5), (6,9,3)}, {8, (1,9,0), (5,3,2), (4,6,7)}, {9,(2,1,4), (3,7,6), (8,0,5)}. ■

**2.13. Lemma:** $X_3|_{NR}DK_{19}$.

**Proof:** Let the vertices of $DK_{19}$ be labelled by the elements of $Z_{19}$. The nineteen parallel classes of a near-resolvable $X_3$-decompostion of $DK_{19}$ are $\{i, (i+1, i+7, i+11), (i+2, i+14, i+3), (i+4, i+9, i+6), (i+18, i+12, i+8), (i+17, i+5, i+16), (i+15, i+10, i+13)\}$, $i \in Z_{19}$. ■

For the remaining cases, the next lemma is the key to showing sufficiency. Note that it is much like the method used in Lemma 2.8.

**2.14. Lemma:** If there exists a PBD($v$,I,1) and for every $i \in$ I, $X_3|_{NR}DK_i$, then $X_3|_{NR}DK_v$.

**Proof:** The PBD($v$,I,1) gives us a decomposition of $K_v$ into complete subgraphs $K_k$, $k \in$ I, and hence a decomposition of $DK_v$ into $DK_k$, $k \in$ I. For every $x$ of $V(DK_v)$ consider those $DK_k$ which contain $x$. These subgraphs have only the vertex $x$ in common and between them contain all vertices of $DK_v$. Since $X_3|_{NR}DK_k$, in each of these subgraphs we have a near $X_3$-factor covering all vertices but $x$. Together these give us a near $X_3$-factor of $DK_v$ which misses vertex $x$. All such near $X_3$-factors are edge-disjoint and thus yield $X_3|_{NR}DK_v$. ■

We are now ready to prove Theorem 2.10.

**Proof of Theorem 2.10.:**

Let us consider two cases.

Case 1: Let $v \equiv 1$ or 4 (mod 12). Hanani [18] has shown that there exists a PBD($v$,{4},1) if and only if $v \equiv 1$ or 4 (mod 12). Hence, $K_4 | K_v$ if $v \equiv 1$ or 4 (mod 12). Then from Lemmas 2.14 and 2.3, it follows that $X3|_{NR}DK_v$ when $v \equiv 1$ or 4 (mod 12).

Case 2: Let $v \equiv 7$ or 10 (mod 12). Brouwer [11] showed the existence of a PBD($v$,{4,7},1) with a unique block of size 7 if and only if $v \equiv 7$ or 10 (mod 12), $v \neq 10$ or 19. By applying Lemmas 2.3, 2.11, and 2.14, it follows that $X3|_{NR}DK_v$ when $v \equiv 7$ or 10 (mod 12), and $v \neq 10$ or 19. Since the factorizations for $v=10$ and $v=19$ have been shown in Lemmas 2.12 and 2.13 respectively, our proof is complete.

Hence $X3|_{NR}DK_v$ if and only if $v \equiv 1 (\text{mod} 3)$. ■

It has been shown by Colbourn and Colbourn [15] that given any decomposition of $2K_v$ into 3-cycles, the 3-cycles can be oriented to give a decomposition of $DK_v$ into transitive triples. Together with Hanani's result, stated in Theorem 1.3, this provides another proof that $TT3|_R DK_v$ if and only if $v \equiv 0 (\text{mod } 3)$, $v \neq 6$, and $TT3|_{NR}DK_v$ if and only if $v \equiv 1 (\text{mod } 3)$.

This concludes the work which has been done on resolvable and near-resolvable oriented 3-cycle decompositions. We now move on to discuss resolvable and near-resolvable oriented 4-cycle decompositions.

## Chapter - 3 Resolvable Oriented 4-Cycle Decompositions

In [19], Harary, Wallis and Heinrich completely solved the problem of when $DK_v$ could be decomposed into each of the four oriented 4-cycles. Their constructions did not generally result in resolvable decompositions, leaving open the question of resolvable decompositions of $DK_v$ into oriented 4-cycles. (We use the symbol $X_4$ to stand for any one of the four oriented 4-cycles.)

To begin we note that if $X_4|_R DK_v$, then $v \equiv 0 \pmod 4$, since each parallel class is made up of 4-cycles. From now on, we let $v = 4n$, where $n$ is a positive integer.

In this Chapter we establish the following theorem.

**3.1. Theorem:**

a) $A \nmid_R DK_{4n}$.

b) $B|_R DK_{4n}$ for all $n$, $n \neq 1$ except possibly when $n=5$ and $n=13$.

c) $C|_R DK_{4n}$ for all $n$.

d) $D|_R DK_{4n}$ for all even $n$.

Then to complete our discussion of resolvable decompositions of $DK_{4n}$ into the oriented 4-cycles, we discuss the following result of Bennett and Zhu.

**3.2. Theorem:** (Bennett and Zhu [9]) $D|_R DK_{4n}$ when $n$ is odd, $n \neq 1$, except possibly when $n = 3$.

Combining Theorems 3.1 and 3.2 gives Theorem 1.9 as stated in Chapter 1.

**3.3. Theorem:** There is no resolvable decomposition of $DK_{4n}$ into A.

**Proof:** Observe that each vertex of A has even in-degree and even out-degree. In $DK_{4n}$, each vertex has odd in-degree, and odd out-degree. Hence it is impossible for A to divide $DK_{4n}$, and in particular A cannot divide $DK_{4n}$ resolvably. ■

**3.4. Lemma:** If $C_4|G$ then $X_4|DG$.

**Proof:** Each oriented 4-cycle is self-converse. So each oriented 4-cycle divides $DC_4$. Hence the result follows. ■

**3.5. Corollary:** $X_4|_R DK_{4,4}$.

**Proof:** This follows immediately from Lemma 3.4, as $K_{4,4}$ has a $C_4$-factorization as shown if Figure 3. ■

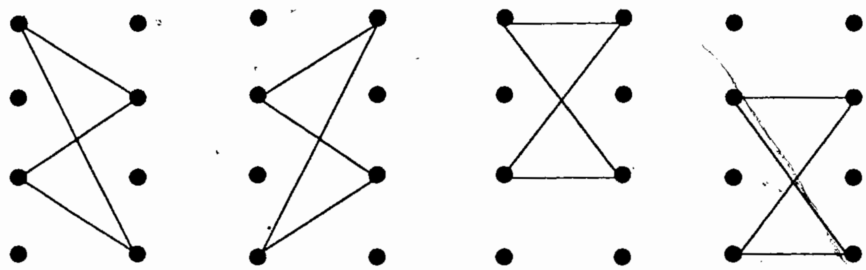


Figure 3

In this and Chapter 4, the following notation is useful. Let G and H be graphs. Then G∗H is the graph with vertex set V(G)×V(H) and edge set $E(G*H)=\{\{(x_1,x_2)(y_1,y_2)\}: x_1y_1\in E(G)$ and $x_2y_2\in E(H)$ or $x_2=y_2$ and $x_1y_1\in E(G)\}$; and G•H is the graph with vertex set V(G)×V(H) and edge set $E(G\bullet H)=\{\{(x_1,x_2)(y_1,y_2)\}: x_1y_1\in E(G)$ and $x_2y_2\in E(H)$, or $x_1=y_1$ and $x_2y_2\in E(H)$, or $x_2=y_2$ and $x_1y_1\in E(G)\}$. We use $(n)G$ to denote $n$ vertex disjoint copies of the graph G.

**3.6 Lemma:** Let $G = H*K_2$ have $2m$ vertices, where $m$ is even, with vertex set $X = S \cup T$ where $S=\{s_1,s_2,...,s_m\}$, and $T=\{t_1,t_2,...,t_m\}$ and the two copies of H are on the vertex sets $S$ and $T$ respectively. Then each 1-factor F of H induces a $C_4$-factor of G.

**Proof:** Let F be a 1-factor of H. Without loss of generality let $F=\{s_1s_2, s_3s_4,\ldots,s_{m-1}s_m\}$. Then the resulting $C_4$-factor is $\{(s_{2i-1},s_{2i},t_{2i-1},t_{2i}): 1 \le i \le \frac{m}{2}\}$. ■

From Lemma 3.6 we obtain the following corollary.

**3.7. Corollary:** If H has a 1-factorization, then $X_4|_R D(H*K_2)$.

We now determine exactly when $DK_4$ can be resolvably decomposed into the oriented 4-cycles B, C, and D.

**3.8. Lemma:** $C|_R D(K_{4n} - F)$.

**Proof:** Consider $K_{4n} - F$ on vertex set $X = S \cup T$ where $S=\{s_1,s_2,\ldots,s_{2n}\}$ and $T=\{t_1,t_2,\ldots,t_{2n}\}$, so that $F = \{s_it_i: 1 \le i \le 2n\}$. Observe that $K_{4n} - F \cong K_{2n}*K_2$. Then from Corollary 3.7, since $K_{2n}$ has a 1-factorization, it follows that $C|_R D(K_{4n} - F)$. ■

Note that if $H = \{H_1,H_2,\ldots,H_n\}$ is a C-factor in the above C-factorization of $D(K_{4n} - F)$, then so too is $H' = \{H_1',H_2',\ldots,H_n'\}$, where $H_i'$ is the converse of $H_i$.

**3.9. Lemma :** $C|_R DK_4$

**Proof:** The decomposition is as shown in Figure 4.

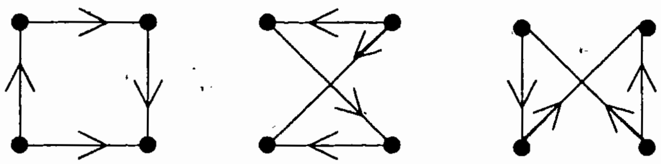


Figure 4

■

**3.10. Theorem:** $C|_R DK_{4n}$.

**Proof:** Let $\mathcal{H}$ be the set of C-factors in the C-factorization of $D(K_{4n} - F)$ as described in Lemma 3.8. Choose any C-factor $H \in \mathcal{H}$ together with its converse H'.

Then $H \cup H' \cup D(F) \cong (n)DK_4$. From Lemma 3.9, $C|_R DK_4$ and hence we have a C-factorization of $H \cup H' \cup D(F)$ which when combined with $\mathcal{H} - \{H, H'\}$ yields a C-factorization of $DK_{4n}$. Therefore $C|_R DK_{4n}$. ■

We now turn our attention to the oriented 4-cycles B and D. In view of Lemma 3.11, constructions in these cases will be somewhat more difficult.

**3.11. Lemma:** Neither the oriented cycle B nor the oriented cycle D divide $DK_4$ resolvably.

**Proof:** It can be shown by exhaustion that $DK_4$ cannot be decomposed into B or D. Hence $DK_4$ cannot be resolvably decomposed into B or D. ■

**3.12. Lemma:** $B|_R DK_8$ and $D|_R DK_8$.

**Proof:** Observe that the graphs X, Y, and Z as shown in Figure 5 partition the edges of $K_8$.

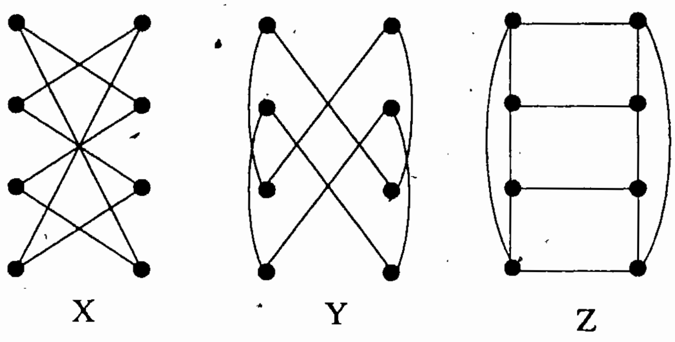
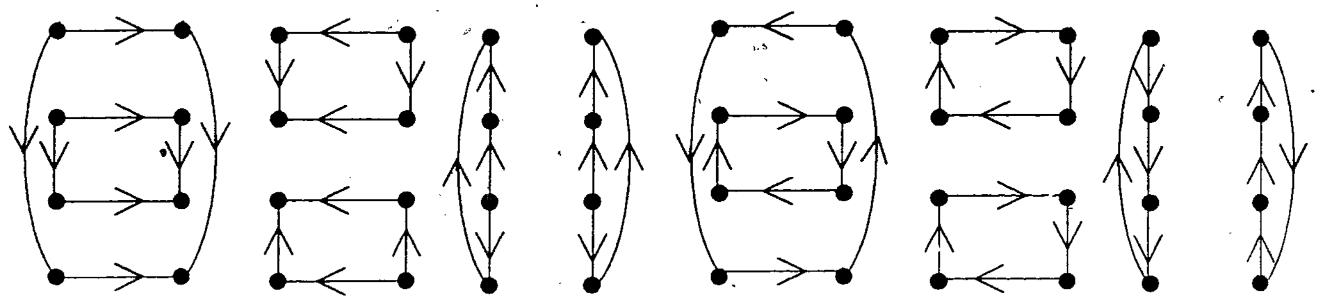



Figure 5

Each of X and Y determine two B- (or D-) factors of $DK_8$ in the obvious way. The graph Z is the cube $Q_3$. Since B and D both divide $DQ_3$ resolvably, as shown in Figure 6, it follows that $B|_R DK_8$ and $D|_R DK_8$. ■



B-factorization D-factorization

Figure 6

**3.13. Lemma:** Both $B|_R DK(n,8)$ and $D|_R DK(n,8)$.

**Proof:** Let the vertex set of $DK(n,8)$ be $S=\bigcup_{i=1}^{n} S_i$ where $|S_i| = 8$ and where each $S_i$ is an independent set. Further, partition each $S_i$ as $S_i= T_i \cup T_{i+n}$ where $|T_i| = |T_{i+n}| = 4$, for $i= 1,2,...,n$. Consider $K_{2n} - F$ with vertex set $V=\{1,2,...,2n\}$ and $F=\{i(i+n): i= 1,2,...,n\}$, and let vertex $i$ correspond to $T_i$ for $i= 1,2,...,2n$. Observe that for any pair $i$ and $j$, the vertex set $T_i \cup T_j$ induces a subgraph, $DK_{4,4}$, unless $i \equiv j (\mathrm{mod}\ n)$, in which case the induced subgraph is $\overline{K_8}$. It is well known that $K_{2n} - F$ has a 1-factorization. Each 1-factor of $K_{2n} - F$ corresponds to a $DK_{4,4}$-factor of $DK(n,8)$. Since $B|_R DK_{4,4}$ and $D|_R DK_{4,4}$, from Lemma 3.5, each factor gives four B-(or D-) factors of $DK(n,8)$. Therefore $B|_R DK(n,8)$ and $D|_R DK(n,8)$. ■

**3.14 Theorem:** When $n$ is even, $B|_R DK_{4n}$ and $D|_R DK_{4n}$ .

**Proof:** Let $n=2k$. Then $DK_{4n}=DK_{8k}=(k)DK_8 \cup DK(k,8)$. Since both $DK_8$ and $DK(k,8)$ have a B-factorization and a D-factorization (from Lemmas 3.12 and 3.13) it follows that $B|_R DK_{4n}$ and $D|_R DK_{4n}$. ■

We next consider the case when $n$ is odd, considering B and D separately.

**3.15. Lemma:** $B|_R DK_{12}$.

**Proof:** Let the vertex set of $DK_{12}$ be $\{0,1,2,...,10,\infty\}$. Then the eleven

B-factors of a resolvable decomposition of $DK_{12}$ are: $\{(i+5, i+8, i+1, i+7), (i+10, i+6, i+4, i+9), (i+2, \infty, i, i+3)\}$ for $i = 0,1,2,...,10$ and addition is modulo 11. ■

For the next theorem we require the following lemma.

**3.16. Lemma:** If $t \geq 2$, then $K_{2t+2} - F$ has a $C_{2t}$-decomposition.

**Proof:** Let the vertex set of $K_{2t+2}$ be $\{0,1,2,...,2t-1,\infty_1,\infty_2\}$ and let $F = \{i(t+i): i=0,1,2,...t-1\} \cup \{\infty_1\infty_2\}$. When $t$ is odd the $C_{2t}$-decomposition is given by: $\{(2t-1+i,\ 1+i,\ 2t-2+i,\ 2+i,..., \left(\frac{t-1}{2}\right)+i, \infty_1,\ 2t-\left(\frac{t+1}{2}\right)+i,\ \frac{t+1}{2}+i,\ 2t-\left(\frac{t+3}{2}\right)+i, \left(\frac{t+5}{2}\right)+i..., t-1+i, \infty_2): i= 0,1,2,...,t-1\} \cup \{(0,1,2,...,2n-1)\}$. When $t$ is even, the $C_{2t}$-decomposition is given by:

$\{(2t-1+i,\ 1+i,\ 2t-2+i,\ 2+i,...,\ 2t-\left(\frac{t}{2}\right)+i, \infty_1, \left(\frac{t}{2}\right)+i,\ 2t-(\left(\frac{t}{2}\right)+1)+i, \left(\frac{t}{2}\right)+1+i, 2t-(\left(\frac{t}{2}\right)+2)+i,...,t-1+i,\infty_2): i= 0,1,2,...,t-1\} \cup \{(0,1,2,...,2t-1)\}$. Observe that each $2t$-cycle misses the endpoints of a distinct edge of the 1-factor. ■

**3.17. Corollary:** If $t \geq 2$, then $K_{2t+3}$ has a $(C_{2t} \cup C_3)$-factorization.

**3.18. Theorem:** When $n \equiv 3 (\mathrm{mod}\ 4)$, $B|_R DK_{4n}$.

**Proof:** Let $n = 4t + 3$. Observe that $K_{4n} \cong K_{2n} \bullet K_2$.

Suppose we have a decomposition of $K_{2n}$ into edge-disjoint subgraphs S, $P_1, P_2, F_1, F_2, ..., F_{8t}$, such that: S is a factor of $K_{2n}$ consisting of one copy of $K_6$ (denoted $S_0$) and $2t$ copies of $K_4$ (denoted $S_i$, $i = 1,2,...,2t$); each of $P_1$ and $P_2$ is a set of $4t$ independent edges covering $V(K_{2n}) - V(S_0)$; and $F_1, F_2, ..., F_{8t}$, are 1-factors of $K_{2n}$. Then $K_{4n} \cong (S \bullet K_2) \cup (\bigcup_{i=1}^{2} P_i * K_2) \cup (\bigcup_{i=1}^{8t} F_i * K_2)$. Each of $F_i * K_2$, for $i = 1,2,..., 8t$, is a $C_4$-factor of $K_{4n}$. So for each $i = 1,2,..., 8t$, $D(F_i * K_2)$ can be decomposed into two B-factors which are also B-factors of $DK_{4n}$. Denote these by $F_i^{(1)}$ and $F_i^{(2)}$.

This leaves $(S \bullet K_2) \cup (P_1 * K_2) \cup (P_2 * K_2)$. Now $(S \bullet K_2) \cong (\bigcup_{i=0}^{2t} S_i) \bullet K_2 = \bigcup_{i=0}^{2t} (S_i \bullet K_2)$. Note that $D(S_i \bullet K_2) \cong DK_{12}$ which has a decomposition into eleven B-factors by Lemma 3.15. Denote these by $S_0^{(1)}$,

$S_0^{(2)},\ldots,S_0^{(11)}$. For $i=1,2,\ldots,2t$, $D(S_i \bullet K_2) \cong DK_8$ which has a decomposition ito seven B-factors, $S_i^{(1)}, S_i^{(2)}, \ldots S_i^{(7)}$. Then for each $j= 1,2,\ldots,7$, $T_j = \bigcup_{i=0}^{2t} S_i^{(j)}$ is a B-factor of $DK_{4n}$. Now $P_1 * K_2$ and $P_2 * K_2$ are each a set of $4t$ vertex-disjoint 4-cycles on $V(\bigcup_{i=1}^{2t} S_i \bullet K_2)$. Hence for $j= 1,2$, $D(P_j * K_2)$ can be decomposed into two B-factors on $V(\bigcup_{i=1}^{2t} S_i \bullet K_2)$, which we denote $P_j^{(1)}$ and $P_j^{(2)}$. Then we obtain f~~our additional~~ B-factors of $DK_{4n}$. These are $S_0^{(8)} \cup P_1^{(1)}$, $S_0^{(9)} \cup P_1^{(2)}$, $S_0^{(10)} \cup P_2^{(1)}$, and $S_0^{(11)} \cup P_2^{(2)}$. Thus we have $2(8t) + 7 + 4 = 16t + 11$ B-factors as required.

Therefore, to complete the proof of the theorem, all we need is to provide such a decomposition of $K_{2n}$.

Without loss of generality we can specify the factor S as described. We must then choose $P_1$ and $P_2$, two sets of $4t$ independent edges covering $V(K_{2n}) - V(S_0)$, so that $G \cong K_{2n} - (S \cup P_1 \cup P_2)$ has a 1-factorization. Arbitrarily pair the $S_i$, $i=1,2,\ldots,2t$; say as $\{(S_i, S_{i+t}): i=1,2,\ldots,2t\}$. Let $L_i^{(1)}$ and $L_i^{(2)}$ be two edge-disjoint 1-factors of the $K_{4,4}$, $K_{S_i,S_{i+t}}$, for $i=1,2,\ldots, 2t$. Let $P_1 = \bigcup_{i=1}^{t} L_i^{(1)}$ and let $P_2 = \bigcup_{i=1}^{t} L_i^{(2)}$. We claim that $G \cong K_{2n} - (S \cup P_1 \cup P_2)$ has a 1-factorization. Consider $K_{2t+1}$ with vertex set $\{v_0, v_1, v_2, \ldots, v_{2t}\}$ where $v_i$ corresponds to $S_i$ for $i=0,1,2,\ldots,2t$. If $t \geq 3$, from Corollary 3.17, $K_{2t+1}$ can be decomposed into $t$ factors where each factor consists of a $(2t-2)$-cycle and a 3-cycle. In accord with that construction, we can denote these factors by $C^{(i)} \cup C_{\{v_0, v_i, v_{i+t}\}}$ for $i=1,2,\ldots,t$, where $C^{(i)}$ is a $(2t-2)$-cycle on $V(K_{2t+1} - \{v_0, v_i, v_{i+t}\})$. For each factor $C^{(i)} \cup C_{\{v_0, v_i, v_{i+t}\}}$, $i= 1,2,\ldots,t$, we obtain eight 1-factors of G as follows. In G, the cycle $C^{(i)}$ corresponds to a $C(2t-2, 4)$ which has a 1-factorization made up of eight 1-factors. In G, $C_{\{v_0, v_i, v_{i+t}\}}$ corresponds to the graph H shown in Figure 7. Clearly $H \cong H_1 \cup H_2$, where $H_1 \cong H_2$ and $H_1$ is as shown in Figure 8, has a 1-factorization made up of eight 1-factors. Therefore the subgraph of G corresponding to $C^{(i)} \cup C_{\{v_0, v_i, v_{i+t}\}}$ has a 1-factorization and thus so does G.

Two edge disjoint 1-factors

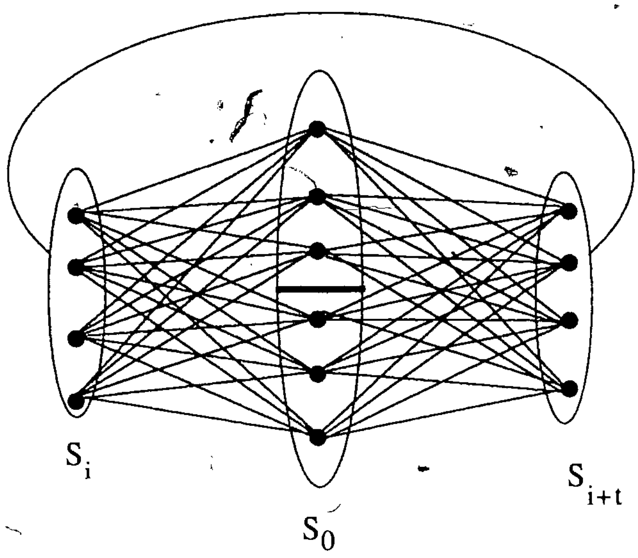



Figure 7

One 1-factor

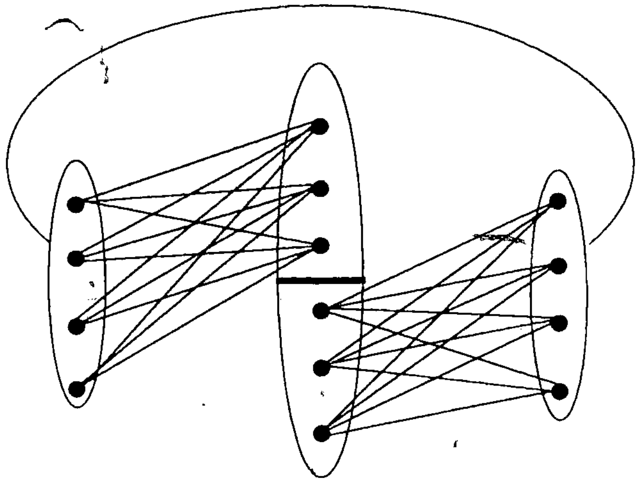


Figure 8

This completes the proof for $t \geq 3$. In Lemma 3.15 we showed that $B|_R DK_{12}$. When $t = 1$ or 2, choose S, $P_1$, $P_2$, and the $F_i$ as described above. When $t = 1$, $G \cong H$ so we are done, and when $t = 2$ we factor G as shown in Figure 9. Therefore $B|_R DK_{4n}$ when $n \equiv 3 (\text{mod } 4)$. ■

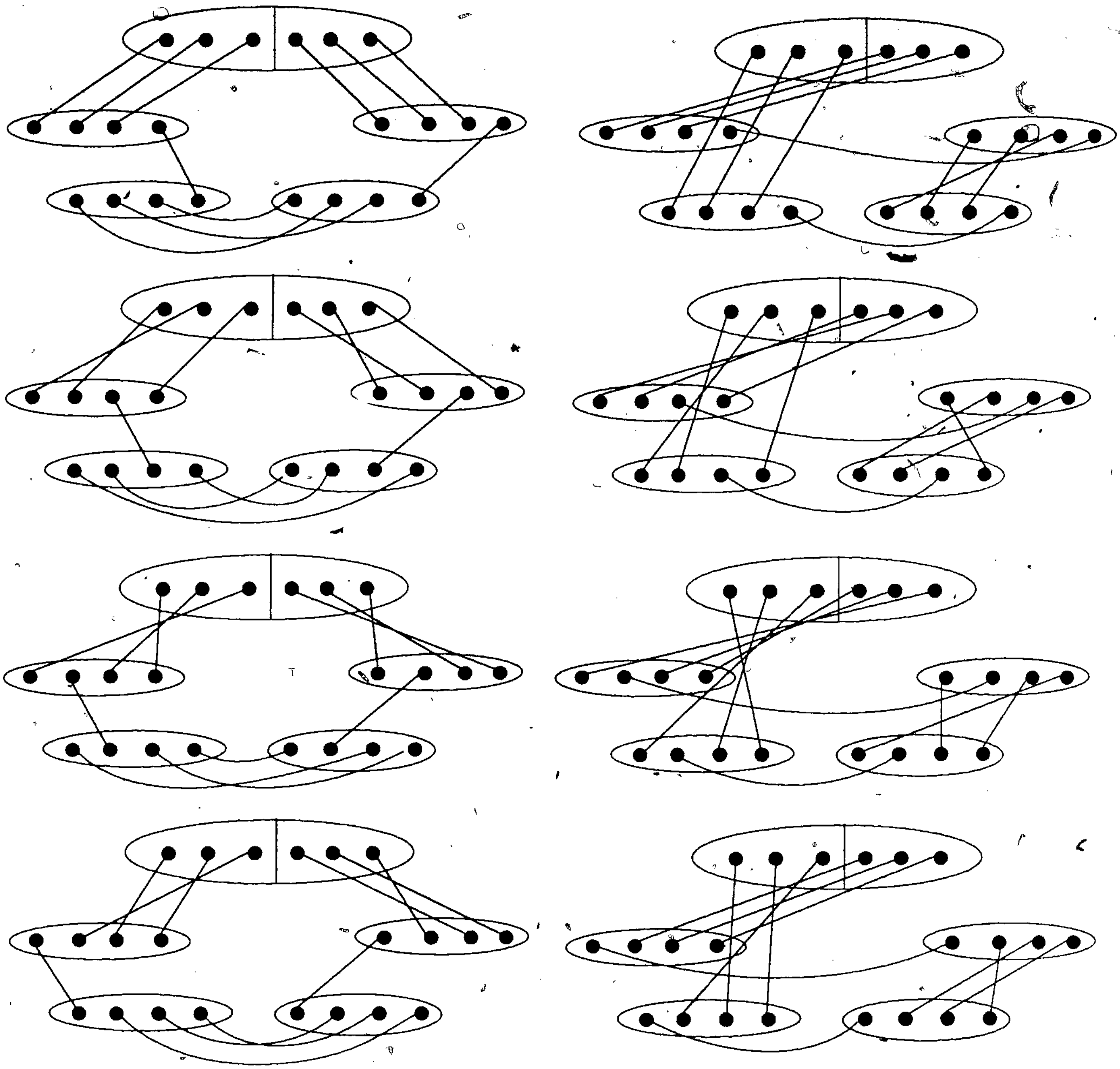


Figure 9

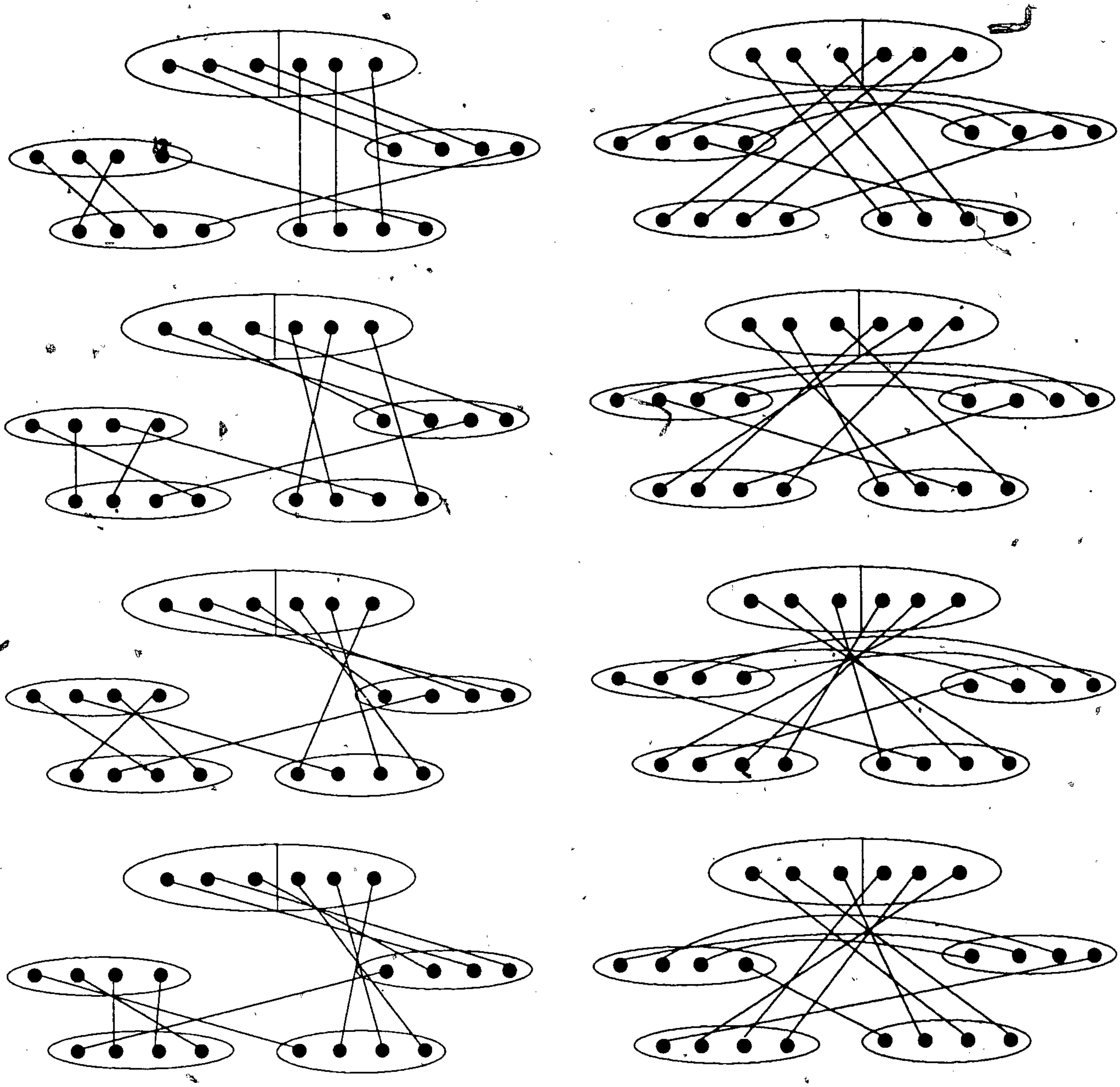


Figure 9, continued

When $n \equiv 1 \pmod 4$, we could follow the same proof as for Theorem 3.17, except that no simple construction for the 1-factorization of G has been found. Thus we appeal to the following result of Chetwynd and Hilton [13] to prove that a 1-factorization of G does indeed exist.

**3.19. Theorem:** (Chetwynd and Hilton [13]) A $k$-regular graph G with an even number of vertices has a 1-factorization whenever $k \geq \frac{1}{2}(\sqrt{7} - 1)|V(G)|$.

**3.20. Theorem:** $B|_R DK_{4n}$ when $n \equiv 1 \pmod 4$, $n \geq 17$.

**Proof:** In the proof of Theorem 3.16 we showed that $B|_R DK_{4n}$ if the graph G, as described, has a 1-factorization. Since $|V(G)|=2n$ and G is regular of degree $2n-6$. Theorem 3.17 guarantees that G has a 1-factorization whenever $2n-6 \geq \frac{1}{2}(\sqrt{7}-1)(2n)$. This holds provided $n \geq 17$. ■

In addition, for the special case when $n=9$ we have the following result.

**3.21. Lemma:** $B|_R DK_{36}$.

**Proof:** Let $DK_{36} \cong D(K_{18} \bullet K_2)$. Partition the vertex set of $K_{18}$ into sets $S_1$, $S_2$, and $S_3$, where $|S_i| = 6$. Then $K_{18} \cong \bigcup_{i=1}^{3} K_{S_i} \cup K_{S_1,S_2,S_3}$ and $DK_{36} \cong D(\bigcup_{i=1}^{3} K_{S_i} \bullet K_2 \cup K_{S_1,S_2,S_3} * K_2) \cong \bigcup_{i=1}^{3} D(K_{S_i} \bullet K_2) \cup D(K_{S_1,S_2,S_3} * K_2)$. Now $\bigcup_{i=1}^{3} D(K_{S_i} \bullet K_2) \cong (3)DK_{12}$, and since $DK_{12}$ can be decomposed into eleven B-factors by Lemma 3.15, $\bigcup_{i=1}^{3} D(K_{S_i} \bullet K_2)$ can be decomposed into eleven B-factors of $DK_{36}$. By Corollary 3.7, if $K_{S_1,S_2,S_3} \cong K_{6,6,6}$ has a 1-factorization, then $D(K_{S_1,S_2,S_3} * K_2)$ has a B-factorization. We claim that such a 1-factorization exists and although it has been shown elsewhere, for completeness we include a proof here.

Let $S_i = \bigcup_{j=1}^{2} S_i^{(j)}$ for $i=1,2,3$, where $|S_i^{(j)}| = 3$. Consider $K_6 - F$ with vertex set $\{v_1^{(1)}, v_1^{(2)}, v_2^{(1)}, v_2^{(2)}, v_3^{(1)}, v_3^{(2)}\}$ where $F = \{v_i^{(1)} v_i^{(2)} : i = 1,2,3\}$. Let $S_i^{(j)}$ correspond to $v_i^{(j)}$ for $i = 1,2,3$, $j=1,2$. $K_6 - F$ has a 1-factorization. This 1-factorization corresponds to an R-factorization of $K_{6,6,6}$ where $R \cong (3)K_{3,3}$. Clearly $K_{3,3}$ has a 1-factorization into three 1-factors and hence $(3)K_{3,3}$ has a 1-factorization into three 1-factors of $K_{6,6,6}$.

Therefore $D(K_{S_1,S_2,S_3} * K_2)$ has a B-factorization and it follows that $B|_R DK_{36}$.

■

This theorem still leaves unresolved the question of the existence of resolvable B-decompositions of $DK_{20}$ and $DK_{52}$, as well as the existence of resolvable

D-decompositions of $DK_{4n}$ when $n$ is odd. The latter question is answered by Bennett and Zhu [9]. In their study of resolvable Mendelsohn designs, they have established the following theorem.

**3.22. Theorem:** (Bennett and Zhu [9]) A $(4n,4,1)$-resolvable Mendelsohn design exists for all $n$ except possibly when $n$=3.

A $(4n,4,1)$-resolvable Mendelsohn design is equivalent to a resolvable D-decomposition of $DK_{4n}$. Hence resolvable decompositions of $DK_{4n}$ exist when $n$ is odd.

The proof of Theorem 3.1 follows from the above theorems and lemmas.

**Proof of Theorem 3.1:**

(a) See Theorem 3.3.

(b) See Theorems 3.14, 3.18, 3.20 and Lemma 3.21.

(c) See Theorem 3.10.

(d) See Theorems 3.14 and 3.22. ■

## Chapter 4 - Near-Resolvable Oriented 4-cycle Decompositions

We now turn to near-resolvable oriented 4-cycle decompositions of $DK_v$. Since each parallel class of such a decomposition omits exactly one vertex of $DK_v$, it is clear that $v \equiv 1 \pmod 4$ is a necessary condition for the decomposition to exist. In what follows we let $v=4n+1$ and determine the values of $n$ for which $DK_{4n+1}$ has a near-resolvable decomposition into each of the four oriented 4-cycles. Recall that the oriented 4-cycle A with vertex set $\{x,y,z,w\}$ has arcs $(x,y)$, $(z,y)$, $(z,w)$, and $(x,w)$; while B has arcs $(x,y)$, $(y,z)$, $(x,w)$, and $(w,z)$; C has arcs $(x,y)$, $(y,z)$, $(z,w)$, and $(x,w)$; and D has arcs $(x,y)$, $(y,z)$, $(z,w)$, and $(w,x)$. Also recall that $X_4$ is used to represent any one of the four oriented 4-cycles.

**4.1. Lemma:** $X_4|_{NR}DK_9$.

**Proof:** Let the vertices of $DK_9$ be labelled by the elements of $Z_9$. The nine near $X_4$-factors of $DK_9$ are $\{i, (i+1, i+5, i+2, i+3), (i+8, i+4, i+7, i+6)\}$. ■

**4.2. Lemma:** $X_4|_{NR}DK_{17}$.

**Proof:** Let the vertices of $DK_{17}$ be labelled by the elements of $Z_{17}$. The seventeen near $X_4$-factors of $DK_{17}$ are $\{i, (i+1, i+9, i+14, i+7), (i+2, i+6, i+4, i+5), (i+16, i+8, i+3, i+10), (i+15, i+11, i+13, i+12)\}$. ■

**4.3. Lemma:** $C(2k,4)$ has a $C_4$-factorization.

**Proof:** Since $C_{2k}$ has a 1-factorization into two 1-factors, $F_1$ and $F_2$ then $C(2k,4)$ has a $(k)K_{4,4}$-factorization. Then since $K_{4,4}$ can be decomposed into two $C_4$-factors, $C(2k,4)$ has a $C_4$-factorization. ■

**4.4. Corollary:** $X_4|_R DC(2k,4)$.

**4.5. Theorem:** $X_4|_{NR}DK_{4n+1}$ when $n$ is even.

**Proof:** Let $n=2k$. When $k \leq 2$, suitable factorizations are given in Lemmas 4.1

and 4.2. So assume $k \geq 3$. Partition the vertex set X of $DK_{4n+1}$ as $X = \bigcup_{i=1}^{2k} S_i \cup \{\infty\}$, where $|S_i| = 4$, $1 \leq i \leq 2k$. Consider $K_{2k+1}$ with vertex set $\{v_0, v_1, v_2, \ldots, v_{2k}\}$ and associate $v_0$ with vertex $\infty$ of $DK_{4n+1}$, and $v_i$ with $S_i$, for $i = 1,2,\ldots,2k$. From Corollary 3.17, $K_{2k+1}$ can be decomposed into $k$ factors, $L_1, L_2, \ldots, L_k$, so that each factor $L_i$ consists of a $(2k-2)$-cycle, $C^{(i)}$, where $V(C^{(i)}) = V(K_{2k+1}) - \{v_0, v_i, v_{i+k}\})$, and a 3-cycle $C_{\{v_0, v_i, v_{i+k}\}}$. These factors induce an R-factorization of $DK_{4n+1}$ where $R \cong DC(2k-2, 4) \cup DK_9$. From Lemma 4.1, $DK_{S_i \cup S_{i+k} \cup \{\infty\}} \cong DK_9$ can be decomposed into nine near $X_4$-factors, $H_i^{(1)}, H_i^{(2)}, \ldots H_i^{(8)}, H_i^{(\infty)}$, where $H_i^{(\infty)}$ misses vertex $\infty$. According to Corollary 4.4, $DC(2k-2, 4)$ has a decomposition into eight $X_4$-factors, $H_i^{(9)}, H_i^{(10)}, \ldots, H_i^{(16)}$, and so for each $i=1,2,\ldots,k$, we obtain eight near $X_4$-factors of $DK_{4n+1}$ by taking $H_i^{(j)} \cup H_i^{(j+8)}$, $j+1,2,\ldots,8$. The final near $X_4$-factor is $\bigcup_{i=1}^{k} H_i^{(\infty)}$. Hence $X_4 |_{NR} DK_{4n+1}$ when $n$ is even. ■

We now consider the case when $n$ is odd.

**4.6. Lemma:** $A|_{NR}DK_5$, and $D|_{NR}DK_5$.

**Proof:** A decomposition of $DK_5$ into near A-factors is shown in Figure 10, and a decomposition into near D-factors is shown in Figure 11. ■

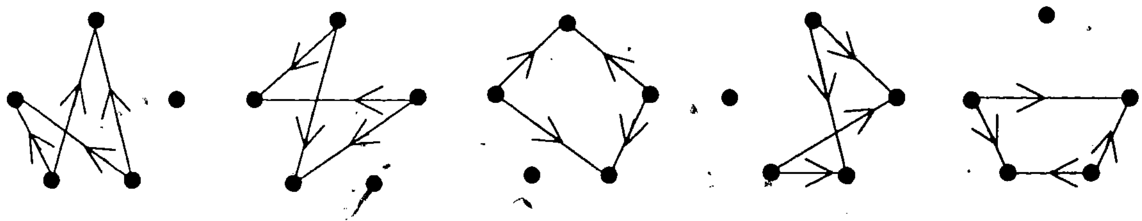


Figure 10

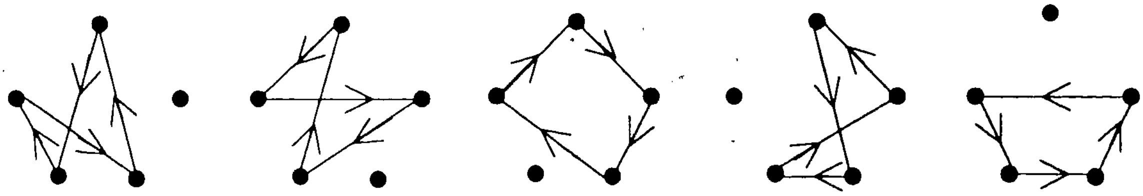


Figure 11

**4.7. Theorem:** When $n$ is odd, $A|_{NR}DK_{4n+1}$ and $D|_{NR}DK_{4n+1}$.

**Proof:** The case $n=1$ is shown in Lemma 4.6. Let $n$ be odd, $n\geq 3$. Partition the vertex set X of $DK_{4n+1}$ such that $X = \bigcup_{i=1}^{n} S_i \cup \{\infty\}$ where $|S_i| = 4$ for $i = 1,2,...n$.

Consider the graph $K_n$ with vertex set $\{v_1,v_2,...,v_n\}$ and let $S_i$ correspond to $v_i$ for $i=1,2,...,n$. Since $n$ is odd, $K_n$ has a near 1-factorization $\mathcal{F} = \{F_1, F_2,..., F_n\}$ where $F_i$ misses vertex $v_i$. Let each near 1-factor of $K_n$ correspond to an R-factor of $DK_{4n+1}$ where $R \cong \binom{\frac{n-1}{2}}{} DK_{4,4} \cup DK_5$. In particular, for each $F_i$, $i=1,2,...,n$, let $R_i = \bigcup_{v_jv_k \in F_i} DK_{S_j,S_k} \cup DK_{S_i \cup \{\infty\}}$. By Lemma 4.1, $DK_{S_i \cup \{\infty\}}$ can be decomposed into five near A-factors $A_i^{(1)}$, $A_i^{(2)}$, $A_i^{(3)}$, $A_i^{(4)}$, and $A_i^{(\infty)}$, where $A_i^{(\infty)}$ misses vertex $\infty$. Each $DK_{S_j,S_k}$ can be decomposed into four A-factors from Corollary 3.5. Hence for each $i=1,2,...,n$ we obtain four A-factors of $DK_{4n+1}$ by taking the four A factors of $\bigcup_{v_jv_k \in F_i} DK_{S_j,S_k}$ together with the four A-factors $A_i^{(1)}$, $A_i^{(2)}$, $A_i^{(3)}$, and $A_i^{(4)}$. This yields $4n$ A-factors of $DK_{4n+1}$. The final A-factor of $DK_{4n+1}$ is $\bigcup_{i=1}^{n} A_i^{(\infty)}$. A similar argument shows that $D|_{NR}DK_{4n+1}$. ■

Thus all near-resolvable X-factorizations of $DK_{4n+1}$, $X \in \{A,D\}$ are possible. We note that the existence of near-resolvable decompositions of $DK_{4n+1}$ into D-factors for all $n$ (except when $n$=8, 14, 23, or 33) can be deduced as a corollary of Bennett's work on Mendelsohn designs in [7].

We now turn to the remaining cases when $X_4 \in \{B,C\}$ and $n$ is odd.

**4.8. Lemma:** There is no near B- or near C-factorization of $DK_5$.

**Proof:** It can be shown by exhaustion that $B/DK_5$ and $C/DK_5$. So clearly there can be no near-resolvable decomposition of $DK_5$ into B or into C. ■

**4.9. Lemma:** $B|_{NR}DK_{13}$, and $C|_{NR}DK_{13}$.

**Proof:** Let the vertices of $DK_{13}$ be labelled by the elements of $Z_{13}$. Then thirteen near B-factors of $DK_{13}$ are $\{\{i, (i+3, i+2, i+5, i+12), (i+7, i+4, i+6, i+11), (i+9, i+1, i+8, i+10)\}: i = 0,1,2,...,12\}$, and thirteen near C-factors of $DK_{13}$ are $\{\{i, (i+3, i+2, i+6, i+8), (i+4, i+10, i+11, i+1), (i+9, i+5, i+12, i+7)\}: i = 0,1,2,...,12\}$.

■

**4.10. Lemma:** Both $B|_{NR}D(K_{4,1} \cup C_4)$ and $C|_{NR}D(K_{4,1} \cup C_4)$.

**Proof:** A near B-factorization of $D(K_{4,1} \cup C_4)$ is shown in Figure 12 and a near C-factorization is given in Figure 13.

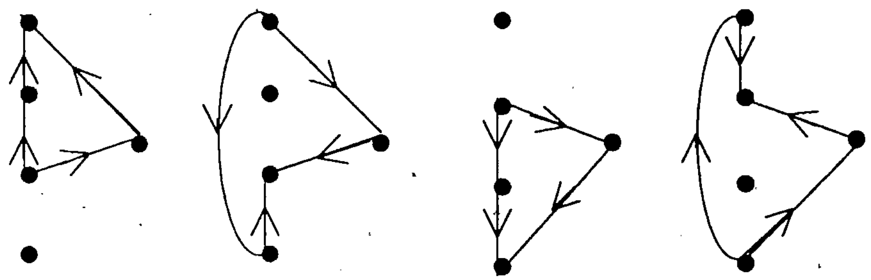


Figure 12

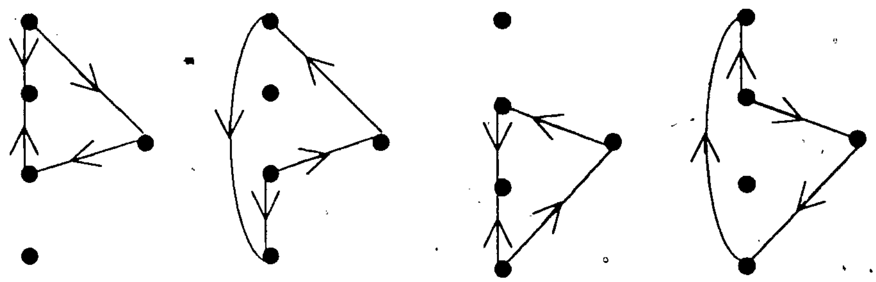


Figure 13

■

**4.11. Lemma:** Let $n$=2$k$+1. Partition the vertex set X of $DK_{4n+1}$ so that X=S∪T where |S| = 4($k$+1) and |T| = 4$k$+1. Then if

1)$DK_S \cong DK_{4(k+1)}$ has a decomposition into 4$k$+3 B- (or C-) factors, such that the union of some two of these factors is isomorphic to ($k$+1)$DC_4$, and

2)$DK_T \cong DK_{4k+1}$ has a near-resolvable decomposition into 4$k$+1 near

B- (or C-) factors,

then $DK_{4n+1}$ has a near B- (or C-) factorization.

**Proof:** Suppose we can partition the vertex set of $DK_{4n+1}$ as described above. Let $M_S^{(1)}, M_S^{(2)}, \ldots M_S^{(4k+1)}, M_S^{(4k+2)}, M_S^{(4k+3)}$, be the B-factors of $DK_S$. Without loss of generality let $M_S^{(4k+2)} \cup M_S^{(4k+3)} \cong (k+1)DC_4 \cong \bigcup_{i=1}^{k+1} DC_{S_i}$, where $S = \bigcup_{i=1}^{k+1} S_i$, and $|S_i| = 4$, $i=1,2,\ldots,k+1$. Let $M_T^{(1)}, M_T^{(2)}, \ldots, M_T^{(4k+1)}$ be the near B-factors of $DK_T$. Then $4k+1$ near B-factors of $DK_{4n+1}$ are given by $M_S^{(i)} \cup M_T^{(i)}$ for $i = 1,2,\ldots,4t+1$. Let DH be the graph obtained by removing these B-factors from $DK_{4n+1}$. Then $DH \cong DC_4 \cup DK_{S,T}$.

Further partition T so that $T = \bigcup_{i=1}^{k} T_i \cup \{\infty\}$, where $|T_i| = 4$ for $i = 1,2,\ldots,k$.

Consider $K_{k+1,k+1}$ with bipartition $Y \cup Z$, where $Y=\{y_1,y_2,\ldots,y_{k+1}\}$ and $Z=\{z_1,z_2,\ldots,z_k,z_\infty\}$. Associate $S_i$ with $y_i$ for $i = 1,2,\ldots,k+1$, $T_i$ with $z_i$ for $i= 1,2,\ldots,k$, and the vertex $\infty$ with $z_\infty$. $K_{k+1,k+1}$ has a 1-factorization $F_1, F_2, \ldots, F_{k+1}$ such that $y_i z_\infty \in F_i$. This 1-factorization corresponds to an R-factorization of DH where $R \cong (k)DK_{4,4} \cup D(K_{4,1} \cup C_4)$. Specifically, let $F_i$ correspond to

$R_i = \bigcup_{\substack{y_j z_k \in F_i \\ j \neq i}} DK_{S_j,T_k} \cup D(K_{S_i,\{\infty\}} \cup C_{S_i})$. Since $DK_{S_j,T_k} \cong DK_{4,4}$ can be factored into four B-factors from Corollary 3.5, $\bigcup_{\substack{y_j z_k \in F_i \\ j \neq i}} DK_{S_j,T_k}$ can be factored into four B-factors. Denote these $L_i^{(1)}, L_i^{(2)}, L_i^{(3)}$, and $L_i^{(4)}$. Also $D(K_{S_i,\{\infty\}} \cup C_{S_i}) \cong D(K_{4,1} \cup C_4)$ can be factored into four near B-factors, $N_i^{(1)}, N_i^{(2)}, N_i^{(3)}$, and $N_i^{(4)}$, as shown in Lemma 4.10. Hence for each $R_i$ we obtain four near B-factors of $DK_{4n+1}$, $L_i^{(j)} \cup N_i^{(j)}$, for $j = 1,2,3,4$. Thus we have a total of $(4k+1) + 4(k+1) = 4(2k+1) + 1 = 4n+1$ near B-factors of $DK_{4n+1}$ as required. The argument for C follows in the same way. ■

**4.12. Theorem:** $Cl_{NR}DK_{4n+1}$ when $n$ is odd, $n>1$.

**Proof:** Let $n=2k+1$. We proceed by induction on $k$. When $k=1$, $Cl_{NR}DK_{13}$ as shown in Lemma 4.9. Let $k>1$ and suppose $Cl_{NR}DK_{4n+1}$ for all odd $n$ when $n \leq 2(k-1)+1 = 2k-1$. That is, $Cl_{NR}DK_{4n+1}$ when $n$ is odd and $4n+1 \leq 8k-3$. We must show that $\mathrm{C}|_{NR}\mathrm{DK}_{4(2k+1)+1}$. This will follow if conditions 1 and 2 of Lemma 4.11. are satisfied.

1) $\mathrm{DK}_{4(k+1)}$ has a decomposition into C-factors such that the union of two of the C-factors is $(k+1)\mathrm{DC}_4$ as given in Lemma 3.10, provided $k \geq 1$.

2) Since $4k+1 < 8k-3$ when $k>1$, $\mathrm{DK}_{4k+1}$ has a near C-factorization, when $k$ is odd, by the induction hypothesis. When $k$ is even, $\mathrm{DK}_{4k+1}$ has a near C-factorization from Lemma 4.5.

Then from Lemma 4.11, $\mathrm{C}|_{NR}\mathrm{DK}_{4(2k+1)+1}$ and therefore $\mathrm{C}|_{NR}\mathrm{DK}_{4n+1}$ when $n$ is odd and $n>1$. ■

When $k$ is even, $\mathrm{DK}_{4(k+1)}$ has either no known B-factorization (when $k = 4$ or $k = 12$, from Theorem 3.1) or the decompositions given in Theorems 3.18, 3.20 and in Lemma 3.21, do not necessarily satisfy condition 1) of Lemma 4.11. Hence to establish the existence of near B-factorizations of $\mathrm{DK}_{4n+1}$ we require a different argument. (The case when $n = 5$ is still open.)

**4.13. Lemma:** $\mathrm{B}|_{NR}\mathrm{DK}_{29}$.

**Proof:** By Lemma 4.11, $\mathrm{DK}_{29}$ has a near B-factorization if conditions 1) and 2) of Lemma 4.11 are satisfied.

1) $\mathrm{DK}_{16}$ has a decomposition into B-factors such that the union of two of the B-factors is $(4)\mathrm{DC}_4$ as given in Lemma 3.14.

2) $\mathrm{DK}_{13}$ has a near B-factorization, as shown in Lemma 4.9.

Hence $\mathrm{B}|_{NR}\mathrm{DK}_{29}$. ■

**4.14. Lemma:** Partition the vertex set X of $\mathrm{DK}_{4n+1}$, $n=2k+1$, such that: $X=S\cup T\cup\{\infty\}$, where $|S| = |T| = 2n$; and $S = \bigcup_{i=0}^{k-1} S_i$, where $|S_0| = 6$ and $|S_i| = 4$ for $i=1,2,\ldots,k-1$. If we can decompose $K_S$ into factors $R_0,R_1,R_2,\ldots,R_{k-1}$, such that $R_i \cong K_{S_i} \cup \mathcal{F}_i$, where $\mathcal{F}_0$ is a family of six edge disjoint 1-factors on $V(K_S-K_{S_0})$ and $\mathcal{F}_i$

is a family of four edge disjoint 1-factors on $V(K_S - K_{S_i})$ for $i=1,2,\ldots,k-1$, then $DK_{4n+1}$ has a near B-factorization.

**Proof:** Let $n=2k+1$. Partition $V(DK_{4n+1})$ as described above and, in addition, let $T = \bigcup_{i=1}^{k-1} T_i$ where $|T_0| = 6$ and $|T_i| = 4$ for $i = 1,2,\ldots,k-1$. Suppose $K_S \cong \bigcup_{i=0}^{k-1} R_i$ and consider $R_0 \cong K_{S_0} \cup \mathcal{F}_0$. For this factor of $K_S$ we obtain twelve near B-factors of $DK_{4n+1}$ as follows. From Lemma 4.9, $DK_{S_0 \cup T_0 \cup \{\infty\}} \cong DK_{13}$ can be decomposed into thirteen near B-factors $M_0^{(1)}, M_0^{(2)},\ldots, M_0^{(12)}, M_0^{(\infty)}$ where $M_0^{(\infty)}$ misses vertex $\infty$. Each of the 1-factors in $\mathcal{F}_0$ gives two B-factors on $V(K_{4n+1} - K_{S_0 \cup T_0 \cup \{\infty\}})$ as described in Lemma 3.6, for a total of twelve B-factors of $D(K_{4n+1} - K_{S_0 \cup T_0 \cup \{\infty\}})$, $N_0^{(1)}, N_0^{(2)},\ldots, N_0^{(12)}$. Then $M_0^{(i)} \cup N_0^{(i)}$ for $i=1,2,\ldots,12$, gives twelve near B-factors of $DK_{4n+1}$. Now consider $R_i \cong K_{S_i} \cup \mathcal{F}_i$, where $i= 1,2,\ldots, k-1$. $DK_{S_i \cup T_i \cup \{\infty\}} \cong DK_9$ can be decomposed into nine near B-factors, $M_i^{(1)}, M_i^{(2)},\ldots, M_i^{(8)}, M_i^{(\infty)}$, where $M_i^{(\infty)}$ misses vertex $\infty$. Also, $\mathcal{F}_i$ is a family of four edge disjoint 1-factors and each of these 1-factors gives two B-factors on $V(K_{4n+1} - K_{S_i \cup T_i \cup \{\infty\}})$, for a total of eight B-factors of $D(K_{4n+1} - K_{S_i \cup T_i \cup \{\infty\}})$, $N_i^{(1)}, N_i^{(2)},\ldots, N_i^{(8)}$. So for each $R_i$, $i=1,2,\ldots,k-1$, we obtain eight near B-factors of $DK_{4n+1}$, $M_i^{(j)} \cup N_i^{(j)}$, for $j= 1,2,\ldots,8$. The remaining near B-factor of $DK_{4n+1}$ is $\bigcup_{i=0}^{k-1} M_i^{(\infty)}$. Thus we have $12 + 8(k-1) + 1 = 8k + 5 = 4(2k + 1) + 1 = 4n + 1$ near B-factors of $DK_{4n+1}$ as required. ■

**4.15. Lemma:** $B|_{NR}DK_{45}$.

**Proof:** Let $R_0 \cong K_6 \cup \mathcal{F}_0$ and $R_i \cong K_4 \cup \mathcal{F}_i$ for $i=1,2,3,4$, where $\mathcal{F}_i$ is as shown in Figure 14. Observe that $\bigcup_{i=0}^{4} R_i \cong K_{22}$. From Lemma 4.14 we conclude that $DK_{45}$ has a near B-factorization. ■

$\mathcal{F}_0$

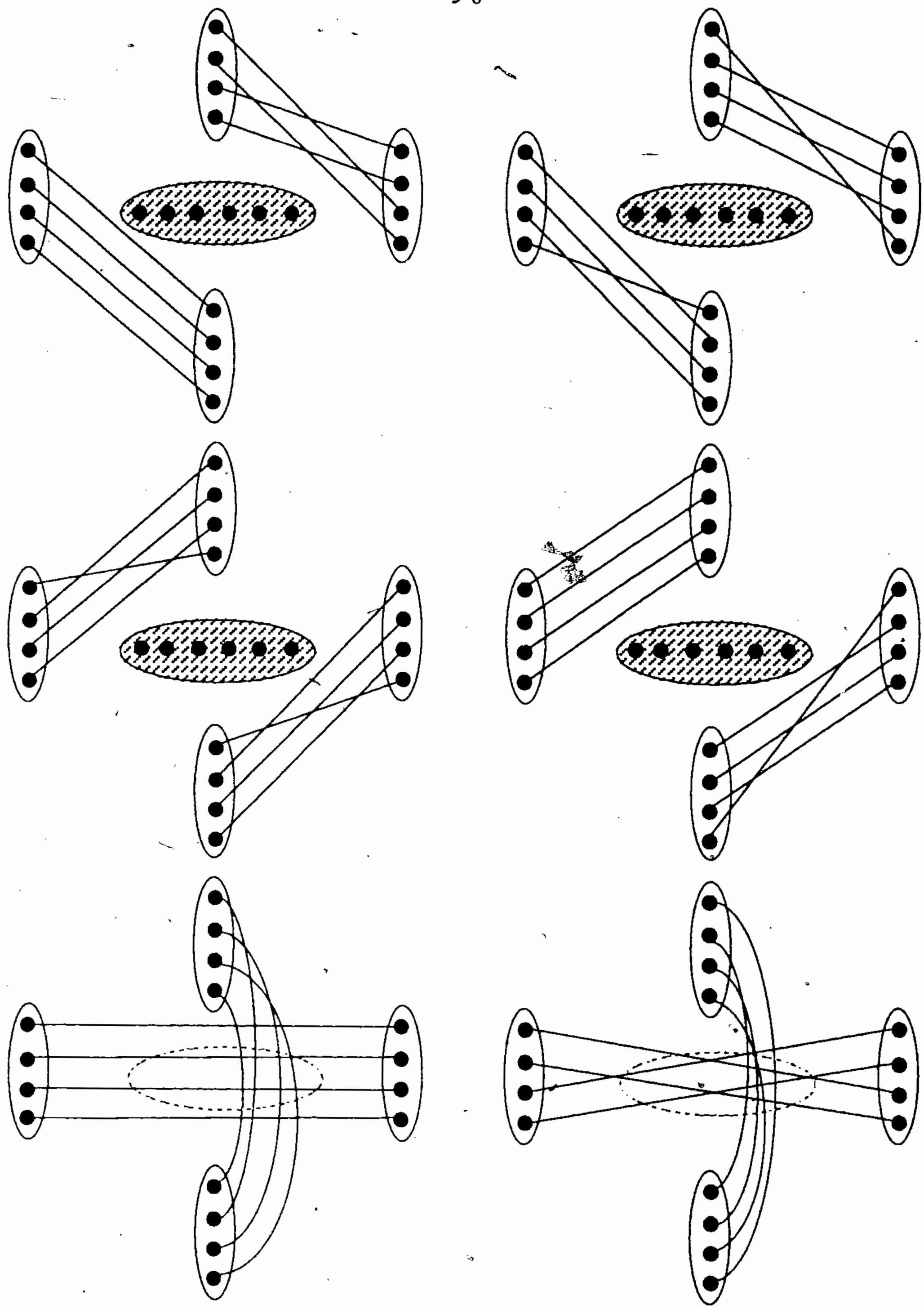
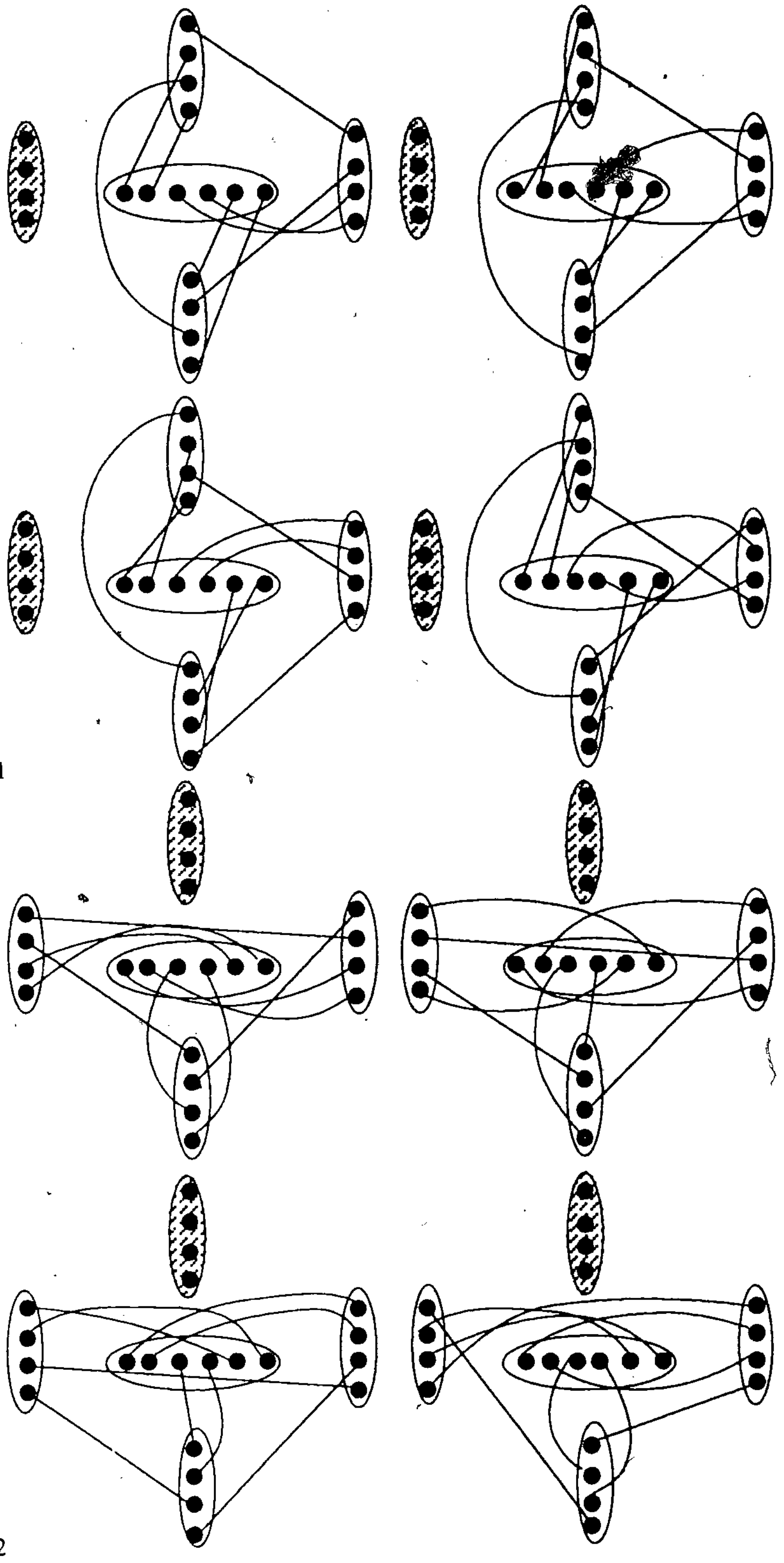


Figure 14



$\mathcal{F}_1$

$\mathcal{F}_2$

Figure 14, continued

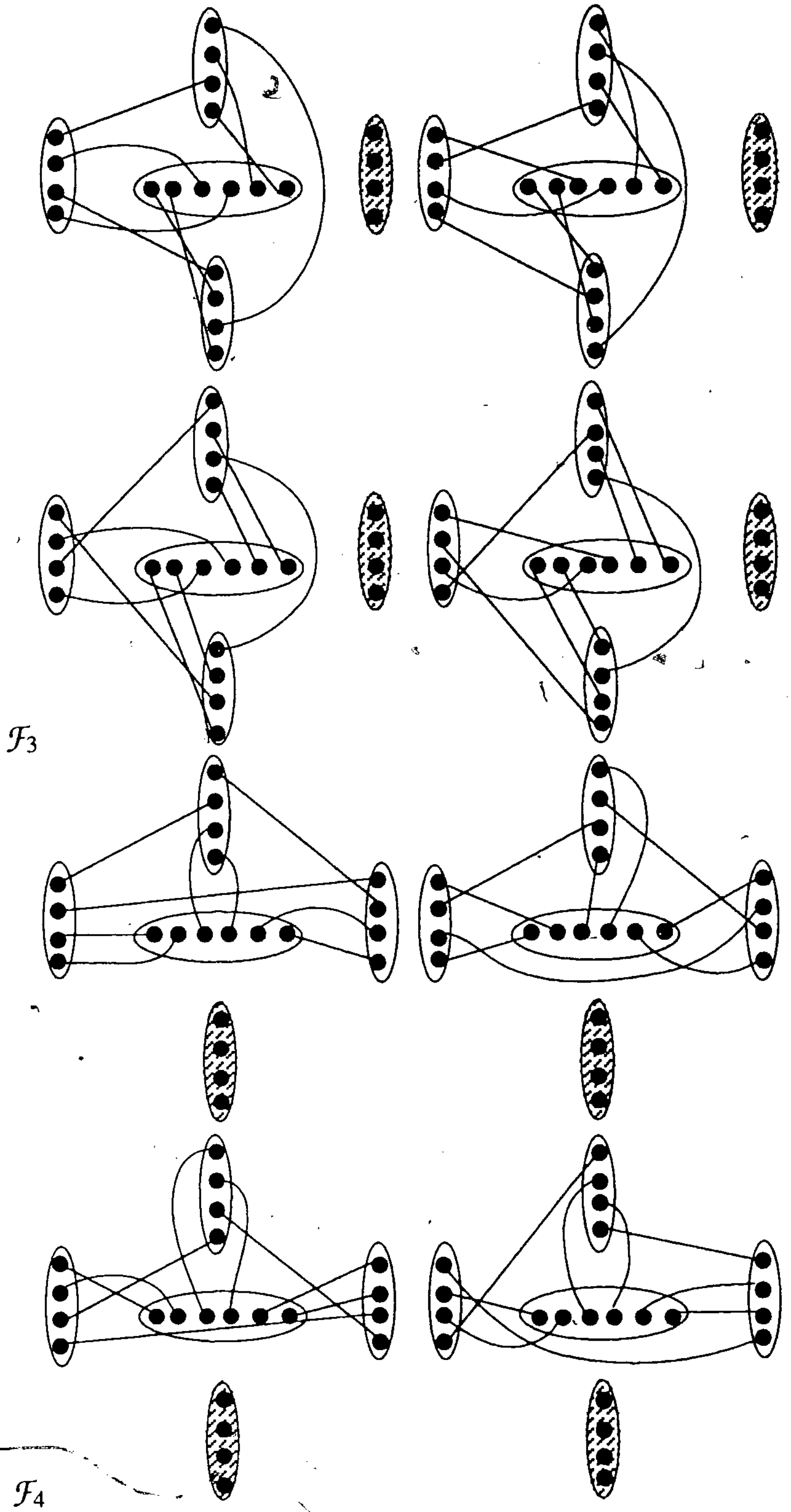



Figure 14, continued

Before constructing the remaining near B-factorizations, we need the following simple lemma.

**4.16. Lemma:** The graph $2K_{2p}$, $p\geq 2$ can be decomposed into $(2p-1)$-cycles.

**Proof:** Label the vertices of $2K_{2p}$, $\{0,1,2,\ldots,2p-2\} \cup \{\infty\}$. Then the $(2p-1)$-cycles of a decomposition are $C^{(i)} = (p+i, (p-1)+i, (p+1)+i, (p-2)+i, (p+2)+i, \ldots, (2p-3)+i, 2+i, (2p-2)+i, 1+i, \infty)$ for $i=0,1,2,\ldots,2p-2$, (where arithmetic is modulo $2p-1$), and $C^{(\infty)} = (0,1,2,\ldots,2p-2)$. ■

For the following theorem it is useful to colour the edges of the $(2p-1)$-cycles of $2K_{2p}$ either thin, dashed, or thick, as shown in Figure 15.

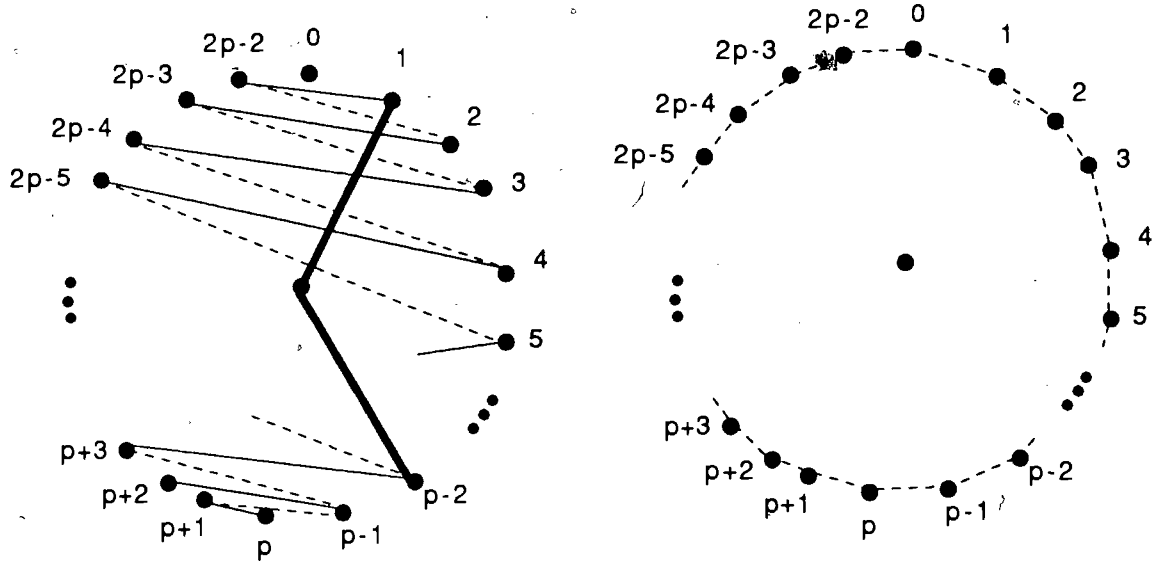



Figure 15

**4.17. Theorem:** $B|_{NR}DK_{4n+1}$ when $n\equiv 1 \pmod 4$, $n\geq 9$.

**Proof:** Let $n = 4t + 1$. Since $n\geq 9$, then $t\geq 2$. Note that $2n = 4(2t-1) + 6$. Partition the vertex set X of $DK_{4n+1}$ such that $X=S\cup T\cup\{\infty\}$, where $|S| = |T| = 2n$. Further partition S so that $S = \bigcup_{i=0}^{2t-1} S_i$, where $|S_0| = 6$ and $|S_i| = 4$ for $i = 1,2,\ldots,2t-1$, and T so that $T = \bigcup_{i=1}^{2t-1} T_i$ where $|T_0| = 6$ and $|T_i| = 4$ for $i = 1,2,\ldots,2t-1$. We will show that

$K_S \cong \bigcup_{i=0}^{2t-1} R_i$, where the $R_i$ are as described in Lemma 4.14.

Consider $2K_{2t}$ with vertex set $\{v_0,v_1,v_2,\ldots,v_{2t-2}\} \cup \{v_\infty\}$, and associate $v_\infty$ with $S_0$ and associate $v_i$ with $S_{i+1}$ for $i= 0,1,2,\ldots,2t-2$. From Lemma 4.16, when $t\geq 2$, $2K_{2t}$ can be decomposed into $2t$ $(2t-1)$-cycles, $C^{(0)}, C^{(1)}, C^{(2)},\ldots, C^{(2t-2)}, C^{(\infty)}$ where $C^{(i)}$ misses $v_i$. For each $C^{(i)}$, $i = 0,1,\ldots, 2t-2$, let $\mathcal{F}_{i+1}$ (a family of four edge disjoint 1-factors of $K_S-K_{S_{i+1}}$) be as shown in Figure 16. Note that if $v_jv_k$ is a thin edge in $C^{(i)}$, then we use one 1-factor between $S_{j+1}$ and $S_{k+1}$, and if $v_jv_k$ is dashed, we use three 1-factors between $S_{j+1}$ and $S_{k+1}$. It is important to observe that the union of these four 1-factors is $K_{4,4}\cong K_{S_{j+1},S_{k+1}}$. For the thick edges, let $K_{S_\infty,S_m}\cong K_{6,4}\cong W_m\cup Y_m$ where $W_m \cong Y_m \cong K_{3,4}$, for $m=0,1,2,\ldots,2p-2$. The edge $v_{j\infty}$ lies in precisely two of the cycles $C^{(i)}$. In one instance the four 1-factors defined partition $K_{3,4}\cong W_{j+1}$ and in the other they partition $K_{3,4}\cong Y_{j+1}$. Finally, corresponding to $C^{(\infty)}$ we, define $\mathcal{F}_0$ as shown in Figure 17. Each edge $v_jv_k$ in $C^{(\infty)}$ is dashed so we use the three remaining 1-factors between $S_{j+1}$ and $S_{k+1}$. Let $R_i\cong K_{S_i} \cup \mathcal{F}_i$ for $i=0,1,2,\ldots,2t-1$. Clearly, $K_S\cong \bigcup_{i=0}^{2t-1} R_i$. Hence from Lemma 4.14, we have $BI_{NR}DK_{4n+1}$ when $n\equiv 1 \pmod 4$, $n\geq 9$. ■

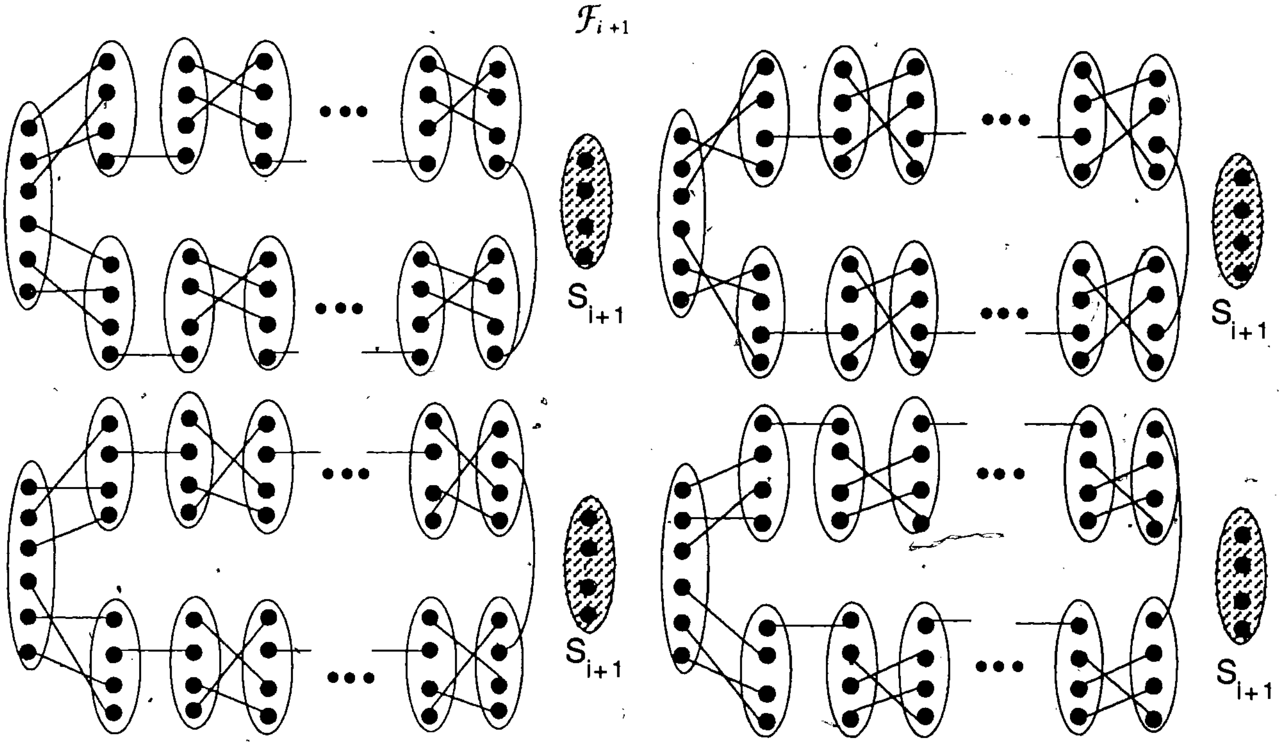



Figure 16

$\mathcal{F}_0$

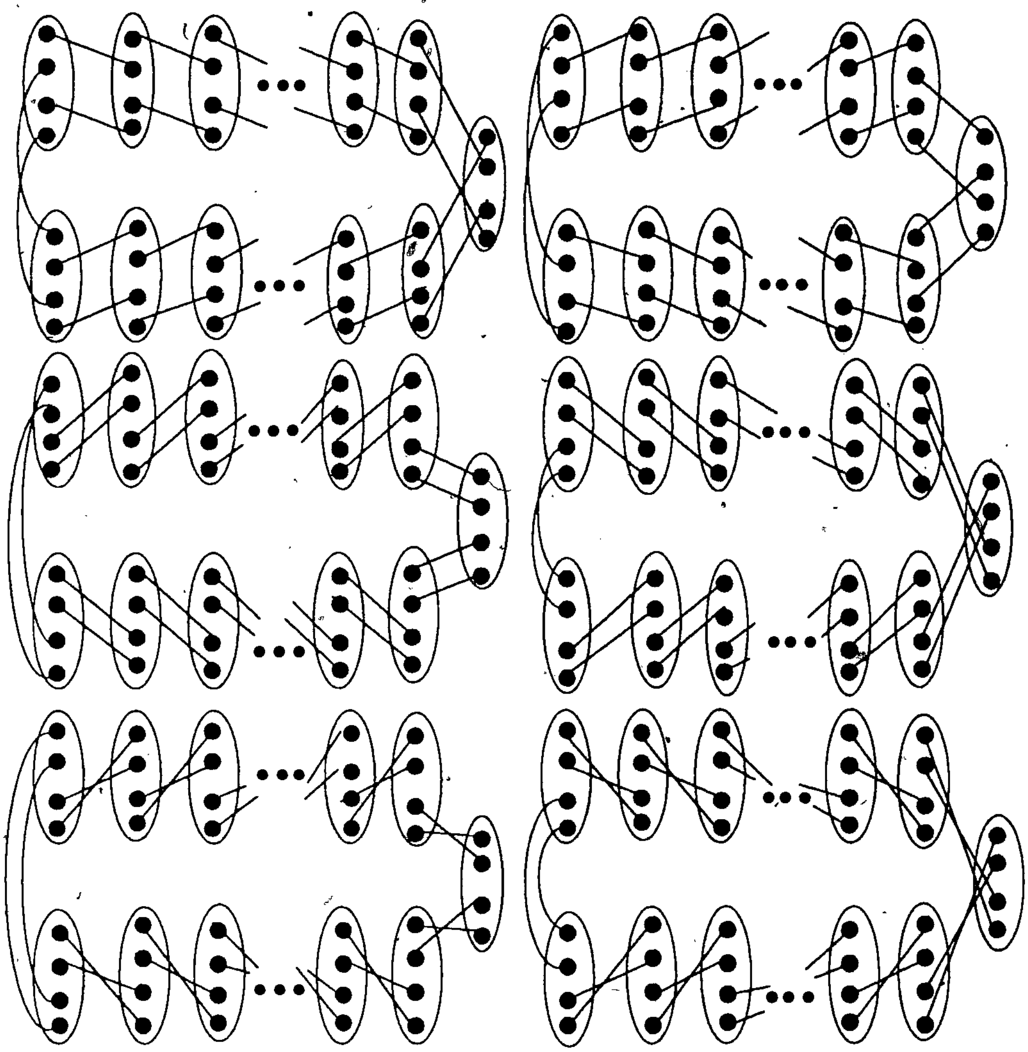


Figure 17

**4.18. Theorem:** $B|_{NR}DK_{4n+1}$ when $n\equiv3 \pmod 4$.

**Proof:** Let $n = 4t + 3$. We proceed by induction on $t$. When $t=0$, $B|_{NR}DK_{13}$ from Lemma 4.9; when $t=1$, $B|_{NR}DK_{29}$ from Lemma 4.13; and when $t=2$, $B|_{NR}DK_{45}$ from Lemma 4.15. Let $t > 2$ and suppose $B|_{NR}DK_{4n+1}$ when $n \leq 4(t-1) + 3 = 4t - 1$ and $n\equiv3 \pmod 4$. We must show that $DK_{4(4t+3)+1}$ has a near B-factorization. Note that from Lemma 3.14, $DK_{4[(2t+1)+1]}\cong DK_{4(2t+2)}$ has a B-factorization such that the union of two of the B-factors is $(2t+2)DC_4$. Clearly $2t+1<4t-1$ when $t>2$. If $t>2$ and

$2t+1 \equiv 3 \pmod 4$, $DK_{4(2t+1)+1}$ has a near B-factorization, either by the induction hypothesis or the induction base. If $2t+1 \equiv 1 \pmod 4$ and $t>2$, then $B|_{NR}DK_{4(2t+1)+1}$ by Theorem 4.17. Hence conditions 1) and 2) of Lemma 4.11 are satisfied. Therefore $DK_{4(4t+3)+1}$ has a near B-factorization and we may conclude that $B|_{NR}DK_{4n+1}$ when $n \equiv 3 \pmod 4$. ■

We have now proven Theorem 1.10 which we restate here.

**1.10. Theorem:** a) A and D divide $DK_{4n+1}$ near resolvably for all $n \geq 1$.

b) B and C divide $DK_{4n+1}$ near resolvably for all $n>1$ (with the possible exception that B may notdivide $DK_{21}$ near resolvably).

**Proof:** a) See Theorems 4.5 and 4.7.

b) See Theorems 4.5, 4.8, 4.12, 4.17, and 4.18.

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