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The Block-Intersection Graph of Pairwise Balanced Designs

by

Donovan Ross Hare

B. Sc. (Honours), University of Victoria, 1986

M. Sc., University of Alberta, 1987

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

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of

Mathematics and Statistics

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THE BLOCK INTERSECTION GRAPH OF PAIRWISE BALANCED DESIGNS

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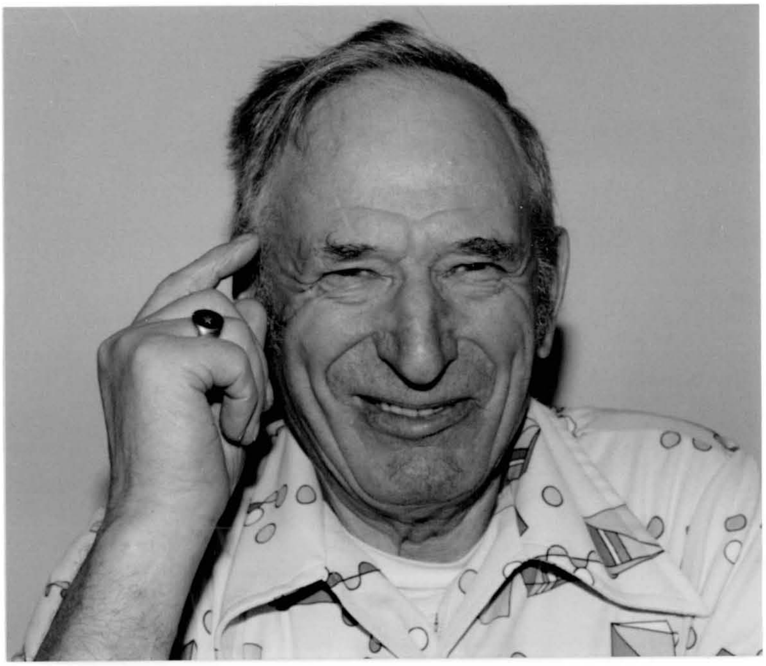
Abstract

The focus of this thesis is to investigate certain graph theoretic properties of a class of graphs which arise from combinatorial design theory. The block-intersection graph of a pairwise balanced design has as its vertices the blocks of the design and has as its edges precisely those pairs of blocks which have non-empty intersection. These graphs have a particular local structure which is exploited in the proofs of the results. In Chapter 1 an overview is given. Definitions, an example, and some motivational material (a brief history, connections to other work) are included here. Cycles of these graphs are investigated in Chapter 2. In particular, it is shown that these graphs are hamiltonian. In Chapter 3 the connectivity of the block-intersection graph is determined for balanced incomplete block designs and for 'large' pairwise balanced designs. Chapter 4 contains the proofs of a number of results which pertain to coloring the block-intersection graph. More specifically, it is shown that the neighborhood of a vertex of the block-intersection graph of 'large' balanced incomplete block designs can always be colored in an optimal way.

TO MY GRANDFATHERS



"BESTEFAR"

ALFRED
ANGELTVEDT

"POP"

GERALD
HARE

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For the extra work that was created by being in other departments during the research of the thesis I wish to thank the Graduate Secretary, Mrs. Sylvia Holmes. I also wish to thank the Department of Mathematics and Statistics, University of Victoria, for their hospitality during the preparation of the thesis. In particular, my appreciation is extended to Dr. Chris Bose for letting me use his NeXT computer to draw the figures.

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Chapter 1

Introduction

This thesis investigates a certain class of graphs which arise from combinatorial design theory. We present in this chapter the background material for the problems being studied. The first section gives the necessary definitions as well as an example of a block-intersection graph. Some of the basic properties of the block-intersection graph are discussed in the following section. A brief history is given in the third section of the three particular areas of the block-intersection graph investigated in the subsequent chapters. The contributions the thesis makes to these areas are stated in this section as well. The final section presents a couple of results about matchings that will be used in later the remaining chapters.

1.1 Definitions

In this section we give definitions of the less familiar combinatorial objects and properties. For the basic graph theoretic terminology the reader is refered to [5]. We will start with designs, followed by hypergraphs and then move to some non-standard graph theory definitions. Some definitions will be given in the other chapters, however, they have been invented to facilitate the reading of the proofs and so are not included here.

1.1.1 Block Designs

Let K be a finite set of positive integers, and let λ and v be positive integers such that $v > \max K$ (here $\max K$ is the maximum element in K; similarily for $\min K$). A *pairwise balanced design*, denoted $PBD(v, K, \lambda)$, is a pair $(V, \mathcal{B})$ where V is a finite set whose elements are called ***points***, $\mathcal{B}$ is a collection of subsets of V, called *blocks*, such that $|V| = v$, the blocks have their cardinalities from K and any pair of distinct points is contained in exactly λ blocks. If $K = \{k\}$, then $(V, \mathcal{B})$ is called a *balanced incomplete block design* and we denote it by $BIBD(v, k, \lambda)$. When $k = 3$ and $\lambda = 1$, the pair $(V, \mathcal{B})$ is called a *Steiner triple system* (denoted by $STS(v)$).

The *replication number* of a point in a design is the number of blocks that contain the point in the design. For a $BIBD(v, k, \lambda)$, a counting argument shows that the replication of any point is

$$r = \lambda \left(\frac{v-1}{k-1} \right),$$

and that the number of blocks is $b = \dfrac{vr}{k}$.

1.1.2 Hypergraphs

A ***hypergraph*** is an ordered pair $(V, \mathcal{A})$, where V is a finite nonempty set whose elements are called ***vertices*** and $\mathcal{A}$ is a collection of nonempty subsets of V whose members are called ***edges***. We assume there are no isolated vertices and hence $V = \cup\mathcal{A} = \cup\{E : E \in \mathcal{A}\}$. By referring to the hypergraph $\mathcal{A}$ we shall mean $(\cup\mathcal{A}, \mathcal{A})$.

Let $v \geq 2$. A hypergraph $\mathcal{A}$ is *v-uniform* if for all $E \in \mathcal{A}$, $|E| = v$. Note that a 2-uniform graph is an ordinary graph. A hypergraph $\mathcal{A}$ is ***linear*** if for all $E, F \in \mathcal{A}$, $E \neq F$, we have $|E \cap F| \leq 1$. Hence every $PBD(v, K, 1)$ is a linear hypergraph. A linear hypergraph is also called a ***nearly disjoint collection of sets*** (such as in [17]).

Let $\mathcal{A}$ be a hypergraph. A hypergraph $\mathcal{H}$ is a ***subgraph*** of $\mathcal{A}$ if $\mathcal{H} \subseteq \mathcal{A}$. A vertex x of $\mathcal{A}$ is ***incident*** with an edge E of $\mathcal{A}$ (and vice versa) if $x \in E$. Two vertices (edges) of a hypergraph are ***adjacent*** if they are incident with a common edge (vertex) of the hypergraph. The ***degree*** of a vertex x of $\mathcal{A}$ is the number of edges of $\mathcal{A}$ incident with x and is denoted by $d_{\mathcal{A}}(x)$. (If $\mathcal{A}$ is a $PBD(v, K, \lambda)$, then the replication number of a point of $\mathcal{A}$ is its degree.) Let $\alpha : \cup\mathcal{A} \to \mathcal{A}$ be defined by

$$\alpha(x) = \{E : x \in E \in \mathcal{A}\}.$$

Then the ***dual*** of $\mathcal{A}$ is the hypergraph $(\mathcal{A}, \{\alpha(x) : x \in \cup\mathcal{A}\})$.

A ***strong r-coloring*** of a hypergraph is an assignment of r distinct colors to its vertices so that no edge is incident with two vertices of the same color.

The *strong chromatic number* of $\mathcal{A}$ is the smallest integer r for which there exists a strong r-coloring of $\mathcal{A}$. The strong chromatic number of $\mathcal{A}$ is denoted by $\gamma(\mathcal{A})$.

An *r-edge coloring* of a hypergraph is an assignment of r distinct colors to its edges so that no two adjacent edges receive the same color. The *chromatic index* of a hypergraph is the least r for which there exists an r-edge coloring. The chromatic index of $\mathcal{A}$ is denoted by $\chi'(\mathcal{A})$.

1.1.3 Graphs

The *length* of a path or a cycle in a graph is the number of its edges. An *n-path* (*n-cycle*) is a path (cycle) of length n. If $P = v_0e_1v_1e_2v_2\cdots e_nv_n$ is a path, then the *reverse* of P is $v_ne_nv_{n-1}e_{n-1}v_{n-2}\cdots v_2e_2v_1e_1v_0$. Moreover, P is said to *start* in v_0 and *end* in v_n. If P_1 and P_2 are paths in a graph G, and P_1 ends in a vertex adjacent in G to the vertex that P_2 starts with, then P_1P_2 denotes the juxtaposition of P_1 and P_2 as alternating sequences of vertices and edges so that the edge from the end of P_1 to the beginning of P_2 is added giving a single path. A graph G is *edge-pancyclic* if for every edge e of G and every integer n, $3 \leq n \leq |V(G)|$, there is an n-cycle in G using e.

Let G be a graph and H be subgraph of G. An *H-factor* $\mathcal{F}$ of G, $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$, is a collection of edge-disjoint subgraphs of G such that $H_i \cong H$ for $i = 1, 2, \ldots, k$, and $G = H_1 \cup H_2 \cup \cdots \cup H_k$.

Let G be an n-regular graph with m vertices. Then G is *strongly regular* if there are integers p, q such that:

- if x and y are adjacent vertices in G, then there are exactly p vertices adjacent to both x and y; and
- if x and y are non-adjacent vertices in G, then there are exactly q vertices adjacent to both x and y.

We say that G has *parameters* (m, n, p, q).

There are many ways of defining graphs from designs. The interested reader is referred to the survey found in [4]. The subject of this thesis is the following graph: the *block-intersection graph* of a $PBD(v, K, 1)$, $(V, \mathcal{B})$, denoted by $B(\mathcal{B})$, has vertex-set $\mathcal{B}$ and has two vertices adjacent if and only if their corresponding blocks have non-empty intersection.

Example 1.1 *A* $(9,3,1)$*-design and its block-intersection graph.*

Let $V = \{1, 2, \ldots, 9\}$ and $\mathcal{B} = \{\{1,2,3\}, \{4,8,9\}, \{5,6,7\}, \{1,5,8\}, \{2,7,9\}, \{3,4,6\}, \{1,4,7\}, \{2,6,8\}, \{3,5,9\}, \{1,6,9\}, \{2,4,5\}, \{3,7,8\}\}$. This is an ***affine plane*** of order 3 where blocks are regarded as lines in the plane. The block-intersection graph of $\mathcal{B}$ is drawn in Figure 1.1.

For balanced incomplete block designs, block-intersection graphs have been used as effective isomorphism invariants to distinguish non-isomorphic designs that have the same parameters (see [4,8]). More generally, we can define the *intersection graph* of a linear hypergraph $\mathcal{A}$ as the graph whose

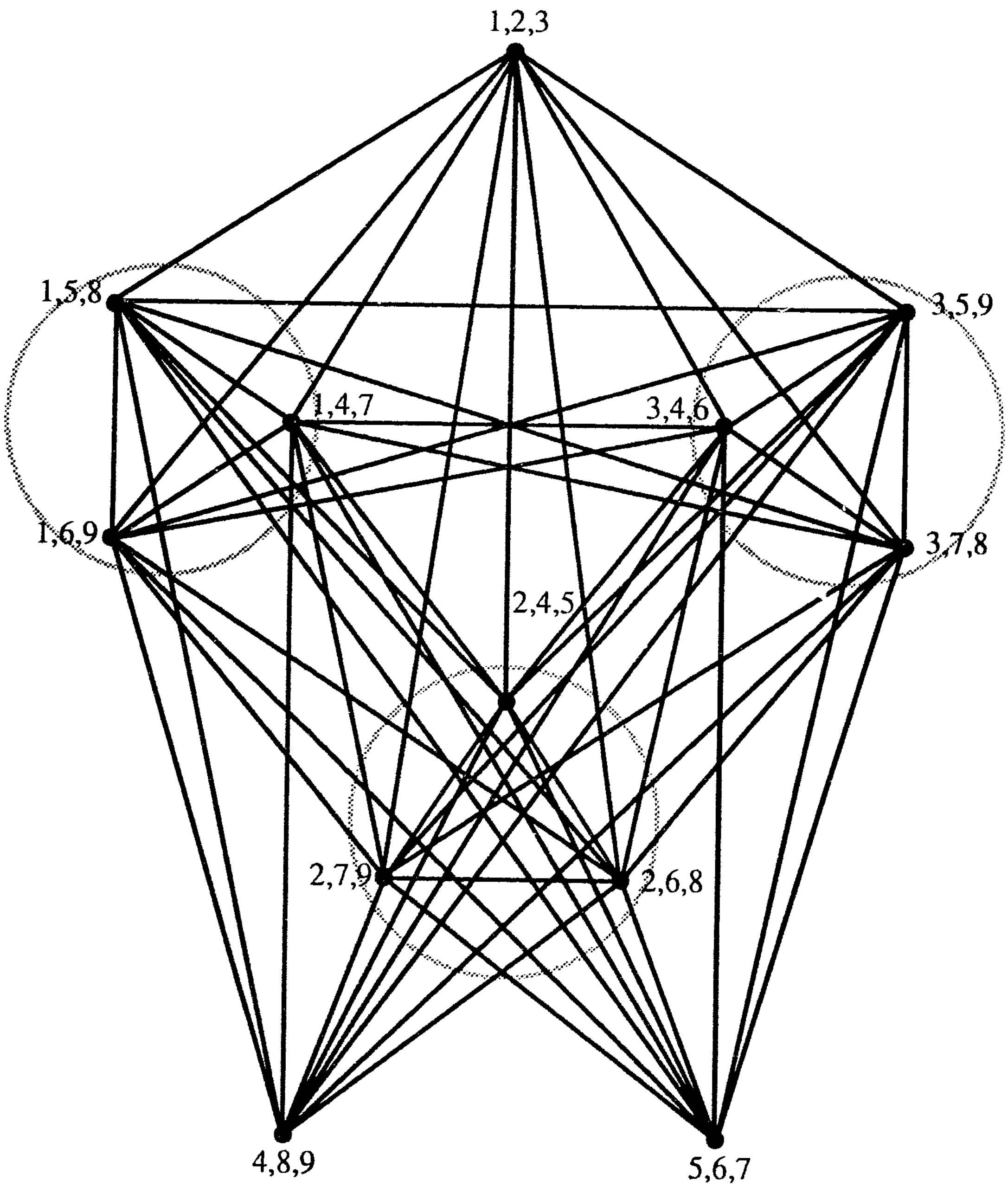

Figure 1.1: The block-intersection graph of a $(9, 3, 1)$-design.

vertex-set is $\mathcal{A}$ and whose edge-set is precisely those pairs of edges of $\mathcal{A}$ that have non-empty intersection.

1.2 Basic Properties of the Block-Intersection Graph

To begin our investigation of the block-intersection graph we look at various parameters of the graph as well as its basic local structure. As much as possible throughout the thesis, lower case letters are used for positive integers or for points of a design, capital letters are used for blocks of a design (or equivalently vertices of the block-intersection graph) or sets of points of the design, and script style letters denote designs or sets of blocks of a design. The following notation is also used throughout unless otherwise stated:

- $(V, \mathcal{B})$ denotes a $PBD(v, K, 1)$;
- $\ell = \min K$ and $u = \max K$;
- $B^* = \{b_1, b_2, \ldots, b_k\}$ is an arbitrary but fixed block in $\mathcal{B}$;
- for $i = 1, 2, \ldots, k$, $\mathcal{B}_i$ is the set $\{B \in \mathcal{B} : B \cap B^* = \{b_i\}\}$;
- $\mathcal{B}^*$ is the neighborhood of B^* in G ($\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$); and
- G is the block-intersection graph of $\mathcal{B}$.

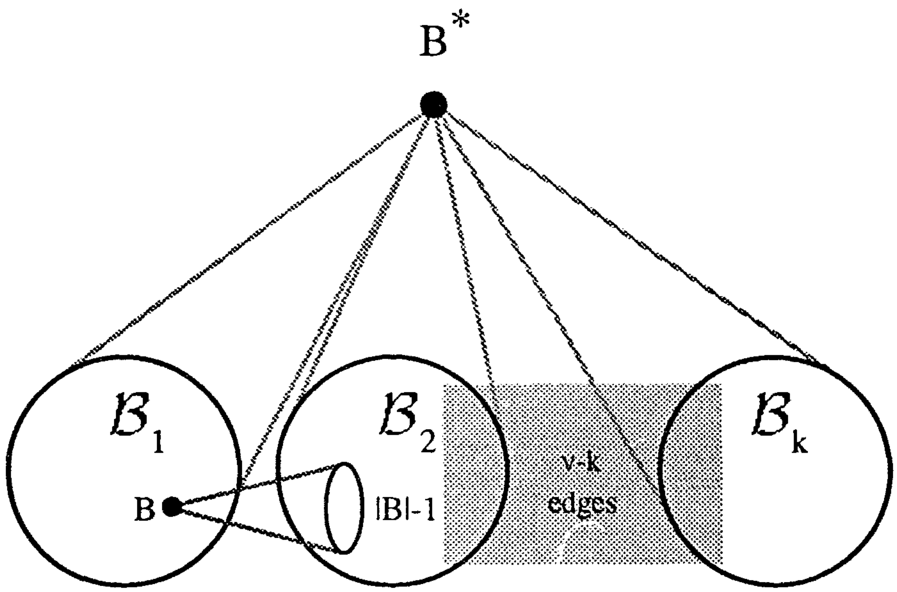

Figure 1.2: The structure of a neighborhood in $B(\mathcal{B})$.

Knowledge of the basic structure of the neighborhood of a vertex in the block-intersection graph will be exploited throughout the thesis. The next lemma describes this structure.

Lemma 1.2 *In the graph G, for $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, and $B \in \mathcal{B}_i$, the number of neighbors of B in $\mathcal{B}_j$ is $|B| - 1$ and the number of edges between $\mathcal{B}_i$ and $\mathcal{B}_j$ is $v - k$. Furthermore,*

$$\frac{v-k}{u-1} \leq |\mathcal{B}_i| \leq \frac{v-k}{\ell-1}.$$

Proof: The basic structure of a neighborhood is illustrated in Figure 1.2.

Let $B \in \mathcal{B}_i$. For each $x \in B \setminus \{b_i\}$ there is a unique block B_x of $\mathcal{B}_j$ containing $\{x, b_j\}$. Moreover, if $\{x, y\} \subset B \setminus \{b_i\}$, $x \neq y$, then $B_x \neq B_y$.

Thus the number of edges in G between B and $\mathcal{B}_j$ is $|B|-1$ and hence the number of edges between $\mathcal{B}_i$ and $\mathcal{B}_j$ is

$$\sum_{B\in\mathcal{B}_i}(|B|-1).$$

On the other hand, there are $v-k$ pairs $\{x,b_i\}$, $x\in V\setminus B^*$, all of these pairs are contained in blocks from $\mathcal{B}_i$, and every $B\in\mathcal{B}_i$ contains $|B|-1$ of the pairs. Thus

$$\sum_{B\in\mathcal{B}_i}(|B|-1)=v-k$$

and the first part of the lemma is proved. For all $B\in\mathcal{B}_i$, $\ell\le|B|\le u$ and so

$$(\ell-1)|\mathcal{B}_i|\le\sum_{B\in\mathcal{B}_i}(|B|-1)\le(u-1)|\mathcal{B}_i|.$$

The lemma is thus proved. ∎

Let $B\in\mathcal{B}$ such that $|B|=u$, and let $x\in V\setminus B$ (note that $v>u$). Since there exists a unique block $B_y\in\mathcal{B}$ containing $\{x,y\}$ for each $y\in B$, and x is the only point in the intersection of any two of these blocks, $v\ge u(\ell-1)+1$. Thus if $\ell\ge 3$, then $u\le\frac{1}{2}v$. Since $d_G(B^*)=\sum_{i=1}^{k}|\mathcal{B}_i|$, if $\ell\ge3$, Lemma 1.2 gives the following bounds on the minimum and maximum degree of G:

$$\ell\left(\frac{v-\ell}{u-1}\right)\le\delta(G)\le\Delta(G)\le u\left(\frac{v-u}{\ell-1}\right). \tag{1.1}$$

Let B_1 and B_2 be distinct vertices of G and let $x\in B_1$ and $y\in B_2\setminus B_1$ (note that the definition of pairwise balance design rules out the possibility that we only have one vertex in G). Then there exists $B_3\in\mathcal{B}$ such that

$\{x, y\} \subseteq B_3$. Thus G has diameter at most 2. Moreover, let $z \in B_1$, $z \neq x$. Then there exists $B_4 \in \mathcal{B}$ such that $\{y, z\} \subseteq B_4$. Hence $B_1B_3B_4B_1$ is a 3-cycle in G and therefore G has girth 3.

By a result of P. Seymour [17], the cardinality of the largest independent set in G is at least $\frac{|\mathcal{B}|}{v}$. Since the blocks in an independent set are mutually disjoint, the cardinality of the largest independent set in G is at most $\frac{v}{\ell}$. Statements about cycles, connectivity and coloring of G will be made in the next section.

For the rest of this section we restrict our attention to the case when $\mathcal{B}$ is a $BIBD(v, k, 1)$. Since $\ell = u$ here, equation 1.1 requires that G is $k(r-1)$-regular when $\ell \geq 3$ (when $\ell = u = 2$, G is $2(v-2)$-regular). Moreover, G has $b = \frac{v(v-1)}{k(k-1)}$ vertices and hence $\frac{v(v-1)(v-k)}{2(k-1)^2}$ edges. If $b = v$, then $\mathcal{B}$ is called a *symmetric* $BIBD(v, k, 1)$. In this case, $r = k$ and every pair of blocks of $\mathcal{B}$ intersect in a point. Thus $G \cong K_b$. For this reason, we assume throughout the rest of the thesis that if $\mathcal{B}$ is a $BIBD(v, k, 1)$, then $\mathcal{B}$ is ***not*** symmetric. Thus we have $r > k$ by Fisher's inequality (see [3, page 18]). If $r = k + 1$, then $\mathcal{B}$ is a $BIBD(k^2, k, 1)$ and is called an ***affine plane*** of order k.

Let B_1 and B_2 be distinct vertices in G. If B_1 and B_2 are adjacent, then let $b \in B_1 \cap B_2$. The point b is in $r - 2$ other blocks of $\mathcal{B}$ and for each $x \in B_1 \setminus \{b\}$ and each $y \in B_2 \setminus \{b\}$ there is a unique block containing $\{x, y\}$. Thus if B_1 and B_2 are adjacent, then they have $r - 2 + (k-1)^2$ common neighbors. If B_1 and B_2 are not adjacent, then there are k^2 blocks which intersect both of them (one for every pair $\{x, y\}$ with $x \in B_1$ and $y \in B_2$). Thus B_1 and B_2 have k^2 common neighbors in this case. Therefore, G is a strongly regular graph with parameters $(b, k(r-1), r-2+(k-1)^2, k^2)$.

If $\mathcal{B}$ is an $STS(v)$, then Seymour's result mentioned above gives $\frac{1}{6}(v-1)$ as a lower bound for the cardinality of the largest independence set in G (also called a ***partial parallel class*** in other contexts). C. Lindner and K. Phelps [14] have improved this lower bound to $\frac{1}{4}(v-1)$ for all $v \geq 9$ except for three cases which were settled by G. Lo Faro [15].

1.3 History and Results of the Thesis

The study of block designs goes primarily back to the middle of the nineteenth century with the work by Kirkman and Steiner, although Euler studied Latin squares in the late eighteenth century. Much of the focus of combinatorial design theory has been in proving the exisitence of different types of designs. It is not the intention of the thesis to add anything to this study. It is fair to say that nothing more than the basic definitions of the various designs mentioned is used in the proofs of the results of the thesis. It is the structure of the block-intersection graph that is of interest here.

We do state, however, a pinacle in the history of pairwise balanced designs: Wilson [18] proved that a $PBD(v, K, \lambda)$ exists if you have 'enough' points and if you satisfy the necessary conditions. More specifically if

$\alpha(K) = \gcd\{k-1 : k \in K\}$ and $\beta(K) = \gcd\{k(k-1) : k \in K\}$, then the statement is the following.

Theorem 1.3 (Wilson) *There exists a least integer $v(K, \lambda)$ so that for all integers v, $v > v(K, \lambda)$, satisfying $\lambda(v-1) \equiv 0 \pmod{\alpha(K)}$ and $\lambda v(v-1) \equiv 0 \pmod{\beta(K)}$, there exists a $PBD(v, K, \lambda)$.*

We move now to the history of the areas studied in the thesis. The next three sections will describe as well the advances made by the thesis.

1.3.1 Cycles

In March 1987 at a meeting of the American Mathematical Society, R. L. Graham asked if the block-intersection graphs of Steiner Triple systems are hamiltonian. P. Horák and A. Rosa [13] were the first to show that if $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$, $k \geq 3$, then $B(\mathcal{B})$ is hamiltonian. B. Alspach, K. Heinrich and B. Mohar [2] subsequently proved that if $(V, \mathcal{B})$ is a $PBD(v, K, 1)$ such that $\max K \leq 2 \min K$, then $B(\mathcal{B})$ is hamiltonian. Even more recently, B. Alspach and D. Hare [1] proved that if $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$, $k \geq 3$, then $B(\mathcal{B})$ is edge-pancyclic and that the same is true for transversal designs.

One of the main results of this thesis is that if $(V, \mathcal{B})$ is a $PBD(v, K, 1)$ with $\min K \geq 3$, then $B(\mathcal{B})$ is edge-pancyclic. The proof of this theorem generalizes the proof in [1] for balanced incomplete block designs. Also included is a proof that the line graph of the complete graph is edge-pancyclic.

1.3.2 Connectivity

The connectivity of the block-intersection graph is by far the least studied of the three properties. The only reference to connectivity is in the proof of the above mentioned result by Horák and Rosa. They showed that the block-intersection graph of a $BIBD(v, k, 1)$ was hamiltonian by showing that the graph's independence number is at most v/k and that the graph's vertex connectivity is at least v/k. For this graph, we show in the thesis that its vertex connectivity is equal to its minimum degree, namely $\dfrac{k(v-k)}{k-1}$. Minimal vertex-cuts are also characterized and vertex-disjoint paths are constructed. The vertex connectivity for the graph of a pairwise balanced design is also investigated and it is shown that if the design has 'enough' points, then the block-intersection graph has vertex connectivity equal its minimum degree. The edge connectivity for the block-intersection graph of any pairwise balanced design is shown to be equal to the minimum degree of the graph as well. Moreover, minimal edge-cuts are characterized and edge-disjoint paths are found.

1.3.3 Coloring

Coloring the block-intersection graph of a pairwise balanced design is the topic of the last chapter of the thesis. The color classes of a coloring of the graph correspond to sets of pairwise disjoint blocks of the design. The chromatic number of the block-intersection graph of a design, although far from being characterized, has some very nice applications (see [8]).

One application is in the area of statistics. Balanced incomplete block designs are used in the design of statistical experiments where each point represents an item in the experiment and each block represents a test involving the items it contains. The entire design is an experiment where every pair of items is in exactly one test together. If two tests have no common items, then they can be performed simultaneously. It is advantageous to group tests together in a way that minimizes the total time for the experiment. The minimum number of groups possible is precisely the chromatic number of the design's block-intersection graph.

Since pairwise balanced designs are linear hypergraphs, in certain contexts, the chromatic index of the design is used instead of the chromatic number of the design's block-intersection graph. The two parameters are in fact equivalent.

Not a lot is known about the chromatic number of the block-intersection graph of a pairwise balanced design. Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$, $G = B(\mathcal{B})$, $u = \max K$, and $\ell = \min K$. By using Brook's Theorem (see [5]) and an upper bound for the maximum degree Δ of the graph (which is achieved when $\ell = u$), we have $\chi(G) \leq \Delta(G) \leq \dfrac{u(v-u)}{\ell - 1}$. The best upper bound so far for this case was found by W. I. Chang and E. L. Lawler [6]. They show that $\chi(G) \leq \left\lceil \frac{3}{2}v - 2 \right\rceil$.

Much of the work for this general case has revolved around the infamous Erdös-Faber-Lovász Conjecture [9,10,11]. If the conjecture is proven, then $\chi(G) \leq v$. In the first part of Chapter 4, the relationship between this conjecture and the chromatic number of the block-intersection graph is explained. The section is mainly expository except for the proof of the equivalence between the Erdös-Faber-Lovász Conjecture and a new conjecture.

More is known about coloring the block-intersection graph of a balanced incomplete block design. Let $(V, \mathcal{B})$ be a $BIBD(v, k, 1)$ and let $G = B(\mathcal{B})$. Since each point is in r blocks, at least r colors are needed to color G. Those designs for which $\chi(G) = r$ are called *resolvable*. Using this and the upper bound given by Brook's Theorem we have

$$\frac{v-1}{k-1} \leq \chi(G) \leq \Delta(G) = \frac{k(v-k)}{k-1}.$$

Again if the Erdös-Faber-Lovász Conjecture is true, then $\chi(G) \leq v$. C. J. Colbourn and M. J. Colbourn [7] have shown this to be true for the case when $\mathcal{B}$ is a cyclic design (designs generated by translations of a difference family). N. Pippenger and J. Spencer [16] have recently proven an asymptotic result for Steiner triple systems. If $\mathcal{G}$ is the set of all Steiner triple systems, then they prove that

$$\lim_{\mathcal{B}} \frac{\chi(B(\mathcal{B}))}{\frac{1}{2}|\cup \mathcal{B}|} = 1$$

where $|\cup \mathcal{B}| \to \infty$ following the filter of cofinite sets over $\mathcal{G}$.

The last section of the thesis proves a result on coloring the neighborhood of any vertex of the block-intersection graph of a balanced incomplete block design. Other new results are obtained on K_ℓ-factors of ℓ-partite graphs. These are used to color the neighborhood of a vertex with r colors.

1.4 Other Needed Results

In this section we state a result without proof that will be used over and over in the thesis. We also present a small lemma that will be needed later.

Theorem 1.4 (Hall's Theorem, see [5]) *Let $H = (X, Y)$ be a bipartite graph. Then H contains a matching that saturates every vertex in X if and only if*

$$|N_H(S)| \geq |S| \textit{ for all } S \subseteq X.$$

Lemma 1.5 *If $H = (X, Y)$ is a bipartite graph with $|X| = |Y| = n$ and $\delta(H) \geq \frac{1}{2}n$, then H has a perfect matching.*

Proof: Let $S \subseteq X$. If $0 < |S| \leq \frac{1}{2}n$, then for all $x \in S$, $|S| \leq \frac{1}{2}n \leq \delta(H) \leq d_H(x) \leq |N_H(S)|$. If $|S| > \frac{1}{2}n$, then every $y \in Y$ has a neighbor in S since $d_H(y) \geq \frac{1}{2}n$ and $|X| = n$. Thus $|S| \leq n = |Y| = |N_H(S)|$. Therefore by Hall's Theorem, H has a perfect matching. ∎

Chapter 2

Cycles

In this chapter we focus our attention on the cycles of the block-intersection graph of a pairwise balanced design. We start with the case when the design only has blocks of cardinality two. We then present a result for the case when the design has blocks of cardinality three or more. The chapter ends with a discussion about the remaining case.

2.1 Blocks of Cardinality Two Only

The first theorem in this chapter is presented for two reasons. First, it answers the edge-pancyclicity question for the specific case when we only have blocks of cardinality two. Although it has not yet come to the author's attention as to whether this result was known previously, it would not be suprising if it were. More importantly, though, is the second reason. The proof of the first theorem provides an outline for the much longer proof of the second theorem of this chapter. The proofs differ in how paths are joined to the neighborhood of a vertex. This process is a trivial matter in the first proof, but a complex one in the second. The basic outlines are the same. Although the statement of the next theorem is strictly in terms of graphs we will use block designs in the proof. It is hoped that by presenting the proof of the first theorem in this way the second proof will thus be more readable.

Theorem 2.1 *The line graph of the complete graph K_v, $L(K_v)$, is edge-pancyclic.*

Proof: If $\mathcal{B}$ is a $BIBD(v,2,1)$, then $\mathcal{B}$ can be viewed as K_v where points of $\mathcal{B}$ correspond to vertices of K_v and blocks of $\mathcal{B}$ correspond to edges of K_v. Thus $B(\mathcal{B}) \cong L(K_v)$.

Let B^*C^* be an edge of $G = B(\mathcal{B})$ and let $B^* = \{b_1, b_2\}$ and $C^* = \{b_1, c\}$. Define $\mathcal{B}_i = \{B \in \mathcal{B} : B \cap B^* = \{b_i\}\}$ for $i = 1, 2$. Since each b_i is in exactly $v-2$ blocks in $\mathcal{B}$ other than B^*, $|\mathcal{B}_i| = v-2$. Moreover, each $B \in \mathcal{B}_i$ has exactly one neighbor in $\mathcal{B}_j$, $i, j \in \{1, 2\}$, $i \neq j$.

Let $p \in \{3, 4, \ldots, |V(G)|\}$. We need to construct a p-cycle which uses B^*C^*.

If $3 \leq p \leq v-1$, then choose $p-2$ vertices other than C^* from $\mathcal{B}_1$. These vertices in any order along with one of the vertices B^* and C^* at each end and the edge B^*C^* form a p-cycle.

If $v \leq p \leq 2v-3$, then form a $(v-1)$-cycle Q in $\mathcal{B}_1$ using B^*C^* as just described and let A be the vertex on Q adjacent to B^* that is not C^*. Let

$\mathcal{Q}'$ be the path defined by $\mathcal{Q}' = \mathcal{Q} \setminus \{AB^*\}$ and let A^* be the neighbor of A in $\mathcal{B}_2$. Futhermore, let $\mathcal{R}$ be any $(p-v)$-path using vertices of $\mathcal{B}_2$ that starts in A^*. Then $\mathcal{Q}'\mathcal{R}B^*$ is a p-cycle containing the edge B^*C^*.

We now deal with the final case: $p > 2v-3$. Let $\mathcal{D} = \mathcal{B} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \{B^*\})$ and let $\mathcal{L}_1$ be a path of maximum length in $G[\mathcal{D}]$ (the subgraph of G induced by the vertices of $\mathcal{D}$). For $j \geq 1$, let $\mathcal{L}_{j+1}$ be a path of maximum length in $G[\mathcal{D}_{j+1}]$ where

$$\mathcal{D}_{j+1} = \mathcal{D} \setminus (V(\mathcal{L}_1) \cup V(\mathcal{L}_2) \cup \ldots \cup V(\mathcal{L}_j)).$$

Moreover, let s be the first integer j such that $\mathcal{D}_{j+1} = \emptyset$. For each $t \in \{1, 2, \ldots, s\}$, we say $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_t$ is a *truncation* of $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_t$ if for $i = 1, 2, \ldots, t-1$, $\mathcal{N}_i = \mathcal{L}_i$ and $\mathcal{N}_t$ is a subpath of $\mathcal{L}_t$ having the same initial vertex. Choose a truncation so that $|V(\mathcal{N}_1) \cup V(\mathcal{N}_2) \cup \cdots \cup V(\mathcal{N}_t)| = p - (2v-3)$.

For $j = 1, 2, \ldots, t$, choose $a_j \in V$ from the first vertex (a block) of $\mathcal{N}_j$ and $c_j \in V$ from the last vertex of $\mathcal{N}_j$ so that $c_j \neq a_j$ (note that this is possible since any vertex of $\mathcal{N}_j$ has two points). Since each $\mathcal{L}_j$ is a maximum length path in $G[\mathcal{D}_j]$, and $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_t$ is a truncation, the points $a_1, c_1, a_2, c_2, \ldots, a_t, c_t$ are distinct. For $j = 1, 2, \ldots, t$, let $A_j \in \mathcal{B}_2$ and $C_j \in \mathcal{B}_2$ be the blocks such that $a_j \in A_j$ and $c_j \in C_j$. Since $|A_j| = 2 = |C_j|$ and $b_2 \in A_j$ and $b_2 \in C_j$, the blocks $A_1, C_1, A_2, C_2, \ldots, A_t, C_t$ are distinct.

We now join the paths $A_j\mathcal{N}_jC_j$ to a cycle which uses all the vertices of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \{B^*\}$. Form a $(2v-3)$-cycle $\mathcal{S}$ in $G[\mathcal{B}_1 \cup \mathcal{B}_2 \cup \{B^*\}]$ that uses

B^*C^* as in the previous case. Since $\mathcal{B}_2$ is a clique and the edges of the path $\mathcal{R}$ (notation from the previous case) are not specified, we may choose $\mathcal{R}$ so that the edges $A_1C_1, A_2C_2, \ldots, A_tC_t$ be in $\mathcal{S}$. Replacing each edge A_jC_j with the path $A_j\mathcal{N}_jC_j$ transforms the cycle $\mathcal{S}$ into a p-cycle which uses B^*C^*. ∎

2.2 Blocks of Differing Cardinalities

The proof of Theorem 2.1 serves as an outline for the proof of the next theorem. In the previous proof, each vertex contained the same number of points. This is not the case for the graph in question now.

In the rest of this chapter let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$ such that $\ell = \min K \geq 3$. Moreover, let $u = \max K$ and choose $B^* \in \mathcal{B}$ so that $B^* = \{b_1, b_2, \ldots, b_u\}$. For $i = 1, 2, \ldots, u$ define

$$\mathcal{B}_i = \{B \in \mathcal{B} : B \cap B^* = \{b_i\}\}.$$

Finally, let $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_u$ and let $G = B(\mathcal{B})$. We have chosen a vertex B^* in G which contains the maximum number of points in the design because this choice ensures that when we carefully choose paths in the rest of the graph, these paths can be joined to the neighborhood $\mathcal{B}^*$ of B^* to create the desired cycle.

It will be seen in the proof of Theorem 2.3 that each case is divided into first constructing 'short' cycles and then constructing 'long' cycles that contain a particular edge. The following proposition is the core of the

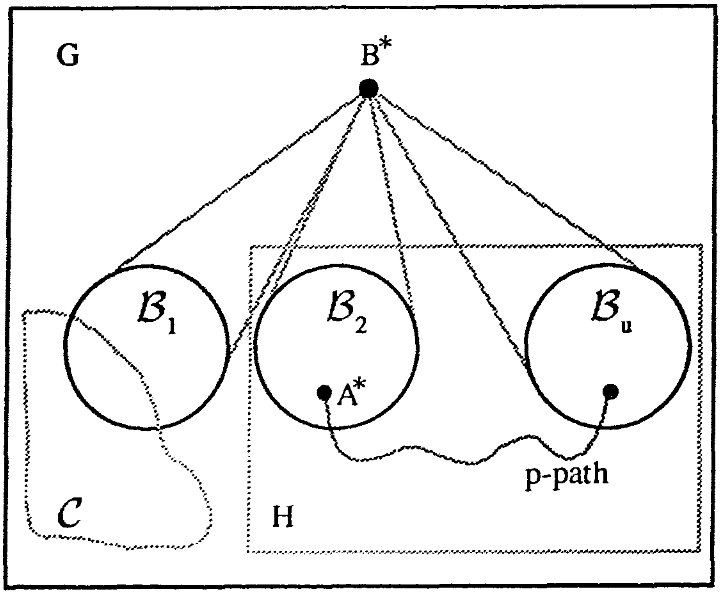

Figure 2.1: Proposition 2.2 guarantees the existence of a p-path in H.

proof of the theorem. It guarantees the existence of long paths which are used in the proof to construct long cycles.

Proposition 2.2 *Suppose* $(V, \mathcal{B})$ *is a* $PBD(v, K, 1)$ *such that* $\ell \geq 3$ *and suppose* B^* *is any block of* $\mathcal{B}$ *such that* $|B^*| = u$. *Let* A^* *be any vertex of* $\mathcal{B}_2$, *let* $\mathcal{C} \subseteq \mathcal{B} \setminus (\mathcal{B}_2 \cup \mathcal{B}_3 \cup \ldots \cup \mathcal{B}_u)$ *and let* $H = G[\mathcal{B} \setminus (\mathcal{C} \cup \mathcal{B}_1 \cup \{B^*\})]$. *For each* p, $1 \leq p \leq |V(H)| - 1$, *there is a path in* H *of length* p *that starts in* A^* *and ends in a vertex of* $\mathcal{B}^* \setminus \mathcal{B}_1$.

Proof: Figure 2.1 illustrates the statement of the proposition.

Let $p \in \{1, 2, \ldots, |V(H)| - 1\}$ and $n = |\mathcal{B}^* \setminus \mathcal{B}_1| - 1$. If $p \leq n$, then the desired path is straightforward to construct. Form a path of length n by starting at A^* and including the rest of the vertices of $\mathcal{B}_2$ in any order ($\mathcal{B}_2$ is a clique in H so this is possible). Next, choose a neighbor in $\mathcal{B}_3$ of the last vertex in $\mathcal{B}_2$ of the path constructed so far, and continue the path from the neighbor using the rest of the vertices of $\mathcal{B}_3$ in any order. Note that since $\ell \geq 3$ and since H is an induced subgraph of G, Lemma 1.2 guarantees that such a neighbor exists. Repeat this last step in $\mathcal{B}_4$, then in $\mathcal{B}_5$, and so on until the path includes all the vertices of $\mathcal{B}^* \setminus \mathcal{B}_1$. The subpath of this path that starts in A^* of length p is the desired path.

Suppose therefore that $p > n$. We will use $p - n$ vertices of H that are not in $\mathcal{B}^* \setminus \mathcal{B}_1$ with the $n + 1$ vertices of $\mathcal{B}^* \setminus \mathcal{B}_1$. To do this, a sequence of paths in the rest of H having $p - n$ vertices are joined to the vertices of $\mathcal{B}^* \setminus \mathcal{B}_1$.

Let $\mathcal{D} = \mathcal{B} \setminus (\mathcal{C} \cup \mathcal{B}^* \cup \{B^*\})$. ($\mathcal{D}$ is the set of all vertices in H not in $\mathcal{B}^*$.) Let $\mathcal{L}_1$ be a path of maximum length in $H[\mathcal{D}]$ (the subgraph of H induced by the vertices of $\mathcal{D}$). For $j \geq 1$, let $\mathcal{L}_{j+1}$ be a path of maximum length in $H[\mathcal{D}_{j+1}]$, where

$$\mathcal{D}_{j+1} = \mathcal{D} \setminus (V(\mathcal{L}_1) \cup V(\mathcal{L}_2) \cup \ldots \cup V(\mathcal{L}_j)).$$

Moreover, let s be the first integer j such that $\mathcal{D}_{j+1} = \emptyset$. Let $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_t$ be a truncation of $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_t$ such that $|V(\mathcal{N}_1) \cup V(\mathcal{N}_2) \cup \cdots \cup V(\mathcal{N}_t)| = p - n$.

For $j = 1, 2, \ldots, t$, choose $a_j \in V \setminus A^*$ from the first vertex (a block) of $\mathcal{N}_j$ and $c_j \in V \setminus A^*$ from the last vertex of $\mathcal{N}_j$ so that $c_j \neq a_j$ (note that this is possible since any vertex of $\mathcal{N}_j$ has at least three points and at most one of them is in A^*). Define Z to be the bipartite graph (X, Y) where $X = \{a_1, c_1, a_2, c_2, \ldots, a_t, c_t\}$ (note that because of the maximality of the $\mathcal{L}_j$ paths, $|X| = 2t$), $Y = \mathcal{B}^* \setminus (\mathcal{B}_1 \cup \{A^*\})$, and for all $x \in X$ and $B \in Y$, $xB \in E(Z)$ if and only if $x \in B$. Since $d_Z(x) = u - 1$ for all $x \in X$, and $d_Z(B) \leq u - 1$ for all $B \in Y$, Z has a matching that saturates X. Let

$$\{a_1 A_1, c_1 C_1, a_2 A_2, c_2 C_2, \ldots, a_t A_t, c_t C_t\}$$

be such a matching. Then for each $j \in \{1, 2, \ldots, t\}$, $\mathcal{P}_j = A_j \mathcal{N}_j C_j$ is a path in H that starts and ends in $\mathcal{B}^* \setminus (\mathcal{B}_1 \cup \{A^*\})$.

We need an orderly way to create the p-path using these paths. Let $\mathcal{M}$ be the multigraph that has vertex-set $\{\mathcal{B}_2, \mathcal{B}_3, \ldots, \mathcal{B}_u\}$ and edge-set $\{e_1, e_2, \ldots, e_t\}$ where $e_j = \{\mathcal{B}_x, \mathcal{B}_y\}$ (there may be loops if $x = y$), $A_j \in \mathcal{B}_x$, and $C_j \in \mathcal{B}_y$, for $j = 1, 2, \ldots, t$. Let q be the number of connected components of $\mathcal{M}$, and for $i = 1, 2, \ldots, q$, let $2o_i$ be the number of odd degree vertices of component i. For each $i = 1, 2, \ldots, q$, if $o_i > 0$, then decompose the edge-set of component i of $\mathcal{M}$ into o_i edge-disjoint open trails, and if $o_i = 0$, then component i has an Euler trail. Let $\mathcal{T}$ be the set of all of these trails.

Note that $E(\mathcal{M}) = \bigcup_{T \in \mathcal{T}} E(T)$ and that if $o_i > 0$ for some $i \in \{1, 2, \ldots, q\}$, then the edge-disjoint trails of component i must begin and end in the $2o_i$ vertices of odd degree. Thus if two trails $T_1, T_2 \in \mathcal{T}$ have a common end-vertex in $\mathcal{M}$, then $T_1 = T_2$.

Each trail $T \in \mathcal{T}$, $T = \mathcal{B}_{i_1} e_{j_1} \mathcal{B}_{i_2} e_{j_2} \cdots e_{j_{r-1}} \mathcal{B}_{i_r}$, is easily transformed into a path $\mathcal{Q}_T$ in H: $\mathcal{Q}_T = \mathcal{Q}_{j_1}\mathcal{Q}_{j_2}\cdots\mathcal{Q}_{j_{r-1}}$ where $\mathcal{Q}_j$ is either $\mathcal{P}_j$ or its reverse, for $j = 1, 2, \ldots, t$. The rest of the proof involves changing all of the $\mathcal{Q}_T$ into cycles and combining these cycles into a single path. Before we continue, though, we need a definition.

We define a ***clique-edge*** to be an edge BC of G such that for some $i \in \{2, 3, \ldots, u\}$, $B, C \in V(\mathcal{B}_i)$. We say a path or cycle $\mathcal{R}$ in G is ***clique-edge extendible avoiding*** $\mathcal{A}$ for some $\mathcal{A} \subset V(\mathcal{R}) \cap \mathcal{B}^*$ if $\mathcal{R}$ satisfies: $B \in (V(\mathcal{R}) \cap \mathcal{B}^*) \setminus \mathcal{A}$ if and only if B is incident with a clique-edge from $E(\mathcal{R})$. Note that a path or cycle $\mathcal{R}$ which is clique-edge extendible avoiding a set $\mathcal{A}$ actually contains a clique-edge if and only if $\mathcal{A} \neq V(\mathcal{R}) \cap \mathcal{B}^*$. With this definition, we have for each $T \in \mathcal{T}$ that $\mathcal{Q}_T$ is clique-edge extendible avoiding the set of its end-vertices (the end-vertices being precisely those vertices on the paths which are not incident with clique-edges from the paths).

We now perform extensions on the $\mathcal{Q}_T$ to turn them into cycles in H which are clique-edge extendible avoiding a special set of vertices. We proceed step by step through the cliques $\mathcal{B}_2, \mathcal{B}_3, \ldots, \mathcal{B}_u$, looking in step j for a path that has an end-vertex in $\mathcal{B}_j$. If such a path exists, then we extend it to a cycle or to a longer path.

Let $\mathcal{R}_1 = \{(\mathcal{Q}_T, \mathcal{A}_T) : T \in \mathcal{T}\}$ where $\mathcal{A}_T$ is the set of end-vertices of $\mathcal{Q}_T$, for all $T \in \mathcal{T}$. It will be shown that at each step j, $1 \leq j \leq u$, the newly formed set $\mathcal{R}_j$ of paths and cycles with their special sets of vertices will have the following properties:

P1. if $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$, then $\mathcal{Q}$ is a path or cycle in H which is clique-edge extendible avoiding $\mathcal{A}$, $A^* \notin \mathcal{A}$, and the paths and cycles are mutually vertex disjoint;

P2. if $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ and $\mathcal{Q}$ is a cycle, then $\mathcal{Q}$ contains a clique-edge;

P3. if $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$ and $\mathcal{Q}$ is a path, then $\mathcal{Q}$ has both its end-vertices in $(\mathcal{B}_{j+1} \cup \mathcal{B}_{j+2} \cup \cdots \cup \mathcal{B}_u) \cap \mathcal{A}$; and

P4. for each $i = 2, 3, \ldots, u$, $|\{(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j : \mathcal{B}_i \cap \mathcal{A} \neq \emptyset\}| \leq 1$, and for all $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j$, $|\mathcal{B}_i \cap \mathcal{A}| \leq 2$. Moreover, $\mathcal{Q}$ is a path with both its end-vertices in $\mathcal{B}_i$ if and only if $|\mathcal{B}_i \cap \mathcal{A}| = 2$.

Property **P4** may be the least understandable of the properties. It ensures that for each $i = 2, 3, \ldots, u$ there is at most one path or cycle having a vertex in $\mathcal{B}_i$ which is not incident with a clique-edge on the path or cycle, and that a path has at most two such vertices in $\mathcal{B}_i$ and it has two if and only if these vertices are the end-vertices of the path. Morever, **P4** implies that every cycle has at most one vertex in $\mathcal{B}_i$ which is not incident with a clique-edge of the cycle since any cycle has at most two such vertices by **P4** but cannot have two (only paths can have two by **P4**).

Properties **P1**–**P4** are true for $\mathcal{R}_1$. Suppose that for some j, $1 < j \leq u$, $\mathcal{R}_{j-1}$ satisfies **P1**–**P4**. We define $\mathcal{R}_j$ using the following cases with the intention that $\mathcal{R}_j$ will also satisfy **P1**–**P4**. Since $\mathcal{R}_{j-1}$ satisfies **P4**, we may use the following three main cases.

Case 1. There does not exist an end-vertex S of a path Q which is clique-edge extendible avoiding $\mathcal{A}$ such that $S \in \mathcal{B}_j \cap \mathcal{A}$ and $(Q, \mathcal{A}) \in \mathcal{R}_{j-1}$.

In this case we let $\mathcal{R}_j = \mathcal{R}_{j-1}$. By the induction hypothesis $\mathcal{R}_{j-1}$ satisfies **P1**–**P4**. Thus $\mathcal{R}_j$ satisfies **P1**, **P2**, and **P4**. Since $\mathcal{R}_{j-1}$ satisfies **P3** and we are in Case 1, $\mathcal{R}_j$ satisfies **P3**.

Case 2. There exist end-vertices $S, F \in \mathcal{B}_j$, $S \neq F$, of a path Q which is clique-edge extendible avoiding $\mathcal{A}$ such that $(Q, \mathcal{A}) \in \mathcal{R}_{j-1}$.

In this case we replace a path avoiding a set with a cycle avoiding a set. Without loss of generality, let Q start in S and end in F. Let $Q' = QS$, let $\mathcal{A}' = \mathcal{A} \setminus \{S, F\}$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(Q, \mathcal{A})\}) \cup \{(Q', \mathcal{A}')\}$. Then the cycle Q' is clique-edge extendible avoiding $\mathcal{A}'$ and $A^* \notin \mathcal{A}'$ since $A^* \notin \mathcal{A}$. Thus $\mathcal{R}_j$ satisfies **P1**. Since $SF \in E(Q')$ is a clique-edge, $\mathcal{R}_j$ satisfies **P2**. Any path (avoiding a set) in $\mathcal{R}_j$ is also in $\mathcal{R}_{j-1}$ and $\mathcal{R}_j$ has no path with an end-vertex in $\mathcal{B}_j$ (there can only be one since $\mathcal{R}_{j-1}$ satisfies **P4** and we have dealt with it here in Case 2), and so $\mathcal{R}_j$ satisfies **P3**. Since $\mathcal{R}_{j-1}$ satisfies **P4**, $S, F \in \mathcal{A}$ and hence $\mathcal{B}_j \cap \mathcal{A}' = \emptyset$. Thus $\mathcal{R}_j$ satisfies **P4**.

Case 3. There exists a path Q which is clique-edge extendible avoiding $\mathcal{A}$ which starts in a vertex $S \in \mathcal{B}_j$ such that $(Q, \mathcal{A}) \in \mathcal{R}_{j-1}$ and

$\mathcal{B}_j \cap \mathcal{A} = \{S\}$.

Let $\mathcal{Q}$ end in a vertex $F \in \mathcal{B}_m$, for some $m \in \{2, 3, \ldots, u\}$. Since $\mathcal{R}_{j-1}$ satisfies **P3**, $m \geq j$, and since $F \neq S$ and $\mathcal{B}_j \cap \mathcal{A} = \{S\}$, $F \notin \mathcal{B}_j$. Hence $m > j \geq 2$. Let $N_m(S) = \{B \in \mathcal{B}_m : S \cap B \neq \emptyset\}$.

We choose a vertex Q from $N_m(S)$ carefully. Let $\{Q_1, Q_2, \ldots, Q_r\} = V(\mathcal{Q}) \cap \mathcal{B}_m$ so that the indices of the Q_i correspond to their order on the path $\mathcal{Q}$ (from S to F). Note that $Q_r = F$ and so $r \geq 1$. If $r > 1$, then since $F \in \mathcal{B}_m \cap \mathcal{A}$ and $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$ by **P1**, F is not incident with a clique-edge from $\mathcal{Q}$ implying that $Q_{r-1}Q_r \notin E(\mathcal{Q})$. Moreover, $Q_1Q_2 \in E(\mathcal{Q})$ since $Q_1 \notin \mathcal{A}$ and hence is incident with a clique-edge in $\mathcal{B}_m$. The other end-vertex of this clique-edge appears next on the path $\mathcal{Q}$ and so must be Q_2 (see Figure 2.2).

Since $\ell \geq 3$, $|N_m(S)| \geq 2$, and so we choose $Q \in N_m(S) \setminus \{Q_1\}$. We will use the fact that $Q \neq Q_1$ in Case 3(f).

We have several subcases depending on where the vertex Q is within $\mathcal{B}_m$.

Case 3(a). For all $(\mathcal{R}, \mathcal{A}) \in \mathcal{R}_{j-1}$, $Q \notin V(\mathcal{R})$.

Let $\mathcal{Q}' = \mathcal{Q}QS$, let $\mathcal{A}' = \mathcal{A} \setminus \{F\}$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$. Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$ and $QF \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$

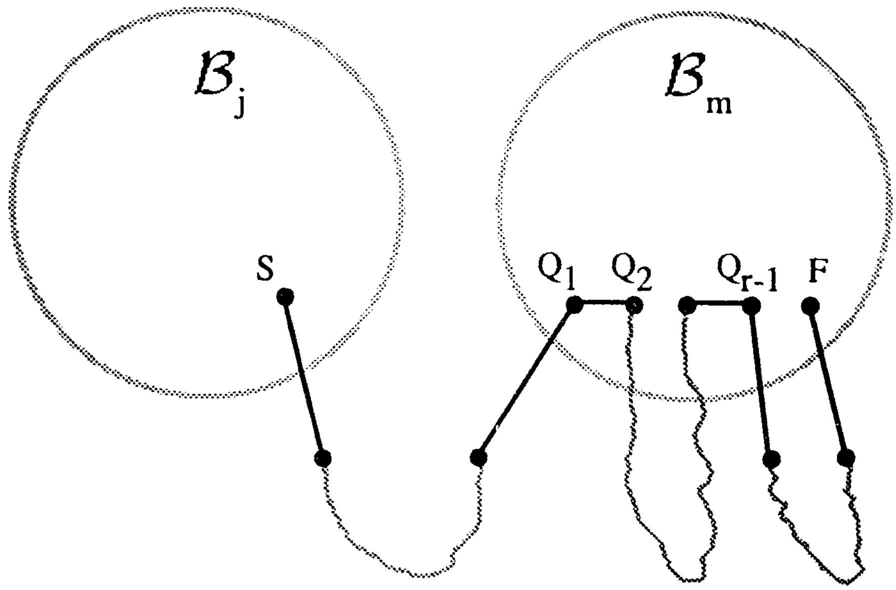

Figure 2.2: Case 3 when $r > 1$.

satisfies **P3**. Finally, $\mathcal{R}_j$ satisfies **P4** because $\mathcal{R}_{j-1}$ does and because $\mathcal{B}_m \cap \mathcal{A}' = \emptyset$.

Case 3(b). $Q \in V(\mathcal{R})$ for some cycle $\mathcal{R}$ avoiding a set $\mathcal{A}_1$ such that $(\mathcal{R}, \mathcal{A}_1) \in \mathcal{R}_{j-1}$.

Since $F \in \mathcal{B}_m \cap \mathcal{A}$, and since $\mathcal{R}_{j-1}$ satisfies **P4**, we conclude that $\mathcal{A}_1 \cap \mathcal{B}_m = \emptyset$. Moreover, since $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, there exists $U \in \mathcal{B}_m$ such that $QU \in E(\mathcal{R})$. Let $\mathcal{R}'$ be the path that starts in U and ends in Q such that $\mathcal{R} = \mathcal{R}'U$ (see Figure 2.3).

The cycle $\mathcal{Q}'$ in this case is $Q\mathcal{R}'S$. If QU is the only clique-edge incident with Q, then let $\mathcal{A}'$ be $(\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1 \cup \{Q\}$. Otherwise, let $\mathcal{A}'$ be $(\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1$. Finally, let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A}), (\mathcal{R}, \mathcal{A}_1)\}) \cup$

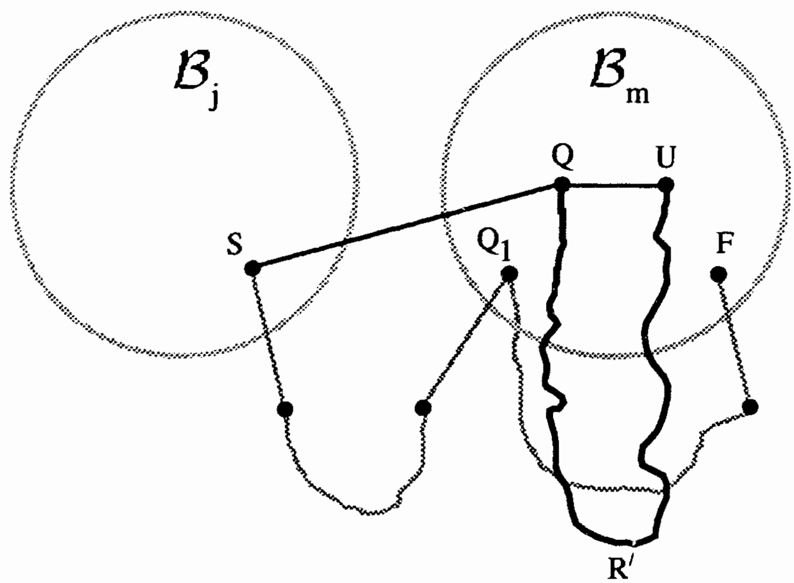

Figure 2.3: Case 3(b).

$\{(\mathcal{Q}', \mathcal{A}')\}$. Note that $A^* \neq Q$ since $Q \in \mathcal{B}_m$ and $m > 2$ whereas $A^* \in \mathcal{B}_2$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, and $FU \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Finally, since $\mathcal{A}' \cap \mathcal{B}_m \subseteq \{Q\}$, $\mathcal{R}_j$ satisfies **P4**.

Case 3(c). $Q \in V(\mathcal{R})$ for some path $\mathcal{R}$ avoiding a set $\mathcal{A}_1$ such that $(\mathcal{R}, \mathcal{A}_1) \in \mathcal{R}_{j-1}$.

In this case we replace the paths $\mathcal{Q}$ and $\mathcal{R}$ with a longer path $\mathcal{Q}'$. As in Case 3(b), there exists $U \in \mathcal{B}_m$ such that $QU \in E(\mathcal{R})$. Let $\mathcal{W}_1$ be

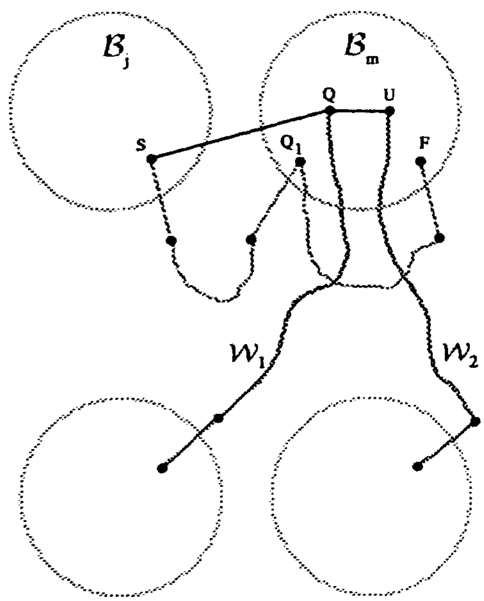

Figure 2.4: Case 3(c).

the path that ends in Q and $\mathcal{W}_2$ be the path that starts in U such that $\mathcal{W}_1\mathcal{W}_2 = \mathcal{R}$ (or the reversed path of $\mathcal{R}$; see Figure 2.4).

We form the longer path $\mathcal{Q}' = \mathcal{W}_1 \mathcal{Q} \mathcal{W}_2$. If QU is the only clique-edge incident with Q in $\mathcal{R}$, then let $\mathcal{A}' = (\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1 \cup \{Q\}$. Otherwise, let $\mathcal{A}' = (\mathcal{A} \setminus \{F\}) \cup \mathcal{A}_1$. Finally, let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A}), (\mathcal{R}, \mathcal{A}_1)\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$. Note again that $A^* \neq Q$ (for the same reasons given in Case 3(b)).

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, $\mathcal{R}$ is clique-edge extendible avoiding $\mathcal{A}_1$, and $FU \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a path that has the same end-vertices as $\mathcal{R}$

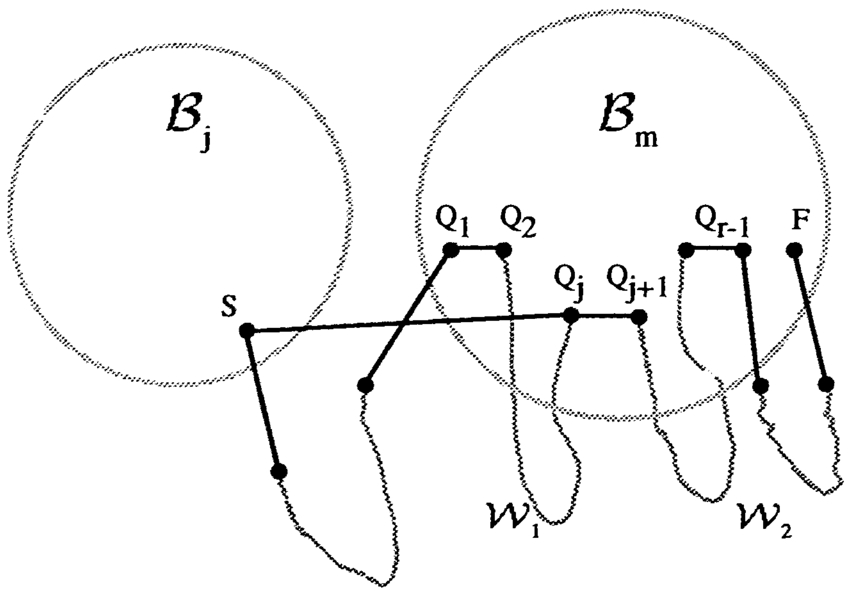

Figure 2.5: Case 3(d).

neither of which are in $\mathcal{B}_j$ (a condition of Case 3 is that $\mathcal{B}_j \cap \mathcal{A} = \{S\}$), $\mathcal{R}_j$ satisfies **P3**. Finally, since $\mathcal{A}' \cap \mathcal{B}_m \subseteq \{Q\}$, $\mathcal{R}_j$ satisfies **P4**.

Case 3(d). $Q = Q_j$ for some $j \in \{2, 3, \ldots, r-3, r-2\}$, and $Q_jQ_{j+1} \in E(\mathcal{Q})$.

In this case we break up $\mathcal{Q}$ into two paths and use these paths to form two cycles $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$. Let $\mathcal{Q} = \mathcal{W}_1\mathcal{W}_2$ so that $\mathcal{W}_1$ starts in S and ends in Q_j, and $\mathcal{W}_2$ starts in Q_{j+1} and ends in F (see Figure 2.5).

Let $\mathcal{Q}_1' = \mathcal{W}_1S$ and if $Q_{j-1}Q_j \notin E(\mathcal{Q})$, then let $\mathcal{A}_1' = (\mathcal{A} \cap V(\mathcal{W}_1)) \cup \{Q_j\}$. Otherwise, let $\mathcal{A}_1' = (\mathcal{A} \cap V(\mathcal{W}_1))$. Moreover, let $\mathcal{Q}_2' = \mathcal{W}_2Q_{j+1}$, $\mathcal{A}_2' = (\mathcal{A} \cap V(\mathcal{W}_2)) \setminus \{F\}$, and $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}_1', \mathcal{A}_1'), (\mathcal{Q}_2', \mathcal{A}_2')\}$. Note that $Q_j \neq A^*$ since $Q_j \in \mathcal{B}_m$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, and $Q_j \in \mathcal{A}'_1$ if and only if $Q_{j-1}Q_j \notin E(\mathcal{Q})$, $\mathcal{Q}'_1$ is clique-edge extendible avoiding $\mathcal{A}'_1$. Also, since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$ and $FQ_{j+1} \in E(\mathcal{Q}'_2)$ is a clique-edge, $\mathcal{Q}'_2$ is clique-edge extendible avoiding $\mathcal{A}'_2$. Since $Q_1Q_2 \in E(\mathcal{Q}'_1)$ is a clique-edge as well (note that $j \geq 2$), $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'_1$ and $\mathcal{Q}'_2$ are cycles, $\mathcal{R}_j$ satisfies **P3**. Finally, since $\mathcal{A}'_1 \cap \mathcal{B}_m \subseteq \{Q_j\}$ and $\mathcal{A}'_2 \cap \mathcal{B}_m = \emptyset$, $\mathcal{R}_j$ satisfies **P4**.

Case 3(e). $Q = Q_j$ for some $j \in \{2, 3, \ldots, r-2, r-1\}$, and $Q_jQ_{j+1} \notin E(\mathcal{Q})$.

In this case, form a longer cycle from $\mathcal{Q}$. Note that since $Q_jQ_{j+1} \notin E(\mathcal{Q})$ and $Q_j \notin \mathcal{A}$, $Q_{j-1}Q_j \in E(\mathcal{Q})$. Let $\mathcal{Q} = \mathcal{W}_1\mathcal{W}_2$ so that $\mathcal{W}_1$ starts in S and ends in Q_{j-1}, and $\mathcal{W}_2$ starts in Q_j and ends in F (see Figure 2.6).

We define $\mathcal{Q}' = \mathcal{W}_1F\overline{\mathcal{W}_2}S$ where $\overline{\mathcal{W}_2}$ is the reverse of $\mathcal{W}_2$. Moreover, let $\mathcal{A}' = (\mathcal{A} \setminus \{F\}) \cup \{Q_j\}$, and $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A}')\}$. Again note that $Q_j \neq A^*$.

Since $\mathcal{Q}$ is clique-edge extendible avoiding $\mathcal{A}$, and $FQ_{j-1} \in E(\mathcal{Q}')$ is a clique-edge and $Q_j \in \mathcal{A}'$, $\mathcal{Q}'$ is clique-edge extendible avoiding $\mathcal{A}'$. Thus $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Finally, since Q_j replaces F in $\mathcal{A}'$, $\mathcal{R}_j$ satisfies **P4**.

Case 3(f). $Q = Q_r$.

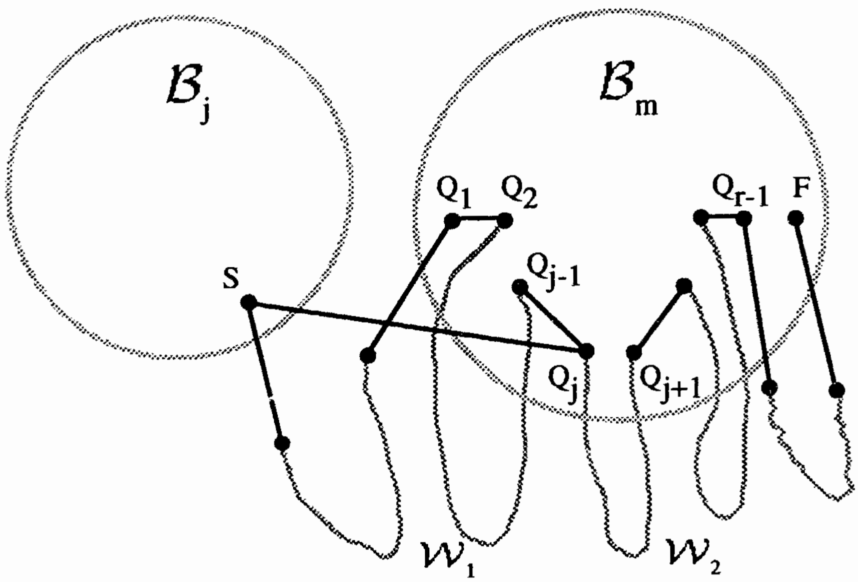

Figure 2.6: Case 3(e).

Since $Q \neq Q_1$, $r \geq 3$. Let $\mathcal{Q}' = \mathcal{Q}S$, and let $\mathcal{R}_j = (\mathcal{R}_{j-1} \setminus \{(\mathcal{Q}, \mathcal{A})\}) \cup \{(\mathcal{Q}', \mathcal{A})\}$. Since $Q_1Q_2 \in E(\mathcal{Q}')$ is a clique-edge, $\mathcal{R}_j$ satisfies **P1** and **P2**. Since $(\mathcal{Q}, \mathcal{A}) \notin \mathcal{R}_j$, and $\mathcal{Q}'$ is a cycle, $\mathcal{R}_j$ satisfies **P3**. Moreover, $\mathcal{R}_j$ satisfies **P4** since $\mathcal{R}_{j-1}$ does.

Thus $\mathcal{R}_j$ satisfies **P1**–**P4** and hence by induction, $\mathcal{R}_u$ satisfies **P1**–**P4**. By **P3**, if $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_u$, then $\mathcal{Q}$ is a cycle.

We extend the cycles in $\mathcal{R}_u$ even further. First, suppose for some $i \in \{2, 3, \ldots, u\}$, there exist $(\mathcal{Q}_1, \mathcal{A}_1), (\mathcal{Q}_2, \mathcal{A}_2) \in \mathcal{R}_u$, $\mathcal{Q}_1 \neq \mathcal{Q}_2$, such that $\mathcal{Q}_1$ has a clique-edge U_1V_1 in $\mathcal{B}_i$, and $\mathcal{Q}_2$ also has a clique-edge U_2V_2 in $\mathcal{B}_i$. We can then replace $(\mathcal{Q}_1, \mathcal{A}_1)$ and $(\mathcal{Q}_2, \mathcal{A}_2)$ in $\mathcal{R}_u$ with $(\mathcal{W}_1\mathcal{W}_2V_1, \mathcal{A}_1 \cup \mathcal{A}_2)$, where $\mathcal{Q}_1 = \mathcal{W}_1U_1V_1$ and $\mathcal{Q}_2 = \mathcal{W}_2U_2V_2$. Let $\mathcal{R}_{u+1}$ be a set resulting from performing this type of replacement as many times as possible on each $\mathcal{B}_i$ (some cycles may be extended more than once). Then $\mathcal{R}_{u+1}$ satisfies **P1–P4**, and the additional property:

P5. for $i = 2, 3, \ldots, u$,

$$|\{\mathcal{Q} : (\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_j \text{ for some } \mathcal{A}, \mathcal{Q} \text{ has a clique-edge in } \mathcal{B}_i\}| \leq 1.$$

Second, if there exists a vertex $B \in \mathcal{B}_i$, for some $i \in \{2, 3, \ldots, u\}$, such that B is not on any cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{u+1}$, then B may be included on a cycle that has a clique-edge in $\mathcal{B}_i$ (if one exists). Since $\mathcal{R}_{u+1}$ satisfies **P5**, there is at most one cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ (with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{u+1}$) that has a clique-edge UV in $\mathcal{B}_i$. If there is one, we extend $\mathcal{Q}$ by replacing UV with UBV (the set $\mathcal{A}$ remains the same). Do this for any such B in $\mathcal{B}^*$ and let $\mathcal{R}_{u+2}$ be the resulting set of pairs of modified cycles and sets. Then $\mathcal{R}_{u+2}$ satisfies **P1–P5**.

From now on, let

$$\mathcal{A}_i = \bigcup_{(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{u+2}} \mathcal{A} \cap \mathcal{B}_i,$$

for $i = 2, 3, \ldots, u$. Note that since $\mathcal{R}_{u+2}$ satisfies **P4**, $|\mathcal{A}_i| \leq 1$. Moreover, $\mathcal{R}_{u+2}$ satsifies the additional property:

P6. for all $i \in \{2, 3, \ldots, u\}$, if there exists a cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ with $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{u+2}$ such that $\mathcal{Q}$ has a clique-edge in $\mathcal{B}_i$, then every vertex $B \in \mathcal{B}_i \setminus \mathcal{A}_i$ is in $V(\mathcal{Q})$.

We are now ready to form the p-path that starts in A^*. Start by letting $U_2 = A^*$. Since $\mathcal{R}_{u+2}$ satisfies **P1**, $U_2 \in \mathcal{B}_2 \setminus \mathcal{A}_2$. If U_2 is on some cycle $\mathcal{Q}$ avoiding a set $\mathcal{A}$ such that $(\mathcal{Q}, \mathcal{A}) \in \mathcal{R}_{u+2}$, then $U_2 \notin \mathcal{A}$ and so there exist a $V_2 \in \mathcal{B}_2 \setminus \mathcal{A}_2$ such that $U_2V_2 \in E(\mathcal{Q})$ since $\mathcal{Q}$ is clique-extendible avoiding $\mathcal{A}$. We then let $\mathcal{S}_2$ be the path defined by $\mathcal{Q} = \mathcal{S}_2U_2$. Note that $\mathcal{B}_2 \subset \mathcal{A}_2 \cup V(\mathcal{S}_2)$ by **P6**. If U_2 is not on some cycle, then let $\mathcal{S}_2$ be the path which starts in U_2 and includes all the vertices of $\mathcal{B}_2 \setminus \mathcal{A}_2$ in any order (note that none of these vertices are on a cycle avoiding a set in $\mathcal{R}_{u+2}$ by **P6**). Let $\mathcal{S}_2$ end in the vertex V_2.

If $\mathcal{B}_3 \subset \mathcal{A}_3 \cup V(\mathcal{S}_2)$, then let $\mathcal{S}_3 = \mathcal{S}_2$ and $V_3 = V_2$. Otherwise, $V(\mathcal{S}_2) \cap (\mathcal{B}_3 \setminus \mathcal{A}_3) = \emptyset$ by **P6**, and so let U_3 be a neighbor of V_2 such that $U_3 \in \mathcal{B}_3 \setminus \mathcal{A}_3$. This is possible since V_2 has $|V_2| - 1 \geq \ell - 1 \geq 2$ neighbors in $\mathcal{B}_3$ and $|\mathcal{A}_3| \leq 1$. Using the above method we form a path that starts at U_3, ends in some vertex $V_3 \in \mathcal{B}_3 \setminus \mathcal{A}_3$, and includes all of the vertices of $\mathcal{B}_3 \setminus \mathcal{A}_3$. Adjoining this path to the end of $\mathcal{S}_2$ we form a path $\mathcal{S}_3$ which starts in U_2 and ends in V_3 and contains all of the vertices of $(\mathcal{B}_2 \setminus \mathcal{A}_2) \cup (\mathcal{B}_3 \setminus \mathcal{A}_3)$.

Suppose that for some $j \in \{3, 4, \ldots, u-1\}$, we have formed a path $\mathcal{S}_j$ which starts in U_2, ends in a vertex V_j, and which contains all of the vertices from $(\mathcal{B}_2 \setminus \mathcal{A}_2) \cup (\mathcal{B}_3 \setminus \mathcal{A}_3) \cup \cdots \cup (\mathcal{B}_j \setminus \mathcal{A}_j)$. If $\mathcal{B}_{j+1} \subset \mathcal{A}_{j+1} \cup V(\mathcal{S}_j)$, then let $\mathcal{S}_{j+1} = \mathcal{S}_j$ and $V_{j+1} = V_j$. Otherwise, $V(\mathcal{S}_j) \cap (\mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1}) = \emptyset$ by **P6**, and so let U_{j+1} be a neighbor of V_j such that $U_{j+1} \in \mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1}$. We extend $\mathcal{S}_j$ to a path $\mathcal{S}_{j+1}$ using the method of the last paragraph. Let $\mathcal{S}_{j+1}$ end in a vertex V_{j+1}.

In both cases, we have formed a path $\mathcal{S}_{j+1}$ which starts in U_2, ends in a vertex $V_{j+1} \in \mathcal{B}^* \setminus \mathcal{B}_1$, and which contains all of the vertices from $(\mathcal{B}_2 \setminus \mathcal{A}_2) \cup (\mathcal{B}_3 \setminus \mathcal{A}_3) \cup \cdots \cup (\mathcal{B}_{j+1} \setminus \mathcal{A}_{j+1})$. Now the path $\mathcal{S}_u$ which we get by induction actually contains all the vertices of $\mathcal{B}^* \setminus \mathcal{B}_1$. This is because each cycle avoiding a set in $\mathcal{R}_{u+2}$ has a clique-edge by **P2** and hence will be used to form some $\mathcal{S}_j$. Moreover, for all $i = 2, 3, \ldots, u$, any element in $\mathcal{A}_i$ is in some cycle avoiding a set in $\mathcal{R}_{u+2}$ and thus in some $\mathcal{S}_j$.

The p-path we want is just $\mathcal{S}_u$. ∎

The majority of the work for Theorem 2.3 is done in the last proposition. All that remains is to form small cycles and to change the long paths into cycles.

Theorem 2.3 *If* $(V, \mathcal{B})$ *is a* $PBD(v, K, 1)$ *with* $\min K \geq 3$, *then the block-intersection graph* $B(\mathcal{B})$ *is edge-pancyclic.*

Proof: Let $CD \in E(G)$ and let $q \in \{3, 4, \ldots, |V(G)|\}$. We will construct a q-cycle in G which uses CD. Let $B^* \in \mathcal{B}$ be such that $|B^*| = u$ and let $b = |\mathcal{B}_1|$. We have five cases depending on where C and D are in relation to $\mathcal{B}^*$.

Case 1. $B^* \in \{C, D\}$.

Suppose without loss of generality that $B^* = C$ and that $D \in \mathcal{B}_1$. If $q \leq b + 1$, then choose $q - 2$ vertices from $\mathcal{B}_1 \setminus \{D\}$ and order them to form a path. Let the last vertex on the path be called A. The path B^*D along with the constructed path and the edge AB^* is a q-cycle that uses CD. If $q > b+1$, then form a $(b+1)$-cycle $\mathcal{Q}$ using CD as just described and let $\mathcal{Q}'$ be that path defined by $\mathcal{Q}' = \mathcal{Q} \setminus \{AB^*\}$. Let A^* be a neighbor of A in $\mathcal{B}_2$. If $q = b+2$, then $\mathcal{Q}'A^*B^*$ is a q-cycle containing CD. If $q > b+2$, then by Proposition 2.2 there is a $(q-(b+2))$-path $\mathcal{R}$ that starts in A^*, ends in $\mathcal{B}^* \setminus \mathcal{B}_1$, and uses only vertices from $\mathcal{B} \setminus (\mathcal{B}_1 \cup \{B^*\})$. The required q-cycle is $\mathcal{Q}'\mathcal{R}B^*$.

Case 2. $C \cap B^* = \{x\} = D \cap B^*$ for some $x \in V$.

Without loss of generality, we may assume that $x = b_1$ so that $C, D \in \mathcal{B}_1$. B^*CDB^* is a 3-cycle using CD. For $4 \le q \le b+1$, choose any $q-3$ vertices from $\mathcal{B}_1 \setminus \{C, D\}$ and order them to form a path letting the last vertex be called A. Continue as in Case 1.

Case 3. $C \cap B^* = \{x\}$ and $D \cap B^* = \{y\}$ for some $x, y \in V$, $x \neq y$.

Without loss of generality, we may assume that $x = b_1$ and $y = b_2$ so that $C \in \mathcal{B}_1$ and $D \in \mathcal{B}_2$. B^*CDB^* is a 3-cycle using CD. For $4 \le q \le b+2$, choose any $q-3$ vertices from $\mathcal{B}_1 \setminus \{C\}$ and order them to form a path $\mathcal{Q}$. Then $B^*\mathcal{Q}CDB^*$ is a q-cycle using CD. For $q > b+2$, let $\mathcal{Q}$ be a path with $b-1$ vertices constructed as just described. By Proposition 2.2 there is a $(q-(b+2))$-path $\mathcal{R}$ that starts in D, ends in $\mathcal{B}^* \setminus \mathcal{B}_1$, and uses only the vertices from

$\mathcal{B} \setminus (\mathcal{B}_1 \cup \{B^*\}$. The required q-cycle is $B^*\mathcal{Q}C\mathcal{R}B^*$.

Case 4. $|C \cap B^*| + |D \cap B^*| = 1$.

Without loss of generality we may suppose that $C \cap B^* = \{b_1\}$ and that $D \cap B^* = \emptyset$ so that $C \in \mathcal{B}_1$ and $D \notin \mathcal{B}^*$. Let B be a neighbor of D in $\mathcal{B}_1$ that is not C. Then $BCDB$ is a 3-cycle in G and B^*CDBB^* is a 4-cycle in G. For $q \in \{5, 6, \ldots, b+2\}$, choose $q-4$ vertices from $\mathcal{B}_1$ none of which is B or C, and order them to form a path. Let the last vertex on this path be called A. The path B^*CDB along with the constructed path and the edge AB^* is a q-cycle that uses CD. If $q > b+2$, then form a $(b+2)$-cycle $\mathcal{Q}$ as just described and let $\mathcal{Q}'$ be the path defined by deleting the edge AB^* from $\mathcal{Q}$. Let A^* be a neighbor of A in $\mathcal{B}_2$. If $q = b+3$, then $\mathcal{Q}'A^*B^*$ is a q-cycle using CD. If $q > b+3$, then by Proposition 2.2 there is a $(q-(b+3))$-path $\mathcal{R}$ that starts in A^*, ends in $\mathcal{B}^* \setminus \mathcal{B}_1$, and uses only vertices from $\mathcal{B} \setminus (\mathcal{B}_1 \cup \{B^*, D\})$. The required q-cycle is $\mathcal{Q}'\mathcal{R}B^*$.

Case 5. $C \cap B^* = \emptyset = D \cap B^*$.

Let $c \in C \setminus D$ and $d \in D \setminus C$. There is a block $B \in \mathcal{B}$ such that $\{c, d\} \subset B$. $BCDB$ is a 3-cycle containing CD. Let C_1 be a neighbor of C in $\mathcal{B}_1$ and let D_1 be a neighbor of D in $\mathcal{B}_1$ such that $D_1 \neq C_1$ (D has $|D|$ neighbors in $\mathcal{B}_1$ since $D \cap B^* = \emptyset$). $D_1C_1CDD_1$ is a 4-cycle in G that uses CD and $B^*C_1CDD_1B^*$ is a 5-cycle in G that uses CD.

For $q \in \{6, 7, \ldots, b+3\}$, choose $q-5$ vertices from $\mathcal{B}_1$ none of which is C_1 or D_1, and order them to form a path. Let the last vertex on this path be called A. The path $B^*C_1CDD_1$ along with the constructed path and the edge AB^* is a q-cycle that uses CD. If $q > b+3$, then form a $(b+3)$-cycle $\mathcal{Q}$ as just described and let $\mathcal{Q}'$ be the path defined by deleting the edge AB^* from $\mathcal{Q}$. Let A^* be a neighbor of A in $\mathcal{B}_2$. If $q = b+3$, then $\mathcal{Q}'A^*B^*$ is a q-cycle using CD. If $q > b+4$, then by Proposition 2.2 there is a $(q-(b+4))$-path $\mathcal{R}$ that starts in A^*, ends in $\mathcal{B}^* \setminus \mathcal{B}_1$, and uses only vertices from $\mathcal{B} \setminus (\mathcal{B}_1 \cup \{B^*, C, D\})$. The required q-cycle is $\mathcal{Q}'\mathcal{R}B^*$.

In each case, a q-cycle is constructed which uses CD. Therefore, G is edge-pancyclic. ∎

2.3 Conclusion

This chapter has shown that the block-intersection graph of any pairwise balanced design with $\lambda = 1$ and mimimum block cardinality at least 3 is edge-pancyclic and, in particular, hamiltonian. If $\lambda > 1$, then characterizing those designs whose graphs are hamiltonian is unsettled. However, if the block-intersection graph of a $PBD(v, k, \lambda)$ is defined so that two blocks are adjacent in the graph if and only if they intersect in precisely λ points, then the graph need not be hamiltonian as the following example demonstrates.

Example 2.4 *A* $(6,3,2)$*-design* $(V, \mathcal{B})$ *such that the block-intersection graph* $B(\mathcal{B})$ *is non-hamiltonian.*

Let $V = \{1,2,3,4,5,6\}$ and $\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}$. Then $B(\mathcal{B})$ is isomorphic to the Petersen graph.

Still open are the related questions when the design has blocks of cardinality 2. It seems very unlikely that the proof of Theorem 2.3 can be generalized in this regard. The specific case when the design has only blocks of cardinality 2 is settled in Theorem 2.1.

Chapter 3

Connectivity

This chapter concerns itself with the connectivity of the block-intersection graph. We start with balanced incomplete block designs and give a number of results which determine the graph's vertex connectivity and also exhibit internally vertex-disjoint paths between nonadjacent vertices. The subsequent section proves some results for the more general case of pairwise balanced designs. The theorems there are not as satisfactory as in the less general case but they give a good start for further study.

3.1 Balanced Incomplete Block Designs

We begin with a theorem which not only determines the vertex connectivity of the block-intersection graph of a balanced incomplete block design but also characterizes all minimal vertex cuts.

Theorem 3.1 *Let* $(V, \mathcal{B})$ *be a* $BIBD(v, k, 1)$ *and let* $G = B(\mathcal{B})$. *If* C *is a vertex cut separating vertices* A^* *and* B^*, *then* $\delta = \delta(G) \leq |C|$. *Further-*

more, we have equality if and only if $\mathcal{C}$ is the set of vertices adjacent to either A^* or B^*.

Proof: For $i = 1, 2$, let $\mathcal{X}_i$ be the set of blocks in $\mathcal{B} \setminus \{A^*, B^*\}$ which intersect exactly i blocks of $\{A^*, B^*\}$. Let $\mathcal{C}_1$ be $\mathcal{C} \cap \mathcal{X}_1$ and let P be the set of points in V which are not in A^* or B^*. Note that $\mathcal{X}_2 \subseteq \mathcal{C} \cap \overline{\mathcal{X}_1}$.

Let x be in P, let $\mathcal{R}_a(x)$ be the set of blocks in $\mathcal{B}$ which contain x and a point in A^*, and let $\mathcal{R}_b(x)$ be the set of blocks in $\mathcal{B}$ which contain x and a point in B^*. Suppose there exists A in $\mathcal{R}_a(x) \setminus \mathcal{R}_b(x)$ and B in $\mathcal{R}_b(x) \setminus \mathcal{R}_a(x)$ such that $\mathcal{C}$ contains neither A nor B. But then A^*ABB^* is an (A^*, B^*)-path in $G - \mathcal{C}$. Hence, $\mathcal{R}_a(x) \setminus \mathcal{R}_b(x) \subseteq \mathcal{C}$ or $\mathcal{R}_b(x) \setminus \mathcal{R}_a(x) \subseteq \mathcal{C}$. Also,

$$\begin{aligned}
|\mathcal{R}_a(x) \setminus \mathcal{R}_b(x)| &= |\mathcal{R}_a(x)| - |\mathcal{R}_a(x) \cap \mathcal{R}_b(x)| \\
&= k - |\mathcal{R}_a(x) \cap \mathcal{R}_b(x)| \\
&= |\mathcal{R}_b(x)| - |\mathcal{R}_a(x) \cap \mathcal{R}_b(x)| \\
&= |\mathcal{R}_b(x) \setminus \mathcal{R}_a(x)|.
\end{aligned}$$

Therefore, at least half of the blocks in $\mathcal{X}_1$ containing x are in $\mathcal{C}_1$.

The previous paragraph now implies

$$\begin{aligned}
2|\mathcal{C}_1|(k-1) &= 2\sum_{B \in \mathcal{C}_1} |\{(x, B) : x \in P, x \in B\}| \\
&= 2\sum_{x \in P} |\{(x, B) : B \in \mathcal{C}_1, x \in B\}| \\
&\geq \sum_{x \in P} |\{(x, B) : B \in \mathcal{X}_1, x \in B\}|
\end{aligned}$$

$$\begin{aligned} &= \sum_{B\in\mathcal{X}_1} |\{(x,B) : x \in P, x \in B\}| \\ &= |\mathcal{X}_1|(k-1). \end{aligned}$$

Hence, $|\mathcal{C}| \geq |\mathcal{C}_1| + |\mathcal{X}_2| \geq \frac{1}{2}|\mathcal{X}_1| + |\mathcal{X}_2| = \frac{1}{2}(d_G(A^*) + d_G(B^*)) \geq \delta$.

Now suppose $\delta = |\mathcal{C}|$. Then we have equality in both equations of the last paragraph. Hence, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{X}_2$. Thus, if C is a block in a vertex cut $\mathcal{C}$ of size δ which separates blocks A and B, then C intersects A or B. Equality also implies that for every point x in P, exactly half the blocks in $\mathcal{X}_1$ containing x are in $\mathcal{C}$. Thus, the set of blocks in $\mathcal{C}_1$ containing x is either $\mathcal{R}_a(x)$ or $\mathcal{R}_b(x)$, for every point x in P.

Every point in A^* is in $r-(k+1)$ blocks in $\mathcal{X}_1$. Thus, if $r = k+1$, then $\mathcal{C} = \mathcal{X}_2$ and hence $\mathcal{C} = N_G(A^*) = N_G(B^*)$. Therefore suppose $r > k+1$. Let p_1 and p_2 be distinct points in A^* and let p_i be in a block D_i which is in $\mathcal{X}_1$, $i = 1,2$. Suppose $D_1 \in \mathcal{C}$ and $D_2 \notin \mathcal{C}$. Since D_1 is a block in a vertex cut $\mathcal{C}$ of size δ which separates blocks D_2 and B^*, D_1 intersects D_2 or B^* as seen in the previous paragraph. Since D_1 and A^* intersect and $D_1 \in \mathcal{X}_1$, D_1 and B^* do not intersect. Hence, D_1 and D_2 intersect in some point x. But then x is in P and $\mathcal{R}_a(x)$ has blocks in both $\mathcal{C}_1$ and $\mathcal{X}_1 \setminus \mathcal{C}_1$. Therefore, we can conclude that the set of blocks in $\mathcal{X}_1$ which are adjacent to A^* is either contained in $\mathcal{C}$ or disjoint from $\mathcal{C}$. It now follows that $\mathcal{C}$ is $N_G(A^*)$ or $N_G(B^*)$. ∎

Menger's Theorem states that the minimal number of vertices seperating two non-adjacent vertices is equal to the maximal number of internally vertex-disjoint paths between the two vertices. Therefore if C is a minimal vertex cut in a graph G, x is a vertex in one component of $G - C$ and y is a vertex in another component of $G - C$, then the cardinality of C is at least the minimal number of vertices seperating x and y, and hence at least the maximal number of internally vertex-disjoint paths between x and y. Moreover, the cardinality of C is at most the minimum degree of G. Although the vertex connectivity of G is determined in Theorem 3.1, we prove this again in the following theorem using Menger's Theorem by constructing a set of internally vertex-disjoint paths of cardinality $k(r-1)$ between any two non-adjacent vertices. The emphasis, however, is on the lengths of these paths.

Theorem 3.2 *Let $(V, \mathcal{B})$ be a $BIBD(v, k, 1)$ and let $G = B(\mathcal{B})$. Between any two nonadjacent vertices A^* and B^* there exists*

- *k^2 (A^*, B^*)-paths of length 2 and*
- *$k(r-k-1)$ (A^*, B^*)-paths of length 3*

all of which are internally vertex-disjoint.

Proof: Let $A^* = \{a_1, a_2, \ldots, a_k\}$ and $B^* = \{b_1, b_2, \ldots, b_k\}$. For $i = 1, 2, \ldots, k$, define the following sets:

$$\mathcal{A}_i' = \{A \in \mathcal{B} : A \cap A^* = \{a_i\}, A \cap B^* = \emptyset\},$$

$$\mathcal{B}_i' = \{B \in \mathcal{B} : B \cap B^* = \{b_i\}, B \cap A^* = \emptyset\},$$

$$\mathcal{C}_i = \{C \in \mathcal{B} : C \cap A^* = \{a_i\}, C \cap B^* \neq \emptyset\}.$$

Define $Z = (X, Y)$ to be the bipartite graph whose parts are $X = \mathcal{A}_1' \cup \mathcal{A}_2' \cup \cdots \cup \mathcal{A}_k'$ and $Y = \mathcal{B}_1' \cup \mathcal{B}_2' \cup \cdots \cup \mathcal{B}_k'$, and whose edges are $E(Z) = \{AB : A \in X, B \in Y, A \cap B \neq \emptyset\}$. Note that $|X| = |Y| = k(r-k-1)$. We will show that Z has a perfect matching.

Let $S \subset X$, $S \neq \emptyset$, and let $T = \{x : x \in A \setminus A^*, A \in S\}$. Moreover, let $N(S) = N_Z(S)$ and let $\mathcal{R}_a(x)$ and $\mathcal{R}_b(x)$ be defined as in the proof of Theorem 3.1. (Note that as in the previous theorem, $|\mathcal{R}_a(x) \setminus \mathcal{R}_b(x)| = |\mathcal{R}_b(x) \setminus \mathcal{R}_a(x)|$.) Then,

$$\begin{aligned}
(k-1)|S| &= \sum_{A\in S}(|A|-1) \\
&= \sum_{A\in S} |\{x : x \in A \setminus A^*\}| \\
&= |\{(x, A) : A \in S, x \in A \setminus A^*\}| \\
&= \sum_{x\in T} |\{A \in S : x \in A\}| \\
&\leq \sum_{x\in T} |\{A \in X : x \in A\}| \\
&= \sum_{x\in T} |\mathcal{R}_a(x) \setminus \mathcal{R}_b(x)| \\
&= \sum_{x\in T} |\mathcal{R}_b(x) \setminus \mathcal{R}_a(x)| \\
&= \sum_{x\in T}\sum_{i=1}^{k} |\{B \in \mathcal{B}_i' : x \in B\}| \\
&= \sum_{x\in T}\sum_{i=1}^{k} |\{(x, B) : x \in B \in \mathcal{B}_i'\}| \\
&= |\{(x, B) : B \in N(S), x \in B \cap T\}|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{B\in N(S)} |\{(x,B) : x \in B \cap T\}| \\
&\leq \sum_{B\in N(S)} |\{(x,B) : x \in B \setminus B^*\}| \\
&= (k-1)|N(S)|.
\end{aligned}
$$

Therefore $|S| \leq |N(S)|$, and so Z has a perfect matching by Hall's Theorem. Call this matching M.

The desired $\delta = k(r-1)$ internally vertex-disjoint (A^*, B^*)-paths are the following:

- A^*CB^*, for all $C \in \mathcal{C}_i$, $i = 1, 2, \ldots, k$, and
- A^*ABB^*, for all $AB \in M$, $A \in X$, $B \in Y$.

Note that there are k^2 paths of length 2 since $|\mathcal{C}_i| = k$ for $i = 1, 2, \ldots, k$, and $k(r-k-1)$ paths of length 3 since $|M| = |X| = k(r-k-1)$. ∎

3.2 Pairwise Balanced Designs

In this section, we investigate connectivity of the graphs associated with pairwise balanced designs. The results are less definitive than in the last section. Theorems similar to Theorems 3.1 and 3.2 for the more general case in this section are given determining the edge connectivity of these graphs. The last theorem of this section determines the vertex connectivity of the graphs whose associated design has a large number of points compared with the cardinalities of its blocks.

Theorem 3.3 *Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$ and let $G = B(\mathcal{B})$. If $\mathcal{S}$ is an edge cut separating vertices A^* and B^*, then* $\max\{d_G(A^*), d_G(B^*)\} \leq |\mathcal{S}|$. *Furthermore, we have equality if and only if $\mathcal{S}$ is the set of edges incident with either A^* or B^*.*

Proof: Let $\mathcal{A}^*$ and $\mathcal{B}^*$ be a partition of $\mathcal{B}$ such that $A^* \in \mathcal{A}^*$, $B^* \in \mathcal{B}^*$, and $\mathcal{S} = [\mathcal{A}^*, \mathcal{B}^*]$. For every point x in V, let $\mathcal{S}_x$ be the set of edges AB in $\mathcal{S}$ such that $A \cap B = \{x\}$. ($\mathcal{S}$ is partitioned by $\{\mathcal{S}_x : x \bullet V, \mathcal{S}_x \neq \emptyset\}$.)

Suppose x is in V. Let r_x be the replication number of x (in this case r_x need not be the same for different x), let a_x be the number of blocks in $\mathcal{A}^*$ which contain x, and let b_x be the number of blocks in $\mathcal{B}^*$ which contain x. If $a_x \geq 1$ and $b_x \geq 1$, then $|\mathcal{S}_x| = a_x b_x = a_x(r_x - a_x) \geq r_x - 1$.

If there exists a point x_1 which is only in blocks in $\mathcal{A}^*$ and there exists a point x_2 which is only in blocks in $\mathcal{B}^*$, then no block in $\mathcal{B}$ can contain both x_1 and x_2. Therefore, we may assume every point in a block in $\mathcal{A}^*$ is also in a block in $\mathcal{B}^*$. Hence, $|\mathcal{S}_x| \geq r_x - 1$, for every point x which is in some block in $\mathcal{A}^*$. Therefore,

$$
\begin{aligned}
|\mathcal{S}| &= \sum_{x \in V} |\mathcal{S}_x| \\
&\geq \sum_{x \in A^*} |\mathcal{S}_x| \\
&\geq \sum_{x \in A^*} (r_x - 1) \\
&= d_G(A^*).
\end{aligned}
$$

If we have equality, then $\mathcal{S}_x$ is empty for every point x which is not in A^*. Hence, every point x which is in some block in $\mathcal{A}^*$ is in A^*. Therefore, $\mathcal{A}^* = \{A^*\}$, and so $\mathcal{S}$ is the set of edges incident with A^*. ∎

We continue now in a similar vein as the previous section, by constructing internally edge-disjoint paths to give an alternative proof to Theorem 3.3. The paths in this case have lengths 2 and 4.

Theorem 3.4 *Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$ and let $G = B(\mathcal{B})$. Between any two nonadjacent vertices A^* and B^* there exists* $\min\{d_G(A^*), d_G(B^*)\}$ *(A^*, B^*)-paths of lengths 2 and 4 all of which are internally edge-disjoint.*

Proof: Let $A^* = \{a_1, a_2, \ldots, a_s\}$ and $B^* = \{b_1, b_2, \ldots, b_t\}$ be any two nonadjacent vertices, and without loss of generality suppose $d_G(A^*) \leq d_G(B^*)$. For $i = 1, 2, \ldots, s$, and for $j = 1, 2, \ldots, t$, define the following sets:

$$\mathcal{A}'_i = \{A \in \mathcal{B} : A \cap A^* = \{a_i\}, A \cap B^* = \emptyset\},$$

$$\mathcal{B}'_j = \{B \in \mathcal{B} : B \cap B^* = \{b_j\}, B \cap A^* = \emptyset\},$$

$$\mathcal{C}_i = \{C \in \mathcal{B} : C \cap A^* = \{a_i\}, C \cap B^* \neq \emptyset\}.$$

Since $\sum_{i=1}^{s}(\mathcal{A}'_i + \mathcal{C}_i) = d_G(A^*) \leq d_G(B^*) = \sum_{j=1}^{t} \mathcal{B}'_j + \sum_{i=1}^{s} \mathcal{C}_i$, we have $\sum_{i=1}^{s} \mathcal{A}'_i \leq \sum_{j=1}^{t} \mathcal{B}'_j$. Therefore, let $\varphi : \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \cdots \cup \mathcal{A}'_s \to \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \cdots \cup \mathcal{B}'_t$ be an injection. Moreover, for $i = 1, 2, \ldots, s$, and $j = 1, 2, \ldots, t$, let $A_{i,j} \in \mathcal{A}'_i$ be such that $b_j \in A_{i,j}$. Finally, let $\theta : \mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \cdots \mathcal{A}'_s \to \{1, 2, \ldots, t\}$ be defined by $\theta(B) = j$ if and only if $\varphi(B) \in \mathcal{B}'_j$. Then the internally edge-disjoint paths are:

- $A^*A_{i,j}B^*$ for $i = 1, 2, \ldots, s$, and $j = 1, 2, \ldots, t$,
- $A^*AA_{i,\theta(B)}\varphi(B)B^*$ for all $B \in \mathcal{B}'_i$, for $i = 1, 2, \ldots, s$. ∎

We now turn from edge connectivity to vertex connectivity. Although the result given here is probably not the best possible it does for any fixed K determine the vertex connectivity of the block-intersection graph for all but a finite number of $PBD(v, K, 1)$.

Theorem 3.5 *Let $(V, \mathcal{B})$ be a $PBD(v, K, 1)$ and let $G = B(\mathcal{B})$. Define $u = \max\{k : k \in K\}$. If $v \geq \frac{1}{3}u^4$, then the vertex connectivity of G is equal to its minimum degree.*

Proof: If $u = 2$, then $\mathcal{B}$ is a $BIBD(v, 2, 1)$ and the theorem follows from Theorem 3.1. Therefore, assume $u \geq 3$. Suppose $v \geq \frac{1}{3}u^4$ and suppose $\mathcal{C}$ is a vertex cut such that $|\mathcal{C}| < \delta(G) = \delta$.

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{C}$ form a partition of $\mathcal{B}$ such that for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, $A_1 \cap A_2 = \emptyset$. Moreover, let V_1 be the set of points in V which are in blocks in $\mathcal{A}_1$. Without loss of generality, we may assume that $v_1 = |V_1| \leq \frac{1}{2}v$ (otherwise, use $V \setminus V_1$ and $\mathcal{A}_2$).

Choose $A^* \in \mathcal{A}_1$ and let $k = |A^*|$. For all $B \in \mathcal{B}$, let $n_B = |B \cap (V_1 \setminus A^*)|$. Divide the blocks of the vertex cut $\mathcal{C}$ which intersect $V_1 \setminus A^*$ into three parts: $\mathcal{X} = \{B \in \mathcal{C} : n_B = 1, B \cap A^* = \emptyset\}$, $\mathcal{Y} = \{B \in \mathcal{C} : n_B = 1, B \cap A^* \neq \emptyset\}$, and $\mathcal{Z} = \{B \in \mathcal{C} : n_B \geq 2\}$. Furthermore, let $D = \{(x, y) : x \in V_1 \setminus A^*, y \in V \setminus V_1\}$. For each pair $(x, y) \in D$, there is a unique block $B_{(x,y)} \in \mathcal{B}$ such that $\{x,y\} \subset B_{(x,y)}$ and $B_{(x,y)} \in \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. For $\mathcal{T} \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ define $D_{\mathcal{T}} = \{(x,y) \in D : B_{(x,y)} \in \mathcal{T}\}$.

With these definitions, we have

$$\begin{aligned}
|D_{\mathcal{X}}| &\leq (u-1)|\mathcal{X}|, & (3.1)\\
|D_{\mathcal{Y}}| &\leq (u-2)|\mathcal{Y}|, \text{ and} & (3.2)\\
|D_{\mathcal{Z}}| &\leq \sum_{B \in \mathcal{Z}} n_B(u-n_B) & \\
&\leq \sum_{B \in \mathcal{Z}} \binom{n_B}{2} 2(u-n_B) & \\
&\leq 2(u-2) \sum_{B \in \mathcal{Z}} \binom{n_B}{2} & \\
&\leq 2(u-2)\binom{v_1-k}{2} & \\
&= (u-2)(v_1-k)(v_1-k-1). & (3.3)
\end{aligned}$$

Since $|\mathcal{C}| < d_G(A^*) = |N_G(A^*)|$, we also have

$$\begin{aligned}
|\mathcal{X}| &\leq |\mathcal{C} \setminus N_G(A^*)| & \\
&= |\mathcal{C}| - |\mathcal{C} \cap N_G(A^*)| & \\
&< |N_G(A^*)| - |N_G(A^*) \cap \mathcal{C}| & \\
&= |N_G(A^*) \setminus \mathcal{C}|. & (3.4)
\end{aligned}$$

Using equations 3.1 to 3.4 gives

$$\begin{aligned}
(v_1-k)(v-v_1) &= |D| \\
&= |D_{\mathcal{X}}| + |D_{\mathcal{Y}}| + |D_{\mathcal{Z}}|
\end{aligned}$$

$$
\begin{aligned}
&\leq (u-1)|\mathcal{X}| + (u-2)|\mathcal{Y}| + (u-2)(v_1-k)(v_1-k-1) \\
&< (u-1)\left[|N_G(A^*) \setminus \mathcal{C}| + |\mathcal{Y}|\right] + (u-2)(v_1-k)(v_1-k-1) \\
&\leq (u-1)((v_1-k)k) + (u-2)(v_1-k)(v_1-k-1). \qquad (3.5)
\end{aligned}
$$

Rearranging equation 3.5 gives

$$
\begin{aligned}
v - v_1 &< (u-1)k + (u-2)(v_1-k-1) \\
&= (u-2)v_1 + k - u + 2 \\
&\leq (u-2)v_1 + 2,
\end{aligned}
$$

and therefore

$$
v_1 > \frac{v-2}{u-1}. \qquad (3.6)
$$

Using 3.6 and the assumption that $v_1 \leq \frac{1}{2}v$, we have

$$
\begin{aligned}
\left(\frac{v-2}{u-1}\right)\left[v - \left(\frac{v-2}{u-1}\right)\right] &< v_1(v-v_1) \\
&= \sum_{B \in \mathcal{C}} |B \cap V_1||B \cap (V \setminus V_1)| \\
&\leq \frac{1}{4}u^2|\mathcal{C}| \\
&< \frac{1}{4}u^2\delta \\
&\leq \frac{1}{4}u^2[u(v-u)] \\
&= \frac{u^3(v-u)}{4}. \qquad (3.7)
\end{aligned}
$$

Collecting the v terms in inequality 3.7 to the left-hand side gives

$$
4(u-2)v^2 - [8(u-3) + (u-1)^2u^3]v + (u-1)^2u^4 - 16 < 0.
$$

Define $f(v)$ to be the quadratic in v (u is fixed) of the left-hand side of the last equation. Then since $u \geq 3$,

$$\begin{aligned}
f\left(\frac{1}{3}u^4\right) &= \frac{1}{9}u^9 - \frac{2}{9}u^8 - \frac{1}{3}u^7 + u^6 - \frac{14}{3}u^5 + 9u^4 - 16 \\
&= \frac{1}{9}u^7(u-3)(u+1) + \frac{1}{3}u^4(3u-8)(u-2) + \frac{11}{3}u^4 - 16 \\
&\geq 0.
\end{aligned}$$

Moreover, if $v \geq \frac{1}{3}u^4$, then

$$\begin{aligned}
f'(v) &\geq \frac{5}{3}u^5 - \frac{10}{3}u^4 - u^3 - 8u + 24 \\
&\geq \frac{5}{3}u^5 - \frac{10}{3}u^4\left(\frac{u}{3}\right) - u^3\left(\frac{u}{3}\right)^2 - 8u\left(\frac{u}{3}\right)^3 + 24 \\
&= \frac{4}{27}u^5 + 24 \\
&\geq 0.
\end{aligned}$$

Therefore for $v \geq \frac{1}{3}u^4$, $f(v) \geq 0$, a contradiction. Thus $|C| \geq \delta$ and therefore the vertex connectivity of G is equal to its minimum degree. ∎

We conjecture that the theorem holds even without the condition $v \geq \frac{1}{3}u^4$.

Chapter 4

Coloring

This chapter investigates coloring the block-intersection graph of pairwise balanced designs. Coloring problems are difficult in general and (as will be demonstrated in the next section) it is no different here. The first part of this chapter will establish the connection between this coloring problem and the well-known Erdös-Faber-Lovász Conjecture [9,10,11]. In the remaining section, new results will be presented for the balance incomplete block design case.

4.1 Erdös-Faber-Lovász Conjecture

The Erdös-Faber-Lovász Conjecture was first posed by the persons named at a party they attended. It seemed to be an easy problem at first, but no proof has yet been found. That it is a difficult problem is indicated by the increasing bounty placed on its solution. Although it is not stated here in its original form it is a simple matter of translation of terminology to the form presented.

Conjecture 4.1 (Erdös-Faber-Lovász) *If $\mathcal{A}$ is a v-uniform linear hypergraph with v edges, then $\gamma(\mathcal{A}) \leq v$.*

A year later, P. Seymour [17] and N. Hindman [12] independently showed that Conjecture 4.1 is equivalent to the following conjecture.

Conjecture 4.2 *If $\mathcal{A}$ is a linear hypergraph with v vertices, then $\chi'(\mathcal{A}) \leq v$.*

Every $PBD(v, K, 1)$ is a linear hypergraph. The converse is not true. However, when it comes to coloring, we will show that Conjecture 4.2 is equivalent to the following conjecture.

Conjecture 4.3 *If $(V, \mathcal{B})$ is a $PBD(v, K, 1)$, then $\chi(B(\mathcal{B})) \leq v$.*

The proof of the equivalence of Conjecture 4.1 and 4.2 essentially shows that they are dual statements (in this case, dual means the dual of a hypergraph). However, there are several subtleties. We include, therefore, the proof of the equivalence in full detail in Section 4.1.2. In doing this, it is hoped that the connection between the Erdös-Faber-Lovász Conjecture and Conjecture 4.3 becomes transparent. We proceed first, though, to a discussion of the dual of a linear hypergraph.

4.1.1 The Dual of a Linear Hypergraph

The dual of a hypergraph is defined on page 3. If a hypergraph is linear, then it is clear that its dual is also. Let $\mathcal{A}$ be a linear hypergraph and let $\mathcal{A}^*$ be its dual. Define $\alpha : \cup\mathcal{A} \to \mathcal{A}^*$ to be $\alpha(x) = \{A : x \in A \in \mathcal{A}\}$. In the proof that Conjecture 4.1 and 4.2 are equivalent it will be necessary that α is one-to-one. This will not be the case if $\mathcal{A}$ has two adjacent vertices x, y each of degree one. This idea motivates the following definition of a property that some (but not all) linear hypergraphs satisfy.

D. For all $x, y \in \cup\mathcal{A}$, $x \neq y$, there exists $A \in \mathcal{A}$ such that $|A \cap \{x, y\}| = 1$.

With this definition we can characterize when α is one-to-one.

Lemma 4.4 *The function α is one-to-one if and only if $\mathcal{A}$ satsifies* **D**.

Proof: Suppose α is one-to-one and let $x, y \in \cup\mathcal{A}$, $x \neq y$. Since $\alpha(x) \neq \alpha(y)$, there exists $A \in \alpha(x) \setminus \alpha(y)$ or $A \in \alpha(y) \setminus \alpha(x)$. Then $|A \cap \{x, y\}| = 1$ and so $\mathcal{A}$ satisfies **D**.

Conversely, suppose $\mathcal{A}$ satisfies **D** and suppose for some $x, y \in \cup\mathcal{A}$, $\alpha(x) = \alpha(y)$. Since $\mathcal{A}$ is linear, $\alpha(x) = \{A\} = \alpha(y)$ for some $A \in \mathcal{A}$. Therefore $\{x, y\} \subseteq A$ and hence $x = y$ by **D**. Thus α is one-to-one. ∎

4.1.2 The Proof of the Equivalence

Before giving the proof of the equivalence of Conjectures 4.1 and 4.2 we need two lemmas. This section is an expanded version of the proof given in [17].

Lemma 4.5 *Suppose that if $\mathcal{A}$ is a linear hypergraph with v edges satisfying* **D** *and $|A| \le v$ for all $A \in \mathcal{A}$, then $\gamma(\mathcal{A}) \le v$. Then Conjecture 4.1 is true.*

Proof: Let $\mathcal{A}$ be a linear v-uniform hypergraph with v edges and let $A \in \mathcal{A}$. If A has no vertices of degree 1, then let $A' = A$. Otherwise, let A' be any subset of A such that A' contains all the vertices of A of degree 2 or more and only one vertex of A of degree 1. Define $\mathcal{A}' = \{A' : A \in \mathcal{A}\}$. Then $\mathcal{A}'$ is a linear hypergraph with v edges and $|A'| \le v$ for all $A' \in \mathcal{A}'$ since $\mathcal{A}$ is v-uniform. Moreover, $\mathcal{A}'$ satisfies **D** since $\mathcal{A}'$ is linear and since by construction there is at most one vertex of degree 1 incident with any edge. Therefore by the assumption of the lemma, $\gamma(\mathcal{A}') \le v$.

Let $c' : \cup\mathcal{A}' \to \{1, 2, \ldots, v\}$ be a strong v-coloring of $\mathcal{A}'$. Then c' can be easily extended to a strong v-coloring of $\mathcal{A}$. For each $A \in \mathcal{A}$, color each $x \in A \setminus A'$ a different color from $\{1, 2, \ldots, v\} \setminus c'(A)$ (this is possible since $|A \setminus A'| \le v$). Therefore $\gamma(A) \le v$ and Conjecture 4.1 is true. ∎

Lemma 4.6 *If $\mathcal{A}$ is a linear hypergraph with v edges satisfying* **D**, *then $|A| \le v$ for all $A \in \mathcal{A}$*

Proof: Suppose there exists an $A' \in \mathcal{A}$ such that $|A'| > v$. Then there are at most $v - 1$ edges $A \in \mathcal{A}$, $A \ne A'$, such that $A \cap A' \ne \emptyset$. Thus there exist $x, y \in A'$ such that $x, y \notin \cup\{A \in \mathcal{A} : A \ne A'\}$. Hence $\mathcal{A}$ does not satisfy **D**, a contradiction. Therefore the lemma holds. ∎

We are now ready to prove the equivalence of Conjectures 4.1 and 4.2.

Theorem 4.7 *Conjecture 4.1 is equivalent to Conjecture 4.2.*

Proof: By Lemma 4.5 and 4.6, Conjecture 4.1 is equivalent to the following statement.

S. If $\mathcal{A}$ is a linear hypergraph with v edges satisfying **D**, then $\gamma(\mathcal{A}) \leq v$.

We will show that **S** is equivalent to Conjecture 4.2.

Suppose that Conjecture 4.2 is true and let $\mathcal{A}$ be a linear hypergraph on v edges satisfying **D**. Let $\mathcal{A}^*$ be the dual of $\mathcal{A}$. Then $\mathcal{A}^*$ is a linear hypergraph with v vertices and hence let $c^* : \mathcal{A}^* \to \{1, 2, \ldots, v\}$ be a v-edge coloring of $\mathcal{A}^*$. Define $c : \cup\mathcal{A} \to \{1, 2, \ldots, v\}$ by $c(x) = c^*(\alpha(x))$. Then c is a strong v-coloring of $\mathcal{A}$ since if $x, y \in A$, $x \neq y$, for some $A \in \mathcal{A}$, then $\alpha(x) \neq \alpha(y)$ by Lemma 4.4, and $\alpha(x) \cap \alpha(y) \neq \emptyset$ implies $c(x) = c^*(\alpha(x)) \neq c^*(\alpha(y)) = c(y)$. Therefore, $\gamma(\mathcal{A}) \leq v$ and **S** is true.

Conversely, suppose **S** is true and let $\mathcal{A}$ be a linear hypergraph with v vertices. Let $\mathcal{A}^*$ be the dual of $\mathcal{A}$. Then $\mathcal{A}^*$ has at most v edges and if $A, B \in \cup\mathcal{A}^*$, $A \neq B$, then there exists $x \in A \setminus B$ or $x \in B \setminus A$. In either case, $|\alpha(x) \cap \{A, B\}| = 1$ and so $\mathcal{A}^*$ satisfies **D**. Thus by **S**, let $c^* : \cup\mathcal{A}^* \to \{1, 2, \ldots, v\}$ be a strong v-coloring of $\mathcal{A}^*$. Since $\cup\mathcal{A}^* = \mathcal{A}$, and since if $A, B \in \mathcal{A}$ and $A \cap B \neq \emptyset$, then A and B are in some edge of $\mathcal{A}^*$ implies $c^*(A) \neq c^*(B)$, c^* is a v-edge coloring of $\mathcal{A}$. Therefore, $\chi'(\mathcal{A}) \leq v$ and Conjecture 4.2 is true. ∎

4.1.3 Pairwise Balanced Designs

We finish this section by proving the equivalence between the Erdös-Faber-Lovász Conjecture and Conjecture 4.3.

Theorem 4.8 *Conjecture 4.2 is equivalent to Conjecture 4.3.*

Proof: A v-edge coloring of a hypergraph corresponds to a v-vertex coloring of its intersection graph and vice versa. Thus it is clear that Conjecture 4.2 implies Conjecture 4.3.

Suppose Conjecture 4.3 is true. Let $\mathcal{A}$ be a linear hypergraph with v vertices. Let

$$\mathcal{B} = \mathcal{A} \cup \{\{x,y\} : \{x,y\} \not\subseteq A \text{ for all } A \in \mathcal{A}, x, y \in \cup\mathcal{A}, x \neq y\}$$

Then $(\cup\mathcal{B}, \mathcal{B})$ is a $PBD(v, K, 1)$ where $K = \{2\} \cup \{|A| : A \in \mathcal{A}\}$. Hence $\chi'(\mathcal{B}) \leq v$ and since $\mathcal{A}$ is a subgraph of $\mathcal{B}$, $\chi'(\mathcal{A}) \leq v$. Thus Conjecture 4.2 is true. ∎

4.2 Balanced Incomplete Block Designs

In this section we fix our attention on coloring the block-intersection graph of balanced incomplete block designs. Specifically, a result is obtained on the existence of optimal colorings of the neighborhoods of vertices in these graphs. The next two sections may not seem like they have anything to do with this problem, however, they in fact provide the core of the proof for the result. The results in them are of interest in themselves and answer some questions about the general structure of the complement of such neighborhoods.

4.2.1 K_3-factors

In this section, K_3-factors of certain tripartite graphs will be investigated. The first result is very specific and yet requires some effort to prove. It and the subsequent theorem provide the base case for an induction proof of the more general result found in the next section.

Proposition 4.9 *Let $G = (X, Y, Z)$ be a tripartite graph such that $|X| = |Y| = |Z| = 7$ and such that every vertex has least 5 neighbors in each of the other two parts. Then G has a K_3-factor.*

Proof: Let G be defined as above. The induced bipartite graph (X, Y) has minimum degree at least 5 and hence has a perfect matching by Lemma 1.5. Thus let $X = \{x_1, x_2, \ldots, x_7\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$ be such that $x_i y_i \in E(G)$ for $i = 1, 2, \ldots, 7$.

Define $H = (Z, C)$ to be the bipartite graph whose parts are Z from G and $C = \{x_i y_i : i \in \{1, 2, \ldots, 7\}\}$, and whose edge-set is $E(H) = \{zc : z \in Z, xy = c \in C, \{x, y, z\}$ is a K_3 in $G\}$. We will try to find a perfect matching in H. If there is a perfect matching in H, then it corresponds to a K_3-factor of G. Otherwise, H has a particular structure which will be used to find a K_3-factor of G.

For $i = 1, 2, \ldots, 7$, since x_i and y_i have at least 5 neighbors each in Z (in the graph G), and $|Z| = 7$, x_i and y_i have at least 3 common neighbors in Z. Thus for all $c \in C$, $d_H(c) \geq 3$. Moreover, each $z \in Z$ has at least 5 neighbors in each of X and Y (in the graph G). Let $A_z = \{i : x_i z \in E(G)\}$ and $B_z = \{j : y_j z \in E(G)\}$. Then $A_z, B_z \subset \{1, 2, \ldots, 7\}$ and $|A_z|, |B_z| \geq 5$. Thus $|A_z \cap B_z| = |A_z| + |B_z| - |A_z \cup B_z| \geq 3$. Thus for all $z \in Z$, $d_H(z) \geq 3$.

Let $S \subset Z$, $S \neq \emptyset$. Since $d_H(z) \geq 3$ for all $z \in Z$, if $|S| \leq 3$, then $|S| \leq 3 \leq |N_H(S)|$. Since $d_H(c) \geq 3$ for all $c \in C$, if $|S| \geq 5$, then $|S| \leq 7 = |N_H(S)|$.

Therefore, if for all $S \subset Z$ such that $|S| = 4$, we have $|S| \leq |N_H(S)|$, then by Hall's Theorem, there exists a perfect matching $M = \{z_1c_1, z_2c_2, \ldots, z_7c_7\}$. Without loss of generality, label Z and C so that $c_i = x_iy_i$. Then $\{\{x_i, y_i, z_i\} : i = 1, 2, \ldots, 7\}$ is a K_3-factor of G.

Otherwise, there exists $S \subset Z$ such that $|S| = 4$ and $|S| > |N_H(S)|$. Since $|N_H(S)| \geq 3$, we have $|N_H(S)| = 3$. Thus $d_H(z) = 3$ for all $z \in S$. Moreover, since $N_H(C \setminus N_H(S)) = Z \setminus S$ and $|Z \setminus S| = 3$, $d_H(c) = 3$ for all $c \in C \setminus N_H(S)$. Therefore, H must have the structure of Figure 4.1 (there may be some additional edges between $Z \setminus S$ and $N_H(S) = N(S)$).

Without loss of generality, let $N_H(S) = \{x_1y_1, x_2y_2, x_3y_3\}$. We will now modify the matching between X and Y so as to produce a K_3-factor of G. We have two cases.

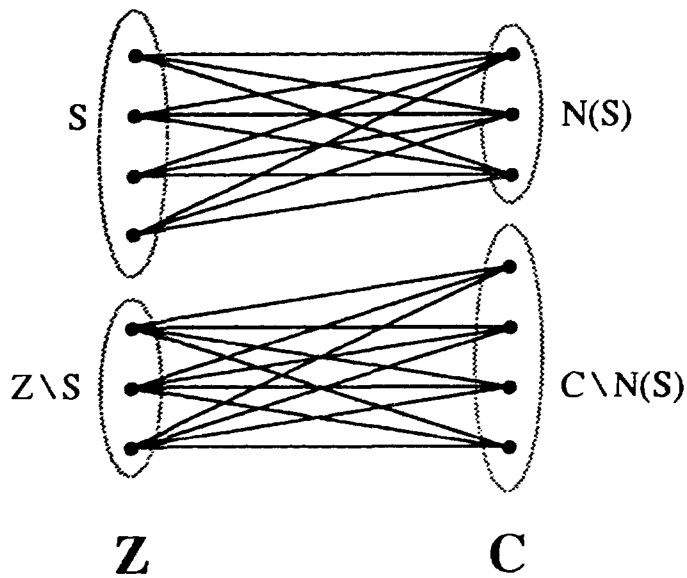

Figure 4.1: The structure of the graph $H = (Z, C)$.

Case 1. There exist i, j, $1 \leq i \leq 3$, $4 \leq j \leq 7$, such that $x_i y_j \in E(G)$ and $x_j y_i \in E(G)$.

Without loss of generality, we may assume $i = 3$ and $j = 4$. Since x_4 has at least 5 neighbors in Z (in the graph G), it has at least 2 in S. Similarly, y_4 has at least 2 neighbors in S. Let z_3 be a neighbor of x_4 in S and z_4 be a neighbor of y_4 in S such that $z_3 \neq z_4$. For the remaining vertices of Z, label them so that $S \setminus \{z_3, z_4\} = \{z_1, z_2\}$ and $Z \setminus S = \{z_5, z_6, z_7\}$. Since $N_H(\{z_1\}) = \{x_1y_1, x_2y_2, x_3y_3\} = N_H(\{z_2\})$, $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$ both induce a K_3 in G. Similarly, $\{x_5, y_5, z_5\}$, $\{x_6, y_6, z_6\}$ and $\{x_7, y_7, z_7\}$ all induce a K_3 in G. Moreover, x_3 is adjacent to each vertex of S in G because x_3y_3 is adjacent to each vertex of S in H. Thus $x_3z_4 \in E(G)$. Similarly, $y_3z_3 \in E(G)$, and hence $\{x_3, y_4, z_4\}$ and $\{x_4, y_3, z_3\}$ each induce a K_3 in G. Therefore G has a K_3-factor.

Case 2. There do not exist i, j, $1 \leq i \leq 3$, $4 \leq j \leq 7$, such that $x_iy_j \in E(G)$ and $x_jy_i \in E(G)$.

For each $i = 1, 2, 3$, let $A_i = \{j : x_iy_j \in E(G), 4 \leq j \leq 7\}$ and $B_i = \{j : x_jy_i \in E(G), 4 \leq j \leq 7\}$. Then for each $i = 1, 2, 3$, $|A_i| \geq 2$, $|B_i| \geq 2$ and by the assumption of Case 2, $A_i \cap B_i = \emptyset$. Thus, $|A_i| = 2 = |B_i|$ for all $i = 1, 2, 3$, and hence $x_ly_m \in E(G)$ for all $l, m \in \{1, 2, 3\}$.

Since x_4 has at least 5 neighbors in Y (in the graph G), it has at least one neighbor in $\{y_1, y_2, y_3\}$. Without loss of generality, let $x_4y_2 \in E(G)$. Similarly, y_4 has at least one neighbor in $\{x_1, x_2, x_3\}$ and by the assumption of Case 2 we may let $y_4x_3 \in E(G)$ without loss of generality.

As in Case 1, let z_3 be a neighbor of x_4 in S and z_4 be a neighbor of y_4 in S so that $z_3 \neq z_4$. Label the remaining vertices of Z as in Case 1. Then $\{x_1, y_1, z_1\}$, $\{x_2, y_3, z_2\}$, $\{x_3, y_4, z_4\}$, $\{x_4, y_2, z_3\}$, $\{x_5, y_5, z_5\}$, $\{x_6, y_6, z_6\}$ and $\{x_7, y_7, z_7\}$ is a K_3-factor of G. ∎

Proposition 4.9 gives the base case for an induction proof for the existence of K_3-factors in tripartite graphs with a certain structure. The following theorem makes this statement clear.

Theorem 4.10 *Let $t \geq 7$ and let $G = (X, Y, Z)$ be a tripartite graph such that $|X| = |Y| = |Z| = t$ and such that every vertex has at least $t - 2$ neighbors in each of the other two parts. Then G has a K_3-factor.*

Proof: By Proposition 4.9, the theorem is true for $t = 7$. Suppose therefore, that the theorem is true for some $t \geq 7$. Let $G' = (X', Y', Z')$ be a tripartite graph such that $|X'| = |Y'| = |Z'| = t + 1$ and such that every vertex has at least $t - 1$ neighbors in each of the other two parts.

Choose $x \in X'$ and let $y \in Y'$ be a neighbor of x. Since x and y each have $t - 1$ neighbors in Z' and $2(t - 1) > t + 1$, x and y have a common neighbor $z \in Z'$. Let $G = (X, Y, Z)$ be the tripartite subgraph of G' induced by $X \cup Y \cup Z$ where $X = X' \setminus \{x\}$, $Y = Y' \setminus \{y\}$ and $Z = Z' \setminus \{z\}$. Then $|X| = |Y| = |Z| = t$ and every vertex has at least $t - 2$ neighbors in each of the other two parts. Hence by induction, G has a K_3-factor, K. Then $K' = K \cup \{\{x, y, z\}\}$ is a K_3-factor for G'. ∎

For $t = 5$, Theorem 4.10 is false as the counter-example given in Figure 4.2 shows (found by Robert D. Fleming). In the figure, the vertices in set A are joined to all the vertices in Y and in Z. Similarly for the vertices in B and C. This graph has no K_3-factor since A' and B' require 4 neighbors in Z in a K_3-factor, but they collectively only have 3 (namely the vertices in C). For $t < 5$, it is straightforward to construct such a graph

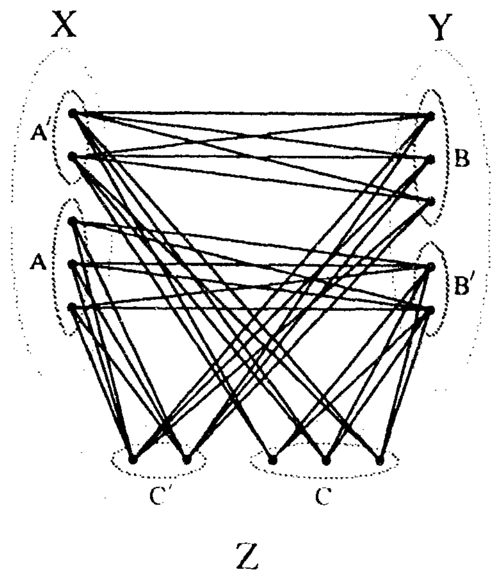

Figure 4.2: Counter-example for $t = 5$.

that does not even contain a K_3. Thus, the only unresolved case is $t = 6$ (whose status has eluded the author).

4.2.2 K_ℓ-factors

We now consider K_ℓ-factors of a certain class of ℓ-partite graphs for $\ell > 3$. Again, these results will be used later to color the neighborhood of a vertex in the block-intersection graph of a $BIBD(v, k, 1)$.

Theorem 4.11 *For each $\ell \geq 2$, there is a least integer $M(\ell)$ such that if $t \geq M(\ell)$ and $G = (X_1, X_2, \ldots, X_{\ell+1})$ is an $(\ell + 1)$-partite graph such that $|X_i| = t$, $i = 1, 2, \ldots, \ell + 1$, and such that every vertex of G has at least $t - \ell$ neighbors in each of the other parts, then G has a $K_{\ell+1}$-factor. Moreover, $M(\ell + 1) \leq 2\ell^2$.*

Proof: The theorem is true for $\ell = 2$ by Theorem 4.10 in which case $6 \leq M(2) \leq 7$. Suppose $\ell > 2$ and the statement is true for all integers i satisfying $2 \leq i \leq \ell$.

Let $t \geq \max\{M(\ell), 2\ell^2\}$. Then for any ℓ subsets $A_1, A_2, \ldots, A_\ell$ of $\{1, 2, \ldots, t\}$ such that $|A_i| \geq t - \ell$ for $i = 1, 2, \ldots, \ell$, we have $|A_1 \cap A_2 \cap \cdots \cap A_\ell| \geq \frac{1}{2}t$. This is the case since

$$\begin{aligned} |\overline{A_1 \cap A_2 \cap \cdots \cap A_\ell}| &= |\overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_\ell}| \\ &\leq \sum_{i=1}^{\ell} |\overline{A_i}| \\ &\leq \ell^2 \end{aligned}$$

giving $|A_1 \cap A_2 \cap \cdots \cap A_\ell| \geq t - \ell^2 \geq t - \frac{1}{2}t = \frac{1}{2}t$.

Let G be an $(\ell+1)$-partite graph $(X_1, X_2, \ldots, X_{\ell+1})$ such that $|X_i| = t$, $i = 1, 2, \ldots, \ell+1$, and such that every vertex of G has at least $t - \ell$ neighbors in each of the other parts. Let $G' = (X_1, X_2, \ldots, X_\ell)$ be the ℓ-partite subgraph of G induced by $X_1 \cup X_2 \cup \cdots \cup X_\ell$. Since $t \geq M(\ell)$, G' has a K_ℓ-factor $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$. Let H be the bipartite graph with bipartition $(X_{\ell+1}, \mathcal{G})$ (where the elements of $\mathcal{G}$ are considered to be vertices of H) and if $x \in X_{\ell+1}$ and $G_i \in \mathcal{G}$, then $xG_i \in E(H)$ if and only if $\{x\} \cup V(G_i)$ induces a $K_{\ell+1}$ in G. We will show that $\delta(H) \geq \frac{1}{2}t$. Without loss of generality, for $j = 1, 2, \ldots, t$, label the vertices of G_j, $x_j^1, x_j^2, \ldots, x_j^\ell$, so that $x_j^i \in X_i$ for $i = 1, 2, \ldots, \ell$. Moreover, label the vertices of $X_{\ell+1}$, $v_1, v_2, \ldots, v_t$. Let $k \in \{1, 2, \ldots, t\}$ and $A_i^k = \{j : x_j^i v_k \in E(G)\}$ for $i = 1, 2, \ldots, \ell$. Then for each $i \in \{1, 2, \ldots, \ell\}$, $|A_i^k| \geq t - \ell$ and $A_i^k \subseteq \{1, 2, \ldots, t\}$, and thus by the choice of t, $d_H(x_k) = |A_1^k \cap A_2^k \cap \cdots \cap A_\ell^k| \geq \frac{1}{2}t$.

On the other hand, let $G_j \in \mathcal{G}$ and let $B_i = \{k : v_k x_j^i \in E(G)\}$ for $i = 1, 2, \ldots, \ell$. Then $|B_i| \geq t - \ell$ and $B_i \subseteq \{1, 2, \ldots, t\}$. Therefore $d_H(G_j) = |B_1 \cap B_2 \cap \cdots \cap B_\ell| \geq \frac{1}{2}t$ again by the choice of t.

Therefore $\delta(H) \geq \frac{1}{2}t$ and hence H has a perfect matching. A perfect matching of H corresponds to a K_ℓ-factor of G. The above argument is valid for all $t \geq \max\{M(\ell), 2\ell^2\}$ and hence $M(\ell+1)$ exists and $M(\ell+1) \leq \max\{M(\ell), 2\ell^2\}$. Therefore $M(\ell)$ exists for all $\ell \geq 2$.

The last statement of the theorem is seen to be true by noticing $M(3) \leq \max\{M(2), 2 \cdot 2^2\} = 8$ and by using induction. ∎

4.2.3 Coloring the Neighborhood of a Block

We finish this chapter with a coloring result of a local nature. If B is a block of a $BIBD(v, k, 1)$, then the closed neighborhood of B in the design's block-intersection graph contains cliques of size r. The next result states that if the design has 'enough' points, then one can (suprisingly) color the closed neighborhood of any vertex of the block-intersection graph in r colors. It may be possible to use this result to get a better upper bound on the chromatic number of the entire graph but nothing has yet been obtained.

Corollary 4.12 *For each $k \geq 3$, there exists a least integer $N(k)$ such that if $(V, \mathcal{B})$ is a $BIBD(v, k, 1)$ and $v \geq N(k)$, then the closed neighborhood* *of any vertex in* $G = B(\mathcal{B})$ *has chromatic number* r. *Moreover,* $N(k) \leq 2k^3 - 10k^2 + 17k - 8$ *for all* $k \geq 4$.

Proof: Let $\ell = k - 1$. Then by Theorem 4.11, $M(\ell)$ exists. Define $n(k) = (M(\ell) + 1)\ell + 1$. Let $(V, \mathcal{B})$ be a $BIBD(v, k, 1)$ such that $v \geq n(k)$. Then $r(k-1) + 1 = v \geq n(k)$ implies $r - 1 \geq M(\ell)$ since $\ell = k - 1$.

Let $B^* \in \mathcal{B}$. Label the points of B^*, $b_1, b_2, \ldots, b_k$, and define $\mathcal{B}_i = \{B \in \mathcal{B} : B \cap B^* = \{b_i\}\}$ Then for each $i = 1, 2, \ldots, k$, $B(\mathcal{B}_i) \cong K_{r-1}$, and for each $j = 1, 2, \ldots, k$, $j \neq i$, each $B \in \mathcal{B}_j$ has $k - 1$ neighbors in $\mathcal{B}_i$. Thus the complement of $B(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k)$ is a k-partite graph $(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ such that each part has size $r - 1$ and such that each vertex has exactly $(r-1) - (k-1) = r - k$ neighbors in each of the other parts. By Theorem 4.11, this graph has a K_k-factor, $\{V_1, V_2, \ldots, V_{r-1}\}$. In the original graph G, each V_i is an independent set. By letting the V_i be the color classes for a coloring of G and giving B^* some other color we obtain an r-coloring for the closed neighborhood of B^*. Since each $B(\{B^*\} \cup \mathcal{B}_i) \cong K_r$, the chromatic number of the closed neighborhood of a block is r.

Thus $N(k)$ exists and $N(k) \leq n(k)$ for all $k \geq 3$. Moreover, by Theorem 4.11 $M(\ell) \leq 2(\ell - 1)^2$, for $\ell \geq 3$, and hence $N(k) \leq (2(k-2)^2 + 1)(k - 1) + 1 = 2k^3 - 10k^2 + 17k - 8$ for all $k \geq 4$. ∎

For $k = 3$, $N(3) \leq 17$ and so for $v \geq 19$, $v \equiv 1, 3 \pmod 6$, the closed neighborhood of any block of an $STS(v)$ is $\frac{1}{2}(v-1)$-chromatic.

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