Hamilton Decompositions of Complete 3– Uniform Hypergraphs

by

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B. Sc. (Hons) University of Victoria 1988

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics and Statistics

SIMON FRASER UNIVERSITY

April 12, 1991

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ISBN 0-315-78220-X

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Approval

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Degree: Master of Science (Mathematics)

Title of Thesis: Hamilton Decompositions of Complete 3-Uniform Hypergraphs

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April 9, 1991

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Abstract

The problem of finding a Hamilton decomposition of the complete 3–uniform hypergraph K_n^3 has been solved for n a prime [4], and for $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$ [2]. We find here a Hamilton decomposition of $K_n^3 - I$, $n \equiv 0 \pmod 3$, and a Hamilton decomposition of K_n^3, $n \equiv 1 \pmod 6$, and thereby complete the solution of the problem.

Acknowledgements

I would like to thank my supervisor, Katherine Heinrich, for all her help, Rob Ballantyne for saving the day so many times, and the Natural Sciences and Engineering Research Council and Simon Fraser University for financial support.

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Chapter 1

Introduction

1.1 Definitions and Notation

Definition 1.1 The complete graph on n vertices will be denoted by K_n, the graph on n vertices in which every two vertices are joined by λ distinct edges will be denoted by λK_n, and the graph on n vertices with no edges will be denoted by $\overline{K}_n$. A 1*-factor* in a graph G is a spanning subgraph of G in which every vertex has degree 1. We will denote the complete graph on n vertices, less a 1–factor, by $K_n - I$.

Definition 1.2 A *cycle* of *length* k in a graph G is a sequence

$$(x_1, x_2, x_3, \ldots, x_{k-1}, x_k)$$

of distinct vertices, together with the edges

$$\{x_i, x_{i+1}\},\ 1 \leq i \leq k,$$

where addition on the subscripts is modulo k. This cycle will be denoted by C_k.

A *C_k–factor* in a graph G is a spanning subgraph of G in which every vertex has degree 2 and is in a cycle of length k.

A *Hamilton cycle* of a graph G on n vertices is a cycle of length n. If the edges of G can be partitioned into Hamilton cycles, then G is said to have a *Hamilton decomposition.*

Definition 1.3 A *hypergraph* $\mathcal{H}(V, \mathcal{E})$ is a set of vertices $V = V(\mathcal{H}) = \{1, 2, \ldots, n\}$ and a set of hyperedges $\mathcal{E} = \mathcal{E}(\mathcal{H}) = \{E_1, E_2, \ldots, E_m\}$, where $E_i \subseteq V$ and $|E_i| \geq 0$, $1 \leq i \leq m$.

If $|E_i| = h$, we call E_i an *h–edge.* If $|E_i| = h$, for all $E_i \in \mathcal{E}$, then we call $\mathcal{H}$ *h–uniform.* For convenience, we will often write the 3–edge $\{a, b, c\}$ as abc.

The *complete h–uniform hypergraph* on n vertices, denoted K_n^h, is a hypergraph on the n vertices of V, in which every h–subset of V determines a hyperedge, or h–edge. It follows that K_n^h has $\binom{n}{h}$ hyperedges.

Definition 1.4 A *1–factor* of the hypergraph $\mathcal{H}(V, \mathcal{E})$ is a spanning subgraph of $\mathcal{H}(V, \mathcal{E})$, in which each of the n vertices of $\mathcal{H}(V, \mathcal{E})$ has degree 1.

We will denote the complete 3–uniform hypergraph on n vertices, less a 1–factor, by $K_n^3 - I$, and the complete 3–uniform hypergraph on n vertices, plus a 1–factor, by $K_n^3 + I$.

Definition 1.5 A *cycle* of *length* k of $\mathcal{H}$ is a sequence of the form

$$(x_1, E_1, x_2, E_2, \ldots, x_k, E_k, x_1),$$

where $\{x_1, x_2, \ldots, x_k\}$ are distinct vertices, and $E_1, E_2, \ldots, E_k$ are h-edges of $\mathcal{H}$, satisfying

(i) $x_i, x_{i+1} \in E_i$, $1 \leq i \leq k$,

(ii) $E_i \neq E_j$ for $i \neq j$.

For convenience, cycles in 3–uniform hypergraphs will be written as

$$(x_1y_1x_2, x_2y_2x_3, x_3y_3x_4, \ldots, x_{k-1}y_{k-1}x_k, x_ky_kx_1),$$

where $x_iy_ix_{i+1}$ is a 3–edge, $\{x_1, x_2, \ldots, x_k\}$ are distinct vertices, and all 3–edges in the cycle are different.

This cycle is known as a Berge cycle, having been introduced by C. Berge in his book *Graphs and Hypergraphs* [1].

Definition 1.6 Hamilton cycles and Hamilton decompositions of a hypergraph are defined as in the case of graphs: a *Hamilton cycle* in a hypergraph $\mathcal{H}$ on n vertices is a cycle of length n; and a *Hamilton decomposition* of $\mathcal{H}$ is a partition of the hyperedges of $\mathcal{H}$ into Hamilton cycles.

Definition 1.7 Let A and B be two graphs. We form the wreath product of A and B, denoted $AwrB$, by replacing each vertex in A by a copy of B, and making two vertices in different copies of B adjacent if and only if the corresponding two vertices in A were adjacent.

1.2 Introduction

In this thesis we consider the problem of constructing Hamilton decompositions of the complete 3–uniform hypergraph K_n^3. The problem has been solved by Bermond [2] for $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$, and Bermond *et al.* [4] have conjectured that both K_n^3, $n \equiv 1 \pmod 6$, and $K_n^3 - I$, $n \equiv 0 \pmod 3$, have a Hamilton decomposition.

In Chapter 2 we discuss known results for decompositions of K_n and K_n^3 into cycles, as well as other types of decompositions of K_n^h.

In Chapter 3 we outline Bermond's constructions for Hamilton decompositions of K_n^3, $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$, and then construct a Hamilton decomposition of $K_n^3 - I$, $n \equiv 0 \pmod 3$, and a Hamilton decomposition of K_n^3, $n \equiv 1 \pmod 6$.

Chapter 2

Survey of Results

The problem of decomposing the complete graph into cycles has been extensively studied since the late 1800's when Walecki [17] proved that K_{2n+1} and $2K_{2n}$ are Hamilton decomposable. The question then was when are the necessary conditions sufficient for the existence of a decomposition of a graph into cycles of some length k. This question has been answered completely for K_n for many small values of k. C. Rodger [21] has published a survey paper of decompositions into cycles of odd length. S. Marshall's Masters thesis [18] is another recent survey of work done decomposing graphs into cycles, and the papers *Cycle and circuit designs: odd case* by Bermond and Sotteau [6] and *Balanced cycle and circuit designs: even case* by Bermond, Huang and Sotteau [5] together are a good survey of the results on decomposing the graph K_n into cycles of length less than n.

In comparison, there are few results on the decompositions of hypergraphs into cycles. One of the reasons for this is that even the notion of a cycle in a hypergraph is not an obvious one. The definition of a cycle, that of a Berge cycle, given in Chapter 1 is the most common, but there are many others (see [7] for examples), all of which are further restrictions on the Berge cycle. Even using this simplest definition of a cycle, the problem of finding a Hamilton decomposition of K_n^h has not been solved. The only result for general h is that the complete h–uniform hypergraph K_n^h has a Hamilton decomposition if n is prime [4]. There are no other results than this for h greater than three, but the following results are known for K_n^3.

Lemma 2.1 [4] *The complete h–uniform hypergraph K_n^h has a Hamilton decomposition if n is prime.*

Lemma 2.2 [2] *The complete 3–uniform hypergraph K_n^3 has a Hamilton decomposition if $n \equiv 2 \,(mod\ 3)$.*

Lemma 2.3 [2] *The complete 3–uniform hypergraph K_{2n}^3 has a Hamilton decomposition if K_n^3 has a Hamilton decomposition.*

Lemmas 2.2 and 2.3 imply that if $n \equiv 4 \,(\text{mod } 6)$ then K_n^3 has a Hamilton decomposition.

For the remaining cases of $n \equiv 1 \,(\text{mod } 6)$ and $n \equiv 0 \,(\text{mod } 3)$ in K_n^3, Bermond *et al.* [4] have made the following conjectures:

Conjecture 2.4 For $n \equiv 1 \,(\text{mod } 6)$, there exists a partition of the 3–edges of K_n^3 into Hamilton cycles.

Conjecture 2.5 For $n \equiv 0 \,(\text{mod } 3)$, there exists a partition of the 3–edges of K_n^3 into a 1–factor and Hamilton cycles.

These two conjectures will be proved in the next chapter, thereby completing the problem of decomposing K_n^3 into (Berge) cycles of length n. If the definition of a cycle in a hypergraph is restricted as follows, we have a new problem and another conjecture.

Definition 2.6 A *cycle* is of *type* t if and only if the cardinality of the intersection of any two consecutive hyperedges in the cycle is equal to t.

Bermond *et al.* [4] have made the following conjecture about decompositions of K_n^3 into Hamilton cycles of type t, $t \in \{1,2\}$:

Conjecture 2.7 For $n \equiv 1, 2 \pmod 3$ there is a partition of the 3–edges of K_n^3 into Hamilton cycles of type t, $t \in \{1,2\}$. For $n \equiv 0 \pmod 3$ there is a partition of the edges of K_n^3 into a 1–factor and Hamilton cycles of type t, $t \in \{1,2\}$.

Little work has been done to decompose hypergraphs into cycles of a given length k; one reason for this may be that there is not a unique definition of a cycle in a hypergraph. However, other types of decompositions of hypergraphs have been studied by such people as Z. Lonc [13, 14, 15, 16], A.F. Mouyart and F. Sterboul [19, 20], and E. Eliad–Badt [8].

Definition 2.8 Let K and H be two h–uniform hypergraphs. K is said to admit an *H–decomposition* if the hyperedges of K can be partitioned into subhypergraphs isomorphic to H.

The necessary condition for an H–decomposition of K, where H and K are two given h–uniform hypergraphs, is usually that the number of h–edges of H divides the number of h–edges of K.

Some results in this area are that necessary and sufficient conditions are known for K_n^3 to admit a K_4^3-decomposition [9], and more generally, that necessary and sufficient conditions have been established for the existence of an H-decomposition of K_n^3, if H is any 3-uniform hypergraph on 4 vertices [3].

Eliad-Badt [8] and Lonc [14], [13] have considered decompositions of hypergraphs into different analogues of stars. The simplest such subhypergraph is known as a star.

Definition 2.9 The *star* S_m^3 is denoted by $x : a_1 \ldots a_m$. Its edges are xa_ia_j, $i \neq j$, $i, j \in \{2, 3, \ldots, m\}$.

A typical result in this area is that the hypergraph λK_n^3 admits an S_m^3-decomposition whenever the necessary conditions are satisfied [3, 8]. Again the necessary conditions are dependent on the number of 3-edges in λK_n^3 and S_m^3.

Chapter 3

Hamilton decompositions of K_n^3

If the 3–edges of K_n^3 can be partitioned into Hamilton cycles as defined in Definition 1.6, then these Hamilton cycles form a Hamilton decomposition of K_n^3. Such a decomposition requires

$$\frac{1}{n}\binom{n}{3} = \frac{(n-1)(n-2)}{6}$$

Hamilton cycles, since each Hamilton cycle uses n 3–edges and there are $\binom{n}{3}$ 3–edges altogether. This condition in turn implies that we must have $n \equiv 1, 2 \pmod 3$ for a Hamilton decomposition of K_n^3 to exist.

Bermond [2], using an idea of Brouwer, has constructed a Hamilton decomposition for K_n^3, $n \equiv 2 \pmod 3$, and then, by showing that a Hamilton decomposition for K_{2n}^3 can be constructed from a Hamilton decomposition for K_n^3, has also solved the problem for $n \equiv 4 \pmod 6$.

For the remaining cases, $n \equiv 1 \pmod 6$ and $n \equiv 0 \pmod 3$, Bermond *et al.* [4] put forward Conjectures 2.4 and 2.5, which we will prove in section 3.2.

3.1 $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$

First of all, we give Bermond's proofs for Hamilton decompositions of K_n^3, $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$.

3.1.1 $n \equiv 2 \pmod 3$

We prove the following lemma by constructing a Hamilton decomposition of K_n for n odd, and of $2K_n$ for n even. These will be used in Bermond's construction of a Hamilton decomposition of K_n^3, $n \equiv 2 \pmod 3$.

Lemma 3.1 Walecki [17] *A Hamilton decomposition of K_n exists if n is odd, and a Hamilton decomposition of $2K_n$ exists if n is even.*

Proof.

First, suppose that n is odd. Then the graph in Figure 3.1 is one Hamilton cycle of K_n,

$$C_0 = (\infty, a_1, a_2, \ldots, a_{n-1}) = (\infty, 1, 2, n-1, 3, \ldots, \frac{n+3}{2}, \frac{n+1}{2}).$$

A Hamilton decomposition of K_n is given by the $\frac{n-1}{2}$ Hamilton cycles:

$$C_i = (\infty, a_1 + i, a_2 + i, \ldots, a_{n-1} + i),\ 0 \leq i \leq \frac{n-3}{2},$$

where addition is modulo $n-1$.

Next, suppose that n is even. The graph on n vertices in Figure 3.2 is one Hamilton cycle of K_n,

$$C_0 = (\infty, b_1, b_2, \ldots, b_{n-1}) = (\infty, 1, 2, n-1, \ldots, \frac{n}{2}, \frac{n+2}{2}).$$

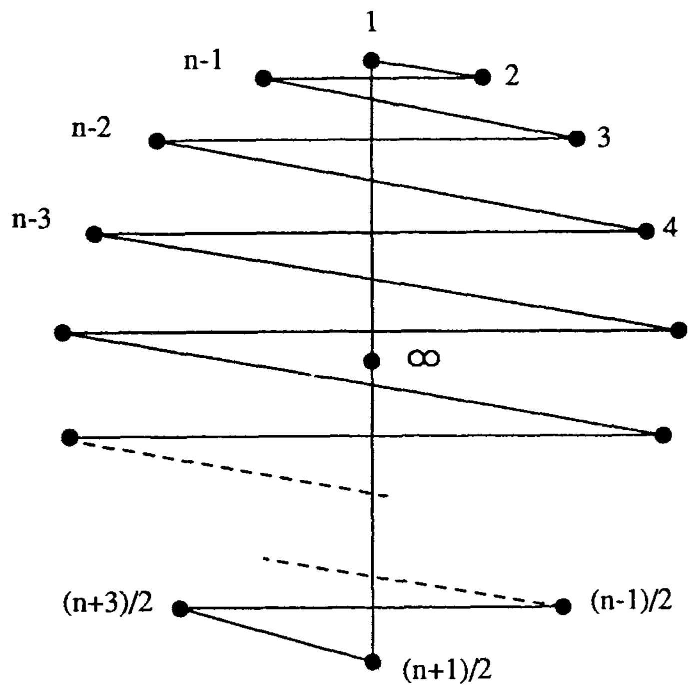

Figure 3.1: A Hamilton cycle in K_n, n odd.

A Hamilton decomposition of $2K_n$ is given by the $n-1$ Hamilton cycles:

$$C_i = (\infty, b_1 + i, b_2 + i, \ldots, b_{n-1} + i),\ 0 \leq i \leq n-2,$$

where again addition is modulo $n-1$.

□

In order to construct a Hamilton decomposition of K_n^3, $n \equiv 2 \pmod 3$, Brouwer first constructed a 'choice design of order n'. We give here a more general version of his definition.

Definition 3.2 A *choice design* of *order* n on a given 3–uniform hypergraph $\mathcal{H}$ on n vertices is a choice of one vertex from each 3–edge of $\mathcal{H}$ to represent

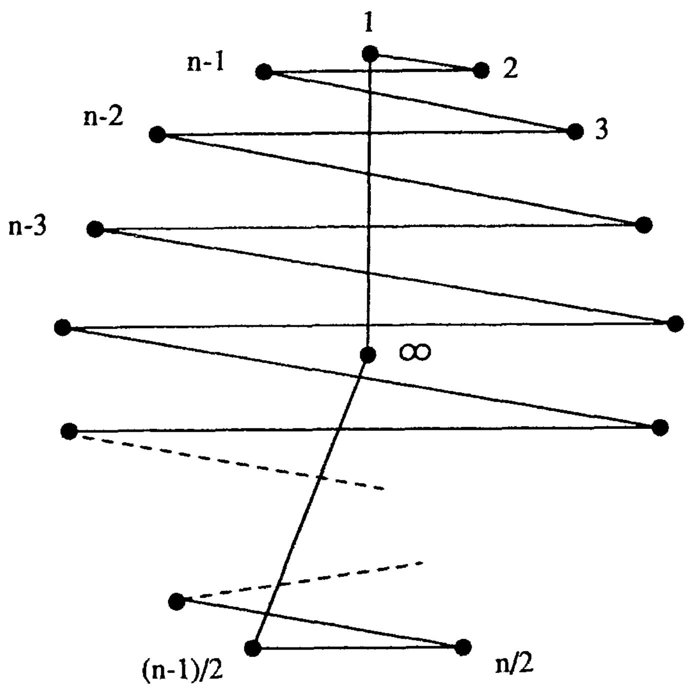

Figure 3.2: A Hamilton cycle in $2K_n$, n even.

that vertex.

In Bermond's proof and in the proofs in Section 3.2, we will construct choice designs subject to certain specified conditions.

Theorem 3.3 (Bermond [2]) *A choice design of order n on K_n^3 that satisfies the following condition exists if $n \equiv 2 \,(mod\, 3)$.*

(i) Among the $(n-2)$ 3-edges containing a given pair a and b, $(n-2)/3$ of them have neither a nor b chosen as their representative.

Proof.

We will use the notation $ab\underline{*}$ to be the set of all 3–edges containing a and b that have neither a nor b as their representative.

The sufficiency of $n \equiv 2 \pmod 3$ is proven by induction. A choice design of order 5 will be constructed, and then a method for constructing a choice design of order $k+3$ from a choice design of order k will be given.

A choice design of order 5 is:

$$1\underline{2}3\ 12\underline{4}\ \underline{1}25\ \underline{1}34\ 1\underline{3}5\ 14\underline{5}\ 2\underline{3}4\ 23\underline{5}\ \underline{2}45\ 3\underline{4}5,$$

where the chosen vertex of each 3–edge is underlined. Note that, as needed, among the three 3–edges containing any given pair a and b, exactly one has neither a nor b chosen as its representative.

Now assume that we have a choice design of order n on the vertices $\{1, 2, \ldots, n\}$. We want to construct a choice design of order $n+3$ on the vertices $\{1, 2, \ldots, n\} \cup \{\alpha, \beta, \gamma\}$.

(1) If $\{i, j, k\} \subseteq \{1, 2, \ldots, n\}$, then choose the representative of ijk as in the choice design of order n.

(2) If $\{i, j\} \subseteq \{1, 2, \ldots, n\}$, then choose the representatives of $ij\alpha$, $ij\beta$, and $ij\gamma$ as follows in Table 3.1.

hyperedges :	$ij\alpha$	$ij\beta$	$ij\gamma$	with $i < j$
representatives :	i	j	γ	if $i + j \equiv 0 \pmod 3$
	j	β	i	if $i + j \equiv 1 \pmod 3$
	α	i	j	if $i + j \equiv 2 \pmod 3$

Table 3.1: Choosing representatives for 3–edges of K^3_{n+3} (1)

(3) If $i \in \{1, 2, \ldots, n\}$, then choose the representatives of $i\alpha\beta$, $i\beta\gamma$, and $i\beta\gamma$ as follows in Table 3.2.

hyperedges :	$i\alpha\beta$	$i\alpha\gamma$	$i\beta\gamma$	
representatives :	i	γ	γ	if $i \equiv 0 \pmod 3$
	β	α	i	if $i \equiv 1 \pmod 3$
	α	i	β	if $i \equiv 2 \pmod 3$

Table 3.2: Choosing representatives for 3–edges of K^3_{n+3} (2)

(4) Choose γ in $\alpha\beta\gamma$.

To prove that this construction works, we must prove that $|ab\underline{*}| = \frac{n+1}{3}$ for all $a, b \in \{1, 2, \ldots, n\} \cup \{\alpha, \beta, \gamma\}$. From now on, assume that $i, j \in \{1, 2, \ldots, n\}$.

Let $p = \frac{n-2}{3}$. We first show that $|ij\underline{*}| = p + 1$.

There are p 3–edges $ij\underline{k}$, where $i, j, k \in \{1, 2, \ldots, n\}$. Depending on the value of $i + j \pmod 3$, exactly one of $ij\underline{\alpha}$, $ij\underline{\beta}$, $ij\underline{\gamma}$ will occur.

Now assume that $i \equiv 0 \pmod 3$. Let $i = 3q, 1 \leq q \leq p$.

Among 3–edges of the form $i\underline{j}\alpha$ with $j > i$, there are $p + 1 - q$ values of j so that $i + j \equiv 1 \pmod 3$. If $j < i$, then there are $q - 1$ values of j so that $i + j \equiv 0 \pmod 3$. The 3–edge $i\alpha\underline{\gamma}$ contributes one more. Thus,

$$|i\alpha\underline{*}| = p + 1 - q + q - 1 + 1 = p + 1, \text{ for } i \equiv 0 \pmod 3.$$

Among 3–edges of the form $i\underline{j}\beta$ with $j > i$, there are $p - q$ values of j so that $i + j \equiv 0 \pmod 3$. If $j < i$, then there are q values of j so that $i + j \equiv 2 \pmod 3$. The 3–edge $i\beta\underline{\gamma}$ contributes one more. Thus,

$$|i\beta\underline{*}| = p - q + q + 1 = p + 1, \text{ for } i \equiv 0 \pmod 3.$$

Among 3–edges of the form $i\underline{j}\gamma$ with $j > i$, there are $p - q + 1$ values of j so that $i + j \equiv 2 \pmod 3$. If $j < i$, then there are q values of j so that $i + j \equiv 1 \pmod 3$. Thus,

$$|i\gamma\underline{*}| = p - q + 1 + q = p + 1, \text{ for } i \equiv 0 \pmod 3.$$

When $i \equiv 1, 2 \pmod 3$ the calculations are similar. Thus

$$|i\alpha\underline{*}| = |i\beta\underline{*}| = |i\gamma\underline{*}| = p + 1$$

for all $i \in \{1, 2, \ldots, n\}$.

There are p values of $i \in \{1, 2, \ldots, n\}$ such that $i \equiv 0 \pmod 3$, so there are p 3–edges $\alpha\beta\underline{i}$. The 3-edge $\alpha\beta\underline{\gamma}$ contributes one more. Thus,

$$|\alpha\beta\underline{*}| = p + 1.$$

There are $p + 1$ values of $i \in \{1, 2, \ldots, n\}$ such that $i \equiv 2 \pmod 3$, so there are $p + 1$ 3–edges $\alpha\gamma\underline{i}$. Thus,

$$|\alpha\gamma\underline{*}| = p + 1.$$

Finally, there are $p + 1$ values of $i \in \{1, 2, \ldots, n\}$ such that $i \equiv 1 \pmod 3$, so there are $p + 1$ 3–edges $\beta\gamma\underline{i}$. Thus,

$$|\beta\gamma\underline{*}| = p + 1.$$

□

The proof of the following lemma shows how to construct a Hamilton decomposition of K_n^3, $n \equiv 2 \pmod 3$, given a choice design of order n that satisfies the condition of Theorem 3.3.

Lemma 3.4 [2] *Given a choice design of order* n *that satisfies condition (i) of Theorem 3.3., a Hamilton decomposition of* K_n^3 *can be constructed.*

Proof.

By Lemma 3.1, Hamilton decompositions of K_n, n odd, and $2K_n$, n even, exist. These are used together with the above choice design of order n in the following construction of a Hamilton decomposition of K_n^3. Since $n \equiv 2 \pmod 3$, we can let $n = 3m + 2$.

First assume that n and hence m is odd. The graph K_n has

$$\frac{1}{3m+2}\binom{3m+2}{2} = \frac{3m+1}{2}$$

Hamilton cycles in a Hamilton decomposition, and the hypergraph K_n^3 will have

$$\frac{1}{3m+2}\binom{3m+2}{3} = \frac{m(3m+1)}{2}$$

Hamilton cycles in a Hamilton decomposition. Each Hamilton cycle of K_n will be used to construct m Hamilton cycles of K_n^3. Choose a Hamilton cycle H in the Hamilton decomposition of K_n. For every edge ab in H, $|ab\underline{*}| = m$. Now choose an element of this set, say $ab\underline{c}$, and add c to the edge ab to get the 3-edge acb. Doing this for each edge of H creates a Hamilton cycle of K_n^3. Since there are m 3-edges in $ab\underline{*}$ for each edge $ab \in H$, we can construct a further $m-1$ Hamilton cycles of K_n^3 from H, giving m Hamilton cycles all together. Thus if $H = \{(x_1, x_2, \ldots, x_n)\}$, we build the following m Hamilton cycles of K_n^3:

$$(x_1 y_1^j x_2, x_2 y_2^j x_3, \ldots, x_{n-1} y_{n-1}^j x_n, x_n y_n^j x_1), \ 1 \le j \le m,$$

where

$$\{x_i x_{i+1} y_i^j : 1 \leq j \leq m\} = x_i x_{i+1}\underline{*}.$$

Constructing m Hamilton cycles in this way from each Hamilton cycle of K_n gives a Hamilton decomposition of K_n^3.

Continuing with the above example, a Hamilton decomposition of K_5^3 is

$$F = (1, 2, 3, 4, 5) \text{ and } G = (1, 3, 5, 2, 4).$$

The first edge of the first cycle in this example is the edge (12). There is one 3–edge in the above choice design that is in the set $12\underline{*}$, namely, $12\underline{4}$, since $m = \frac{5-2}{3} = 1$. Thus a 4 is inserted between the 1 and the 2 to give the 3–edge 142. Continuing in this way, we obtain

$$F_1 = (142, 253, 314, 425, 531) \text{ and } G_1 = (123, 345, 512, 234, 451),$$

which is a Hamilton decomposition of K_5^3.

Now assume that n and hence m is even and choose a Hamilton cycle H' in the Hamilton decomposition of $2K_n$. A Hamilton decomposition of the graph $2K_n$ has

$$\frac{2}{3m+2}\binom{3m+2}{2} = 3m+1$$

Hamilton cycles, and each of these will be used to construct $\frac{m}{2}$ Hamilton cycles of K_n^3. For each edge ab in H', divide the set $ab\underline{*}$ into two parts $(ab\underline{*})_1$ and $(ab\underline{*})_2$, so that

$$|(ab\underline{*})_1| = |(ab\underline{*})_2| = \frac{m}{2}.$$

Letting $H' = \{(x_1, x_2, \ldots, x_n)\}$, we use it to build the following $\frac{m}{2}$ Hamilton cycles of K_n^3:

$$(x_1y_1^jx_2, x_2y_2^jx_3, \ldots, x_{n-1}y_{n-1}^jx_n, x_ny_n^jx_1),\ 1 \leq j \leq \frac{m}{2},$$

where $\{x_ix_{i+1}y_i^j : 1 \leq j \leq \frac{m}{2}\} = (x_ix_{i+1}\underline{*})_1$, if H' is the first cycle that the edge x_ix_{i+1} appears in, and $\{x_ix_{i+1}y_i^j : 1 \leq j \leq \frac{m}{2}\} = (x_ix_{i+1}\underline{*})_2$ if H' is the second cycle that the edge x_ix_{i+1} appears in.

Building each cycle of the Hamilton decomposition of $2K_n$ into $\frac{m}{2}$ Hamilton cycles of K_n^3 in this way yields a Hamilton decomposition of K_n^3. □

3.1.2 $n \equiv 4 \pmod 6$

The following theorem is also from Bermond's paper [2]; the proof was obtained with D. Sotteau.

Theorem 3.5 *If there is a Hamilton decomposition of K_n^3, then there is a Hamilton decomposition of K_{2n}^3.*

To prove this theorem we need the following definition and two lemmas.

Definition 3.6 The complete symmetric directed graph on n vertices will be denoted by K_n^*.

The following lemma follows directly from Lemma 3.1, by taking two copies of each cycle and orienting them in opposite directions.

Lemma 3.7 *The digraph K_{2n+1}^* can be decomposed into $2n$ directed Hamilton cycles.*

Lemma 3.8 (Tillson [24]) *If* $2n \geq 8$, *then* K_{2n}^{*} *can be decomposed into* $2n-1$ *directed Hamilton cycles.*

Proof of Theorem 3.5.

Let the vertex set of K_{2n}^3 be $X \cup X'$, where $|X| = |X'| = n$. First, associate four Hamilton cycles of K_{2n}^3 with each Hamilton cycle of K_n^3 in the following way.

Case 1: n even.

Associate with the cycle

$$(x_1y_1x_2, x_2y_2x_3, \ldots, x_ny_nx_1)$$

the following:

$$(x_1y_1x_2, x_2y_2x_3, \ldots, x_{n-1}y_{n-1}x_n, x_ny_nx_1',$$
$$x_1'y_1x_2', x_2'y_2x_3', \ldots, x_{n-1}'y_{n-1}x_n', x_n'y_nx_1),$$

and

$$(x_1y_1x_2', x_2'y_2x_3, x_3y_3x_4', \ldots, x_{n-1}y_{n-1}x_n', x_n'y_nx_1',$$
$$x_1'y_1x_2, x_2y_2x_3', \ldots, x_{n-1}'y_{n-1}x_n, x_ny_nx_1),$$

and the two cycles obtained by interchanging the vertices of X and X'.

Case 2: n odd.

Associate with

$$(x_1y_1x_2, x_2y_2x_3 \ldots, x_ny_nx_1)$$

the following:

$$(x_1y_1x_2, x_2y_2x_3, \ldots, x_{n-2}y_{n-2}x_{n-1}, x_{n-1}y'_{n-1}x'_n, x'_ny_nx'_1$$
$$x'_1y_1x'_2, x'_2y_2x'_3 \ldots, x'_{n-2}y_{n-2}x'_{n-1}, x'_{n-1}y'_{n-1}x_n, x_ny_nx_1),$$

$$(x'_1y'_1x'_2, x'_2y'_2x'_3, \ldots, x'_{n-2}y'_{n-2}x'_{n-1}, x'_{n-1}y_{n-1}x'_n, x'_ny'_nx_1,$$
$$x_1y'_1x_2, x_2y'_2x_3, \ldots, x_{n-2}y'_{n-2}x_{n-1}, x_{n-1}y'_{n-1}x_n, x_ny'_nx'_1),$$

$$(x_1y_1x'_2, x'_2y_2x_3, \ldots, x_{n-2}y_{n-2}x'_{n-1}, x'_{n-1}y_{n-1}x_n, x_ny_nx'_1,$$
$$x'_1y_1x_2, x_2y_2x'_3, \ldots, x'_{n-2}y_{n-2}x_{n-1}, x_{n-1}y_{n-1}x'_n, x'_ny_nx_1),$$

and

$$(x'_1y'_1x_2, x_2y'_2x'_3, \ldots, x'_{n-2}y'_{n-2}x_{n-1}, x_{n-1}y_{n-1}x_n, x_ny'_nx_1,$$
$$x_1y'_1x'_2, x'_2y'_2x_3, \ldots, x_{n-2}y'_{n-2}x'_{n-1}, x'_{n-1}y_{n-1}x'_1, x'_ny'_nx'_1).$$

These cycles contain every 3–edge not of the form x, x', y or x, x', y'. We use Lemmas 3.7 and 3.8 to decompose these remaining 3–edges. With the directed Hamilton cycle $x_1, x_2, \ldots, x_n$ of a decomposition of K_n^*, we associate the following Hamilton cycle of K_n^3:

$$(x_1x'_1x_2, x_2x'_2x_3, \ldots, x_{n-2}x'_{n-2}x_{n-1}, x_{n-1}x'_{n-1}x_n, x_nx'_nx'_1,$$
$$x'_1x_1x'_2, x'_2x_2x'_3, \ldots, x'_{n-2}x_{n-2}x'_{n-1}, x'_{n-1}x_{n-1}x'_n, x'_nx_nx_1).$$

□

3.2 $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 6$

3.2.1 $n \equiv 0 \pmod 3$

There cannot exist a Hamilton decomposition of K_n^3 when $n \equiv 0 \pmod 3$ since the necessary condition for the existence of a Hamilton decomposition (that $\binom{n}{3}/n$ is an integer) is not satisfied. This is similar to the case of K_{2n}: it is not possible to have a Hamilton decomposition of K_{2n}, because

$$\frac{1}{2n}\binom{2n}{2} = \frac{2n-1}{2},$$

which is not an integer. However, if a 1-factor is removed from K_{2n}, then the resulting graph does have a Hamilton decomposition. In an analogous way, we shall remove a 1–factor from K_n^3, $n \equiv 0 \pmod 3$, and then construct a Hamilton decomposition of the remaining 3–edges. (See Section 1.1 for a definition of a 1–factor in a hypergraph.) Since $n \equiv 0 \pmod 3$, let $n = 3s$. A 1–factor of K_n^3 obviously exists; it will contain s 3–edges.

The hypergraph $K_n^3 - I$ has $(\binom{3s}{3} - s)$ 3–edges. The necessary condition for the existences of a Hamilton decomposition of $K_n^3 - I$ is that

$$\frac{\binom{3s}{3} - s}{3s}$$

is an integer. Since

$$\frac{\binom{3s}{3} - s}{3s} = \frac{3s(s-1)}{2},$$

the necessary condition is satisfied.

A Hamilton decomposition of $K_n^3 - I$, $n \equiv 0 \pmod 3$ is the 'next best thing' to a Hamilton decomposition of K_n^3 in the following sense.

Lemma 3.9 *A 1–factor contains the fewest number of 3–edges that can be* *removed from K_n^3, $n \equiv 0 \,(mod\ 3)$, so that the resulting graph satisfies the necessary condition for the existence of a Hamilton decomposition.*

Proof.

Suppose we remove x 3–edges from K_n^3. The resulting hypergraph has $\binom{3s}{3} - x$ 3–edges. The necessary condition for the existence of a Hamilton decomposition is that

$$\frac{1}{3s}\left(\binom{3s}{3} - x\right)$$

is an integer.

Since

$$\frac{\binom{3s}{3} - x}{3s} = \frac{9s^2 - 9s + 2}{6} - \frac{x}{3s},$$

and $\frac{9s^2-9s}{6}$ is an integer, we need $\frac{2}{6} - \frac{x}{3s}$ to be an integer.

Therefore, the possible solutions for x are

$$x = k \cdot 3s + s, k \in Z^{\geq 0}.$$

When $k = 0$, x is a minimum. So the smallest possible number of 3–edges that can be removed so that the resulting hypergraph satisfies the necessary condition is $x = s$. A 1–factor has s 3–edges. □

Theorem 3.10 *If $n \equiv 0 \,(mod\ 3)$, then there is a Hamilton decomposition of $K_n^3 - I$.*

Without loss of generality, we consider a specific 1–factor, namely,

$$T = \{123, 456, \ldots, (n-2)(n-1)n\},$$

and the hypergraph $K_n^3 - T$ constructed by removing the 1–factor T from K_n^3.

We will use another choice design, similar to that used by Bermond [2] for $n \equiv 2 \pmod 3$, to find Hamilton decompositions of $K_n^3 - T$.

This time, however, instead of building up the Hamilton decomposition of the hypergraph from Hamilton decompositions of K_n and $2K_n$, we will use Hamilton decompositions of $K_n - T^*$ and $2(K_n - T^*)$, where T^* is a C_3–factor of K_n and

$$T^* = \{(1,2,3),(4,5,6),\ldots,(n-2,n-1,n)\}.$$

We do this because the number of Hamilton cycles in a Hamilton decomposition of $K_n^3 - T$ is divisible by the number of Hamilton cycles in a Hamilton decomposition of $K_n - T^*$ if n is odd, and by the number of Hamilton cycles in a Hamilton decomposition of $2(K_n - T^*)$ if n is even. Hamilton decompositions of $K_n - T^*$, n odd, and $2(K_n - T^*)$, n even, will be constructed later. Once we have the choice designs, the Hamilton cycles in the Hamilton decompositions of $K_n - T^*$, n odd, and $2(K_n - T^*)$, n even, will be extended to Hamilton cycles of $K_n^3 - T$.

The hypergraph $K_{3s}^3 - T$ has

$$\binom{3s}{3} - s$$

3–edges, and so any Hamilton decomposition of it has

$$\left(\frac{1}{3s}\right)\left(\binom{3s}{3} - s\right) = \frac{3s(s-1)}{2}$$

Hamilton cycles.

The graph $K_{3s} - T^*$ has

$$\binom{3s}{2} - 3s = \frac{9s(s-1)}{2}$$

edges. If $n = 3s$ is odd, a decomposition of the edges of $K_{3s} - T^*$ into $\frac{3(s-1)}{2}$ Hamilton cycles will be given, and if n is even, a decomposition of the edges of $2(K_n - T^*)$ into $3(s-1)$ Hamilton cycles will be given.

Thus we want a choice design that will allow each Hamilton cycle of $K_n - T^*$ to be built up into s Hamilton cycles of $K_n^3 - T$, for odd n, and each Hamilton cycle of $2(K_n - T^*)$ to be extended to $\frac{s}{2}$ Hamilton cycles of $K_n^3 - T$, for even n.

The following grouping of the elements of $V = V(K_n^3 - T) = \{1, 2, \ldots, 3s\}$ in Figure 3.3 will be used in the definition and the construction of the choice design. Group the elements of V into s groups, where the i^{th} group G_i is $G_i = \{3i-2, 3i-1, 3i\}$, $1 \le i \le s$.

	1	4	7	···	3i-2	···	3s-2
	2	5	8	···	3i-1	···	3s-1
	3	6	9		3i		3s
Group number	1	2	3	····	i	····	s

Figure 3.3: A $3 \times s$ array of the elements of V.

We will write $G(a)$ to indicate the group number containing a. Let $\binom{V}{3}$ be the set of all 3–edges from V, and $\binom{V}{3} - T$ be the set of 3–edges of $K_{3s}^3 - T$. Notice that $T = \{G_i : 1 \le i \le s\}$.

We define two types of 3–edges in $\binom{V}{3} - T$: Type (1) 3–edges are 3–edges abc in which a and b are in the same group, and c is in a different group;

and Type (2) 3–edges are 3–edges abc in which a, b, and c are all in different groups.

Lemma 3.11 *There exists a choice design on the 3–edges of $K^3_{3s} - T$, where the vertices of $V(K^3_{3s} - T)$ are grouped into groups $G_i = \{3i-2, 3i-1, 3i\}$, $1 \le i \le s$, and $T = \{123, 456, \ldots, (n-2)(n-1)n\}$, that satisfies the following two conditions.*

(i) If $abc \in \binom{V}{3} - T$ and a and b are in the same group, then c is not *chosen as the representative of this 3–edge.*

(ii) Given b and c in different groups, the set $bc\underline{*}$ contains s elements.

Proof.

We first construct choice designs for odd and even $3s$, and then prove that they satisfy the two conditions above.

Case 1 $3s$ odd:

Let $3s = 6t + 3$, so that $s = 2t + 1$.

Choosing representatives for 3–edges of Type (1):

The partition of V in Figure 3.3 has $s = 2t + 1$ groups, where the i^{th} group G_i is

$$G_i = \{3i-2, 3i-1, 3i\},\ 1 \le i \le 2t+1, \text{ and } V = \bigcup_{i=1}^{2t+1} G_i.$$

Order the elements of a given 3–edge as abc so that $a, b \in G_i$, with $b \equiv a + 1 \pmod 3$, and $c \in G_j$, $i \neq j$.

If

$$j - i \equiv (2l - 1) \pmod{2t+1},$$

for $1 \leq l \leq t$, choose a as the representative of the 3–edge. Otherwise choose b.

Choosing the representative for 3–edges of Type (2):

Order the 3-edge as abc so that $G(a) < G(b) < G(c)$. Then,

$$\begin{aligned} &\text{if} \quad a+b+c \equiv 0 \pmod 3, \quad \text{choose } a,\\ &\text{if} \quad a+b+c \equiv 1 \pmod 3, \quad \text{choose } b, \text{ and}\\ &\text{if} \quad a+b+c \equiv 2 \pmod 3, \quad \text{choose } c. \end{aligned}$$

We must now prove that this is indeed a choice design as defined.

Condition (i) follows immediately by the choice of representatives for Type (1) 3–edges.

The verification that condition (ii) holds is a little more involved. Let b and c be elements in different groups, G_i and G_j, respectively. There are four 3–edges of Type (1) containing both b and c: b_1bc, b_2bc, c_1cb, and c_2cb, where $\{b, b_1, b_2\} = G_i$, and $\{c, c_1, c_2\} = G_j$.

To determine the representatives of these four 3–edges, we consider $(j - i) \bmod (2t+1)$. Suppose that $j - i \equiv (2l-1) \pmod{2t+1}$, $1 \leq l \leq t$.

If b is the representative for the 3–edge b_1bc, then $b_1 \equiv b + 1 \pmod 3$, implying that $b \equiv b_2 + 1 \pmod 3$, and hence, that b_2 is the representative for the 3–edge b_2bc.

If b_1 is the representative for the 3–edge b_1bc, then $b \equiv b_1 + 1 \pmod 3$, implying that $b_2 \equiv b + 1 \pmod 3$, and hence, that b is the representative for the 3–edge b_2bc.

In either case, b is the representative in one of the 3–edges b_1bc and b_2bc, and the element not equal to b or c is chosen in the other 3–edge.

A similar argument holds if $j - i \equiv 2l \pmod{2t+1}$, $1 \leq l \leq t$.

On repeating this argument for the 3–edges c_1cb and c_2cb, we can conclude that if b and c are in different groups, then among the four 3–edges of Type (1) that contain both b and c, exactly two of them are elements of the set $bc\underline{*}$.

Now suppose abc is a 3–edge of Type (2), with b and c fixed. The question is: 'How many 3–edges abc of Type (2) are in the set $bc\underline{*}$?'

With b and c fixed, the 3–edges abc of Type (2) are created by allowing a to run through the three levels of each of the remaining $(2t-1)$ groups. Thus, exactly once in each group, the value of

$$a + b + c \pmod 3$$

will force a to be chosen as the representative of the 3–edge. So there are exactly $(2t-1)$ 3–edges of Type (2) in the set $bc\underline{*}$. Thus $|bc\underline{*}| = (2t+1)$, satisfying condition (ii).

Case 2: $3s$ even:

Let $3s = 6r$, so that $s = 2r$.

Again we construct a choice design and then prove that it satisfies the conditions of the definition.

Choosing representatives;

We again partition V, this time into $2r$ groups, where the i^{th} group G_i is

$$G_i = \{3i-2, 3i-1, 3i\},\ 1 \leq i \leq 2r,\ \text{and } V = \bigcup_{i=1}^{2r} G_i.$$

Then every 3–edge from V except for the 3–edges within a group G_i is a 3–edge of $K^3_{6r} - T$. Again there are 3–edges of Types (1) and (2).

Choosing the representatives for the 3–edges of Type (1):

Order the elements of a given 3–edge abc as in Case 1 so that a and b lie in the same group G_i, with $b \equiv a + 1 \pmod 3$, and so that c lies in group G_j, $i \neq j$.

If

$$a + b + c \equiv 1, 2 \pmod 3, \text{ choose } a,$$

and if

$$a + b + c \equiv 0 \pmod 3, \text{ choose } b$$

as the representative of the 3–edge.

Choosing the representatives for the 3–edges of Type (2):

Choose the representatives for the 3–edges of Type (2) as in Case 1.

We now verify that we do indeed have the required choice design.

Condition (i) follows immediately, but condition (ii) again takes a little more work. Let b and c be elements in different groups, G_i and G_j, respectively. Let $G_i = \{b, b_1, b_2\}$ and $G_j = \{c, c_1, c_2\}$, where $b_1 \equiv b + 1 \pmod 3$, and $b_2 \equiv b + 2 \pmod 3$ in G_i, and $c_1 \equiv c + 1 \pmod 3$, and $c_2 \equiv c + 2 \pmod 3$ in G_j.

Then the four 3–edges of Type (1) which contain b and c, with their elements in the 'right' order are:

$$bb_1c, \; b_2bc, \; cc_1b, \text{ and } c_2cb.$$

(a) If $b \equiv c \pmod 3$ then $a + b + c \equiv 1, 2 \pmod 3$, for $a \in \{b_1, b_2, c_1, c_2\}$. Thus, in each of the above four 3–edges, the representative would be the first element. This implies that there are exactly two 3–edges of Type (1) in $bc\underline{*}$.

(b) If $b \equiv c - 1 \pmod 3$, then $b + b_1 + c \equiv 2 \pmod 3$; choose b in bb_1c.

If $b \equiv c - 1 \pmod 3$, then $b_2 + b + c \equiv 0 \pmod 3$; choose b in b_2bc.

If $b \equiv c - 1 \pmod 3$, then $c + c_1 + b \equiv 0 \pmod 3$; choose c_1 in cc_1b.

If $b \equiv c - 1 \pmod 3$, then $c_2 + c + b \equiv 1 \pmod 3$; choose c_2 in c_2cb.

Again $bc\underline{*}$ has two elements of Type (1) in it.

(c) Similarly, if $b \equiv c + 1 \pmod 3$ there are exactly two elements of Type (1) in $bc\underline{*}$.

If abc is of Type (2), an argument that is exactly the same as in Case 1 shows that $bc\underline{*}$ has $2r - 2$ elements of Type (2) in it.

Hence $|bc\underline{*}| = 2r$, as needed. □

Before we can prove Theorem 3.10, we must first construct Hamilton decompositions of $K_{3s} - T^*$, s odd, and $2(K_{3s} - T^*)$, s even. To do this, we consider the graph $C_s wr \overline{K}_3$, "C_s wreath $\overline{K}_3$", formed by replacing each vertex in C_s by a copy of $\overline{K}_3$, and then putting an edge between any two vertices in adjacent copies of $\overline{K}_3$, and the graph $K_s wr \overline{K}_3$, "K_s wreath $\overline{K}_3$", formed by replacing each vertex in K_s by a copy of $\overline{K}_3$, and then putting an edge between any two vertices in different copies of $\overline{K}_3$. Clearly, $K_s wr \overline{K}_3 \cong K_{3s} - T^*$.

Lemma 3.12 *A Hamilton decomposition of $K_{3s} - T^*$ exists if s is odd,* *and a Hamilton decomposition of* $2(K_{3s} - T^*)$ *exists if* s *is even.*

Proof.

Case 1: s odd.

Assume s is odd. The graph $K_{3s} - T^*$ has

$$\binom{3s}{2} - 3s = \frac{9s(s-1)}{2}$$

edges, and hence, we want to partition it into

$$\frac{1}{3s}\left(\frac{9s(s-1)}{2}\right) = \frac{3(s-1)}{2}$$

Hamilton cycles. Since $K_{3s} - T^* \cong K_s wr \overline{K}_3$, we will use the graph $K_s wr \overline{K}_3$ to prove the result.

By Lemma 3.1, the graph K_s can be partitioned into $\frac{s-1}{2}$ Hamilton cycles. If we take the wreath product of each of these Hamilton cycles with $\overline{K}_3$, we will have a partition of $K_s wr \overline{K}_3$ into $\frac{s-1}{2}$ copies of $C_s wr \overline{K}_3$. Therefore, if we can partition the $9s$ edges of each copy of $C_s wr \overline{K}_3$ into $9s/(3s) = 3$ Hamilton cycles, we will have constructed a Hamilton decomposition of $K_{3s} - T^*$.

Let $V(C_s wr \overline{K}_3)$ be exhibited in a $3 \times s$ array of vertices,

$$V = V_1 \cup V_2 \cup \ldots \cup V_s, \text{ where } V_i = \{3i-2, 3i-1, 3i\}, 1 \leq i \leq s,$$

and observe that every two adjacent columns of vertices, $V_i \cup V_{i+1}, 1 \leq i \leq (s-1)$ and $V_s \cup V_1$, induce a $K_{3,3}$, as shown in Figure 3.4.

We will first find a Hamilton decomposition of $C_3 wr \overline{K}_3$ and then show how to extend it to a Hamilton decomposition for all $C_s wr \overline{K}_3$, when s is odd.

A Hamilton decomposition of $C_3 wr \overline{K}_3$ is shown in Figure 3.5.

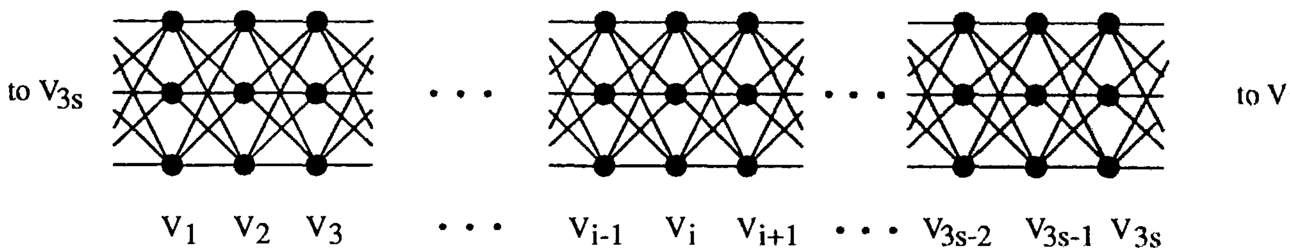

Figure 3.4: A $3 \times s$ array of edges and vertices of $C_s wr \overline{K}_3$.

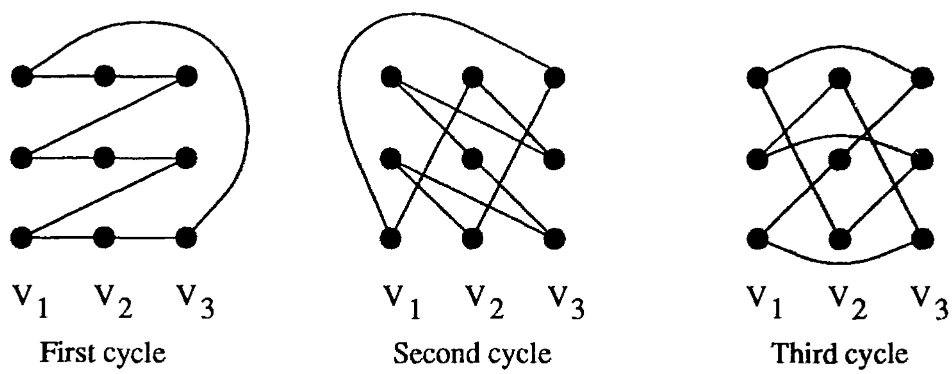

Figure 3.5: A Hamilton decomposition of $C_3 wr \overline{K}_3$.

For odd s greater than 3, we exhibit in Figure 3.6 a decomposition of $G[V_{2i-1}, V_{2i}, V_{2i+1}]$, $2 \leq i \leq \frac{s-1}{2}$, into three subgraphs, each isomorphic to $P_3 \cup P_3 \cup P_3$, with the additional feature that in each subgraph, path j starts in row j and ends in row j, $j \in \{1, 2, 3\}$.

Take the union of the decompositions of $G[V_{2i-1}, V_{2i}, V_{2i+1}]$, $2 \leq i \leq \frac{s-1}{2}$, so as to make a decomposition of $G[V_3, V_4, \ldots, V_s]$ into three subgraphs, each isomorphic to $P_{s-2} \cup P_{s-2} \cup P_{s-2}$, and each with its j^{th} path still starting and ending in row j, $j \in \{1, 2, 3\}$. We want a decomposition of $G[V_1, V_2, \ldots, V_s]$ into three Hamilton cycles. To get the remaining edges of the decomposition, take the Hamilton decomposition of $C_3 wr \overline{K}_3$ and replace the j^{th} vertex of

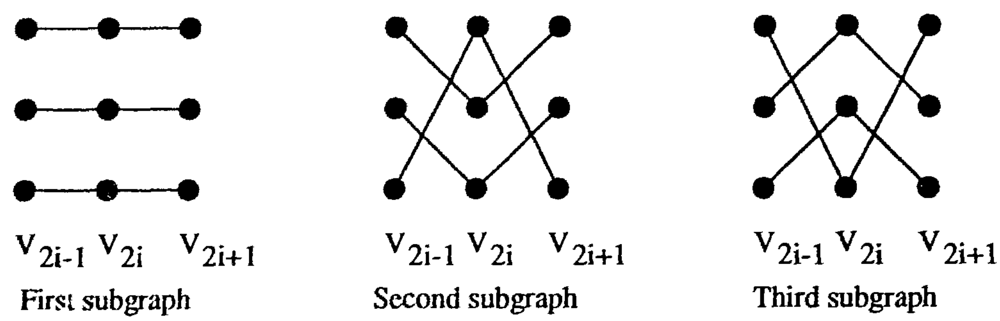

Figure 3.6: A decomposition of $G[V_{2i-1}, V_{2i}, V_{2i+1}]$ into $P_3 \cup P_3 \cup P_3$.

V_3 in the r^{th} cycle in Figure 3.5, by the j^{th} path of the r^{th} subgraph of the decomposition we have just constructed on $V_3, V_4, \ldots, V_s$, where $j \in \{1,2,3\}$ and $r \in \{1,2,3\}$.

Figure 3.7 shows a decomposition of $C_7 wr \overline{K}_3$ into 3 cycles.

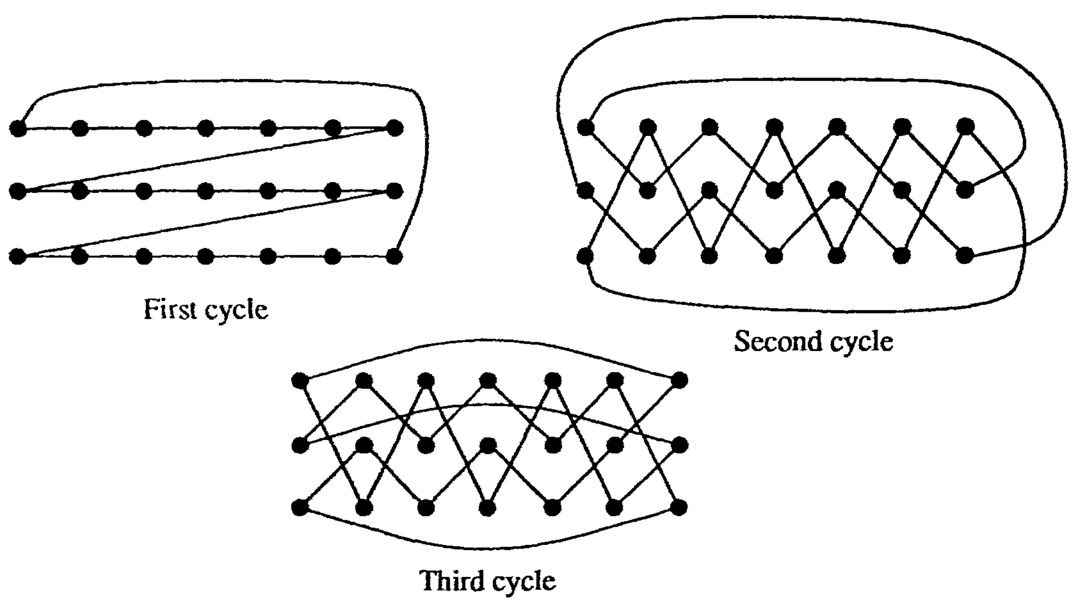

Figure 3.7: Example: A Hamilton decomposition of $C_7 wr \overline{K}_3$.

The Hamilton decomposition of $K_{3s} - T^*$ (or $K_s wr \overline{K}_3$) is completed by taking this Hamilton decomposition on each copy of $C_s wr \overline{K}_3$ in the partition of $K_s wr \overline{K}_3$.

Case 2: n even.

Let $n = 3s$, so that s is even. We want a Hamilton decomposition of the edges of $2(K_{3s} - T^*) \cong 2(K_s wr \overline{K}_3)$.

We will first of all do the case $s = 2$ in Figure 3.8.

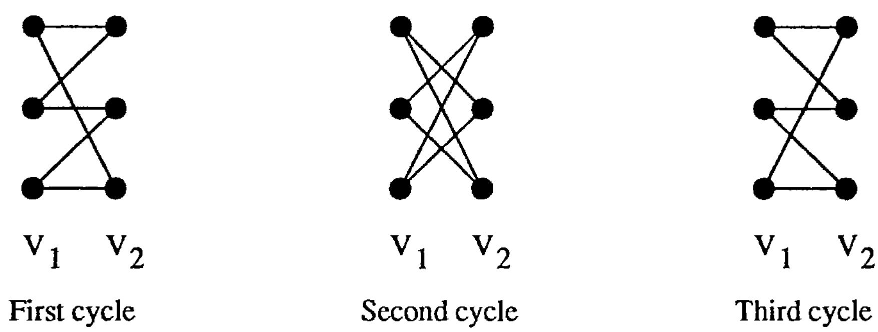

Figure 3.8: A Hamilton decomposition of $2(K_2 wr \overline{K}_3)$.

Now assume $s \geq 4$. The graph $2(K_{3s} - T^*)$ has

$$3s(3s-1) - 6s = 9s(s-1)$$

edges, and we want to decompose these edges into $3(s-1)$ Hamilton cycles. By Lemma 3.1, $2K_s$ can be partitioned into $s-1$ Hamilton cycles. Thus, $2(K_s wr \overline{K}_3)$ can be partitioned into $s-1$ copies of $C_s wr \overline{K}_3$. Again we partition each copy of $C_s wr \overline{K}_3$ into three Hamilton cycles, which gives a partition of $2(K_s wr \overline{K}_3)$ into $3(s-1)$ Hamilton cycles.

Consider a $3 \times s$ array of vertices as in Case 1, with the columns labelled V_1 to V_s.

We again use Figure 3.6 to decompose the edges of $G[V_{2i-1}, V_{2i}, V_{2i+1}]$, $2 \le i \le \frac{s}{2}$, into three subgraphs, each isomorphic to $P_3 \cup P_3 \cup P_3$. As in Case 1, take the union of these $\frac{s-2}{2}$ decompositions to form a decomposition of $G[V_2, V_3, \ldots, V_s]$, and note that path j starts and ends in row j, $j \in \{1, 2, 3\}$.

Form Hamilton cycles in $G[V_1, V_2, \ldots, V_s]$ by replacing the vertices of V_2 by $V_2 \cup V_3 \cup \ldots \cup V_s$ and 'inserting' paths as before. But this time be careful. You have to ensure that the edges from V_1 to V_2 are different in all three subgraphs on $V_1 \cup V_2 \cup \ldots \cup V_s$. □

Proof of Theorem 3.10.

Let $n = 3s$. By Lemma 3.12, if n is odd, $K_n - T^*$ can be decomposed into $\frac{3(s-1)}{2}$ Hamilton cycles. We build each of these cycles $(x_1, x_2, \ldots, x_n)$ into the following s Hamilton cycles of $K_n^3 - T$:

$$(x_1 y_1^j x_2, x_2 y_2^j x_3, \ldots, x_{n-1} y_{n-1}^j x_n, x_n y_n^j x_1),\ 1 \le j \le s,$$

where

$$\{x_i x_{i+1} y_i^j : 1 \le j \le s\} = x_i x_{i+1} \underline{*}.$$

When n is even, for every pair of vertices $a, b \in V$, we arbitrarily divide $ab\underline{*}$ into two equal pieces so that

$$ab\underline{*} = (ab\underline{*})_1 \cup (ab\underline{*})_2,$$

and

$$|(ab\underline{*})_1| = |(ab\underline{*})_2| = \frac{s}{2}.$$

Again by Lemma 3.12, if n is even, $2(K_n - T^*)$ can be decomposed into $3(s-1)$ Hamilton cycles. We build each of these cycles $(x_1, x_2, \ldots, x_n)$ into the following $\frac{s}{2}$ Hamilton cycles of $K_n^3 - T$:

$$(x_1 y_1^j x_2, x_2 y_2^j x_3, \ldots, x_{n-1} y_{n-1}^j x_n, x_n y_n^j x_1),\ 1 \leq j \leq \frac{s}{2},$$

where

$$\{x_i x_{i+1} y_i^j : 1 \leq j \leq \frac{s}{2}\} = (x_i x_{i+1} \underline{*})_1,$$

the first time the edge $(x_i x_{i+1})$ occurs in one of the cycles, and

$$\{x_i x_{i+1} y_i^j : 1 \leq j \leq \frac{s}{2}\} = (x_i x_{i+1} \underline{*})_2,$$

the second time the edge $(x_i x_{i+1})$ occurs in one of the cycles. □

Finding a Hamilton decomposition of $K_n^3 - I$, $n \equiv 0 \pmod 3$, is known as a *packing* problem; there is a similar notion of a *covering* problem.

Definition 3.13 Let $\mathcal{H}(V, \mathcal{E})$ be a hypergraph. Let $E_1, E_2, \ldots, E_q \subseteq \mathcal{E}$. If $E_i \cap E_j = \emptyset, \forall\, 1 \leq i < j \leq q$, then $E_1, E_2, \ldots, E_q$ is an $E_1, E_2, \ldots, E_q$ *packing* into $\mathcal{H}$. If $E_1 \cup E_2 \cup \cdots \cup E_q = \mathcal{E}$, then $E_1, E_2, \ldots, E_q$ is a *covering* of $\mathcal{H}$ by $E_1, E_2, \ldots, E_q$. If $E_1, E_2, \ldots, E_q$ is both a packing and a covering of $\mathcal{E}$, then it is obviously a partition of $\mathcal{E}$.

Corollary 3.14 *The hypergraph $K_n^3 + I$, $n \equiv 0 \,(mod\ 3)$, can be covered with Hamilton cycles.*

Proof. Let S be the following 1–factor of K_n^3, $n \equiv 0 \pmod 3$:

$$S = \{234, 567, \ldots, (n-4)(n-3)(n-2), (n-1)(n)1\},$$

and recall that

$$T = \{123, 456, \ldots, (n-2)(n-1)n\}.$$

Then the hypergraph $K_n^3 + S = K_n^3 - T + (S + T)$. We can write $S + T$ as

$(123, 234, 456, 567, \ldots, (n-4)(n-3)(n-2), (n-2)(n-1)n, (n-1)(n)1)$,

which is a Hamilton cycle of K_n^3.

By Theorem 3.10, there is a Hamilton decomposition of $K_n^3 - T$. Thus, there is a Hamilton decomposition of $K_n^3 + S$, and the result follows. □

3.2.2 $n \equiv 1 \,(\text{mod } 6)$

When $n \equiv 1 \,(\text{mod } 6)$, the necessary condition for the existence of a Hamilton decomposition of K_n^3 (that $\frac{1}{n}\binom{n}{3}$ is an integer) is satisfied. We shall give here a general construction for a Hamilton decomposition of K_n^3, $n \equiv 1 \,(\text{mod } 6)$, from a Hamilton decomposition of K_n. Since Bermond *et al.* [4] have shown that for n prime there exists a Hamilton decomposition of K_n^3, the first unsolved case when $n \equiv 1 \,(\text{mod } 6)$ is $n = 25$, so we shall use $n = 25$ as an example throughout the proof, and shall give the choice design for constructing a Hamilton decomposition of K_{25}^3 from a Hamilton decomposition of K_{25}.

Consider the 3-edges of K_n^3 as triangles of K_n, where $n = 6k + 1$. Let the vertices of K_n be $V = \{1, 2, \ldots, n\}$ and let calculation on the vertices be modulo n on the residues $1, 2, \ldots, n$. We associate each triangle $\{a, b, c\}$ of K_n with the triples of differences (α, β, γ), so that

$$\{\pm(a-b), \pm(b-c), \pm(c-a)\} = \{\pm\alpha, \pm\beta, \pm\gamma\}.$$

Definition 3.15 Each edge ij in the graph K_{6k+1} has a *length* l,

$$l = min\{(i-j)\,(\text{mod } n),\ (j-i)\,(\text{mod } n)\},$$

associated with it, where $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor = 3k$.

For instance, in K_7 the edge lengths are 1, 2, and 3. (The edge 12 has length 1, the edge 13 has length 2, and the edge 14 has length 3.)

Since $6k+1$ is odd, there will be $6k+1$ edges of the same length l in K_{6k+1}, for each $l \in \{1, 2, \ldots, 3k\}$.

More that one triangle of K_n is associated with each triple of differences, so that equivalence classes of the triangles of K_n can be constructed using the following equivalence relation $\mathcal{R}$:

$$\{a, b, c\}\mathcal{R}\{a', b', c'\} \leftrightarrow \exists i \in \{1, \ldots, n\} \text{ such that } \{a', b', c'\} = \{a+i, b+i, c+i\},$$

where addition in modulo n.

For example, with $n = 25$, the triangles $\{5, 10, 15\}$ and $\{18, 23, 3\}$ are in the same equivalence class, determined by the triple of differences $(5, 5, 10)$.

From now on, use the following notation to denote addition in the triples of differences:

$$\begin{aligned} i * j &= i + j, & &\text{if } i + j \leq \tfrac{n-1}{2}, \text{ and} \\ i * j &= n - (i + j), & &\text{if } i + j > \tfrac{n+1}{2}. \end{aligned}$$

Note that each triangle can have more than one triple of differences associated with it. It follows from an observation by Bermond, Germa and Sotteau [3], that if n is odd, as in this case with $n \equiv 1 \pmod 6$, it is possible to choose α, β, and γ in a triple of differences so that

$$0 < \alpha \leq \beta \leq \gamma = \alpha * \beta < \frac{n}{2}, \qquad (3.1)$$

giving a unique triple of differences for each triangle. This is obviously true if you consider that α, β, and γ are simply the lengths of the edges of the triangles in the equivalence class associated with (α, β, γ). Henceforth, we will assume that all triples of differences are in this form.

The following lemmas provide a few facts about equivalence classes of triangles when $n \equiv 1 \,(\text{mod}6)$.

Lemma 3.16 *For $n \equiv 1\,(mod\ 6)$ there are n triangles in each equivalence class.*

Proof. By definition, $\{a, b, c\}$ and $\{a', b', c'\}$ are in the same equivalence class if and only if there exists $i \in \{1, 2, \ldots, n\}$ such that $\{a', b', c'\} = \{a + i, b + i, c + i\}$. There are exactly n possibilities for i, and since $n \not\equiv 0\,(\text{mod}3)$, $\{a, b, c\} = \{a + i, b + i, c + i\}$ if and only if $i = n$. □

Lemma 3.17 *In any triple of differences (α, β, γ) in K_n, n odd, that satisfies equation 3.1 above, if two of α, β, and γ are equal then (α, β, γ) is the triple of differences of exactly one equivalence class of triangles, and if α, β, and γ are all different, then (α, β, γ) is the triple of differences of exactly two equivalence classes. (If $\alpha = \beta = \gamma$ in a triple of differences, then $n \equiv 0\,(mod\ 3)$.)*

Proof.

Case 1: Suppose $\alpha = \beta \neq \gamma$.

Then any triangle in an equivalence class associated with this triple of differences must have two edges of length α, and so we can write it as $\{a, a +$

$\alpha, a+2\alpha\}$, for some $a \in \{1, 2, \ldots, n\}$. If there were a second equivalence class associated with the triple of differences (α, α, γ), then any triangle in it would also have to have two edges of length α, and could be written as $\{b, b+\alpha, b+2\alpha\}$, for some $b \in \{1, 2, \ldots, n\}$. Then obviously,

$$\{a, a+\alpha, a+2\alpha\}\mathcal{R}\{b, b+\alpha, b+2\alpha\},$$

since

$$\{a, a+\alpha, a+2\alpha\} = \{b+(a-b), b+(a-b)+\alpha, b+(a-b)+2\alpha\},$$

and hence there is only one equivalence class associated with the triple of differences (α, α, γ).

Case 2: Now suppose $\alpha \neq \beta = \gamma$.

Then, as in Case 1, any triangle in an equivalence class associated with (α, β, γ) must have exactly two edges of length γ. A proof similar to that of Case 1 gives the result.

Case 3: Now suppose $\alpha \neq \beta \neq \gamma$.

Then any triangle in an associated equivalence class of the triple of differences (α, β, γ) has exactly one edge of length α and exactly one of length β. First, we shall show that the two triangles

$$\{a, a+\alpha, a+\alpha+\beta\} \text{ and } \{a, a+\beta, a+\alpha+\beta\}$$

are in different equivalence classes.

Both triangles have a third edge of length $\alpha * \beta$, so they do both have the associated triple of differences (α, β, γ).

Suppose $\{a, a+\alpha, a+\alpha+\beta\}\mathcal{R}\{a, a+\beta, a+\alpha+\beta\}$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\{a, a+\alpha, a+\alpha+\beta\} = \{a+i, a+\beta+i, a+\alpha+\beta+i\}.$$

We consider all possibilities (calculations are modulo n).

i) If $a = a+i$ then $i = 0$ and either $a+\alpha = a+\beta+i = a+\beta$ which implies $\alpha = \beta$, or $a+\alpha = a+\alpha+\beta+i = a+\alpha+\beta$ which implies $\beta = 0$.

ii) If $a = a+\beta+i$ then $i = n-\beta$. But then either $a+\alpha = a+i = a-\beta$ so that $\alpha = -\beta$ and $\gamma = 0$, or $a+\alpha+\beta = a+i = a-\beta$, and hence, $\alpha+2\beta = 0$, in which case, $\gamma = \alpha+\beta = n-\beta > \frac{n}{2}$, or $\gamma = n-(\alpha+\beta) = \beta$.

iii) Finally, if $a = a+\alpha+\beta+i$, then $i = n-\alpha-\beta$. Then either $a+\alpha = a+i = a-\alpha-\beta$, implying $2\alpha+\beta = 0$ and $\gamma = n-\alpha > \frac{n}{2}$, or $\gamma = \alpha$, or $a+\alpha = a+\beta+i = a-\alpha$, so that $2\alpha = 0$, and hence $\alpha = 0$, since n is odd.

Thus there are at least two equivalence classes.

If there were a third such equivalence class, then any triangle in it would also have to have an edge of length α and an edge of length β. Thus it would have to contain either the triangle $\{b, b+\alpha, b+\alpha+\beta\}$ or the triangle $\{c, c+\beta, c+\alpha+\beta\}$, $1 \le b, c \le n$. □

Again using $n = 25$ as an example, the triangles of K_{25} that are in the equivalence class determined by the triple of differences $(5, 5, 10)$, are

$$\{\{1+i, 6+i, 11+i\} : 1 \le i \le 25\},$$

while the triangles of K_{25} that are in the equivalence classes determined by the triple of differences $(5, 6, 11)$ are

$$\{\{1+i, 6+i, 12+i\} : 1 \leq i \leq 25\}, \text{ and}\{\{1+i, 7+i, 12+i\} : 1 \leq i \leq 25\}.$$

Corollary 3.18 *Let $n = 6k+1$. There are $3k^2 + k$ triples of differences in K_n.*

Proof. For any α, $1 \leq \alpha \leq 3k$, there is exactly one triple of differences that is either of the form (α, α, β) or (β, α, α). All other triples of differences are of the form $(\alpha, \beta, \alpha * \beta)$, $1 \leq \alpha < \beta < \alpha * \beta < \frac{n}{2}$.

There are $\frac{1}{n}\binom{n}{3}$ equivalence classes. By Lemma 3.17, there are $3k$ equivalence classes each of which is associated with one triple of differences. The rest of the equivalence classes can be paired so that each pair is associated with one triple of differences. Therefore, in total, there are

$$\frac{\frac{1}{n}\binom{n}{3} - 3k}{2} + 3k = 3k^2 + k$$

triples of differences. □

Lemma 3.19 *The length i, $2k+1 \leq i \leq 3k$, in K_{6k+1} cannot be the first element of a triple of differences.*

Proof. Suppose we have the triple of differences $(\alpha = i, \beta, \gamma)$, where $2k+1 \leq i \leq 3k$. Then $2k < \alpha \leq \beta$ and $\gamma = \alpha * \beta = n - (\alpha + \beta) \leq 2k$, which is a contradiction since γ must be greater than or equal to β. □

We again want to construct a choice design on the 3-edges of K^3_{6k+1}. We do this by first choosing representatives for the triples of differences, and then by transfering this to a choice of representatives on the 3–edges. As before with $n \equiv 2 \pmod 3$, we will use a Hamilton decomposition of K_{6k+1} together with the choice design on the 3–edges of K^3_{6k+1} to construct a Hamilton decomposition of K^3_{6k+1}.

Let us first consider the number of Hamilton cycles in Hamilton decompositions of K_{6k+1} and of K^3_{6k+1}. There are

$$\frac{1}{6k+1}\binom{6k+1}{2} = 3k$$

Hamilton cycles in a Hamilton decomposition of K_{6k+1}, and

$$\frac{1}{6k+1}\binom{6k+1}{3} = k(6k-1)$$

Hamilton cycles in a Hamilton decomposition of K^3_{6k+1}. If we let one of the $3k$ Hamilton cycles of K_{6k+1} correspond to k Hamilton cycles of K^3_{6k+1}, and the remaining $3k-1$ Hamilton cycles of K_{6k+1} each correspond to $2k$ Hamilton cycles of K^3_{6k+1}, then we will have

$$1 \times k + (3k-1) \times 2k = k(6k-1)$$

Hamilton cycles of K^3_{6k+1} altogether, as needed.

In a Hamilton decomposition of K_{6k+1}, we can always assume that one of the Hamilton cycles is

$$H_1 = (1,\, 2,\, \cdots,\, 6k+1),$$

and choose it to be the Hamilton cycle that is extended to exactly k Hamilton cycles of K^3_{6k+1}. Thus we want a choice design on the 3–edges of K^3_{6k+1} in which $|a(a+1)\underline{*}| = k$ for all $a \in \{1, 2, \ldots, 6k+1\}$. Since H_1 contains all the edges of length 1 in K_{6k+1}, the other edges of lengths l, where $2 \leq l \leq 3k$, will all occur in other Hamilton cycles of K_{6k+1}. If the choice design has $|ab\underline{*}| = 2k$ for all $a, b \in \{1, 2, \ldots, 6k+1\}$, $a \neq b, b+1, b-1$, then all of the Hamilton cycles of K_{6k+1} except H_1 will be extended to exactly $2k$ Hamilton cycles of K^3_{6k+1}, as needed.

We will construct this choice design of order $6k+1$ on K^3_{6k+1} from a 'representative design' on the triples of differences of K_{6k+1}.

Definition 3.20 A *representative design* on the triples of differences of K_{6k+1} is a way of choosing elements from the triples of differences (α, β, γ) of K_{6k+1} so that the following are satisfied. Let $C(\delta)$ be the set of all triples of differences for which δ is a representative.

i) The triples of differences that correspond to two equivalence classes of triangles in K_{6k+1} have two representatives.

ii) The triples of differences that correspond to one equivalence class of triangles in K_{6k+1} have one representative.

iii) $|C(1)| = k$ and $|C(\delta)| = 2k$, $2 \leq \delta \leq 3k$.

Recall that if the three numbers in the triple of differences are distinct, then there are two associated equivalence classes — thus we choose two representatives for each of these triples of differences and associate one representative with one equivalence class, and the other representative with the other equivalence class. Also, if two of the numbers in a triple of differences are the same, then there is only one associated equivalence class, and hence we choose only one representative for this triple of differences.

The following lemma states that a representative design on the triples of differences of K_{6k+1} exists. This is the major construction of this section, and will be proved after we prove in Lemma 3.22 that a representative design on the triples of differences of K_{6k+1} leads to the right choice design of order $6k+1$, and in Theorem 3.23 that this choice design leads to a Hamilton decomposition of K^3_{6k+1}.

Lemma 3.21 *There is a representative design on the triples of differences of K_{6k+1} in which the number 1 is chosen as a representative in exactly k triples of differences, and each number i, $2 \le i \le 3k$, is chosen as a representative in exactly $2k$ triples of differences.*

Lemma 3.22 *There is a choice design on the 3–edges of K^3_{6k+1} in which*

$$|a(a+1)\underline{*}| = k,\ 1 \le a \le 6k+1,\ \textit{and}$$
$$|ab\underline{*}| = 2k,\ 1 \le a,b \le 6k+1,\ a \ne b, b+1, b-1.$$

Proof.

By Lemma 3.21 there is a representative design on the triples of differences in K_{6k+1} in which the number 1 is chosen as a representative in exactly k triples of differences, and each number i, $2 \le i \le 3k$, is chosen as a representative in exactly $2k$ triples of differences. We shall first transfer the representative design on the triples of differences to a way of choosing one representative from each of the triangles in K_{6k+1}. Then, by noting that each triangle in K_{6k+1} is a 3–edge in K^3_{6k+1}, we shall have the needed choice design.

Transfer the choice of representative(s) on a triple of differences (α, β, γ) to a choice of one representative for each triangle in its associated equivalence class(es) in the following way:

Case 1: $\alpha \neq \beta \neq \gamma$.

In triples of differences (α, β, γ) with two representatives, arbitrarily choose one of the representatives, call it l, and one of the associated equivalence classes. All of the triangles in this equivalence class will have exactly one edge of length l. Choose as the representative in each of these triangles the vertex that is *not* an end–vertex of that edge.

For example, with $6k + 1 = 25$, if the triple of differences is $(\underline{1}, \underline{2}, 3)$,with 1 as one of its representatives, then the triangles and their representatives in one of the corresponding equivalence classes would be

$$\{\{j, j+1, \underline{j+3}\} : 1 \leq j \leq n\},$$

since the vertex $j + 3$ is not an end–vertex of the edge of length 1.

Now do the same thing with the other equivalence class and the other representative of the triple of differences. In our example we would get

$$\{\{j, j+2, \underline{j+3}\} : 1 \leq j \leq n\},$$

since the other representative of the triple of differences was a 2.

Case 2: $\alpha = \beta \neq \gamma$ or $\alpha \neq \beta = \gamma$.

In triples of differences of the form (α, α, β), those with exactly one representative with exactly one associated equivalence class of triangles in K_{6k+1}, if β is chosen as the representative, then we choose the representatives of the triangles in the associated equivalence class as in Case 1.

However, if α is chosen as the representative, then the situation is different because the triangles in the associated equivalence class have two edges of length α. However, we simply pick one of these edges, choose the vertex that is not in this edge as the representative of the triangle, and then be consistent with this choice when choosing representatives for all the other triangles in the equivalence class. More precisely, if we pick a to represent the triangle $\{a, a+b, a+2b\}$, then we pick $a+i$ to represent the triangle $\{a+i, a+b+i, a+2b+i\}$, $1 \leq i < 6k+1$, and if we pick $a+2b$ to represent the triangle $\{a, a+b, a+2b\}$, then we pick $a+2b+i$ to represent the triangle $\{a+i, a+b+i, a+2b+i\}$, $1 \leq i < 6k+1$. For example, in K_{25} if the triple of differences is of the form $(\underline{1}, 1, 2)$, do either

$$\begin{aligned} &\{\underline{1},2,3\},\{\underline{2},3,4\},\ldots,\{\underline{25},1,2\} \\ \text{or}\quad &\{1,2,\underline{3}\},\{2,3,\underline{4}\},\ldots,\{25,1,\underline{2}\}. \end{aligned}$$

If the triple of differences is of the form (α, γ, γ), choose the representatives of the triangles in the equivalence class as above, but now each triangle has two edges of length γ and one of length α.

Now let each triangle in K_{6k+1} be a 3-edge in K^3_{6k+1}. For each $a \in \{1, 2, \ldots, 6k+1\}$, there are k triangles of the form $\{a, a+1, \underline{c}\}$, $a, a+1 \neq c$, because $|C(1)| = k$. So $|a(a+1)\underline{*}| = k$, $1 \leq a \leq 6k+1$. Also, for each $a \in \{1, 2, \ldots, 6k+1\}$, there are $2k$ triangles of the form $\{a, b, \underline{c}\}$, $a \neq b \neq c$, $a \neq b+1, b-1$, because the edge ab has length l, where $2 \leq l \leq 3k$, and $|C(l)| = 2k$. So $|ab\underline{*}| = 2k$, $1 \leq a, b \leq 6k+1$, $a \neq b, b+1, b-1$. □

Theorem 3.23 *There is a Hamilton decomposition of K_n^3, $n \equiv 1\,(mod\ 6)$.*

Proof. By Lemma 3.22 there is a choice design on the 3–edges of K_n^3 such that

$$|a(a+1)\underline{*}| = k,\ 1 \le a \le 6k+1,\ \text{and}$$
$$|ab\underline{*}| = 2k,\ 1 \le a, b \le 6k+1,\ a \ne b, b \pm 1.$$

Take the Hamilton cycle $H_1 = (1, 2, \ldots, 6k, 6k+1)$ in a Hamilton decomposition of K_{6k+1}. Since H_1 contains all the edges of length 1 in K_{6k+1}, we use the choice design to build it up into the following k Hamilton cycles of K_{6k+1}^3:

$$(1\,y_1^j\,2, 2\,y_2^j\,3, \ldots, 6k\,y_{6k}^j\,(6k+1), (6k+1)\,y_{6k+1}^j\,1),\ 1 \le j \le k,$$

where

$$\{i(i+1)y_i^j : 1 \le j \le k\} = i(i+1)\underline{*}.$$

Then all other cycles $(x_1, x_2, \ldots, x_n)$ in the Hamilton decomposition of K_{6k+1} contain no edges of length 1, and hence, from the choice design, are each built up into the following $2k$ cycles:

$$(x_1y_1^jx_2, x_2y_2^jx_3, \ldots, x_{6k}y_{6k}^jx_{6k+1}, x_{6k+1}y_{6k+1}^jx_1),\ 1 \le j \le 2k,$$

where

$$\{x_ix_{i+1}y_i^j : 1 \le j \le 2k\} = x_ix_{i+1}\underline{*}.$$

□

In the proof of Lemma 3.21, we will begin the representative design on the triples of differences by constructing sets $F(\alpha)$, $1 \le \alpha \le 2k$, where

$F(\alpha)$ is the set of triples of differences with first entry α, together with the representative(s) of those triples of differences. As before, an element of a triple of differences will be underlined if it is one of the representatives. By Lemma 3.19, $F(\alpha) = \emptyset$, $2k+1 \leq \alpha \leq 3k$. Since a triple of differences has a least element, the sets $F(1), \cdots, F(2k)$ partition the set of triples of differences in K_{6k+1}.

The following lemma gives the size of $F(\alpha)$, $1 \leq \alpha \leq 2k$.

Lemma 3.24 *For* $1 \leq \alpha \leq 2k$, $|F(\alpha)| = \lfloor \frac{n-\alpha}{2} \rfloor - \alpha + 1$.

Proof.

Let a triple of differences be denoted by (α, β, γ), where α, β, and γ satisfy equation 3.1, that is,

$$0 < \alpha \leq \beta \leq \gamma = \alpha * \beta < \frac{n}{2}.$$

Given α, we want to count the number of possible β. If β satisfies $\alpha \leq \beta \leq 3k - \alpha$, then $\gamma = \alpha + \beta$, and trivially $\beta \leq \gamma < \frac{n}{2}$. Therefore, all values of β from α to $\lfloor \frac{n}{2} \rfloor - \alpha$ are possible.

If $\beta \geq 3k - \alpha + 1$, then $\gamma = n - (\alpha + \beta)$, since $\alpha + \beta > \frac{n}{2}$. We also need $\beta \leq \gamma$, which gives $\beta \leq n - (\alpha + \beta)$, which in turn implies $\beta \leq \lfloor \frac{n-\alpha}{2} \rfloor$. Therefore, all values of β from $\lfloor \frac{n}{2} \rfloor - \alpha + 1$ to $\lfloor \frac{n-\alpha}{2} \rfloor$ are possible.

So, given α, the possible β range between α and $\lfloor \frac{n-\alpha}{2} \rfloor$. Thus

$$|F(\alpha)| = \left\lfloor \frac{n-\alpha}{2} \right\rfloor - \alpha + 1.$$

□

Recall that $C(\delta)$ is the set of all triples of differences that contain δ as a representative. Let $NC(\delta)$ be the set of all triples of differences that contain δ, but not as a representative.

We construct the sets $F(1), F(2), \cdots, F(2k)$ in that order. The order is important because any given α, $1 \leq \alpha \leq 2k$, only occurs in $F(1), \cdots, F(\alpha)$. (Triples of differences are written with their least element first, and $F(\alpha)$ contains only those triples of differences with first element α.) Now we are ready to prove Lemma 3.21.

Proof of Lemma 3.21.

Constructing a choice design on the triples of differences of K_{6k+1}:

For $1 \leq j \leq 2k$, we construct $F(j)$ and check that $|F(j)| = \lfloor \frac{n-j}{2} \rfloor - j + 1$. Then we check that $|C(1)| = k$ and $|C(j)| = 2k$, $2 \leq j \leq 3k$, as needed.

1) $F(1)$:

We choose the representatives for the triples of differences in $F(1)$, so that k of them will have 1 as a representative:

i) $(\underline{1}, 1, 2) \in C(1)$.

ii) $(\underline{1}, \underline{i}, i+1) \in C(1)$, where $2 \leq i \leq k$.

iii) $(1, \underline{i}, \underline{i+1}) \in NC(1)$, where $k+1 \leq i \leq 3k-1$.

iv) $(1, \underline{3k}, 3k) \in NC(1)$.

Then,

$$|F(1)| = 1 + (k-1) + (2k-1) + 1 = 3k = \lfloor \tfrac{(6k+1)-1}{2} \rfloor - 1 + 1, \text{ and}$$
$$|C(1)| = 1 + (k-1) = k.$$

Notice that each i, $2 \leq i \leq k+1$, has been chosen exactly once as a representative in $F(1)$, and each i, $k+2 \leq i \leq 3k$, has been chosen exactly twice.

In sections 2)–4), we will check that the triples of differences satisfy $j < i < j*i < \frac{n}{2}$, or $j = i < j*i < \frac{n}{2}$, or $j < i = j*i < \frac{n}{2}$, where it is not immediately obvious.

2) $F(j)$, $2 \leq j \leq k+1$:

i) $(\underline{j}, j, j*j) \in C(j)$.

ii) $(\underline{j}, \underline{i}, j*i) \in C(j)$, where $j+1 \leq i \leq 2k$, except when $j = k+1$ and $i = 2k$. In this case we choose $(\underline{k+1}, 2k, \underline{3k}) \in C(k+1)$.
(Note that if $j+i \leq 3k$, then $j*i = j+i > i$, and if $j+i > 3k$, then $j*i = n-(j+i) \geq (6k+1)-(3k+1) \geq 3k > i$.)

iii) $(j, \underline{i}, \underline{j*i}) \in NC(j)$, where $2k+1 \leq i \leq \lceil \frac{6k-1-j}{2} \rceil$.
(In this case note that if $j+i \leq 3k$, then $j*i = j+i > i$. If $j+i > 3k$, then $j*i = n-(j+i)$, and since $i \leq \lceil \frac{6k-1-j}{2} \rceil$, $2i \leq 6k-j$. Thus $n-(j+i) \geq i+1$ and so $j*i > i$.)

iv) $(j, \underline{i}, i) \in NC(j)$, where $i = \frac{6k+1-j}{2}$, if j is odd.
(Here observe that $i = \frac{6k+1-j}{2} \geq \frac{6k+1-(k+1)}{2} = \frac{5k}{2} > 2k > j$.)
Then, for $2 \leq j \leq k+1$:

$$\begin{aligned} \text{If } j \text{ is even, } |F(j)| &= 1+(2k-j)+\left(\left\lceil \frac{6k-1-j}{2} \right\rceil - 2k\right), \\ &= 1-j+\frac{6k-j}{2} \\ &= \left\lfloor \frac{6k+1-j}{2} \right\rfloor - j + 1 \text{ and,} \\ \text{if } j \text{ is odd, } |F(j)| &= 1+(2k-j)+\left(\left\lceil \frac{6k-1-j}{2} \right\rceil - 2k\right)+1, \end{aligned}$$

$$\begin{aligned} &= 2 - j + \frac{6k-1-j}{2} \\ &= \left\lfloor \frac{6k+1-j}{2} \right\rfloor - j + 1. \end{aligned}$$

Also,

$$\begin{array}{rlll} |C(j) \cap F(j)| & = 1 + 2k - j, & 2 \le j \le k+1, & \text{see 2(i), 2(ii).} \\ C(j) \cap F(1) & = \{(\underline{1}, \underline{j}, j+1)\}, & 2 \le j \le k, & \text{1(ii).} \\ & \cup\{(1, \underline{j}, \underline{j+1})\}, & j = k+1, & \text{1(iii).} \\ C(j) \cap F(i) & = \{(\underline{i}, \underline{j}, i+j)\}, & 2 \le i < j,\ 2 \le j \le k+1. & \text{2(ii).} \end{array}$$

Thus,

$$|C(j)| = (1 + 2k - j) + 1 + (j - 2) = 2k, \text{ for } 2 \le j \le k+1.$$

3) $F(j)$, $k + 2 \le j \le 2k - 1$:

i) $(\underline{j}, j, j * j) \in C(j)$.

ii) $(\underline{j}, \underline{i}, j * i) \in C(j)$, where $j + 1 \le i \le 2k - 1$.

(Observe that if $j + i \le 3k$, then $j * i = j + i > i$, and if $j + i > 3k$, then $j * i = n - (i + j) \ge (6k + 1) - (4k - 2) = 2k + 3 > i$.)

iii) $(j, \underline{i}, \underline{j * i}) \in NC(j)$, where $2k \le i \le \lceil \frac{6k-1-j}{2} \rceil$.

(Again, if $j + i \le 3k$, then $j * i > i$. If $j + i > 3k$, then $j * i = n - (i + j)$, and as $2i \le 6k - j$, $n - (i + j) \ge i + 1$, implying that $j * i > i$.)

iv) $(j, \underline{i}, i) \in NC(j)$, where $i = \frac{6k+1-j}{2}$, if j is odd.

(Note that $i = \frac{6k+1-j}{2} \ge \frac{6k+1-(2k-1)}{2} = 2k + 1 > j$.)

Then, for $k + 2 \le j \le 2k - 1$:

$$\text{If } j \text{ is even, } |F(j)| = 1 + (2k - 1 - j) + \left(\left\lceil \frac{6k-1-j}{2} \right\rceil - 2k + 1 \right)$$

$$
\begin{aligned}
&= 1 - j + \frac{6k-j}{2} \\
&= \left\lfloor \frac{6k+1-j}{2} \right\rfloor - j + 1, \text{ and} \\
\text{if } j \text{ is odd, } |F(j)| &= 1 + (2k-1-j) + \left(\left\lceil \frac{6k-1-j}{2} \right\rceil - 2k + 1\right) + 1 \\
&= 2 - j + \frac{6k-1-j}{2} \\
&= \left\lfloor \frac{6k+1-j}{2} \right\rfloor - j + 1.
\end{aligned}
$$

Further,

$$
\begin{aligned}
|C(j) \cap F(j)| &= 1 + (2k-1-j) = 2k - j, && \text{see 3(i), 3(ii).} \\
C(j) \cap F(1) &= \{(1, \underline{j-1}, \underline{j}), (1, \underline{j}, \underline{j+1})\}, && \text{1(i).} \\
C(j) \cap F(i) &= \{(\underline{i}, \underline{j}, i * j)\}, \quad 2 \le i < j. && \text{2(ii), 3(ii).}
\end{aligned}
$$

Thus,

$$
|C(j)| = 2k - j + 2 + j - 2 = 2k, \text{ for } k+2 \le j \le 2k-1.
$$

4) $F(2k)$:

i) $(\underline{2k}, 2k, 2k+1) \in C(2k)$.

Then,

$$
|F(2k)| = 1 = \left\lfloor \frac{6k+1-2k}{2} \right\rfloor - 2k + 1, \text{ and}
$$

$$
\begin{aligned}
|C(2k) \cap F(2k)| &= 1, && \text{see 4(i).} \\
C(2k) \cap F(1) &= \{(1, \underline{2k-1}, \underline{2k}), (1, \underline{2k}, \underline{2k+1})\}, && \text{1(iii).} \\
C(2k) \cap F(i) &= \{(\underline{i}, \underline{2k}, 2k+i)\}, 2 \le i \le k, && \text{2(ii).} \\
C(2k) \cap F(k+1) &= \emptyset, && \text{2(ii).} \\
C(2k) \cap F(i) &= \{(i, \underline{2k}, \underline{i * 2k})\}, k+2 \le i \le 2k-1. && \text{3(iii).}
\end{aligned}
$$

Thus,

$$|C(2k)| = 1 + 2 + k - 1 + 0 + k - 2 = 2k.$$

We must check that we have assigned representatives to every one of the triples of differences. For any j, $1 \leq j \leq 2k$, the elements of $F(j)$ are certainly distinct by construction, and we have checked that

$$|F(j)| = \lfloor \frac{6k+1-j}{2} \rfloor - j + 1.$$

For any i, j, $1 \leq i \neq j \leq 2k$, the elements of $F(i)$ and $F(j)$ are distinct. Therefore, it is enough to prove that the total number of triples of differences in $F(1), F(2), \ldots, F(2k)$ equals the number of triples of differences of K_{6k+1}.

By Corollary 3.18, there are $3k^2 + k$ triples of differences in K_{6k+1}.

$$\begin{aligned}
\sum_{j=1}^{2k} |F(j)| &= \sum_{j=1}^{2k} \left(\left\lfloor \frac{6k+1-j}{2} \right\rfloor - j + 1 \right) \\
&= \frac{-2k(2k+1)}{2} + 2k + 2k \times 3k - \sum_{j=1}^{2k} \left\lceil \frac{j-1}{2} \right\rceil \\
&= 4k^2 + k - (0 + 1 + 1 + 2 + 2 + \cdots + (k-1) + (k-1) + k) \\
&= 4k^2 + k - (k-1)k - k \\
&= 3k^2 + k.
\end{aligned}$$

Thus, every triple of differences in K_{6k+1} has its representative(s), the number 1 has been chosen as a representative in k triples of differences, and the numbers $2, 3, \cdots, 2k$ have each been chosen as representatives in $2k$ triples of differences. We must check that the numbers $2k+1, 2k+2, \cdots, 3k$ have also each been chosen to represent $2k$ triples of differences. By Lemma 3.16, there are $\frac{1}{6k+1}\binom{6k+1}{3}$ equivalence classes. One representative is chosen for each equivalence class. Therefore,

$$\frac{1}{6k+1}\binom{6k+1}{3} = k(6k-1)$$

representatives are chosen.

We have shown in section 1) that 1 is chosen exactly k times, and in 2)–4) that each j, $2 \le j \le 2k$ is chosen exactly $2k$ times. This leaves

$$k(6k-1) - k - 2k(2k-1) = 2k^2$$

equivalence classes for which we have not yet counted a representative. If we show that for each j, $2k+1 \le j \le 3k$, that j is chosen at least $2k$ times, then we must have that each j is chosen exactly $2k$ times and we are done.

By Lemma 3.19, since $j > 2k$, j is either in the second or third position in a triple of differences. For $2k+1 \le j \le 3k$, we list the triples of differences in $C(j)$ and check that $|C(j)| \ge 2k$. We consider $C(2k+1)$ in section 5), then $C(j)$ with $2k+2 \le j \le 3k-1$, in section 6), and finally $C(3k)$ in section 7). In all three cases, we first list the triples of differences with j in their second position, and then the triples of differences with j in their third position. The first position will always be i, with $1 \le i \le 2k$. For each subset of $C(j)$ listed, a short justification that all triples of differences (α, β, γ) in it satisfy equation 3.1 will be given if it is not immediately obvious.

5) $C(2k+1)$ contains the following triples of differences:

i) $(1, \underline{2k+1}, \underline{2k+2})$. The choice of representative is from 1(iii).

ii) $(i, \underline{2k+1}, \underline{2k+1+i})$, where $2 \le i \le k-1$. The choice of representatives is from 2(iii).

(Note that we have $i < 2k+1 < 2k+1+i \leq 3k$ as needed for these to be triples of differences.)

iii) $(i, \underline{2k+1}, \underline{4k-i})$, where $k \leq i \leq 2k-2$. The choice of representatives is from 2(iii) and 3(iii), since $\lceil\frac{6k-1-i}{2}\rceil \geq \lceil\frac{4k+1}{2}\rceil = 2k+1$.
(Note that in this case $i < 2k+1 < 4k-i \leq 3k$ and $4k-i = n-(i+2k+1)$, as needed.)

iv) $(2k-1, \underline{2k+1}, 2k+1)$. The choice of representatives is from 3(iv).

v) $(1, \underline{2k}, \underline{2k+1})$. The choice of representatives is from 1(iii).

Thus $|C(2k+1)| \geq 1+k-2+k-1+1+1 = 2k$.

6) $C(j)$, $2k+2 \leq j \leq 3k-1$, contains the following triples of differences:

i) $(1, \underline{j}, \underline{1+j})$. The choice of representatives is from 1(iii).

ii) $(i, \underline{j}, \underline{i+j})$, where $2 \leq i \leq 3k-j$. The choice of representatives is from 2(iii), since $i \leq 3k-j$ implies that $i \leq k-2$ and $j \leq \lceil\frac{6k-2i}{2}\rceil \leq \lceil\frac{6k-i-2}{2}\rceil \leq \lceil\frac{6k-i-1}{2}\rceil$.
(Since $i \leq k-2$, we have $2 \leq i < j < i+j \leq 3k$, as needed. Note that when $j = 3k-1$, the bounds $2 \leq i \leq 3k-j$ do not hold, but we will count this triple zero times anyway, so this does not matter.)

iii) $(i, \underline{j}, \underline{n-(i+j)})$, where $3k-j+1 \leq i \leq 6k-2j$. The choice of representatives is from 2(iii) and 3(iii), since if i is odd, $i \leq 6k-2j-1$, so that $j \leq \lceil\frac{6k-i-1}{2}\rceil$, and if i is even, $i \leq 6k-2j$, so that $j \leq \lceil\frac{6k-i}{2}\rceil = \lceil\frac{6k-i-1}{2}\rceil$. Also $2 \leq i \leq 2k-4$.
(Note that $2 \leq i \leq 2k-4 < j$, and $i \leq 6k-2j$ implies that $j < n-(i+j) \leq n-(3k-j+1+j) = 3k$.)

iv) $(n-2j, \underline{j}, j)$. The choice of representative is from 2(iv) and 3(iv), since $3 \leq n-2j \leq 2k-3$.

(Also $2 \leq n-2j < j$, as needed for this to be a triple of differences.)

v) $(1, \underline{j-1}, \underline{j})$. The choice of representatives is from 1(iii).

vi) $(i, \underline{j-i}, \underline{j})$, where $2 \leq i \leq j-2k-1$. The choices of representatives is from 2(iii), since $2 \leq i \leq j-2k-1 \leq k-2$ and $2k+1 \leq j-i \leq 3k-1-i \leq \frac{6k-2-2i}{2} \leq \frac{6k-4-i}{2} < \lceil\frac{6k-i-1}{2}\rceil$. (Again note that if $j = 2k+2$, then the bounds $2 \leq i \leq j-2k-1$ do not hold. Again, this does not matter since we count this triple zero times.)

(Since $2i \leq 2j-4k-2 \leq j-k-3$, $i \leq j-i-k-3$, and so, $2 \leq i < j-i < j < 3k$.)

vii) $(i, \underline{n-(i+j)}, \underline{j})$, where $n-2j+1 \leq i \leq 4k-j$. The choice of representatives is from 2(iii) and 3(iii), since $n-(i+j) \geq 2k+1$. Also, $n-(i+j) \leq n-(n-2j+1)-j \leq j-1$ implies that $n \leq 2j+i-1$, which implies that $2n-2(i+j) \leq n-2i-2j+2j+i-1 \leq 6k-i$. Thus, if i is odd, $n-(i+j) < \frac{6k-i}{2} \leq \lceil\frac{6k-i-1}{2}\rceil$, and if i is even, $n-(i+j) \leq \frac{6k-i}{2} \leq \lceil\frac{6k-i-1}{2}\rceil$. (Since $j \leq 3k-1$, $n-2j+1 \geq 4$. Also, $n-(i+j) \geq n-4k = 2k+1 > i$ and $n-(i+j) \leq n-(n-2j+1+j) \leq j-1 < j$. Therefore, $2 \leq i < n-(i+j) < j < 3k$.)

viii) $(4k-j+1, \underline{2k}, \underline{j})$. The choice of representatives is from 3(iii), since $k+2 \leq 4k-j+1 \leq 2k-1$.

(This is a triple of differences since $k+2 \leq 4k-j+1 < 2k < j < 3k$.)

Thus, $|C(j)| \geq 1+(3k-j-1)+(3k-j)+1+1+(j-2k-2)+(4k+j-n)+1 = 2k$.

7) $C(3k)$ contains the following triples of differences:

i) $(1, \underline{3k}, 3k)$. The choice of representatives is from 1(iv).

ii) $(1, \underline{3k-1}, \underline{3k})$. The choice of representative is from 1(iii).

iii) $(i, \underline{3k-i}, \underline{3k})$, where $2 \leq i \leq k-1$. The choice of representatives is from 2(iii), since $2k+1 \leq 3k-i \leq \lceil \frac{6k-2i}{2} \rceil \leq \lceil \frac{6k-i-1}{2} \rceil$.
(Note that since $i \leq k-1$ implies that $3k-i \geq 2k+1$, we have $i < 3k-i < 3k$)

iv) $(\underline{k+1}, 2k, \underline{3k})$. The choice of representatives is from 2(ii).

v) $(i, \underline{3k+1-i}, \underline{3k})$, where $2 \leq i \leq k$. The choice of representatives follows from 2(iii) and the fact that if i is odd, $2k+1 \leq 3k+1-i \leq \lceil \frac{6k+2-2i}{2} \rceil \leq \lceil \frac{6k-1-i}{2} \rceil$, and if i is even, $2k+1 \leq 3k+1-i \leq \lceil \frac{6k+2-2i}{2} \rceil \leq \lceil \frac{6k-i}{2} \rceil = \lceil \frac{6k-i-1}{2} \rceil$. (Since $i \leq k$ implies $3k+1-i \geq 2k+1$, we have $i < 3k+1-i < 3k$, as needed.)

Thus, as needed, $|C(3k)| \geq 2 + (k-2) + 1 + (k-1) = 2k$.

□

Table 3.3 gives a choice of representatives of the triples of differences of K_{25} that will lead to a Hamilton decomposition of K_{25}^3. From these choices of representatives of the triples of differences, we build a choice design on the 3–edges of K_{25}^3 as in Lemma 3.22. We then take a Hamilton decomposition of K_{25} of which one Hamilton cycle is

$$H_1 = (1, 2, 3, \cdots, 23, 24, 25),$$

which exists by Lemma 3.1. This Hamilton cycle H contains all the edges of length 1 from the graph of K_{25}. Then, using the choice design on the 3–edges of K_{25}^3, build H into $k = 4$ Hamilton cycles of K_{25}^3, and all the other Hamilton cycles of the Hamilton decomposition of K_{25} into $2k = 8$ Hamilton cycles of K_{25}^3.

3.3 Summary

Theorems 3.10 and 3.23, together with Bermond's results of Lemmas 2.2 and 2.3, complete the problem of constructing a Hamilton decomposition of K_n^3, when $n \equiv 1, 2 \pmod 3$, and a Hamilton decomposition of $K_n^3 - I$, when $n \equiv 0 \pmod 3$

Triples of differences	Choice(s) of representatives	Triples of differences	Choice(s) of representatives
$(1,1,2)$	1	$(3,7,10)$	3 7
$(1,2,3)$	1 2	$(3,8,11)$	3 8
$(1,3,4)$	1 3	$(3,9,12)$	9 12
$(1,4,5)$	1 4	$(3,10,12)$	10 12
$(1,5,6)$	5 6	$(3,11,11)$	11
$(1,6,7)$	6 7	$(4,4,8)$	4
$(1,7,8)$	7 8	$(4,5,9)$	4 5
$(1,8,9)$	8 9	$(4,6,10)$	4 6
$(1,9,10)$	9 10	$(4,7,11)$	4 7
$(1,10,11)$	10 11	$(4,8,12)$	4 8
$(1,11,12)$	11 12	$(4,9,12)$	9 12
$(1,12,12)$	12	$(4,10,11)$	10 11
$(2,2,4)$	2	$(5,5,10)$	5
$(2,3,5)$	2 3	$(5,6,11)$	5 6
$(2,4,6)$	2 4	$(5,7,12)$	5 7
$(2,5,7)$	2 5	$(5,8,12)$	5 12
$(2,6,8)$	2 6	$(5,9,11)$	9 11
$(2,7,9)$	2 7	$(5,10,10)$	10
$(2,8,10)$	2 8	$(6,6,12)$	6

cont.

cont.

Triples of differences	Choice(s) of representatives	Triples of differences	Choice(s) of representatives
(2, 9, 11)	9 11	(6, 7, 12)	6 7
(2, 10, 12)	10 12	(6, 8, 11)	8 11
(2, 11, 12)	11 12	(6, 9, 10)	9 10
(3, 3, 6)	3	(7, 7, 11)	7
(3, 4, 7)	3 4	(7, 8, 10)	8 10
(3, 5, 8)	3 5	(7, 9, 9)	9
(3, 6, 9)	3 6	(8, 8, 9)	8

Table 3.3: Representatives of triples of differences for K_{25}^3

Bibliography

[1] C. Berge, "Graphs and Hypergraphs". North Holland, Amsterdam, 1979.

[2] J.C. Bermond, Hamiltonian decompositions of graphs, directed graphs and hypergraphs. Annals of Discrete Mathematics **3** (1978) 21–28.

[3] J.C. Bermond, A. Germa and D. Sotteau, Hypergraph-Designs. Ars Combinatoria, **3** (1977) 47–66.

[4] J.C. Bermond, A. Germa, M.C. Heydemann et D. Sotteau, Hypergraphes Hamiltoniens. *Colloques internationaux C.N.R.S.* No.260- *Problèmes Combinatoires et Théorie des Graphes*, Orsay, (1976)

[5] J.C. Bermond, C. Huang and D. Sotteau, Balanced cycle and circuit designs: even case. Ars. Combinatoria **5** (1978) 293–318.

[6] J.C. Bermond and D. Sotteau, Cycle and circuit designs: odd case. *Beitrage zur Graphentheorie und deres Arwendunger,* Proc. Colloq. Oberhof Illmenau (1978) 11–32.

[7] R. Duke, Types of cycles in hypergraphs. Annals of Discrete Mathematics **27** (1985) 399–418

[8] E. Eliad-Badt, Decomposition of the complete hypergraph into stars. Elsevier Science Publishers B.V. (North Holland) (1988)

[9] H. Hanani, Balanced incomplete block designs and related designs. Discrete Mathematics **11** (1975) 255–369

[10] C. Huang and A. Rosa, Another class of balanced graph designs, balanced circuit designs. Discrete Mathematics **12** (1975) 269–293

[11] C. Huang and A. Rosa, On the existence on balanced bipartite designs. Utilitas Mathematics **9** (1973) 55–75

[12] A. Kotzig, On the decomposition of complete graphs into $4m$–gons. Mat.–Fyz. Časopis Sloven. Akad. Vied. **15** (1965) 229–233 (in Russian).

[13] Z. Lonc, Decompositions of Hypergraphs into Delta-Systems and Constellations. JCMCC **7**, (1990) 201–217

[14] Z. Lonc, Decompositions of Hypergraphs into Hyperstars. JCTA **55**, No. 1, (1990) 33–48

[15] Z. Lonc, On some packing, covering and decomposition problems for Hypergraphs. Graphs, Hypergraphs, and Matroids. III Zielona Góra (1989).

[16] Z. Lonc, Solution of a Delta-System Decomposition Problem. Discrete Mathematics **66**, (1987) 157–168

[17] E. Lucas, *Récréations Mathématiques,* Vol. 2, Gauthier–Villars, Paris (1884)

[18] S. Marshall, Cycle Decompositions of Complete Graphs. Masters Thesis, Simon Fraser University, (1989).

[19] A.F. Mouyart and F. Sterboul, Decomposition of the Complete Hypergraph into Delta-Systems I. JCTA **40**, (1985) 290–304

[20] A.F. Mouyart and F. Sterboul, Decomposition of the Complete Hypergraph into Delta-Systems II. JCTA **41**, (1986) 139–149

[21] C. Rodger, Problems of cycle cyctems of odd length. Preprint.

[22] A. Rosa, On the cyclic decompositions of the complete graph into polygons with an odd number of edges. Časopis Pěst. Math. **91** (1966) 53–63

[23] A. Rosa, On cyclic decompositions of the complete graph into $(4m+2)$-gons. Mat.–Fyz. Časopis Sloven. Akad. Vied. **16** (1966) 349–353.

[24] T. Tillson, A hamilton decomposition of K^*_{2m},$2m \geq 8$, J. Combinatorial theory (B) **29** (1980) 68–74.