A NON-SPLITTING THEOREM FOR D.R.E. SETS

by

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Abstract

A set of natural numbers is called d.r.e. (difference recursively enumerable) if it may obtained from some recursively enumerable set by deleting the numbers belonging to another recursively enumerable set. Sacks showed that for each non-recursive recursively enumerable set A there are disjoint recursively enumerable sets B, C which cover A such that A is recursive in neither $A \cap B$ nor $A \cap C$. The thesis constructs a counterexample which shows that Sacks's theorem is not in general true when A is d.r.e. rather than recursively enumerable.

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Dedication

To the memory of
my father

Contents

Chapter 1

Introduction

All sets in this thesis are subsets of ω, the set of all natural numbers. Following the usual conventions of recursive function theory we identify sets with their characteristic functions. A (partial) function is called *(partial) recursive* if it is computable by an algorithm. A subset of the natural numbers A is *recursively enumerable* (*r.e.*) if there is an algorithm which will list the elements of A. One set A is *Turing computable* from another set B, written $A \leq_T B$, if there is an algorithm for computing the characteristic functions of A given an 'oracle' for B. Two sets are *Turing equivalent*, written $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$. The (Turing) *degree* of a set A is $deg(A) =_{def} \{B : B \equiv_T A\}$. A degree $\mathbf{a}$ is less than or equal a degree $\mathbf{b}$ (written $\mathbf{a} \leq \mathbf{b}$) if there is a set $A \in \mathbf{a}$ and a set $B \in \mathbf{b}$ with $A \leq_T B$. A set A is *n-r.e.* if there is a recursive function f such that for all x, $\lim_s f(x, s) = A(x)$, $f(x, 0) = 0$ and $|\{s : f(x, s+1) \neq f(x, s)\}| \leq n$. In particular, a set A is *r.e.* if and only if A is 1-r.e., and A is the difference of two r.e. sets (*d.r.e.*) if and only if A is 2-r.e.. A degree is called *r.e.* (*d.r.e.*, *n-r.e.*) if it is the degree of an r.e. (d.r.e., n-r.e.) set. A d.r.e. (n-r.e.) degree is called *properly* d.r.e. (n-r.e.) if it is not r.e. ((n-1)-r.e.). An r.e. set A is *complete* if for every r.e. set B, $B \leq_T A$. An r.e. degree is *complete* if it includes a complete r.e. set. The degree of the halting problem, $\mathbf{0}'$, is complete. For each $n < \omega$, the n-r.e. degrees $\mathbf{D}_n$ are partially ordered by $\leq$. Let $(\mathbf{R}, \leq)$ denote $(\mathbf{D}_1, \leq)$. The partial order $(\mathbf{D}_n, \leq)$ has a least degree $\mathbf{0}$ (the degree of the computable sets) and a greatest degree $\mathbf{0}'$ (the degree of the halting problem). Every pair of degrees has a least upper bound. For each pair of degrees in $\mathbf{D}_n$, their least upper bound also in $\mathbf{D}_n$. But a pair of degree may fail to have a greatest lower bound. The partial order $(\mathbf{D}_n, \leq)$ is thus an upper semi-lattice. For more notation and background terminology see [12].

All early examples of nonrecursive r.e. sets were complete. This led Post to ask the natural question: is there a nonrecursive r.e. set which is not complete? Friedberg and Muchnik independently solved this problem by exhibiting a pair of $\leq_T$-incomparable r.e. sets. The technical they used became known as the priority method. The easier applications of this method, including the proof of the Friedberg-Muchnik theorem, are characterized as "finite injury" arguments. A construction is made in ω stages to meet certain goals, usually called "requirements". In a finite injury construction each requirement is injured at most a finite number of times and so need be addressed at at most a finite number of stages.

The priority method has become the central technique in the study of recursion theory. The finite injury priority method was first extended by Shoenfield and later by Sacks to prove further theorems and this technique is now known as the infinite injury priority method. Lachlan [9] further refined the infinite injury priority method to prove the famous Nonsplitting Theorem. This was the first example of a $\mathbf{0}'''$-priority argument.

We now sketch the history of the study of the n-r.e. sets and degrees. For the r.e. case, there are two most fundamental results which were all by Sacks.

Sacks Density Theorem. The partial order of r.e. degrees is dense.

Sacks Splitting Theorem for r.e. sets. For each non-recursive recursively enumerable set A there are disjoint recursively enumerable sets B, C which cover A such that A is recursive in neither $A \cap B$ nor $A \cap C$.

These two theorems can be found in Soare's monograph [12], see $VIII.4.1$ and $VII.3.2$. As a corollary of the latter result one can get:

Sacks Splitting Theorem for r.e. degrees. For each nonrecursive r.e. degree $\mathbf{a}$ there exist r.e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$ such that $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_i < \mathbf{a}$ for $i = 0, 1$.

The question of whether the d.r.e. degrees are also dense is answered negatively by a number of authors in [6]:

Nondensity Theorem for n-r.e. degrees. The d.r.e. degrees are not dense.

The splitting conjecture for d.r.e. degrees is confirmed by Cooper [2]:

Splitting Theorem for n-r.e. degrees. For each nonrecursive n-r.e. degree $\mathbf{a}$ there exist n-r.e. degrees $\mathbf{a}_0$, $\mathbf{a}_1$ such that $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_i < \mathbf{a}$ for $i = 0, 1$.

Cooper, Lempp and Watson [7], verifying a claim of Arslanov [1], proved:

Theorem. $(\mathbf{D}_2, \leq) \models (\forall \mathbf{a} > \mathbf{0})(\exists \mathbf{b} < \mathbf{0}')[\mathbf{a} \cup \mathbf{b} = \mathbf{0}']$.

This was the first example of a first-order sentence true in $(\mathbf{D}_2, \leq)$ and false in $(\mathbf{R}, \leq)$. The failure of this particular sentence in $(\mathbf{R}, \leq)$ was shown independently by Cooper and Yates [see 10]. An unrelated sentence true in $(\mathbf{D}_2, \leq)$ but not in $(\mathbf{R}, \leq)$ was found by Downey [8]:

Theorem. $(\mathbf{D}_2, \leq) \models (\exists \mathbf{a}, \mathbf{b} < \mathbf{0}')[\mathbf{a} \cup \mathbf{b} = \mathbf{0}' \wedge \mathbf{a} \cap \mathbf{b} = \mathbf{0}]$.

Lachlan's nondiamond theorem [12, IX.3.1] says that this sentence is false in $(\mathbf{R}, \leq)$.

The goal of this thesis is to discover how strong an analogue of the Sacks splitting theorem for r.e. sets is true for properly d.r.e. sets. What we succeed in doing is to show that a strong analogue of Sacks's theorem fails. More precisely we prove:

Main Theorem. There exists a properly d.r.e. set D such that for all r.e. sets A^0, A^1 with $A^0 \cap A^1 = \emptyset$,

$$D \subseteq A^0 \cup A^1 \Longrightarrow [D \leq_T A^0 \cap D \vee D \leq_T A^1 \cap D].$$

One should note that, if the word "properly" is deleted in the statement of the theorem, then we can obtain a trivial example by letting D be the completement of a maximal set.

The proof of our main theorem above is long and fairly complicated. Therefore, before giving the formal proof of above theorem, we shall outline some of the basic ideas. In Chapter 2 by way of introduction to the complexities of the main theorem we show that the negation of the main theorem is not true effectively. This provider the basic module used for our main construction.

The most exciting application of splitting properties of d.r.e. degrees is Cooper's solution to an old question of Kleene and Post by showing that Turing jump operator is definable in $\mathbf{D}$ (see Cooper [4,5]). Thus splitting theorems are not only important for themselves. There are other applications for them, e.g. Slaman and Woodin [11] showed that the $\mathbf{R}$, is definable in $\mathbf{D}(\leq \mathbf{0}')$, by using the Sacks Splitting Theorem for r.e. sets (and a general definability result). Here we prove that the strongest possible analogue of the Sacks splitting theorem for d.r.e. sets fails. One should note that, if a suitable theorem allowing the splitting of d.r.e. sets into low d.r.e. sets could be found, then it could be used in the manner of Slaman and Woodin [11] to show that $\mathbf{D}_2$ is definable in $\mathbf{D}(\leq \mathbf{0}')$.

Chapter 2

The basic module

Our immediate aim in this chapter is to show that if a properly d.r.e. set D is presented via the indexes of r.e. sets B, C such that $D = B - C$, then one cannot always effectively find A^0, A^1 such that $B = A^0 \sqcup A^1$ and $\deg(A^i - C) <_T \deg(D)$ $(i < 2)$. This means that the strongest possible analogue of the Sacks splitting theorem for r.e. sets fails for d.r.e. sets. The module developed in this chapter will play a crucial role in everything that follows. Formally, we establish:

2.1 Theorem. *Given the indexes of r.e. sets A^0, A^1, we can effectively enumerate B, C with $C \subseteq B$ such that $D = B - C$ is properly d.r.e. and*

$$B = A^0 \sqcup A^1 \Longrightarrow [D \equiv_T A^0 - C \vee D \equiv_T A^1 - C].$$

Proof. Let $\{(W^e, \Phi^e, \Psi^e) : e < \omega\}$ be an effective enumeration of all tuples (W, Φ, Ψ), where W is an r.e. set and Φ, Ψ are p.r. functionals. Let indexes of the r.e. sets A^0, A^1 be given. Our task is to effectively enumerate B, C such that $C \subseteq B$ and $D = B - C$ meets the following requirements:

$$\begin{aligned} \mathcal{R}^e: &\qquad D \neq \Psi^e(W^e) \vee W^e \neq \Phi^e(D); \\ \mathcal{S}: &\qquad B = A^0 \sqcup A^1 \Longrightarrow [D \leq_T A^0 - C \vee D \leq_T A^1 - C]. \end{aligned}$$

Remark. In describing a construction, notations such as A^0, D, Φ^e, and Ψ^e are used to denote the current approximations to these objects. The notations A^0_s, D_s, Φ^e_s, and Ψ^e_s

denote the approximations which exist immediately before stage s. Occasionally, if the notation A^0 would be ambiguous, we use A^0_ω to make it clear that we are referring to the value of A^0 at the end of the construction. The use function for a p.r. functional, which is the length of the initial segment of the 'oracle' used in this computation, is denoted by the corresponding lower case letter.

Before describing the construction of B, C we discuss how to meet a single requirement

$$\mathcal{R}: \qquad D \neq \Psi(W) \vee W \neq \Phi(D)$$

of the same kind as $\mathcal{R}^e$. $\mathcal{R}$ can be met by proceeding as follows:

1. Choose a number, k say, not yet in B. Our intention is to make D and $\Psi(W)$ disagree at k.

2. Wait until $\Psi(W; k) = 0$, and $\Phi(D)$ is defined and agrees with W upto $\psi(W, k)$. At this stage, s say, enumerate k in B and restrain all numbers $< \varphi(D, \psi(W, k))$ $(= r$ say$)$ from C.

3. If a number $< \psi(W, k)$ is enumerated in W, then enumerate in C all numbers $\leq r$ which entered B at a stage $\geq s$. Restrain all numbers $\leq r$ from B and C.

By exploiting this basic module we obtain a finite-injury construction of r.e. sets B, C such that $D = B - C$ is properly d.r.e. We call this strategy as I.

In using this simple idea to solve the problem at hand we can make use of two simplifications:

i) We can assume that a number is enumerated in $A^0 \cup A^1$ only if it has already been enumerated in B, that $A^0 \cap A^1 = \emptyset$, and that a number enumerated in B is immediately enumerated in either A^0 or A^1.

ii) At any point in the construction, given k, we can "request" that some number $> k$ be enumerated in A^0 without any more numbers $\leq k$ being used.

We achieve i) and ii) as follows.

For i) we ensure that $A^0, A^1 \subseteq B$ and that $A^0 \cap A^1 = \emptyset$ by not enumerating in A^i a number which has already been enumerated in A^{1-i} or which has not yet been enumerated in B. Also, whenever b is enumerated in B, we reset and restart the strategy I using only numbers which are large compared with those used so far. As soon as b is enumerated in one of A^0, A^1 we terminate this copy of the strategy I enumerating in C all numbers it has enumerated in B. At the same time we return to the construction described below. Of course, if b is never enumerated in either of A^0 or A^1, then the construction succeeds in a trivial way. So we ignore this possibility in describing the construction below.

For ii) the procedure is similar. When a number $> k$ is requested in A^0, we reset and restart the strategy I using only numbers which are both $> k$ and $>$ any number yet used in the construction. We pursue the strategy I, restraining numbers $\leq k$ from B and C until some $a > k$, which has been enumerated in B but not yet in C, is enumerated in A^0. If no such a is ever found, either B is not covered by $A^0 \cup A^1$ or D is almost a subset of A^1. In either case the construction again succeeds in a trivial way. So again we ignore the possibility in describing the construction below. When a is found, we terminate this copy of I enumerating in C all numbers it has enumerated in B except for a.

The activity just described to ensure i) and ii) will be called the *invisible strategy.* Apart from supplying the numbers requested in A^0, its net effect is to generate pauses in the construction (pauses which are ignored in the description given below), and to enumerate certain numbers in B and then in C, numbers which are large enough to be irrelevant to the *visible* strategy. The latter point means that we must be careful to ensure that the visible strategy has enough numbers to work with. During the construction a number will be said to have been *used* if it is, or has been, the value of one of the parameters k^e, a^e, r^e, $k^{e,i}$, $a^{e,i}$, $r^{e,i}$ mentioned below or has been enumerated in B (and then in C) by the invisible strategy.

Now we describe the basic module for a single requirement

$$\mathcal{R}: \quad D \neq \Psi(W) \vee W \neq \Phi(D).$$

Our main strategy M is aimed to satisfying $\mathcal{R}$ while at the same time building a Turing reduction of D to $A^0 - C$ to satisfy $\mathcal{S}$. To attack $\mathcal{R}$ we choose a target number k, not yet used, at which we would like to make D and $\Psi(W)$ differ. We also choose a number $a \in A^0 - C$, such that $k < a$. The purpose of a is to serve as the use, for numbers $\geq k$ and $\leq a$, of the reduction of D to $A^0 - C$ being constructed.

In the interval (k, a) we are playing an auxiliary strategy S which assumes that all numbers which are enumerated in B by S are enumerated in A^1 rather than A^0.

Now we describe stage s of the basic module under the simplifications assumed above. No action is taken unless

$$\varphi(D, \psi(W,k)) \downarrow \wedge \Psi(W;k) = 0 \wedge (\forall x \leq \psi(W,k))[\Phi(D;x) = W(x)]. \tag{2.1}$$

Note that $D(k) = 0$ by choice of k. If eventually (2.1) never holds, then $\mathcal{R}$ is satisfied and $\mathcal{S}$ is not injured.

When (2.1) is satisfied we begin by executing one step of the strategy S, being played in the interval (k, a). There are two possibilities:

Case 1. If a number is enumerated in B by S at this step, then that number is in A^1. In this case we enumerate a in C and we reset a to a new larger value in $A^0 - C$.

Case 2. Otherwise. Some number b, $k < b < a$, is enumerated in B by S at this step and b is enumerated in A^0. (So the assumption underlying S has been violated.) This event is seen as permission to execute one step of the strategy M. We enumerate k in B and restrain all number $< \varphi(D, \psi(W, k))$ ($= r$ say) from B and C.

At this point we have:

$$D(k) = 1 \neq 0 = \Psi(W;k).$$

The functional being constructed to reduce D to $A^0 - C$ has not been injured since $A^0 - C$ has changed at b. So unless some number $\leq \psi_s(W_s, k)$ is enumerated in W at a stage $\geq s$, $\mathcal{R}$ has been satisfied and no further action is required except for an appropriate restraint on B and C. On the other hand, if some number $\leq \psi_s(W_s, k)$, say x, is enumerated in W at a stage $\geq s$, then we enumerate k and b in C. Then $\Phi(D)$ and W differ at x, since

$$W(x) = 1 \neq 0 = \Phi_s(D_s;x) = \Phi(D;x),$$

and this disagreement is preserved forever. The functional being constructed to reduce D to $A^0 - C$ has not been injured since $A^0 - C$ has changed at b again.

We conclude that, if Case 2 ever occurs, then the basic module is successful and imposes only a finite restraint. Suppose Case 2 never occurs. Now the strategy S comes to the fore because it is reset only a finite number of times and infinitely many steps are executed. Since all numbers enumerated by S in B are in A^1, the strategy S need not be concerned with the requirement $\mathcal{S}$. Thus the basic module is successful in this case too, as same as the basic module to constructe a properly d.r.e. set.

Organization of the construction. As well as enumerating the r.e. sets B, C we shall be implicitly constructing a Turing reduction of D to $A^0 - C$. The main strategy M aims to satisfy all the $\mathcal{R}^e$'s as well as ensuring there is such a reduction. However, if it has to act infinitely often on behalf of $\mathcal{R}^e$, then M fails. Thus, for each e, we have an auxiliary strategy S^e. If e is least such that $\mathcal{R}^e$ requires attention infinitely often from M, then D will be recursive in $A^1 - C$, by default as it were, and the strategy S^e will ensure that the requirements $\mathcal{R}^j$ $(j < \omega)$ are all met.

There are a number of parameters associated with the M's attempt to satisfy R^e:

i) k^e is the argument of attack, i.e., the argument at which we aim to make $\Psi^e(W^e)$ different from D.

ii) a^e is an element of $A^0 - C$ which is $> k^e$, which can be regarded as the 'use' at k^e of the functional which we hope will reduce D to $A^0 - C$. The auxiliary strategy S^e will be pursued in the interval (k^e, a^e).

iii) r^e is the restraint intended to protect the M's attack on $\mathcal{R}^e$.

iv) c^e is a counter which records how far M's attack on $\mathcal{R}^e$ has proceeded.

Similarly, with the S^e's attempt to satisfy $\mathcal{R}^j$ we associate:

i) $k^{e,j}$, the argument of attack.

ii) $c^{j,e}$, a counter which records how far S^e's attack on R^e has proceeded.

$\mathcal{R}^e$ *requires attention from* M if one of following holds:

i) $k^e \uparrow$

ii) $k^e \downarrow \wedge c^e = 0 \wedge \varphi^e(D, \psi^e(W^e, k^e)) \downarrow \wedge$
$\Psi^e(W^e; k^e) = 0 \wedge (\forall x \leq \psi^e(W^e, k^e))[\Phi^e(D; x) = W^e(x)]$

iii) $k^e \downarrow \wedge c^e = 1 \wedge (\exists x \leq \psi^e_t(W^e_t, k^e))[x \in W^e - W^e_t]$,
where t is the last stage in which c^e was set equal 1.

$\mathcal{R}^i$ *requires attention from* S^e if one of following holds:

i) $k^{e,i} \uparrow$

ii) $k^{e,i} \downarrow \wedge c^{e,i} = 0 \wedge \varphi^i(D, \psi^i(W^i, k^{e,i})) \downarrow \wedge \varphi^i(D, \psi^i(W^i, k^{e,i})) < a^e \wedge$
$\Psi^i(W^i; k^{e,i}) = 0 \wedge (\forall x \leq \psi^i(W^i, k^{e,i}))[\Phi^i(D; x) = W^i(x)]$

iii) $k^{e,i} \downarrow \wedge c^{e,i} = 1 \wedge (\exists x \leq \psi^i_t(W^i_t, k^{e,i}))[x \in W^i - W^i_t]$,
where t is the most recent stage in which $c^{e,i}$ was set equal 1.

The construction is now given by:

Stage s. Let e be least such that R^e requires attention from M. (In discussing the construction we shall say that $\mathcal{R}^e$ *receives attention from* M at stage s.) Cancel the values if any of k^j, a^j, r^j, c^j, $k^{j,i}$, $r^{j,i}$, and $c^{j,i}$ for all $j > e$ and all i. For the rest there are three cases:

Case 1. $c^e \uparrow$. Let b be the least number which exceeds every number used so far in the construction. Request a new number $a \in A^0 - C$ such that $a > 2b$. Note that the numbers in $[b, 2b]$ are all unused. Set $k^e = b$, $r^e = a^e = a$, and $c^e = 0$.

Case 2. $c^e = 0$. From the definition of requiring attention we know that

$$\varphi^e(D, \psi^e(W^e, k^e)) \downarrow \wedge \Psi^e(W^e, k^e) = 0 \wedge (\forall x \leq \psi^e(W^e, k^e))[\Phi^e(D; x) = W^e(x)].$$

Let i be the least number such that $\mathcal{R}^i$ requires attention from S^e. (In discussing the construction we shall say that $\mathcal{R}^i$ *receives attention from* S^e at stage s.) Cancel the values if any of $k^{e,j}$, $r^{e,j}$, and $c^{e,j}$ for all $j > i$. For the rest there are three subcases.

Case 2.1. $c^{e,i} \uparrow$. Set $k^{e,i}$ equal to the least x if any such that $x < a^e$ and x is greater than any number $< a^e$ which has been used by the visible strategy. If $k^{e,i}$ becomes defined, set $c^{e,i} = 0$. Otherwise, leave $c^{e,i}$ undefined. Whether such x exists or not, enumerate $a^e \in C$.

Request a new number $a \in A^0 - C$ such that $a > 2b$, where b is the least number exceeding all those used in the construction so far. Set $a^e = a$. (This process beginning with the enumeration of a^e in C is called *resetting* a^e.) Set $r^e = a^e$.

Case 2.2. $c^{e,i} = 0$. From the definition of requiring attention:

$$\varphi^i(D, \psi^i(W^i, k^{e,i})) \downarrow \wedge \varphi^i(D, \psi^i(W^i, k^{e,i})) < a^e$$

and

$$\Psi^i(W^i; k^{e,i}) = 0 \wedge (\forall x \leq \psi^i(W^i, k^{e,i}))[\Phi^i(D; x) = W^i(x)].$$

Set $r^{e,i} = 1 + \varphi^i(D, \psi^i(W^i, k^{e,i}))$. Enumerate $k^{e,i}$ in B. Again the case splits:

Case 2.2.1. $k^{e,i}$ is enumerated in A^0. Cancel the values if any of $k^{e,j}$, $r^{e,j}$, and $c^{e,j}$ for all j. Enumerate $k^e \in B$. Set $r^e = 1 + \varphi^e(D, \psi^e(W^e, k^e))$ and $c^e = 1$.

(Note that a^e is neither reset nor enumerated in C.)

Case 2.2.2. $k^{e,i}$ is enumerated in A^1. Reset a^e as in Case 2.1. Set $r^e = a^e$, $r^{e,i} = 1 + \varphi^i(D, \psi^i(W^i, k^{e,i)}))$, and $c^{e,i} = 1$.

Case 2.3. $c^{e,i} = 1$. From the definition of requiring attention:

$$(\exists x \leq \psi^i_t(W^i_t, k^{e,i}))[x \in W^i - W^i_t],$$

where t is the most recent stage in which $c^{e,i}$ was set equal 1. Enumerate $k^{e,i} \in C$, set $c^{e,i} = 2$. Reset a^e. Set $r^e = a^e$.

Case 3. $c^e = 1$. From the definition of requiring attention:

$$(\exists x \leq \psi^e_t(W^e_t, k^e))(x \in W^e - W^e_t)),$$

where t is the most recent stage in which c^e was set equal to 1. Cancel the values if any of $k^{e,j}$, $r^{e,j}$, and $c^{e,j}$ for all j. Enumerate in C the two numbers which were enumerated in B at stage t. Set $c^e = 2$.

(Note that $r^e_{t+1} = 1 + \varphi^e_t(D_t, \psi^e_t(W^e_t, k^e))$ from stage t. Since r^e is constant, between stages t and s no number $\leq r^e_{t+1}$ is enumerated in $B \cup C$. Thus the effect of the action at this stage is to ensure that $D_{s+1} \upharpoonright r = D_t \upharpoonright r$. Since W^e has gained a member $x \leq \psi^e_t(W^e_t, k^e))$ at a stage $\geq t$ and $< s$, W^e and $\Phi^e(D)$ now disagree at x. Further, $r^e_{s+1} = r^e_{t+1}$ is large enough to preserve the disagreement.)

To conclude stage s we enumerate one number in one of the sets W^j $(j < \omega)$ and one axiom in one of the p.r. functionals Φ^j, Ψ^j $(j < \omega)$.

Verification. We will show that for all e, $\mathcal{R}^e$ is satisfied, and D is recursive in $A^0 - C$ or $A^1 - C$.

Suppose that, for all $i < e$, $\mathcal{R}^i$ requires attention from M at at most a finite number of stages. Let s_0 be the least number such that no $\mathcal{R}^i$ $(i < e)$ requires attention from M at any stage $\geq s_0$.

At stage s_0, k^e is defined and c^e is set equal to 0. Notice that k^e is neither cancelled nor reset at any stage $> s_0$. There are now three cases.

Case 1. There exists $s > s_0$ such that c^e is set equal 1 at stage s. Let s_1 denote the least such s. From the construction we see that Case 2.2.1 obtains at stage s_1. Let i_1 be the unique i such that $\mathcal{R}^i$ receives attention from the S^e at stage s_1. There are two subcases.

Case 1.1. There is a stage $s > s_1$ at which $\mathcal{R}^e$ receives attention from M. Let s_2 be the least such s. It is clear that at the stages $> s_1$ and $< s_2$ only $\mathcal{R}^i$'s with $i > e$ receive attention from M. Thus a^e, r^e, and k^e do not change between stages s_1 and s_2. For brevity, let a, r, k denote the values of a^e, r^e, and k^e at the end of stage s_1. Any number enumerated in B or C at a stage $> s_1$ and $< s_2$ is either a value of k^i or $k^{i,j}$ established after stage s_1 or is enumerated by the invisible strategy. It follows that any number enumerated in B or C at a stage $> s_1$ and $< s_2$ is $> \max(a, r)$ by Cases 1 and 2.1 of the description of stage s. Case 3 obtains at stage s_2. Notice that the t for Case 3 is s_1. From the description of Case 3, in stage s_2 there exists x such that

$$x \leq \psi^e_{s_1}(W^e_{s_1}, k) \wedge x \in W^e - W^e_{s_1}.$$

From Case 2 at stage s_1 we have

$$\Phi^e_{s_1}(D_{s_1}; x) = W^e_{s_1}(x) = 0 \neq 1 = W^e_{s_2}(x).$$

In stage s_2, the numbers k and $k^{e,i_1}_{s_1}$ which were enumerated in B in stage s_1, are enumerated in C. This means that

$$D_{s_2+1} \upharpoonright r = D_{s_1} \upharpoonright r.$$

From Case 2.2.1 at stage s_1,

$$r = 1 + \varphi^e_{s_1}(D_{s_1}, \psi^e_{s_1}(W^e_{s_1}, k^e)) > \varphi^e_{s_1}(D_{s_1}, x).$$

Therefore

$$\Phi^e_{s_2+1}(D_{s_2+1}; x) = \Phi^e_{s_1}(D_{s_1}; x) = 0 \neq 1 = W^e_{s_2}(x).$$

Moreover, at stage s_2, c^e is set equal 2 and so at no subsequent stage does any $\mathcal{R}^i$ with $i \leq e$ receive attention. It follows that no number $\leq r$ is enumerated in B or C at a stage $> s_2$. Hence, at the end of the construction, $\Phi^e(D)$ and W^e disagree at x. So in this case $\mathcal{R}^e$ is satisfied and receives attention from M at at most finitely many stages.

Case 1.2. Otherwise. At no stage $> s_1$ does $\mathcal{R}^e$ require attention from M. Let t denote s_1, and r, k the values of r^e, k^e at the end of stage t. By induction on stages, $r^e = r$, $k^e = k$, and $c^e = 1$ at all stages $> t$. Since $\mathcal{R}^e$ never requires attention from M at a stage $> t$ through Case 3, we have

$$(\forall x \leq \psi^e_t(W^e_t, k^e))[W^e(x) = W^e_t(x)].$$

By the same token, k^e, which is enumerated in B in stage t, is never enumerated in C. Thus at the end of the construction

$$\Psi^e(W^e; k) = \Psi^e_t(W^e_t; k^e) = 0 \neq 1 = D_t(k^e) = D(k).$$

So in this case $\mathcal{R}^e$ is satisfied and receives attention from M at at most finitely many stages.

Case 2. Not Case 1 and $\mathcal{R}^e$ requires attention from M at only a finite number of stages. Let k denote $k^e_{s_0+1}$. By induction on stages $k^e = k$ and $c^e = 0$ at all stages $> s_0$, and k is never enumerated in B. Since eventually $\mathcal{R}^e$ never requires attention from M, by Case 2 we see that at all sufficiently large stages

$$\varphi^e(D, \psi^e(W^e, k)) \uparrow \vee \Psi^e(W^e; k) \neq 0 \vee (\exists x \leq \psi^e(W^e, k))[\Phi^e(D, x) \neq W^e(x)].$$

Thus $\mathcal{R}^e$ is satisfied.

Case 3. Otherwise. Then there are an infinite number of stages at which $\mathcal{R}^e$ requires attention from the M-strategy. Let these stages which are $> s_0$ be numbered s_1, s_2, ...in order of magnitude. At each stage s_k with $k > 0$ Case 2 holds but Case 2.2.1 does not. Thus a^e is reset at each of these stages and so increases to ∞ as the construction unfolds. At the stages s_k with $k > 0$ the S^e-strategy is pursued on the interval (k^e, a^e). The restraint r^e ensures that after stage s_0 the S^e-strategy is never injured.

If $k > 0$, $c^{e,i} \uparrow$, and $\mathcal{R}^i$ receives attention from S^e at stage s_k, then there exists a suitable value for $k^{e,i}$ in stage s_k because, when a^e is set or reset in stage s_{k-1}, then a whole block of numbers $(b, 2b]$ is left unused, where $k^e \leq b < 2b < a^e = r^e$ at the end of stage s_{k-1}. Further, at stages $> s_{k-1}$ and $< s_k$ no number $\leq r^e$ is used. Since there is a copy of I operating in the interval (k^e, a^e) at the stages s_1, s_2, ..., it is easy to see that, for each i, $\mathcal{R}^i$ receives attention at stage s_k for only finitely many k and is satisfied.

Any number enumerated in B or C at a stage $s > s_{k-1}$, $s \notin \{s_k : k > 0\}$, is $> a^e = (a^e_{s_{k-1}})$. Any number enumerated in B in a stage s_k is also enumerated in A^1. Hence $D \leq_{\mathrm{T}} A^1 - C$.

From the discussion above we conclude that there are two possibilities:

i) For some e, $\mathcal{R}^e$ receives attention infinitely often from M, $D \leq_{\mathrm{T}} A^1 - C$, and all the requirements $\mathcal{R}^i$ $(i < \omega)$ are satisfied by S^e.

ii) For every e, $\mathcal{R}^e$ receives attention at at most a finite number of stages and is satisfied.

The requirement $\mathcal{S}$ is satisfied in the former case because $D \leq_{\mathrm{T}} A^1 - C$. It only remains to show that $\mathcal{S}$ is also satisfied in the latter case. So from now on assume that each e, $\mathcal{R}^e$ receives attention from M at at most a finite number of stages. For each e, as $s \to \infty$, k^e_s has a limit k^e_ω, and a^e_s has a limit a^e_ω. Further, $a^e_\omega \in A^0 - C$. Assume that an $(A^0_\omega - C_\omega)$-oracle is given. Then for every e we can compute $s(e)$ such that

$$(A^0_{s(e)} - C_{s(e)}) \upharpoonright (a^e_{s(e)} + 1) = (A^0_\omega - C_\omega) \upharpoonright (a^e_{s(e)} + 1). \tag{2.2}$$

The key point in the argument is:

Claim. *For each e,*

$$D_{s(e)} \upharpoonright (a^e_{s(e)} + 1) = D_\omega \upharpoonright (a^e_{s(e)} + 1).$$

To prove the claim we argue as follows. If there are no stages $\geq s(e)$ in which D changes $\leq a^e_{s(e)}$, then the conclusion is clear. So let s_0, s_1,..., s_p be the stages $\geq s(e)$, listed in increasing order, in which D changes $\leq a^e_{s(e)}$. For each $i \leq p$ let e_i denote the number such that $\mathcal{R}^{e_i}$ receives attention from M at stage s_i. If $e < e'$, then, at any stage $\geq s(e)$, either $k^{e'} \uparrow$ or $k^{e'} \geq a^e_{s(e)}$. Thus no number $\leq a^e_{s(e)}$ is enumerated in either B or C at a stage $\geq s(e)$ in which some $\mathcal{R}^{e'}$, $e < e'$, receives attention. Thus each e_i is $\leq e$. Further, we see that

$$e \geq e_0 \geq e_1 \geq \ldots \geq e_p$$

because as soon as $\mathcal{R}^{e'}$ receives attention at a stage $\geq s(e)$ from M all the parameters whose (first) superscript is $> e'$ are cancelled. Any subsequent values of these parameters are $> a^e_{s(e)}$. By the same token $k^{e_i}_{s_i} = k^{e_i}_{s(e)}$. It follows that $a^{e_i}_{s_i} = a^{e_i}_{s(e)} \leq a^e_{s(e)}$. Otherwise a^{e_i} is reset at some stage $\geq s(e)$ and $< s_i$, and $a^{e_i}_{s(e)}$ is enumerated in C contradicting (2.2).

At stage s_0, since D is changed and a^{e_0} is not enumerated in C, either Case 2.2.1 or Case 3 holds. If Case 3 holds, let t_0 be the greatest stage $< s_0$ in which c^{e_0} is set equal 1. Then in stage t_0 Case 2.2.1 holds, k^{e_0} $(= k^{e_0}_{s(e)})$ and a number x of the form $k^{e_0,i}$ are enumerated in B, the latter being enumerated in A^0. By definition of s_0, $t_0 < s(e)$. At stage s_0, x is enumerated in C. This contradicts (2.2). Hence Case 2.2.1 holds at stage s_0. Thus in stage s_0, k^{e_0} $(= k^{e_0}_{s(e)})$ and a number x of the form $k^{e_0,i}$ are enumerated in B, the latter being enumerated in A^0. Since (2.2) holds it must be the case that $e_1 = e_0$ and that Case 3 holds at stage s_1; otherwise x stays in $A^0 - C$ forever. So the two numbers $\leq a^e_{s(e)}$, which were enumerated in B in stage s_0, are enumerated in C in stage s_1. Now we can apply the argument which we made for s_0 to s_2. We see that two numbers $\leq a^e_{s(e)}$ are enumerated in B at stage s_2 and enumerated in C in stage s_3. And so on. After the last of the the stages s_i, D is the same for arguments $\leq a^e_{s(e)}$ as it was just before stage $s(e)$. This completes the proof of the claim.

From the claim it is clear that D is recursive in $A^0 - C$, which completes the proof of the theorem.

Chapter 3

The modules

Recall that our final goal in this thesis is to prove:

3.1 Theorem. *There exists a properly d.r.e. set D such that for all r.e. sets A^0, A^1 with $A^0 \cap A^1 = \emptyset$,*

$$D \subseteq A^0 \cup A^1 \Longrightarrow [D \leq_T A^0 \cap D \vee D \leq_T A^1 \cap D].$$

In present chapter we will move closer to achieving this end by describing the modules for the construction which is described in detail in the next chapter.

Our task is to effectively enumerate B, C such that $C \subseteq B$ and $D = B - C$ meets for all e, the following requirements:

$$\begin{array}{ll} \mathcal{R}^e: & D \neq \Psi^e(W^e) \vee W^e \neq \Phi^e(D); \\ \mathcal{S}^e: & B = A^{e,0} \sqcup A^{e,1} \Longrightarrow [D \leq_T A^{e,0} - C \vee D \leq_T A^{e,1} - C], \end{array}$$

where $\{(A^{e,0}, A^{e,1})\}_{e<\omega}$ is an effective enumeration of all pairs of r.e. sets.

The priority ranking of the requirements is $\mathcal{S}^0, \mathcal{R}^0, \mathcal{S}^1, \mathcal{R}^1, \mathcal{S}^2, \mathcal{R}^2, \cdots$. We already know how to attack $\mathcal{R}^0$ while maintaining our strategy for $\mathcal{S}^0$. The next thing to understand is how to attack $\mathcal{R}^i$ for $i > 0$ while according priority to the requirements $\mathcal{S}^0, \mathcal{S}^1, \cdots, \mathcal{S}^i$.

3.1 The $\mathcal{R}$-requirements below two $\mathcal{S}$-requirements: $\mathcal{S}^0, \mathcal{S}^1$

The question addressed here is how a requirement:

$$\mathcal{R}: \quad D \neq \Psi(W) \vee W \neq \Phi(D)$$

should be attacked while priority is being given to two $\mathcal{S}$-requirements: $\mathcal{S}^0, \mathcal{S}^1$.

We choose a target number k, which is unused, at which we would like to make D and $\Psi(W)$ differ. We assume at any point, $\varphi(D, \psi(W, k)) \downarrow$ implies

$$\Psi(W; k) = 0 \wedge (\forall x \leq \psi(W, k))[\Phi(D; x) = W(x)].$$

Choose markers q_j $(j \leq 4)$ such that

$$k < q_0 < q_1 < q_2 < q_3 < q_4.$$

In the interval (k, q_0) we are playing the main strategy of the basic module (described below) which assumes that both $A^{0,0} \cup A^{0,1}$ and $A^{1,0} \cup A^{1,1}$ cover B and that arbitrarily large elements of $A^{i,0} - C$ $(i = 0, 1)$ are available on demand. In the interval (q_0, q_1) we are playing a strategy Q_0 which assumes that eventually all numbers enumerated in B enter $A^{0,1}$. In the interval (q_1, q_2) we are playing a strategy Q_1 which assumes eventually all numbers enumerated in B enter $A^{1,1}$. For $i = 0, 1$, a strategy Q_{2+i} is played in the interval (q_{2+i}, q_{3+i}) which assumes that the last number to enter B via one of Q_0, Q_1 or the main strategy will never be enumerated in $A^{i,0} \cup A^{i,1}$.

The main strategy tries to define markers a^0 and a^1 such that

$$k < a^0 < a^1$$

and $a^i \in A^{i,0} - C$. It proceeds as follows. If the marker a^0 is not defined, then look for $x \in (q_0, q_1) \cap (A^{0,0} - C)$ and set a^0 equal to the greatest such x. Reset all the q_i's so that $a^0 < q_0$. If no such x exists, then play one move of Q_0 and reset $q_1, \cdots, q_4$. Suppose a^0 is defined. Then play a move of the strategy Q_1. Once a^0 is defined look for $x \in (q_1, q_2) \cap (A^{1,0} - C)$ and set a^1 equal to the greatest such x. Reset all the q_i's so that $a^1 < q_0$. If no such x exists, then play one move of Q_1 and reset q_2, q_3, q_4.

The role of Q_2 and Q_3 is as follows. When some number x is enumerated in B by either the main strategy or one of Q_0, Q_1, then Q_2 is reset and is played on the interval (q_2, q_3) until x has been enumerated in $A^{0,0} \cup A^{0,1}$. While Q_2 is being played, q_0, q_1, q_2 are fixed but q_3 moves steadily to the right. Once x has been enumerated in $A^{0,0} \cup A^{0,1}$, q_3 also becomes fixed. If the main strategy enumerates x in B without prior permission from $A^{0,0}$ and x turns up in $A^{0,0} \cup A^{0,1}$, we make another move in the main strategy. (In fact, if $x \in A^{0,0}$, then a number will be enumerated in B and otherwise a^1 is destroyed.) Then Q_3 is reset and played on (q_3, q_4) with q_4 moving steadily to the right until x enters $A^{1,0} \cup A^{1,1}$.

So the strategies Q_i $(i \leq 3)$ are only significant if at some stage, the main strategy is unable to find the markers a^0, a^1 it wants for k or if one of the numbers enumerated in B fails to turn up in $(A^{0,0} \cup A^{0,1}) \cap (A^{1,0} \cup A^{1,1})$. If, for particular $i \leq 3$, infinitely many moves of Q_i are played without Q_i being reset, then the basic module is successful because each Q_i enjoys a substantial advantage. For instance, Q_1 may ignore $\mathcal{S}^1$. Thus below we may assume that the main strategy always finds a^0 and a^1 eventually and that $B \subseteq A^{i,0} \cup A^{i,1}$ $(i \leq 1)$.

We now describe the main strategy in detail. In the interval (k, a^0) we are playing a strategy S^0 which assumes that all numbers enumerated in B by S^0 are enumerated in $A^{0,1}$ rather than $A^{0,0}$. In the interval (a^0, a^1) we are playing a strategy S^1 which assumes that if a number enters B by S^1 without prior permission from $A^{0,0}$, and enters $A^{0,0}$, then that number is in $A^{1,1}$. Under the assumption of strategy S^1, we are playing a list of substrategies $S^{1,i}$. $S^{1,i}$ assumes either we get a reduction of D to $A^{0,1} - C$ or the functionals reducing D to $A^{0,0} - C$ and $A^{1,1} - C$ are preserved and $\mathcal{R}^i$ is satisfied. For each i, $S^{1,i}$ has a substrategy $S^{1,i,0}$ which assumes that each number which is enumerated in B by $S^{1,i,0}$ enters $A^{0,1}$.

We now describe stage s of the basic module. No action is taken unless $\varphi(D, \psi(W, k)) \downarrow$. If eventually $\varphi(D, \psi(W, k)) \downarrow$ never holds, then $\mathcal{R}$ is satisfied and none of $\mathcal{S}^0, \mathcal{S}^1$ is injured.

When $\varphi(D, \psi(W, k)) \downarrow$ is satisfied we begin by executing one step of the strategy S^1. We think of (a^0, a^1) as the universe on which S^1 operates. Whenever we play a move of S^1 which does not enumerate a number in B, we may enumerate a^1 in C. Then a^1 is reset by the main strategy thus extending the universe of S^1. Let the current aim of S^1 be to satisfy $\mathcal{R}^i$. Executing one step of S^1 means executing one step of $S^{1,i}$. $S^{1,i}$ will proceed as follows.

Choose a target number k^i for $\mathcal{R}^i$, which is unused, at which we would like to make D and $\Psi^i(W^i)$ differ. The strategy $S^{1,i}$ also tries to define numbers a_0^i, b_1^i such that

$$a^0 < k^i < a_0^i < b_1^i < a^1$$

$a_0^i \in A^{0,0} - C$, $b_1^i \in A^{1,1} - C$ and k^i greater than all parameters used for $\mathcal{R}^j$ for all $j < i$, (here k^i, a_0^i, and b_1^i are parameters for $\mathcal{R}^i$). The search for a_0^i and b_1^i proceeds as follows. Choose markers q_j^i $(j = 0, 1, 2)$ such that

$$k^i < q_0^i < q_1^i < q_2^i < a^1.$$

In the interval (q_0^i, q_1^i) we are playing a strategy Q_0^i which assumes that all numbers enumerated in B enter $A^{0,1}$. In the interval (q_1^i, q_2^i) we are playing a strategy Q_1^i which assumes that all numbers enumerated in B enter $A^{1,0}$. If the marker a_0^i is not defined, then we look for $x \in (q_0^i, q_1^i) \cap (A^{0,0} - C)$ and we set a_0^i equal to the greatest such x. We reset all the q_j^i's so that $a_0^i < q_0^i$. If no such x exists, then we play one move of Q_0^i and reset q_1^i, q_2^i and a^1, and put a^1 into C. Suppose a_0^i becomes defined. Then we play a move of the strategy Q_1^i. If the marker b_1^i is not defined, then we look for $x \in (q_1^i, q_2^i) \cap (A^{1,1} - C)$ and we set b_1^i equal to the greatest such x. If no such x exists, then we play one move of Q_1^i and reset q_2^i, a^1 and put a^1 into C. We need not enquire into the nature of the strategies Q_j^i $(j < 2)$. It is sufficient to notice that, if, for particular $j \leq 1$, infinitely many moves of Q_j^i are played without Q_j^i being reset, then a substantial advantage has been gained because one of $\mathcal{S}^0, \mathcal{S}^1$ may be dismissed from consideration. The module is successful by the last section.

To focus attention on what is essential we assume that the assumptions for Q_0^i and Q_1^i both fail. Thus we succeed in finding a_0^i, b_1^i, such that

$$a^0 < k^i < a_0^i < b_1^i < a^1$$

and $a_0^i \in A^{0,0} - C$, $b_1^i \in A^{1,1} - C$.

No action is taken unless $\varphi^i(D, \psi^i(W^i, k^i)) \downarrow < a^1$. If eventually $\varphi^i(D, \psi^i(W^i, k^i)) \downarrow < a^1$ never holds, then $\mathcal{R}^i$ is satisfied and none of $\mathcal{S}^0, \mathcal{S}^1$ is injured. When $\varphi^i(D, \psi^i(W^i, k^i)) \downarrow < a^1$ is satisfied we begin by executing one step of the strategy $S^{1,i,0}$, being played in the interval (k^i, a_0^i). There are two possibilities:

Case 1^1. If a number is enumerated in B by $S^{1,i,0}$ at this step, then that number is in $A^{0,1}$. In this case we enumerate a_0^i, b_1^i and a^1 in C and we reset them to new larger values. (Note that k, a^0 and k^i are all unchanged.)

Case 2^1. Some number p^1, $k^i < p^1 < a_0^i$, is enumerated in B by $S^{1,i,0}$ at this step and p^1 is enumerated in $A^{0,0}$. Enumerate k^i into B and set $c^i = 1$. Now there are two subcases.

Subcase 2^1.1. p^1 is enumerated in $A^{1,0}$. (The assumption underlying S^1 is violated.) This event is seen as permission to execute one step of the strategy S^0 being played in the interval (k, a^0).

Subcase 2^1.2. p^1 is enumerated in $A^{1,1}$. (The assumption underlying $S^{1,i}$ is confirmed.) In this case we enumerate a^1 in C and we reset it to a new larger value. (Note that k, a^0, k^i, a_0^i and b_1^i are all unchanged.)

For the strategy S^0, again there are two possibilities.

Case 1^0. If a number is enumerated in B by S^0 at this step, then that number is in $A^{0,1}$. In this case we enumerate a^0 in C. We reset a^0, a_0^i, b_1^i, a^1 in that order and reset the strategy S^1.

Case 2^0. Some number p^0, $k < p^0 < a^0$, is enumerated in B by S^0 at this step and p^0 is enumerated in $A^{0,0}$. We enumerate k into B and set $c = 1$.

At this point we have:

$$D(k) = 1 \neq 0 = \Psi(W; k).$$

The functionals being constructed to reduce D to $A^{i,0} - C$ $(i = 0, 1)$ have not been injured since $A^{i,0} - C$ has changed at p^i for each $i \leq 1$. So unless some number $\leq \psi_s(W_s, k)$ is enumerated in W at a stage $\geq s$, $\mathcal{R}$ has been satisfied and no further action is required except for an appropriate restraint on B and C. On the other hand, if some number $\leq \psi_s(W_s, k)$, say x, is enumerated in W at a stage $\geq s$, then we enumerate k, k^i and p^t $(t = 0, 1)$ in C. Then $\Psi(D)$ and W differ at x, since

$$W(x) = 1 \neq 0 = \Phi_s(D_s; x) = \Phi(D; x),$$

and this disagreement is preserved forever.

The possible of outcomes

First, assume the main strategy is executed at only finitely many stages. Let i be least such that Q_i acts infinitely many stages. Here the strategy Q_i comes to the fore because it is reset only a finitely number of times and infinitely many steps are executed. Since Q_i may ignore either $\mathcal{S}_0$ or $\mathcal{S}_1$ according as i is even or odd, it is an easy matter within Q_i to satisfy the other $\mathcal{S}$-requirement and all the $\mathcal{R}$-requirements.

For the rest suppose the main strategy is executed at infinitely many stages. In each case we assume that none of the previous cases holds. There are several cases.

Case 1. Case 2^0 eventually occurs. $\mathcal{R}$ is satisfied, the only cost being a finite restraint which is eventually fixed. The functionals which are implicitly being constructed to reduce D to $A^{i,0} - C$ $(i = 0, 1)$ are not injured.

Case 2. Case 1^0 holds infinitely often. Now the strategy S^0 comes to the fore because it is reset only a finite number of times and infinitely many steps are executed. Since Case 1 never holds all numbers enumerated by S^0 in B are in $A^{0,1}$. Thus the strategy S^0 need not be concerned with the requirement $\mathcal{S}^0$. The module is successful in this case too, even though $\mathcal{R}$ has not been satisfied, because one of the $\mathcal{S}$-requirements of higher priority has been eliminated.

Case 3. Otherwise. Eventually S^0 is never pursued. Thus eventually every move played in the main strategy is a move in the strategy S^1. Recall that S^1 consists of the substrategies $S^{1,i}$.

Case 3.1. There exists i such that $S^{1,i}$ is addressed at infinitely many stages. Fix i to be the least such number. These are two subcases.

Case 3.1.1 The strategy $S^{1,i,0}$ is pursued at infinitely many stages. In this case eventually all numbers enumerated in (a^0, a^i_0) enter $A^{0,1}$, a^0 is fixed, and a^i_0 increases to ∞. Thus $\mathcal{S}^0$ can be ignored and there will be no difficulty in satisfying the other requirements.

Case 3.1.2. Otherwise. One of Q^i_0, Q^i_1 is pursued infinitely often. Let l be the least such that Q^i_l is active infinitely often. Then eventually in the interval (q^i_l, q^i_{l+1}) every number enumerated in B is enumerated in $A^{0,1}$ if $l = 0$, and in $A^{1,0}$ if $l = 1$. Moreover, q^i_l is fixed while q^i_{l+1} increases to ∞. So we have a similar situation to that of the previous case.

Case 3.2. For each i, $S^{1,i}$ is active at most finitely often. In this case the assumption of S^1, that all numbers enumerated in B from (a^0, a^1), without $A^{0,0}$-permission, enter $A^{0,0}$ and then $A^{1,1}$, is eventually not violated. For each i, eventually $\mathcal{R}^i$ is satisfied at the cost of a finite restraint. At the same time we obtain reductions of D to $A^{0,0} - C$ and $A^{1,1} - C$.

3.2 The general module

In general, an $\mathcal{R}$-requirement will have to be satisfied while priority is being given a finite number of the $\mathcal{S}$-requirements say $\mathcal{S}^0, \cdots, \mathcal{S}^n$. To focus attention on what is essential we shall assume that for each $i \leq n$

$$A^{i,0} \cap A^{i,1} = \emptyset \wedge A^{i,0} \cup A^{i,1} \supseteq B$$

and that arbitrarily large elements of $A^{i,j} - C$ $(i \leq n, j \leq 1)$ are available on demand. We will first describe the basic module under these simplifications.

We choose a target number k, which is unused, at which we would like to make D and $\Psi(W)$ differ. We also choose numbers a^i, $i \leq n$, such that

$$k < a^0 < a^1 < \cdots < a^n$$

and $a^i \in A^{i,0} - C$. We think of a^i as the use at k of a functional which is implicitly being constructed to reduce D to $A^{i,0} - C$. We set the counter $c = 0$.

We now proceed as follows. For $0 < i \leq n$ in the interval (a^{i-1}, a^i) we are playing a strategy S^i which assumes that any number x enumerated in B, without prior permission from any $A^{j,0} - C$ with $j < i$, satisfies

$$(x \in \cap_{j<i} A^{j,0}) \Longrightarrow x \in A^{i,1}.$$

In the interval (k, a^0) we are playing a strategy S^0 which assumes that all numbers enumerated in B are enumerated in $A^{0,1}$ rather than $A^{0,0}$.

For $i < n$, S^i becomes active when there exists t less than the current stage and $x_n, \cdots, x_{i+1} \in B - B_t$ such that

- $a^n = a^n_t \downarrow$

- $\psi(W,k) = \psi_t(W_t,k) \downarrow$ *and* $\varphi(D,\psi(W,k)) = \varphi_t(D_t,\psi_t(W_t,k)) \downarrow$
- x_l *is designated by* S^l, $a^{l-1} < x_l < a^l$, *and* $x_l \in \bigcap_{j\leq l} A^{j,0}$ *for all* l, $i+1 \leq l \leq n$.

The witnesses x_l $(i+1 \leq l \leq n)$ only activates S^i for one stage, except that, if in the first stage of activation S^i enumerates $y \in B$, then S^i becomes active for m more stages, where

$$m = \max\{k \leq i : (\forall j < k)[y \in A^{j,0}]\} + 1.$$

When S^i $(0 \leq i \leq n)$ enumerates a number in B it may designate it. The strategy S^i has the right to assume that only a finite number of designated number are enumerated in $\bigcap_{j\leq i} A^{j,0}$. When S^i designates a number x at stage t the main strategy is set until x is cleared or some S^j with $j > i$ is reset. S^i may not enumerate any number in C which was already in B at stage t and so may not ask for a^i to be reset. Apart from this restriction S^i may request that a^i be reset and main strategy will comply. S^i $(i > 0)$ is responsible for producing a Turing reduction to $A^{j,0} - C$ for each $j < i$ of the restriction of D to $(a^{i-1}, a^i]$.

We now describe stage s of the basic module under the simplifications made above. No action is taken unless $\varphi(D,\psi(W,k)) \downarrow$. Note that $D(k) = 0$ by choice of k. If eventually $\varphi(D,\psi(W,k)) \downarrow$ never holds, then $\mathcal{R}$ is satisfied and none of $\mathcal{S}^0, \cdots, \mathcal{S}^n$ is injured. When $\varphi(D,\psi(W,k)) \downarrow$ is satisfied we begin by executing one step of the strategy S^n, being played in the interval (a^{n-1}, a^n). There are two possibilities:

Case 1^n. The assumption underlying S^n is not violated. In this case we enumerate a^n in C and we reset a^n to a new larger value in $A^{n,0} - C$. (Note that $k, a^0, \cdots, a^{n-1}$ are all unchanged.)

Case 2^n. Otherwise. Some designated number j^n, $a^{n-1} < j^n < a^n$, is enumerated in B by S^n at this step and j^n is enumerated in $\bigcap_{j\leq n} A^{j,0}$. (So the assumption underlying S^n has been violated.) This event is seen as permission to execute one step of the strategy S^{n-1} being played in the interval (a^{n-2}, a^{n-1}).

Again there are two possibilities:

Case 1^{n-1}. The assumption underlying S^{n-1} is not been violated. In this case we enumerate a^{n-1} and a^n in C. We reset a^{n-1} and then a^n. We reset the strategy S^n since it has been violated.

Case 2^{n-1}. Otherwise. A designated number $j^{n-1}, a^{n-2} < j^{n-1} < a^{n-1}$, is enumerated in B by S^{n-1} at this step and $j^{n-1} \in \bigcap_{j \leq n-1} A^{j,0}$. (So the assumption underlying S^{n-1} has been violated.) This event is seen as permission to execute one step of the strategy S^{n-2} being played in (a^{n-3}, a^{n-2}).

The pattern of cases should now be clear. For $i > 0$, Case 2^i splits into Case 1^{i-1} and Case 2^{i-1}. It remains to describe Case 2^0.

Case 2^0. For each $i \leq n$, we have j^i which was enumerated in B by strategy S^i such that $a^{i-1} < j^i < a^i$ and such that j^i has just been enumerated in $\bigcap_{j \leq i} A^{i,0}$ and is not yet in C. We enumerate k in D and set $c = 1$.

At this point we have:

$$D(k) = 1 \neq 0 = \Psi(W; k).$$

The functionals being constructed to reduce D to $A^{i,0} - C$ $(i \leq n)$ have not been injured since $A^{i,0} - C$ has changed at j^i for each $i \leq n$. So unless some number $\leq \psi_s(W_s, k)$ is enumerated in W at a stage $\geq s$, $\mathcal{R}$ has been satisfied and no further action is required except for an appropriate restraint on B and C. On the other hand, if some number $\leq \psi_s(W_s, k)$, say x, is enumerated in W at a stage $\geq s$, then we enumerate k and j^i $(i \leq n)$ in C. Then $\Psi(D)$ and W differ at x, since

$$W(x) = 1 \neq 0 = \Phi_s(D_s; x) = \Phi(D; x),$$

and this disagreement is preserved forever.

We conclude that, if Case 2^0 ever occurs, then the basic module is successful and imposes only a finite restraint. Suppose Case 2^0 never occurs. There is a least i, say $i = j$, such that Case 1^i occurs infinitely often. In this case $a^0, \cdots, a^{j-1}$ are eventually fixed. Now the strategy S^j comes to the fore because it is reset only a finite number of times and infinitely many steps are executed. S^j enjoys a small but significant advantage: after some point any number x enumerated in B, without prior permission from any $A^{i,0} - C$ with $i < j$, satisfies

$$(x \in \bigcap_{i<j} A^{i,0}) \Longrightarrow x \in A^{j,1}.$$

The main difficulty of the construction which follows is to exploit the small advantage we have exhibited by nesting similar strategies one within the other to achieve the desired goal.

As we have noted above, in the actual construction we cannot assume that $A^{i,0} \cup A^{i,1} \supseteq B$ and that elements of $A^{i,j}$ $(j \leq 1)$(not yet in C) are always available when we need them. This necessitates the introduction of additional nodes into the tree of strategies which is the "priority tree" described in the next section. For example, when attacking $\mathcal{R}^0$ while giving priority to $\mathcal{S}^0$, apart from the main strategy (which is constructing a reduction of D to $A^{0,0} - C$) and the back-up strategy (which assumes that eventually "all" numbers enumerated in B fall into $A^{0,1-j}$), we need

- *a strategy based on the assumption that the last number to enter B will never enter $A^{0,0} \cup A^{0,1}$, and*
- *a strategy based on the assumption that the element of $A^{0,j}$ not yet in C currently being sought for the main strategy will never be found.*

Further down the priority tree analogous strategies must be introduced to allow for the possibility that $A^{i,0} \cup A^{i,1}$ may not cover B or that suitable elements of $A^{i,j}$ not yet in C may not present themselves.

These aspects of the construction, although they introduce new complications, do not really cause any serious difficulty. It might be added that, although it was not nade explicit, the module described in 3.1 already takes into account the fact that some of the pairs $(A^{i,0}, A^{i,1})$ may not cover B. If it is known that $A^{i,0} \cup A^{i,1} \supseteq B$ for all i, then the level of the complexity of the problem is reduced dramatically.

Chapter 4

Construction

In this chapter we describe the construction. Before giving the construction we should describe the priority tree T which we shall picture as growing downwards. This tree provides a convenient way of organizing the various substrategies which make up the whole construction. The nodes of the tree are finite strings of pairs (n, j), where $n \leq 6$ and $j < \omega$. Such strings will be denoted by lower case Greek letters. With each node α there is associated a natural number $i(\alpha)$ and a strategy for satisfying $\mathcal{R}^{i(\alpha)}$. If at the end of the construction the true path contains α, then all the requirements $\mathcal{R}^i$ with $i < i(\alpha)$ are satisfied by strategies associated with nodes $\beta \subset \alpha$. (For readers not familiar with the concept we shall explain the concept of the true path later in next section.) With a node α there may also associated a strategy for satisfying $\mathcal{S}^{i(\alpha)}$ by implicitly constructing a Turing reduction of D to $A^{i(\alpha),0} - C$. We say "may" because the outcomes of a strategy at some node above α may already have guaranteed that $D \leq_T A^{i(\alpha),1} - C$ in which case there is no need for us to be concerned with $\mathcal{S}^{i(\alpha)}$ at node α. A number $j \leq i(\alpha)$ is called *active at* α if none of the "outcomes" of the nodes above α guarantees that $D \leq_T A^{j,1} - C$. Thus j is active at α if $j \leq i(\alpha)$ and we must still be concerned with building a reduction of D to $A^{j,0} - C$.

At node α the attack on $\mathcal{R}^{i(\alpha)}$ will proceed in the way described for the basic module of the last section. Now $\mathcal{R}^{i(\alpha)}$ plays the role of $\mathcal{R}$ and the requirements $\mathcal{S}^j$, $j \leq i(\alpha)$ and j active, play the role taken by $\mathcal{S}^0, \cdots, \mathcal{S}^n$ in the last section. The outcomes at node α corresponding to the outcomes of the module described in last section are:

$(2, i(\alpha))$: the α-attack on $\mathcal{R}^{i(\alpha)}$ is successful trivially

$(3, i(\alpha))$: the α-attack on $\mathcal{R}^{i(\alpha)}$ is successful non-trivially

$(0, j)$: the α-attack on $\mathcal{R}^{i(\alpha)}$ is not successful and j is the least number active at α such that $A^{j,0}$-permission is given only a finite number of times.

To take account of the possibility that either

$$A^{i(\alpha),0} \cup A^{i(\alpha),1} \not\supseteq B$$

or that suitable elements in $A^{j,i} - C$, $j \leq i(\alpha)$ and $i \leq 1$, are not always available eventually, we allow three further kinds of outcome:

$(6, i(\alpha))$: $A^{i(\alpha),0} \cup A^{i(\alpha),1} \not\supseteq B$

$(5, j)$: at some point in the construction we wait forever for a sufficiently large element of $A^{j,1} - C$

$(4, j)$: at some point in the construction we wait forever for a sufficiently large element of $A^{j,0} - C$.

We also find it convenient to include in our design certain pseudo-outcomes:

$(1, j)$: correcting the functional which reduces D to $A^{j,1} - C$.

The additional nodes arising from these pseudo-outcomes help us keep track of tasks that need to be performed during the construction.

We now give some definitions which will be needed for the description of the priority tree T. Let λ denote $\langle(2, -1)\rangle$. Then $\lambda \in T$. Suppose $\alpha \in T$ is given, let $i(\alpha)$ denote the least i such that (n, i) does not occur in α for $n \in \{2, 3\}$. j is called *active at* α if $j \leq i(\alpha)$, none of $(4, j), (5, j), (6, j)$ occurs on α and for every n such that $\alpha(n) = (0, j)$ there exist $m > n$ and $l < j$ such that $\alpha(m) \in \{(0, l), (4, l), (5, l)\}$. j is called *pseudo-active at* α if $j \leq i(\alpha)$, none of $(4, j), (5, j), (6, j)$ occurs on α, j is not active at α, and there exists $i < j$ such that i is active at α. For $n \leq 6$, α is called an *n-node* if $\alpha(l(\alpha) - 1) = (n, j)$ for some j. Sometime we also call that α an (n, j)-node. For each node $\alpha \neq \lambda$, α^- denotes the immediate predecessor of α, i.e. $\alpha^- = \alpha \restriction (l(\alpha) - 1)$. When α is a 0-node, let $j(\alpha)$ denote the unique number j such that $\alpha = \alpha^{-\wedge}\langle(0,j)\rangle$.

Remark. For each 0-node α, $j(\alpha) \leq i(\alpha^-)$.

We have already stipulated that $\lambda \in T$; λ is the root of the tree. We complete the definition of T by specifying what the outcomes of α are for each $\alpha \in T$. The immediate successors of α in T are just the strings of the form $\alpha^\wedge\langle(m,j)\rangle$, with (m,j) an outcome of α. We note that "outcomes" of the form $(1,i)$ are not really outcomes of the strategy at α but serve to include nodes on the tree which are useful for book-keeping purposes. If α is a 1-node, then α has no immediate successor. If α is an n-node for $n \in \{2,3\}$ then the outcomes of α are:

$$\{(0,j),(1,i),(2,i(\alpha)),(3,i(\alpha)),(4,j),(5,i),(6,i(\alpha)) : j \text{ is active at } \alpha, i \text{ is pseudo-active at } \alpha\}.$$

For the rest the outcomes of α are:

$$\{(0,j),(1,i),(2,i(\alpha)),(3,i(\alpha)),(4,j),(5,i) : j \text{ is active at } \alpha, i \text{ is pseudo-active at } \alpha\}.$$

This completes the description of T. Let $\Lambda = \{\langle j,i\rangle : j \leq 6, i < \omega\}$, be the set of symbols for the possible outcomes. Define a linear ordering $<_\Lambda$ on Λ as follows. Let $n = (n_0, n_1)$ and $m = (m_0, m_1)$ be two outcomes of α. If $n_0, m_0 \notin \{1,5\}$, then $n <_\Lambda m$ if and only if $n_0 < m_0$, or $n_0 = m_0$ and $n_1 < m_1$; if $m_0 = 1$, then $n <_\Lambda m$ if and only if $n_0 \leq 1$ and $n_1 < m_1$; if $m_0 = 5$, then $n <_\Lambda m$ if and only if $n_0 < 4$, or $n_0 \in \{4,5\}$ and $n_1 < m_1$. As described in Soare [12], two orderings $<_L, \leq$ are defined as follows:

4.1 Definition. Let $\alpha, \beta \in T$.

i) α is to the *left* of β ($\alpha <_L \beta$) if

$$(\exists a, b \in \Lambda)(\exists \gamma \in T)[\gamma^\wedge\langle a\rangle \subseteq \alpha \wedge \gamma^\wedge\langle b\rangle \subseteq \beta \wedge a <_\Lambda b].$$

ii) $\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$.

iii) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

4.2 Definition. *i)* β is an *active extension of* α if β has the form $\alpha^\wedge\langle(0,j)\rangle$. β is a *pseudo-active extension of* α if β has the form $\alpha^\wedge\langle(1,j)\rangle$. We introduce the notations: $\mathcal{A}(\alpha)$ for the set of active extensions of α, and $\mathcal{B}(\alpha)$ for the set of active or pseudo-active extensions of α.

ii) Let $m_0, \cdots, m_k \in \omega$ and $\beta \subseteq \alpha$ be nodes in T. $(m_0, \cdots, m_k)$ *occurs on* α *below* β, if there exist numbers $n_0 < \cdots < n_k$ such that $n_0 \geq |\beta|$ and for each i with $0 \leq i \leq k$, $\alpha(n_i) = (0, m_i)$.

For each $\alpha \in T$, there are a number of parameters associated with the α-strategy's attempt to satisfy $\mathcal{R}^{i(\alpha)}$:

i) k^α is the point of attack, i.e., the argument at which we aim to make $\Psi^{i(\alpha)}(W^{i(\alpha)})$ different from D.

ii) a^β for $\beta \in \mathcal{B}(\alpha)$. If $\beta = \alpha^\wedge\langle(i,j)\rangle$, then a^β, if defined, is an element of $A^{j,i} - C$ which is $> k^\alpha$, can be regarded as the 'use' at k^α of the functional which we hope will reduce D to $A^{j,i} - C$.

iii) p^β for β an active extension of α. If $\beta = \alpha^\wedge\langle(0,j)\rangle$, then p^β, if defined, is a number which has been enumerated in $A^{j,0}$ and which can be regarded as j-permission (i.e., permission from the intended reduction of D to $A^{j,0} - C$) for k^α to be enumerated in B.

iv) c^α is a counter which records how far the α-strategy's attack on $\mathcal{R}^{i(\alpha)}$ has proceeded.

v) r^α is the restraint intended to protect the α-strategy's attack on $\mathcal{R}^{i(\alpha)}$.

Each parameter belongs to a node. $k^\alpha, c^\alpha, r^\alpha, a^\alpha, p^\alpha$ belong to α. Partition ω effectively into infinite sets N_α ($\alpha \in T$). The members of N_α are called α-*numbers.* Now we describe what actions should be taken at the various nodes of T. If x is enumerated in B during the construction then $x = k^\beta$ for some β. When $x = k^\beta$ enters B, then we set $\alpha(x) = \beta$. For $\alpha \in T$, at any particular stage we say that α is *ready* if $\varphi^{i(\alpha)}(D, \psi^{i(\alpha)}(W^{i(\alpha)}, k^\alpha)) \downarrow$,

$$\Psi^{i(\alpha)}(W^{i(\alpha)}; k^\alpha) = 0 \wedge (\forall x \leq \psi^{i(\alpha)}(W^{i(\alpha)}, k^\alpha))[\Phi^{i(\alpha)}(D; x) = W^{i(\alpha)}(x)],$$

and for all $i \leq 1$ and for all i-nodes δ such that $\delta \subseteq \alpha$ or $\alpha^\wedge\langle(2, i(\alpha))\rangle <_L \delta$, either $a^\delta \uparrow$, or $a^\delta \downarrow$ and

$$\varphi^{i(\alpha)}(D, \psi^{i(\alpha)}(W^{i(\alpha)}, k^\alpha)) < a^\delta.$$

Two key notions for the construction described below are:

4.3 Definition. Let β be a 0-node and $j_0 < \cdots < j_m = j(\beta)$ be an enumeration of all $i \leq j(\beta)$ which are active at β^-. At a particular instant in stage s, x is a *designated number* for β if the following conditions hold:

i) $a_s^\gamma \downarrow$ for each $\gamma \in \mathcal{B}(\beta^-)$, $r^{\beta^-} \downarrow$, $\alpha(x) \downarrow$, and $\alpha(x) \supseteq \beta$.

ii) x entered B after r^{β^-} attains its cuurent value

iii) $(j_m, \cdots, j_0)$ occurs on $\alpha(x)$ below β^-.

iv) Let $\langle \delta_{j_m}, \delta_{j_{m-1}}, \cdots, \delta_{j_0} \rangle$ be the unique $(m+1)$-tuple in T such that $\beta = \delta_{j_m} \subset \cdots \subset \delta_{j_0} \subseteq \alpha(x)$ and δ_{j_i} is a $(0, j_i)$-node $(i \leq m)$. When x enters B, the following hold:

- $r^{\delta_j^-} \downarrow$ *for all* $j \in \{j_0, \cdots, j_m\}$.
- $p^\xi \downarrow$ *for all* ξ *such that* $\xi \in \mathcal{A}(\delta_j^-)$ *for some* $j \in \{j_0, \cdots, j_m\}$ *with* $\delta_j <_L \xi$.

It is worth noting that for some of the j and ξ just mentioned, $r^{\delta_j^-}$ and p^ξ may be defined earlier in the same stage at which x enters B.

4.4 Definition. Let $\xi \subset \beta$ be 0-nodes. Let $j_0 < \cdots < j_m$ be an enumeration of all $j \leq j(\xi)$ which are active at ξ^-. ξ is *preferred to* β if one of the following cases holds:

i) $\beta = (\mu\gamma)[\gamma \in \mathcal{A}(\beta^-)]$ *and* $\langle j_m, \cdots, j_0 \rangle$ *occurs on* β^- *below* ξ^-.

ii) There exists $\theta \supseteq \max\{\delta : \delta \in \mathcal{A}(\beta^-) \wedge \delta <_L \beta\}$, $\langle j_m, \cdots, j_0 \rangle$ *occurs on* θ *below* ξ^-.

Remarks. 1. If ξ is preferred to β, then there exists a 0-node θ such that θ is preferred to β, $\xi \subseteq \theta \subset \beta$ and $j(\theta) < j(\beta)$ is active at ξ^-.

2. The intuition for the notion "preferred to" is: Suppose ξ is preferred to β and $\beta = (\mu\gamma)[\gamma \in \mathcal{A}(\beta^-)]$. If k^{β^-} is enumerated in B, then k^{β^-} may be designated for ξ.

Suppose ξ is preferred to β and $\theta \supseteq \max\{\delta : \delta \in \mathcal{A}(\beta^-) \wedge \delta <_L \beta\}$ is such that $\langle j_m, \cdots, j_0 \rangle$ occurs on θ below ξ^-. If k^θ is enumerated in B, then k^θ may be designated for ξ.

3. ξ is "preferred to" β in the sense that before we define p^β we wish to complete the ξ^--attack to the point at which r^{ξ^-} is defined and p^θ is defined for all $\theta \in \mathcal{A}(\xi^-)$, $\theta > \xi$. This is appropriate because some 0-node η with $j(\eta) < j(\beta)$ lies in $\xi \subseteq \eta \subset \beta$.

4.5 Lemma. *i) Let ξ, η, θ, β be 0-nodes such that $\theta = (\mu\delta)[\delta \in \mathcal{A}(\eta^-) \wedge \eta <_L \delta]$ and $\xi \subset \eta \subset \beta$. If ξ is preferred to β, then ξ is preferred to θ.*

ii) Let π, σ, ζ be 0-nodes such that $\pi \subset \sigma \subset \zeta$, π is preferred to ζ and there is no (n, j)-node β with $\sigma \subseteq \beta \subset \zeta$ and $j < j(\zeta)$. Then π is preferred to σ.

Proof. The first part is clear from Definition 4.4. Let π, σ, ζ be 0-nodes such that $\pi \subset \sigma \subset \zeta$, π is preferred to ζ and there is no (n, j)-node β with $\sigma \subseteq \beta \subset \zeta$ and $j < j(\zeta)$. Let $j_0 < \cdots < j_m$ be an enumeration of all $j \leq j(\pi)$ which are active at π^-. Suppose $\zeta = (\mu\delta)[\delta \in \mathcal{A}(\zeta^-)]$. Then $\langle j_m, \cdots, j_0 \rangle$ occurs on ζ^- below π^-. Let $\langle \tau_{j_m}, \cdots, \tau_{j_0} \rangle$ be the unique $(m+1)$-tuple on ζ^- such that τ_j is a $(0, j)$-node. Then $j_0 < j(\zeta)$ and so $\tau_{j_0} \subset \sigma$. Hence π is preferred to σ. Suppose $\zeta \neq (\mu\delta)[\delta \in \mathcal{A}(\zeta^-)]$. Let $\epsilon = \max\{\delta : \delta \in \mathcal{A}(\zeta^-) \wedge \delta <_L \zeta]$. Then there exists $\theta \supseteq \epsilon$ such that $\langle j_m, \cdots, j_0 \rangle$ occurs on θ. Let $\langle \tau_{j_m}, \cdots, \tau_{j_0} \rangle$ be the unique $(m+1)$-tuple on θ such that τ_j is a $(0, j)$-node. Let i be the least j such that $\tau_j \subset \zeta^-$. By Remark 1 above, $i < j(\zeta)$. Hence $\tau_i \subset \sigma$. Also, for all $j < i$, j is active at ζ^- if and only if j is active at σ^-. Thus π is preferred to σ. ∎

Before we describe the construction, we define a class C of 0-nodes as follows. At any point in stage s, $\xi \in C$ if the following conditions hold:

i) $r^{\xi^-} \downarrow$,

ii) there exists some $x \in B$ which is designated for ξ,

iii) for each (n, j)-node α such that

- $\xi \subset \alpha \wedge j < j(\xi)$,
- *there exists a number y with $\alpha(y) \supseteq \alpha$ which entered B after r^{ξ^-} attained its current value,*

there exists a 0-node η such that $\xi \subset \eta \subseteq \alpha$, $j(\eta) \leq j$, and $\eta \in \mathcal{C}$.

Remarks. 1. It will turn out that, if $\xi \in \mathcal{C}_{s+1}$, then $a_s^\xi \downarrow = a_{s+1}^\xi$.

2. The reader should note that the truth-value of $\xi \in \mathcal{C}$ is defined by recursion on $|\xi|$. Since $r^{\xi^-} \downarrow$ for only a finitely number of $\xi \in T$, the induction is sound.

3. $\xi \in \mathcal{C}$ means that the attack on $\mathcal{R}^{i(\xi^-)}$ which is associated with ξ^- need not be reset because of activity at nodes extending ξ.

CONSTRUCTION

At stage s of the construction we begin at node λ and carry out certain instructions passing down the tree. We now give the instructions for node α. When at node α we first carry out the action prescribed for the first of the following cases which holds, with one exception which is mentioned below, and then the instructions for ending a stage.

Case 1. $c^\alpha \downarrow$ *and there exist* i, x *such that* $i \leq i(\alpha)$ and, either

$$x \in B - C \wedge x \notin A_s^{i,0} \cup A_s^{i,1} \wedge$$
$$[(i(\alpha^-) < i \wedge \alpha(x) <_L \alpha^\wedge\langle(6,i)\rangle) \vee (i(\alpha^-) \geq i \wedge (\alpha(x) <_L \alpha \vee \alpha(x) \supset \alpha))],$$

or

there exists a 0-node β such that $j(\beta) = i$, $\alpha(x) \supseteq \beta \supset \alpha$, x is a designated number for β, $p^\gamma \uparrow$ for each 0-node γ such that either $\gamma \subseteq \beta$ or $\beta >_L \gamma \in \mathcal{A}(\beta^-)$, *$r^\theta \uparrow$ for each node θ with $\theta^\wedge\langle(2, i(\theta))\rangle \subset \beta$, and either*

i) $x \in A_s^{i,1}$, and there is no θ-designated number for each 0-node θ such that $\alpha \subset \theta \subset \beta$ and $j(\theta) < j(\beta)$, or

ii) $x \in A_s^{i,0}$ and for each 0-node $\xi \subset \beta$ which is preferred to β, the following hold:

- r^{ξ^-} is defined,
- for each node $\zeta \in \mathcal{A}(\xi^-)$ such that $\xi <_L \zeta$, p^ζ is currently defined.

Choose the least such i and then the least x. It will turn out that, if $\alpha^\wedge\langle(6,i)\rangle \notin T$, then $x \in A_s^{i,0} \cup A_s^{i,1}$. If $x \notin A^{i,0} \cup A^{i,1}$, pass to $\alpha^\wedge\langle(6,i)\rangle$. Otherwise, choose the $\leq$-least β. There are two cases:

Case 1.1. $x \in A^{i,1}$. Destroy c^τ and r^τ for $\tau >_L \alpha$. Destroy a^γ, r^{γ^-} for each 0-node γ such that $\gamma \subseteq \beta$ or $\beta <_L \gamma$. Let $\mathcal{C}[s]$ be the class $\mathcal{C}$ at this point. Also, destroy a^γ and r^{γ^-} for each 0-node γ such that $\gamma \supset \beta$ and there is y with $\alpha(y) \supseteq \gamma$ which entered B after a^γ was set unless either there exists a 0-node ξ such that $\xi \subseteq \gamma$ and $\xi \in \mathcal{C}[s]$ or there exists π such that $\beta \subseteq \pi \subset \pi^\wedge\langle(2, i(\pi))\rangle \subset \gamma$ and r^π is defined.

Case 1.2. $x \in A^{i,0}$. Then pass to β and go directly to Case 3 at β. We call this *jumping from α to β.*

Case 2. *α is an i-node for $i \leq 1$, $a^\alpha \uparrow$, and $a^\delta \downarrow$, where δ is the maximal 0-node such that $\delta \subset \alpha$ if any.* Let $\alpha = \alpha^{-\wedge}\langle(i,j)\rangle$. Set a^α equal to the greatest x in $(k^{\alpha^-}, a^\delta) \cap (A^{j,i} - \mathcal{C})$ such that $\alpha(x) \supset \delta$, and for all $t \leq s$ and all β

- $\beta <_L \alpha \vee \alpha \subset \beta \Longrightarrow a_t^\beta < x$,
- $\beta \leq \alpha \vee \alpha \subset \beta \Longrightarrow k_t^\beta < x$,
- $\beta^\wedge\langle(2, i(\beta))\rangle <_L \alpha \Longrightarrow r_t^\beta < x$,
- $x \notin \{a_s^\epsilon : a_s^\epsilon \downarrow \wedge \alpha^- <_L \epsilon\}$,
- $\alpha^{-\wedge}\langle(6, i(\alpha^-))\rangle \in T \Longrightarrow x \notin \{a_s^\epsilon : a_s^\epsilon \downarrow \wedge \alpha^{-\wedge}\langle(6, i(\alpha^-))\rangle \subset \epsilon\}$,

and if v is the last stage at which a^α was defined, then x exceeds every values taken by a parameter at a stage $\leq v$. Destroy a^δ for each 0-node δ such that either $\delta \subset \alpha$ or $\alpha <_L \delta$. Read λ for δ and ∞ for a^δ when δ is undefined. Destroy c^τ and r^τ for $\tau >_L \alpha$.

(The construction will only pass to α if a suitable x exists.)

Case 3. $\alpha = \alpha^{-\wedge}\langle(0,j)\rangle$, *$p^\alpha \uparrow$, and there exists $x \in A^{j,0} - \mathcal{C}$, such that x is a designated number for α.* Fix the least such x. (In fact, it will turn out that there is at most one possibility for x.) Set p^α equal to x. Further, let $\delta \in \mathcal{A}(\alpha^-)$ be the maximal node if any such that $\delta < \alpha$. There are two subcases:

- Case 3.1. *δ does not exist.* Enumerate k^{α^-} in B, and set $c^{\alpha^-} = 1$. Destroy c^τ and r^τ for $\tau >_L \alpha$ unless p^π is defined for $\pi \subseteq \tau$ with $\pi^- = \tau \cap \alpha$. Let $\mathcal{C}[s]$ be the class $\mathcal{C}$ at this point. For each 0-node γ such that either $\gamma \subseteq \alpha^-$ *or* $\alpha^{-\wedge}\langle(2, i(\alpha^-))\rangle <_L \gamma$, destroy a^γ and r^{γ^-} unless either there exists a 0-node ξ such that $\xi \subseteq \gamma$ and $\xi \in \mathcal{C}[s]$ or there exists π such that $\pi^\wedge\langle(2, i(\pi))\rangle \subset \gamma$ and r^π is defined.

- Case 3.2. *δ exists.* Pass to δ.

Case 4. $c^\alpha\uparrow$, $c^\beta\downarrow$, $a^\gamma\downarrow$, *where β denotes α^- and γ is the maximal 0-node $\subseteq\alpha$ if any.* Let x be the least unused α-number in (k^β, a^γ) if any such that for all $t\le s$,

- $\delta<\alpha\Longrightarrow k^\delta<x$
- $\delta<_L\alpha\Longrightarrow a^\delta_t<x$
- $\delta<_L\alpha\wedge\delta^\wedge\langle(2,i(\delta))\rangle\subseteq\alpha\Longrightarrow r^\delta<x$

and if v is the last stage at which c^α was defined, then x exceeds every values taken by a parameter at a stage $\le v$. In this definition read -1 for k^β when $\alpha=\lambda$ and ∞ for a^γ when γ is undefined. If x exists, set $k^\alpha=x$ and $c^\alpha=0$. In any case, destroy a^δ for each 0-node δ such that either $\delta\subseteq\alpha$ or $\alpha<\delta$. Destroy c^τ and r^τ for $\tau>_L\alpha$.

Case 5. $c^\alpha=0$, $a^\delta_s\downarrow$ *for the maximal 0-node $\delta\subseteq\alpha$ if any, and there exists $\beta\in\mathcal{B}(\alpha)$ such that $a^\beta\uparrow$.* Let $\gamma=\alpha^\wedge\langle(i,j)\rangle$ be the $\le$-least such β. If there exists x in $(k^\alpha,a^\delta)\cap(A^{j,i}-C)$ such that $\alpha(x)\supset\delta$, and for all $t\le s$ and all β,

- $\beta<_L\gamma\vee\gamma\subset\beta\Longrightarrow a^\beta_t<x$,
- $\beta\le\gamma\vee\gamma\subset\beta\Longrightarrow k^\beta_t<x$,
- $\beta^\wedge\langle(2,i(\beta))\rangle<_L\gamma\Longrightarrow r^\beta_t<x$,
- $x\notin\{a^\epsilon_s : a^\epsilon_s\downarrow\wedge\alpha<_L\epsilon\}$,
- $\alpha^\wedge\langle(6,i(\alpha))\rangle\in T\Longrightarrow x\notin\{a^\epsilon_s : a^\epsilon_s\downarrow\wedge\alpha^\wedge\langle(6,i(\alpha))\rangle\subset\epsilon\}$,

and if v is the last stage at which a^γ was defined, then x exceeds every values taken by a parameter at a stage $\le v$, then pass to γ. Otherwise pass to $\alpha^\wedge\langle(4+i,j)\rangle$. Read λ for δ and ∞ for a^δ when δ is undefined.

Case 6. $c^\alpha=0$, *and α is not ready.* Pass to $\alpha^\wedge\langle(2,i(\alpha))\rangle$.

Case 7. $c^\alpha=0$, $r^\alpha\uparrow$, *and α is ready.* Define

$$r^\alpha=Max(\{\varphi^{i(\alpha)}(D,\psi^{i(\alpha)}(W^{i(\alpha)},k^\alpha))\}\cup\{a^\beta:\beta\in\mathcal{B}(\alpha)\}).$$

Perform the actions prescribed by the first of Cases 8-9 which holds.

Case 8. $\mathcal{A}(\alpha) = \emptyset$. Enumerate k^α in B, and set $c^\alpha = 1$. Destroy c^τ and r^τ for $\tau >_L \alpha$ unless p^π is defined for $\pi \subseteq \tau$ with $\pi^- = \tau \cap \alpha$. Destroy c^β and r^β for all $\beta \supset \alpha$ such that $\alpha^\wedge\langle(2, i(\alpha))\rangle \leq \beta$. Let $\mathcal{C}[s]$ be the class $\mathcal{C}$ at this point. For each 0-node γ such that either $\gamma \subseteq \alpha$ or $\alpha <_L \gamma$, destroy a^γ and r^{γ^-} unless either there exists a 0-node ξ such that $\xi \subseteq \gamma$ and $\xi \in \mathcal{C}[s]$ or there exists π such that $\pi^\wedge\langle(2, i(\pi))\rangle \subset \gamma$ and r^π is defined.

Case 9. $\mathcal{A}(\alpha) \neq \emptyset$. Pass to the maximal node in $\mathcal{A}(\alpha)$.

Case 10. $c^\alpha_s = 1$ *and no number* $\leq \psi^{i(\alpha)}_t(W^{i(\alpha)}_t, k^\alpha)$ *has entered* $W^{i(\alpha)}$ *since the stage* t, *in which* r^α *attained its present value.* Pass to $\alpha^\wedge\langle(3, i(\alpha))\rangle$.

Case 11. $c^\alpha_s = 1$ *and a number* $\leq \psi^{i(\alpha)}_t(W^{i(\alpha)}_t, k^\alpha)$ *has entered* $W^{i(\alpha)}$ *since stage* t, *where* t *is the stage in which* r^α *attained its present value.* Enumerate k^α into C. Put into C every y such that $\alpha(y) \supseteq \beta$ for some $\beta \in \mathcal{A}(\alpha)$ and y was enumerated in B since r^α was set. Set $c^\alpha = 2$. Destroy a^δ for each 0-node δ such that either $\delta \subseteq \alpha$ or $\alpha <_L \delta$. Destroy c^τ and r^τ for $\tau >_L \alpha$.

Case 12. $c^\alpha = 2$. Pass to $\alpha^\wedge\langle(3, i(\alpha))\rangle$.

Case 13. None of above Cases holds at α. Destroy c^γ and r^γ for $\gamma >_L \alpha$; a^γ for $\gamma \subseteq \alpha$ or $\gamma >_L \alpha$.

Ending a stage

Let α be the last node which is visited; α is said to *receive attention* at this stage. After completing the instructions for the particular cases which hold at the various nodes, to end the stage we carry out the following:

(E1) if $\delta <_L \gamma$ and a^δ is destroyed in the main part of the stage, then a^γ and c^γ are destroyed.

(E2) if a^δ has been destroyed in the stage, then a^δ is to be enumerated into C.

(E3) if a^δ has been destroyed and $\delta \in \mathcal{A}(\delta^-)$, then p^δ is destroyed if defined.

(E4) if p^δ has been destroyed, then c^β, if defined, is destroyed for all $\beta \supseteq \delta$.

(E5) r^δ is destroyed if either a^β has been destroyed for some $\beta \in \mathcal{B}(\delta)$ or c^δ has been destroyed.

Remarks. 1. In the construction, if k^α becomes defined, then it retains its value until reset to a new value.

2. For a 0-node α, r^{α^-}, p^α may become defined in a stage and be destroyed before the end of the stage.

3. When the construction jumps to β, the instructions of the construction ignore the Cases 1 and 2 at β. The conditions for jumping ensure that Case 3 holds at β hence p^β becomes defined.

4. If a 0-node is visited at a stage, then the construction passes to that node by one of the following cases:

i) Case 1.2, i.e. by jumping.

ii) Case 3.2.

iii) Case 5.

iv) Case 9.

5. Let γ be a 0-node such that $r_s^{\gamma^-}\downarrow$. The first node in $\mathcal{A}(\gamma^-)$ which is visited in stage s, if any, is reached by jumping.

6. If α is a 1-node, then a^α can only be destroyed at the end of a stage through (E1).

7. Because the class is just referred at a point in some stage in the construction, in stage s we just define $\mathcal{C}_s$, and $\mathcal{C}[s]$ if one of the Cases 1.1, 3.1 and 8 holds at stage s.

In the construction the class $\mathcal{C}$ plays an important role although no analogue of it occurred in Chapter 3. Roughly speaking, $\xi \in \mathcal{C}$ means that we wish to protect a^ξ. As mentioned above with each 0-node ξ is associated a strategy S^ξ which either reduces D to $A^{j(\xi),1} - C$ or gets $A^{j(\xi),0}$-permission for the ξ^--attack to enumerate k^{ξ^-} into B. Let J^ξ be the set of all $j < j(\xi)$ which are active at ξ^-. Then the assumption of S^ξ is that every ξ-designated number which enters $\cap\{A^{j,0} : j \in J^\xi\}$, enters $A^{j(\xi),1}$. It will turn out that in any particular instant, there is at most one ξ-designated number. So, if there is no ξ-designated number, the assumption for the strategy S^ξ is not violated and we can think that S^ξ still active. In this case we also can destroy a^ξ if we need to. If there is a ξ-designated number, the module for the ξ^--attack requires that a^ξ not be destroyed unless and until the ξ-designated number enters $A^{j,1}$ for some $j \in J^\xi \cup \{j(\xi)\}$. Now the class $\mathcal{C}$ allows a^ξ to be destroyed more often than would otherwise be the case. $\xi \notin \mathcal{C}$ and there is a ξ-designated number means that, at the particular instant, the strategy S^ξ appears to be invalidated because there is a strategy at $\beta \supset \xi$ of higher priority which is currently active. Let α witness $\xi \notin \mathcal{C}$. Suppose either α is not a 0-node or there is no α-designated number. In this case we can take $\beta = \alpha$. Suppose α is a 0-node and an α-designated number exists. Note that $\alpha \notin \mathcal{C}$. By the induction hypothesis, there is a strategy at $\beta \supset \alpha$ currently active which has higher priority than S^α. The strategy at β also has higher priority than S^ξ which intuitively just keeping ξ out of $\mathcal{C}$.

Chapter 5

Verification, Part I

We need to verify that the above construction satisfies both the $\mathcal{R}^e$ and the $\mathcal{S}^e$ requirements. In present chapter we investigate the properties of the construction. We say that the α-attack *has been started at* stage s if r^α becomes defined at stage s and that the α-attack *is destroyed* if r^α is destroyed. The α-attack *is completed* if c^α is set equal 1. We start with lemmas about the parameters of the construction:

5.1 Lemma. *i) k^α is defined whenever c^α is defined, and c_s^α, a_s^β are defined for each $\beta \in \mathcal{B}(\alpha)$ and for all s such that r^α is defined at some point in stage s.*

ii) Let $c^\alpha \downarrow$. Then c^α is monotonic non-decreasing ≤ 2 unless destroyed through one of the following circumstances:

(a) a node $\beta <_L \alpha$ receives attention and if either Case 3.1 or Case 8 occurs at β, then there is no π such that

$$\pi \subseteq \alpha \wedge \pi^- = \beta \cap \alpha \wedge p^\pi \downarrow;$$

(b) a node β receives attention, Case 8 holds at β, $\beta \subset \alpha$, and $\beta^\wedge\langle(2, i(\beta))\rangle \leq \alpha$;

(c) p^γ is destroyed for some $\gamma \subseteq \alpha$;

(d) a^γ is destroyed for some $\gamma <_L \alpha$.

iii) Let $\delta \subset \alpha$. If c^δ is destroyed, then so is c^α.

iv) Let $r^\alpha \downarrow$. Then r^α retains the same value unless destroyed simultaneously with c^α, or unless a^β is destroyed, for some $\beta \in \mathcal{B}(\alpha)$.

v) Let β be a 0-node and $a^\beta_v \downarrow$. If β is visited at stage v, then r^{β^-} is defined at some point during stage v before β is visited.

vi) Let $\alpha \subset \beta$. Then

(a) β is visited at stage s implies $c^\alpha_s \downarrow$.

(b) $c^\beta_s \downarrow$ implies $c^\alpha_s \downarrow$.

vii) A number which is enumerated in B at a stage s in which α receives attention has the form k^β_s, where $\beta = \alpha$ if $\mathcal{A}(\beta) = \emptyset$, and $\beta = \alpha^-$ otherwise. In the latter case, p^α becomes defined at stage s. If k^β_s is enumerated in B in stage s, then $c^\beta_s = 0$ and $c^\beta_{s+1} = 1$.

viii) Let $\alpha \in \mathcal{B}(\alpha^-)$ and $a^\alpha_s \downarrow$. If a^α is destroyed in stage s, then one of the following holds:

(a) a^γ is destroyed for some $\gamma <_L \alpha$;

(b) a node β receives attention such that either $\alpha \subseteq \beta$ or $\beta <_L \alpha$, and one of Case 3.1, Case 8, and Case 11 holds for β;

(c) a node β receives attention such that either $\alpha \subset \beta$ or $\beta <_L \alpha$, and Case 2 holds for β;

(d) a node β receives attention such that $\alpha \subseteq \beta$ or $\beta < \alpha$, and one of Case 1.1 or Case 4 holds for β;

(e) a node β receives attention such that $\alpha \subseteq \beta$ or $\beta <_L \alpha$, and no case in the construction holds.

ix) Let $a^\alpha_s \downarrow$. Then in stage s, a^α_s is destroyed if and only if it is enumerated in C.

x) If $\delta \in C_{s+1}$, or $C[s]$ exists and $\delta \in C[s]$, then $a^\delta_s \downarrow = a^\delta_{s+1} \downarrow$.

xi) Let $p^\alpha \downarrow$. Then p^α retains the same value unless destroyed simultaneously with a^α.

xii) Let $a^\alpha \downarrow$ and $\alpha \in \mathcal{B}(\alpha^-) - \mathcal{A}(\alpha^-)$. Then a^α retains the same value unless destroyed simultaneously with a^β, for some $\beta \in \mathcal{A}(\alpha^-)$ and $\beta <_L \alpha$.

xiii) *Let α, β and s satisfy $a^\alpha_s \downarrow$, $a^\beta_s \downarrow$ and either $\alpha \supset \beta$ or $\alpha <_L \beta$. If a^α is destroyed at stage s, so is a^β.*

xiv) *If α, $\beta \in \mathcal{B}(\alpha^-)$ and $\alpha <_L \beta$, then a^β is defined implies a^α is defined.*

xv)
- Let α be a 0-node, and some node $\supseteq \alpha$ be visited in stage s. Then $p^\alpha_s \uparrow$.
- Let α be a 0-node, and some node $\supset \alpha$ be visited in stage s. Then $p^\alpha \uparrow$ throughout stage s.
- If r^β is defined at some point in stage s, then there no node $\supseteq \beta^\wedge\langle(2, i(\beta))\rangle$ is visited at stage s.

xvi) *If a node $\supseteq \alpha^\wedge\langle(3, i(\alpha))\rangle$ is visited at stage s, then $c^\alpha_s \geq 1$.*

xvii) *If $\beta^- = \zeta^-$, $\beta <_L \zeta$, and ζ is an i-node with $i \leq 3$ and ζ is visited in stage s, then $a^\beta_s \downarrow$.*

xviii) *Let α, β satisfy $\alpha <_L \beta$, $\alpha^- \subset \beta$, and $(\alpha \cap \beta)^\wedge\langle(3, i(\alpha \cap \beta))\rangle \not<_L \beta$. Then at any stage, $a^\beta \downarrow$ implies $a^\alpha \downarrow$.*

xix) *Let θ, β be 0-nodes such that $\theta <_L \beta$, $\theta^- = \beta^-$ and both are visited at stage s. Let α be the maximal node in $\mathcal{A}(\beta^-)$ such that $\alpha <_L \beta$. Then p^β becomes defined in stage s, α is visited at stage s after p^β is defined, $\alpha = \theta$, and p^α cannot becomes defined in stage s.*

xx) *If α is visited at stage s, then for each node $\beta \subset \alpha$, either β is visited at stage s or there exists a jump from some node $\subset \beta$ to some node $\delta \supset \beta$ such that $\delta = \alpha$ or $\delta^- \subset \alpha$, $\delta = (\mu\theta)[\theta \in \mathcal{A}(\delta^-) \wedge \alpha <_L \theta]$. Moreover, if we jump to δ at stage s, then p^δ becomes defined at stage s.*

xxi) *Let $a^\delta_s \downarrow$. If some node $\supseteq \delta$ receives attention at stage s, then at stage s one of the following holds:*

- a^δ_s is destroyed.
- k^{δ^-} enters B and p^δ becomes defined.
- some y enters B with $\alpha(y) \supseteq \delta$.

xxii) *Let $a^\delta_s \downarrow$ and r^π be defined at some point in stage s for some π with $\pi^\wedge\langle(2, i(\pi))\rangle \subset \delta$. Then a^δ_s is not destroyed unless and until r^π is destroyed.*

xxiii) *Let ξ, δ, β, u satisty:*

(a) $\xi \subset \delta$, δ is a 0-node,

(b) $a^\xi_u \downarrow$, $a^\delta_u \downarrow$,

(c) β receives attention at stage u,

(d) $\xi \subseteq \beta \Longrightarrow \beta <_L \delta \vee \delta \subset \beta$,

(e) If $\mathcal{C}[u]$ exists, then there is no σ such that $\xi \subset \sigma \subseteq \delta$ and $\sigma \in \mathcal{C}[u]$,

(f) There is no σ such that $\xi \subset \sigma^\wedge\langle(2, i(\sigma))\rangle \subset \delta$ and $r^\sigma \downarrow$ in stage u,

(g) There is no σ such that $\xi \subset \sigma \subset \delta$ and p^σ is defined at stage u, and $p^\delta \downarrow$ implies $\delta^- \neq \beta^-$,

(h) a^ξ is destroyed at stage u.

Then a^δ is destroyed at stage u.

xxiv) *(a) Let γ be a 0-node and v be a stage such that $r^{\gamma^-}_{v+1} \downarrow$, and $\gamma \in \mathcal{C}_v - \mathcal{C}_{v+1}$. Then there exist α and π such that $\alpha \supseteq \pi \in \mathcal{A}(\gamma^-)$, $\pi \leq \gamma$, and α receives attention in stage v.*

(b) Let γ be a 0-node and v be a stage such that $\mathcal{C}[v]$ exists, $\gamma \in \mathcal{C}_v - \mathcal{C}[v]$, and $r^{\gamma^-} \downarrow$ when we define $\mathcal{C}[v]$. Then there exist α and π such that $\alpha \supseteq \pi \in \mathcal{A}(\gamma^-)$, $\pi \leq \gamma$, and α receives attention in stage v.

xxv) *Let one of the Cases 1.1, 3.1 and 8 hold at stage v. Let γ be a 0-node such that $r^{\gamma^-}_{v+1} \downarrow$. Then $\gamma \in \mathcal{C}[v]$ if and only if $\gamma \in \mathcal{C}_{v+1}$.*

Proof. We just verify $v) - vii)$, $x)$, $xiii)$, $xv) - xix)$, $xxi) - xxv)$, the rest are obvious.

$v)$ If β is visited in stage v, then either β^- is also visited and Case 9 holds at β^- or the construction jumps to a node δ such that $\beta^- = \delta^-$. In the former case $r^{\beta^-} \downarrow$ is required for Case 9. In the latter case there is a number designated for δ which means that $r^{\delta^-} \downarrow$.

$vi)$ It is sufficient to show firstly that, if β is visited, then c^{β^-} is defined, and secondly that c^{β^-} is defined whenever c^β is defined. By $iii)$, whenever c^{β^-} becomes undefined, so does c^{β}. Suppose β is visited at a particular stage. Every case which allows the construction to pass from β^- to β requires that $c^{\beta^-}\downarrow$. Thus, if β^- is also visited, then $c^{\beta^-}\downarrow$. Otherwise, there is a jump to δ with $\delta^- = \beta^-$ and $r^{\delta^-}\downarrow$. So again $c^{\beta^-}\downarrow$ by i). Since c^{β} becomes defined only if β is visted this enough.

vii) A number can enter B only through Case 3.1 or Case 8 at the node which receives attention in the given stage. Examining those cases we see that we need only verify that, if k^{β}_s enters B in stage s, then $c^{\beta}_s = 0$. This is clear if $\mathcal{A}(\beta) = \emptyset$. Otherwise, Case 3.1 occurs and there exists x designated for α. So p^{α} becomes defined at stage s and r^{β} is defined at some point in stage s. By i), $c^{\beta}_s\downarrow$. Towards a contradiction assume that $c^{\beta}_s \geq 1$. Let $u < s$ be the last stage at which c^{β} was set equal to 1. At stage u, p^{α} became defined. Hence at a stage v, $u \leq v < s$, p^{α} becomes undefined. Then a^{α} and r^{β} are destroyed at stage v. But when r^{β} becomes defined, $c^{\beta} = 0$. This is a contradiction.

x) When r^{δ^-} becomes defined, a^{δ} is already defined. When a^{δ} becomes undefined, so does r^{δ^-}. This is sufficient since either $\delta \in \mathcal{C}_{s+1}$ or $\delta \in \mathcal{C}[s]$ requires $r^{\delta^-}\downarrow$ at some point in stage s.

xiii) Let a^{α}_s be destroyed at stage s. Suppose $\beta \subset \alpha$. By *xii*) we may suppose that α is a 0-node. By the construction a^{β}_s is also destroyed at stage s if defined. Now suppose $\alpha <_L \beta$. By (E1), a^{β}_s is also destroyed at stage s if defined.

xv) Suppose α is not visited at stage s. Then there is a jump to some node $\supset \alpha$, and by Case 1, $p^{\alpha}_s\uparrow$. Suppose α is visited at stage s.Then the construction passes to α by one of Cases 1.1, 3.2, 5, and 9. $p^{\alpha}_s\uparrow$ if there is a jump to α. Suppose α is visited by Case 3.2. Let π be the maximal node in $\mathcal{A}(\alpha^-)$ which is visited at stage s. Since Case 3.2 holds at the 0-node δ which is visited immediately before α, by Case 3 some number $z \in B$ is designated for δ at stage s and so $r^{\alpha^-} = r^{\delta^-}$ is defined at some point in stage s. Since $z \in B$ when the construction passes to δ, $z \in B_s$. It follows that $r^{\delta^-}_s\downarrow$. By i), $a^{\beta}_s\downarrow$ for all $\beta \in \mathcal{B}(\delta^-)$. So, if in stage s the construction passes to δ^-, then r^{δ^-} and a^{β} ($\beta \in \mathcal{B}(\delta^-)$) are all defined, and Case 1.2 must hold. It follows that there is a jump to π and then the construction passes to α by repetition of Case 3.2. Clearly, $\alpha <_L \pi$. By the conditions for Case 1, $p^{\alpha}\uparrow$.

Suppose α is visited by Case 5. $a^{\alpha}_s\uparrow$. But when p^{α} becomes defined, $a^{\alpha}\downarrow$, and p^{α} is destroyed only by $(E3)$. Hence $p^{\alpha}\downarrow$ implies $a^{\alpha}\downarrow$. Thus $p^{\alpha}_s\uparrow$ if $a^{\alpha}_s\uparrow$. Suppose α is visited by Case 9. Then α is the maximal 0-node in $\mathcal{A}(\alpha^-)$. Towards a contradiction assume that $p_s^\alpha \downarrow$. Let p^α have been set equal to p_s^α at stage $v < s$. Hence $r_t^{\alpha^-} \downarrow$. Note that at stage s, r^{α^-} becomes defined. Hence r^{α^-} became undefined between stages t and s, say at stage w. Suppose one of a^γ, $\gamma \in \mathcal{A}(\alpha^-)$, was destroyed at stage w. By $xii)$, we can choose $\gamma \in \mathcal{A}(\alpha^-)$. Then $\gamma \leq \alpha$. Hence a^α and then p^α were destroyed at stage w, contradicts the choice of t. Suppose c^{α^-} was destroyed at stage w. Then c^{α^-} becomes defined between stages w and s, and at the same time a^α and then p^α are destroyed, a contradiction.

The second part of $xv)$ is immediately from the first part unless p^α becomes defined in stage s. Suppose p^α becomes defined at stage s. By Case 3, there is no node $\supset \alpha$ is visited in stage s. This is enough. The third part of $xv)$ is clear.

$xvi)$ We prove $xvi)$ by induction on stages. Suppose some node $\supseteq \alpha^\wedge\langle(3, i(\alpha))\rangle$ is visited at stage s. Note that $\alpha^\wedge\langle(3, i(\alpha))\rangle$ is visited at stage s only if $c_s^\alpha \geq 1$. So $xvi)$ is clear unless there is a jump from some node $\supseteq \alpha$ to some node $\beta \supset \alpha^\wedge\langle(3, i(\alpha))\rangle$. Note that $c_s^{\beta^-} \downarrow$ and $\beta^- \supseteq \alpha^\wedge\langle(3, i(\alpha))\rangle$. Let t be the last stage before s in which c^{β^-} became defined. By the induction hypothesis, $c_t^\alpha \geq 1$. Then $c_s^\alpha \geq 1$ unless c^α is destroyed between stages t and s. However, if c^α is destroyed, then c^{β^-} is also destroyed by $iii)$ which contradicts the choice of t.

$xvii)$ Suppose $\zeta \in \mathcal{B}(\beta^-)$. If the construction passes to ζ other than by Case 5, then $a_s^\zeta \downarrow$ and so $a_s^\beta \downarrow$ by $xiii)$. Also, if the construction passes to ζ by Case 5, then $a_s^\beta \downarrow$. Suppose ζ is an n node for $n \in \{2, 3\}$. By the construction, ζ cannot be visited unless Case 5 does not hold at β^-. Hence $a_s^\beta \downarrow$. This is sufficient.

$xviii)$ Fix α. We proceed $xviii)$ by induction on $l(\beta)$. Suppose a^α is destroyed. By (E1), a^β is destroyed. Let ζ be the least node such that $\alpha \cap \beta \subset \zeta \subseteq \beta$. Then ζ is an n-node for $n \in \{0, 1, 2, 3\}$. Suppose $\beta = \zeta$. Then $\zeta \in \mathcal{B}(\alpha^-)$. By $xiv)$, when a^β becomes defined, a^α is already defined, and if a^α is destroyed a^β is destroyed simultaneously. Suppose $\beta \neq \zeta$. Let a^β become defined at stage v. Suppose that some 0-node ξ with $\zeta \subseteq \xi \subset \beta$ is visited and $a_v^\xi \downarrow$. By the induction hypothesis, $a_v^\alpha \downarrow$. Suppose there is no such ξ. Then ζ^- and ζ are visited at stage v. Let $\delta \subseteq \alpha^-$ be the maximal 0-node if any. Suppose δ exists. Then δ is visited since ζ^- $(= \alpha^-)$ is visited. Towards a contradiction assume $a_v^\delta \uparrow$. Then δ cannot be visited unless Case 5 holds at δ^- at stage v. Note that at stage v, if Case 1 holds at δ, then

Case 1 holds at δ^-. Then Case 1 cannot hold at δ at stage v. Hence Case 2 holds when δ is visited at stage v and then δ receives attention, a contradiction. Now by Case 5, $a_v^\alpha \downarrow$. Otherwise ζ cannot be visited at stage v. This is sufficient.

xix) From the construction the only possible θ is visited after β is visited is p^β becomes defined at stage s, and Case 3.2 holds at β. By Case 3.2, α is visited. Towards a contradiction assume $\alpha \neq \theta$. Clearly, $\theta <_L \alpha$. Then θ is not visited at stage s unlees p^α becomes defined. Therefore p^α is also defined at stage s. Let p^α be set equal to x at stage s. Let x enter B at stage u. Then $u < s$ and $r_{u+1}^{\alpha^-} \downarrow = r_s^{\alpha^-} \downarrow$ because x is d esignated for α at stage s. By the same token p^β was defined at some point in stage u. Observe that $p_{u+1}^\beta \downarrow = p_s^\beta \downarrow$. Otherwise, let p^β become undefined at stage v, $u \leq v < s$. Then a^β becomes undefined at stage v and so does r^{β^-}, contradiction. Hence p^β cannot become defined at stage s because $p_s^\beta \downarrow$.

xxi) Let α receive attention at stage s. Clearly a^δ is destroyed at stage s unless Case 3 or Case 8 holds. Suppose one of Case 3 and Case 8 holds at α. In either case some y enters B at stage s. If $\alpha(y) \supseteq \delta$, it is enough. Suppose $\alpha(y) \not\supseteq \delta$. Hence $\alpha^- = \alpha(y)$, $\alpha = \delta$, Case 3 holds at α and p^δ becomes defined at stage s. This is sufficient for xxi).

$xxii$) Towards a contradiction consider the least s, π and then δ which witness that $xxii$) fails. Then $r_{s+1}^\pi \downarrow$ and a^δ is destroyed at stage s. Let β receive attention at stage s. By xv), $\beta \not\supseteq \pi^\wedge\langle(2, i(\pi))\rangle$. By $viii$), at stage s, one of the following holds:

a) a^γ is destroyed for some $\gamma <_L \pi^\wedge\langle(2, i(\pi))\rangle$.

b) $\beta <_L \pi^\wedge\langle(2, i(\pi))\rangle$ and one of Case 2, Case 3.1, Case 8 and Case 11 holds.

c) $\beta <_L \pi^\wedge\langle(2, i(\pi))\rangle$ and no case in the construction hold.

d) $\beta < \pi^\wedge\langle(2, i(\pi))\rangle$ and one of Case 1.1, Case 4 holds for β.

Suppose a) holds. Then $\gamma <_L \pi$ or $\gamma \supseteq \tau$ for some $\tau \in \mathcal{B}(\pi)$. If $\gamma <_L \pi$. c^π is destroyed at stage s by (E1). If $\gamma \supseteq \tau$ for some $\tau \in \mathcal{B}(\pi)$. Then a^τ is destroyed at stage s by $xiii$). Hence in either case r^π is destroyed at stage s by (E5), contradiction. Suppose one of $b) - d)$ holds. Let $\beta <_L \pi$. BY construction, c^π is destroyed at stage s unless one of Case 3.1, and Case 8 holds. However, in these two cases, a^δ cannot become undefined because r^π is not destroyed. Suppose $\beta \subseteq \pi$. Then one of Case 1.1 and Case 4 holds. By i), $c_s^\pi \downarrow$ since $r_{s+1}^\pi \downarrow$.

By *vi*), $c_s^\beta \downarrow$ since $c_s^\pi \downarrow$. Hence Case 4 cannot hold. Suppose Case 1.1 holds. Then a^δ cannot become undefined because r^π is not destroyed. Suppose $\beta \supseteq \tau$ for some $\tau \in \mathcal{B}(\pi)$. Then by *xxi*), one of the following holds:

i) a^τ is destroyed.

ii) k^π enters B and p^τ becomes defined at stage s.

iii) some y enters B with $\alpha(y) \supseteq \tau$.

Since r^π is not destroyed, *i*) cannot hold. Suppose *ii*) holds. By Case 3.1, a^δ is not destroyed at stage s. Suppose *iii*) holds. By Case 3.1 and Case 8, a^δ cannot become undefined because is not destroyed at stage s. This is sufficient.

xxiii) By *viii*), it is clear that a_s^δ is destroyed at stage u unless at stage u one of the following four cases holds:

A) $\beta \supseteq \xi$ or $\beta <_L \xi$, and one of the Cases 3.1, 8, and Case 11 holds at β;

B) $\beta \supset \xi$ or $\beta <_L \xi$, and Case 2 holds at β;

C) $\xi \subseteq \beta$ or $\beta < \xi$, and one of Cases 1.1, 4 holds at β;

D) $\xi \subseteq \beta$ or $\beta <_L \xi$, and no case in the construction holds.

Suppose Case 4 holds at β at stage u with $\xi \subseteq \beta$ or $\beta < \xi$. By (*d*), in either case, $\delta \subseteq \beta$ or $\beta < \delta$. By Case 4 of the construction, a^δ is destroyed at stage u.

Suppose Case 1.1 holds at stage u. By (*d*) we see that there are three possibilities: $\beta \subset \xi$, $\beta <_L \delta$, and $\delta \subset \beta$. Let $\beta \subset \xi$. Since a^ξ is destroyed at stage u, by *xiii*) for each 0-node $\pi \subset \xi$, if $a_u^\pi \downarrow$, then a^π becomes undefined at stage u. From this fact and by (*e*) there is no $\tau \subseteq \delta$ such that $\tau \in \mathcal{C}[u]$. By (*f*), there is no γ such that $\xi \subseteq \gamma \subset \gamma^\wedge\langle(2, i(\gamma))\rangle \subseteq \delta$ and r^γ is defined at stage u. Hence by Case 1.1, a^δ is destroyed at stage u. Let $\beta <_L \delta$. It is clear by Case 1.1 that there exists a 0-node $\pi \supset \beta$, $a_u^\pi \downarrow$ and a^π is destroyed at stage u. By *xii*), a^δ is destroyed at stage u. Let $\delta \subset \beta$. Clearly a^δ is destroyed by Case 1.1.

Suppose Case 2 holds at stage u. Let $\beta <_L \xi$. Clearly a^δ is destroyed at stage u. Let $\beta \supset \xi$. By (*d*), $\beta <_L \delta$ or $\delta \subset \beta$. By Case 2, in either case a^δ is destroyed at stage u.

Similarly, if Case 11 holds or no case holds at β at stage u, we see that a^δ is destroyed at stage u.

Suppose Case 3.1 holds at stage u. Since a^ξ is destroyed, either $\beta \supset \xi$ or $\beta^- \hat{} \langle (2, i(\beta^-)) \rangle <_L \xi$. Let $\beta^- \hat{} \langle (2, i(\beta^-)) \rangle <_L \xi$. Then $\beta^- \hat{} \langle (2, i(\beta^-)) \rangle <_L \delta$. By (e), (f) and Case 3.1, a^δ is destroyed at stage u. Let $\beta \supset \xi$. By (d), either $\beta \supset \delta$ or $\beta <_L \delta$. Also, if a^ξ is destroyed at stage u, by Case 3.1 a^τ is also destroyed at stage u for each 0-node $\tau \subset \xi$ with $a^\tau_u \downarrow$. By (e), there is no $\tau \subseteq \delta$ with $\tau \in \mathcal{C}[u]$. By (f), there is no γ such that $\xi \subseteq \gamma \hat{} \langle (2, i(\gamma)) \rangle \subset \delta$ and r^γ is defined at stage u. Hence a^δ is destroyed at stage u if $\beta \supset \delta$ or $\beta^- \hat{} \langle (2, i(\beta^-)) \rangle \leq \delta$. Suppose $\delta \supseteq \theta$ for some $\theta \in \mathcal{A}(\beta^-)$ with $\beta <_L \theta$. By Case 3, there is a designated number for β at stage u. Then p^θ is defined at stage u, which contradicts (g). For Case 8 the argument is similar to that for Case 3.1 but simpler; we leave it to the reader.

xxiv) Without loss of generality it suffices to consider γ with $|\gamma|$ maximal such that

- $\gamma \in \mathcal{C}_v$, $\gamma \notin \mathcal{C}_{v+1}$
- $r^{\gamma^-}_{v+1} \downarrow$.

Since $\gamma \in \mathcal{C}_v$, $r^{\gamma^-}_v \downarrow$, and r^{γ^-} is not destroyed in stage v, the first two conditions in the definition of $\mathcal{C}_{v+1}$ for γ hold. Because $\gamma \notin \mathcal{C}_{v+1}$, there exists an (n, j)-node β such that

- $\gamma \subset \beta \wedge j < j(\gamma)$,
- *there exists* $y \in B_{v+1}$ *with* $\alpha(y) \supseteq \beta$ *which entered* B *after* r^{γ^-} *attained its value at stage* v,
- *for each 0-node* η *such that* $\gamma \subset \eta \subseteq \beta$ *and* $j(\eta) \leq j$, $\eta \notin \mathcal{C}_{v+1}$.

If y enters B at stage v, then desired conclusion is immediate. Suppose $y \in B_v$. Because $\gamma \in \mathcal{C}_v$, there exists a (least) 0-node η such that $\gamma \subset \eta \subseteq \beta$, $j(\eta) \leq j$ and $\eta \in \mathcal{C}_v$. Suppose r^{η^-} is not destroyed at stage v. Note that $\eta \notin \mathcal{C}_{v+1}$. This contradicts the maximality of $|\gamma|$. Therefore r^{η^-} is destroyed at stage v. By *iv)* and *xii)*, at stage v either c^{η^-} is destroyed or a^π is destroyed for some $\pi \in \mathcal{A}(\eta^-)$. Towards a contradiction assume that a^π is destroyed at stage v for some $\pi \in \mathcal{A}(\eta^-)$. By *xiii)* and *iv)*, a^γ and then r^{γ^-} are destroyed at stage v, a contradiction. Hence c^{η^-} is destroyed at stage v. By *ii)*, the destruction of c^{η^-} at stage v is caused by one of the following:

A) a node $\alpha <_L \eta^-$ receives attention;

B) c^ϵ is set equal to 1 for some ϵ with $\epsilon^\wedge\langle(2, i(\epsilon))\rangle \leq \eta^-$;

C) p^ϵ is destroyed for some $\epsilon \subseteq \eta^-$;

D) a^ϵ is destroyed for some $\epsilon <_L \eta^-$.

Consider stage v. Suppose A) holds. There are two cases:

Case 1. $\alpha <_L \gamma^-$. Since c^{γ^-} is not destroyed at stage v one of Cases 3.1, 8 holds and p^π is defined for the unique π such that $\pi \subset \gamma$ and $\pi^- = \alpha \cap \gamma$. But in this case, c^{η^-} also cannot become undefined by Case 3.1 and Case 8.

Case 2. Otherwise. Then $\alpha \supseteq \pi$ for some π such that $\pi \in \mathcal{B}(\gamma^-)$ and $\pi \leq \gamma$, which is the desired conclusion.

Suppose B) holds. By the construction we know that at stage v, k^ϵ enters B. Let α receive attention at stage v. Clearly, $\epsilon \subseteq \alpha$. If $\gamma \subseteq \alpha$, we are done. The only other case is $\alpha < \gamma$. Then $\epsilon^\wedge\langle(2, i(\epsilon))\rangle \leq \gamma$ and so $\epsilon^\wedge\langle(2, i(\epsilon))\rangle \leq \gamma^-$. In this case, c^{γ^-} and r^{γ^-} are destroyed at stage v, contradiction.

Suppose C) holds. Then either $\epsilon \subseteq \gamma^-$ or $\gamma \subset \epsilon$. Let $\epsilon \subseteq \gamma^-$. When p^ϵ becomes undefined, c^{γ^-} becomes undefined by ($E4$), a contradiction. Let $\gamma \subseteq \epsilon$. Since p^ϵ is destroyed, a^ϵ must have been destroyed in stage v. Hence a^γ is destroyed at stage v by $xiii$), which means that r^{γ^-} is also destroyed, contradiction.

Suppose D) holds. Clearly, $\epsilon <_L \eta$. So when a^ϵ is destroyed at stage v, a^η and then a^γ are destroyed by $xiii$). This yields the destruction of r^{γ^-} at stage v, contradiction. This completes the proof of (a). The argument for (b) is similar. We leave it to the reader.

xxv) Towards a contradiction consider a 0-node γ with $|\gamma|$ maximal such that

- $\gamma \in \mathcal{C}[v]$, $\gamma \notin \mathcal{C}_{v+1}$,
- $r^{\gamma^-}_{v+1} \downarrow$.

As in the proof of $xxiv$) there is a 0-node $\eta \supset \gamma$ such that $\eta \in \mathcal{C}[v]$ and $\eta \notin \mathcal{C}_{v+1}$. By the choice of γ we know that r^{η^-} is destroyed in stage v after $\mathcal{C}[v]$ is defined. By iv) either c^{η^-} or one of a^π ($\pi \in \mathcal{A}(\eta^-)$) is destroyed in stage v. But the destruction of a^π for ($\pi \in \mathcal{A}(\eta^-)$)

implies the destruction of a^γ by $xiii$). By iv), r^{γ^-} is also destroyed, a contradiction. Hence c^{η^-} is destroyed after $C[v]$ is defined. By Cases 1.1, 3.1, and 8, no c's can become undefined after $C[v]$ is defined in the main part of the construction. Therefore c^{η^-} is destroyed after the main part of the construction, i.e., it is destroyed by one of $(E1)$ and $(E4)$. Suppose c^{η^-} is destroyed by $(E1)$ because a^δ has been destroyed in the main part of the construction for some $\delta <_L \eta^-$. Whether $\delta <_L \gamma^-$ or not, r^{γ^-} is destroyed, a contradiction. Suppose c^{η^-} is destroyed by $(E4)$ because p^δ has been destroyed for some $\delta \subseteq \eta^-$. Note that a^δ is also destroyed since $p's$ can only be destroyed by $(E3)$. Whether $\delta \subseteq \gamma^-$ or not, r^{γ^-} is destroyed in stage v, a contradiction.

Towards a contradiction consider a 0-node γ with $|\gamma|$ maximal such that $\gamma \in C_{v+1}$ and $\gamma \notin C[v]$. Note that $r^{\gamma^-}_{v+1} \downarrow$ since $\gamma \in C_{v+1}$. As in the proof of $xxiv$) there is a 0-node $\eta \supset \gamma$ such that $\eta \in C_{v+1}$ and $\eta \notin C[v]$. This contradicts the maximality of $|\gamma|$. This completes the proof of xxv). ■

5.2 Lemma. *Let $t \leq s$. The following are true whenever the parameters mentioned are defined provided that c^ξ_v is also defined for each pair (ξ, v) such that k^ξ_v is mentioned:*

i) $\delta < \alpha \Longrightarrow k^\delta_t < k^\alpha_s$.

ii) $\delta <_L \alpha \vee \alpha \subset \delta \Longrightarrow a^\delta_t < a^\alpha_s$.

iii) $\alpha \leq \beta \vee \beta \subset \alpha \Longrightarrow k^\alpha_t < a^\beta_s$.

iv) $\delta <_L \alpha \Longrightarrow a^\delta_t < k^\alpha_s$.

v) $\alpha^\wedge\langle(2, i(\alpha))\rangle \leq \beta \Longrightarrow r^\alpha_t < k^\beta_s \vee (\exists\pi)[\pi \subseteq \beta \wedge p^\pi_t \downarrow = p^\pi_s \downarrow]$.

vi) $k^\alpha_t < r^\alpha_s$.

vii) Let ϵ be a 0-node and β be an i-node for $i \leq 1$. Then $\epsilon \subseteq \alpha(p^\epsilon)$, $\beta <_L \beta^{-\wedge}\langle(3, i(\beta))\rangle <_L \alpha(a^\beta)$, and $\alpha(a^\beta) \supset \delta$ where δ is the maximal 0-node $\subset \beta$ if any.

viii) $a^\gamma_t < p^\beta_s < a^\beta_s$ *for all β, γ such that $\gamma^- = \beta^-$ and $\gamma <_L \beta$.*

ix) Let $a^\alpha_s < a^\beta_s$ and a^α be destroyed at stage s. Then a^β is destroyed at stage s.

Proof. *i*) Suppose $\delta <_L \alpha$. When c^δ becomes defined at a stage v, δ receives attention and c^α becomes undefined by Case 4. Whether c^δ_v is destroyed or not, if k^α is defined subsequently, then by Case 4 we have $k^\delta_v < k^\alpha$. So $k^\delta_t < k^\alpha_s$ whenever c^δ_t, c^α_s are both defined. Now suppose $\delta \subset \alpha$. Whenever α is visited c^δ is defined by 5.1 *vi*). So by Case 4, when c^α becomes defined, we have $k^\delta < k^\alpha$. But, if c^δ is destroyed, so is c^α by 5.1 *iii*). Also, $k^\delta_t \leq k^\delta_s$. This is sufficient.

ii) When a^δ becomes defined at stage v, a^α becomes undefined by Case 2. Now if a^α becomes defined at a stage $> v$, then by Case 2 we have $a^\delta_{v+1} < a^\alpha$. This is sufficient.

iii) When c^α becomes defined at stage v, a^β becomes undefined. If a^β becomes defined subsequently, $k^\alpha_{v+1} < a^\beta$. This is enough.

iv) Let $\delta <_L \alpha$. When a^δ becomes defined at stage v, c^α becomes undefined. If c^α is defined subsequently, then by Case 4 we have $a^\delta_v < k^\alpha$ whether or not a^δ_v has been destroyed. So $a^\delta_t < k^\alpha_s$ whenever a^δ_t, k^α_s are both defined.

v) Let r^α be set equal to r^α_t at stage v. Note that whenever a node $<_L \beta$ receives attention c^β is destroyed unless p^π is defined for some $\pi \subseteq \beta$. There are two cases.

Case 1. $\alpha <_L \beta$. If c^β is destroyed at stage v, then whenever c^β becomes defined at stage $> v$, k^β is given a vlaue $> r^\alpha_{v+1} = r^\alpha_t$. Otherwise, we have $\pi \subseteq \beta$ such that p^π is defined when α is visited in stage v. Either $p^\pi_t \downarrow$, or p^π is destroyed at some stage $\geq v$ and $< t$. In the latter case c^β is destroyed in the same stage by (E3) and so we again have the desired conclusion.

Case 2. $\alpha \subset \beta$. Case 7 holds at α in stage v. If Case 8 also holds at α, then c^β is destroyed since $\alpha^\wedge\langle(2, i(\alpha))\rangle \leq \beta$, and we finish as in Case 1. Otherwise, Case 9 holds and a node $<_L \beta$ receives attention. Again we finish as in Case 1. This completes the proof of *v*).

vi) To see $k^\alpha_t < r^\alpha_s$, notice that, whenever α is visited at a stage v, $t \leq v \leq s$, and Case 7 holds, then c^α_v is defined. So by Case 7, when r^α becomes defined, we have $k^\alpha_v < r^\alpha_v$. But destruction of c^α_v implies the simultaneous destruction of r^α_v by (E5). Also, $k^\alpha_t \leq k^\alpha_v$. This is sufficient.

vii) $\epsilon \subseteq \alpha(p^\epsilon)$ is clear from Case 3. Also it is obvious that $\beta <_L \beta^{-\wedge}\langle(3, i(\beta))\rangle$ if $\beta \in \mathcal{B}(\beta^-)$.

Let $a^\beta\downarrow$. Towards a contradiction assume that $\beta^{-\wedge}\langle(3,i(\beta))\rangle \not<_L \alpha(a^\beta)$. Let γ denote $\alpha(a^\beta)$, and n be the current value of a^β. Note that γ is not a 1-node. Let t be the stage at which k^γ was set equal n and s be the stage at which a^β was set equal n. Then $t<s$ since n enters B after stage t and a^β can be set equal n only if n is already in B. We examine four cases:

Case 1. $\gamma\le\beta^-$. By i), $n=k^\gamma_t\le k^{\beta^-}_s$, and by iii), $n=a^\beta_s>k^{\beta^-}_s$, a contradiction.

Case 2. $\gamma\supset\beta^-$ and $\gamma<_L\beta$. By iii), $n=k^\gamma_t<a^\beta_s=n$, a contradiction.

Case 3. $\gamma\supseteq\beta$. Let δ be the greatest 0-node such that $\delta\subseteq\gamma$. If k^γ becomes defined at stage t, a^δ_t is defined. Since $\beta\subseteq\delta\subseteq\gamma$, by Case 4 of the construction, $n=k^\gamma_{t+1}<a^\delta_t$. But $a^\delta_t\le a^\beta_s=n$ by ii), which contradicts $n<a^\delta_t$.

Case 4. Otherwise. Then $\gamma\supseteq\zeta$ and $\beta<_L\zeta$, where either $\zeta\in\mathcal{A}(\beta^-)$ or $\zeta=\beta^{-\wedge}\langle(n,i(\beta))\rangle$ where $n\in\{2,3\}$. Let $v<s$ be the greatest number, if any, such that $a^\beta_v\downarrow$. By Case 2 of the construction, $a^\beta_{s+1}>k^\gamma_w$ for $w\le v$. Hence $v<t<s$, and at stage t, ζ is not visited by 5.1 $xvii$). Since ζ is not visited and γ is visited, at stage t there is a jump to some 0-node η, $\zeta\subseteq\eta$. By the condition for a jump, $a^\eta_t\downarrow$. By 5.1 $xviii$), $a^\beta_t\downarrow$. This contradicts the choice of v. This completes the proof of $\beta^{-\wedge}\langle(3,i(\beta))\rangle<_L\alpha(a^\beta)$.

Let δ be the maximal 0-node $\subset\beta$ if any. From Case 2 it is clear that $\delta\subset\alpha(a^\beta)$. This completes the proof of vii).

$viii$) Suppose that n is the greatest stage $<s$ at which p^β becomes defined. Then $a^\beta_n\downarrow$ and p^β is set equal k^δ_n for some $\delta\supseteq\beta$ by Case 3. Let $v<n$ be the stage in which k^δ is set equal to k^δ_n. Let ϵ be the greatest 0-node $\subseteq\delta$, which exists since $\delta\supseteq\beta$ and β is a 0-node. By Case 4, $a^\epsilon_v\downarrow$. By ii), $a^\epsilon_v\le a^\beta_n$. So $p^\beta_n=k^\delta_v<a^\epsilon_v\le a^\beta_n$ by iii).

Let $\gamma^-=\beta^-$ and $\gamma<_L\beta$. By vii), $\alpha(p^\beta)\supseteq\beta$. Notice that $p^\beta=k^{\alpha(p^\beta)}$. With the above notation, if $a^\gamma_v\downarrow=a^\gamma_n$, then by Case 4, $k^\delta_{v+1}>a^\gamma_v$. Towards a contradiction assume $a^\gamma_v\ne a^\gamma_n$. When a^γ gets the value a^γ_n, a^β and c^δ, if defined, are both destroyed which means that k^δ_v must already have entered B. This makes it impossible for p^β to be set equal k^δ_v since a^β_n is set after k^δ_v enters B. This is sufficient.

ix) By ii), $\alpha<_L\beta$ or $\beta\subset\alpha$. By 5.1 xii), it is clear. ∎

Before we show the next lemma, a class $\mathcal{D}$ of 0-nodes is defined which is related to the class $\mathcal{C}$: A 0-node $\xi\in\mathcal{D}$ at some point in stage s if it the following conditions:

i) $a^\xi\downarrow$,

ii) there exists $x \in B$ with $\alpha(x) \supseteq \xi$ which entered B since a^ξ was set,

iii) $\xi \notin \mathcal{C}$.

Remark. If $\xi \in \mathcal{D}$ at some point in stage s, then $a^\xi_s\downarrow$.

5.3 Lemma. *i) Let β be a 0-node which is visited at stage t and $a^\beta_t\downarrow$. Then for each γ such that $\gamma = \beta$ or γ is preferred to β, at some point during stage t before β is visited, the following hold:*

(a) r^{γ^-} is defined.

(b) p^ξ is defined for each $\xi \in \mathcal{A}(\gamma^-)$ with $\gamma <_L \xi$.

ii) There are no nodes τ, π and stages $t(\pi), s, s(\pi)$ and number $z(\pi)$ such that

(a) $\tau <_L \pi$,

(b) $t(\pi) < s < s(\pi)$,

(c) τ receives attention at stage s,

(d) p^π is set to $z(\pi)$ at stage $s(\pi)$ and $z(\pi)$ entered B at stage $t(\pi)$.

iii) Let α, i and s satisfy: $i \leq i(\alpha)$, $\alpha^\wedge\langle(6,i)\rangle \notin T$ and $(6,i)$ does not occur in α. If α is visited, but there is no jump to α at stage s, and $y \in B_s - C$ with $\alpha(y) <_L \alpha$ or $\alpha(y) \supset \alpha$, then $y \in A^{i,0}_s \cup A^{i,1}_s$.

iv) Let β be the maximal 0-node $\subseteq \alpha$. If α is visited at stage s, then $a^\beta_s\downarrow$ or $\beta = \alpha$ and a^β becomes defined at stage s.

v) Suppose that at some point in a stage, ξ, α satisfy

- *$\xi \subset \alpha$*
- *$\xi \in \mathcal{C}$*
- *there is no 0-node ϵ such that $\xi \subset \epsilon \subseteq \alpha$ and $\epsilon \in \mathcal{C}$*
- *there exists a number y with $\alpha(y) \supseteq \alpha$ which entered B since r^{ξ^-} was set.*

If α is an (n, j)-node, then $j(\xi) \leq j$.

vi) *Let δ be a 0-node and $a^\delta_s \downarrow$.*

(a) *Let u, $\delta \in \mathcal{D}_{u+1}$ and ξ satisfying*

- $\xi \subset \delta \in \mathcal{D}_{u+1}$, $u < s$;
- ξ is the maximal node $\subset \delta$ such that $\xi \in \mathcal{C}_{u+1}$;
- $c^{\xi^-}_s \downarrow$, $c^{\xi^-}_{s+1} \downarrow$, $a^\xi_u = a^\xi_s$;
- *there exists $y \in B_{u+1}$ with $\alpha(y) \supseteq \delta$ which entered B since $r^{\xi^-}_{u+1}$ was set;*
- a^ξ is destroyed at s.

Then a^δ is destroyed at stage s.

(b)
- $p^\delta_s \downarrow \wedge c^{\delta^-}_s \downarrow \Longrightarrow \delta \in \mathcal{C}_s$.
- $[a^\delta, c^{\delta^-}, p^\delta$ all defined when $\mathcal{C}[s]$ is defined $] \Longrightarrow \delta \in \mathcal{C}[s]$.

(c) *Let p^δ be defined at some point in stage s and $(\exists \pi \in \mathcal{A}(\delta^-))[\pi <_L \delta]$. Then p^δ is destroyed at stage s implies that a^π is destroyed for some $\pi \in \mathcal{A}(\delta^-)$ with $\pi <_L \delta$.*

(d) *Suppose there exists $y \in B_s$ with $\alpha(y) \supseteq \delta$ which entered B since a^δ was set equal a^δ_s, and some node $\supseteq \delta$ is visited at stage s. Let α be the first node $\supseteq \delta$ which is visited in stage s. Then p^α becomes defined in stage s, and $\alpha \neq \delta$ implies $j(\alpha) < j(\delta)$.*

(e) *Let z be designated for δ at some point in stage s. Then z is the first number to enter B since a^δ_s was set.*

(f) *Suppose that in stage s $\alpha \supseteq \delta$ receives attention, one of Cases 3.1 and 8 holds, and $\delta \in \mathcal{C}[s]$. Then $r^{\delta^-}_{s+1} \downarrow$.*

vii) *There are no stage w and node δ such that*

$$\delta \in \mathcal{D}_w \wedge [\delta \in \mathcal{C}_{w+1} \vee (\mathcal{C}[w] \downarrow \wedge \delta \in \mathcal{C}[w])].$$

viii) *At each stage s, one of the Cases 1-12 holds at the node which receives attention at stage s.*

ix) *If $c^\beta_s \downarrow \geq 1$, $r^\beta_s \downarrow$ and $r^\beta_{s+1} \uparrow$, then $c^\beta_{s+1} \uparrow$.*

x) Let δ, β and s satisfy that $a_s^\delta \downarrow$, $\delta \supset \beta$, $\delta <_L \beta^\wedge\langle(2, i(\beta))\rangle$, $r_s^\beta \downarrow$ and $c_s^\beta \downarrow \geq 1$. If $a_{s+1}^\delta \uparrow$, then $a_{s+1}^\xi \uparrow$ for each $\xi \in \mathcal{A}(\beta)$.

xi) Let $\delta \supset \beta^\wedge\langle(2, i(\beta))\rangle$. If a^δ is destroyed in stage s and $r^\beta \downarrow$ at some point in stage s, then $r_{s+1}^\beta \uparrow$.

xii) Let x be enumerated in C at stage s and α receive attention. Then $x = a_s^\delta$ or $x = k_s^\delta$. Further, if $x = k_s^\delta$ and is not of the form a_s^γ for any γ, then Case 11 holds at α and either $\delta = \alpha$ or $\delta \supseteq \beta$ for some $\beta \in \mathcal{A}(\alpha)$.

xiii)
- $(\forall \delta, \beta, t, s)[\delta \neq \beta \wedge a_t^\delta \downarrow \wedge a_s^\beta \downarrow \Longrightarrow a_t^\delta \neq a_s^\beta]$,
- If $a_s^\delta \downarrow$, then $a_s^\delta \notin C_s$.

Proof. i) Suppose γ is visited at stage t.

Case 1. r^{γ^-} becomes defined in stage t. Then Case 9 holds at γ^- and $\gamma = \max \mathcal{A}(\gamma^-)$.

Case 2. $r_t^{\gamma^-} \downarrow$. If we jump to $\delta \in \mathcal{A}(\gamma^-)$, then p^δ becomes defined and either this is the end of the stage or the construction passes $\epsilon = \max\{\pi \in \mathcal{A}(\gamma^-) : \pi <_L \delta\}$. By 5.1 xix), p^ϵ does not become defined and so no further node in $\mathcal{A}(\gamma^-)$ is visited. So γ is either δ or ϵ. Since p^δ becomes defined there exists x which is designated for δ. From the definition of designation, $p^\pi \downarrow$ for all $\pi > \delta$, $\pi \in \mathcal{A}(\gamma^-)$.

Suppose γ is not visited at stage t. Let θ be the first node $\supset \gamma$ which is visited at stage t. θ exists since $\gamma \subset \beta$ and β is visited at stage t. Note that at stage t we jump to θ. Note that we have $\theta = \beta$ or $\beta \supseteq \max\{\delta : \delta \in \mathcal{A}(\theta^-) \wedge \delta <_L \theta\}$. By Lemma 4.5, γ is preferred to θ if $\beta \neq \theta$. Hence in either case, γ is preferred to θ. By Case 1 in the construction, a) and b) hold when the construction jumps to θ.

ii) Towards a contradiction assume ii) fails. Let τ, π, $t(\pi)$, s, $s(\pi)$ be chosen to satisfy $(a) - (d)$ and to minimize first $|\tau \cap \pi|$ and then s. Let ζ, θ be the least nodes respectively such that $\tau \cap \pi \subset \theta \subseteq \tau$ and $\tau \cap \pi \subset \zeta \subseteq \pi$.

Claim 1.

a) $\zeta = \pi^-$.

b) θ is not visited at stage s.

Proof of the Claim 1: a) Towards a contradiction assume that a) fails. Then $\tau <_L \pi^-$. At stage s, τ receives attention, so c^{π^-} is destroyed unless p^ζ is defined at stage s by 5.1 $ii(a)$). Suppose p^ζ is defined at stage s. At stage $s(\pi)$, π is visited. By 5.1 xv), p^ζ is destroyed between stages s and $s(\pi)$ and c^{π^-} is destroyed at the same time. But by 5.1 vi), $c^{\pi^-}_{s(\pi)}\downarrow$. So at some stage u, $s < u < s(\pi)$, c^{π^-} becomes defined and at the same stage a^π becomes undefined by Case 4. At stage $s(\pi)$, p^π is set equal to $z(\pi)$. But by Case 3, $z(\pi)$ is designated for π at stage $s(\pi)$ which means that $a^\pi_{s(\pi)}$ and $a^\pi_{t(\pi)}$ are both defined and equal. This contradicts a^π being destroyed at stage u. This is enough.

b) Towards a contradiction assume that b) fails. By a), $\pi^- = \tau \cap \pi$. Because in stage s, θ is visited. By i), p^π is defined at some point in stage s before θ is visited. So p^π becomes undefined at some stage u, $s \leq u < s(\pi)$, and at the same stage a^π becomes undefined since p^π only becomes undefined through (E3). This contradicts a^π remaining unchanged between stages $t(\pi)$ and $s(\pi)$. This completes the proof of the Claim 1.

Let ϵ be the first node $\supseteq \theta$ which is visited at stage s. By Claim 1 b), $\epsilon \neq \theta$ and there is a jump from some node $\subseteq \tau \cap \pi$ to ϵ, and then p^ϵ becomes defined when ϵ is visited at stage s. Let p^ϵ be set equal to $z(\epsilon)$ at stage s which entered B at stage $t(\epsilon) < s$. By the minimality of s, $t(\epsilon) \leq t(\pi)$. Clearly $t(\epsilon) \neq t(\pi)$.

Let v be the least stage such that $v > t(\epsilon)$ and at which some node $\supseteq \sigma$ is visited for some $\sigma \in \mathcal{A}(\zeta^-)$ and $\zeta \leq \sigma$. Note that v exists and $\leq t(\pi)$. Fix such node σ.

Claim 2. At stage v, there is no jump from a node $\subseteq \tau \cap \pi$ to a node $\supset \tau \cap \pi$.

Proof of the Claim 2: Towards a contradiction assume that at stage v there is a jump from some node $\subseteq \tau \cap \pi$ to some node $\supset \tau \cap \pi$. Let η be the first node $\supset \tau \cap \pi$ which is visited at stage v. Then η is a 0-node and p^η becomes defined at stage v. Note that $\sigma \leq \eta$ otherwise nodes $\supseteq \sigma$ cannot be visited at stage v. Let the number which is set equal to p^η at stage v have entered B at stage $t(\eta) < v$. Clearly $t(\eta) \neq t(\epsilon)$. But $t(\eta) < t(\epsilon)$ contradicts the minimality of s since $t(\eta) < t(\epsilon) < v\ (= s(\eta))$. $t(\epsilon) < t(\eta)$ contradicts the minimality of v. This completes the proof of the Claim 2.

Now let t be the maximal stage $\leq t(\pi)$ at which there is no jump from a node $\subseteq \tau \cap \pi$ to a node $\supset \tau \cap \pi$ and some node σ is visited at stage t with $\sigma \supseteq \delta \in \mathcal{A}(\tau \cap \pi)$ and $\pi \leq \sigma$ for some δ.

Claim 3. No node $\supseteq \theta$ is visited at a stage w such that $t \leq w < s$.

Proof of the Claim 3: Towards a contradiction assume that there exist a stage w and a (least) node β such that $t \leq w < s$, $\tau \cap \pi \subset \beta$, $\beta <_L \pi$ and β is visited at stage w. By the minimality of s, $w \leq t(\pi)$. Clearly $w \neq t(\pi)$. Let u be the least stage $> w$ at which some (first) node η is visited with $\tau \cap \pi \subset \eta$ and $\pi \leq \eta$. Note that u exists and $t \leq w < u \leq t(\pi)$. Then by the choice of t, at stage u, there is a jump to η. Let p^{η} be set equal to $z(\eta)$ at stage u which entered B at stage $t(\eta) < u$. By the choice of u, $t(\eta) \leq w$. Clearly $t(\eta) \neq w$. Hence $t(\eta) < w < u$ $(= s(\eta))$, contradicts the minimality of s and yields the conclusion of the Claim 3.

Below we show that Case 1 in the construction holds at $\tau \cap \pi$ at stage t. To see that Case 1 holds at stage t at $\tau \cap \pi$ reading $j(\epsilon)$ for i, $z(\epsilon)$ for x, and ϵ for β we need to show that for each 0-node γ which is preferred to ϵ, the following hold:

A) r^{γ^-} is defined at some point in stage t when $\tau \cap \pi$ is visited,

B) p^{σ} is defined at some point in stage t when $\tau \cap \pi$ is visited for each node $\sigma \in \mathcal{A}(\gamma^-)$ such that $\gamma <_L \sigma$.

By Claim 3, for each 0-node γ such that $\tau \cap \pi \subset \gamma \subset \epsilon$ and γ is preferred to ϵ, the following hold:

- $r_t^{\gamma^-} \downarrow$,
- $p_t^{\sigma} \downarrow$ *for each node* $\sigma \in \mathcal{A}(\gamma^-)$ *such that* $\gamma <_L \sigma$.

Fix a 0-node $\gamma \subseteq \tau \cap \pi$ which is preferred to ϵ. If γ is visited at stage t, then A) and B) hold by i). Suppose γ is not visited at stage t, then because $\tau \cap \pi$ and δ are visited at stage t there exists a 0-node η such that $\gamma \subset \eta$, $\eta^- \subset \tau \cap \pi$, $\eta = (\mu\beta)[\beta \in \mathcal{A}(\eta^-) \wedge \tau \cap \pi <_L \beta]$, and we jump from some node $\subset \gamma$ to η. By Lemma 4.5, γ is preferred to η, thus $A), B)$ hold by i).

Since δ is not a 6-node, it is not visited at stage t, contradicts the choice of t. This completes the proof of ii).

iii) Towards a contradiction assume there are i and y which witness that iii) fails. Let y enter B at stage u. Let ϵ receive attention at stage u. Note that ϵ exists, and either $\epsilon = \alpha(y)$

or $\epsilon^- = \alpha(y)$. Let γ be the $(n, i-1)$-node $\subseteq \alpha$ for $n \in \{2,3\}$. Note that $\gamma \neq \alpha$. Since $(6,i)$ does not occur on α, $\alpha <_L \gamma^\wedge\langle(6,i)\rangle$. Then there exists a jump from some node $\subseteq \gamma$ to some node $\pi >_L \alpha$ and $\pi \supset \gamma$ at stage s, because α is visited at stage s. Note that $\alpha(y) <_L \pi$. Let v be the least stage $> u$ at which some node $>_L \alpha(y)$ and $\supset \gamma$ is visited. Then $v \leq s$ and there exists a jump from some node $\subseteq \gamma$ to τ such that $\gamma \subset \tau$ and $\alpha(y) <_L \tau$. Hence p^τ set equal to $z(\tau)$ at stage v. Let $z(\tau)$ enter B at stage $t(\tau)$. Then by the choice of v, $t(\tau) < u$. Taking $\pi = \tau$, $t(\pi) = t(\tau)$, $s(\pi) = v$, $\tau = \epsilon$ and $s = u$ we have a contradiction with ii).

iv) First we show that, if $\alpha = \beta$ and $a_s^\beta \uparrow$, then a^β becomes defined at stage s. Suppose $a_s^\beta \uparrow$. Then by the construction the only possibility of visiting β at stage s is that Case 5 holds for β^- and we pass to β. Towards a contradiction assume that Case 1 in the construction holds at β at stage s. Choose the least i and then the x such that they witness that Case 1 holds at β at stage s. By iii), $x \in A_s^{i,o} \cup A_s^{i,1}$ since β is a 0-node. Suppose x is a δ-designated number for some $\delta \supset \beta$ with $i = j(\delta)$. Then Case 1 holds on β^- at stage s. However, a jump from β^- to β is impossible since $a_s^\beta \uparrow$. This is a contradiction. By Case 2, a^β becomes defined.

Let $\beta \subset \alpha$. Consider stage s. Suppose β is visited. Then $a_s^\beta \downarrow$ is defined by the previous case. Now suppose β is not visited. There is a jump from some node $\subset \beta$ to $\eta \supset \beta$. Since α is visited either $\eta = \alpha$ or $\alpha <_L \eta$. In either case there is a 0-node γ such that $\beta \subset \gamma \subseteq \alpha$, a contradiction.

v) Towards a contradiction assume that $j < j(\xi)$. Because at some point in a stage $\xi \in \mathcal{C}$ and there exists a number y with $\alpha(y) \supseteq \alpha$ which entered B after r^{ξ^-} was set, there exists an η such that $\xi \subset \eta \subseteq \beta$ and $\eta \in \mathcal{C}$ at this moment, a contradiction with the choice of ξ. Hence $j(\xi) \leq j$.

vi) The proof is by induction first on s and then on $|\delta|$ and finally on $|\xi|$.

For a) we apply $xxiii$) of 5.1. So our task is to verify that the hypotheses $(a) - (h)$ of $xxiii$) are all satisfied. Let β receive attention at stage s. Now $(a) - (c)$ and (h) are immediate from our present hypotheses.

Note that $j(\xi) \leq j$ for each (n,j)-node with $\xi \subset \alpha \subseteq \delta$ by v).

Claim 1. Let t be the last stage $\leq u$ at which some y entered B with $\alpha(y) \supseteq \delta$. Then for all v, $t < v \leq s$, if τ is visited at a stage v, it is impossible that $\xi \subset \tau$ and either $\tau \subseteq \delta$ or $\delta <_L \tau$.

Proof of Claim 1: We proceed by induction on v.

Consider the first node τ visited at stage v such that $\xi \subset \tau$ and either $\tau \subseteq \delta$ or $\delta <_L \tau$. Suppose ξ is visited at stage v. By (d) of the induction hypothesis, p^ξ becomes defined at stage v. In stage v, the construction either stops at ξ or passes from ξ to a node $\zeta <_L \xi$. Hence τ is not visited, contradiction. So ξ is not visited at stage v. Thus there is a jump to τ and p^τ becomes defined at stage v. Suppose $\delta <_L \tau$. Let p^τ be set equal to $z(\tau)$ at stage v. Then $z(\tau)$ entered B at a stage $t(\tau) < v$. Clearly, $t(\tau) \neq t$. If $t(\tau) > t$, then the hypothesis of induction on v implies that in stage $t(\tau)$ no node $\supseteq \tau$ is visited. This contradicts $\alpha(z(\tau)) \supseteq \tau$. Therefore $t(\tau) < t$. Now *ii*) yields a contradiction. We take τ for π, $t(\tau)$ for $t(\pi)$, $z(\tau)$ for $z(\pi)$, t for s, the node $\supseteq \eta$ receives attention in stage t for τ, and v for $s(\pi)$. We conclude that $\delta <_L \tau$ is impossible.

Suppose $\xi \subset \tau \subseteq \delta$. By (d) of the induction hypothesis, $j(\tau) < j(\xi)$. By *v*), $j(\xi) \leq j(\tau)$, a contradiction. This completes the proof of Claim 1.

In Claim 1 taking $v = s$ we see that $\xi \subset \beta$ implies $\beta \not\subseteq \delta$ and $\delta \not<_L \beta$. Hence $\xi \subset \beta$ implies $\beta <_L \delta$ or $\delta \subset \beta$. If $\xi = \beta$, then by the induction hypothesis, taking ξ for δ, we have from (d) that p^ξ becomes defined in stage s. Since a^ξ is destroyed in stage s, so is c^ξ, contradiction. Hence hypothesis (d) of *xxiii*) of 5.1 holds. To check that the hypotheses (f) and (g) of *xxiii*) hold we argue as follows.

At stage t, a node $\supseteq \delta$ receives attention. Applying *xv*) of 5.1 at stage t we see that r^σ is undefined throughout stage t for each σ such that $\sigma^\wedge\langle(2, i(\sigma))\rangle \subseteq \delta$. By Claim 1, σ is not visited at any stage $> t$ and $\leq s$. Hence (f) of *xxiii*) of 5.1 holds. Similarly we see that (g) of *xxiii*) of 5.1 holds.

It only remains to establish that hypotheses (e) of *xxiii*) of 5.1 holds. Towards this goal we prove:

Claim 2. Let ξ', η, v satisfy:

- $\xi' \subseteq \xi \subset \eta$, $\xi' \in \mathcal{C}_v$, $\eta \in \mathcal{D}_v$, $u < v \leq s$

- *for each (n,j)-node ϵ with $\xi' \subset \epsilon \subseteq \eta$, $j(\xi') \leq j$*

Then $\eta \notin \mathcal{C}_{v+1}$ if $v < s$ and $\eta \notin \mathcal{C}[v]$ if $\mathcal{C}[v]$ is defined.

Proof of Claim 2: Towards a contradiction consider the least v and then the maximal $|\eta|$ for which the Claim 2 fails, i.e., $\eta \in \mathcal{D}_v \cap (\mathcal{C}_{v+1} \cup \mathcal{C}[v])$.

Subclaim 1. At stage v, there is no τ which is visited with $\xi' \subset \tau \subseteq \eta$.

Proof of Subclaim 1: Consider the first node τ visited at stage v such that $\xi' \subset \tau \subseteq \eta$. Suppose ξ' is visited at stage v. Applying the induction hypothesis with ξ' for δ, by (d) we see that $p^{\xi'}$ becomes defined at stage v. In stage v, the construction either stops at ξ' or passes from ξ' to a node $<_L \xi'$. Hence τ is not visited, contradiction. So ξ' is not visited at stage v. Let $\zeta \supset \xi'$ be the first node visited at stage v. Thus there is a jump to ζ and p^{ζ} becomes defined at stage v. Applying the induction hypothesis in the same way as before, $j(\zeta) < j(\xi')$. Note that either $\tau = \zeta$, or $\tau^- = \zeta^-$ and $\tau <_L \zeta$. In either case $j(\tau) \leq j(\zeta) < j(\xi')$ which contradicts the assumption of Claim 2. This completes the proof of Subclaim 1.

Below we reduce a contradiction in the case $\mathcal{C}[v]$ exists and $\eta \in \mathcal{C}[v]$. For $\eta \in \mathcal{C}_{v+1}$ $(v < s)$ is similar. We leave it to the reader. Because $\eta \in \mathcal{C}[v]$, there exists a η-designated number when $\mathcal{C}[v]$ is defined. Now we split into two cases.

Case 1. A number x designated for η enters B at stage v. In this case we will deduce a contradiction. Let τ be the first node $\supseteq \xi'$ which is visited at stage v. Applying the induction hypothesis exactly as above we see that p^{τ} becomes defined at stage v and that $\tau = \xi'$ implies that $j(\tau) < j(\xi')$. If $\tau = \xi'$, this implies that no node $\supseteq \eta$ is visited in stage v, contradiction. Hence $\xi' \subset \tau$ and $j(\tau) < j(\xi')$. From Subclaim 1, $\tau \not\subseteq \eta$.

To deduce a contradiction assume that $\eta <_L \tau$. Then $\tau^- \subset \eta$. Let θ be the least node such that $\tau^- \subset \theta \subseteq \eta$. Then θ is a 0-node and $j(\theta) < j(\tau) < j(\xi')$. This contradicts the choice of ξ' and η. Hence $\eta \subset \tau$, and p^{τ} becomes defined at stage v. Let p^{τ} be set equal to $z(\tau)$ at stage v which entered B at stage $t(\tau)$. There are two subcases:

Subcase 1. There is no (n,k)-node ϵ such that $\eta \subset \epsilon \subset \tau$ and $k < j(\tau)$. Clearly $j(\tau)$ is active at η^-. Whether $\tau = (\mu\sigma)[\sigma \in \mathcal{A}(\tau^-)]$ or not, x cannot be an η-designated number because the condition *iii*) in 4.3 fails.

Subcase 2. Otherwise. Let ϵ be an (n,k)-node such that $\eta \subset \epsilon \subset \tau$ and k is least possible. Clearly $n \in \{0,4,5\}$ and $k < j(\tau) < j(\xi')$. Let $n = 0$. From the choice of ϵ, any $j < j(\epsilon)$ which is active at ϵ^- is also active at τ^-. From 4.4 one can easily check that ϵ is preferred to τ. Since in stage v the construction jumps from a node $\subset \xi'$ to τ, $\xi' \subset \epsilon \subset \tau$, and ϵ is preferred to τ we have:

- $r_v^{\epsilon^-} \downarrow$
- $p_v^{\sigma} \downarrow$ *for each* $\sigma \in \mathcal{A}(\epsilon^-)$ *with* $\epsilon <_L \sigma$.

Let w be the least stage such that

- $r_{w+1}^{\epsilon^-} \downarrow = r_v^{\epsilon^-}$
- $p_{w+1}^{\sigma} \downarrow = p_v^{\sigma}$ *for each* $\sigma \in \mathcal{A}(\epsilon^-)$ *with* $\epsilon <_L \sigma$.

By 5.1 *xix*), ϵ is visited at stage w. Note that p^ϵ cannot become defined at any stage $\geq w$ and $\leq v$. Otherwise τ cannot be visited at stage v. Now by 5.1 *xxi*), either a^ϵ is destroyed at stage w or some y enters B at stage w with $\alpha(y) \supseteq \pi$. By choice of w, a^ϵ cannot become undefined between stages w and v. Thus y enters B at stage w with $\alpha(y) \supseteq \epsilon$. Suppose y is an ϵ-designated number at stage w. Then y is an ϵ-designated number at stage v and at stage v, the construction jumps to ϵ rather than τ since if γ is preferred to ϵ, then γ is preferred to τ. Therefore y is not ϵ-designated at stage w. Since y enters B either Case 3.1 or Case 8 occurs at the node which receives attention in stage w. Since a^ϵ is not destroyed in stage w there is a maximal 0-node $\zeta \subset \epsilon$ such that $\zeta \in \mathcal{C}$ when μ receives attention. By v), $j(\zeta) \leq j(\epsilon)$. Hence $\zeta \subset \xi'$. Note that $j(\zeta) < j(\tau)$. By (d) of the induction hypothesis, τ cannot be visited at stage v unless and until a^ζ is destroyed between stages w and v, say at m. Towards a contradiction assume that one of c_m^ζ, c_{m+1}^ζ is undefined. Then by 5.1 *vi*), one of c_m^ξ, c_{m+1}^ξ is undefined. Since $c_s^\xi \downarrow$, c^ξ becoems defined at a stage z, $m \leq z < s$. Thus by Case 4, a^ξ, a^ϵ are destroyed at stage z. Whether $z < v$ or $v \leq z$, this is a contradiction. Hence both c_m^ζ, c_{m+1}^ζ are defined. Now applying (a) of the induction hypothesis by taking $s = m$, $u = w$, $\xi = \zeta$ and $\delta = \epsilon$, a^ϵ is destroyed at stage m, which a contradicts the choice of w. This completes the case $n = 0$.

Let $n \in \{4,5\}$. Clearly x cannot be an η-designated number because the condition *iii*) in 4.3 fails.

Case 2. Otherwise. Just before stage v there exists $x \in B$ which is designated for η. $r_v^{\eta^-} \downarrow$ because $r^{\eta^-} \downarrow$ in stage v and η^- is not visited at stage v by Subclaim 1. Because $\eta \in \mathcal{D}_v$ and the first two conditions in the definition of $\mathcal{C}_v$ hold for η, there exists an (m,k)-node γ such that

- $\eta \subset \gamma \wedge k < j(\eta)$
- *there exists $z \in B_v$ with $\alpha(z) \supseteq \gamma$ which entered B after r^{η^-} was set equal $r_v^{\eta^-}$*
- *for each 0-node τ such that $\eta \subset \tau \subseteq \gamma$ and $j(\tau) \leq k$, $\tau \notin \mathcal{C}_v$.*

Because $\eta \in \mathcal{C}[v]$, there exists a (least) 0-node τ such that $\eta \subset \tau \subseteq \gamma$, $j(\tau) \leq k$ and $\tau \in \mathcal{C}[v]$. $\tau \in \mathcal{D}_v$ because $a_v^\tau \downarrow$, $\tau \notin \mathcal{C}_v$ and z with $\alpha(z) \supseteq \gamma$ entered B after r^{η^-} attained its value at stage $v-1$ and hence also after a^τ attained its value at stage $v-1$.

To complete the proof of Claim 2 we will show that τ satisfies all the assumptions about η. Since $\eta \subset \tau$ and $\tau \in \mathcal{C}[v]$, the choice of η is thereby contradicted.

Subclaim 2. For each (n,j)-node α such that $\xi' \subset \alpha \subseteq \tau$, $j(\xi') \leq j$.

Proof of the Subclaim 2: If $\xi' \subset \alpha \subseteq \eta$, then desired conclusion is immediate from the hypothesis of Claim 2. Let $\eta \subset \alpha \subset \tau$. By choice of γ, there exists a number z with $\alpha(z) \supseteq \gamma \supset \alpha \supset \eta$ which entered B after r^{η^-} attained its value at stage $v-1$. Note that $\eta \in \mathcal{C}[v]$ and by the choice of τ, there is no ϵ such that $\eta \subset \epsilon \subset \tau$ and $\epsilon \in \mathcal{C}[v]$. Hence by v), $j(\eta) \leq j$. It follows by the hypothesis of Claim 2 that $j(\xi') \leq j(\eta) \leq j$.

Next we want to show that $j(\xi') \leq j(\tau)$. First we show that:

Subclaim 3. For each 0-node β and w, if $\beta \in \mathcal{D}_w$ and $\beta \in \mathcal{C}[w]$, then there exists x with $\alpha(x) \supseteq \beta$ which enters B at stage w.

Proof of the Subclaim 3: Fix w we proceed it by reverse induction on $|\beta|$. Suppose a β-designated number enters B at stage w. It is enough for Subclaim 3. Otherwise. Let y be a β-designated number at stage w. Hence $y \in B_w$, and then $r_w^{\beta^-}$ is equal the value of r^{β^-} in stage w. Because $\beta \in \mathcal{D}_w$ and the first two conditions in the definition of $\mathcal{C}$ hold for β just before stage w, there exists an (n,j)-node α such that

- $\beta \subset \alpha \wedge j < j(\beta)$

- *there exists $z \in B_w$ with $\alpha(z) \supseteq \alpha$ which entered B after r^{β^-} attained its value at stage $w-1$*
- *for each 0-node π such that $\beta \subset \pi \subseteq \alpha$ and $j(\pi) \leq j$, $\pi \notin \mathcal{C}_w$.*

Because $\beta \in \mathcal{C}[w]$, there exists a 0-node π such that $\beta \subset \pi \subseteq \alpha$ and $\pi \in \mathcal{C}[w]$. Then by the induction hypothesis, there exists x with $\alpha(x) \supseteq \pi \supset \beta$ which enters B at stage w. This completes the proof of Subclaim 3.

Remark. By Subclaim 3, we know that, there is x with $\alpha(x) \supseteq \tau$ which enters B at stage v.

Now we return to the proof of Subclaim 2 and, in particular, to showing that $j(\xi') \leq j(\tau)$. Towards a contradiction assume that $j(\tau) < j(\xi')$. Let t be the stage in which $r^{\xi'^-}$ was set equal to $r_v^{\xi'^-}$. Suppose there exists $y \in B_v - B_t$ such that $\alpha(y) \supseteq \tau$. Because $\xi' \in \mathcal{C}_v$, there exists a 0-node γ such that $\xi' \subset \gamma \subseteq \tau$, $j(\gamma) \leq j(\tau)$ and $\gamma \in \mathcal{C}_v$. Note that $\gamma \neq \tau$ since $\tau \notin \mathcal{C}_v$. Since Subclaim 2 has already been proved for $\alpha \neq \tau$, $j(\gamma) \leq j(\tau) < j(\xi') \leq j(\gamma)$, a contradiction. Hence no such y exists. Since $\tau \in \mathcal{D}_v$, there exists $t_0 < v$ such that at stage t_0 a number z enters B with $\alpha(z) \supseteq \tau$ and $a_{t_0}^{\tau} = a_v^{\tau} \downarrow$. Fix such t_0 and z with t_0 least possible. From above $t_0 < t$. We will complete the proof of Subclaim 2 by showing $r^{\xi'^-}$ cannot become defined in stage t thus contradicting the choice of t. There are two cases.

Case 1. $\tau \notin \mathcal{C}[t_0]$. Since a^{τ} is not destroyed at stage t_0 by whichever of Case 3.1 and Case 8 enumerates z in B, there exists a maximal $\sigma \subset \tau$ such that $\sigma \in \mathcal{C}[t_0]$. Note that by $xv)$ of 5.1 there is no π such that r^{π} is defined at some point in stage t_0 and $\pi^\wedge\langle(2, i(\pi))\rangle \subset \tau$. By $v)$, $j(\sigma) \leq j(\theta)$ for each 0-node θ such that $\sigma \subset \theta \subseteq \tau$. In particular, taking $\theta = \tau$ we have $j(\sigma) \leq j(\tau)$, which gives $j(\sigma) < j(\xi')$. Since $j(\xi') \leq j(\epsilon)$ for each 0-node ϵ with $\xi' \subseteq \epsilon \subset \tau$, we have $\sigma \subset \xi'$. By inspection of Cases 3.1 and 8, c^{σ^-} which is defined since $\sigma \in \mathcal{C}[t_0]$ remains defined throughout the main part of stage t_0. Further, at the end of the stage c^{σ^-} is not destroyed by $(E1)$ since no a^{ϵ} is destroyed with $\epsilon <_L \sigma^-$. Also, by $xv)$ of 5.1, c^{σ^-} is not destroyed by $(E4)$. We conclude that $c_{t_0+1}^{\sigma^-} \downarrow$. Since $\sigma \in \mathcal{C}[t_0]$, r^{σ^-} and a^{σ} are defined just after z enters B and are not destroyed in the main part of stage t_0. By inspection of $(E1)$ a^{σ} is not destroyed at the end of stage t_0. Towards a contradiction suppose r^{σ^-} is destroyed in the ending of stage t_0. This must be by $(E5)$. Hence a^{ϵ} has been destroyed for some $\epsilon \in \mathcal{B}(\sigma^-)$. By *xii*) and *xiii*) of 5.1 we can choose such $\epsilon \in \mathcal{A}(\sigma^-)$ with $\sigma <_L \epsilon$. Fix the least possible ϵ. Since $\sigma \in \mathcal{C}[t_0]$, $p^\epsilon \downarrow$. By $(E3)$, $p^\epsilon_{t_0+1} \uparrow$. By (c) of the induction hypothesis, a^π is destroyed in stage t_0 for some $\pi \in \mathcal{A}(\sigma^-)$ with $\pi <_L \epsilon$. By the choice of ϵ, $\pi \leq_L \sigma$. Finally, by *xiii*) of 5.1 this implies that a^σ is destroyed in stage t_0, contradiction. Hence $r^{\sigma^-}_{t_0+1}$ is defined. By *xxv*) of 5.1, $\sigma \in \mathcal{C}_{t_0+1}$.

Suppose $a^\sigma_{t_0} = a^\sigma_t$. Let α be the first node $\supseteq \sigma$ which is visited in stage t. α exists since ξ'^- is visited in stage t. By the induction hypothesis, taking σ for δ and t for s, we have from (d) that p^α becomes defined in stage t, and $\alpha \neq \sigma$ implies $j(\alpha) < j(\sigma)$. Since ξ'^- is visited in stage t, either $\alpha \subseteq \xi'^-$ or $\xi' <_L \alpha$. Suppose $\alpha \subseteq \xi'^-$. By *xv*) of 5.1, $r^{\xi'^-}$ cannot become defined in stage t. Suppose $\xi' <_L \alpha$. Then there exists a 0-node θ such that $\sigma \subset \theta \subset \xi'$, $\theta^- = \alpha^-$ and $\theta <_L \alpha$. Then $j(\sigma) \leq j(\theta) < j(\alpha) < j(\sigma)$, a contradiction. Hence $a^\sigma_{t_0} \neq a^\sigma_t$. Let h be the least stage such that $t_0 < h < t$ and a^σ is destroyed in stage h. Since a^τ is not destroyed in stage h, by (a) of the induction hypothesis one of c^σ_h, c^σ_{h+1} is undefined. Since $\sigma \subset \xi' \subseteq \xi$, by *iii*) of 5.1, one of c^ξ_h, c^ξ_{h+1} is undefined. Hence c^ξ becomes defined in a stage k with $h \leq k < s$. Note that $\xi \subseteq \tau$. By Case 4 when c^ξ becomes defined, a^τ, a^ξ become undefined, if defined. Hence $k < v$ contradicts $a^\tau_{t_0} = a^\tau_v$. But $v \leq k$ implies $u < k < s$ which contradicts $a^\xi_u = a^\xi_s$.

Case 2. $\tau \in \mathcal{C}[t_0]$. By the argument in case 1 which shows that $\sigma \in \mathcal{C}_{t_0+1}$, we see that $\tau \in \mathcal{C}_{t_0+1}$. Let β be the maximal 0-node which is preferred to τ. By Remark 1 after 4.4, $\beta \subset \xi'$ since Subclaim 2 has already been proved for $\alpha \neq \tau$. Note that $a^\tau_{t_0} = a^\tau_v \downarrow$. Suppose p^τ becomes defined between stages t_0 and t. Since p^τ is destroyed implies a^τ is destroyed simultaneously, $p^\tau_v \downarrow$. Hence by *xv*) of 5.1, no node $\supseteq \tau$ is visited in stage v. This contradicts Subclaim 3. Hence $p^\tau_t \uparrow$.

Subclaim 4. $r^{\tau^-}_v = r^{\tau^-}_{t_0+1}$.

Proof of Subclaim 4: Towards a contradiction consider the least stage w such that $t_0 < w < v$ and r^{τ^-} is destroyed in stage w. By *iv*) of 5.1, either c^{τ^-} is destroyed or a^π is destroyed at stage w for some $\pi \in \mathcal{A}(\tau^-)$. Suppose $c^{\tau^-}_{w+1} \neq c^{\tau^-}_{t_0} \downarrow$. By Subclaim 3, some x enters B at stage v with $\alpha(x) \supseteq \tau$. By *vi*) and *vii*) of 5.1, c^{τ^-} becomes defined in a stage $> w$ and $< v$. But when c^{τ^-} becomes defined, a^τ is destroyed. This contradicts $a^\tau_{t_0} = a^\tau_v$. Hence $c^{\tau^-}_{w+1} \downarrow = c^{\tau^-}_{t_0}$ and some a^π ($\pi \in \mathcal{A}(\tau^-)$) is destroyed in stage w. Note that $\tau \in \mathcal{C}_{t_0+1}$ so

$p^{\epsilon}_{t_0+1}\downarrow$ for each $\epsilon \in \mathcal{A}(\tau^-)$ with $\tau <_L \epsilon$. Hence by *xiii*) of 5.1 and (*c*) of the induction hypothesis by taking w for s, a^τ is destroyed at stage w. This is a contradiction and yields the conclusion of Subclaim 4.

Subclaim 5. Case 1 holds at ξ'^- in stage t.

Proof of Subclaim 5: Let y be a designated number for τ at stage t. y exists since $\tau \in C_{t_0+1}$ and $r^{\tau^-}_t = r^{\tau^-}_{t_0+1}$. Write α for ξ'^-. We will show that Case 1 holds at α reading y for x, τ for β, t for s and $j(\tau)$ for i. First, $j(\tau) < j(\xi') \leq i(\xi'^-) = i(\alpha)$. Suppose $c^\alpha_t\uparrow$. By *vi*) of 5.1, $c^{\xi^-}_t\uparrow$. Hence c^{ξ^-} becomes defined at a stage w with $t < w < s$. But $w < v$ contradicts $a^\tau_{t_0+1} = a^\tau_v\downarrow$, and $v \leq w < s$ contradicts that $a^\xi_u = a^\xi_s\downarrow$. Therefore $c^\alpha_t\downarrow$. Clearly $\alpha(y) \supseteq \tau \supset \alpha$. Since $j(\tau) \leq i(\alpha^-)$, Case 1 holds at α if $y \notin A^{j(\tau),0}_t \cup A^{j(\tau),1}_t$.

We now suppose $y \in A^{j(\tau),0}_t \cup A^{j(\tau),1}_t$. Fix a 0-node γ such that either $\gamma \subseteq \tau$ or $\tau >_L \gamma \in \mathcal{A}(\tau^-)$. Towards a contradiction assume p^γ is defined when α is visited. Suppose $\gamma \subseteq \alpha$. By *xv*) of 5.1, $p^\gamma_t\uparrow$ since α is visited at stage t. If p^γ becomes defined in stage t, then r^α cannot become defined in stage t. Suppose $\xi' \subseteq \gamma \subset \tau$. Recall that for each (m,j)-node ϵ such that $\xi' \subseteq \epsilon \subset \tau$, $j(\tau) < j(\xi') \leq j$. Hence $j(\tau)$ is active at γ^-, and $\gamma^-{}^\wedge\langle(0,j(\tau))\rangle <_L \gamma$. Suppose $p^\gamma_v\downarrow$. By *xv*) of 5.1, no node $\supseteq \tau$ is visited in stage v. This contradicts Subclaim 3. Therefore p^γ is destroyed at a stage $\geq t$ and $< v$. Hence by (*c*) of the induction hypothesis, a^π is destroyed for some $\pi \in \mathcal{A}(\gamma^-)$ with $\pi <_L \gamma$. Note that $\pi <_L \tau$ so a^τ is also destroyed at a stage $\geq t$ and $< v$, a contradiction. Clearly, p^γ does not becomes defined in stage t before α is visited since $\alpha \subset \gamma$. For $\gamma = \tau$, we have already seen that $p^\tau\uparrow$ above. Suppose $\gamma <_L \tau$ with $\gamma^- = \tau^-$. Fix γ to be the maximal node in $\mathcal{A}(\gamma^-)$ such that $\gamma <_L \tau$ and $p^\gamma_t\downarrow$. Let θ be the leftmost 0-node in $\mathcal{A}(\gamma^-)$ with $\gamma <_L \theta$. Then $p^\theta_t\uparrow$. Let p^γ be set equal to p^γ_t at stage w. Since there is a γ-designated number at stage w, p^θ is defined in stage w. Hence p^θ is destroyed at a stage $\geq w$ and $< t$. By (*c*) of the induction hypothesis, a^γ and hence p^γ are destroyed when p^θ is destroyed, a contradiction. Therefore $p^\gamma_t\uparrow$.

Now fix a node θ such that $\theta^\wedge\langle(2,i(\theta))\rangle \subset \tau$. Towards a contradiction assume $r^\theta\downarrow$. By *xv*) of 5.1, $\theta \not\subset \alpha$. Also $\theta \neq \alpha$ since ξ' is not a 2-node. Therefore $\alpha \subset \theta$, and so $r^\theta_t\downarrow$. By *xv*) of 5.1 since some node $\supseteq \tau$ is visited at stage v, r^θ is destroyed at a stage w, $t \leq w < v$. Hence either c^θ is destroyed or some a^π ($\pi \in \mathcal{A}(\theta)$) is destroyed at stage w. Recall that $r^{\tau^-}_{t_0+1} = r^{\tau^-}_v$. By *vi*) of 5.1, $c^\theta_{w+1}\downarrow$. Therefore some a^π ($\pi \in \mathcal{A}(\theta)$) is destroyed at stage w.

Note that $\pi <_L \tau$ since $\theta^\wedge\langle(2, i(\theta))\rangle \subset \tau$. By *xiii*) of 5.1, a^τ is also destroyed in stage w, a contradiction. Note that there is not 0-node θ such that $\xi'^- \subset theta \subset \tau$ and $j(\theta) < j(\tau)$. Thus Case 1 holds at α if $y \in A^{j(\tau),1}$.

Suppose $y \in A^{j(\tau),0}$. For any 0-node γ, if γ is preferred to τ, then $\gamma \subseteq \beta \subseteq \xi'^-$. We want to show that

- *r^{γ^-} is defined,*
- *p^π is defined for each $\pi \in \mathcal{A}(\pi^-)$ with $\gamma <_L \pi$.*

If γ is visited at stage t, then $a^\gamma_t \downarrow$ by *vi*) and the desired conclusion then follows by *i*). Suppose γ is not visited at stage t. Let the construction jump $\theta \supset \gamma$ from a node $\subset \gamma$. By 4.4, γ is preferred to θ. Again the desired conclusion follows by *i*).

Thus Case 1 holds at α reading z for x, τ for β and $j(\tau)$ for i. In any case $r^{\xi'^-}$ cannot become defined at stage t. This contradiction completes the proof of Subclaim 2.

From Subclaim 2, Claim 2 fails with τ for η at stage v. But we chose a particular counterexample (η, v) to Claim 2 minimizing v and then maximizing $|\eta|$. Further, $\tau \supset \eta$ and so we have contradicted the choice of (η, v). This completes the proof of Claim 2.

Claim 3. For any 0-node σ and any stage v such that $\xi \subset \sigma \subseteq \delta$ and $u < v \leq s$, $\sigma \notin \mathcal{C}_v$ and $\sigma \notin \mathcal{C}[v]$ if $\mathcal{C}[v]$ exists.

Proof of Claim 3: Fix a 0-node σ such that $\xi \subset \sigma \subseteq \delta$. Suppose $a^\sigma_u \uparrow$. Then $a^\sigma_v \uparrow$. Otherwise a^ξ would be destroyed at a stage $\geq u$ and $< v$, contradicting the hypothesis of (a). Hence $\sigma \notin \mathcal{C}_v$, and $\sigma \notin \mathcal{C}[v]$ if $\mathcal{C}[v]$ exists by *x*) of 5.1.

Suppose $a^\sigma_u \downarrow$. Then $a^\sigma_s = a^\sigma_u$ otherwise a^ξ is destroyed at a stage $\geq u$ and $< s$, contradiction.

Subclaim 1. For each v such that $u < v \leq s$, if $\sigma \notin \mathcal{C}_v$, then $\sigma \in \mathcal{D}_v$.

Proof of Subclaim 1: From the hypothesis of (a) some $y \in B_{u+1}$ with $\alpha(y) \supseteq \delta$ entered B since $r^{\xi^-}_{u+1}$ was set. Clearly y entered B after a^σ_u was set since a^ξ and r^{ξ^-} are destroyed when a^σ becomes defined. Hence $\sigma \in \mathcal{D}_v$ because $\sigma \notin \mathcal{C}_v$ and $a^\sigma_v = a^\sigma_u$. This completes the proof of Subclaim 1.

Subclaim 2. There exists a function $v \mapsto \xi_v$ $(u < v \leq s)$ such that for all v, w with $u < v, w \leq s$,

- $\xi_v \in \mathcal{C}_v$
- $\xi_{u+1} = \xi$
- $v < w \implies \xi_w \subseteq \xi_v$
- a^{ξ_v} *is not destroyed at a stage* $\geq v$ *and* $< s$
- *for each* (m, j)*-node* ϵ *such that* $\xi_v \subseteq \epsilon \subseteq \sigma$, $j(\xi_v) \leq j$.

Proof of Subclaim 2: First note that for each node $\zeta \subset \xi$, $c^{\zeta}_u \downarrow$ and c^{ζ} cannot become undefined at a stage $\geq u$ and $< s$.

For $v = u + 1$, let $\xi_{u+1} = \xi$. Clearly, all conditions hold. Suppose we have ξ_v for a particular $v < s$ which satisfies the above conditions. If $\xi_v \in \mathcal{C}_{v+1}$, we can let $\xi_{v+1} = \xi_v$. Suppose $\xi_v \notin \mathcal{C}_{v+1}$. Towards a contradiction assume that $r^{\xi_v^-}_{v+1} \uparrow$. Since $\xi_v \in \mathcal{C}_v$, $r^{\xi_v^-}_v \downarrow$. But $c^{\xi_v^-}_{v+1} = c^{\xi_v^-}_v$, so a^{π} is destroyed at stage v for some $\pi \in \mathcal{A}(\xi_v^-)$. Let π be the leftmost such node. Suppose $\pi \leq \xi_v$. By *xiii)* of 5.1, a^{ξ_v} is destroyed at stage v, a contradiction. Therefore $\xi_v <_L \pi$. Since $\xi_v \in \mathcal{C}_v$, $p^{\pi}_v \downarrow$. p^{π} is destroyed at stage v by $(E3)$ since a^{π} is destroyed at stage v. By (c) of the induction hypothesis, a^{ϵ} is destroyed at stage v for some $\epsilon \in \mathcal{A}(\xi_v^-)$ with $\epsilon <_L \pi$. This contradicts the choice of π. Therefore $r^{\xi_v^-}_{v+1} \downarrow$. By *xxv)* of 5.1, $\xi_v \notin \mathcal{C}[v]$. By *xxiv)* of 5.1, there exist α, π such that $\alpha \supseteq \pi \in \mathcal{A}(\xi_v^-)$, $\pi \leq \xi_v$ and α receives attention at stage v. Note that $a^{\pi}_v \downarrow$ since $a^{\xi_v}_v \downarrow$. By *xxi)* of 5.1, since a^{ξ_v} and hence a^{π} are not destroyed in stage v, some z enters B at stage v. Towards a contradiction assume that $\alpha(z) = \pi^-$. Then $\alpha = \pi$. Hence p^{ξ_v} is defined when $\mathcal{C}[v]$ is defined. By (b) of the induction hypothesis, $\xi_v \in \mathcal{C}[v]$, a contradiction. Therefore $\alpha(z) \supseteq \pi$.

Since a^{ξ_v} is not destroyed at stage v, by Case 3 and Case 8 there is a maximal 0-node $\zeta \subset \xi_v$ such that $\zeta \in \mathcal{C}[v]$. Towards a contradiction suppose $\pi \in \mathcal{C}[v]$. Then $\pi \neq \xi_v$ from above. Also, $p^{\xi_v} \downarrow$ when $\mathcal{C}[v]$ is defined since $\pi <_L \xi_v$. From (b), $\xi_v \in \mathcal{C}[v]$. This contradiction confirms that $\pi \notin \mathcal{C}[v]$. By *v)* for each (n, j)-node ϵ with $\zeta \subset \epsilon \subseteq \pi$, $j(\zeta) \leq j$. So the same is true with ξ_v instead of π. By (f) of the induction hypothesis, $r^{\zeta^-}_{v+1} \downarrow$. By *xxv)* of 5.1, $\zeta \in \mathcal{C}_{v+1}$. We claim that there is no 0-node τ such that $\zeta \subset \tau \subset \pi$ and $\tau \in \mathcal{C}_{v+1}$. Otherwise, $\tau \in \mathcal{C}_{v+1}$ implies $r^{\tau^-}_{v+1} \downarrow$ and so $\tau \in \mathcal{C}[v]$ by *xxv)* of 5.1, contradicting the choice of ζ. By (a) of the induction hypothesis with $\xi = \zeta$, $\delta = \pi$, $u = v$, a^{ζ} is not destroyed at a stage $\geq v + 1$

and $< s$ otherwise a_v^{π}, $a_v^{\xi_v}$ are also destroyed in the same stage, contradiction. By letting ξ_{v+1} equal ζ we have the desired conclusions. This completes the proof of Subclaim 2.

Now we prove Claim 3. Towards a contradiction consider the least stage v such that $u < v \leq s$ and $\sigma \in \mathcal{C}_v \cup \mathcal{C}[v]$. Suppose $\sigma \in \mathcal{C}_v$. By the maximality of ξ in the hypothesis of (a), $v > u+1$. Then $\sigma \notin \mathcal{C}_{v-1}$. By Subclaim 1, $\sigma \in \mathcal{D}_{v-1}$. Now by Claim 2, reading σ for η and ξ_v for ξ', we have $\sigma \notin \mathcal{C}_v$, a contradiction. Therefore $\sigma \notin \mathcal{C}_v$ which implies $\sigma \in \mathcal{C}[v]$. By Subclaim 1, $\sigma \in \mathcal{D}_v$. Applying Claim 2 again, $\sigma \notin \mathcal{C}[v]$, a contradiction.

Finally, (e) of $xxiii)$ of 5.1 now follows from Claim 3. This completes the proof of (a).

(b) Let p^δ be defined at some point in stage s and x be the value of p^δ in stage s. Let x enter B at stage $v < s$ and p^δ be set equal to x at stage $t \leq s$. Clearly, x is a designated number for δ at the end of stage v. Towards a contradiction assume that $\delta \notin \mathcal{C}_{v+1}$. Note that $r_{v+1}^{\delta^-}\downarrow = r_t^{\delta^-}$ otherwise p^δ cannot be set equal to x at stage t. It follows by $iv)$ of 5.1, $a_t^\delta = a_{v+1}^\delta \downarrow$. Also $a_s^\delta \downarrow = a_t^\delta$ because a^δ is destroyed implies p^δ is destroyed at the same stage if p^δ is defined. Since a^δ cannot become defined in stage v, we have $a_s^\delta = a_v^\delta \downarrow$. By $xxv)$ of 5.1, $\delta \notin \mathcal{C}[v]$. By Case 3.1 and Case 8, since a^δ is not destroyed at stage v, there is a maximal 0-node ξ such that $\xi \in \mathcal{C}[v]$ and $\xi \subset \delta$. By $v)$, for each 0-node ϵ with $\xi \subset \epsilon \subseteq \delta$, $j(\xi) \leq j(\epsilon)$. Towards a contradiction assume $a_t^\xi \downarrow = a_v^\xi$. Let $\alpha \supseteq \xi$ be the first node visited in stage t. Applying (d) with $\delta = \xi$, $s = t$, and $y = x$, we infer that p^α becomes defined in stage t and $\alpha \neq \xi$ implies $j(\alpha) < j(\xi)$. Since δ is visited, $\alpha \not\subset \delta$ by $xv)$ of 5.1. If $\alpha \neq \delta$, then there exists $\theta \in \mathcal{A}(\alpha^-)$ with $\xi \subset \theta \subseteq \delta$ and $\theta <_L \alpha$. Therefore whether $\alpha = \delta$ or not, there exists θ such that $j(\theta) < j(\xi)$ and $\xi \subset \theta \subseteq \delta$ which contradicts our finding above. Hence a^ξ is destroyed between stages v and t. Towards a contradiction assume $c_w^{\xi^-}\uparrow$ for some w with $v < w < s$. Since $c_s^{\delta^-}\downarrow$ by the hypothesis of (b), c^{ξ^-} becomes defined at some stage $\geq w$ and $< s$ by $vi)$ of 5.1. But when c^{ξ^-} becomes defined a^δ is destroyed, contradiction. Therefore $c_w^{\xi^-}\downarrow$ for each w with $v < w \leq s$. In particular, $c_{v+1}^{\xi^-}\downarrow$. By (f) of the induction hypothesis, $r_{v+1}^{\xi^-}\downarrow$. Hence $\xi \in \mathcal{C}_{v+1}$ by $xxv)$ of 5.1. Also, for each 0-node σ, $\xi \subset \sigma \subset \delta$, $\sigma \notin \mathcal{C}_{v+1}$. Otherwise $r_{v+1}^{\sigma^-}\downarrow$ and $\sigma \in \mathcal{C}[v]$ by $xxv)$ of 5.1 which contradicts the maximality of ξ. Note that $\delta \in \mathcal{D}_{v+1}$ since $\delta \notin \mathcal{C}_{v+1}$ and $a_v^\delta \downarrow = a_{v+1}^\delta$. Let a^ξ be destroyed at stage u with u least possible such that $v < u < t$. By (a) with $s = u$ and $u = v$, a^δ is destroyed at stage u. Hence r^{δ^-} is destroyed at stage u, contradiction. Therefore $\delta \in \mathcal{C}_{v+1}$.

Towards a contradiction assume $r_s^{\delta^-} \neq r_{v+1}^{\delta^-}$. Let m be the least stage $> v$ such that $r_{m+1}^{\delta^-} \uparrow$. We have $m < s$. Note that $c_z^{\delta^-} \downarrow$ for all z, $v \leq z \leq s$, and in particular for $z = m+1$. If not, since $c_s^{\delta^-} \downarrow$, c^{δ^-} becomes defined at some stage $\geq v$ and $< s$, a^δ would be destroyed in the same stage, contradiction. Therefore, one of a^π ($\pi \in \mathcal{A}(\delta^-)$) is destroyed at stage m. We choose such π least possible. By $xiii$) of 5.1, $\delta <_L \pi$ otherwise a^δ is destroyed at stage m, contradiction. Since $\delta \in \mathcal{C}_{v+1}$, $p_{v+1}^\pi \downarrow$. Then $p_m^\pi \downarrow$. By (c) of the induction hypothesis, a^ϵ is destroyed at stage m for some $\epsilon <_L \pi$ with $\epsilon \in \mathcal{A}(\delta^-)$. This contradicts the minimality of π. Therefore $r_s^{\delta^-} = r_{v+1}^{\delta^-}$.

Towards a contradiction assume $\delta \notin \mathcal{C}_s$. From above $a_s^\delta \downarrow = a_v^\delta$ and $c_z^{\delta^-} \downarrow$ for $v \leq z \leq s$. Let $w < s$ be the least stage such that $w > v$ and $\delta \notin \mathcal{C}_{w+1}$. Note that w exists and $\delta \in \mathcal{C}_w$. By $xxiv$) of 5.1, there exist α, π such that $\alpha \supseteq \pi \in \mathcal{A}(\delta^-)$, $\pi \leq \delta$ and α receives attention at stage w. $a_w^\pi \downarrow$ by 5.1 $xiii$) since $a_w^\delta \downarrow$. Note that a^π is not destroyed at stage w otherwise r^{δ^-} is destroyed at stage w, a contradiction. By xxi) of 5.1, some y enters B at stage w. Hence $\mathcal{C}[w]$ is defined. Note that either $\alpha(y) = \alpha$ or $\alpha(y) = \alpha^-$. $\delta \notin \mathcal{C}[w]$ by xxv) of 5.1 since $r_{w+1}^{\delta^-} \downarrow$ and $\delta \notin \mathcal{C}_{w+1}$. Towards a contradiction assume that $\alpha(y) = \pi^-$. Then $\alpha = \pi$. In stage w, p^π becomes defined and p^δ is defined when $\mathcal{C}[w]$ is defined. Note that a^δ and c^{δ^-} are defined when $\mathcal{C}[w]$ is defined. $\delta \in \mathcal{C}[w]$ by (b) of the induction hypothesis, a contradiction. Therefore $\alpha(y) \supseteq \pi$. Because a^δ is not destroyed at stage w and $\delta \notin \mathcal{C}[w]$, by Case 3.1 and Case 8 there exists a maximal 0-node σ such that $\sigma \subset \delta$ and $\sigma \in \mathcal{C}[w]$. By ($f$) of the induction hypothesis, $r_{w+1}^{\sigma^-} \downarrow$. Hence $\sigma \in \mathcal{C}_{w+1}$ by xxv) of 5.1. By v), $j(\sigma) \leq j(\epsilon)$ for each 0-node ϵ with $\sigma \subset \epsilon \subseteq \pi$. Suppose $w < t$. Towards a contradiction assume that $a_t^\sigma = a_w^\sigma$. Let $\tau \supseteq \sigma$ be the first node which is visited at stage t. Note that τ exists because δ is visited in stage t. By (d) of the induction hypothesis, p^τ becomes defined at stage t and $\tau \neq \sigma$ implies $j(\tau) < j(\sigma)$. Hence either $\tau = \delta$ or $\tau^- \subset \delta$ with $\delta <_L \tau$. In either case there exists a 0-node ϵ such that $\sigma \subset \epsilon \subseteq \delta$ and $j(\epsilon) < j(\sigma)$, a contradiction. Hence $a_w^\sigma \neq a_t^\sigma$. Let $m > w$ be the least stage at which a^σ is destroyed. Note that $m < t$. $c_m^\sigma \downarrow$ and $c_{m+1}^\sigma \downarrow$ by vi) of 5.1 since $c_m^{\delta^-} \downarrow$ and $c_{m+1}^{\delta^-} \downarrow$. Note that for each 0-node ϵ such that $\sigma \subset \epsilon \subset \pi$, $\epsilon \notin \mathcal{C}_{w+1}$ otherwise $\epsilon \in \mathcal{C}[w]$, contradiction. Towards a contradiction assume $\pi \in \mathcal{C}[w]$. Since $\delta \notin \mathcal{C}[w]$, $\pi <_L \delta$. Also $p^\delta \downarrow$ when $\mathcal{C}[w]$ is defined since $\pi \in \mathcal{C}[w]$. By (b) of the induction hypothesis, $\delta \in \mathcal{C}[w]$, contradiction. Hence $\pi \notin \mathcal{C}[w]$. By xxv) of 5.1,

$\pi \notin C_{w+1}$. Since $a^{\pi}_{w+1} \downarrow$ and y with $\alpha(y) \supseteq \pi$ enters B in stage w, $\pi \in \mathcal{D}_{w+1}$. By (a) of the induction hypothesis with $\xi = \sigma$, $\delta = \pi$, $s = m$ and $u = w$, a^{δ} is destroyed at stage m. Thus a^{δ} is destroyed at stage m, contradiction. Suppose $t \leq w$. Note that $p^{\delta} \downarrow$ when $\mathcal{C}[w]$ is defined. By (b) of the induction hypothesis, $\delta \in \mathcal{C}[w]$, a contradiction. Therefore $\delta \in \mathcal{C}_s$.

To complete (b) assume $\mathcal{C}[s]$ exists and that a^{δ}, c^{δ^-}, p^{δ} are all defined when $\mathcal{C}[s]$ is defined. Towards a contradiction assume $\delta \notin \mathcal{C}[s]$. Note that $a^{\delta}_s \downarrow$ and $r^{\delta^-}_s \downarrow$ since $\delta \in \mathcal{C}_s$. Examining the instructions of Cases 1.1, 3.1 and 8 we see that r^{δ^-} cannot become undefined in stage s before $\mathcal{C}[s]$ is defined. Otherwise either c^{δ^-} or a^{δ} is destroyed at the same time. Then the first two conditions in the definition for δ to belong to $\mathcal{C}[s]$ hold. Because $\delta \notin \mathcal{C}[s]$ and $\delta \in \mathcal{C}_s$, there exists a 0-node $\eta \supset \delta$ such that $\eta \in \mathcal{C}_s$ and $\eta \notin \mathcal{C}[s]$. Towards a contradiction assume $r^{\eta^-} \uparrow$ when $\mathcal{C}[s]$ is defined. Note that $r^{\eta^-}_s \downarrow$. Let ζ receive attention at stage s. By $xv)$ of 5.1, $\zeta \not\supset \delta$ since p^{δ} is defined when $\mathcal{C}[s]$ is defined. Suppose Case 1.1 holds at ζ. Let β be the 0-node from Case 1.1. There are three cases: $\zeta <_L \eta^-$, $\eta \subseteq \beta$ and $\beta <_L \eta$. Since $\delta \not\subset \zeta$ and $\delta \subset \eta$, either $\zeta <_L \delta$ or $\delta \subseteq \beta$. Suppose that either $\delta \subseteq \beta$ or $\zeta <_L \delta^-$. By Case 1.1 r^{δ^-} is destroyed before $\mathcal{C}[s]$ is defined, contradiction. Hence $\delta^- \subseteq \zeta$ and $\zeta <_L \delta$. Let π be the unique node in $\mathcal{C}(\delta^-)$ such that $\pi \subseteq \zeta$. By Case 1.1 $r^{\delta^-} = r^{\pi^-}$ is destroyed before $\mathcal{C}[s]$ is destroyed, contradiction. For Cases 3.1 and 8 the arguments are similar. We leave them to the reader. Therefore, $r^{\eta^-} \downarrow$ when $\mathcal{C}[s]$ is defined. By $xxiv)$ of 5.1, there exists $\alpha \supset \delta$ which receives attention in stage s. Since p^{δ} is defined at some point in stage s, this contradicts $xv)$ of 5.1. This completes the proof of (b).

(c) p^{δ} can be destroyed only by $(E3)$ which requires the destruction of a^{δ}. Also, p^{δ} can only be defined when a^{δ} is already defined. Thus a^{δ} is defined whenever p^{δ} is. Suppose that p^{δ} is destroyed at stage s.

Let ϵ be the maximal node in $\mathcal{A}(\delta^-)$ with $\epsilon <_L \delta$. We will show that a^{ϵ} is destroyed in stage s. Let β receive attention at stage s. By $viii)$ of 5.1 since a^{δ} is destroyed one of the following conditions holds for stage s:

A) a^{γ} is destroyed for some $\gamma <_L \delta$;

B) either $\delta \subseteq \beta$ or $\beta <_L \delta$, and one of Case 3.1, Case 8, and Case 11 holds at β;

C) either $\delta \subset \beta$ or $\beta <_L \delta$, and Case 2 holds at β;

D) $\delta \subseteq \beta$ or $\beta < \delta$, and either Case 1.1 or Case 4 holds at β;

E) $\delta \subseteq \beta$ or $\beta <_L \delta$, and Case 13 holds at β.

Suppose A) holds. Then $\gamma <_L \epsilon$ or $\epsilon \subseteq \gamma$. By *xiii*) of 5.1, a^ϵ is destroyed. In the remaining cases we can suppose that a^δ is not destroyed in the ending of stage s. Suppose B) holds. Towards a contradiction assume $\delta \subseteq \beta$. $p^\delta_s \downarrow$ contradicts *xv*) of 5.1. Suppose p^δ becomes defined in stage s. Since ϵ exists, $\delta \not\subseteq \beta$, a contradiction. Therefore $\beta <_L \delta$. Then either $\beta <_L \epsilon$ or $\epsilon \subseteq \beta$. If Case 11 holds, a^ϵ is destroyed. Suppose one of Cases 3.1, 8 holds. Let z enter B at stage s. Note that a^δ is defined when $\mathcal{C}[s]$ is defined. Suppose $c^{\delta^-} \uparrow$ when $\mathcal{C}[s]$ is defined. $\delta^- \not\subseteq \alpha(z)$ by *vi*) of 5.1. Also, it is clear that $\alpha(z) \neq \delta^-$. Therefore $\beta <_L \delta^-$. Note that $\epsilon \notin \mathcal{C}[s]$ since c^{δ^-} and r^{δ^-} are undefined. If $\alpha(z) <_L \delta^-$, then a^ϵ is destroyed in stage s since a^δ is destroyed at stage s. If $\alpha(z) \subset \delta^-$, then a^δ is not destroyed by Case 3.1 and Case 8. Since the cases $\alpha(z) <_L \delta^-$ and $\alpha(z) \subset \delta^-$ are exhaustive, we have the desired conclusion if $c^{\delta^-} \uparrow$ when $\mathcal{C}[s]$ is defined. Suppose $c^{\delta^-} \downarrow$ when $\mathcal{C}[s]$ is defined. By (b), $\delta \in \mathcal{C}[s]$. Hence a^δ cannot be destroyed in the main part of the construction in stage s.

The cases C) through E) may be treated in similar fashion. We leave them to the reader.

(d) Suppose the hypothesis holds. We have to show that p^α becomes defined, and $\alpha \neq \delta$ implies $j(\alpha) < j(\delta)$. Let x be the first number to enter B after a^δ is set equal to a^δ_s such that $\alpha(x) \supseteq \delta$. Let x enter B at stage $t < s$.

Case 1. $\delta \in \mathcal{C}[t]$. Hence $r^{\delta^-} \downarrow$ and $p^\sigma \downarrow$ for each $\sigma \in \mathcal{A}(\delta^-)$ with $>_L \delta$ when $\mathcal{C}[t]$ is defined. $c^{\delta^-} \downarrow$ since $r^{\delta^-} \downarrow$ when $\mathcal{C}[t]$ is defined. Since Case 3.1 or Case 8 holds at the node which receives attention no a^ϵ is destroyed in stage t before $\mathcal{C}[t]$ is defined. By (b) of the induction hypothesis $\sigma \in \mathcal{C}[t]$ for each $\sigma \in \mathcal{A}(\delta^-)$ with $\delta <_L \sigma$. Hence r^{δ^-} is not destroyed in the main part of stage t. Also r^{δ^-} cannot become undefined in the ending of stage t because $\alpha(x) \supseteq \delta$. Therefore $r^{\delta^-}_{t+1} \downarrow$. Towards a contradiction assume $c^{\delta^-}_v \uparrow$ for some v with $t < v \leq s$. Since $\alpha \supseteq \delta$ is visited at stage s, c^{δ^-} becomes defined at a stage $\geq v$ and $< s$ by *vi*) of 5.1. By Case 4, when c^{δ^-} becomes defined a^δ is destroyed, contradiction. Therefore $c^{\delta^-}_v$ for all v with $t < v \leq s$. Towards a contradiction assume that $r^{\delta^-}_v \uparrow$ for some (least) v with $t + 1 < v \leq s$. By *iv*) and *xii*) of 5.1 since $c^{\delta^-}_v \downarrow$, $a^\sigma_v \uparrow$ for some $\sigma \in \mathcal{A}(\delta^-)$. Choose such σ least possible. Suppose $\sigma \leq \delta$. a^δ is destroyed by *xiii*) of 5.1, contradiction.

Therefore $\delta <_L \sigma$. $p^\sigma \downarrow$ when $\mathcal{C}[t]$ is defined since $\delta \in \mathcal{C}[t]$. Hence $p^\sigma_{v-1} \downarrow$ since $a^\sigma_{v-1} \downarrow$. By (c) of the induction hypothesis, a^ϵ is destroyed at stage $v-1$ with $\epsilon <_L \sigma$, contradicting the minimality of σ. Therefore $r^{\delta^-}_s \downarrow = r^{\delta^-}_{t+1}$. Since $r^{\delta^-}_s \downarrow$ and $p^\pi_s \downarrow$ for each $\pi \in \mathcal{A}(\delta^-)$ with $\delta <_L \pi$, δ cannot be visited unless there is a jump to δ. Suppose $\alpha = \delta$. Then (d) is clear. Suppose $\alpha \neq \delta$. Then δ is not visited at stage s and there is a jump from a node $\theta \subset \delta$ to α at stage s. So p^α becomes defined at stage s. Towards a contradiction assume $j(\delta) \leq j(\alpha)$. Let p^α be set equal to $z(\alpha)$ at stage s. There are two subcases:

Subcase 1.1. There is no (n,k)-node ϵ such that $\delta \subset \epsilon \subset \alpha$ and $k < j(\delta)$. Then for each 0-node $\gamma \subset \delta$, if γ is preferred to δ, then γ is preferred to α. By the argument used in the proof of Subclaim 5 of the proof of (a), we can show that Case 1 holds at θ in stage s with $i = j(\delta)$ and $\beta = \delta$. This contradicts $\alpha \neq \delta$.

Subcase 1.2. Otherwise. There are two subcases again.

Subcase 1.2.1. There is a 0-node τ such that $\delta \subseteq \tau \subset \alpha$ and τ is preferred to α. Fix such τ with $j(\tau)$ least possible. By the Remark following 4.4, $j(\tau) < j(\alpha)$. Note that for each 0-node $\beta \subset \tau$, if β is preferred to τ, then β is preferred to α. Since at stage s, the construction jumps over τ, by Case 1 we have

- $r^{\tau^-}_s \downarrow$
- $p^\pi_s \downarrow$ *for each* $\pi \in \mathcal{A}(\tau^-)$ *with* $\tau <_L \pi$.

Let v be the least stage such that

- $r^{\tau^-}_{v+1} \downarrow = r^{\tau^-}_s$
- $p^\pi_{v+1} \downarrow = p^\pi_s$ *for each* $\pi \in \mathcal{A}(\tau^-)$ *with* $\tau <_L \pi$.

Note that τ is visited at stage v by $xix)$ of 5.1. Because a^τ is not destroyed at stage v, some y enters B at stage v by $xxi)$ of 5.1. Since r^{τ^-} is not destroyed at any stage $\geq v$ and $< s$, we have $a^\tau_s = a^\tau_v \downarrow$. So, if p^τ becomes defined in stage v, p^τ_s is defined. By $xv)$ of 5.1 this contradicts α beong visited in stage s. Therefore p^τ does not become defined in stage v. Hence $\alpha(y) \supseteq \tau$. Suppose $\tau \in \mathcal{C}[v]$. Recall that every γ which is preferred to τ is preferred to α. Suppose that there is not ϵ-designated number for each 0-node ϵ such that $\theta \subset \epsilon \subset \tau$ and $j(\epsilon) < j(\tau)$. By the argument used in the proof of Subclaim 5 of the proof of (a), we can show that Case 1 holds at θ in stage s with $i = j(\tau)$ and $\beta = \tau$. This contradicts $\alpha \neq \tau$. Suppose there is a ϵ-designated number when θ is visited for some ϵ such that $\theta \subset \epsilon \subset \tau$ and $j(\epsilon) < j(\tau)$. Choose such ϵ with $j(\epsilon)$ least possible. Towards a contradiction assume $\delta \subseteq \epsilon \subset \tau$. First we show that for each 0-node σ such that $\delta \subseteq \sigma \subset \tau$ and $j(\sigma) < j(\tau)$, there is an (n, j)-node η with $\sigma \subset \eta \subset \tau$ and $j < j(\sigma)$. Otherwise, σ is preferred to α, contradicts the minimality of τ. Form this it is clear that $\epsilon \notin \mathcal{C}[v]$. Let u be the stage at which a^ϵ is set equal to a^ϵ_s. Suppose $u < v$. Since a^ϵ is not destroyed at stage v, there is a maximal 0-node $\xi \subset \epsilon$ such that $\xi \in \mathcal{C}[v]$. Note that $\xi \subset \delta$. By v), there is no 0-node π such that $\xi \subset \pi \subseteq \epsilon$ and $j(\pi) < j(\xi)$. Towards a contradiction assume $a^\xi_v = a^\xi_s$. Then by (d) of the induction hypothesis, α cannot be visited in stage s. Therefore $a^\xi_v \neq a^\xi_s$. Let $h > v$ be the least stage such that $a^\xi_{h+1} \uparrow$. By (f), $r^{\xi^-}_{v+1} \downarrow$. By xxv) of 5.1, $\xi \in \mathcal{C}_{v+1}$ and $\pi \notin \mathcal{C}_{v+1}$ for each 0-node π such that $\xi \subset \pi \subseteq \epsilon$. Therefore, $\epsilon \in calD_{v+1}$. By vi) of 5.1 since for all w, $v \leq w < s$, $c^{\tau^-}_w \downarrow$, $c^{\xi^-}_h \downarrow$ and $c^{\xi^-}_{h+1} \downarrow$. By ($a$) of the induction hypothesis, a^ϵ is destroyed at stage h, contradiction. Suppose $v \leq u$. Then $v < u$. By the argument used in the proof of Subclaim 5 of the proof of (a), we can show that Case 1 holds at ϵ^- in stage u with $i = j(\tau)$ and $\beta = \tau$. This contradicts that a^ϵ becomes defined at stage u. Therefore, $\epsilon \subset \delta$. By (d) of the induction hypothesis, it is impossible since $j(\epsilon) < j(\tau), j(\alpha)$.

Therefore $\tau \notin \mathcal{C}[v]$. Because a^τ is not destroyed at stage v, there is a maximal 0-node $\xi \subset \tau$ with $\xi \in \mathcal{C}[v]$. By (f) of the induction hypothesis, $r^{\xi^-}_{v+1} \downarrow$. Note that $\tau \in \mathcal{D}_{v+1}$ since $a^\tau_{v+1} \downarrow$. Also, for each 0-node ζ with $\xi \subset \zeta \subset \tau$, $\zeta \notin \mathcal{C}_{v+1}$, otherwise $\zeta \in \mathcal{C}[v]$ by xxv) of 5.1, which would contradict the maximality of ξ. By v), for each 0-node ζ with $\xi \subseteq \zeta \subseteq \tau$, $j(\xi) \leq j(\zeta)$. Towards a contradiction assume that $\xi \supseteq \delta$. Note that $j(\xi) \leq j(\tau)$. Suppose there is no (n, k)-node ϵ such that $\xi \subset \epsilon \subset \tau$ and $k < j(\xi)$. Then $j(\xi) < j(\tau)$ and ξ is preferred to τ. Hence ξ is preferred to α, contradicting the choice of τ. Suppose such ϵ exists. Since $\xi \in \mathcal{C}[v]$ there exists a 0-node π such that $\xi \subset \pi \subseteq \epsilon$ and $\pi \in \mathcal{C}[v]$. This contradicts the maximality of ξ. Therefore $\xi \subset \delta$, and so $j(\xi) \leq j(\delta)$.

Towards a contradiction assume $a^\xi_v = a^\xi_s$. Let β be the first node $\supseteq \xi$ which is visited at stage s. By (d) of the induction hypothesis, p^β becomes defined, and $\beta \neq \xi$ implies $j(\beta) < j(\xi)$. Then $\beta \not\subset \alpha$ by xv) of 5.1 since α is visited at stage s. Recall that there is no node $\supset \theta$ and $\subset \alpha$ which is visited in stage s by the choice of α. Suppose $\beta \neq \alpha$. Then

$\xi \subset \theta$. But θ is visited in stage s. So $\beta^- \subset \theta$ and $\theta <_L \beta$. Let ζ be the unique 0-node such that $\zeta \in \mathcal{A}(\beta^-)$ and $\zeta \subseteq \theta$. Then $\xi \subset \zeta \subset \tau$ and so $j(\xi) \leq j(\zeta) < j(\beta) < j(\xi)$, contradiction. Suppose $\beta = \alpha$. Then $j(\tau) < j(\alpha) = j(\beta) < j(\xi) \leq j(\tau)$, contradiction. Therefore $a_v^\xi \neq a_s^\xi$. Let w be the least stage $> v$ with $a_{w+1}^\xi \uparrow$. Note that both $c_w^{\xi^-}$, $c_{w+1}^{\xi^-}$ are defined. Otherwise. c^{ξ^-} becomes defined at some stage $\geq w$ and $< s$ by *vi)* of 5.1 since α is visited at stage s. But when c^{ξ^-} becomes defined a^τ is destroyed. This contradicts $a_s^\tau = a_v^\tau \downarrow$ which we found above. Now applying (a) of the induction hypothesis with $\delta = \tau$, $u = v$ and $s = w$, a^τ is destroyed at stage w, contradiction.

Subcase 1.2.2. Otherwise. Let ϵ be an (n,k)-node such that $\delta \subset \epsilon \subset \alpha$ with k least possible. ϵ exists, $k < j(\delta)$, and $n \in \{0,4,5\}$. In fact, $n \neq 0$ otherwise ϵ is preferred to α, contradicting the hypothesis of the case. Let p^α be set equal to $z(\alpha)$ at stage s, and $z(\alpha)$ have entered B at stage v.

Suppose $a_v^\delta = a_s^\delta \downarrow$. By the choice of t we have $t \leq v$. Recall that $r_{t+1}^{\delta^-} = r_s^{\delta^-} \downarrow$. So $z(\alpha)$ enters B after $r_s^{\delta^-}$ is set. If $\delta \in \mathcal{C}[v]$, then there exists a 0-node η with $\delta \subset \eta \subseteq \epsilon$ such that $j(\eta) \leq k$ (and $\eta \in \mathcal{C}[v]$). This puts us back in Case 1.2.1. Hence $\delta \notin \mathcal{C}[v]$. Since a^δ is not destroyed at stage v there exists a maximal $\xi \in \mathcal{C}[v]$ with $\xi \subset \delta$ by Cases 3.1 and 8. By (f) of the induction hypothesis, $r_{v+1}^{\xi^-} \downarrow$. By *xxv)* of 5.1, $\xi \in \mathcal{C}_{v+1}$ and $\delta \notin \mathcal{C}_{v+1}$. Hence $\delta \in \mathcal{D}_{v+1}$ since $a_{v+1}^\delta \downarrow$. Similarly there is no 0-node τ such that $\xi \subset \tau \subset \delta$ and $\tau \in \mathcal{C}_{v+1}$. By *v)*, $j(\xi) \leq j$ for each (m,j)-node π such that $\xi \subset \pi \subseteq \epsilon$. In particular, $j(\xi) \leq k$ and $j(\xi) \leq j(\tau)$ for each 0-node τ such that $\xi \subset \tau \subseteq \epsilon$. By the minimality of k, $k \leq j(\tau)$ for each 0-node τ such that $\epsilon \subset \tau \subset \alpha$. Also by hypothesis $j(\delta) \leq j(\tau)$. Hence $j(\xi) \leq j(\tau)$ for each 0-node τ such that $\xi \subset \tau \subseteq \alpha$. Towards a contradiction assume that $a_s^\xi = a_v^\xi$. Let ζ be the first node such that $\supseteq \xi$ which is visited at stage s. ζ exists since α is visited at stage s. By (d) of the induction hypothesis, p^ζ becomes defined at stage s and $\xi \neq \zeta$ implies $j(\zeta) < j(\xi)$. Clearly, $\zeta \not\subset \alpha$ otherwise α cannot be visited at stage s. Whether $\zeta = \alpha$ or not, there exists a 0-node τ such that $\xi \subset \tau \subseteq \alpha$ and $j(\tau) < j(\xi)$, contradiction. Hence $a_s^\xi \neq a_v^\xi$. Let w be the least stage such that $v < w < s$ and $a_{w+1}^\xi \uparrow$. Towards a contradiction assume one of $c_w^{\xi^-}$, $c_{w+1}^{\xi^-}$ is undefined. Since α is visited at stage $s > w$, c^{ξ^-} becomes defined at a stage $\geq w$ and $< s$ by *vi)* of 5.1. But when c^ξ becomes defined, a^δ becomes undefined, a contradiction. Hence $c_w^{\xi^-} \downarrow$ and $c_{w+1}^{\xi^-} \downarrow$. By (a) of the induction hypothesis with $u = v$ and $s = w$, a^δ is destroyed at stage w, contradiction.

Suppose $a^\delta_s \neq a^\delta_v$. Then $v < t$ and a^δ becomes defined after stage v, say at stage u. Note that δ^- is visited at stage u because δ is visited at stage u and $a^\delta_u \uparrow$. Since Subcase 1.2.1 fails any τ preferred to α satisfies $\tau \subset \delta$. Repeating the argument used to show Subclaim 5 in the proof of (a), we can show that Case 1 holds at δ^- in stage u with $i = j(\alpha)$ and $\beta = \alpha$. Hence δ is not visited at stage u, contradiction. This completes the case $\delta \in \mathcal{C}[t]$.

Case 2. $\delta \notin \mathcal{C}[t]$. There exists a maximal 0-node ξ such that $\xi \subset \delta$ and $\xi \in \mathcal{C}[t]$ by Cases 3.1 and 8, since a^δ is not destroyed at stage t. $j(\xi) \leq j(\zeta)$ for each 0-node with $\xi \subset \zeta \subseteq \delta$ by v). By (f) of the induction hypothesis, $r^{\xi^-}_{t+1} \downarrow$. $\delta \in \mathcal{D}_{t+1}$ since $a^\delta_{t+1} \downarrow$. Also, $\zeta \notin \mathcal{C}_{t+1}$ for each 0-node ζ with $\xi \subset \zeta \subset \delta$. Suppose $a^\xi_s \downarrow = a^\xi_t$. Let β be the first node $\supseteq \xi$ which is visited at stage s. Note that β exists since $\alpha \supset \xi$ is visited at stage s. By (d) of the induction hypothesis, p^β becomes defined at stage s, and $\beta \neq xi$ implies $j(\beta) < j(\xi)$. By xv) of 5.1, $\beta \not\subset \alpha$. Hence either $\beta = \alpha$ or $\alpha <_L \beta$. In either case there exists a 0-node γ such that $\xi \subset \gamma \subseteq \alpha$ and $j(\gamma) < j(\xi)$, contradiction. Suppose $a^\xi_s \neq a^\xi_t$. Let v be the least stage, $t \leq v < s$, such that a^ξ is destroyed at stage v. Note that $c^{\xi^-}_v$ and $c^{\xi^-}_{v+1}$ are both defined. By a) of the induction hypothesis with $u = t$ and $s = v$, a^δ is destroyed at stage v, contradiction. This completes the proof of (d).

e) Let z be designated for δ at stage s and d be the first number to enters B after a^δ_s is set. Let d enter B at stage $v \leq s$. If $v = s$, (e) is immediate because in a given stage at most one number enters B. So suppose $v < s$.

Claim. A number which enters B at a stage $> v$ and $\leq s$ is not δ-designated at stage s.

Proof of Claim: Let $v < u \leq s$ and x enter B in stage u. It is clear that x is not δ-designated number unless $\alpha(x) \supseteq \delta$. Suppose $\alpha(x) \supseteq \delta$. Let α be the first node $\supseteq \delta$ which is visited at stage u. By (d) of the induction hypothesis, p^α becomes defined at stage u and $\alpha \neq \delta$ implies $j(\alpha) < j(\delta)$. Towards a contradiction assume $\alpha = \delta$. By Case 3, $\alpha(x) \not\supseteq \delta$. Therefore $\alpha \neq \delta$. Towards a contradiction assume that there is no (n, j)-node ϵ such that $\delta \subset \epsilon \subset \alpha$ and $j < j(\alpha)$. Then condition iii) in 4.3 does not hold for $\alpha(x)$ and so x is not a designated number for δ at stage s. Therefore there is an (n, j)-node ϵ such that $\delta \subset \epsilon \subset \alpha$ and $j < j(\alpha)$. Choose such ϵ such that j is least possible. Suppose $n \neq 0$. It is clear that x is not δ-designated at stage s. So $n = 0$. Note that ϵ is preferred to α. Let p^α be set equal to $z(\alpha)$ at stage u, and $z(\alpha)$ have entered B at stage $t(\alpha)$. By the conditions for jumping,

- $r_u^{\epsilon^-}\downarrow$
- $p_u^\sigma \downarrow$ *for each* $\sigma \in \mathcal{A}(\epsilon^-)$ *with* $\epsilon <_L \sigma$.

Let t be the least stage such that

- $r_{t+1}^{\epsilon^-}\downarrow = r_u^{\epsilon^-}$
- $p_{t+1}^\sigma \downarrow = p_u^\sigma$ *for each* $\sigma \in \mathcal{A}(\epsilon^-)$ *with* $\epsilon <_L \sigma$.

Note that at stage t, ϵ is visited. Because a^ϵ is not destroyed at stage t, by $xxi)$ of 5.1, some y enters B at stage t. p^ϵ cannot become defined at stage t; otherwise α is not visited at stage u by $xv)$ of 5.1. Hence $\alpha(y) \supseteq \epsilon$.

Suppose $\epsilon \in \mathcal{C}[t]$. Note that every γ which is preferred to ϵ is also preferred to α. Hence at stage u, the construction jumps to ϵ rather than α, contradiction.

Suppose $\epsilon \notin \mathcal{C}[t]$. Because a^ϵ is not destroyed at stage t, by Case 3.1 and Case 8 there exists a maximal 0-node ξ such that $\xi \in \mathcal{C}[t]$ and $\xi \subset \epsilon$. By $v)$ we know that $j(\xi) \leq j(\tau)$ for each 0-node $\xi \subset \tau \subseteq \epsilon$. Hence $\xi \subset \delta$. Towards a contradiction assume $a_u^\xi \downarrow = a_t^\xi$. Let β be the first node $\supseteq \xi$ which is visited at stage u. By (d) of the induction hypothesis, p^β becomes defined and $\beta \neq \xi$ implies $j(\beta) < j(\xi)$. Since α is visited at stage u, $\beta \not\subset \alpha$ by $xv)$ of 5.1. Hence either $\beta = \alpha$, or $\alpha <_L \beta$ and $\beta^- \subset \delta$. Suppose $\alpha = \beta$. Then $j(\beta) < j(\xi) \leq j(\epsilon) < j(\alpha)$, contradiction. Suppose $\beta \neq \alpha$. Then there exists a 0-node θ such that $\theta^- = \beta^-$, $\theta <_L \beta$ and $\xi \subset \theta \subseteq \delta$. But $j(\xi) \leq j(\theta) < j(\beta) < j(\xi)$, contradiction. Therefore $a_u^\xi \neq a_t^\xi$. Let a^ξ be destroyed at stage w, $t < w < u$, with w least possible. From $vi)$ of 5.1, $c_u^{\xi^-}\downarrow$. If $c_u^{\xi^-}$ becomes defined at a stage $\geq w$ and $< u$, then a^ϵ is destroyed. But since $r_{t+1}^{\epsilon^-} = r_u^{\epsilon^-}\downarrow$ we have $a_{t+1}^\epsilon = a_u^\epsilon \downarrow$ which yields a contradiction since $t < w$. Also, $\zeta \notin \mathcal{C}_{t+1}$ for each 0-node ζ with $\xi \subset \zeta \subseteq \epsilon$ by $xxv)$ of 5.1, and $\epsilon \in \mathcal{D}_{t+1}$ since $a_{t+1}^\epsilon \downarrow$. By (f) of the induction hypothesis, $r_{t+1}^{\xi^-}\downarrow$. Hence $\xi \in \mathcal{C}_{t+1}$ by $xxv)$ of 5.1. By (a) of the induction hypothesis with $\delta = \epsilon$, $u = t$ and $s = w$, a^ϵ is destroyed at stage w, contradiction. This completes the proof of Claim.

Now (e) is immediate by the Claim above.

(f) Towards a contradiction assume $r_{s+1}^{\delta^-}\uparrow$. By $iv)$ of 5.1 either c^{δ^-} or a^ϵ for some $\epsilon \in \mathcal{A}(\delta^-)$ becomes undefined in stage s. From Cases 3.1 and 8, we know that c^{δ^-} is not destroyed in the main part of the stage s. Also, c^{δ^-} cannot become undefined in the ending of stage s (through $(E1)$ or $(E4)$). Hence $c^{\delta^-}_{s+1}\downarrow$, so a^ϵ is destroyed in stage s for some $\epsilon\in\mathcal{A}(\delta^-)$. Choose ϵ to be least possible such that a^ϵ is destroyed at stage s with $\epsilon\in\mathcal{A}(\delta^-)$. Clearly, $\epsilon\not<_L\delta$. Suppose $\epsilon=\delta$. a^δ cannot become undefined in the main part of the construction since $\delta\in\mathcal{C}[s]$. So a^ζ is destroyed by $(E1)$ because some a^τ is destroyed in the main part of the construction with $\tau<_L\delta$. This is impossible. Therefore $\delta<_L\epsilon$. Note that $p^\epsilon\downarrow$ when $\mathcal{C}[v]$ is defined. By (c) of the induction hypothesis, a^τ is destroyed at stage v for some $\tau<_L\epsilon$, contradicting the minimality of ϵ. Therefore $r^{\delta^-}_{s+1}\downarrow$. This completes the proof of (f) and then $vi)$.

vii) Towards a contradiction consider a pair (w,δ) with least w and then greatest $|\delta|$ such that $\delta\in\mathcal{D}_w$ and $\delta\in\mathcal{C}_{w+1}\cup\mathcal{C}[w]$. $a^\delta_w\downarrow$ by $x)$ of 5.1. Because $\delta\in\mathcal{D}_w$, there exists a least stage $v<w$ such that $a^\delta_v\downarrow=a^\delta_w$ and some x enters B at stage v with $\alpha(x)\supseteq\delta$. Suppose $\delta\in\mathcal{C}_{w+1}$. Let y be the number designated for δ at stage $w+1$. By $x)$ of 5.1, $a^\delta_{w+1}=a^\delta_w$. By (e) of $vi)$, $y=x$. Therefore $r^{\delta^-}_{w+1}=r^{\delta^-}_{v+1}\downarrow$. Since the first two conditions for δ to belong to $\mathcal{C}_w$ hold, there exists an (n,j)-node α such that

- $\delta\subset\alpha\wedge j<j(\delta)$
- *there exists $y\in B_w$ with $\alpha(y)\supseteq\alpha$ which entered B after r^{δ^-} attained its value at stage $w-1$*
- *for each 0-node ϵ with $\delta\subset\epsilon\subseteq\alpha$ and $j(\epsilon)\le j$, $\epsilon\notin\mathcal{C}_w$.*

Because $\delta\in\mathcal{C}_{w+1}$, there exists a 0-node ϵ such that $\delta\subset\epsilon\subseteq\alpha$, $j(\epsilon)\le j$ and $\epsilon\in\mathcal{C}_{w+1}$. By $x)$ of 5.1, $a^\epsilon_{w+1}=a^\epsilon_w\downarrow$. Note that y entered B after a^ϵ attained the value a^ϵ_w because when a^ϵ becomes defined, a^δ and then r^{δ^-} are destroyed. Hence $\epsilon\in\mathcal{D}_w$. This contradicts the maximality of δ.

The case in which $\delta\in\mathcal{C}[w]$ can be treated similarly. This completes the proof of $vii)$.

viii) By induction on stages we show that when α is visited, one of Cases 1-12 in the construction holds at α. It is obvious that at stage 0, k^λ becomes defined. Consider stage $s>0$. The conclusion is clear if $c^\alpha_s\downarrow\ge 1$. Suppose α is a 0-node, and $a^\alpha_s\uparrow$. By $iv)$ of 5.3, a^α becomes defined at stage s. Suppose α is a 1-node, and $a^\alpha_s\uparrow$. Since $a^\alpha_s\uparrow$, the construction can only pass to α through Case 5 at α^-, in which Case 2 applies at α and a^α becomes defined. Suppose $a^\alpha_s \downarrow$ or α is not an i-node for $i \leq 1$. Let $c^\alpha_s \uparrow$. Let γ be the maximal 0-node $\subseteq \alpha$. If $\gamma \neq \alpha$, $a^\gamma_s \downarrow$ by *iv*). If $\alpha = \gamma$, then $a^\gamma_s \downarrow$ by hypothesis above. By *vi*) of 5.1, $c^{\alpha^-}_s \downarrow$. Hence Case 4 holds at α. It remains to consider the situation in which $c^\alpha_s \downarrow = 0$ and in which a^α_s if α is a 0-node. There are three cases:

Case A. $r^\alpha_s \downarrow$. Then $a^\beta_s \downarrow$ for each $\beta \in \mathcal{B}(\alpha)$ by *i*) of 5.1. Since $c^\alpha_s = 0$, there is a maximal $\delta \in \mathcal{A}(\alpha)$ such that $p^\delta_s \uparrow$. Let $t < s$ be the greatest such that at stage t, r^α becomes defined or p^β becomes defined for some $\beta \in \mathcal{A}(\alpha)$ with $\delta <_L \beta$. For $\gamma \in \mathcal{A}(\alpha)$ the parameters p^γ become defined in $\leq_L$-decreasing order, and whenever p^γ becomes defined the $\leq_L$-immediate-predecessor of γ is visited if it exists. Also, if p^β with $\beta \in \mathcal{A}(\alpha)$ becomes defined, then r^α is already defined. When r^α becomes defined, then construction passes to the node $\max\{\beta : \beta \in \mathcal{A}(\alpha)\}$. So δ is visited at stage t. Note that p^δ cannot become defined at stage t. Otherwise, suppose p^δ becomes defined at stage t, then $p^\delta_s \downarrow$ since $a^\delta_t = a^\delta_s \downarrow$, contradicting to the choice of t. Hence some node $\supseteq \delta$ receives attention at stage t. By *xxi*) of 5.1, since a^δ is not destroyed at stage t, some x enters B at stage t. Note that $\alpha(x) \supseteq \delta$ since $p^\delta \uparrow$ in stage t. Note that $a^\delta_t = a^\delta_s$.

Claim. $\delta \in C[t]$.

Proof of Claim: Towards a contradiction assume $\delta \notin C[t]$. Since a^δ is not destroyed at stage t there exists a maximal 0-node ξ such that $\xi \subset \delta$ and $\xi \in C[t]$. By *v*), for all 0-node ϵ with $\xi \subset \epsilon \subseteq \delta$, $j(\xi) \leq j(\epsilon)$. By (f) of *vi*), $r^{\xi^-}_{t+1} \downarrow$. Hence $\xi \in C_{t+1}$ by *xxv*) of 5.1. By the same token, $\epsilon \notin C_{t+1}$ for each 0-node ϵ with $\xi \subset \epsilon \subseteq \delta$. Hence $\delta \in \mathcal{D}_{t+1}$ since $a^\delta_{t+1} \downarrow$. Towards a contradiction assume $a^\xi_s = a^\xi_t$. Let β be the first node $\subseteq \xi$ which is visited at stage s. By (d) of *vi*), p^β becomes defined at stage s, and $\beta \neq \xi$ implies $j(\beta) < j(\xi)$. By *xv*) of 5.1, $\beta \not\subset \alpha$. Note that α is visited at stage s. Then either $\beta = \alpha$, or $\alpha <_L \beta$ and $\beta^- \subset \alpha$. In either case there exists a 0-node θ such that $\xi \subset \theta \subseteq \alpha$ and $j(\theta) < j(\xi)$, contradicting the fact we have found above. Therefore $a^\xi_t \neq a^\xi_s$. Let a^ξ be destroyed at a stage v, $t \leq v < s$, with v least possible. Note that $c^{\xi^-}_s \downarrow$ and so both $c^{\xi^-}_v$ and $c^{\xi^-}_{v+1}$ are defined, otherwise when c^{ξ^-} becomes defined a^δ is destroyed, contradiction. Now applying (a) of *vi*), a^δ is destroyed at stage v, contradiction. Therefore $\delta \in C[t]$.

Let y be the δ-designated number at stage t when $\mathcal{C}[t]$ is defined. Then y is the δ-designated number at stage s. By the argument used to prove Subclaim 5 in the proof of part (a) of $vi)$, Case 1 holds at α with $i = j(\delta)$ and $\beta = \delta$. This completes the proof of Case A.

CASE B. $r_s^\alpha \uparrow$ and $(\forall \beta \in \mathcal{B}(\alpha))[a_s^\beta \downarrow]$. Either Case 6 or Case 7 holds if no earlier case holds.

CASE C. Otherwise. Then for some $\beta \in \mathcal{B}(\alpha)$, $a_s^\beta \uparrow$. Let δ be the maximal 0-node $\subseteq \alpha$. If $\delta \neq \alpha$, $a_s^\delta \downarrow$ by $iv)$. If $\alpha = \delta$, then $a_s^\delta \downarrow$ by hypothesis above. Hence Case 5 holds at α in stage s. This completes the proof of $viii)$.

$ix)$ Let $c_s^\beta \downarrow \geq 1$. Suppose r^β is destroyed at stage s. Then in stage s either c^β or one of a^γ $(\gamma \in \mathcal{A}(\beta))$ is destroyed by $iv)$ and $xii)$ of 5.1.

Claim. $p_s^\gamma \downarrow$ for all $\gamma \in \mathcal{A}(\beta)$.

Proof of the Claim: We may suppose that $\mathcal{A}(\beta) \neq \emptyset$. Let $t < s$ be the greatest stage in which c^β is set equal 1. Then Case 3.1 occurs in stage t and immediately before Case 3.1 all the p^γs with $\gamma \in \mathcal{A}(\beta)$ are defined. If at some stage u, $t \leq u < s$, p^γ is destroyed for some $\gamma \in \mathcal{A}(\beta)$, then a^γ is destroyed in the same stage since p^α can only be destroyed by $(E3)$. It follows by $(E4)$ that, if one of the p^γ's is destroyed at stage u, $t \leq u < s$, then so is r^β. But r^β must be redefined at a stage v, $u < v < s$, and $c_v^\beta = 0$ since Case 7 holds, contradicting the choice of t. This completes the proof of the Claim.

Now $ix)$ is immediate if $c_{s+1}^\beta \uparrow$. Suppose for some $\gamma \in \mathcal{A}(\beta)$, a^γ is destroyed. By (c) of $vi)$, we can choose γ to be the least node in $\mathcal{A}(\beta)$. Let α receive attention at stage s. By $xv)$ of 5.1, $\alpha \not\supseteq \gamma$. Then $\alpha <_L \gamma$ or $\alpha \subset \gamma$ by $viii)$. First suppose a^τ is destroyed in the main part of the stage s with $\tau <_L \gamma$. Then $\tau <_L \beta$. So $c_{s+1}^\beta \uparrow$ by $(E1)$. For the rest we may assume that a^γ is not destroyed in the ending of stage s.

Suppose $\alpha \subseteq \beta = \gamma^-$. Then one of Cases 1.1 and 4 holds at α. But $c_s^\alpha \downarrow$ since $c_s^\beta \downarrow$ by $vi)$ of 5.1. Therefore Case 1.1 holds at α. Let ζ be the 0-node β mentioned in Case 1.1. Clearly, $\gamma \not\subseteq \zeta$ since $p_s^\gamma \downarrow$. Suppose a^γ is destroyed before $\mathcal{C}[s]$ is defined. Then $\zeta <_L \gamma$, and so $\zeta <_L \beta$. By $(E1)$, c^β is destroyed in stage s since a^ζ is destroyed in the main part of stage s. Suppose a^γ is not destroyed in stage s before $\mathcal{C}[s]$ is defined. If $c^\beta \uparrow$ when $\mathcal{C}[s]$ is defined, the result is clear. Suppose $c^\beta \downarrow$ when $\mathcal{C}[s]$ is defined. By (b) of $vi)$, $\gamma \in \mathcal{C}[s]$. Then

a^γ is not destroyed in the rest of stage s, contradiction.

Suppose $\alpha <_L \gamma$. Then $\alpha <_L \beta$. c^β is destroyed at stage s unless one of Cases 3.1 and 8 holds and $p^\pi \downarrow$ when α receives attention for the unique π such that $\pi^- = \alpha \cap \beta$ and $\pi \subseteq \beta$. For the rest we assume that Case 3.1 or Case 8 holds at stage s. a^π is destroyed in the main part of the construction in stage s since a^γ is. Hence p^π is destroyed in stage s by $(E3)$. By $(E4)$, c^β is destroyed in stage s. This is sufficient and completes the proof of ix).

x) Let $\delta \supseteq \pi$ for some $\pi \in \mathcal{B}(\beta)$. $a^\pi_s \downarrow$ since $r^\beta_s \downarrow$. Suppose a^δ is destroyed in stage s. By $xiii$) of 5.1, a^π is destroyed at stage s. By xii) we can assume that $\pi \in \mathcal{A}(\beta)$. By the last Claim above we know that $p^\gamma_s \downarrow$ for all $\gamma \in \mathcal{A}(\beta)$. By $(E3)$ and (c) of vi), all a^γ is destroyed for $\gamma \in \mathcal{A}(\beta)$ and so for all $\gamma \in \mathcal{B}(\beta)$ by $xiii$) of 5.1.

xi) Suppose δ, β satisfy the hypothesis of xi) with δ $<_L$-least possible. Towards a contradiction assume $r^\beta_{s+1} \downarrow$. Let γ be the maximal 0-node in $\mathcal{A}(\beta)$. Let α receive attention at stage s. By xv) of 5.1, $\alpha \not\supseteq \beta^\wedge\langle(2, i(\beta))\rangle$. Then by $viii$) of 5.1, there are three cases: $\alpha <_L \beta$, $\alpha \subseteq \beta$ or $\alpha \supseteq \zeta$ for some $\zeta \in \mathcal{A}(\beta)$. First suppose a^τ is destroyed in the main part of the stage s with $\tau <_L \delta$. Then $\tau <_L \gamma$ or $\gamma \subseteq \tau$ by the minimality of δ. a^γ is destroyed in stage s by $xiii$) of 5.1. By iv) of 5.1, $r^\beta_{s+1} \uparrow$, contradiction. For the rest we may assume that a^δ is not destroyed in the ending of stage s.

Suppose $\alpha \subseteq \beta$. Then one of Cases 1.1 and 4 holds at stage s. But $c^\alpha_s \downarrow$ since $c^\beta_s \downarrow$ by i) and vi) of 5.1. Therefore Case 1.1 holds. Let η be the 0-node which plays the part of β in Case 1.1. Then $\eta \not\supseteq \beta^\wedge\langle(2, i(\beta))\rangle$ by the choice of η since $r^\beta \downarrow$. Suppose a^δ is destroyed in stage s before $\mathcal{C}[s]$ is defined. Then either $\eta <_L \delta$ or $\delta \subseteq \eta$. Hence a^η is destroyed and either $\eta <_L \gamma$ or $\gamma \subseteq \eta$ since $\eta \not\supseteq \beta^\wedge\langle(2, i(\beta))\rangle$. By $xiii$) and iv) of 5.1, a^γ and r^β are destroyed in stage s, contradiction. Suppose a^δ is defined when $\mathcal{C}[s]$ is defined. Note that $r^\beta \downarrow$ when $\mathcal{C}[s]$ is defined since r^β_{s+1} and r^β cannot become defined after $\mathcal{C}[s]$ is defined. By Case 1.1, a^δ is not destroyed in stage s, contradiction.

Suppose $\alpha <_L \beta$. Since c^β is not destroyed at stage s, one of Cases 3.1 and 8 holds and $p^\pi \downarrow$ when α receives attention for the unique π such that $\pi^- = \alpha \cap \beta$ and $\pi \subseteq \beta$. Since a^δ is destroyed in the main part of the construction in stage s, then p^π is destroyed in stage s by $(E3)$. Hence c^β is destroyed in stage s by $(E4)$, and so is r^β by $(E5)$, contradiction.

Suppose $\alpha \supseteq \tau$ for some $\tau \in \mathcal{A}(\beta)$. Note that $a^\tau_s \downarrow$. By $xxi)$ of 5.1, either a^τ is destroyed in stage s or some x enters B at stage s. Suppose $a^\tau_{s+1} \uparrow$. Then $r^\beta_{s+1} \uparrow$ by $iv)$ of 5.1, contradiction. Suppose some x enters B at stage s. By Cases 3.1, and 8, a^δ is not destroyed in stage s since $r^\beta \downarrow$ when $C[s]$ is defined. This is sufficient and completes the proof of $xi)$.

$xii)$ Suppose $x \neq a^\delta_s$ for all δ. From the construction the only case which can enumerate a number in C which is not a^δ_s for some δ is Case 11 holds. Then either $x = k^\alpha_s$ or $\alpha(x) \supseteq \beta$ for some $\beta \in \mathcal{A}(\alpha)$ and x entered B after r^{β^-} attained its value at stage s. Note that for $\gamma \supseteq \beta$, when c^γ becomes defined, a^β and r^{β^-} are destroyed. Hence $x = k^\gamma_s$ for some $\gamma \supseteq \beta$ if x is not k^α_s.

$xiii)$ Let $a^\delta_t \downarrow$, $a^\beta_s \downarrow$ and $\delta \neq \beta$. Let a^β have been set equal to a^β_s at stage v. Without loss we can assume that $t \leq v$. From Case 2, clearly $a^\delta_t \neq a^\beta_s$ if $\delta <_L \beta$, $\delta \subset \beta$, $\beta \subset \delta$, $\beta^- <_L \delta$, or $\beta^{-\wedge}\langle(6, i(\beta^-))\rangle \in T \wedge \beta^{-\wedge}\langle(6, i(\beta^-))\rangle \subseteq \delta$. Suppose $\beta <_L \delta$. Let ϵ be the unique node $\subseteq \delta$ such that $\epsilon^- = \beta \cap \delta$. For the rest we can assume ϵ is an (n, j)-node for $n \leq 5$. Let a^δ be set equal to a^δ_t at stage $u < t$. Suppose $a^\beta_u \downarrow$. By Case 2 at stage v, $a^\delta_t < a^\beta_{v+1}$. Suppose $a^\beta_u \downarrow$. Also we can assume that $a^\gamma_u \downarrow$ for all $\gamma \in \mathcal{B}(\beta^-)$ with $\gamma <_L \beta$. Otherwise, a^γ must become defined at a stage $> u$ and $< v$. When a^γ becomes defined, a^δ is defined is destroyed and enumerated in C. Hence in this case a^β cannot be set equal to a^δ_{u+1} in stage v. Now let a^δ be set equal to k at stage u and k enter B at stage $z < u$. Towards a contradiction assume a^β is set equal to k at stage v. It follows that $k \notin C_v$.

Claim. Let $z < w \leq u$ and a node $\supset \beta^-$ and $\subseteq \delta$ be visited in stage w. Then there is a jump from a node $\subseteq \beta^-$ to a node $\supset \beta^-$ at stage w.

Proof of Claim: Fix such a stage w. Towards a contradiction assume there is no such jump. Then Case 1 fails at β^- in stage w since ϵ is not 6-node. By $iii)$, $k \in A^{j(\beta),0}_w \cup A^{j(\beta),1}_w$. Hence $k \in A^{j(\beta),j}_w$ if β is a $(j, j(\beta))$-node. By $iv)$, $a^\gamma_w \downarrow$ where γ is the maximal 0-node $\subseteq \beta^-$ if any. Note that a^β will be set equal to k at stage $v > u$ and for any θ, if a^θ is destroyed, a^θ enters C. Hence Case 5 holds at β^- in stage w and the construction passes to β, contradiction and completes the proof of Claim.

By the Claim, there is a jump from a node $\subseteq \beta^-$ to a node $\theta \supset \beta^-$ at stage u. Since a^δ becomes defined, $\delta <_L \theta$. Let w be the least stage such that $z < w \leq u$, at which there is a jump from a node $\subseteq \beta^-$ to a node $\tau \supset \beta^-$ with $\delta <_L \tau$, and some node $\supset \beta^-$ and

$\subseteq \delta$ is visited. If $a^{\delta}_{w} \downarrow$, then the value assigned to a^{δ} in stage u must exceed all the value of parameters at stages $\leq w$. In particular, this implies that $k < a^{\delta}_{u+1}$. Hence $a^{\delta}_{w} \uparrow$ and so $\tau^{-}\delta^{-}$. Let ζ be the unique 0-node such that $\zeta^{-} = \tau^{-}$ and $\zeta \subset \delta$. ζ exists otherwise no node $\supset \beta^{-}$ and $\subseteq \delta$ is visited in stage w. Since a^{δ} is set equal to k at stage u, $\alpha(k) \supseteq \zeta$. Suppose $p^{\tau} \downarrow$ when $\mathcal{C}[z]$ is defined. By (c) of vi) of 5.3, a^{ζ} is destroyed at a stage $\geq z$ and $< w$. But $a^{\zeta}_{w} \downarrow$. So at a stage $h > z$ and $< w$, a^{ζ} becomes defined. But h satisfies the requirements for w, contradicting the minimality of w. Therefore $p^{\tau} \uparrow$ when $\mathcal{C}[z]$ is defined. Hence $\zeta \notin \mathcal{C}[z]$. From the argument above we know a^{ζ} is not destroyed in stage z. By Cases 3.1 and 8, there is a maximal $\xi \subset \zeta$ such that $\xi \in \mathcal{C}[z]$. By v), $j(\xi) \leq j(\theta)$ for all 0-node θ such that $\xi \subset \theta \subseteq \zeta$. By ($f$) of vi), $r^{\xi^{-}}_{z+1} \downarrow$. So $\xi \in \mathcal{C}_{z+1}$ by xxv) of 5.1. By xxv) of 5.1 again, $\theta \notin \mathcal{C}_{z+1}$ for each 0-node θ such that $\xi \subset \theta \subseteq \zeta$. Hence $\zeta \in \mathcal{D}_{z+1}$. Suppose $a^{\xi}_{w+1} = a^{\xi}_{z}$. By ($d$) of vi), there is a 0-node θ such that $\xi \subset \theta \subseteq \zeta$ and $j(\theta) < j(\xi)$, contradiction. Suppose $a^{\xi}_{w+1} \neq a^{\xi}_{z}$. Let h be the least stage such that $z < h < w$ and a^{ξ} is destroyed in stage h. Note that $c^{\xi^{-}}_{h} \downarrow$ and $c^{\xi^{-}}_{h+1} \downarrow$ by the argument above used to show a^{ζ} is not destroyed in stage z. By (a) of vi), a^{ζ} is destroyed in stage h, contradicting the minimality of w by the argument above. Therefore, $a^{\delta}_{t} \neq a^{\beta}_{s}$.

Suppose $a^{\delta}_{s} \downarrow$. Let $k = a^{\delta}_{s}$. Towards a contradiction assume $k \in C_s$. Suppose k enters C at stage t. From above we know that $k \neq a^{\beta}_{t}$ for all $\beta \neq \delta$. By xii), $k = k^{\beta}_{t}$ for some β and Case 11 holds in stage t. By vii) of 5.2, $\delta <_L \beta$. Let α receive attention at stage t. Suppose $\alpha = \beta$. Clearly a^{δ} was set equal to k before stage t by Case 2. But when a^{δ} becomes defined, c^{β} is destroyed and then k^{β}_{t} cannot enter C by Case 11 at a stage in which β receives attention. Suppose $\alpha \neq \beta$. From Case 11, $\beta \supseteq \pi \in \mathcal{A}(\alpha)$ for some π. Since $\delta <_L \beta$, either $\pi \subset \delta$ or $\delta <_L \pi$. Also, a^{δ} was set equal to k before stage t. When a^{δ} becomes defined, a^{π} and r^{α} are destroyed. So k^{β}_{t} cannot enter C by Case 11. This completes the proof of $xiii$) and then Lemma 5.3. ∎

Define the *true path* P to be the subset of all $\alpha \in T$ such that α is visited infinitely often and there are at most finitely many stages in which some $\beta <_L \alpha$ is visited.

In the next lemma when we say that a parameter is *eventually fixed* we mean the parameter is defined only finitely often. If we say that a parameter $p_s \longrightarrow \infty$ as $s \longrightarrow \infty$ we mean that there are infinitely many stages in which p becomes defined and that the values assigned to s tend to ∞.

5.4 Lemma. (True Path Lemma)
For each $\alpha \in P$ the following hold:

i) For all $\beta <_L \alpha$, eventually a^β, p^β, k^β, c^β and r^β are fixed.

ii) If α is a 0-node then p^α is eventually always undefined and $a^\alpha_s \to \infty$ as $s \to \infty$.

iii) k^α, c^α are eventually fixed.

iv)
- α is not a 1-node;
- α is a 2-node $\Longrightarrow c^{\alpha^-}_\omega = 0$;
- α is a 3-node $\Longrightarrow c^{\alpha^-}_\omega \geq 1$.

v) There is $w \in \Lambda$ such that $\alpha^\wedge\langle w\rangle \in P$.

vi) If α is a 2-node or 3-node then a^β is eventually always defined for all $\beta \in \mathcal{B}(\alpha^-)$.

vii) If no $\beta \in \mathcal{A}(\alpha^-)$ is visited infinitely often, then r^{α^-} is eventually fixed.

viii) k^α, c^α are eventually always defined.

ix) For each n there are at most finitely many s such that $(\exists\beta)(\alpha <_L \beta \wedge a^\beta_s \downarrow \wedge a^\beta_s < n)$.

Proof. We will use induction on $(T, \leq)$.

i) Let $\beta <_L \alpha$ be a 0-node. Once defined, a^β retains the same value unless destroyed; and similarly for p^β. a^β can become defined only when β is visited, and thus only finitely often. Further, p^β can become defined only while a^β is defined and a^β is destroyed whenever p^β is. Thus p^β becomes defined only finitely often. The rest is clear because whenever a^β (k^β or c^β or r^β) becomes defined β is visited. This is sufficient.

ii) Consider what happen in the construction once nodes $<_L \alpha$ have been visited and the parameters belonging to those nodes have been changed for the last time. There are two cases:

Case 1. $(\exists\beta \in \mathcal{A}(\alpha^-))[\beta <_L \alpha]$. By Case 3, when p^α becomes defined, a node $<_L \alpha$ is visited. Hence p^α cannot become defined infinitely often. By *xv)* of 5.1, if $p^\alpha_s \downarrow$, α is not visited at stage s. Hence p^α is eventually undefined.

Case 2. Otherwise. Note that when p^α becomes defined, c^{α^-} is set equal 1. From *vii*) of 5.1, $c_s^{\alpha^-} = 0$ if c^{α^-} is set equal 1 at stage s. Bu *iii*) of the induction hypothesis c^{α^-} is eventually fixed. Hence p^α becomes defined only finitely often. But by *xv*) of 5.1,

$p^\alpha \uparrow$ when α is visited. Therefore p^α is eventually permanently undefined.

Each time a^α is redefined it is given a strictly larger value. By *iv*) of 5.3, if α is visited at stage s and $a_s^\alpha \uparrow$, then a^α becomes defined at stage s. Towards a contradiction assume a^α is destroyed only finitely often. Consider what happens in the construction once a^α has been destroyed for the last time. Since a^α is not destroyed, by *xxi*) of 5.1 at each stage in which α is visited some y with $\alpha(y) \supseteq \alpha$ enters B. Further, $k^{\alpha(y)}$ was set equal to y before a^α was set. So there are only finitely many possible y. When $k^{\alpha(y)}$ enters B, $c^{\alpha(y)}$ is set equal to 1. $k^{\alpha(y)}$ cannot enter B at a later stage unless $c^{\alpha(y)}$ becomes 0 again which means that $k^{\alpha(y)}$ is reset to a larger value. We conclude that α is visited only finitely often. This contradiction completes the proof.

iii) By *ii*) of 5.1, once defined, c^α is monotonic non-decreasing ≤ 2 unless destroyed because one of the following holds:

- *a node* $<_L \alpha$ *is visited;*
- $c_\omega^{\alpha^-} \geq 1 \iff \alpha$ *is a 3-node.*

a) some γ with $\gamma^\wedge\langle(2, i(\gamma))\rangle \leq \alpha$ receives attention, and c^γ is set equal to 1;

b) p^γ is destroyed for some $\gamma \subseteq \alpha$;

c) a^γ is destroyed for some $\gamma <_L \alpha$.

By the induction hypothesis, this yields the desired conclusion for c^α. Once defined, k^α retains the same value until reset to a new value. If k^α is reset in a stage, then c^α becomes defined in the same stage.

iv) Towards a contradiction assume α is a 1-node. When α is visited, a^α becomes defined if it is not already defined and retains the same value unless destroyed. Whenever a^α is destroyed, by Lemma 5.1 *xii*), a^β is destroyed for some $\beta \in \mathcal{A}(\alpha^-)$ with $\beta <_L \alpha$. Notice that $\beta <_L \alpha$. By *i*), this happens at most finitely often. Eventually a^α is defined never to be destroyed again. Whenever a^α is defined, α is not visited, a contradiction.

Suppose α is a 2-node. By construction, α is visited at stage s only if $c_s^{\alpha^-} = 0$. This is sufficient. Similar for the case α is a 3-node.

v) First we show that for some immediate successor δ of α, there exist infinitely many stages at which some node $\supseteq \delta$ is visited. Towards a contradiction suppose there is no such δ. Clearly, there are only a finite number of stages in which nodes $\supset \alpha$ are visited. Consider what happens in the construction once the nodes $<_L \alpha$ and $\supset \alpha$ have been visited and the parameters belonging those nodes have been changed for the last time. Note that a parameter belonging to β can only be assigned a new value in a stage in which β is visited. By the induction hypothesis, *ii*) and *iii*) we can also assume that all parameters k, c and p belonging to nodes $\subseteq \alpha$ have been changed for the last time.

Suppose α is a 0-node. Clearly, when α is visited a^α becomes defined if it is not already defined and retains the same value unless destroyed. By 5.1 *viii*), eventually a^α is destroyed only if α receives attention.

Now we show that, whether α is a 0-node or not, c^α is eventually always defined. Suppose not. By the induction hypothesis, choose the least stage t after which, r^δ for $\delta^\wedge\langle(2, i(\delta))\rangle < \alpha$, a^γ for $\gamma <_L \alpha$, and k^γ for $\gamma < \alpha$ are all fixed. Let n be a number greater than all such parameters. Let s be the last stage if any such that $c_s^\alpha \downarrow$. Choose the least α-number $m > max\{n, k_s^\alpha\}$. Notice this α-number is unused because any used α-number is one of the (previous) values of k^α. Let γ be the maximal 0-node $\subseteq \alpha$. By *ii*), $a_v^\gamma \to \infty$ as $v \to \infty$. Choose the least $v > t$ such that $a_v^\gamma > m$. Let u be the least stage $\geq v$ at which α is visited and $a_u^\alpha \downarrow$ if α is a 0-node. Such u exists. Towards a contradiction assume one of Cases 1-3 holds at α in stage u. Note that α receives attention. Case 1.1. cannot hold; otherwise a^β is destroyed for some 0-node $\beta \supset \alpha$, contradiction. Case 2 cannot hold since if α is a 0-node, then $a_u^\alpha \downarrow$. Case 3 cannot holds at α; otherwise c^{α^-} is set equal 1, contradiction. Therefore none of Cases 1-3 hold at α in stage u. Note that if $\alpha \neq \gamma$, then $a_u^\gamma \downarrow$ by *iv*) of 5.3. By *vi*) of 5.1, $c_u^{\alpha^-} \downarrow$. Hence Case 4 holds at α when α is visited in stage u. c^α becomes defined at stage u, contradiction.

Consider a stage s at which α is visited and $a_s^\alpha \downarrow$. From the argument of the last paragraph there is such an s and none of Cases 1-4 holds at α in stage s. But by inspection of the construction in each of these cases either the construction passes below α or one of c^α,

c^{α^-} is increased. This contradicts our findings above. We conclude that for some immediate successor δ of α there are infinitely many stages in which a node $\supseteq \delta$ is visited. Fix δ to be the least such node.

We will prove that δ is visited infinitely often. Towards a contradiction suppose there is a stage t after which δ is never visited. Consider a stage $v > t$ at which a node $\supset \delta$ is visited. In stage v there is a jump from a node $\subseteq \alpha$ to a node $\tau = \tau_v \supset \delta$ via Case 1.2. We node that $a_v^\tau \downarrow$ and $r_v^{\tau^-} \downarrow$ since Case 1.2 requires that there be a number designated for τ. Let $u = u_v < v$ be the stage at which a^τ was set equal to a_v^τ. Consider the least v, if any, such that $t < u$. In stage u, a^σ is destroyed for all σ such that either $\sigma \subset \tau$ or $\tau <_L \sigma$. Note that $r_{u+1}^{\tau^-} \uparrow$. Let s be the least stage $> s$ in which r^{τ^-} becomes defined. Clearly, $s < v$. Since in stage s the construction jumps from a node $\subseteq \alpha$ to τ_s and then moves left, $\tau <_L \tau_s$. Clearly, a^{τ_s} becomes defined at a stage $> u$ and $< s$. This contradicts the choice of v. Hence $u_v \leq t$ for all v.

Consider stages $v > w > t$ such that in both stage v and stage w a node $\supset \delta$ is visited. In stage w, p^{τ_w} becomes defined. Either $p_v^{\tau_w} \downarrow$ or p^{τ_w}, and hence also a^{τ_w}, is destroyed at a stage $\geq w$ and $< v$. In the latter case $\tau_v = \tau_w$ would imply $u_v \geq w > t$, contradiction. In the former case $\tau_v \neq \tau_w$ because p^{τ_w} is already defined. Therefore $\tau_v \neq \tau_w$. Since there are only a finite number of values possible for τ_v, the proof is completed.

vi) By *i)*, a^β is fixed for all $\beta \in \mathcal{B}(\alpha^-)$. Let t be a stage after which a^β is fixed for all $\beta \in \mathcal{B}(\alpha^-)$. Towards a contradiction consider $\gamma = \alpha^-{}^\wedge\langle(i,j)\rangle$, the least node in $\mathcal{B}(\alpha^-)$ such that $a_{t+1}^\gamma \uparrow$. Consider a stage $s > t$ at which α is visited. Since α is a 2- or 3-node, α^- is visited at stage s. We want to show such s does not exist.

Since the construction passes from α^- to α one of Case 6, 10, and Case 12 occurs at α^-. Towards a contradiction suppose $c_s^{\alpha^-} \geq 1$ and let v be the greatest stage $< s$ in which c^{α^-} is set equal to 1. By *vii)* of 5.1, $r_{v+1}^{\alpha^-} \downarrow$. Since $a_s^\gamma \uparrow$, $r_s^{\alpha^-} \uparrow$ by *i)* of 5.1. Let u be the least stage $> v$ such that $r_{u+1}^{\alpha^-} \uparrow$. By *ix)* of 5.3 one of $c_u^{\alpha^-}$ and $c_{u+1}^{\alpha^-}$ is undefined. Since $u < s$, the choice of v is contradicted. Therefore $c_s^{\alpha^-} \not\geq 1$.

By *vi)* of 5.1, $c_s^{\alpha^-} \downarrow$ and so $c_s^{\alpha^-} = 0$. Let δ be the maximal 0-node $\subseteq \alpha^-$. By *iv)* of 5.3 since α^- and α are visited at stage s, $a_s^\delta \downarrow$ if δ exists. Now Case 5 holds at α^- in stage s. By Case 5 and the choice of t, pass to $\alpha^-{}^\wedge\langle(4+i,j)\rangle$, and then α is not visited in stage s, contradiction.

vii) If no $\beta \in \mathcal{A}(\alpha^-)$ is visited infinitely often, it follows that $\alpha^{-\wedge}\langle(2, i(\alpha^-))\rangle \leq \alpha$. If $\beta \in \mathcal{B}(\alpha^-)$, then a^β is fixed eventually by *i)*. So after a^β is fixed for all $\beta \in \mathcal{B}(\alpha^-)$, and c^{α^-} is fixed, r^{α^-} changes at most once by *iv)* of 5.1. This is sufficient.

viii) After c^α is fixed for the last time, there is a stage at which some node $\supset \alpha$ is visited. By *vi)* of 5.1, $c^\alpha \downarrow$. By *i)* of 5.1, $k^\alpha \downarrow$ whenever $c^\alpha \downarrow$.

ix) Let $\beta >_L \alpha$ and $\beta \in \mathcal{B}(\beta^-)$. Suppose that a^β becomes defined for the last time in stage t. Consider a stage $s > t$ in which α is visited and such that $a^\beta_s = a^\beta_{s+1} = a^\beta_{t+1} \downarrow$. Since a^β is not destroyed, by inspection of the construction Case 3.1 or Case 8 holds in stage s and some y enters B with $\alpha(y) <_L \beta$ or $\alpha(y) \supseteq \beta$. Since $a^\beta_s = a^\beta_{s+1}$, $y = k_t^{\alpha(y)}$ because, when $k^{\alpha(y)}$ is reset by Case 4, a^β is destroyed. Hence there are only a finite number of possible values of y. From the proof of *ii)* each y is enumerated in B at most once. Hence there are only a finite number of possibility for s. We conclude that a^β never becomes defined for the last time. This is sufficient because the value assignated to a^β as the construction proceeds are strictly increasing. ∎

Below, when $\alpha \in P$, we shall use α^+ to denote the unique immediate successor of α in P.

Chapter 6

Verification, Part II

In the present chapter, we verify that all requirements are satisfied.

6.1 Lemma. *For each $i < \omega$, $\mathcal{R}^i$ is satisfied.*

Proof. We proceed by induction on i. Fix i. The set

$$\{\beta \in T : (2,i),\ (3,i) \text{ do not occur in } \beta\}$$

is finite. Hence there exists $\alpha \in P$ such that $i(\alpha) = i$ and α^+ is either a 2- or 3-node. From the definition of T, α is uniquely determined by i. We will show that the strategy associated with α succeeds for $\mathcal{R}^i$.

By 5.4, fix the least stage t after which:

A. δ is not visited for $\delta <_L \alpha$ and a^δ, c^δ, k^δ, p^δ are fixed for $\delta <_L \alpha$,

B. k^δ is fixed for $\delta \subset \alpha$,

C. p^δ is fixed for $\delta \subseteq \alpha$,

D. $\delta^\wedge\langle(3, i(\delta))\rangle \subseteq \alpha \wedge c^\delta_\omega \geq 1 \Longrightarrow (\forall s > t)[c^\delta_s \downarrow\geq 1]$,

E. r^δ is fixed for δ with $\delta^\wedge\langle(2, i(\delta))\rangle \leq \alpha$.

We first show that $c^{\alpha}_{t+1} \uparrow$.

Let β denote the node which receives attention at stage t. Towards a contradiction suppose that c^{α} becomes defined in stage t. By Case 4 of the construction the only parameters (apart from c^{α}) belonging to a node $\leq \alpha$, which change in stage t, change through the destruction of a^{δ} for $\delta \subseteq \alpha$. By *xv*) of 5.1, no p^{δ} with $\delta \subseteq \alpha$ is defined at any point in stage t. Examining $(E1) - (E5)$ we see that $A - E$ all hold for stage t, contradicting the choice of t.

Suppose $c^{\alpha}_t \downarrow$. Suppose A fails in stage t. A number of cases must be examined. We will treat the case in which c^{δ} changes for some $\delta <_L \alpha$. The other cases will be left to the reader. Suppose $c^{\delta}_{t+1} \downarrow$. Then δ is visited and c^{α} is destroyed unless $p^{\pi}_{t+1} \downarrow$, where $\pi \subseteq \alpha$ and $\pi \in \mathcal{A}(\alpha \cap \delta)$. But from *ii*) of 5.4, $p^{\pi}_{t+1} \uparrow$ by choice of t. Therefore $c^{\alpha}_{t+1} \uparrow$. Suppose that c^{δ} is destroyed in stage t. From *ii*) of 5.1 we have the following cases:

Case 1. $\beta <_L \delta$. We see that $c^{\alpha}_{t+1} \uparrow$ by repeating the argument above.

Case 2. a^{γ} is destroyed for some $\gamma <_L \delta$. By $(E1)$, c^{α} is also destroyed.

Case 3. Case 8 holds at β, $\beta \subset \delta$, and $\beta^\wedge\langle(2, i(\beta))\rangle \leq \delta$. Either $\beta <_L \alpha$, or $\beta \subset \alpha$ and $\beta^\wedge\langle(2, i(\beta))\rangle \leq \alpha$. We may suppose that the latter holds. Then c^{α} is destroyed in the main part of the stage.

Case 4. p^{γ} is destroyed for some $\gamma \subseteq \delta$. Then a^{γ} is destroyed. Either $\gamma <_L \alpha$ which takes us to Case 2, or $\gamma \subset \alpha$. By $(E4)$ c^{α} is destroyed.

The cases in which either B or C fails in stages t is quite easy.

Suppose D fails in stage t. Then $c^{\delta}_t = 0$ and $c^{\delta}_{t+1} = 1$. Either Case 3.1 holds at β which is the least node in $\mathcal{A}(\delta)$ or Case 8 holds at $\delta = \beta$. If Case 3.1 holds, then $\beta <_L \alpha$ which takes us to A. Suppose Case 8 holds. By Case 8, c^{α} is destroyed.

Suppose E fails in stage t. Suppose r^{δ} is destroyed in stage t. By *iv*) and *xii*) of 5.1, either c^{δ} or a^{γ} is destroyed for some $\gamma \in \mathcal{A}(\delta)$. By *iii*) of 5.1 and $(E1)$ since $\gamma <_L \alpha$, in either case c^{α} is destroyed. Therefore $c^{\alpha}_{t+1} \uparrow$.

Let $s_0 > t$ be least such that c^{α} is defined at stage s_0. By Case 4, $c^{\alpha}_{s_0+1} \downarrow = 0$ and k^{α} becomes defined at stage s_0. Note by *ii*) of 5.1, c^{α} is not destroyed after stage s_0. Hence k^{α} is eventually always defined and constant. To see $\mathcal{R}^i$ is satisfied, there are two subcases.

CASE 1. $c^\alpha_\omega = 0$. Then eventually for stages at which α^+ is visited, $c^\alpha = 0$. By *iv*) of 5.4, α^+ is a 2-node. Note that $B_\omega(k^\alpha_\omega) = 0$. Suppose $\Psi^i_\omega(W^i_\omega; k^\alpha_\omega)\uparrow$. Then clearly $\mathcal{R}^i$ is satisfied. Similarly if $\Psi^i_\omega(W^i_\omega; k^\alpha_\omega)\downarrow\neq 0$. Suppose $\Psi^i_\omega(W^i_\omega; k^\alpha_\omega)\downarrow= 0$. Let $n = \psi^i_\omega(W^i_\omega, k^\alpha_\omega)$ and $s_1 > s_0$ be such that for all $s \geq s_1$

$$\Psi^i_s(W^i_s \upharpoonright n; k^\alpha_\omega)\downarrow= 0 \wedge W^i_s \upharpoonright n = W^i_\omega \upharpoonright n.$$

Towards a contradiction assume that for all $x \leq n$, $\Phi^i_\omega(D_\omega; x) = W^i_\omega(x)$. Let $m = \phi^i_\omega(D_\omega, n)$. Let $s_2 > s_1$ be such that for all $s \geq s_2$ and all $x \leq n$,

$$\Phi^i_s(D_s \upharpoonright m; x) = W^i_s(x) = W^i_\omega(x) \wedge \phi^i_s(D_s, n) = m \wedge D_s \upharpoonright m = D_\omega \upharpoonright m.$$

By *ii*) of 5.4 let $v \geq s_2$ be the least stage in which $\alpha^\wedge\langle(2,i)\rangle$ is visited and for all $j \leq 1$ and all j-node β such that either $\beta \subseteq \alpha$ or $\alpha^\wedge\langle(2,i)\rangle <_L \beta$

$$a^\beta_v\downarrow\Longrightarrow a^\beta_v > m.$$

Note that α is visited at stage v and none of Cases 1-5 holds at α; otherwise $\alpha^\wedge\langle(2,i)\rangle$ is not visited. But we can see that α is ready at stage v by the choice of n, m and v. Then c^α is set equal to 1 at stage v, contradiction. Therefore

$$(\exists x \leq n)[\Phi^i_\omega(D_\omega; x) \neq W^i_\omega(x)],$$

and so $\mathcal{R}^i$ is satisfied.

CASE 2. Otherwise. Since $c^\alpha_\omega \geq 1$, $\alpha^+ = \alpha^\wedge\langle(3,i)\rangle$. Then there exists $s > s_0$ such that c^α is set equal 1 at stage s. Let s_1 denote the least such s. Let z denote the stage $\leq s_1$ in which r^α attained its value at stage s_1.

CASE 2.1. There is a stage $s > s_1$ at which c^α was set equal 2. Let s_2 be the least such s. When c^α is set equal 1 at stage s_1, α-attack is completed. Note that by Cases 3.1 and 8, $r^\alpha_{s_1+1}\downarrow$. By *ix*) of 5.3, r^α is not destroyed after stage s_1, and then is not destroyed after stage z by the choice of z.

Claim 1. There is no $x \leq r^\alpha_{z+1}$ which enters C at stage $v \in \{s : s \geq z \wedge s \neq s_2\}$.

Proof of Claim 1: Fix v such that $v \geq z$ and $\neq s_2$. Towards a contradiction assume $x \leq r^\alpha_{z+1}$ enters C at stage v. By *xii*) of 5.3, $x = a^\delta_v$ or $x = k^\delta_v$ for some δ.

Towards a contradiction assume $x = a^\delta_v$. By the choice of t and by $xiii)$ of 5.3, $\delta \not<_L \alpha$ and a^δ is destroyed in stage v. Suppose $\alpha^\wedge\langle(2,i)\rangle <_L \delta$. Suppose $a^\delta_z \downarrow$. Since α is ready at stage z, $r^\alpha_{z+1} < a^\delta_z$, contradiction. Suppose $a^\delta_z \uparrow$. Since $a^\delta_v \downarrow$, let a^δ was set equal to a^δ_v at a stage u with $z \le u < v$. $u \ne z$ by the choice of z. When a^β becomes defined at stage $u > z$, $r^\alpha_{z+1} < a^\beta$ by Case 2, contradiction. Suppose $\delta \subseteq \alpha$. Let γ be the maximal 0-node $\subseteq \alpha$. By $iv)$ of 5.3, when r^α becomes defined at stage z, $a^\gamma_z \downarrow$. Since α is ready at stage z, $r^\alpha_{z+1} < a^\gamma_z$. By $ii)$ of 5.2, $a^\gamma_z \le a^\delta_v$ since $\delta \subseteq \alpha$, contradiction. Suppose $\delta \supseteq \alpha^\wedge\langle(2,i)\rangle$. By $xi)$ of 5.3, if a^δ is destroyed at stage v in which r^α is defined, then r^α is destroyed in the same stage, contradiction. Suppose $\delta \supseteq \pi$ for some $\pi \in \mathcal{B}(\alpha)$. By $xiii)$ and $iv)$ of 5.1, a^π and r^α are destroyed at stage v, contradiction. Therefore $x \ne a^\delta_v$ for some δ.

Suppose $x = k^\delta_v$. There are two cases:

Case 1. δ receives attention at stage v. Note that $c^\delta_v \downarrow$, c^δ is set equal to 2 in stage v, and $v \ne s_2$. Therefore $\delta \ne \alpha$. $\delta \not< \alpha$ by the choice of t. Suppose $\alpha^\wedge\langle(2,i)\rangle \le \delta$. Then $v \ne z$ and so $v > z$. By $xv)$ of 5.1, there is no $\pi \subseteq \delta$ with $p^\pi_v \downarrow$. By $v)$ of 5.2, $r^\alpha_{z+1} < k^\delta_v$. Suppose $\delta \supseteq \pi$ for some $\pi \in \mathcal{A}(\alpha)$. By Case 11, a^π_v is destroyed in stage v. Hence r^α is destroyed at stage v by $iv)$ of 5.1, contradiction.

Case 2. Otherwise. By $xii)$ of 5.3, there exists $\beta \subset \delta$ such that Case 11 holds at β in stage v. Note that $c^\beta_v \downarrow$. By the argument of Case 1 it is sufficient to show that $k^\beta_v < k^\delta_v$. Suppose $c^\delta_v \downarrow$. By $i)$ of 5.2, $k^\beta_v < k^\delta_v$. Suppose $c^\delta_v \uparrow$. Let $w < v$ be the greatest stage such that $c^\delta_w \downarrow$. Note that k^δ_v entered B before stage w and c^δ_w is destroyed at stage w. $c^\beta_w \downarrow$ by $vi)$ of 5.1. By $i)$ of 5.2, $k^\beta_w < k^\delta_w = k^\delta_v$. Towards a contradiction assume that $k^\beta_w \ne k^\beta_v$. Then at a stage u, $w < u < v$, c^β is destroyed. Hence r^β is destroyed at stage u. By Case 11, when Case 11 holds for β at stage v, we just need to enumerate the numbers which entered B after r^β was set equal to its present value. But k^δ_v entered B before stage w and so before stage u. Hence k^δ_v does not enter C at stage v, contradiction. This completes the proof of Claim 1.

Claim 2. If x enters B at a stage $v \ge z$ with $x \le r^\alpha_{+1}$, then $v \le s_1$ and either $\alpha(x) = \alpha$ or $\alpha(x) \supseteq \theta$ for some $\theta \in \mathcal{A}(\alpha)$.

Proof of Claim 2: Fix x, v such that $z \le v$ and $x \le r^\alpha_{z+1}$ enters B at stage v. Then $x = k^\delta_v$ for some δ by $vii)$ of 5.1. By the same token, $c^\delta_v \downarrow= 0$, $c^\delta_{v+1} \downarrow= 1$ and δ receives attention at stage v. For the rest we suppose $\delta \neq \alpha$. $\delta \not< \alpha$ by the choice of t. Suppose $\alpha^\wedge\langle(2,i)\rangle \leq \delta$. By choice of z, $v \neq z$. By xv) of 5.1, $p_v^\pi \uparrow$ for all 0-node $\pi \subseteq \delta$. By v) of 5.2, $r_{z+1}^\alpha < k_v^\delta$, contradiction. Therefore, $\delta \supseteq \theta$ for some $\theta \in \mathcal{A}(\alpha)$. Towards a contradiction assume $s_1 < v$. Recall that r^α is not destroyed after stage z. Since p^θ is defined at some point in stage s_1, p^θ is not destroyed at any stage $\geq s_1$. Otherwise, a^θ and r^α are destroyed, contradiction. Then by xv) of 5.1, δ cannot be visited at stage v, contradiction. This completes the proof of Claim 2.

Since Case 11 holds at α in stage s_2, there exists $x \leq \psi_z^i(W_z^i, k^\alpha)$ such that $x \in W_{s_2}^i - W_z^i$. At stage z we have

$$\Phi_z^i(D_z; x) = W_z^i(x) = 0 \neq 1 = W_{s_2}^i(x).$$

In stage s_2, by Case 11, all numbers $x < r_{z+1}^\alpha$ which entered B at a stage $\geq z$ and $< s_2$ with $\alpha(x) \supseteq \theta$ for some $\theta \in \mathcal{A}(\alpha)$ are enumerated in C. Hence by Claims 1 and 2

$$D_{s_2+1} \upharpoonright r_{z+1}^\alpha = D_z \upharpoonright r_{z+1}^\alpha.$$

By definition, $r_{z+1}^\alpha \geq \varphi_z^i(D_z, \psi_z^i(W_z^i, k^\alpha)) \geq \varphi_z^i(D_z, x)$. Therefore,

$$\Phi_{s_2+1}^i(D_{s_2+1}; x) = \Phi_z^i(D_z; x) = 0 \neq 1 = W_{s_2}^i(x).$$

By Claims 1 and 2, at the end of the construction, $\Phi^i(D)$ and W^i disagree at x.

Case 2.2. Otherwise.

Let r, k the values of r^α, k^α at the end of stage s_1. We know $r^\alpha = r, k^\alpha = k$ for all stages $> s_1$. Since α never requires attention at a stage $> s_1$, we have

$$(\forall x \leq \psi_z^i(W_z^i, k^\alpha))[W^i(x) = W_z^i(x)].$$

But k^α is enumerated in B at stage s_1 and is never enumerated in C. Thus

$$\Psi^i(W^i; k^\alpha) = \Psi_z^i(W_z^i; k^\alpha) = 0 \neq 1 = D_{s_1+1}(k^\alpha) = D(k^\alpha).$$

This completes the proof of 6.1. ■

6.2 Definition. i is called *active on P* if there exists $\alpha \in P$ such that for all $\beta \supset \alpha$, $\beta \in P$, i is active at β.

6.3 Lemma. *If i is not active on P, then there exists $\alpha \in P$ such that i is not active at β for any $\beta \in P$, $\alpha \subset \beta$.*

Proof. It is obvious if one of $(4,i),(5,i)$, and $(6,i)$ occurs in P. Otherwise, for all $\alpha \in P$, there exists $\beta \supseteq \alpha$, $\beta \in P$, such that i is not active at β. Notice that for all k, α, β, if k is not active at α, $\alpha \subset \beta$, and k is active at β, then there exists $j < k$ such that one of $(0,j),(4,j),(5,j)$ occurs on β below α. Also, for any β, if there exist $n < m$ such that $\beta(n) = \beta(m) = (0,j)$ then there exist k, l such that $n < l < m$, $k < j$ and $\beta(l) \in \{(0,k),(4,k),(5,k)\}$. This is sufficient. ∎

6.4 Lemma. *For each $i < \omega$, if $A^{i,0} \sqcup A^{i,1} \supseteq B$, then*

i) $(4,i)$ *occurs on* $P \Longrightarrow D \leq_T A^{i,1} \cap D$.

ii) $(5,i)$ *occurs on* $P \Longrightarrow D \leq_T A^{i,0} \cap D$.

iii) Otherwise,

(a) i *is not active on* $P \Longrightarrow D \leq_T A^{i,1} \cap D$.

(b) i *is active on* $P \Longrightarrow D \leq_T A^{i,0} \cap D$.

Proof. Fix i such that $A^{i,0} \sqcup A^{i,1} \supseteq B$. Note that $(6,i)$ does not occur on P. Let J_i be the set of all $j \leq i$ which are not active on P. Let I_i be the set of all $j \leq i$ with $j \notin J_i$. Let α_0 be the least $\alpha \in P$ such that

i) $(\forall j)(\forall \beta \in P)[j \in J_i \wedge \alpha \subseteq \beta^+ \Rightarrow j \textit{ is not active at } \beta]$;

ii) $(\forall j)(\forall \beta \in P)[j \in I_i \wedge \alpha \subseteq \beta^+ \Rightarrow j \textit{ is active at } \beta]$;

iii) $(\forall j \leq i)[(2,j) \in ran(\alpha) \vee (3,j) \in ran(\alpha)]$.

For each $j \leq i$, let β_j denote the $(k, j-1)$-node on α_0 for $k \in \{2,3\}$. Then for $n < m \leq i$, $\beta_n \subset \beta_m$. Recall that λ is the $(2,-1)$-node. Let s_0 be the least stage after which

- *no $\delta <_L \alpha_0$ is visited*
- *a^δ and p^δ are fixed for all $\delta <_L \alpha_0$*

- *k^δ, c^δ and p^δ are fixed for all δ such that $\delta \leq \alpha_0$*
- *r^δ is fixed for all δ such that $\delta^\wedge\langle(2, i(\delta))\rangle \leq \alpha_0$.*

Remarks: 1. If α_0 is an (n, j)-node, then $j \geq i$ and, if $j = i$, then $n \in \{2, 3\}$.

2. After stage s_0, we cannot jump to some node $\theta \subseteq \alpha_0$, otherwise p^θ becomes defined can can never subsequently be destroyed, contradiction.

Case 1. $(4, i)$ occurs in P. Let δ_0 be the least $\delta \in P$ such that $(4, i)$ occurs in δ, and γ_0 be the maximal 0-node $\subset \delta_0$ if any.

Case 1.1. γ_0 undefined. In this case after stage s_0 every sufficiently number which enters B is enumerated in $A^{i,1} \cup C$, hence $D \leq_T A^{i,1} \cap D$. Otherwise, for some sufficiently large x which is enumerated in B, $x \notin C$ and $x \in A^{i,0}$. Consider a stage in which $x \in A^{i,0}$ and δ_0 is visited. Since δ_0 is a 4-node, δ_0^- is visited that stage and no Cases 1-4 hold. Hence Case 5 requires to pass $\delta_0^\wedge\langle(0, i)\rangle$, which is $<_L \delta_0$, contradiction.

Case 1.2. γ_0 exists. There are an infinite number of stages at which some node $\supset \gamma_0$ is visited. We want to show that $D \leq_T A^{i,1} - C$. Fix $x > s_0$. Choose s to be the least stage such that

- *δ_0 is visited at stage s*
- $x < a_s^{\gamma_0} \downarrow$
- $(A_s^{i,1} - C_s) \restriction (x + 1) = (A_\omega^{i,1} - C_\omega) \restriction (x + 1)$
- $D_s \restriction (x + 1) \subseteq A_s^{i,0} \cup A_s^{i,1}$.

Claim. $x \in D$ if and only if $x \in D_s - E_s$ where

$$E_s = \{a_s^\epsilon : a_s^\epsilon \downarrow \wedge [\delta_0 <_L \epsilon \vee [\delta_0^{-\wedge}\langle(6, i(\delta_0^-))\rangle \in T \wedge \delta_0^{-\wedge}\langle(6, i(\delta_0^-))\rangle \subset \epsilon]]\}.$$

Proof of Claim: Note that if $x \in E_s$, $x \notin D$ by *ix*) of 5.4. Towards a contradiction assume $x \in D - D_s$. Let x enter B at a stage $t \geq s$. Then $x = k_t^\delta$ for some δ with $c_t^\delta \downarrow = 0$ and $c_{t+1}^\delta = 1$ by *vii*) of 5.1. By the choice of s_0 we have $\gamma_0 \subseteq \delta$ or $\gamma_0 <_L \delta$. Suppose $\gamma_0 <_L \delta$. Then $a_s^{\gamma_0} < k_t^\delta = x$ by *iv*) of 5.2, contradiction. Therefore $\gamma_0 \subseteq \delta$. Since c^δ increases at a stage $> s_0$, either $\delta_0^- <_L \delta$ or $\delta_0^- \subset \delta$. Since $A^{i,0} \cup A^{i,1} \supseteq D$, let $v > t$ be the least stage such that $x \in A_v^{i,0} \cup A_v^{i,1}$. Note that we can assume that for each stage u, $A_u^{i,0} \cup A_u^{i,1} \subseteq B_u$. Hence $x \notin A_s^{i,0} \cup A_s^{i,1}$ since $x \notin B_s$. Towards a contradiction assume $x \in A_v^{i,0}$. Let $u \geq v$ be the least stage at which δ_0 is visited. u exists since $\delta_0 \in P$. $a_u^{\gamma_0} \downarrow$ by $iv)$ of 5.3. Note that $x \notin C$ and $x \notin E_s$. Then δ_0^- is visited at stage u, $c_u^{\delta_0^-} \downarrow = 0$ and none of Cases 1-4 holds at δ_0^-; otherwise δ_0 cannot be visited at stage u. By 5.2, $k_u^{\delta_0^-} = k_{s_0+1}^{\delta_0^-} < k_t^\delta < a_u^{\gamma_0} \downarrow$. Since $x > s_0$, by Case 5, we pass to $\delta_0^- {}^\wedge \langle (0, i) \rangle$ which $<_L \delta_0$, contradiction. Therefore $x \in A_v^{i,1}$. Thus $x \in (A^{i,1} - C) - (A_s^{i,1} - C_s)$, contradicting the choice of s. Therefore, if $x \in D$, then $x \in D_s$.

Towards a contradiction assume $x \in D_s - (D \cup E_s)$. Let x enter C at a stage $v \geq s$ and x have entered B at a stage $t > s_0$. Note that $t < s$. Suppose $\alpha(x) \supseteq \gamma_0$. By choice of s, $x \in A_s^{i,0} \cup A_s^{i,1}$. Hence $x \notin A_{s_0}^{i,0} \cup A_{s_0}^{i,1}$ since $x \notin B_{s_0}$. Towards a contradiction assume $x \in A_s^{i,0}$. Repeating the argument of the last paragraph with s playing the part of u we obtain a contradiction. Therefore $x \in A_s^{i,1}$. Thus $x \in (A_s^{i,1} - C_s) - (A_\omega^{i,1} - C_\omega)$, contradicting the choice of s. Below we suppose that $\alpha(x) \not\supseteq \gamma_0$. By $xii)$ of 5.3, $x = a_v^\delta$ or $x = k_v^\delta$ for some δ. By $vii)$ of 5.2, $\delta \not\supseteq \gamma_0$. By the choice of s_0, $\delta \not<_L \gamma_0$. Suppose $x = a_v^\delta$. Since either $\delta \subset \gamma_0$ or $\gamma_0 <_L \delta$, $x < a_s^{\gamma_0} < a_v^\delta = x$ by $ii)$ of 5.2, contradiction.

Suppose $x = k_v^\delta$. Let α receive attention at stage v. Note that $c_{v+1}^\alpha > c_v^\alpha \downarrow = 1$. By the choice of s_0, $\gamma_0 <_L \alpha$. Then $a_s^{\gamma_0} < k_v^\alpha$ by $iv)$ of 5.2. For the rest there are two cases:

Case 1. $\alpha = \delta$. Then $x < a_s^{\gamma_0} < k_v^\alpha = k_v^\delta = x$, contradiction.

Case 2. Otherwise. By $xii)$ of 5.3, $\delta \supseteq \pi$ for some $\pi \in \mathcal{A}(\alpha)$ and k^δ entered B after r_v^α was set. Note that $c_v^\alpha \downarrow$. First we show that $k_v^\alpha < k_v^\delta$. Suppose $c_v^\delta \downarrow$. By $i)$ of 5.2, $k_v^\alpha < k_v^\delta$. Suppose $c_v^\delta \uparrow$. Let $w < v$ be the greatest stage such that $c_w^\delta \downarrow$. Note that k_v^δ entered B before stage w and c_w^δ is destroyed at stage w. $c_w^\alpha \downarrow$ by $vi)$ of 5.1. By $i)$ of 5.2, $k_w^\alpha < k_w^\delta = k_v^\delta$. Towards a contradiction assume that $k_w^\alpha \neq k_v^\alpha$. Then at a stage u, $w < u < v$, c^α is destroyed. Hence r^α is destroyed at stage u. By Case 11, when Case 11 holds for α at stage v, we just need to enumerate the numbers which entered B after r^α was set equal to present value. But k_v^δ entered B before stage w and so before stage u. Hence k_v^δ cannot enter C at stage v, contradiction. Therefore $k_w^\alpha = k_v^\alpha$. Hence $x < a_s^{\gamma_0} < k_v^\alpha < k_v^\delta = x$, contradiction. Therefore, if $x \in D_s$, then $x \in D$. This completes the proof of Claim.

Now $D \leq_T A^{i,1} - C$ is immediate by Claim.

Case 2. $(5, i)$ occurs in P. This is similar to Case 1.

Case 3. Otherwise. There are two cases:

Case 3.1. $i \in J_i$. Let $\gamma \subseteq \alpha_0$ be the maximal $(0, i)$-node. There are two subcases.
Case 3.1.1. $I_i = \emptyset$.

Claim. If x enters B at some stage $> s_0$ with $\alpha(x) \supseteq \gamma$, then $x \in A^{i,1}$.
Proof of Claim: Let x enter B at stage $t > s_0$. Let α receive attention at stage t.
Subclaim 1. When α is visited in stage t,

- $r^{\gamma^-} \downarrow$
- *$p^\pi \downarrow$ for each $\pi \in \mathcal{A}(\gamma^-)$ with $\gamma <_L \pi$.*

Proof of Subclaim 1: Suppose γ is visited in stage t. Since x enters B at stage t, $a_t^\gamma \downarrow$ by *iv*) of 5.3. We have the desired conclusion by *i*) of 5.3. Suppose γ is not visited at stage t. Then there is a jump from some node $\subset \gamma$ to $\beta \supset \gamma$. Since $I_i = \emptyset$, γ is preferred to β. By *i*) of 5.3, we have the conclusion of Subclaim 1.

Note that r^{γ^-} is not destroyed at stage t before $C[t]$ is defined since $\gamma \subseteq \alpha$. Hence x is γ-designated when $C[t]$ is defined. By (e) of *vi*) of 5.3, x is the first number after a_t^γ was set. Hence x is the first number to enter B after the value of r^{γ^-} in stage t was set. Then $\gamma \in C[t]$. By (f) of *vi*) of 5.3, $r_{t+1}^{\gamma^-} \downarrow$. Therefore $\gamma \in C_{t+1}$ by *xxv*) of 5.1. By *i*) of 5.4, a^γ is destroyed infinitely often. Let $v > t$ be the least stage such that $a_{v+1}^\gamma \uparrow$. Let η receive attention at stage v. By *viii*) of 5.1 and the choice of s_0, $\eta \subset \gamma$ or $\gamma \subseteq \eta$.
Subclaim 2. There is no node $\supseteq \gamma$ which is visited at a stage $> t$ and $\leq v$.

Proof of Subclaim 2: Towards a contradiction assume there is a stage u at which some node $\supseteq \gamma$ is visited. Let $\tau \supseteq \gamma$ be the first node which is visited in stage u. By (d) of *vi*) of 5.3, p^τ becomes defined at stage u, and $\tau \neq \gamma$ implies $j(\tau) < j(\gamma)$. But by *i*) of 5.4 and the choice of s_0, $\tau \neq \gamma$. Therefore $j(\tau) < j(\gamma)$. This is a contradiction since $I_i = \emptyset$. This completes the proof of Subclaim 2.

Now from Subclaim 2, $\eta \not\supseteq \gamma$. Therefore $\eta \subset \gamma$.

Hence one of Case 1.1 and Case 4 holds at η. Clearly, Case 4 cannot occur at η after stage s_0. Let π be the 0-node β in Case 1.1.

Subclaim 3. $\pi = \gamma$.

Proof of Subclaim 3: By the choice of s_0, $\pi \not<_L \gamma$. Since a^γ is destroyed at stage v, either $\pi \subset \gamma$ or $\gamma \subseteq \pi$. Towards a contradiction assume $\pi \subset \gamma$. Because there is no (n, j)-node $\alpha \supset \gamma$ with $j < i = j(\gamma)$, $\gamma \in \mathcal{C}$ if and only if $r^{\gamma^-} \downarrow$ and there is some $z \in B$ designated for γ. If r^{γ^-} is destroyed at a stage $u > t$ and $< v$, then a^π is destroyed for some $\pi \in \mathcal{A}(\gamma^-)$ by *iv*) of 5.1. Consider the least such u and then the least such π. By *xiii*) of 5.1, $\gamma <_L \pi$. We have $p^\pi_u \downarrow$. By (*c*) of *vi*) of 5.3, a^σ is destroyed in stage u for some $\sigma \in \mathcal{A}(\gamma^-)$ with $\sigma <_L \pi$, contradiction. Hence $r^{\gamma^-}_v = r^{\gamma^-}_{t+1} \downarrow$. Since x remains designated for γ until the definition of $\mathcal{C}[v]$, $\gamma \in \mathcal{C}[v]$. Hence by Case 1.1, a^γ is not destroyed in the main part of the construction of stage v. By the choice of s_0, a^γ is not destroyed in stage v by $(E1)$. This contradicts the choice of v. Therefore $\gamma \subseteq \pi$. If $\gamma \subset \pi$, then $\eta \subset \gamma \subset \pi$ and at the beginning of stage v, x is designated for γ. So (i) in Case 1 fails which means that Case 1.1 fails, contradiction. Therefore $\gamma = \pi$.

Now by Case 1.1, we have $x \in A^{j(\gamma),1}$. This completes the proof of Claim.

There are an infinite number of stages at which some node $\supset \gamma$ is visited. We want to show that $D \leq_T A^{i,1} - C$. Fix $x > s_0$. Choose s to be the least stage such that

- *γ is visited at stage s*
- $x < a^\gamma_s \downarrow$
- $(A^{i,1}_s - C_s) \upharpoonright (x+1) = (A^{i,1}_\omega - C_\omega) \upharpoonright (x+1)$
- $D_s \upharpoonright (x+1) \subseteq A^{i,0}_s \cup A^{i,1}_s$.

Repeating the argument used for the case $(4, i)$ occurs in P, we can show that $x \in D$ if and only if $x \in D_s$. Hence $D \leq_T A^{i,1} - C$.

Case 3.1.2. $I_i \neq \emptyset$. Let $i_0 < \cdots < i_m$ be an enumeration of I_i. Note that for each $e \geq i(\alpha_0)$, there exists an $\alpha^e \in T$ such that

- $\alpha^e \supseteq \alpha_0$, $(\alpha^e)^- \in P$, $e = i(\alpha^e) < i(((\alpha^e)^-)^+)$,

- $\alpha^e = (\alpha^e)^{-\wedge}\langle(1,i)\rangle <_L ((\alpha^e)^-)^+$.

To see how to find this α^e we can let ξ be the least $\delta \in P$ such that $i(\delta) = e+1$, then α^e can be chosen as $\delta^{-\wedge}\langle(1,i)\rangle$. Note that by Lemma 5.4 *vi*), $a_\omega^{\alpha^e}$ is defined.

Define $\mathcal{E} \subset T$ as

$$\{\alpha : \alpha \supset \alpha_0 \wedge \alpha(l(\alpha)-1) = (1,i) \wedge (\forall j \in I_i)[j \textit{ is active at } \alpha^-]\}.$$

Note that for each $e \geq i(\alpha_0)$, $\alpha^e \in \mathcal{E}$. To show that $D \leq_T A^{i,1} - C$, fix g, find the least (η, s) such that

- $\eta \in \mathcal{E}$, $s > s_0$
- $g \leq a_s^\eta \downarrow$
- α_0 *is visited at stage* s
- $B_s \upharpoonright (a_s^\eta + 1) \subseteq A_s^{i,0} \cup A_s^{i,1}$
- $(A_s^{i,1} - C_s) \upharpoonright (a_s^\eta + 1) = (A_\omega^{i,1} - C_\omega) \upharpoonright (a_s^\eta + 1)$.

Clearly, for each $e \geq \max\{n, i(\alpha_0)\}$ there exist arbitrarily large s such that (α^e, s) satisfies the conditions specified for (η, s). Note that a^η is not destroyed at any stage $\geq s$. We will prove that $g \in D$ if and only if $g \in D_s$.

Remarks. 1. By *viii*) of 5.3, $a_s^\eta \notin C_s$. By Case 2 of the construction $a_s^\eta \in A_s^{i,1}$. Hence by the choice of (η, s), a^η is not destroyed at any stage $\geq s$.

2. For each $\delta < \eta$. k^δ is constant at stages $\geq s$. This is because that when k^δ is set to a new value, Case 4 holds at δ and a^η is destroyed.

3. For each $\delta <_L \eta$. a^δ is constant at stages $\geq s$. This is because that when a^δ becomes defined a^η is destroyed, and when a^δ is destroyed so is a^η by *xiii*) of 5.1.

Claim 1. Let β, u satisfy

- $\alpha_0 \subset \beta < \eta$, $s \leq u$,
- α_0 *is visited in stage* u,
- *for all* $j \in I_i$, j *is active at* β,

- $a_u^{\beta^\wedge\langle(0,i_m)\rangle}\downarrow$,
- *some y with $\alpha(y) \supseteq \beta^\wedge\langle(0,i_m)\rangle$ enters B at a stage $> s_0$ and $< u$, and after $a^{\beta^\wedge\langle(0,i_m)\rangle}$ is set equal to $a_u^{\beta^\wedge\langle(0,i_m)\rangle}$.*

Let x be the first number with $\alpha(x) \supseteq \beta^\wedge\langle(0,i_m)\rangle$ to enter B at a stage $> s_0$ and $< u$, and after $a^{\beta^\wedge\langle(0,i_m)\rangle}$ is set equal to $a_u^{\beta^\wedge\langle(0,i_m)\rangle}$. Then $x \in A_u^{i,1}$ and $k_u^\beta \in B_u$.
Proof of Claim 1: Let x enter B at stage $t < u$. Let π_n denote $\beta^\wedge\langle(0,i_n)\rangle$ for $n \leq m$. Recall that γ is the greatest $(0,i)$-node $\subset \alpha_0$. By *vi*) of 5.1, $c_t^\beta \downarrow$. By the choice of t, $k_u^\beta = k_t^\beta \downarrow$.
Subclaim 1. $\pi_m \in \mathcal{C}_{t+1}$, $\gamma \in \mathcal{C}_{t+1}$, and x is the γ- and π_m-designated number when $\mathcal{C}[t]$ is defined.

Proof of Subclaim 1: By the choice of η, $\pi_m <_L \eta$. By the choice of t we know that a^{π_m} is not destroyed at a stage $\geq t$ and $< u$, and so is not destroyed at a stage $\geq t$. First we show that $\pi_m \in \mathcal{C}[t]$. Towards a contradiction assume $\pi_m \notin \mathcal{C}[t]$. Since a^{π_m} is not destroyed in stage t, there is a maximal 0-node $\xi \subset \pi_m$ such that $\xi \in \mathcal{C}[t]$. By *v*) of 5.3, $j(\xi) \leq i_m$. Hence $\xi \subset \alpha_0$. By (*f*) of *vi*) of 5.3, $r_{t+1}^{\xi^-}\downarrow$. By *xxv*) of 5.1, $\xi \in \mathcal{C}_{t+1}$, and $\tau \notin \mathcal{C}_{t+1}$ for all τ with $\xi \subset \tau \subseteq \pi_m$. Since $a_{t+1}^{\pi_m}\downarrow$, $\pi_m \in \mathcal{D}_{t+1}$. By *ii*) of 5.4, a^ξ is destroyed infinitely often. Let $v > t$ be the least stage at which a^ξ is destroyed. By the choice of s_0 and *viii*) of 5.4, $c_v^{\xi^-}$ and $c_{v+1}^{\xi^-}$ are defined. By (*a*) of *vi*) of 5.3, a^{π_m} is destroyed at stage v, contradiction. Therefore $\pi_m \in \mathcal{C}[t]$. By (*f*) of *vi*) of 5.3, $r_{t+1}^{\pi_m^-}\downarrow$. By *xxv*) of 5.1, $\pi_m \in \mathcal{C}_{t+1}$. By (*e*) of *vi*) of 5.3, x is designated for π_m at the end of stage t.

To see that $\gamma \in \mathcal{C}_{t+1}$, first we show that before x enters B at stage t,

- $r^{\gamma^-}\downarrow$
- $p^\pi \downarrow$ *for each $\pi \in \mathcal{A}(\gamma^-)$ with $\alpha <_L \pi$.*

Suppose γ is visited at stage t. Note that $a_t^\gamma \downarrow$ by *iv*) of 5.3, since x enters B at stage t. By *i*) of 5.3, we have the desired conclusion. Suppose γ is not visited at stage t. Let τ be the first node $\supset \gamma$ which is visited at stage t. Clearly, there is a jump from a node $\subset \gamma$ to τ at stage t. γ is preferred to τ by *xix*) of 5.1 since x is π_m-designated at the end of stage t. Hence by *i*) of 5.3, we have the desired conclusion. Therefore, x is a designated number for γ when $\mathcal{C}[t]$ is defined. By (*e*) of *vi*) of 5.3, x is the first number y with $\alpha(y) \supseteq \gamma$ to enter

B after a^γ was set equal a_t^γ. Hence $\gamma \in \mathcal{C}[t]$ since $\pi_m \in \mathcal{C}[t]$. By (f) of $vi)$ of 5.3, $r_{t+1}^{\gamma^-} \downarrow$. By $xxv)$ of 5.1, $\gamma \in \mathcal{C}_{t+1}$. This completes the proof of Subclaim 1.

Subclaim 2. Let $v > t$ be any stage such that $a_v^\gamma = a_t^\gamma \downarrow$, and some node $\supseteq \gamma$ is visited in stage v. Let α be the first node $\supseteq \gamma$ which is visited in stage v. Then $\alpha \supseteq \pi_n$ for some $n \leq m$.

Proof of Subclaim 2: We proceed by induction on v. Suppose $a_v^\gamma = a_t^\gamma$ and some node $\supseteq \gamma$ is visited at stage v. Let α be the first node $\supseteq \gamma$ which is visited in stage v. Since $a_v^\gamma = a_t^\gamma$, by (d) of $vi)$ of 5.3, p^α becomes defined at stage v and $\alpha \neq \gamma$ implies $j(\alpha) < j(\gamma)$. By the choice of s_0, $\alpha \neq \gamma$. Therefore $\alpha \not\subseteq \beta$ by the choice of β. Towards a contradiction assume $\pi_m <_L \alpha$. Let p^α be set equal to $z(\alpha)$, where $z(\alpha)$ entered B at stage $t(\alpha) < v$. By the induction hypothesis, $t(\alpha) < t$. Now the three stages $t(\alpha), t, v$ contradict $ii)$ of 5.3. Therefore $\pi_m \not<_L \alpha$. By the same token, if $w > t$, $a_w^\gamma = a_t^\gamma \downarrow$, and $\tau \supseteq \gamma$ is visited in stage w, then $\tau \not\subseteq \beta$ and $\pi_m \not<_L \tau$.

Towards a contradiction assume $\alpha <_L \beta$. By $xxi)$ of 5.1, since a^{π_m} and so a^α are not destroyed in stage v, some z enters B at stage v. By $ii)$ of 5.4, a^γ is destroyed infinitely often. Let $w \geq v$ be the least stage in which a^γ is destroyed. Below we show that a^{π_m} is destroyed in stage w, contradiction. First we want to show that c^β, r^β are destroyed in stage v. By $xv)$ of 5.1, in stage t $p^\pi \uparrow$ for all π with $\gamma \subseteq \pi \subseteq \beta$, and $r^\zeta \uparrow$ for all ζ such that $\gamma \subseteq \zeta \subset \zeta^\wedge\langle(2, i(\zeta))\rangle \subseteq \beta$. Suppose $\alpha(z) <_L \beta$. Then in stage v, c^β and r^β are destroyed since there is no $\pi \subseteq \beta$ such that $p^\pi \downarrow$ in stage v. Suppose $\alpha(z) \subset \beta$. Note that Case 3.1 holds at the node which receives attention in stage v. Let ζ be the least node such that $\alpha(z) \subset \zeta \subseteq \beta$. Note that ζ is not a 0-node, otherwise $p^\zeta \downarrow$ when z enters B at stage v. By $xv)$ of 5.1, $p_{t+1}^\zeta \uparrow$ and ζ is not visited at any stage $> t$ and $\leq v$. Hence c^β and r^β are destroyed in stage v by Case 3.1. Let ϵ receive attention at stage w. From above, if $\epsilon \supseteq \gamma$, then $\epsilon <_L \pi_m$ or $\epsilon \supseteq \pi_m$. Also since no node $\supseteq \gamma$ and $\subseteq \beta$ is visited at any stage $> t$ and $\leq w$, $p^\pi \uparrow$ for all π with $\gamma \subseteq \pi \subseteq \beta$, and $r^\pi \uparrow$ for all π such that $\gamma \subseteq \pi \subset \pi^\wedge\langle(2, i(\pi))\rangle \subseteq \beta$ in stage w. The remarks above show that c^β and r^β are destroyed in stage v before $\mathcal{C}[v]$ is defined. Therefore $\pi_m \notin \mathcal{C}[w]$ if $\mathcal{C}[w]$ is defined. Hence if $\mathcal{C}[w]$ is defined, $\pi \notin \mathcal{C}[w]$ for all π such that $\gamma \subset \pi \subseteq \pi_m$. By $xxiii)$ of 5.1, a^{π_m} is destroyed in stage w, contradiction. This completes the proof of Subclaim 2.

Let $v > t$ be the least stage in which a^γ is destroyed. By the choice of s_0 and (c) of $vi)$ of 5.3, $r_v^{\gamma^-} = r_{t+1}^{\gamma^-}\downarrow$. By Subclaim 2, $v < u$. Let τ receive attention at stage v. By the choice of s_0 and $viii)$ of 5.1, either $\tau \subset \gamma$ or $\gamma \subseteq \tau$.

Subclaim 3. Suppose y enters B at stage $h < v$, $r_v^{\gamma^-} = r_{h+1}^{\gamma^-}$, and $\alpha(y) \supseteq \gamma$, then $h \geq t$ and either $\alpha(y) = \beta$ or $\alpha(y) \supseteq \pi_n$ for some $n \leq m$.

Proof of Subclaim 3: Since y enters B after r^{γ^-} is set equal $r_v^{\gamma^-}$, y enters B after a^γ is set equal to a_v^γ. By (e) of $vi)$ of 5.3, x is the first number z with $\alpha(z) \supseteq \gamma$ to enter B after a^γ is set equal to a_t^γ. Note that $a_v^\gamma = a_t^\gamma$. Hence $h \geq t$. Let α receive attention at stage h. By Subclaim 2, $\alpha \supseteq \pi_n$ for some $n \leq m$. Now the subclaim 3 is clear.

Subclaim 4. $\gamma \in \mathcal{C}_v$.

Proof of the Subclaim 3: Towards a contradiction assume that $\gamma \notin \mathcal{C}_v$. Let w be the least stage $> t$ such that $\gamma \notin \mathcal{C}_{w+1}$. Note that $w < v$ and so $r_{w+1}^{\gamma^-}\downarrow$. By $xxiv)$ of 5.1 and the choice of s_0, some $\alpha \supseteq \gamma$ receives attention at stage w. By Subclaim 2, $\alpha \supseteq \theta$ for some $\theta \in \mathcal{A}(\beta)$ with $\theta \leq \pi_m$. By $xxi)$ of 5.1, since a^{π_m} and a^θ are not destroyed in stage w, some number z enters B at stage w. Note that either $\alpha(z) = \beta$ or $\alpha(z) \supseteq \theta$. Suppose $\alpha(z) = \beta$. Then, when $\mathcal{C}[w]$ is defined r^β, c^β, and a^ϵ, p^ϵ for all $\epsilon \in \mathcal{A}(\beta)$, are all defined. Hence $\epsilon \in \mathcal{C}[w]$ for all $\epsilon \in \mathcal{A}(\beta)$ by (b) of $vi)$ of 5.3. By subclaim 3, all $y \neq z$ with $\alpha(y) \supseteq \gamma$ which entered B after $r_w^{\gamma^-}$ was set, $\alpha(y) \supseteq \pi_n$ for some $n \leq m$ and y entered B at a stage $\geq t$. Therefore $\gamma \in \mathcal{C}[w]$. By (f) of $vi)$ of 5.3, $r_{w+1}^{\gamma^-}\downarrow$. Then $\gamma \in \mathcal{C}_{w+1}$ by $xxv)$ of 5.1, contradiction. Suppose $\alpha(z) \supseteq \theta$. Towards a contradiction assume $\theta \notin \mathcal{C}[w]$. Since a^θ is not destroyed in stage w, there is a maximal 0-node $\xi \subset \theta$ such that $\xi \in \mathcal{C}[w]$. By $v)$ of 5.3, $j(\xi) \leq j(\theta)$. Hence $\xi \subset \gamma$. By (f) of $vi)$ of 5.3, $r_{w+1}^{\xi^-}\downarrow$. By $xxv)$ of 5.1, $\xi \in \mathcal{C}_{w+1}$ and $\pi \notin \mathcal{C}_{w+1}$ for each 0-node π such that $\xi \subset \pi \subseteq \theta$. Hence $\theta \in \mathcal{D}_{w+1}$. Let $w' > w$ be the least stage at which a^ξ is destroyed. By choice of s_0 and (a) of $vi)$ of 5.3, a^θ is destroyed at stage w', contradiction. Therefore $\theta \in \mathcal{C}[w]$.

Subsubclaim. Let $\alpha(y) \supseteq \gamma$ and y enter B at a stage $> t$ and $< w$. Then there exists $\sigma \in \mathcal{A}(\beta)$ such that $\alpha(y) \supseteq \theta$ and $\theta \leq \sigma \leq \pi_m$.

Proof of Subsubclaim: Let k be a stage such that $t < k < w$ and some y with $\alpha(y) \supseteq \gamma$ enters B at stage k. By Subclaim 2 there is a unique $\sigma \in \mathcal{A}(\beta)$ such that $\alpha(y) \supseteq \sigma$ and

$\sigma \leq \pi_m$. Towards a contradiction assume that $\sigma < \theta$. By $xv)$ of 5.1, $p^\theta_w \uparrow$. Hence $p^\theta \uparrow$ when y enters B at stage k since a^θ is not destroyed after stage t. Hence $\sigma \notin \mathcal{C}[k]$. Since a^σ is not destroyed in stage k, there is a maximal 0-node $\xi \subset \sigma$ such that $\xi \in \mathcal{C}[k]$. By the argument used in the last paragraph we obtain a contradiction. This yields the Subsubclaim.

Since $\theta \in \mathcal{C}[w]$, when $\mathcal{C}[w]$ is defined $p^\sigma \downarrow$ for each $\sigma \in \mathcal{A}(\beta)$ with $\theta <_L \sigma$. By Cases 3.1 and 8, $c^\beta \downarrow$ and $a^\theta \downarrow$ when $\mathcal{C}[w]$ is defined. Then by (b) of $vi)$ of 5.3, $\sigma \in \mathcal{C}[w]$. Combining this with the result above we have $\sigma \in \mathcal{C}[w]$ for all $\sigma \in \mathcal{A}(\beta)$, $\theta \leq \sigma \leq \pi_m$. We want to check that $\gamma \in \mathcal{C}[w]$. From above, $r^{\gamma^-}_v = r^{\gamma^-}_{t+1} \downarrow$ and so r^{γ^-} has the value $r^{\gamma^-}_{t+1}$ throughout stage w. Since $\gamma \in \mathcal{C}_{t+1}$ by Subclaim 1, x is designated for γ at the beginning of stage $t+1$ and hence throughout stage w. To check the final condition for $\gamma \in \mathcal{C}[w]$ we consider an (n, j)-node $\epsilon \supset \gamma$ such that $j < j(\gamma) = i$ and such that some y with $\alpha(y) \supseteq \epsilon$ entered B after r^{γ^-} was set equal $r^{\gamma^-}_v$. By subclaim 3, subsubclaim and the choice of β, y entered B at a stage $\geq t$ and $\alpha(y) \supseteq \sigma$ for some $\sigma \in \mathcal{A}(\beta)$, $\theta \leq \sigma \leq \pi_m$. By choice of β, $\sigma \subseteq \epsilon$. So σ may taken for η in (iii) of teh definition of $\mathcal{C}[w]$. Note that $j(\sigma) \leq j$ may fail. In this case we can apply condition (iii) to σ and ϵ to obtain σ' with $\sigma \subset \sigma' \subseteq \epsilon$ and $j(\sigma') \leq j$ as required. Therefore $\gamma \in \mathcal{C}[w]$. Hence $\gamma \in \mathcal{C}_{w+1}$ by $xxv)$ of 5.1 since $r^{\gamma^-}_{w+1} \downarrow$, contradiction. This completes the proof of Subclaim 4.

Towards a contradiction assume $\gamma \subseteq \tau$. By Subclaim 2, $\tau \supseteq \pi$ for some $\pi \in \mathcal{A}(\beta)$ with $\pi \leq \pi_m$. By $xxi)$ of 5.1, since a^{π_m} and a^π are not destroyed in stage v, some number z enters B at stage v. Repeating the argument used for the proof of Subclaim 4, taking $\theta = \pi$ and $w = v$, we see that $\gamma \in \mathcal{C}[v]$ (the only difference is that here we cannot assume $r^{\gamma^-}_{v+1} \downarrow$). Then by Cases 3.1 and 8, a^γ is not destroyed in the main part of the construction in stage v. But clearly a^γ is not destroyed through $(E1)$. Thus a^γ is not destroyed in stage v at all, contradiction. Therefore $\tau \subset \gamma$. Then either Case 1.1 or Case 4 holds at τ in stage v. By the choice of s_0, Case 1.1 holds. Let ϵ be the 0-node β in Case 1.1.

Subclaim 5. $\epsilon = \gamma$.

Proof of Subclaim 5: Towards a contradiction assume $\epsilon \neq \gamma$. Recall that $r^{\gamma^-}_v = r^{\gamma^-}_{t+1} \downarrow$ and $r^{\gamma^-} \downarrow$ when $\mathcal{C}[v]$ is defined. By Subclaim 4, $\gamma \in \mathcal{C}_v$. By $xxiv)$ of 5.1, $\gamma \in \mathcal{C}[v]$ since $\tau \subset \gamma$. Suppose $\epsilon \subset \gamma$. Then a^γ is not destroyed in stage v by Case 1.1, contradiction. Clearly, $\gamma \not<_L \epsilon$. Since a^ϵ is destroyed by Case 1.1, by the choice of s_0, $\epsilon \not<_L \gamma$. Therefore $\gamma \subset \epsilon$.

Since there is a γ-designated number when τ is visited in stage v, condition (i) on (x, β) in Case 1 implies that $j(\epsilon) \leq j(\gamma) = i$. Hence $\epsilon \not\subseteq \beta$. Towards a contradiction assume that $\pi_m <_L \epsilon$. By the choice of β, $\epsilon^- \not\subseteq \beta$ since $j(\epsilon) \leq i$. Then $\pi_m <_L \epsilon^-$. Since some node $\supseteq \pi_m$ received attention in stage t, either c^{ϵ^-} is destroyed at stage t or $p^\zeta \downarrow$ for the unique ζ such that $\zeta^- = \pi_m \cap \epsilon$ and $\zeta \subset \epsilon$. Note that $c_v^{\epsilon^-} \downarrow$. By Subclaim 2, $c_v^{\epsilon^-} = c_t^{\epsilon^-} \downarrow$. Therefore $p^\zeta \downarrow$ at some point in stage t. By the choice of ϵ, we know $p_v^\zeta \uparrow$. Hence p^ζ is destroyed at a stage $\geq t$ and $< v$. But when p^ζ is destroyed, so is c^{ϵ^-} since $\zeta \subseteq \epsilon^-$, contradiction. Hence $\epsilon <_l \pi_m$ or $\pi_m \subseteq \epsilon$. Since a^ϵ is destroyed in stage v, so is a^{π_m}, contradiction. Therefore $\epsilon = \gamma$.

By Subclaim 5, since Case 1.1 holds at τ and $\epsilon = \gamma$, $x \in A_v^{i,1}$ since x is the unique γ-designated number at the begining of stage v by (e) of $vi)$ of 5.3. Therefore $x \in A_u^{i,1}$ since $v < u$ by Subclaim 2.

Towards a contradiction assume that $k_t^\beta \notin B_u$. Let $v > t$ be the least stage in which some node $\supseteq \alpha_0$ and $\subseteq \beta$ is visited. Clearly, $v \leq u$. Let α be the the greatest node $\subset \alpha_0$ which is visited in stage v.

Subclaim 6. $r_v^\beta = r_{t+1}^\beta \downarrow$.

Proof of Subclaim 6: Towards a contradiction assume $r_v^\beta \neq r_{t+1}^\beta$. Let $w > t$ be the least stage in which r^β is destroyed. Clearly, $w < v$. By $iv)$ and $xii)$ of 5.1, in stage w either c^β or one of a^π ($\pi \in \mathcal{A}(\beta)$) is destroyed. Suppose a^π is destroyed in stage w for some $\pi \in \mathcal{A}(\beta)$. Choose such π least possible. Since a^{π_m} is not destroyed in stage w, $\pi_m < \pi$ by $xiii)$ of 5.1. Since $\pi_m \in C_{t+1}$, $p_{t+1}^\pi \downarrow$. Hence $p_w^\pi \downarrow$ by choice of w. Also, p^π is destroyed in stage w since a^π is. By (c) of $vi)$ of 5.3, a^ϵ is destroyed in stage w for some $\epsilon \in \mathcal{A}(\beta)$ with $\epsilon <_L \pi$, which contradicts the minimality of π. Therefore, $a_{w+1}^\pi \downarrow$ for all $\pi \in \mathcal{A}(\beta)$. Hence c^β is destroyed in stage w. Let τ receive attention in stage w. By $ii)$ of 5.1, there are four cases:

Case 1. p^ϵ is destroyed in stage w for some $\epsilon \subseteq \beta$. By the choice of s_0, $\epsilon \not\subseteq \alpha_0$. Suppose $\alpha_0 \subset \epsilon \subseteq \beta$. Note that $p^\epsilon \uparrow$ throughout stage t by $xv)$ of 5.1 since x with $\alpha(x) \supseteq \pi_m$ enters B at stage t. So p^ϵ becomes defined at a stage $> t$ and $\leq w$, contradicting the minimality of v.

Case 2. a^ϵ is destroyed in stage w for some $\epsilon <_L \beta$. This is impossible from the remarks before Claim 1.

Cases 1 and 2 are the only ones pertaining to the ending of the stages. For the rest we can assume that c^β is destroyed in the main part of the construction.

Case 3. Case 8 holds at τ with $\tau \subset \beta$ and $\tau^\wedge\langle(2, i(\tau))\rangle \leq \beta$. Note that $c^\tau_w = 0$ and $c^\tau_{w+1} = 1$. By the choice of s_0, $\tau \not\subset \alpha_0$. Therefore $\alpha_0 \subseteq \tau \subset \beta$. Since τ is visited in stage w, this contradicts the minimality of v.

Case 4. $\tau <_L \beta$. By the choice of s_0, $\alpha_0 \subset \tau$. Let $\theta \supseteq \alpha_0$ be the first node which is visited in stage w. Since no node $\supseteq \alpha_0$ and $\subseteq \beta$ is visited in stage w, there is a jump to θ and $\theta <_L \beta$. Clearly, $\tau \cap \beta = \theta \cap \beta$. By *xxi*) of 5.1, since a^θ is not destroyed, some z enters B at stage w. Let $\epsilon \subseteq \beta$ be the node such that $\epsilon^- = \tau \cap \beta$. Since c^β is destroyed and either Case 3.1 or Case 8 holds, p^ϵ is undefined throughout stage w. By *xv*) of 5.1, $r^\zeta_{t+1} \uparrow$ for all ζ with $\zeta^\wedge\langle(2, i(\zeta))\rangle \subseteq \beta$. By choice, for such ζ, r^ζ remains undefined throughout stage w. Since a^{π_m} is not destroyed at stage w, there is a maximal $\xi \subset \pi_m$ such that $\xi \in \mathcal{C}[w]$. By (f) of *vi*) of 5.3, $r^{\xi^-}_{w+1} \downarrow$. Note that $a^\xi_{w+1} \downarrow$. Towards a contradiction assume that $\alpha_0 \subset \xi$. Since $\xi \in \mathcal{C}[w]$, a^ξ_w and $r^{\xi^-}_w$ are defined. Since $w < v$, $a^\xi_w = a^\xi_t \downarrow$ and $r^{\xi^-}_w = r^{\xi^-}_{t+1}$. Since x entered B in stage t, by *v*) of 5.3, we have $j(\xi) \leq j(\pi_m) = i_m$. This contradicts the hypothesis on β. Therefore $\xi \subseteq \alpha_0$. Since $\xi \in \mathcal{C}[w]$, it follows by (f) of *vi*) of 5.3 that $r^{\xi^-}_{w+1} \downarrow$. By *v*) of 5.3, $j(\xi) \leq j(\zeta)$ for all 0-nodes ζ with $\xi \subset \zeta \subseteq \tau \cap \beta$. Let h be the least stage $> w$ at which a^ξ is destroyed.

Subsubclaim. If $\zeta \supseteq \xi$ is visited at a stage $> w$ and $\leq h$, then $\zeta <_L \beta$.

Proof of Subsubclaim: Towards a contradiction consider the least k, $w < k \leq h$, such that some node $\sigma \supseteq \xi$ is visited in stage k and $\sigma \not<_L \beta$. Let σ be the first node $\supseteq \xi$ which is visited in stage k. By (d) of *vi*) of 5.3 5.3, p^σ becomes defined in stage k and $\sigma \neq \xi$ implies $j(\sigma) < j(\xi)$. By the choice of s_0, $\sigma \neq \xi$. Therefore in stage k the construction jumps to σ which implies that $r^{\sigma^-}_k \downarrow$. Since $c^\beta_{w+1} \downarrow$, $c^\beta_k \uparrow$ by the choice of k. Hence $\beta \not\subset \sigma$ by *vi*) of 5.1. Suppose $\beta <_L \sigma^-$. Then c^{σ^-} and r^{σ^-} are destroyed in stage w, unless $p^\delta \downarrow$ for some $\delta \subseteq \sigma^-$. But p^δ is destroyed before stage k by *xv*) of 5.1. By $(E4)$, when p^δ is destroyed, so is c^{σ^-}. Hence $c^{\sigma^-}_k \uparrow$ and $r^{\sigma^-}_k \uparrow$, contradiction. The only remaining is $\sigma^- \subset \beta$ which we divide into three subcases. Suppose $\xi \subset \sigma \subseteq \tau \cap \beta$ or $\tau \cap \beta <_L \sigma$. Then there is a 0-node ζ such that $\xi \subset \zeta \subseteq \tau \cap \beta$ and $j(\zeta) < j(\xi)$, contradiction. Suppose $\sigma^- \supset \tau \cap \beta$. Since $\sigma^- \subset \beta$ we have $\tau <_L \sigma^-$. Then at stage w, c^{σ^-} is destroyed since c^β is destroyed and $p^\epsilon \uparrow$. Hence $c^{\sigma^-}_k \uparrow$ and

$r_k^{\sigma^-}\uparrow$ by the choice of k, contradiction. Finally, suppose $\sigma^- = \tau \cap \beta$. Let π be the least node such that $\tau \cap \beta \subset \pi \subseteq \tau$. Note that ϵ is a 0-node and $\epsilon \leq \sigma$. So π is a 0-node and $\pi <_L \epsilon$. Since the construction jumps to σ in stage k, $a_k^\sigma \downarrow$ and $p_k^\zeta \uparrow$ for all $\zeta \in \mathcal{A}(\sigma^-)$ with $\zeta \leq \sigma$. In particular, $p_k^\epsilon \uparrow$. By xiv) of 5.1, $a_k^\epsilon \downarrow$. By choice of k, $a_k^\epsilon = a_w^\epsilon \downarrow$. If p^ϵ is destroyed in a stage, so is a^ϵ. Hence $p^\epsilon \uparrow$ throughout stage w. Hence $\pi \notin \mathcal{C}[w]$. By v) of 5.3, $j(\xi) \leq j(\pi)$. But $j(\pi) < j(\sigma) < j(\xi)$, contradiction. This completes the proof of Subsubclaim.

Suppose $\mathcal{C}[h]$ is defined. Fix any 0-node ζ such that $\xi \subset \zeta \subseteq \pi_m$. Towards a contradiction suppose $\zeta \in \mathcal{C}[h]$. By x) of 5.1, $a_h^\zeta \downarrow$. By Subsubclaim, since z enters B at stage w, $a_h^\zeta = a_w^\zeta \downarrow$. Note that $\zeta \notin \mathcal{C}[w]$ by the choice of ξ. Hence $\zeta \notin \mathcal{C}_{w+1}$ by xxv) of 5.1. We now consider two cases according as $\xi \subseteq \alpha_0$ or not. First suppose $\zeta \subseteq \alpha_0$. Then $\zeta \in \mathcal{D}_{w+1}$. If $\zeta \notin \mathcal{C}_h$, then $\zeta \in \mathcal{D}_h$, which contradicts vii) of 5.3. So we may assume $\zeta \in \mathcal{C}_h$. Let k be the least stage such that $\zeta \in \mathcal{C}_{k+1}$, we have $\zeta \in \mathcal{D}_k$ which again contradiction. This finishes the case $\zeta \subseteq \alpha_0$. Now suppose $\alpha_0 \subset \zeta$. By the choice of v, $a_t^\zeta = a_w^\zeta = a_h^\zeta$. Hence x entered B after a^ζ was set equal to a_h^ζ. Let y be the ζ-designated number when $\mathcal{C}[h]$ is defined. By (e) of vi) of 5.3, y is the first number to enter B after a^ζ is set equal to a_h^ζ. Hence x enters B after r^{ζ^-} is set equal to $r_h^{\zeta^-}$. Recall that c^β is destroyed in stage w. By the subsubclaim c^β, and hence also r^β, are undefined throughout stage h. Hence $\pi_m \notin \mathcal{C}[h]$. Recall also that $\alpha(x) \supseteq \pi_m$. Since $\zeta \in \mathcal{C}[h]$, taking $\xi = \zeta$, $\alpha = \pi_m$ and $y = x$ in iii) of the definition of $\mathcal{C}$, there exists a 0-node σ such that $\zeta \subset \sigma \subset \pi_m$ and $j(\sigma) \leq j(\pi_m) = i_m$. This contradicts the choice of β. Therefore, $\zeta \notin \mathcal{C}[h]$.

Towards a contradiction assume that $p^\sigma \downarrow$ at some point in stage h for some σ with $\xi \subset \sigma \subset \pi_m$. By the Subsubclaim, $p^\sigma \downarrow$ at some point in stage w. This contradicts xv) of 5.1 if $\sigma \subseteq \tau \cap \beta$. Suppose $\tau \cap \beta \subset \sigma \subset \pi_m$. By the choice of v, $p^\sigma \downarrow$ at some point in stage t. This contradicts xv) of 5.1 again. Thus $p^\sigma \uparrow$ throughout stage h for all σ with $\xi \subset \sigma \subset \pi_m$. By the same argument we see that for each σ such that $\xi \subset \sigma^\wedge\langle(2, i(\sigma))\rangle \subseteq \beta$, $r^\sigma \uparrow$ throughout stage h. By $xxiii$) of 5.1, a^{π_m} is destroyed in stage h, contradiction. This completes the proof of Subclaim 6.

Recall that we are aiming for a contradiction from the assumption $k_t^\beta \notin B_v$. From Subsubclaim 6, $k_v^\beta = k_t^\beta$. Since $k_v^\beta \notin B_v$, $p_v^\zeta \uparrow$ for some $\zeta \in \mathcal{A}(\beta)$. Let π be the maximal 0-node in $\mathcal{A}(\beta)$ such that $p_v^\pi \uparrow$. From Subsubclaim 1, $\pi_m \in \mathcal{C}_{t+1}$. Hence $p_{t+1}^\theta \downarrow$ for all

$\theta \in \mathcal{A}(\beta)$ with $\pi_m <_L \theta$. By (c) of $vi)$ of 5.3, no p^θ is destroyed after stage t since a^{π_m} is not destroyed after stage t. Hence $\pi \leq \pi_m$.

Subclaim 7. Case 1 holds at α in stage v with $i = j(\pi)$ and $\beta = \pi$.

Proof of Subclaim 7: Let w the least stage $\geq t$ at which some node $\supseteq \pi$ is visited. If $\pi = \pi_m$, then $w = t$. Otherwise, p^ζ for $\zeta \in \mathcal{A}(\beta)$ immediately to the right of π must be defined before stage v, and π is visited in that stage. Therefore $w < v$ and some node $\supseteq \pi$ receives attention at stage w. By $xxi)$ of 5.1, some number z enters B at stage w since a^π is not destroyed. Clearly, either $\alpha(z) = \beta$ or $\alpha(z) \supseteq \pi$. But $\alpha(z) = \beta$ contradicts $k_t^\beta \notin B_v$. Therefore $\alpha(z) \supseteq \pi$. Towards a contradiction assume $\pi \notin \mathcal{C}[w]$. Since a^π is not destroyed in stage w, there is a maximal 0-node $\xi \subset \pi$ such that $\xi \in \mathcal{C}[w]$. By $v)$ of 5.3, $j(\xi) \leq j(\pi)$. So $\xi \subset \alpha_0$ by the choice of β. By (f) of $vi)$ of 5.3, $r_{w+1}^{\xi^-} \downarrow$. By $xxv)$ of 5.1, $\xi \in \mathcal{C}_{w+1}$ and $\theta \notin \mathcal{C}_{w+1}$ for each 0-node θ with $\xi \subset \theta \subseteq \pi$. Let $h > w$ be the least stage in which a^ξ is destroyed. By (a) of $vi)$ of 5.3, a^π is destroyed at stage h, contradiction. Therefore, $\pi \in \mathcal{C}[w]$. Let y be the π-designated number when $\mathcal{C}[w]$ is defined. By (f) of $vi)$ of 5.3, $r_{w+1}^{\pi^-} \downarrow$. By $xxv)$ of 5.1, $\pi \in \mathcal{C}_{w+1}$. By $xv)$ of 5.1, $p^\theta \uparrow$ throughout stage w for each 0-node $\theta \subseteq \beta$, and $r^\theta \uparrow$ throughout stage w for all θ with $\theta^\wedge\langle(2, i(\theta))\rangle \subseteq \beta$. By the choice of v and $xv)$ of 5.1, $p_v^\theta \uparrow$ for each 0-node $\theta \subseteq \beta$, and $r_v^\theta \uparrow$ for all θ with $\theta^\wedge\langle(2, i(\theta))\rangle \subseteq \beta$. Now we fix $\zeta \in \mathcal{A}(\beta)$ with $\zeta <_L \pi$. Towards a contradiction assume $p_v^\zeta \downarrow$. Let h be the greatest stage in which p^ζ becomes defined. Then $p_h^\theta \downarrow$ for all $\theta \in \mathcal{A}(\beta)$ with $\zeta <_L \theta$. Note that $p_v^\pi \uparrow$. So p^π is destroyed at a stage $\geq h$ and $< v$. By (c) of $vi)$ of 5.3, a^ζ and hance also p^ζ are destroyed at a stage $\geq h$ and $< v$, contradiction. Therefore, $p_v^\zeta \uparrow$ for $\zeta \in \mathcal{A}(\beta)$ with $\zeta <_L \pi$. If $y \notin A_v^{j(\pi),0} \cup A_v^{j(\pi),1}$, then Case 1 holds at α. So we may suppose that $y \in A_v^{j(\pi),0} \cup A_v^{j(\pi),1}$. Towards a contradiction assume that there is a 0-node τ such that $\alpha \subset \tau \subset \pi$, $j(\tau) < j(\pi)$, and there is a τ-designated number at stage v. Clearly, $\tau \subset \alpha_0$. Let σ be the first node $\supseteq \tau$ which is visited in stage v. Such σ exists by the choice of v. By (d) of $vi)$ of 5.3, p^σ becomes defined and $\sigma \neq \tau$ implies $j(\sigma) < j(\tau)$. By the choice of α, $\sigma \supset \alpha_0^-$ and either $\sigma \subseteq \beta$, or $\beta <_L \sigma$ and $\sigma^- \subset \beta$. In either case, there is a 0-node θ such that $\alpha_0 \subseteq \theta \subseteq \beta$ and $j(\theta) < j(\tau) < j(\pi) < i$, contradiction. Thus there is no 0-node τ, $\alpha \subset \tau \subset \pi$, such that $j(\tau) < j(\pi)$ and there is a τ-designated number when α is visited in stage v.

If $y \in A_v^{j(\pi),1}$, then (i) of Case 1 holds at α in stage v. It remains to consider the case $y \in A_v^{j(\pi),0}$. For this we need:

Subsubclaim. For each 0-node θ which is preferred to π, when α is visited in stage v,

- $r^{\theta^-} \downarrow$
- $p^\zeta \downarrow$ *for each* $\zeta \in \mathcal{A}(\theta^-)$ *with* $\theta <_L \zeta$.

Proof of Subsubclaim: Fix θ such that θ is preferred to π. By the remark after 4.4, there is a 0-node θ' such that $\theta \subseteq \theta' \subset \pi$ and $j(\theta') < j(\pi)$. Therefore, $\theta \subset \alpha_0$. Suppose θ is visited at stage v. Then $a_v^\theta \downarrow$, otherwise no node $\supseteq \alpha_0$ is visited in stage v by $iv)$ of 5.3. By $i)$ of 5.3, we have the conclusion of the Subsubclaim. Suppose θ is not visited in stage v. Then either $\theta \subset \alpha$ or $\alpha \subset \theta$. Let σ be the first node $\supset \theta$ which is visited in stage v. Then there is a jump from a node $\subset \theta$ to σ. We show that θ is preferred to σ. Suppose $\theta \subset \alpha$. Since α and some node $\supseteq \alpha_0$ are visited in stage v, $\alpha <_L \sigma$, $\sigma^- \subset \alpha$, and

$$\alpha \supseteq \max\{\delta : \delta \in \mathcal{A}(\sigma^-) \wedge \delta <_L \sigma\}.$$

Hence by 4.5, θ is preferred to σ. Suppose $\alpha \subset \theta$. Then $(\alpha_0)^- \subset \sigma$, and either $\sigma \subseteq \beta$, or $\beta <_L \sigma$ and

$$\beta \supseteq \max\{\delta : \delta \in \mathcal{A}(\sigma^-) \wedge \delta <_L \sigma\}.$$

Note that in either case θ is preferred to σ by 4.5 since θ is preferred to π. By $i)$ of 5.3, we have the conclusion of Subsubclaim.

Now Case 1 holds at α again by taking $x = y$, $\beta = \pi$ and $i = j(\pi)$.

By Subclaim 7, no node $\supseteq \alpha_0$ and $\subseteq \beta$ is visited in stage v, contradiction. Therefore $k_t^\beta \in B_v$ and so $k_t^\beta \in B_u$. This completes the proof of Claim 1.

To see that $g \in D$ if and only if $g \in D_s$, we just verify the following:

Claim 2.

i) If $x \leq a_s^\eta$ is enumerated in B at a stage $\geq s$, then x is enumerated in C.

ii) If $x \leq a_s^\eta$ is enumerated in C at a stage $\geq s$, then x is enumerated in B at a stage $\geq s$.

Proof of Claim 2. Recall that a^η is not destroyed at a stage $\geq s$.

i) Let $x \leq a^\eta_s$ be enumerated in B at stage $u \geq s$. Then $x = k^\delta_u$ for some δ by *vii*) of 5.1. By *iv*) of 5.2, either $\delta < \eta$ or $\eta \subseteq \delta$. We have $k^\delta_u = k^\delta_s$; otherwise c^δ becomes defined by Case 4 at a stage $\geq s$ and $< u$, which would destroyed a^η. $\eta \not\subseteq \delta$ since η is a 1-node.

Subclaim. Let β be a any node such that

- $\alpha_0 \subseteq \beta < \eta$,
- $(\forall j \in I_i)[j \text{ } \textit{is active at } \beta]$,
- $k^\beta_s \downarrow$,
- k^β_s *enters* B *at a stage* $\geq s$.

Then $k^\beta_s \in C$.

Proof of the Subclaim: Let k^β_s enters B at a stage $w \geq s$. Note that when k^β enters B, $p^{\beta^\wedge\langle(0,i_m)\rangle}$ is defined. Let y be the value of $p^{\beta^\wedge\langle(0,i_m)\rangle}$ when k^β_u enters B. Since y is designated for $\beta^\wedge\langle(0,i_m)\rangle$ when $p^{\beta^\wedge\langle(0,i_m)\rangle}$ is set equal to y, by (e) of *vi*) of 5.3 y is the first number z with $\alpha(z) \supseteq \beta^\wedge\langle(0,i_m)\rangle$ to enter B since $a_u^{\beta^\wedge\langle(0,i_m)\rangle}$ was set. Towards a contradiction assume that $y \in B_s$. By Claim 1, $k^\beta_s \in B_s$, contradiction.

Therefore $y \notin B_s$. Let h be the least stage $\geq s$ such that $y \in B_h$ and α_0 is visited in stage h. $a^{\beta^\wedge\langle(0,i_m)\rangle}$ is never destroyed at a stage $\geq s$ since $\beta^\wedge\langle(0,i_m)\rangle <_L \eta$. Hence $a^{\beta^\wedge\langle(0,i_m)\rangle}$ has the constant value $a_s^{\beta^\wedge\langle(0,i_m)\rangle}$ at all stages $\geq s$. Applying Claim 1 with $\beta = \beta$, $u = k$, and $x = y$, we have $y \in A^{i,1}$. By *ii*) and *viii*) of 5.2, $y < a^\eta_s$. By the choice of (η, s), y enters C at some stage $v > u$. By *xii*) of 5.3, $y = a^\zeta_v$ or $y = k^\zeta_v$ for some ζ. By *vii*) of 5.2, $\zeta \supset \beta^\wedge\langle(0,i_m)\rangle$. Suppose $a^\zeta_v = y$ and a^ζ is destroyed in stage v, then a^η is destroyed in stage v, contradiction. Suppose $y = k^\zeta_v$. Then Case 11 holds at some node β in stage v, where $\beta \subseteq \zeta$. Since $p_w^{\beta^\wedge\langle(0,i_m)\rangle} \downarrow$ for all $w > u$, $\beta \subseteq \beta$. Since k^β_s enters B after y, when y enters C at stage v, so does k^β_s. This completes the proof of Subclaim.

Let $\delta \subset \eta$. Then $\delta \supseteq \alpha_0$ by the choice of s_0. By Subclaim, k^δ_u enters C.

Let $\delta <_L \eta$. Let ζ, τ be the least nodes respectively such that $\delta \cap \eta \subset \zeta \subseteq \delta$ and $\delta \cap \eta \subset \tau \subseteq \eta$. There are two subcases:

Case 1. There is no (n,e)-node α such that $\zeta \subseteq \alpha \subseteq \delta$ and $e < i$. Hence for each $j \in I_i$, j is active at δ. By Subclaim, k_u^δ enters C.

Case 2. Otherwise. Let α be the least such node. Towards a contradiction assume that τ is a k-node for $k > 2$. Then $\delta <_L \eta^-$ and c^{η^-} is destroyed at stage u. We now obtain a contradiction by showing that a^η is destroyed at some stage $\geq u$. The arguments is almost identical to that used for the Case 4 in the proof of Subclaim 6 of Claim 1. The parameters β, τ, w, z of that argument correspond to the present η^-, δ, u, and x respectively. Therefore, τ is a k-node for $k \leq 2$, and ζ is a 0-node. By *xviii*) of 5.1, $a_u^\zeta \downarrow$ since $a_u^\eta = a_s^\eta \downarrow$. Suppose α is a 0-node and $a_u^\alpha \downarrow$. Suppose $\alpha \in \mathcal{C}[u]$. Let π be the $(0, i_m)$-node in $\mathcal{A}(\alpha^-)$. Note that $\alpha \leq \pi$. Since $\alpha \in \mathcal{C}[u]$, $a_u^\pi \downarrow$. If $\pi < \pi$, then p^π is defined when $\mathcal{C}[u]$ is defined. Let y be this value of p^π. If p^π is set equal to y at stage w, the y enters B after r^{π^-} is set equal $r_w^{\pi^-}$, i.e., after a^π is set equal to a_w^π. Further, $a_w^\pi = a_u^\pi$. Otherwise p^π would be destroyed at a stage $\geq w$ and $< u$, contradiction. So whether $\alpha < \pi$ or $\alpha = \pi$, there is a number y with $\alpha(y) \supseteq \pi$ which enters B after a^π is set equal a_u^π. Let $w > u$ be the least stage in which α_0 is visited. By Claim 1, $k_u^{\alpha^-} \in B_w$. Note that $k_u^{\alpha^-}$ enters B after stage u. By Subclaim, $k_u^{\alpha^-}$ enters C at a stage $v > s$. Towards a contradiction assume $k_u^{\alpha^-} = a_v^\tau$ for some τ. $\tau <_L \alpha^-$ by *vii*) of 5.2. So $\tau <_L \eta$, contradiction. Therefore, by *xii*) of 5.3, Case 11 holds in stage v at some node $\beta \subseteq \alpha^-$. Note that k_u^δ enters B before $k_u^{\alpha^-}$ enters B. Therefore, by Case 11, k_u^δ also enters C. Suppose either $n \neq 0$, or $n = 0$ and $a_u^\alpha \uparrow$. Then $\zeta \neq \alpha$ and $j(\zeta) > e$. So $\zeta \notin \mathcal{C}[u]$. Since a^ζ is not destroyed in stage u, there is a maximal 0-node $\xi \subset \zeta$ such that $\xi \in \mathcal{C}[u]$. By *v*) of 5.3, $j(\xi) \leq j$ for all (k,j)-nodes θ with $\xi \subset \theta \subseteq \alpha$. In particularly, $j(\xi) \leq e$, and so $\xi \subset \alpha_0$. Let $w > u$ be the least stage in which a^ξ is destroyed. Then $a_w^\xi = a_u^\xi \downarrow$. By (f) of *vi*) of 5.3, $r_{u+1}^{\xi^-} \downarrow$. By *xxv*) of 5.1, $\xi \in \mathcal{C}_{u+1}$ and $\theta \notin \mathcal{C}_{u+1}$ for all 0-nodes θ with $\xi \subset \theta \subseteq \zeta$. Hence $\zeta \in \mathcal{D}_{u+1}$. Clearly, $c_w^{\xi^-} \downarrow$ and $c_{w+1}^{\xi^-} \downarrow$. By (a) of *vi*) of 5.3, a^ζ is destroyed at stage w. By *xiii*) of 5.1, a^η is destroyed at stage w, contradiction. This completes the proof of *i*).

ii) Suppose $x \leq a_s^\eta$ is enumerated in C at a stage $u \geq s$. By *xii*) of 5.3, $x = a_u^\delta$ or $x = k_u^\delta$ for some δ.

Let $x = a_u^\delta$. By *ii*) of 5.2, either $\delta <_L \eta$ or $\eta \subset \delta$. By *xiii*) of 5.3, a^δ is destroyed at stage u. By *xiii*) of 5.1, a_u^η is also destroyed at stage u, contradiction.

Let $x = k_u^\delta$. By *iv*) of 5.2, $\delta < \eta$ or $\eta \subseteq \delta$. But $\eta \not\subseteq \delta$ since η is 1-node. Towards a contradiction suppose $k_u^\delta \in B_s$. There are two cases:

Case 1. δ receives attention at stage u. Then Case 11 holds at δ in stage u.

Suppose $\delta \subset \eta$. Note that $\alpha_0 \subseteq \delta$. Let $y = p_u^{\delta\hat{}\langle(0,i_m)\rangle)}$. Then y is designated for $\delta\hat{}\langle(0,i_m)\rangle)$ at stage w in which $p^{\delta\hat{}\langle(0,i_m)\rangle)}$ is set equal to $p_u^{\delta\hat{}\langle(0,i_m)\rangle)}$. Note that $a_u^{\delta\hat{}\langle(0,i_m)\rangle)} = a_w^{\delta\hat{}\langle(0,i_m)\rangle)} \downarrow$ since $p^{\delta\hat{}\langle(0,i_m)\rangle)}$ is not destroyed in any stage $\geq w$ and $< u$. Further, $c_u^\delta = c_w^\delta \downarrow$ by the same token. We also observe that $y \in B_w$ and $x = k_u^\delta = k_w^\delta \notin B_w$ since a k^δ cannot be enumerated in B until $p^\epsilon \downarrow$ for every $\epsilon \in \mathcal{A}(\delta)$. By (e) of *vi*) of 5.3, y is the first number z with $\alpha(z) \supseteq \delta\hat{}\langle(0,i_m)\rangle)$ which enters B after $a^{\delta\hat{}\langle(0,i_m)\rangle)}$ is set equal to $a_u^{\delta\hat{}\langle(0,i_m)\rangle)}$. Note that $y \in B_s$. $y = p_u^{\delta\hat{}\langle(0,i_m)\rangle)} < a_u^{\delta\hat{}\langle(0,i_m)\rangle)} < a_u^\eta = a_s^\eta$ by *ii*) and *viii*) of 5.2. By Claim 1, $y \in A_s^{i,1}$. By the choice of (η, s), y enters C at a stage $v < s$. Towards a contradiction assume that $y = a_v^\tau$ for some τ. By *vii*) of 5.2, $\tau \supset \delta\hat{}\langle(0,i_m)\rangle)$. Since any value assigned to a p is not in C, $w \leq v$. By *xiii*) of 5.3, a^τ is destroyed in stage v, which contradicts *xiii*) of 5.1. Therefore $y = k_v^\tau$ for some τ and Case 11 holds at some $\xi \subseteq \tau$. By *vii*) of 5.2, $\tau \supseteq \delta\hat{}\langle(0,i_m)\rangle)$. Again, $w \leq v$. Here $w \neq v$ since, if Case 11 holds in stage w, it will be at a node $<_L \delta\hat{}\langle(0,i_m)\rangle)$. By *xv*) of 5.1, $\xi \subseteq \delta$. Let ζ be the immediate successor of ξ such that $\zeta \subseteq \tau$. By Case 11, $\zeta \in \mathcal{A}(\xi)$. By *ix*) of 5.3, $p_v^\zeta \downarrow$ since it must have been defined during the greatest stage $< v$ in which c^ξ was set equal 1. Since c^δ remains defined at all stages $\geq w$ and $< u$, p^ζ is not destroyed at any stage $\geq v$ and $< u$. Since no node $\supseteq \zeta$ can be visited during these stages, $x = k_u^\delta \in B_v$. Since y is enumerated in C by Case 11 at ξ, y must have entered B after $r_v^{\xi^-}$ was set. Since x entered B after y and before stage v, Case 11 requires that x also be enumerated in C at stage v. This contradicts the choice of x.

Now suppose $\delta <_L \eta$. When Case 11 holds at δ in stage u, a^η is destroyed, contradiction.

Case 2. δ does not receive attention at stage u. Let ξ receive attention at stage u. By *xii*) of 5.3, Case 11 holds at ξ in stage u such that for some $\zeta \in \mathcal{A}(\xi)$, $\zeta \subset \delta$ and k_u^δ entered B since r_u^ξ was set. Note k_u^ξ is enumerated in C at stage u. By *iii*) of 5.2, $k_u^\xi < a_s^\eta$. Applying to ξ the argument used for δ in Case 1 above, $\xi \subset \eta$ and k_u^ξ is enumerated in B at stage $\geq s$. It is obvious that $\alpha_0 \subset \xi$. Let h the greatest stage at which r^ξ was set equal r_u^ξ. Clearly, $r_{h+1}^\xi = r_u^\xi$ and so $a_u^\epsilon = a_h^\epsilon \downarrow$ for all $\epsilon \in \mathcal{A}(\xi)$. By choice of s, a^ϵ is not destroyed at any stage

$\geq h$ for any $\epsilon \in \mathcal{A}(\xi)$ with $\epsilon \leq \xi^\wedge\langle(0, i_m)\rangle$. Further, $p_u^\epsilon \downarrow$ for all $\epsilon \in \mathcal{A}(\xi)$ since $p^\epsilon \downarrow$ when k^ξ enters B, and a^ϵ cannot become undefined at a stage $\geq h$ and $< u$. By (e) of $vi)$ of 5.3, p_u^ζ is the first number y with $\alpha(y) \supseteq \zeta$ to enter B after a_u^ζ is set. Note that k_u^δ entered B after r_u^ξ was set, so k_u^δ entered B after a_u^ζ was set. There are two cases:

Case 2.1. $j(\zeta) < i_m$. Let $y = p_u^{\xi^\wedge\langle(0,i_m)\rangle}$. By (e) of $vi)$ of 5.3, y is the first number z with $\alpha(z) \supseteq \xi^\wedge\langle(0, i_m)\rangle$ to enter B after $a_u^{\xi^\wedge\langle(0,i_m)}$ is set since y is $\xi^\wedge\langle(0, i_m)$-designated when $p^{\xi^\wedge\langle(0,i_m)}$ is set equal to y. By Claim 1, y enters B at a stage $\geq s$, otherwise k_u^ξ enters B at a stage $< s$, contradiction. Towards a contradiction assume that k_u^δ enters B at a stage $v < s$. Clearly, $h \leq v$. Then when k_u^δ enters B, $p^{\xi^\wedge\langle(0,i_m)\rangle}$ is undefined. Hence $\zeta \notin \mathcal{C}[v]$. Since a^ζ is not destroyed in stage v, there is a maximal 0-node $\epsilon \subset \zeta$ such that $\epsilon \in \mathcal{C}[v]$. By $v)$ of 5.3, $j(\epsilon) \leq j(\zeta)$. Hence $\epsilon \subset \alpha_0$ by the choice of η. By (f) of $vi)$ of 5.3, $r_{v+1}^{\epsilon^-} \downarrow$. By $xxv)$ of 5.1, $\epsilon \in \mathcal{C}_{v+1}$ and $\theta \notin \mathcal{C}_{v+1}$ for all 0-nodes θ with $\epsilon \subset \theta \subseteq \zeta$. Let w be the least stage $> v$ at which a^ϵ is destroyed. By (a) of $vi)$ of 5.3, a^ζ is destroyed, contradiction.

Case 2.2. Otherwise. Let $y = p_u^\zeta$. To show k_u^δ enters B at a stage $\geq s$ we just need to show that y enters B at a stage $\geq s$. Towards a contradiction assume $y \in B_s$. Let τ be the least $(0, i_m)$-node $\supseteq \zeta$ on $\alpha(y)$. τ exists since y is a ζ-designated number. Note that for all $j \in I_i$, j is active at τ^-. Clearly, when y entered B, it was designated for τ. By (e) of $vi)$ of 5.3, y is the first number z with $\alpha(z) \supseteq \tau$ enter B after a^τ is set equal to a_u^τ. By Claim 1, $y \in A_s^{i,1}$. By $viii)$ and $ii)$ of 5.2, $y < a_u^\zeta < a_u^\eta = a_s^\eta$. By Case 11, y is enumerated in C at stage u because $y = p_u^\zeta$ and the value given to p^ζ is required to have entered B since $r^{\zeta^-} = r^\xi$ was set. By the choice of s, $y \in C_s$. By $xiii)$ of 5.3 this is a contradiction because y cannot be enumerated in C both at a stage $< s$ and at stage u. This completes the proof of $ii)$.

Case 3.2. $i \in I_i$. Let $i_0 < \cdots < i_m$ be an enumeration of I_i. Note that for each $e \geq i(\alpha_0)$, there exists an $\gamma^e \in T$ such that

- $\gamma^e \supseteq \alpha_0$, $(\gamma^e)^- \in P$, $i(\gamma^e) = e < i(((\gamma^e)^-)^+)$,
- $\gamma^e = (\gamma^e)^{-\wedge}\langle(0, i)\rangle <_L ((\gamma^e)^-)^+$.

To see how to find this γ^e we can let ξ be the least $\delta \in P$ such that $i(\delta) = e + 1$, then γ^e can be chosen as $\delta^{-\wedge}\langle(0, i)\rangle$. Note that by $vi)$ of 5.4, $a_\omega^{\gamma^e}$ is defined.

Define $\mathcal{F} \subset T$ as

$$\{\alpha : \alpha \supset \alpha_0 \wedge \alpha(l(\alpha)-1) = (0,i) \wedge (\forall j \in I_i)[j \textit{ is active at } \alpha]\}.$$

Note that for each $e \geq i(\alpha_0)$, $\gamma^e \in \mathcal{F}$. To show that $D \leq_T A^{i,0} - C$, fix g, find the least (γ, s) such that

- $\gamma \in C$, $s \leq s_0$
- $a_s^\gamma \downarrow \geq g$
- *each node* $\subseteq \alpha_0$ *is visited at stage* s
- $B_s \upharpoonright (a_s^\gamma + 1) \subseteq A_s^{i,0} \cup A_s^{i,1}$
- $(A_s^{i,1} - C_s) \upharpoonright (a_s^\gamma + 1) = (A_\omega^{i,1} - C_\omega) \upharpoonright (a_s^\gamma + 1)$.

Note that, for each $e \geq \max\{n, i(\alpha_0)\}$ there exist arbitrarily large s such that (α^e, s) satisfies the conditions specified for (γ, s).

It can be shpwn that $g \in D$ if and only if $g \in D_s$. It is sufficient to establish the following two claims hold:

Claim 1. Let β, u satisfy

- $\alpha_0 \subset \beta < \gamma$, $s \leq u$,
- α_0 *is visited in stage* u,
- *for all* $j \in I_i$, j *is active at* β,
- $a_u^{\beta\wedge\langle(0,i_m)\rangle} \downarrow$.

If some y entered B at a stage $> s_0$ and $< u$ with $\alpha(y) \supseteq \beta\wedge\langle(0, i_m)\rangle$ since $a_u^{\beta\wedge\langle(0,i_m)\rangle}$ was set, then $k^\beta \in B_u$.

Claim 2.

i) If $x \leq a_s^\gamma$ is enumerated in C at a stage $\geq s$, then x is enumerated in B at a stage $\geq s$.

ii) If $x \leq a_s^\gamma$ is enumerated in B at a stage $\geq s$, then x is enumerated in C.

The proofs of these claims are similar to the proofs of Claims 1 and 2 in the treatment of Case 3.1.2 above. Since the proofs are easier in the present case, we leave them to the reader. Now from Claim 2 it is clear that $g \in D$ if and only if $g \in D_s$. This completes the proof of 6.4. ■

Chapter 7

Further notes and conjectures

First, the question as to what splitting property might hold for d.r.e. sets is addressed by:

Conjecture 1. There exists a properly d.r.e. set D such that for all d.r.e. sets A^0, A^1,

$$[A^0 \sqcup A^1 = D \Longrightarrow D \leq_T A^0 \vee D \leq_T A^1].$$

Definition. Fix n. A degree $\mathbf{a} \in \mathbf{D}_n$ is called a *minimal cover* in $\mathbf{D}_n$ if there exists a degree $\mathbf{b} \in \mathbf{D}_n$ with $\mathbf{b} < \mathbf{a}$ such that there is no degree $\mathbf{c} \in \mathbf{D}_n$ and $\mathbf{b} < \mathbf{c} < \mathbf{a}$.

As was mentioned in Chapter 1 the d.r.e. degrees are not dense, see [6]. That theorem was proved by showing that $\mathbf{0}'$ is a minimal cover in $\mathbf{D}_2$. We believe that that result can be strengthened:

Conjecture 2. Every high d.r.e. degree is a minimal cover in $\mathbf{D}_2$.

Verification of this would be interesting because it would yield an elementary property separating high degrees from the low_2 degrees in the upper semilattice of d.r.e. degrees. Cooper asked whether there are such properties.

Conjecture 3. There is no d.r.e. degree which is a miminal cover of an r.e. degree in $\mathbf{D}_2$.

An *isolated degree* is a d.r.e. degree $\mathbf{d}$ such that among the r.e. degrees $< \mathbf{d}$ there is a greatest one. Such degrees exist by unpublished work of Cooper and the author independently.

Conjecture 4. The isolated degrees are dense in the r.e. degrees.

Finally, the question about definability of d.r.e. degrees is still open:

Conjecture 5. The set of d.r.e. degrees is definable in the upper semilattice of Turing degrees.

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