# THE GENERAL NON-SYMMETRICAL PUNCH AND CRACK PROBLEMS

by

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Abstract

The problems of displacement of the surface of an elastic medium by a circular punch whose face is non-symmetrical and of the penny-shaped crack whose faces are subjected to arbitrary tractions are reduced to a system of integral equations for the components of displacement discontinuity. This reduction is accomplished by the use of the Somigliana formula. For the punch problems the Fourier components of stress and displacement discontinuity on the surface of the medium are found. The special case of constant unidirectional tangential displacement is examined. For the penny-crack problem expressions for the stresses on the plane of the crack beyond the crack tip are found. Stress intensity factors are derived and Barenblatt and BCS models for axisymmetric normal and shear cracks are constructed.

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## 1. Introduction

An integral equation method will be used to investigate two types of problems involving an isotropic linear elastic medium. The first problem is that of a frictionless circular punch or die(whose face is of arbitrary shape and whose radius is a) which indents the surface of a half space by a prescribed amount. The second problem considered is that of a penny-shaped crack of radius c (imbedded in the medium ) whose upper and lower surfaces are loaded by equal and opposite arbitrary tractions.

The axisymmetric case where the punch is given a displacement normal to the surface of the half space has been previously investigated by Green and Zerna [1] and by others [2,3,4]. Keer [5] has found a solution for the case of the non axisymmetric normal punch problem by the use of a potential method.

The case where the crack is subjected to axisymmetric normal tractions has been dealt with by, among others, Sneddon [6,7], and Collins [8]. Sneddon used transforms to reduce the problem to a set of dual integral equations while Collins used the Papkovitch-Neuber potentials. Again, Keer [5] used a potential method to solve the non-axisymmetric normally loaded crack problem.

Another method has been introduced [9], originally to calculate the densities of arrays of dislocations. This method has been shown to be useful in plain-strain boundary value problems [10] and has also been used for certain axisymmetric half-space problems [11]. This dislocation method has been extended recently to deal with certain crack problems in which the crack is viewed as a layer of dislocation, and an integral equation is obtained for the displacement discontinuity across the layer [12,13,14]. This method will be applied here to both the penny-crack and punch problems.

In Section 2 the Somigliana formula is derived. This formula gives an expression for the displacement field in a linear elastic material in which an internal surface is subjected to certain displacement discontinuities. In Section 3 the crack and punch problems are formulated and the governing integral equations are seen to partially decouple into normal and tangential expressions. With the use of a Fourier decomposition the integral equations are further simplified in Section 4. Up to this point both the general punch and crack problems may be treated mathematically the same except for a limit of integration.

In Section 5 the integral equations for both the normal and tangential punch problems are shown to have the same form and the equations are then solved. From this solution expressions are found in Section 6 for the Fourier coefficients of displacement discontinuities and stresses on the surface of the half-space. The special case of a constant uni-directional displacement over a region $r \leqslant a$, $z = 0$ is examined in Section 7. Expressions for the displacement discontinuities and stresses on the surface of the half-space are found.

In Section 8 the integral equations for both the normal and tangential crack problems are shown to also have the same form and the equations are then solved. From this solution the Fourier coefficients for the displacement discontinuities and stresses on the plane of the crack are found in Section 9. The Fourier series are then summed giving expressions for the displacement discontinuities and stresses on the plane of the crack. For the case of normal loading a stress intensity factor, $K_I(\theta)$ at the crack edge is found.

The results from Section 9 are used in Section 10 to construct a Barenblatt Model for the crack under normal axisymmetric loading. Also a BCS Model is constructed for the crack under an axisymmetric torsional load. Finally in Section 10 the problem of a penny-crack under unidirectional shear loading is examined. This final application is due to Lardner in [15].

## 2. The Somigliana Formula

The derivation of the Somigliana formula for the displacement field $u_i(\underline{r})$ in an infinite linear elastic material subjected to certain deformations begins with the determination of the displacement field in terms of a Green's tensor.

Consider a finite body $B$ consisting of linear elastic material for which the elastic modulus components are $C_{ijk\ell}$. The body is in equilibrium under given body forces whose components are $f_i$, under given tractions $t_i$ acting on a part $S_1$ of its boundary, and under given displacements $U_i$ over a part $S_2$ of its boundary. Since $B$ is in equilibrium and because of the boundary values the components of displacement $u_i(\underline{r})$ must satisfy

$$C_{ijk\ell}u_{\ell,jk} + f_i = 0 \qquad \text{in } B \tag{2.1a}$$

$$C_{ijk\ell}u_{\ell,k}n_j = t_i \qquad \text{on } S_1 \tag{2.1b}$$

$$u_i = U_i \qquad \text{on } S_2 \tag{2.1c}$$

Here $n_j$ are the components of a unit vector normal to the surface $S_1$. The comma notation is used to denote partial differentiation with respect to the indicated component of $\underline{r}$. Repeated indices are summed over 1, 2, 3.

To construct a Green's tensor for this boundary value problem let $\underline{r}'$ represent a point inside $B$. With $\underline{r}'$ as its center construct a sphere $S_\varepsilon$ with radius $\varepsilon$. Let $B_\varepsilon = B - S_\varepsilon$. Finally let the Green's tensor $G_{ij}(\underline{r},\underline{r}')$ satisfy:

$$C_{ijk\ell}G_{\ell m,jk}(\underline{r},\underline{r}') = 0 \qquad \text{for } \underline{r} \text{ in } B_\varepsilon \tag{2.2a}$$

$$C_{ijk\ell}G_{\ell m,k}(\underline{r},\underline{r}')n_j = 0 \qquad \text{for } \underline{r} \text{ on } S_1 \tag{2.2b}$$

$$G_{\ell m}(\underline{r},\underline{r}') = 0 \qquad \text{for } \underline{r} \text{ on } S_2 \tag{2.2c}$$

$$\int_{\partial S_\varepsilon} C_{ijk\ell}G_{\ell m,k}(\underline{r},\underline{r}')n_j dS = \delta_{im} \tag{2.2d}$$

$$\varepsilon^2 G_{\ell m}(\underline{r},\underline{r}') \to 0 \text{ as } \varepsilon \to 0 \qquad \text{for } \underline{r} \text{ on } \partial S_\varepsilon \tag{2.2e}$$

Apply the Gauss divergence theroem on $u_i(\underline{r})$ and $G_{ij}(\underline{r},\underline{r}')$:

$$\int_{\partial B+\partial S_\varepsilon} [u_i(\underline{r})G_{\ell m,k}(\underline{r},\underline{r}') - G_{im}(\underline{r},\underline{r}')u_{\ell,k}(\underline{r})]n_j C_{ijk\ell} dS =$$

$$= \int_{B_\varepsilon} [u_i(\underline{r})G_{\ell m,k}(\underline{r},\underline{r}') - G_{im}(\underline{r},\underline{r}')u_{\ell,k}(\underline{r})]_{,j} C_{ijk\ell} dV. \tag{2.3}$$

Letting $\varepsilon \to 0$, using equations (2.1) and (2.2) together with the fact that $u_i(\underline{r})$ are continuous inside B, (2.3) becomes

$$u_m(\underline{r}') = \int_B G_{\ell m}(\underline{r},\underline{r}')f_\ell(\underline{r})dV + \int_{S_1} G_{\ell m}(\underline{r},\underline{r}')t_\ell dS -$$

$$- \int_{S_2} U_i(\underline{r})G_{\ell m,k}(\underline{r},\underline{r}')n_j C_{ijk\ell} dS. \tag{2.4}$$

If the medium extends to infinity some restrictions on the displacements and the Green's tensor must be imposed. In addition to the requirement that $G_{\ell m}(\underline{r},\underline{r}')$ satisfy equations (2.2) we must also require that ($S_R$ is a sphere of a large radius R)

$$\int_{\partial S_R} C_{ijk\ell} G_{\ell m,k}(\underline{r},\underline{r}') n_j dS$$

be bounded and must balance the unit force at $\underline{r}'$. It is sufficient if we suppose $G_{\ell m}(\underline{r},\underline{r}') \sim O(1/r)$ and $G_{\ell m,k}(\underline{r},\underline{r}') \sim O(1/r^2)$ as $|\underline{r}| \to \infty$. The displacements $u_i(\underline{r})$ must satisfy equations (2.1) and in addition

$$u_i(\underline{r}) \to 0 \quad \text{as } |\underline{r}| \to \infty$$

$$r u_{i,j}(\underline{r}) \to 0 \quad \text{as } |\underline{r}| \to \infty.$$

Then with the addition of the above restrictions on $G_{ij}(\underline{r},\underline{r}')$ and $u_i(\underline{r})$ equation (2.4) is valid even if B extends to infinity.

Now suppose that the body B contains a crack occupying a surface whose two faces are labeled $A_+$ and $A_-$, and that a deformation be caused in the body by the application of tractions only upon the surfaces $A_\pm$. The tractions are required to be equal and opposite at corresponding points of $A_+$ and $A_-$. There then will be a displacement discontinuity $\Delta u_i(\underline{r})$ across the crack. This discontinuity has components

$$\Delta u_i(\underline{r}) = u_i^+(\underline{r}) - u_i^-(\underline{r}). \tag{2.5}$$

Since tractions and displacements on the external boundaries are assumed to be zero along with the body forces $f_i$, equation (2.4) can be written as

$$u_m(\underline{r}') = \int_{A_+ + A_-} [G_{\ell m}(\underline{r},\underline{r}')t_\ell(\underline{r}) - U_i(\underline{r})G_{\ell m,k}(\underline{r},\underline{r}')C_{ijk\ell}n_j]dS. \quad (2.6)$$

Denoting $\sigma_{ij}(\underline{r})$ as the strees tensor, $t_\ell = \sigma_{\ell j}n_j$ and

$$\sigma_{\ell j}n_j\Big|_{A_+} = -\sigma_{\ell j}n_j\Big|_{A_-}. \quad (2.7)$$

With the help of (2.5) and (2.7) equation (2.6) may be simplified to

$$u_m(\underline{r}') = -\int_A \Delta u_i(\underline{r})G_{\ell m,k}(\underline{r},\underline{r}')C_{ijk\ell}n_j dS. \quad (2.8)$$

Equation (2.8) is known as the Sǫmigliana formula [16].

The work to follow will be concerned with linear elastic isotropic material. In such a case

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \quad (2.9)$$

where $\lambda$ and $\mu$ are the Lamé constants. In addition the components of the Green's tensor for an infinite medium are given by [21].

$$G_{\ell m}(\underline{r},\underline{r}') = \frac{1}{8\pi\mu}[\delta_{\ell m}R_{,kk} - \frac{1}{2(1-\nu)}R_{,\ell m}], \quad (2.10)$$

where $\nu = \lambda/2(\lambda+\mu)$ and $R = |\underline{r}-\underline{r}'|$. Differentiation of $R$ is with respect to the $x_i$.

Suppose now that the surface $A$ is planar. For convenience we place a Cartesian coordinate system inside the body in such a way that the surface $A$ lies within the x-y plane. The unit normal to $A$ pointing from $A_+$ to $A_-$ is then in the negative z-direction. Equation (2.8) then may be written

$$u_m(\underline{r}') = \frac{1}{8\pi\mu} C_{i3k\ell} \int_A \Delta u_i(\underline{r}) \left[\delta_{\ell m} R_{,ppk} - \frac{1}{2(1-\nu)} R_{,\ell mk}\right] dS. \qquad (2.11)$$

The stress components are given by

$$\sigma_{ij}(\underline{r}') = C_{ijk\ell} u_{k,\ell}(\underline{r}'). \qquad (2.12)$$

By the use of equations (2.11) and (2.12) the stresses can therefore be expressed in terms of the components of displacement discontinuity. It is appropriate at this stage to point out that on the plane $z' = 0$ the component $\sigma_{zz}(\underline{r}')$ depends only upon $\Delta u_z$ while the components $\sigma_{xz}(\underline{r}')$ and $\sigma_{yz}(\underline{r}')$ depend upon both $\Delta u_x$ and $\Delta u_y$.

## 3. Formulation of the Punch and Crack Problems

The remarks at the end of Section 2 indicate that for the system under consideration the general problem can be decomposed into two sub-problems: (1) normal loading, and (2) shear or tangential loading on the surface A (z'=0).

The general problems for a frictionless circular punch of radius a (whose face is of arbitrary shape) which displaces the surface on an elastic half-space is decomposed as follows:

Problem A: normal indentation, where

$$\Delta u_z(\underline{r}') \text{ specified, } \sigma_{zz}(\underline{r}') \neq 0, \qquad |\underline{r}'|<a,\ z'=0$$

$$\sigma_{zz}(\underline{r}') = 0, \qquad |\underline{r}'|>a,\ z'=0$$

$$\Delta u_x(\underline{r}') = \Delta u_y(\underline{r}') = 0$$

$$\sigma_{xz}(\underline{r}') = \sigma_{yz}(\underline{r}') = 0.$$

Problem B: tangential displacement, where

$$\left.\begin{array}{l}\Delta u_x(\underline{r}') \text{ and } \Delta u_y(\underline{r}') \text{ specified,}\\ \sigma_{xz}(\underline{r}') \text{ and } \sigma_{yz}(\underline{r}') \neq 0\end{array}\right\} \qquad |\underline{r}'|<a,\ z'=0$$

$$\sigma_{xz}(\underline{r}') = \sigma_{yz}(\underline{r}') = 0 \qquad |\underline{r}'|>a,\ z'=0$$

$$\Delta u_z(\underline{r}') = 0$$

$$\sigma_{zz}(\underline{r}') = 0$$

The general problem of a penny-shaped crack of radius c subjected to prescribed tractions on the crack surface is decomposed as follows:

Problem C: crack under normal loading, where

$$\sigma_{zz}(r') \text{ specified, } \Delta u_z(r') \neq 0, \qquad |\underline{r}'|<c,\ z'=0$$

$$\sigma_{xz}(r') = \sigma_{yz}(r') = 0$$

$$\Delta u_x(r') = \Delta u_y(r') = 0.$$

$$\Delta u_z(r') = 0 \qquad |\underline{r}'| > c$$

Problem D: crack under shear loading, where

$$\left.\begin{array}{l}\sigma_{xz}(\underline{r}') \text{ and } \sigma_{yz}(\underline{r}') \text{ specified} \\ \Delta u_x(\underline{r}'),\ \Delta u_y(\underline{r}') \neq 0 \\ \sigma_{zz}(\underline{r}') = 0,\ \Delta u_z(r') = 0\end{array}\right\} \qquad |\underline{r}'|<c,\ z'=0$$

$$\Delta u_x(r') = \Delta u_y(r') = 0 \qquad |\underline{r}'| > c$$

The information in all four of the above problems may be translated into integral equations involving the unknown components of displacement discontinuity, $\Delta u_i(\underline{r}')$. In order to derive these integral equations expressions for certain stress components must be calculated. Using equations (2.11) and (2.12) together with (2.9) expressions for $\sigma_{xz}(\underline{r}')$, $\sigma_{yz}(\underline{r}')$, and $\sigma_{zz}(\underline{r}')$ may be found.

In problems B and D, $\Delta u_z(\underline{r}') = 0$. Hence,

$$u_m(\underline{r}') = \frac{1}{8\pi}\int_A \Big\{\Delta u_x(\underline{r})\left[\delta_{1m}R_{,pp3} + \delta_{3m}R_{,pp1} - \frac{1}{1-\nu}R_{,13m}\right] + $$

$$+ \Delta u_y(\underline{r})\left[\delta_{2m}R_{,pp3} + \delta_{3m}R_{,pp2} - \frac{1}{1-\nu}R_{,23m}\right]\Big\} dS. \qquad (3.1)$$

And

$$\begin{aligned}\sigma_{xz}(\underline{r}') &= \mu\left[u_{x,z'} - u_{z,x'}\right] \\ \sigma_{yz}(\underline{r}') &= \mu\left[u_{y,z'} - u_{z,y'}\right].\end{aligned} \qquad (3.2)$$

Substituting (3.1) into (3.2) gives

$$\sigma_{xz}(\underline{r}') = -\frac{\mu}{8\pi}\int_A \left\{-\Delta u_x(\underline{r})\left[R_{,pp22} + \frac{2}{1-\nu} R_{,1133}\right] + \right.$$

$$\left. + \Delta u_y(\underline{r})\left[R_{,pp12} - \frac{2}{1-\nu} R_{,1233}\right]\right\} dS. \tag{3.3}$$

$$\sigma_{yz}(\underline{r}') = -\frac{\mu}{8\pi}\int_A \left\{\Delta u_x(\underline{r})\left[R_{,pp12} - \frac{2}{1-\nu} R_{,1233}\right] - \right.$$

$$\left. - \Delta u_y(\underline{r})\left[R_{,pp11} + \frac{2}{1-\nu} R_{,2233}\right]\right\} dS, \tag{3.4}$$

where we have used the fact that $\nabla^4 R = 0$ and that $R_{,x} = -R_{,x'}$. Note that the indices m and p (= 1,2,3) have been used interchangably with x,y, and z.

In problems A and C, $\Delta u_x(\underline{r}') = \Delta u_y(\underline{r}') = 0$. Hence

$$u_m(\underline{r}') = \int_A \Delta u_z(\underline{r})\left[\nu R_{,ppm} + 2(1-\nu)\delta_{3m} R_{,pp3} - R_{,33m}\right] dS. \tag{3.5}$$

And

$$\sigma_{zz}(\underline{r}') = 2(\lambda+\mu)\left\{\nu\left[u_{x,x'}(\underline{r}') + u_{y,y'}(\underline{r}')\right] + (1-\nu)u_{z,z'}(r')\right\}. \tag{3.6}$$

Substituting (3.5) into (3.6) gives

$$\sigma_{zz}(\underline{r}') = -\frac{\lambda+\mu}{4\pi(1-\nu)}\int_A \Delta u_z(\underline{r})\left\{\nu(1-2\nu)R_{,pp33} + \right.$$

$$\left. + (1-2\nu)\left[(2-\nu)R_{,pp33} - R_{,3333}\right]\right\} dS$$

$$= -\frac{\lambda+\mu}{4\pi(1-\nu)}\int_A \Delta u_z(\underline{r})\,(1-2\nu)\left[2R_{,pp33} - R_{,3333}\right] dS.$$

Since $R_{,ppqq} = 0 = R_{,\alpha\alpha\beta\beta} + 2R_{,\alpha\alpha33} + R_{,3333}$, $(\alpha,\beta = 1,2)$

$$\sigma_{zz}(\underline{r}') = \frac{\mu}{4\pi(1-\nu)}\int_A \Delta u_z(\underline{r})\,R_{,\alpha\alpha\beta\beta}\, dS. \qquad (3.7)$$

Lardner has shown [15] that for the penny crack problem equations (3.3), (3.4), and (3.7) may be simplified further by an integration by parts. First, however, note that so far the above equations have been used to represent components of stress for both the punch and the crack problems. The equations representing the punch problems differ from those representing the crack problems only in the interpretation of the area of integration, A. We notice that if equations (3.3), (3.5), and (3.7) are to be used to represent the crack problems, the area A of integration is a disc of radius c, since the components of displacement discontinuity, $\Delta u_i(\underline{r})$ are all zero for $|\underline{r}| > c$. If the same equations are to be used to represent the punch problems, the area A of integration is the whole x-y plane. In the punch case the $\Delta u_i(\underline{r}) \to 0$ as $|\underline{r}| \to \infty$.

From the above remarks it is clear then that in either type of problem $\Delta u_i(\underline{r})$ vanishes on the boundary of A. Hence in an integration by parts the boundary terms will vanish. We first integrate (3.3) and (3.4) by parts to give

$$\sigma_{xz}(\underline{r}') = -\frac{\mu}{8\pi}\int_A \Big\{\Delta u_{x,y}R_{,pp2} + \frac{2}{1-\nu}\Delta u_{x,x}R_{,133} - \Delta u_{y,x}R_{,pp2} + \frac{2}{1-\nu}\Delta u_{y,y}R_{,133}\Big\}\, dS$$

$$\sigma_{yz}(\underline{r}') = -\frac{\mu}{8\pi}\int_A \Big\{-\Delta u_{x,y}R_{,pp1} + \frac{2}{1-\nu}\Delta u_{x,x}R_{,233} + \Delta u_{y,x}R_{,pp1} + \frac{2}{1-\nu}\Delta u_{y,y}R_{,233}\Big\}\, dS.$$

If we define the quantities

$$a(\underline{r}) = \Delta u_{x,x}(\underline{r}) + \Delta u_{y,y}(\underline{r})$$

$$b(\underline{r}) = (1-\nu)\left[\Delta u_{x,y}(\underline{r}) - \Delta u_{y,x}(\underline{r})\right],$$

and perform the differentiations on R, we obtain

$$\sigma_{xz}(\underline{r}') = \frac{-\mu}{4\pi(1-\nu)}\int_A \Big\{a(\underline{r})\frac{\partial}{\partial x}\Big[\frac{1}{R} - \frac{z'^2}{R^3}\Big] + b(\underline{r})\frac{\partial}{\partial y}\Big[\frac{1}{R}\Big]\Big\}\, dS$$

$$\sigma_{yz}(\underline{r}') = \frac{-\mu}{4\pi(1-\nu)}\int_A \left\{a(\underline{r})\frac{\partial}{\partial y}\left[\frac{1}{R} - \frac{z'^2}{R^3}\right] - b(\underline{r})\frac{\partial}{\partial x}\left[\frac{1}{R}\right]\right\} dS.$$

It is convenient at this point to switch to cylindrical polar coordinates. Using the usual transformations and recalling that $\frac{\partial}{\partial x'}(R) = -\frac{\partial}{\partial x}(R)$, on the plane $z' = 0$,

$$\frac{4\pi(1-\nu)}{\mu}\sigma_{rz}(r',\theta',0) = \int_A \left\{\alpha(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right) + \beta(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta \tag{3.8}$$

$$\frac{4\pi(1-\nu)}{\mu}\sigma_{\theta z}(r',\theta',0) = \int_A \left\{\alpha(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right) - \beta(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta, \tag{3.9}$$

where

$$\alpha(r,\theta) = \Delta u_{r,r}(r,\theta) + \frac{1}{r}\Delta u_r(r,\theta) + \frac{1}{r}\Delta u_{\theta,\theta}(r,\theta)$$

$$\beta(r,\theta) = (1-\nu)\left[\frac{1}{r}\Delta u_{r,\theta}(r,\theta) - \Delta u_{\theta,r}(r,\theta) - \frac{1}{r}\Delta u_\theta(r,\theta)\right] .$$

Returning to equation (3.7) integrate by parts to obtain

$$\sigma_{zz}(\underline{r}') = \frac{-\mu}{4\pi(1-\nu)}\int_A \Delta u_{z,\alpha}(\underline{r})\; R_{,\alpha\beta\beta}\; dS$$

$$= \frac{-\mu}{4\pi(1-\nu)}\int\limits_A \left\{\Delta u_{z,x}(\underline{r})\frac{\partial}{\partial x}\left[\frac{1}{R} + \frac{z'^2}{R^3}\right] + \Delta u_{z,y}(\underline{r})\frac{\partial}{\partial y}\left[\frac{1}{R} + \frac{z'^2}{R^3}\right]\right\} dS$$

Switching to polar coordinates we have on the plane $z' = 0$,

$$\sigma_{zz}(r',\theta',0) = \frac{-\mu}{4\pi(1-\nu)}\int\limits_A \left\{\Delta u_{z,r}\ (r,\theta)\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2}\Delta u_{z,\theta}(r,\theta)\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta, \tag{3.10}$$

where in terms of polar variables on the plane $z = 0$,

$$R = \left[r'^2 + r^2 - 2rr'\cos(\theta-\theta')\right]^{1/2}.$$

We may now summarize these results as they apply to the four stated problems.

Problem A: (normal punch)

$$\sigma_{zz}(r',\theta',0) = \frac{-\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^{\infty}\left\{\Delta u_{z,r}(r,\theta)\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2}\Delta u_{z,\theta}(r,\theta)\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta \tag{3.11}$$

Problem B: (shear punch)

$$\sigma_{rz}(r',\theta',0) = \frac{\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^{\infty}\left\{\alpha(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right) + \beta(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta \tag{3.12}$$

$$\sigma_{\theta z}(r',\theta',0) = \frac{\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^{\infty}\left\{\alpha(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right) - \beta(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right)\right\} r\,dr\,d\theta \tag{3.13}$$

Problem C: (normal crack)

$$\sigma_{zz}(r',\theta',0) = \frac{-\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^c\left\{\Delta u_{z,r}(r,\theta)\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2}\Delta u_{z,\theta}(r,\theta)\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right\}rdrd\theta \tag{3.14}$$

Problem D: (shear crack)

$$\sigma_{rz}(r',\theta',0) = \frac{\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^c\left\{\alpha(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right) + \beta(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right)\right\}rdrd\theta \tag{3.15}$$

$$\sigma_{\theta z}(r',\theta',0) = \frac{\mu}{4\pi(1-\nu)}\int_0^{2\pi}\int_0^c\left\{\alpha(r,\theta)\frac{1}{r'}\frac{\partial}{\partial\theta'}\left(\frac{1}{R}\right) - \beta(r,\theta)\frac{\partial}{\partial r'}\left(\frac{1}{R}\right)\right\}rdrd\theta \tag{3.16}$$

Upon using the specified boundary values in the four stated problems integral equations for the unknown displacement discontinuity components can be found. In problems A and B (punch problems) the left hand sides of equations (3.11), (3.12), and (3.13) are known to be zero for $r'>a$. In problems C and D (Crack problems) the left hand sides of equations (3.14), (3.15), and (3.16) are all specified for $r'<c$.

## 4. Reduction of the Integral Equations

A further simplification of the integral equations is necessary before they can be solved. A reduction to one variable may be carried out by expanding all the quantities in equations (3.11)-(3.16) in a Fourier series. Using a notation developed by Lardner [15]:

$$\Delta u_m(r,\theta) = \frac{1}{2} f_o^m(r) + \sum_{n=1}^{\infty} \left\{ f_n^m(r)\cos n\theta + g_n^m(r)\sin n\theta \right\} \tag{4.1}$$

$$\alpha(r,\theta) = \frac{1}{2}\alpha_0(r) + \sum_{n=1}^{\infty} \left\{ \alpha_n(r)\cos n\theta + \tilde{\alpha}_n(r)\sin n\theta \right\} \tag{4.2}$$

$$\beta(r,\theta) = \frac{1}{2}\tilde{\beta}_0(r) + \sum_{n=1}^{\infty} \left\{ \tilde{\beta}_n(r)\cos n\theta + \beta_n(r)\sin n\theta \right\} \tag{4.3}$$

$$\frac{1}{R} = \frac{1}{2} I_0(r,r') + \sum_{n=1}^{\infty} \left\{ I_n(r,r')\cos n(\theta-\theta') \right\} \tag{4.4}$$

$$\frac{4\pi(1-\nu)}{\mu}\sigma_{mz}(r,\theta) = \frac{1}{2} P_0^m(r) + \sum_{n=1}^{\infty} \left\{ P_n^m(r)\cos n\theta + Q_n^m(r)\sin n\theta \right\} \tag{4.5}$$

Substitution of (4.1), (4.4), and (4.5) into (3.11) or (3.14) gives

$$\frac{1}{2}P_0^z(r') + \sum_{n=1}^{\infty} \left\{ P_n^z(r')\cos n\theta' + Q_n^z(r')\sin n\theta' \right\} =$$

$$= -\int_0^{2\pi}\int_0^t \left\{ \left[\frac{1}{2} f_0^{z'}(r) + \sum_{n=1}^{\infty} (f_n^{z'}(r)\cos n\theta + \right.\right.$$

$$+ g_n^{z'}(r)\sin n\theta\,)\Big]\left[\frac{1}{2}\frac{\partial}{\partial r} I_0(r,r') + \sum_{n=1}^{\infty} \frac{\partial}{\partial r} I_n(r,r')\cos n(\theta-\theta')\right]$$

$$+ \frac{1}{r^2}\Big[\sum_{n=1}^{\infty}(-nf_n^z(r)\sin n\theta \;+ ng_n^z(r)\cos n\theta\;)\Big]\Big[\sum_{n=1}^{\infty}-nI_n(r,r')\sin n(\theta-\theta')\Big]\Big\} r\,dr\,d\theta, \tag{4.6}$$

where the limits of the r-integral are either 0 to $\infty$ or 0 to c depending upon whether the problem is one of the punch or crack respectively.

The $\theta$ integration may be performed by recalling (no sum here)

$$\begin{aligned}
&\int_0^{2\pi}\cos m\theta\,\cos n(\theta-\theta')\,d\theta = \pi\cos n\theta'\delta_{mn}\\
&\int_0^{2\pi}\sin m\theta\,\cos n(\theta-\theta')\,d\theta = \pi\sin n\theta'\delta_{mn}\\
&\int_0^{2\pi}\sin m\theta\,\sin n(\theta-\theta')\,d\theta = \pi\cos n\theta'\delta_{mn}\\
&\int_0^{2\pi}\cos m\theta\,\sin n(\theta-\theta')\,d\theta = -\pi\sin n\theta'\delta_{mn}
\end{aligned} \tag{4.7}$$

The right hand side of (4.6) then becomes

$$\begin{aligned}
-\int_0^t\Big\{&\frac{\pi}{2}\, f_0^{z'}(r)\frac{\partial}{\partial r}I_0(r,r') + \pi\sum_{n=1}^{\infty}\Big[f_n^{z'}(r)\frac{\partial}{\partial r}I_n(r,r')\cos n\theta' +\\
&+ g_n^{z'}(r)\frac{\partial}{\partial r}I_n(r,r')\sin n\theta' +\\
&+ \frac{n^2}{r^2}I_n(r,r')(f_n^z(r)\cos n\theta' + g_n^z(r)\sin\theta')\Big]\Big\} r\,dr.
\end{aligned} \tag{4.8}$$

Comparing coefficients of the left hand side of (4.6) with the right hand side of (4.8) gives

$$\int_0^t f_0^{z'}(r)\frac{\partial}{\partial r}I_0(r,r')rdr = -\frac{1}{\pi}P_0^z(r') \tag{4.9}$$

$$\int_0^t \left\{ f_n^{z'}(r)\frac{\partial}{\partial r}I_n(r,r') + \frac{n^2}{r^2}f_n^z(r)I_n(r,r')\right\}rdr = -\frac{1}{\pi}P_n^z(r') \tag{4.10}$$

$$\int_0^t \left\{ g_n^{z'}(r)\frac{\partial}{\partial r}I_n(r,r') + \frac{n^2}{r^2}g_n^z(r)I_n(r,r')\right\}rdr = -\frac{1}{\pi}Q_n^z(r') \tag{4.11}$$

Notice that equation (4.11) may be obtained from (4.10) by replacing $f_n^z$ by $g_n^z$ and $P_n^z$ by $Q_n^z$.

Similarly substituting (4.2)-(4.5) into (3.12) and (3.13) or (3.15) and (3.16) gives

$$\frac{1}{2}P_0^r(r') + \sum_{n=1}^{\infty}\left[P_n^r(r')\cos n\theta' + Q_n^r(r')\sin n\theta'\right] =$$

$$= \int_0^{2\pi}\int_0^t \Big\{ \Big[\frac{1}{2}\alpha_0(r) + \sum_{n=1}^{\infty}(\alpha_n(r)\cos n\theta + \tilde{\alpha}_n(r)\sin n\theta)\Big] \cdot$$

$$\cdot \Big[\frac{1}{2}\frac{\partial}{\partial r'}I_n(r,r') + \sum_{n=1}^{\infty}\frac{\partial}{\partial r'}I_n(r,r')\cos n(\theta-\theta')\Big] + \Big[\frac{1}{2}\tilde{\beta}_0(r) + \sum_{n=1}^{\infty}\tilde{\beta}_n(r)\cos n\theta +$$

$$+ \beta_n(r)\sin n\theta)\Big]\Big[\frac{1}{r'}\sum_{n=1}^{\infty} nI_n(r,r')\sin n(\theta-\theta')\Big]\Big\}rdrd\theta, \tag{4.12}$$

$$\frac{1}{2}P_0^\theta(r') + \sum_{n=1}^{\infty}\left[P_n^\theta(r')\cos n\theta' + Q_n^\theta(r')\sin n\theta'\right] =$$

$$= \int_0^{2\pi}\int_0^t \Big\{ -\Big[\frac{1}{2}\tilde{\beta}_0(r) + \sum_{n=1}^{\infty}\tilde{\beta}_n(r)\cos n\theta + \beta_n(r)\sin n\theta\Big] \cdot$$

$$\cdot \Big[\frac{1}{2}\frac{\partial}{\partial r'}I_0(r,r') + \sum_{n=1}^{\infty}\frac{\partial}{\partial r'}I_n(r,r')\cos n(\theta-\theta')\Big] + \Big[\frac{1}{2}\alpha_0(r) + \sum_{n=1}^{\infty}(\alpha_n(r)\cos n\theta +$$

$$+ \tilde{\alpha}_n(r)\sin n\theta)\Big]\Big[\frac{1}{r'}\sum_{n=1}^{\infty} nI_n(r,r')\sin n(\theta-\theta')\Big]\Big\}rdrd\theta \tag{4.13}$$

After performing the $\theta$ integrations and comparing coefficients in (4.12) and (4.13) we obtain

$$\frac{1}{\pi} P_0^r(r') = \int_0^t \alpha_0(r) \frac{\partial}{\partial r'} I_0(r,r') r dr \qquad (4.14)$$

$$\frac{1}{\pi} P_0^\theta(r') = -\int_0^t \tilde{\beta}_0(r) \frac{\partial}{\partial r'} I_0(r,r') r dr \qquad (4.15)$$

$$\frac{1}{\pi} P_n^r(r') = \int_0^t \left[ \frac{\partial}{\partial r'} I_n(r,r') \alpha_n(r) + \frac{n}{r'} I_n(r,r') \beta_n(r) \right] r dr \qquad (4.16)$$

$$\frac{1}{\pi} Q_n^\theta(r') = -\int_0^t \left[ \frac{\partial}{\partial r'} I_n(r,r') \beta_n(r) + \frac{n}{r'} I_n(r,r') \alpha_n(r) \right] r dr \qquad (4.17)$$

$$\frac{1}{\pi} P_n^\theta(r') = \int_0^t \left[ -\frac{\partial}{\partial r'} I_n(r,r') \tilde{\beta}_n(r) + \frac{n}{r'} I_n(r,r') \tilde{\alpha}_n(r) \right] r dr \qquad (4.18)$$

$$\frac{1}{\pi} Q_n^r(r') = \int_0^t \left[ \frac{\partial}{\partial r'} I_n(r,r') \tilde{\alpha}_n(r) - \frac{n}{r'} I_n(r,r') \tilde{\beta}_n(r) \right] r dr \qquad (4.19)$$

Notice that equations (4.18) and (4.19) may be obtained from (4.16) and (4.17) respectively by replacing $P_n^r$ by $P_n^\theta$, $Q_n^\theta$ by $-Q_n^r$, $\alpha_n$ by $-\tilde{\beta}_n$, and $\beta_n$ by $\tilde{\alpha}_n$.

Then to summarize,

In Problems A or C use:

$$\int_0^t \left[ f_n^{z'}(r) \frac{\partial}{\partial r} I_n(r,r') + \frac{n^2}{r^2} f_n^z(r) I_n(r,r') \right] r dr = -\frac{1}{\pi} P_n^z(r') \qquad (4.20)$$

In Problems B or D use:

$$\int_0^t \left[\frac{\partial}{\partial r'} I_n(r,r')\beta_n(r) + \frac{n}{r'} I_n(r,r')\alpha_n(r)\right] r dr = -\frac{1}{\pi} Q_n^{\theta}(r') \qquad (4.21)$$

$$\int_0^t \left[\frac{\partial}{\partial r'} I_n(r,r')\alpha_n(r) + \frac{n}{r'} I_n(r,r')\beta_n(r)\right] r dr = \frac{1}{\pi} P_n^{r}(r'), \qquad (4.22)$$

where for the punch or crack problems the limit of integration, $t$ is replaced by $\infty$ or $c$ respectively.

We could set about now to find the solutions of each of the integral equations (4.20)-(4.22). The $f_n^z(r)$ may be readily found in equation (4.20), but the solutions of (4.21) and (4.22) would take much more work. However the work can be minimized by the following result. Again Lardner has shown [15] that for the crack problems equations (4.20)-(4.22) can be rearranged to have the same form. The same is true for the punch problems.

Recall that $I_n(r,r')$ is the Fourier component of $\frac{1}{R}$, hence

$$I_n(r,r') = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\phi \, d\phi}{R}, \qquad \phi = \theta - \theta' \qquad (4.23)$$

If we let

$$L_n(r,r') = r^n \frac{\partial}{\partial r}\left[\frac{I_n(r,r')}{r^n}\right] = \frac{\partial}{\partial r} I_n(r,r') - \frac{n}{r} I_n(r,r'). \qquad (4.24)$$

Then

$$\frac{\partial}{\partial r} I_n(r,r') = L_n(r,r') + \frac{n}{r} I_n(r,r'). \qquad (4.25)$$

Also

$$r'\frac{\partial}{\partial r'}I_n(r,r') + r\frac{\partial}{\partial r}I_n(r,r') = -\frac{1}{\pi}\int_0^{2\pi}\frac{\cos n\phi}{R}\,d\phi = -I_n(r,r').$$

So

$$r'\frac{\partial}{\partial r'}I_n(r,r') = -rL_n(r,r') - nI_n(r,r') - I_n(r,r')$$

and

$$r'\frac{\partial}{\partial r}I_n(r,r') = -rL_n(r,r') - (n+1)I_n(r,r'). \tag{4.26}$$

With the use of equation (4.24) the left hand side of (4.20) becomes

$$\int_0^t \left\{ rf_n^{z'}(r)L_n(r,r') + n\,\frac{I_n(r,r')}{r^n}\,\frac{d}{dr}\left[r^n f_n^z(r)\right]\right\}dr.$$

Integrate the second term by parts. The boundary term vanishes in both the punch and crack cases since $f_n^z(r) \to 0$ as $t \to \infty$, and $f_n^z(c) = 0$. With the help of (4.24) equation (4.20) may then be written

$$\int_0^t \left\{ f_n^{z'}(r) - \frac{n}{r}\,f_n^z(r)\right\}L_n(r,r')\,rdr = -\frac{1}{\pi}\,P_n^z(r'). \tag{4.27}$$

With the use of (4.26) the left hand sides of (4.21) and (4.22) become respectively

$$\frac{1}{r'}\int_0^t \left\{ rL_n(r,r')\alpha_n(r) + I_n(r,r')\left[(n+1)\alpha_n(r) - n\beta_n(r)\right]\right\}rdr,$$

$$\frac{1}{r'}\int_0^t \left\{rL_n(r,r')\beta_n(r) + I_n(r,r')\left[(n+1)\beta_n(r) - n\alpha_n(r)\right]\right\}rdr.$$

We again integrate by parts the second terms in each of the above expressions. Both terms have the form

$$\int_0^t I_n(r,r')\gamma(r)rdr = I_n(t,r')t\bar{\gamma}(t) - \int_0^t L_n(r,r')\bar{\gamma}(r)rdr, \qquad (4.28)$$

where we define

$$\bar{\gamma}(r) = \frac{1}{r^{n+1}}\int_0^r s^{n+1}\gamma(s)\, ds. \qquad (4.29)$$

With the above calculations equations (4.21) and (4.22) can finally be written

$$\int_0^t \left[(n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r)\right]L_n(r,r')rdr = \frac{r'}{\pi}P_n^r(r') + A_nI_n(t,r') \qquad (4.30)$$

$$\int_0^t \left[(n+1)\bar{\beta}_n(r) - n\bar{\alpha}_n(r) - r\beta_n(r)\right]L_n(r,r')rdr = -\frac{r'}{\pi}Q_n^\theta(r') + B_nI_n(t,r'), \qquad (4.31)$$

where

$$\begin{aligned} A_n &= t\left[(n+1)\bar{\alpha}_n(t) - n\bar{\beta}_n(t)\right] \\ B_n &= t\left[(n+1)\bar{\beta}_n(t) - n\bar{\alpha}_n(t)\right] \end{aligned} \qquad (4.32)$$

Notice that the kernel of each integral equation (4.27), (4.30), and (4.31) is the same, namely, $L_n(r,r')$ . The method of solution of the integral equations will depend only upon the value of the upper limit of integration. Therefore only one integral equation need be solved for each of the punch and crack general problems.

## 5. Solutions of the Integral Equations: punch problems

The integral equations for the punch problems are

Problem A: normal indentation

$$\int_0^\infty [f_n^{z'}(r) - \frac{n}{r} f_n^z(r)] L_n(r,r') r dr = -\frac{1}{\pi} P_n^z(r') \qquad (5.1)$$

Problem B: tangential displacement

$$\int_0^\infty [(n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r)] L_n(r,r') r dr = \frac{r'}{\pi} P_n^r(r') \qquad (5.2)$$

$$\int_0^\infty [(n+1)\bar{\beta}_n(r) - n\bar{\alpha}_n(r) - r\beta_n(r)] L_n(r,r') r dr = -\frac{r'}{\pi} Q_n^\theta(r'). \qquad (5.3)$$

Notice that the right hand side of equations (5.2) and (5.3) involve only one term since $A_n I_n(t,r')$ and $B_n I_n(t,r') \to 0$ as $t \to \infty$.

If $r' > a$, then the right hand sides of the above equations are known and the integral equations may be solved. In the above three equations let

$$r^n \frac{d}{dr}\phi_n^z(r) = \frac{d}{dr} f_n^z(r) - \frac{n}{r} f_n^z(r) = r^n \frac{d}{dr}\left[\frac{f_n^z(r)}{r^n}\right] \qquad (5.4)$$

$$r^n \frac{d}{dr}\phi_n^r(r) = (n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r) \qquad (5.5)$$

$$r^n \frac{d}{dr}\phi_n^\theta(r) = (n+1)\bar{\beta}_n(r) - n\bar{\alpha}_n(r) - r\beta_n(r) \qquad (5.6)$$

$$S_n^z(r) = -\frac{1}{\pi} P_n^z(r) \tag{5.7}$$

$$S_n^r(r) = \frac{r}{\pi} P_n^r(r) \tag{5.8}$$

$$S_n^\theta(r) = -\frac{r}{\pi} Q_n^\theta(r) \tag{5.9}$$

If $r < a$ the right hand sides of equations (5.4)-(5.6) are known since the displacements are given there. Upon integrating these three equations we obtain for $r < a$:

$$\phi_n^k(r) = \Phi_n^k(r) + C_n^k \quad , \qquad (k = r,\theta,z) \tag{5.10}$$

where the $\Phi_n^k(r)$ are defined as

$$\Phi_n^z(r) = \frac{f_n^z(r)}{r^n} \tag{5.11}$$

$$\Phi_n^r(r) = \int_r^a \left[(n+1)\bar{\alpha}_n(s) - n\bar{\beta}_n(s) - s\alpha_n(s)\right] s^{-n} ds \tag{5.12}$$

$$\Phi_n^\theta(r) = \int_r^a \left[(n+1)\bar{\beta}_n(s) - n\bar{\alpha}_n(s) - s\beta_n(s)\right] s^{-n} ds, \tag{5.13}$$

and the $C_n^k$ are constants to be determined later.

With the above definitions equations (5.1)-(5.3) all have the same form, namely,

$$\int_0^\infty r^{n+1} \frac{d}{dr}\phi_n^k(r) L_n(r,r')\, dr = S_n^k(r') H(a-r'), \tag{5.14}$$

where $H(r)$ is the Heavyside step function. The kernel, $L_n(r,r')$ has an integral representation (see Appendix I, AI.4)

$$L_n(r,r') = -2\int_0^\infty J_{n+1}(r\xi)J_n(r'\xi)\xi d\xi \ .$$

Upon using the integral representation of $L_n(r,r')$ we notice that equation (5.4) is a Hankel transform over r' which may be inverted to give

$$\int_0^\infty r^{n+1}\frac{d}{dr}\phi_n^k(r)J_{n+1}(r\xi)dr = -\frac{1}{2}\int_0^a S_n^k(r)J_n(r\xi)rdr.$$

Integrating the left hand side by parts and recalling equation (5.10) gives

$$\xi\int_0^a r^{n+1}[\Phi_n^k(r) + C_n^k]J_n(r\xi)dr + \xi\int_a^\infty r^{n+1}\phi_n^k(r)J_n(r\xi)dr = \frac{1}{2}\int_0^a S_n^k(r)J_n(r\xi)rdr$$

Multiply both sides by $\xi^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\xi t)$ , $t > a$ and integrate over $\xi$ from 0 to $\infty$. Using results (AII.1) and (AII.2) of Appendix II:

$$\left[\frac{2}{\pi}\right]^{\frac{1}{2}}\int_0^a r^{2n+1}\frac{[\Phi_n^k(r) + C_n^k]}{(t^2-r^2)^{\frac{1}{2}}}dr + \left[\frac{2}{\pi}\right]^{\frac{1}{2}}\int_a^t r^{2n+1}\frac{\phi_n^k(r)}{(t^2-r^2)^{\frac{1}{2}}}dr =$$

$$= 2^{-\frac{3}{2}}\frac{\Gamma(\frac{2n+1}{2})}{\Gamma(n+1)}\int_0^a S_n^k(r)r^{n+1}dr.$$

Now multiply both sides by $\dfrac{tdt}{(p^2-t^2)^{\frac{1}{2}}}$ and integrate over t from a to p. Using result (AII.3) of Appendix II :

$$\left[\frac{2}{\pi}\right]^{\frac{1}{2}} \int_a^p \frac{t}{(p^2-t^2)^{\frac{1}{2}}} dt \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(t^2-r^2)^{\frac{1}{2}}} dr + \left[\frac{\pi}{2}\right]^{\frac{1}{2}} \int_a^p r^{2n+1} \phi_n^k(r) dr =$$

$$2^{-\frac{3}{2}} \frac{\Gamma(\frac{2n+1}{2})}{\Gamma(n+1)} (p^2-a^2)^{\frac{1}{2}} \int_0^a S_n^k(r) r^{n+1} dr.$$

Differentiating with respect to p gives

$$\left[\frac{\pi}{2}\right]^{\frac{1}{2}} p^{2n} \phi_n^k(p) = (p^2-a^2)^{-\frac{1}{2}} \left\{ \frac{\Gamma(\frac{2n+1}{2})}{2^{\frac{3}{2}} \Gamma(n+1)} \int_0^a S_n^k(r) r^{n+1} dr - \right.$$

$$\left. - \left[\frac{2}{\pi}\right]^{\frac{1}{2}} \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(a^2-r^2)^{\frac{1}{2}}} dr \right\} + \left[\frac{2}{\pi}\right]^{\frac{1}{2}} \int_a^p \frac{t\, dt}{(t^2-r^2)^{\frac{3}{2}}} \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(p^2-t^2)^{\frac{1}{2}}} dr.$$

We require the Fourier coefficients of displacement discontinuity to be bounded at $p = a$. In order that this be true, we must have that

$$\frac{\Gamma(\frac{2n+1}{2})}{2^{\frac{3}{2}} \Gamma(n+1)} \int_0^a S_n^k(r) r^{n+1}\, dr = \left[\frac{2}{\pi}\right]^{\frac{1}{2}} \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(a^2-r^2)^{\frac{1}{2}}} dr.$$

Then finally,

$$\phi_n^k(p) = \frac{2}{\pi} p^{-2n} \int_a^p \frac{t\, dt}{(t^2-r^2)^{\frac{3}{2}}} \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(p^2-t^2)^{\frac{1}{2}}} dr. \qquad (5.15)$$

The $t$ integration in (5.15) may be performed to give an alternate expression of $\phi_n^k(p)$:

$$\phi_n^k(p) = \frac{2}{\pi} p^{-2n} (p^2-a^2)^{\frac{1}{2}} \int_0^a r^{2n+1} \frac{[\Phi_n^k(r) + C_n^k]}{(a^2-r^2)^{\frac{1}{2}}(p^2-r^2)} dr \qquad (5.15a)$$

6. Results of Punch Problems

It is now possible to calculate the Fourier components of the stresses on the plane under the punch. For $r' < a$ all expressions have the same form, that of equation (5.14):

$$s_n^k(r') = -2\int_0^\infty\int_0^\infty r^{n+1}\frac{d}{dr}\phi_n^k(r)J_{n+1}(r\xi)J_n(r'\xi)\xi dr d\xi$$

$$= 2r'^{n-1}\frac{d}{dr'}\left\{r'^{1-n}\int_0^\infty\int_0^\infty r^{n+1}\frac{d}{dr}\phi_n^k(r)J_{n+1}(r\xi)J_{n-1}(r'\xi)dr d\xi\right\}$$

After integrating by parts over $r$ and using the solution (5.15) together with the definition (5.10) we have

$$s_n^k(r') = -2r'^{n-1}\frac{d}{dr'}\left\{r'^{1-n}\int_0^\infty\int_0^a\left\{[\Phi_n^k(r) + C_n^k]J_{n-1}(r'\xi)r^{n+1}\left[J_n(r\xi) + \right.\right.\right.$$

$$\left.\left.\left.+\frac{2}{\pi}r^n\int_a^\infty\int_a^p\frac{tJ_n(p\xi)\quad dt dp}{(p^2-t^2)^{\frac{1}{2}}(t^2-r^2)^{\frac{3}{2}}p^{n-1}}\right]\right\}dr d\xi\right\} .$$

Using result (AI.6) of Appendix I,

$$s_n^k(r') = 2\left[\frac{2}{\pi}\right]^{\frac{1}{2}}r'^{n-1}\frac{d}{dr'}\left\{r'^{1-n}\int_0^a[\Phi_n^k(r) + \right.$$

$$\left.+ C_n^k]r^{2n}dr\int_0^\infty J_{n-1}(r'\xi)\frac{d}{dr}\int_r^a\frac{J_{n-\frac{1}{2}}(\xi t)}{t^{n-\frac{3}{2}}(t^2-r^2)^{\frac{1}{2}}}\xi^{\frac{1}{2}}d\xi\right\}.$$

Integrate by parts over r:

$$S_n^k(r') = -2\left[\frac{2}{\pi}\right]^{\frac{1}{2}} r'^{\,n-1} \frac{d}{dr'}\left\{r'^{\,1-n}\int_0^\infty J_{n-1}(r'\xi)\xi^{\frac{1}{2}}d\xi \cdot\right.$$

$$\left.\cdot \int_0^a \frac{d}{dr}\left[r^{2n}(\Phi_n^k(r) + C_n^k)\right]dr\int_r^a \frac{J_{n-\frac{1}{2}}(\xi t)}{t^{n-\frac{3}{2}}(t^2-r^2)^{\frac{1}{2}}}dt\right\}$$

The integration over $\xi$ may be performed with the help of (AII.4) in Appendix II.

$$S_n^k(r') = -\frac{4}{\pi} r'^{\,n-1}\frac{d}{dr'}\left\{\int_{r'}^a \frac{t^{1-2n}}{(t^2-r'^2)^{\frac{1}{2}}}dt \cdot\right.$$

$$\left.\cdot \frac{d}{dt}\int_0^t (t^2-r^2)^{\frac{1}{2}}\frac{d}{dr}\left[r^{2n}(\Phi_n^k(r) + C_n^k)\right]dr\right\}.$$

A final integration by parts gives

$$S_n^k(r') = -\frac{4}{\pi} r'^{\,n-1}\frac{d}{dr'}\int_{r'}^a \frac{t^{1-2n}}{(t^2-r'^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t r^{2n+1}\frac{[\Phi_n^k(r) + C_n^k]}{(t^2-r^2)^{\frac{1}{2}}}drdt \qquad (6.1)$$

Hence equation (6.1) together with definitions (5.7)-(5.9) and (5.11)-(5.13) give expressions for the Fourier coefficients of the stress components.

Equation (6.1) still contains the unknown constants, $C_n^k$. These constants may be evaluated after an examination of the Fourier coefficients for displacement discontinuity. Problems A and B will now be discussed separately.

Fourier Components of Displacement

Discontinuity (normal punch)

Recall that for Problem A the definition of $\phi_n^z(r)$ was given by equation (5.4). Upon integrating that equation we have

$$\phi_n^z(r) = \frac{f_n^z(r)}{r^n} + c_n^z, \qquad r > a . \tag{6.2}$$

Also for $r < a$, from equation (5.10),

$$\phi_n^z(r) = \Phi_n^z(r) + c_n^z. \tag{6.3}$$

In order that $\phi_n^z(r)$ be continuous at $r = a$ the constants of integration of equation (6.2) and (6.3) must be the same. Now since

$$\frac{f_n^z(r)}{r^n} \to 0 \quad \text{as } r \to \infty,$$

and from equation (5.15),

$$\phi_n^z(r) \to 0 \text{ as } r \to \infty,$$

we have that the $c_n^z$ must be identically zero.

From equation (6.2) and (5.15),

$$f_n^z(r') = \frac{2}{\pi} r'^{-n} \int_a^{r'} \frac{t\,dt}{(t^2-r^2)^{\frac{3}{2}}} \int_0^a \frac{r^{n+1} f_n^z(r)\,dr}{(r'^2-t^2)^{\frac{1}{2}}}, \quad r' > a . \tag{6.4}$$

A similar expression for the Fourier sine component of displacement discontinuity, $g_n^z(r')$ is found from equation (6.4) by replacing $f_n^z(r)$ by $g_n^z(r)$.

Fourier Components of Displacement
Discontinuity (shear punch)

The determination of the Fourier coefficients of displacement disconuity for Problem B is not as straightforward as that for Problem A. Equations (5.5) and (5.6) are coupled. We must therefore first find an expression relating the Fourier components of displacement to the $\bar{\alpha}_n$, $\bar{\beta}_n$, $\alpha_n$, and $\beta_n$. Then we must solve simultaneously the equations (5.5) and (5.6) and find an expression relating $\alpha_n$ and $\beta_n$ in terms of the solutions $\phi_n^k$.

Recall the definition of $\alpha(r,\theta)$ and $\beta(r,\theta)$:

$$\begin{aligned} \alpha(r,\theta) &= \Delta u_{r,r} + \frac{1}{r}\Delta u_{\theta,\theta} \\ \beta(r,\theta) &= (1-\nu)\left(\frac{1}{r}\Delta u_{r,\theta} - \Delta u_{\theta,r} - \frac{1}{r}\Delta u_\theta\right). \end{aligned} \tag{6.5}$$

Substituting the expansions (4.1), (4.2) and (4.3) into (6.5) and equating coefficients gives

$$\begin{aligned} \alpha_n(r) &= f^r_{n,r} + \frac{1}{r}f^r_n + \frac{n}{r}g^\theta_n \\ \beta_n(r) &= -(1-\nu)\left(\frac{n}{r}f^r_n + g^\theta_{n,r} + \frac{1}{r}g^\theta_n\right) \end{aligned} \tag{6.6}$$

From the definition (4.29)

$$\begin{aligned} \bar{\alpha}_n(r) &= \frac{1}{r^{n+1}}\int_0^r s^{n+1}\alpha_n(s)\,ds \\ \bar{\beta}_n(r) &= \frac{1}{r^{n+1}}\int_0^r s^{n+1}\beta_n(s)\,ds \end{aligned} \tag{6.7}$$

Now the $\alpha_n(r)$ and $\beta_n(r)$ may be discontinuous at $r=a$. To remind us of this we write (6.7) as

$$\bar{\alpha}_n(r) = \frac{1}{r^{n+1}} \int_0^a s^{n+1}\alpha_n(s)ds + \frac{1}{r^{n+1}} \int_a^r s^{n+1}\alpha_n(s)ds$$

or

$$\bar{\alpha}_n(r) = \frac{a^{n+1}}{r^{n+1}} \bar{\alpha}_n(a) + \frac{1}{r^{n+1}} \int_a^r s^{n+1}\alpha_n(s)ds \tag{6.7a}$$

and

$$\bar{\beta}_n(r) = \frac{a^{n+1}}{r^{n+1}} \bar{\beta}_n(a) + \frac{1}{r^{n+1}} \int_a^r s^{n+1}\beta_n(s)ds.$$

Then substituting (6.6) into (6.7a) gives

$$\bar{\alpha}_n(r) = \frac{a^{n+1}}{r^{n+1}} [\bar{\alpha}_n(a) - f_n^r(a)] + f_n^r(r) + \frac{n}{r^{n+1}} \int_a^r s^n[g_n^\theta(s) - f_n^\theta(s)]ds.$$

If we define

$$h(s) = s^n[f_n^r(s) - g_n^\theta(s)] \tag{6.8}$$

and

$$H(r) = \frac{1}{r^n} \int_a^r h(s)ds, \tag{6.9}$$

then

$$\bar{\alpha}_n(r) = \frac{a^{n+1}}{r^{n+1}} \left[\bar{\alpha}_n(a) - f_n^r(a)\right] + f_n^r(r) - \frac{n}{r} H(r), \tag{6.10}$$

$$\frac{\bar{\beta}_n(r)}{1-\nu} = \frac{a^{n+1}}{r^{n+1}} \left[\frac{\bar{\beta}_n(a)}{1-\nu} + g_n^\theta(a)\right] - g_n^\theta(r) - \frac{n}{r} H(r). \tag{6.11}$$

Adding (6.10) and (6.11) gives

$$Z(r) = \frac{A}{r} + h(r) - 2nr^{n-1}H(r), \tag{6.12}$$

where

$$Z(r) = r^n\left[\bar{\alpha}_n(r) + \frac{\beta_n(r)}{1-\nu}\right] \tag{6.13}$$

and

$$A = a^{n+1}\left[\bar{\alpha}_n(a) + \frac{\bar{\beta}_n(a)}{1-\nu} + g_n^{\theta}(a) - f_n^{r}(a)\right]. \tag{6.14}$$

Using (6.9), equation (6.12) becomes

$$Z(r) = \frac{A}{r} + h(r) - \frac{2n}{r}\int_a^r h(s)ds.$$

In order to solve this equation for $h(r)$, multiply by $r$ and differentiate. Hence

$$\frac{1}{r^{2n}}\frac{d}{dr}(rZ) = \frac{d}{dr}\left[\frac{h}{r^{2n-1}}\right].$$

Integration gives

$$h(r) = -r^{2n-1}\int_r^{\infty}\frac{d}{ds}\,[sZ(s)]\frac{ds}{s^{2n}} \tag{6.15}$$

Hence after integrating (6.15) by parts, (6.12) may be solved for $H(r)$:

$$H(r) = -r^n\int_r^{\infty} Z(s)\frac{ds}{s^{2n}} + \frac{A}{2nr^n} \tag{6.16}$$

Now for brevity define

$$\begin{aligned} r^{n+1}\alpha_n(r) &\equiv f(r), \quad \text{so} \quad r^{n+1}\bar{\alpha}_n(r) = a^{n+1}\bar{\alpha}_n(a) + \int_a^r f(s)ds \\ r^{n+1}\beta_n(r) &\equiv g(r), \quad \text{so} \quad r^{n+1}\bar{\beta}_n(r) = a^{n+1}\bar{\beta}_n(a) + \int_a^r g(s)ds \end{aligned} \tag{6.17}$$

Then from (6.13)

$$Z(r) = \frac{B}{r^n} + \frac{1}{r}\int_a^r P(s)ds,$$

where

$$B = a^{n+1}\left[\bar{\alpha}_n(a) + \frac{1}{1-\nu}\ \bar{\beta}_n(a)\right] \tag{6.18}$$

and

$$P(s) = f(s) + \frac{1}{1-\nu}\, g(s).$$

The integral in (6.16) may then be written:

$$\int_r^\infty \frac{Z(s)}{s^{2n}}ds = \frac{B}{2nr^{2n}} + \frac{1}{2nr^{2n}}\int_a^r P(s)ds + \frac{1}{2n}\int_r^\infty \frac{P(s)}{s^{2n}}ds.$$

Equation (6.16) then can be written

$$\frac{n}{r}H(r) = \frac{A-B}{2r^{n+1}} - \frac{1}{2r^{n+1}}\int_a^r P(s)ds - \frac{r^{n-1}}{2}\int_r^\infty \frac{P(s)}{s^{2n}}ds, \tag{6.19}$$

Then substituting (6.19) into (6.10) and (6.11) and solving for $f_n^r$ and $g_n^\theta$ we have for $r > a$

$$f_n^r(r) = \frac{a^{n+1}}{2r^{n+1}}\left[g_n^\theta(a) + f_n^r(a)\right] + \frac{1}{2r^{n+1}}\int_a^r\left[f(s) - \frac{1}{1-\nu}g(s)\right]ds -$$

$$- \frac{r^{n-1}}{2}\int_r^\infty\left[f(s) + \frac{1}{1-\nu}\, g(s)\right]\frac{ds}{s^{2n}} \tag{6.20}$$

and

$$g_n^\theta(r) = \frac{a^{n+1}}{2r^{n+1}}\left[g_n^\theta(a) + f_n^r(a)\right] + \frac{1}{2r^{n+1}}\int_a^r \left[f(s) - \frac{1}{1-\nu}\, g(s)\right]ds +$$

$$+ \frac{r^{n-1}}{2}\int_r^\infty \left[f(s) + \frac{1}{1-\nu}\, g(s)\right]\frac{ds}{s^{2n}}. \qquad (6.21)$$

The other two coefficients, $g_n^r$ and $f_n^\theta$ may be obtained from (6.20) and (6.21) by replacing $f_n^r$ by $g_n^r$, $g_n^\theta$ by $-f_n^\theta$, $\alpha_n$ by $\tilde{\alpha}_n$, and $\beta_n$ by $-\tilde{\beta}_n$. Therefore through the first of (6.17) equations (6.20) and (6.21) give the Fourier coefficients for the displacement discontinuities in terms of $\alpha_n(r)$ and $\beta_n(r)$.

It remains now to find the $\alpha_n$ and $\beta_n$ in terms of the solutions (5.15). Define the quantities $F_n(r)$ and $G_n(r)$ with the help of (5.5) and (5.6):

$$\left.\begin{aligned} \frac{d}{dr}\phi_n^r &= \left[(n+1)\bar{\alpha}_n - n\bar{\beta}_n - r\alpha_n\right]r^{-n} = -\frac{F_n(r)}{r^{2n}} \\ \frac{d}{dr}\phi_n^\theta &= \left[(n+1)\bar{\beta}_n - n\bar{\alpha}_n - r\beta_n\right]r^{-n} = -\frac{G_n(r)}{r^{2n}} \end{aligned}\right\} \quad r > a \qquad (6.22)$$

From definitions (6.17) equations (6.22) become

$$(n+1)a^{n+1}\,\bar{\alpha}_n(a) + (n+1)\int_a^r f(s)ds - na^{n+1}\bar{\beta}_n(a) -$$

$$- n\int_a^r g(s)ds - f(r) = - rF_n(r),$$

$$(n+1)a^{n+1}\,\bar{\beta}_n(a) + (n+1)\int_a^r g(s)ds - na^{n+1}\,\bar{\alpha}_n(a) -$$

$$- n\int_a^r f(s)ds - g(r) = - rG_n(r).$$

Add and subtract the above two equations to give:

$$\frac{1}{r}\int_a^r \psi(s)ds - \psi(r) + \frac{a^{n+1}}{r}\left[\bar{\alpha}_n(a) + \bar{\beta}_n(a)\right] = - F_n(r) - G_n(r) \equiv -\Psi(r)$$

$$\frac{(2n+1)}{r}\int_a^r \times(s)ds - \times(r) + (2n+1)\frac{a^{n+1}}{r}\left[\bar{\alpha}_n(a) - \bar{\beta}_n(a)\right] = - F_n(r) + G_n(r) \equiv -\chi(r)$$

where

$$\begin{aligned}\psi(r) &= f(r) + g(r)\\ \times(r) &= f(r) - g(r)\ .\end{aligned} \qquad (6.23)$$

The solutions of these two equations are readily found to be

$$\psi(r) = \Psi(r) + \int_a^r \frac{\Psi(s)}{s}ds + a^n\left[\bar{\alpha}_n(a) + \bar{\beta}_n(a)\right] \qquad (6.24)$$

$$\times(r) = \chi(r) + (2n+1)r^{2n}\int_a^r \frac{\chi(s)}{s^{2n+1}}ds + (2n+1)\frac{r^{2n}}{a^n}\left[\bar{\alpha}_n(a) - \bar{\beta}_n(a)\right]. \qquad (6.25)$$

From equation (6.23), and the first of (6.17):

$$\begin{aligned}r^{n+1}\alpha_n(r) &= \tfrac{1}{2}\left[\psi(r) + \times(r)\right]\\ r^{n+1}\beta_n(r) &= \tfrac{1}{2}\left|\psi(r) - \times(r)\right|.\end{aligned} \qquad (6.26)$$

Hence through (6.25) and (6.26) the $\alpha_n$ and $\beta_n$ are given in terms of the solutions (5.15), and the Fourier coefficients of displacement discontinuities given by (6.20) and (6.21) are completely determined in terms of the solutions (5.15).

The constants $C_n^r$ and $C_n^\theta$ are determined in the following way. The Fourier coefficients $f_n^r$ and $g_n^\theta$ must vanish at infinity. In equations (6.20) and (6.21) both the first and third terms do vanish as r tends to infinity. However the second term is not identically zero as r tends to infinity. This condition gives one equation relating $C_n^r$ to $C_n^\theta$. Another equation relating the constants to eachother is obtained by noting that the Fourier coefficients should be continuous at $r = a$. The left hand sides of (6.20) and (6.21) are known for $r = a$. We therefore have a second equation relating $C_n^r$ to $C_n^\theta$. These two equations are then solved simultaneously for $C_n^r$ and $C_n^\theta$. Hence the Fourier components of stress are completely determined by the expression given as equation (6.1). This procedure will be illustrated by an example in the next section.

## 7. Unidirectional Shear Punch Problem

As an application of the results found in Section 6 we consider a half space whose surface is displaced by an amount $C$ in the x-direction over a region $r \lesssim a$. In terms of our present notation, $\Delta u_x = C$ for $r \lesssim a$ and $\sigma_{xz} = 0$ for $r > a$. In polar coordinates these boundary conditions become:

$$\sigma_{rz} = \sigma_{\theta z} = 0 \quad , r > a \tag{7.1}$$

$$\left.\begin{aligned} \Delta u_r &= C\cos\theta \\ \Delta u_\theta &= - C\cos\theta \ . \end{aligned}\right\} , r \lesssim a \tag{7.2}$$

Now for $r \leqslant a$ the only non zero Fourier coefficients of displacement discontinuity are

$$f_1^r = C \quad \text{and} \quad g_1^\theta = - C. \tag{7.3}$$

All other components of displacement are zero. Hence from (6.6), (6.7), (6.23), (5.12), and (5.13):

$$\alpha_n = \beta_n = 0 = \bar{\alpha}_n = \bar{\beta}_n$$

$$f(r) = g(r) = 0$$

$$\Phi_n^\theta = \Phi_n^r = 0.$$

From equation (5.15) the solution becomes for $r > a$,

$$\phi_1^k(p) = \frac{2}{\pi} p^{-2} \int_a^p \frac{t\,dt}{(t^2-r^2)^{\frac{3}{2}}} \int_0^a \frac{r^3 C_1^k\,dr}{(p^2-t^2)^{\frac{1}{2}}},$$

where $k = r,\theta$.

Performing the integrations gives

$$\phi_1^k(r) = \frac{2}{\pi} C_1^k \left\{ \sin^{-1}\left(\frac{a}{r}\right) - \frac{a(r^2-a^2)^{\frac{1}{2}}}{r^2} \right\}$$

and

$$\frac{d}{dr}\phi_1^r(r) = -\frac{4}{\pi} C_1^r \frac{a^3}{r^3(r^2-a^2)^{\frac{1}{2}}} = -\frac{F_1(r)}{r^2}$$

$$\frac{d}{dr}\phi_1^\theta(r) = -\frac{4}{\pi} C_1^\theta \frac{a^3}{r^3(r^2-a^2)^{\frac{1}{2}}} = -\frac{G_1(r)}{r^2}.$$

Then

$$-\Psi(r) = -F_1(r) - G_1(r) = -\frac{4}{\pi} \frac{a^3(C_1^r+C_1^\theta)}{r(r^2-a^2)^{\frac{1}{2}}}$$

$$-\chi(r) = -F_1(r) + G_1(r) = -\frac{4}{\pi} \frac{a^3(C_1^r-C_1^\theta)}{r(r^2-a^2)^{\frac{1}{2}}}.$$

From equation (6.24)

$$\psi(r) = \frac{4}{\pi} a^3 (C_1^r+C_1^\theta) \left\{ \frac{1}{r(r^2-a^2)^{\frac{1}{2}}} + \int_a^r \frac{ds}{s^2(s^2-a^2)^{\frac{1}{2}}} \right\}$$

$$= \frac{4}{\pi} \frac{ar}{(r^2-a^2)^{\frac{1}{2}}} (C_1^r+C_1^\theta).$$

From equation (6.25)

$$\chi(r) = \frac{4}{\pi} a^3 (C_1^r - C_1^\theta)\left\{ \frac{1}{r(r^2-a^2)^{\frac{1}{2}}} + 3r^2 \int_a^r \frac{ds}{s^4(s^2-a^2)^{\frac{1}{2}}} \right\}$$

$$= \frac{4}{\pi} \frac{(C_1^r - C_1^\theta)}{a} \left\{ \frac{2r^3}{(r^2-a^2)^{\frac{1}{2}}} - \frac{a^2 r}{(r^2-a^2)^{\frac{1}{2}}} \right\} .$$

So from (6.23)

$$f(r) = \tfrac{1}{2}(\psi+\chi) = \frac{4}{\pi}\left\{ \frac{r^3 C_1^r}{a(r^2-a^2)^{\frac{1}{2}}} + \left[\frac{ar}{(r^2-a^2)^{\frac{1}{2}}} - \frac{r^3}{a(r^2-a^2)^{\frac{1}{2}}}\right] C_1^\theta \right\}$$

$$g(r) = \tfrac{1}{2}(\psi-\chi) = \frac{4}{\pi}\left\{ \left[ \frac{ar}{(r^2-a^2)^{\frac{1}{2}}} - \frac{r^3}{a(r^2-a^2)^{\frac{1}{2}}}\right] C_1^r + \frac{r^3 C_1^\theta}{a(r^2-a^2)^{\frac{1}{2}}} \right\} .$$

Then in order that $f_1^r \to 0$ as $r \to \infty$ we must have that

$$\frac{1}{r^2}\int_a^r \left[f(s) - \frac{1}{1-\nu} g(s)\right] ds \to 0 \quad \text{as } r \to \infty.$$

Now

$$\frac{1}{r^2}\int_a^r \left[f(s) - \frac{1}{1-\nu} g(s)\right] ds =$$

$$= \frac{4}{\pi} \frac{(r^2-a^2)^{\frac{3}{2}}}{r^2} \left\{ \left[ \frac{1}{3a} + \frac{a}{(r^2-a^2)} + \frac{1}{3a(1-\nu)} \right] (C_1^r - C_1^\theta) \right\} .$$

Hence we must have that

$$C_1^r = C_1^\theta . \tag{7.4}$$

Next we set $r = a$ in either of (6.20) or (6.21) and use equation (7.4):

$$C = -\frac{2}{\pi} C_1^r a \frac{(2-\nu)}{1-\nu} \int_a^\infty \frac{ds}{s(s^2-a^2)^{\frac{1}{2}}} .$$

Hence

$$C_1^r = C_1^\theta = -\frac{(1-\nu)}{(2-\nu)} C \quad . \tag{7.5}$$

Then

$$f(s) = g(s) = -\frac{4}{\pi} \frac{(1-\nu)}{(2-\nu)} \frac{Cas}{(s^2-a^2)^{\frac{1}{2}}} \tag{7.6}$$

Substituting (7.6) into (6.20) and (6.21) gives

$$f_1^r(r) = \frac{2}{\pi} C \left\{ \frac{\pi}{2} - \cos^{-1}\left(\frac{a}{r}\right) + \frac{a\nu}{2-\nu} \frac{(r^2-a^2)^{\frac{1}{2}}}{r^2} \right\}$$

$$g_1^\theta(r) = \frac{2}{\pi} C \left\{ \cos^{-1}\left(\frac{a}{r}\right) - \frac{\pi}{2} + \frac{a\nu}{2-\nu} \frac{(r^2-a^2)^{\frac{1}{2}}}{r^2} \right\} .$$

Then from equation (4.1) for $r > a$:

$$\Delta u_r(r,\theta) = \frac{2}{\pi} C \left\{ \frac{\pi}{2} - \cos^{-1}\left(\frac{a}{r}\right) + \frac{a\nu}{2-\nu} \frac{(r^2-a^2)^{\frac{1}{2}}}{r^2} \right\} \cos\theta$$

$$\Delta u_\theta(r,\theta) = \frac{2}{\pi} C \left\{ \cos^{-1}\left(\frac{a}{r}\right) - \frac{\pi}{2} - \frac{a\nu}{2-\nu} \frac{(r^2-a^2)^{\frac{1}{2}}}{r^2} \right\} \sin\theta .$$

The Fourier components of stress are calculated from (6.1). Recall from (5.8) and (5.9) that

$$S_1^r(r) = \frac{r}{\pi} P_1^r(r)$$

$$S_1^\theta(r) = -\frac{r}{\pi} Q_1^\theta(r) .$$

So for $r < a$

$$\frac{r}{\pi} P_1^r(r) = -\frac{4}{\pi}\frac{d}{dr}\int_r^a \frac{dt}{t(t^2-r^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t \frac{s^3 C_1^r \, ds}{(t^2-s^2)^{\frac{1}{2}}}$$

$$= \frac{4}{\pi} C_1^r \left\{ \frac{(a^2-r^2)^{\frac{1}{2}}}{r} + \frac{a^2}{r^2}\cos^{-1}\left(\frac{r}{a}\right) \right\}$$

Hence

$$\sigma_{zr}(r,\theta,0) = -\frac{\mu}{\pi(2-\nu)} C\left\{ \frac{(a^2-r^2)^{\frac{1}{2}}}{r^2} + \frac{a^2}{r^3}\cos^{-1}\left(\frac{r}{a}\right) \right\}\cos\theta$$

$$\sigma_{z\theta}(r,\theta,0) = \frac{\mu}{\pi(2-\nu)} C\left\{ \frac{(a^2-r^2)^{\frac{1}{2}}}{r^2} + \frac{a^2}{r^3}\cos^{-1}\left(\frac{r}{a}\right) \right\}\sin\theta .$$

## 8. Solutions of the Integral Equations: Crack Problems

The integral equations for the crack problems are obtained from equations (4.30) and (4.31):

Problem C: crack under normal loading

$$\int_0^c \left\{ f_n^{z'}(r) - \frac{n}{r} f_n^z(r) \right\} L_n(r,r')\,r\,dr = -\frac{1}{\pi} P_n^z(r') \tag{8.1}$$

Problem D: crack under shear loading

$$\int_0^c \left\{ (n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r) \right\} L_n(r,r')\,r\,dr = \frac{r'}{\pi} P_n^r(r') + A_n I_n(c,r') \tag{8.2}$$

$$\int_0^c \left\{ (n+1)\bar{\beta}_n(r) - n\alpha_n(r) - r\beta_n(r) \right\} L_n(r,r')\,r\,dr = \frac{-r'}{\pi} Q_n^\theta(r') + B_n I_n(c,r') \tag{8.3}$$

If $r' < c$ then the right hand side of equations (8.1)-(8.3) are known and the integral equations may be solved. In the above three equations let

$$\omega_n^z(r) = f_n^{z'}(r) - \frac{n}{r} f_n^z(r) = r^n \frac{d}{dr}\left[\frac{f_n^z(r)}{r^n}\right] \tag{8.4}$$

$$\omega_n^r(r) = (n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r) \tag{8.5}$$

$$\omega_n^\theta(r) = (n+1)\bar{\beta}_n(r) - n\bar{\alpha}_n(r) - r\beta_n(r) \tag{8.6}$$

$$S_n^z(r') = -\frac{1}{\pi} P_n^z(r') \tag{8.7}$$

$$S_n^r(r') = \frac{r'}{\pi} P_n^r(r') + A_n I_n(c,r') \tag{8.8}$$

$$S_n^\theta(r') = -\frac{r'}{\pi} Q_n^\theta(r') + B_n I_n(c,r') \quad . \tag{8.9}$$

Then equations (8.1)-(8.3) all have the same form, namely:

$$\int_0^c \omega_n^m(r) L_n(r,r') r dr = S_n^m(r') \quad , \; r' < c \tag{8.10}$$

where $\omega_n^m(r)$ is an unknown expression and $S_n^m(r')$ a known function apart from a constant in equations (8.2) and (8.3).

From equation (AI.4)

$$L_n(r,r') = -2\int_0^\infty J_{n+1}(r\xi) J_n(r'\xi) \xi d\xi \quad .$$

Thus equation (8.10) takes the form

$$\int_0^c \int_0^\infty \omega_n^m(r) J_{n+1}(r\xi) J_n(r'\xi) \xi r d\xi dr = -\tfrac{1}{2} S_n^m(r') \quad . \tag{8.11}$$

Now if we let $S_n^m(r')$ be defined for the moment for all $r'$, then equation (8.11) is a Hankel transform which may be inverted to give

$$\int_0^c J_{n+1}(r\xi) \omega_n^m(r) r dr = -\tfrac{1}{2}\int_0^\infty S_n^m(r) J_n(r\xi) r dr \quad .$$

Multiply both sides by $J_{n+\frac{1}{2}}(\xi t)\xi^{\frac{1}{2}}$ and integrate over $\xi$ from 0 to $\infty$. Here $0 < t < c$. Using the results (AII.5), (AII.1) the integrals over $\xi$ may be performed to give

$$t^{n+\frac{1}{2}}\int_t^c \frac{\omega_n^m(r)dr}{r^n(r^2-t^2)^{\frac{1}{2}}} = -\frac{1}{2t^{n+\frac{1}{2}}}\int_0^t \frac{S_n^m(r)r^{n+1}dr}{(t^2-r^2)^{\frac{1}{2}}} .$$

Finally by multiplying by $t^{\frac{1}{2}-n}(t^2-p^2)^{-\frac{1}{2}}$ and integrating over $t$ from $p$ to $c$ we have that

$$\int_p^c \frac{tdt}{(t^2-p^2)^{\frac{1}{2}}}\int_t^c \frac{\omega_n^m(r)dr}{r^n(r^2-t^2)^{\frac{1}{2}}} = -\tfrac{1}{2}\int_p^c \frac{dt}{t^{2n}(t^2-p^2)^{\frac{1}{2}}}\int_0^t \frac{S_n^m(r)r^{n+1}dr}{(t^2-r^2)^{\frac{1}{2}}} .$$

On the left hand side we change the order of integration and switch the limits to give

$$\int_p^c \frac{\omega_n^m(r)dr}{r^n}\int_p^r \frac{tdt}{[(t^2-p^2)(r^2-t^2)]^{\frac{1}{2}}} = \frac{\pi}{2}\int_p^c \frac{\omega_n^m(r)dr}{r^n} . \qquad (8.12)$$

After differentiating equation (8.12) with respect to p the solution is

$$\omega_n^m(p) = \frac{1}{\pi}p^n\frac{d}{dp}\int_p^c \frac{dt}{t^{2n}(t^2-p^2)^{\frac{1}{2}}}\int_0^t \frac{S_n^m(r)r^{n+1}dr}{(t^2-r^2)^{\frac{1}{2}}} , \quad p< c . \qquad (8.13)$$

## 9. Results of Crack Problems

We now use the solution (8.13) to calculate the stress and displacement discontinuity components for the normal and shear crack problems.

### Normal Loading

Recall that for the normal loaded crack $\omega_n^z$ was defined by equation (8.4). Solving that equation for $f_n^z(r)$ involves a straightforward integration and gives

$$f_n^z(r) = -\frac{1}{\pi^2} r^n \int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{P_n^z(s) s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} , \tag{9.1}$$

where we have used equation (8.7) and the fact that $f_n^z(c) = 0$. An identical relation relates the other Fourier component, $g_n^z$ to $Q_n^z$ : (Recall the remarks following equation (4.11))

$$g_n^z(r) = -\frac{1}{\pi^2} r^n \int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{Q_n^z(s) s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} . \tag{9.2}$$

Since $P_n^z(r)$ and $Q_n^z(r)$ are the Fourier components of the stress $\sigma_{zz}(r,\theta,0)$, (see equation (4.5)) we have that

$$\begin{Bmatrix} P_n^z(r) \\ Q_n^z(r) \end{Bmatrix} = \frac{4(1-\nu)}{\mu} \int_0^{2\pi} \sigma_{zz}(r,\theta,0) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} d\theta \tag{9.3}$$

Then the Fourier components of displacement discontinuity are

$$\begin{Bmatrix} f_n^z(r) \\ g_n^z(r) \end{Bmatrix} = -\frac{4(1-\nu)}{\pi^2\mu} r^n \int_r^c \frac{dt}{(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{sds}{(t^2-s^2)^{\frac{1}{2}}} \int_0^{2\pi} \frac{s^n \sigma_{zz}(s,\theta',0)}{t^{2n}} \begin{Bmatrix} \cos n\theta' \\ \sin n\theta' \end{Bmatrix} d\theta' . \tag{9.4}$$

The displacement discontinuity is then found by using equation (9.4) in the Fourier series (4.1):

$$\Delta u_z(r,\theta) = -\frac{4(1-\nu)}{\pi^2\mu} \int_r^c \frac{dt}{(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{sds}{(t^2-s^2)^{\frac{1}{2}}} \int_0^{2\pi} \sigma_{zz}(s,\theta',0) \Bigg\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{rs}{t^2}\right]^n \cos n(\theta-\theta') \Bigg\} d\theta' .$$

After performing the sum the displacement discontinuity can finally be written

$$\Delta u_z(r,\theta) = -\frac{2(1-\nu)}{\pi^2\mu} \int_r^c \frac{dt}{(t^2-r^2)^{\frac{1}{2}}} \int_0^t \int_0^{2\pi} \frac{(t^4-r^2s^2)\sigma_{zz}(s,\theta',0)\, s\, d\theta' ds}{(t^2-s^2)^{\frac{1}{2}}\left[t^4+r^2s^2-2t^2 rs\cos(\theta-\theta')\right]} . \tag{9.5}$$

In order to calculate the Fourier component of stress $p_n^z(r)$ we return to equation (8.1). For $r' > c$ equation (8.1) together with (9.1) give the Fourier component $p_n^z(r)$:

$$p_n^z(r') = \frac{-2}{\pi} \int_0^c \int_0^{\infty} r^{n+1} \frac{d}{dr} \left\{ \int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{p_n^z(s) s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} \right\} J_{n+1}(r\xi) J_n(r'\xi) \xi d\xi dr . \tag{9.6}$$

Equation (9.6) may be written

$$P_n^z(r') = \frac{-2}{\pi r'^{\,n+1}} \frac{d}{dr'} \left\{ r'^{\,n+1} \int_0^c \int_0^\infty r^{n+1} \frac{d}{dr} \left\{ \int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}} \int_0^t \frac{P_n^z(s)\, s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} \right\} \cdot J_{n+1}(r\xi)\, J_{n+1}(r'\xi)\, d\xi\, dr \right\} .$$

After integrating by parts over $r$ and interchanging the orders of integration of $r$ and $t$, the integral over $r$ may be evaluated using the result (AII.6) in Appendix II.

$$P_n^z(r') = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{r'^{\,n+1}} \frac{d}{dr'} \left\{ r'^{\,n+1} \int_0^\infty J_{n+1}(r'\xi)\, \xi^{\frac{1}{2}} d\xi \int_0^c \frac{J_{n+\frac{1}{2}}(\xi t)\, dt}{t^{n-\frac{1}{2}}} \int_0^t \frac{P_n^z(s)\, s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} \right\}$$

Integrate over $\xi$ using the result (AII.5) in Appendix II and perform the $r'$ differentiation:

$$P_n^z(r') = -\frac{2}{\pi}\, r'^{\,-n} \int_0^c \frac{t\, dt}{(r'^2-t^2)^{\frac{3}{2}}} \int_0^t \frac{P_n^z(s)\, s^{n+1} ds}{(t^2-s^2)^{\frac{1}{2}}} .$$

Finally the $t$ integration may be performed by using result (AII.7) to give

$$P_n^z(r') = -\frac{2}{\pi r'^{\,n} (r'^2-c^2)^{\frac{1}{2}}} \int_0^c \frac{(c^2-s^2)^{\frac{1}{2}} P_n^z(s)\, s^{n+1} ds}{(r'^2-s^2)} . \qquad (9.7)$$

In view of the remark after (4.11) an expression for $Q_n^z$ may be found by replacing $P_n^z$ with $Q_n^z$ in equation (9.7). Substituting the expressions for $P_n^z$ and $Q_n^z$ into the expansion

(4.5) and using equation (9.3) the normal stress is given by

$$\sigma_{zz}(r',\theta',0) = -\frac{2}{\pi^2(r'^2-c^2)^{\frac{1}{2}}}\int_0^c \frac{(c^2-s^2)sds}{r'^2-s^2}\int_0^{2\pi}\sigma_{zz}(s,\theta,0)\left\{\tfrac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{s}{r'}\right]^n \cos n(\theta-\theta')\right\}d\theta \quad .$$

Thus after evaluating the sum, the normal stress on the plane of the crack is:

$$\sigma_{zz}(r',\theta',0) = -\frac{1}{\pi^2(r'^2-c^2)^{\frac{1}{2}}}\int_0^c\int_0^{2\pi}\frac{(c^2-s^2)^{\frac{1}{2}}\sigma_{zz}(s,\theta,0)\ sd\theta ds}{r'^2+s^2-2r's\cos(\theta-\theta')} \quad . \tag{9.9}$$

This stress component has the familiar square root singularity at the crack edge. A stress intensity factor, $K_I(\theta')$ at position $\theta'$ on the crack edge may be defined such that

$$\sigma_{zz}(r',\theta',0) \sim K_I(\theta')(r'-c)^{-\frac{1}{2}} \quad \text{as } r'\to c.$$

Then

$$K_I(\theta') = -\frac{1}{\pi^2(2c)^{\frac{1}{2}}}\int_0^c\int_0^{2\pi}\frac{(c^2-s^2)^{\frac{1}{2}}\sigma_{zz}(s,\theta,0)sd\theta ds}{c^2+s^2-2cs\cos(\theta-\theta')} \quad . \tag{9.10}$$

If the traction on the crack is axi-symmetric, ie.

$$\sigma_{zz}(r',\theta,0) = \sigma_{zz}(r',0),$$

then the stress intensity factor becomes the well known result

$$K_I = -\frac{2}{\pi(2c)^{\frac{1}{2}}}\int_0^c \frac{\sigma_{zz}(s,0)\,s\,ds}{(c^2-s^2)^{\frac{1}{2}}} . \tag{9.10a}$$

## Shear Loading

The case of a crack under shear loading has been considered by Lardner [15] , where expressions for the Fourier coefficients of displacement discontinuity and stresses on the plane of the crack have been found. The following is a summary of his findings.

From the definitions (8.5), (8.6), (8.8), and (8.9), together with the solution (8.13) we have

$$\begin{aligned}(n+1)\bar{\alpha}_n(r) - n\bar{\beta}_n(r) - r\alpha_n(r) &= -r^{-n}\left[F_n(r) - A_n M_n(r)\right]\\ (n+1)\bar{\beta}_n(r) - n\bar{\alpha}_n(r) - r\beta_n(r) &= -r^{-n}\left[F_n(r) - B_n M_n(r)\right],\end{aligned} \tag{9.11}$$

where

$$\begin{Bmatrix} F_n(r) \\ G_n(r) \end{Bmatrix} = -\frac{1}{\pi^2}\, r^{2n}\frac{d}{dr}\int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}}\int_0^t \begin{Bmatrix} P_n^r(s) \\ Q_n^\theta(s) \end{Bmatrix}\frac{s^{n+2}ds}{(t^2-s^2)^{\frac{1}{2}}} \tag{9.12}$$

and

$$M_n(r) = \frac{2}{\pi}\frac{r^n}{c^n}\frac{d}{dr}\int_r^c \frac{dt}{t^{2n}(t^2-r^2)^{\frac{1}{2}}}\int_0^t \frac{t^{2n}dt}{(c^2-t^2)^{\frac{1}{2}}} . \tag{9.13}$$

To solve equations (9.11) for $\alpha_n(r)$ and $\beta_n(r)$, we define the quantities

$$\psi(r) = r^{n+1}[\alpha_n(r) + \beta_n(r)]$$
$$\chi(r) = r^{n+1}[\alpha_n(r) - \beta_n(r)] \quad . \tag{9.14}$$

Then adding and subtracting the two equations (9.11) we have

$$\psi(r) - \frac{1}{r}\int_0^r \psi(s)ds = \Psi(r)$$
$$\chi(r) - \frac{(2n+1)}{r}\int_0^r \chi(s)ds = \chi(r), \tag{9.15}$$

where

$$\Psi(r) = F_n(r) - G_n(r) + (2-\nu)nH_n(c)M_n(r)$$
$$\chi(r) = F_n(r) + G_n(r) + n\nu(2n+1)H_n(c)M_n(r), \tag{9.16}$$

and $H_n(c)$ satisfies the equation

$$n(2-\nu)c^{n-1}H_n(c)\left[1 + c^{1-n}\int_0^c \frac{M_n(s)ds}{s}\right] = -\int_0^c \frac{[F_n(s) - G_n(s)]}{s}ds \quad .$$

The constant $H_n(c)$ may be written with the help of (9.12) and (9.13):

$$H_n(c) = -\left[2(2-\nu)n\pi c^n E_n\right]^{-1}\int_0^c s^n(c^2-s^2)^{\frac{1}{2}}[P_n^r(s) - Q_n^\theta(s)]ds \quad , \tag{9.17}$$

where

$$E_n = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

The solutions of (9.15) are readily found to be

$$\psi(r) = \Psi(r) + \int_0^r \frac{\Psi(s)ds}{s}$$

$$\times(r) = -\frac{n\nu(2n+1)}{c^{n+1}}H_n(c)r^{2n} - \chi(r) - (2n+1)r^{2n}\int_r^c \frac{\chi(s)ds}{s^{2n+1}} \quad (9.18)$$

Hence $\alpha_n(r)$ and $\beta_n(r)$ are known (through equations (9.14) and (9.18)) in terms of the prescribed data in the form of equations (9.6) and (9.7).

Next we calculate the Fourier coefficients, $f_n^r(r)$ and $g_n^\theta$ in terms of the known $\alpha_n(r)$ and $\beta_n(r)$. Recall that

$$\alpha(r,\theta) = \Delta u_{r,r}(r,\theta) + \frac{1}{r}\Delta u_r(r,\theta) + \frac{1}{r}\Delta u_{\theta,\theta}(r,\theta)$$

$$\beta(r,\theta) = (1-\nu)\left[\frac{1}{r}\Delta u_{r,\theta}(r,\theta) - \Delta u_{\theta,r}(r,\theta) - \frac{1}{r}\Delta u_\theta(r,\theta)\right]. \quad (9.19)$$

Then substituting the expansions (4.1), (4.2), and (4.3) into (9.19) and comparing coefficients, we have:

$$\alpha_n(r) = f_{n,r}^r + \frac{1}{r}f_n^r + \frac{n}{r}g_n^\theta$$

$$\beta_n(r) = -(1-\nu)\left[\frac{n}{r}f_n^r + g_{n,r}^\theta + \frac{1}{r}g_n^\theta\right]. \quad (9.20)$$

According to the definition (4.29), we have that

$$\bar{\alpha}_n(r) = f_n^r(r) - \frac{n}{r} H_n(r)$$

$$\bar{\beta}_n(r) = (1-\nu)\left[g_n^\theta(r) + \frac{n}{r} H_n(r)\right] \quad , \tag{9.21}$$

where

$$H_n(r) = r^{-n}\int_0^r h_n(s)ds \quad ,$$

and

$$h_n(s) = s^n\left[f_n^r(s) - g_n^\theta(s)\right] \quad . \tag{9.22}$$

Combining equations (9.21), with the help of (9.22) gives

$$h_n(r) - \frac{2n}{r}\int_0^r h_n(s)ds = r^n\left[\bar{\alpha}_n(r) + (1-\nu)^{-1}\bar{\beta}_n(r)\right] \quad ,$$

whose solution is given by

$$h_n(r) = -\, r^{2n-1}\int_r^c \frac{d}{ds}\left\{s^{n+1}\left[\bar{\alpha}_n(s) + (1-\nu)^{-1}\bar{\beta}_n(s)\right]\right\}s^{-2n}ds. \tag{9.23}$$

Hence from the first of (9.22), $H_n(r)$ is known as well. Then from (9.22), $f_n^r(r)$ and $g_n^\theta(r)$ can be written

$$f_n^r(r) = \frac{-1}{2r}\int_r^c \left\{\alpha_n(s)\left[(\tfrac{s}{r})^n + (\tfrac{r}{s})^n\right] - \frac{\beta_n(s)}{1-\nu}\left[(\tfrac{s}{r})^n - (\tfrac{r}{s})^n\right]\right\}sds$$

$$g_n^\theta(r) = \frac{-1}{2r}\int_r^c \left\{\alpha_n(s)\left[(\tfrac{s}{r})^n - (\tfrac{r}{s})^n\right] - \frac{\beta_n(s)}{1-\nu}\left[(\tfrac{s}{r})^n + (\tfrac{r}{s})^n\right]\right\}sds. \tag{9.24}$$

The Fourier coefficients, $f_n^\theta(r)$ and $g_n^r(r)$ may be obtained from equations (9.24) by replacing $f_n^r(r)$ with $g_n^r(r)$ and $g_n^\theta(r)$ with $-f_n^\theta(r)$, $\alpha_n(r)$ with $-\tilde{\beta}_n(r)$ and $\beta_n(r)$ with $\tilde{\alpha}_n(r)$.

Next the stress components, $\sigma_{rz}$ and $\sigma_{\theta z}$ may be calculated. Still summarizing Lardner's results, substitute (9.12) and (9.13) into equations (4.30) and (4.31) with $r' > c$. The integrals over $r$ and $t$ resemble those found in (9.6) and they may be treated in the same way. Hence

$$\begin{Bmatrix} P_n^r(r') \\ \\ Q_n^\theta(r') \end{Bmatrix} = \frac{-2}{\pi r'^{n+1}} \int_0^c \frac{t\,dt}{(r'^2-t^2)^{\frac{3}{2}}} \int_0^t \frac{s^{n+2}}{(t^2-s^2)^{\frac{1}{2}}} \begin{Bmatrix} P_n^r(s) \\ \\ Q_n^\theta(s) \end{Bmatrix} ds \; -$$

$$- c\bar{\alpha}_n(c) \begin{Bmatrix} 1+n\nu \\ -(1-\nu-n\nu) \end{Bmatrix} \left\{ \frac{\pi}{r'} I_n(c,r') + \frac{4}{r'^n c^n} \int_0^c \frac{t\,dt}{(r'^2-t^2)^{\frac{3}{2}}} \int_0^t \frac{s^{2n}ds}{(c^2-s^2)^{\frac{1}{2}}} \right\}. \tag{9.25}$$

The expression in the last bracket of (9.25) may be simplified to the expression

$$\frac{4c^n E_n}{r'^{n+1}(r'^2-c^2)^{\frac{1}{2}}} \; ,$$

where $E_n$ is as defined just after equation (9.17).

Since $-nH_n(c) = c\bar{\alpha}_n(c)$ and from (9.7), after evaluating the $t$ integral in the first term, equation (9.25) becomes

$$\begin{Bmatrix} P_n^r(r') \\ \\ Q_n^\theta(r') \end{Bmatrix} = \frac{-2}{r'^{n+1}(r'^2-c^2)^{\frac{1}{2}}}\left\{\int_0^c \frac{(c^2-s^2)^{\frac{1}{2}}}{(r'^2-s^2)}\begin{Bmatrix} P_n^r(s) \\ \\ Q_n^\theta(s) \end{Bmatrix} s^{n+2}ds + \right.$$

$$\left. + \frac{1}{2-\nu}\begin{Bmatrix} n\nu+1 \\ n\nu-1+\nu \end{Bmatrix}\int_0^c s^n(c^2-s^2)^{\frac{1}{2}}[P_n^r(s) - Q_n^\theta(s)]ds\right\} . \tag{9.26}$$

The other two Fourier conponents may be found from (9.26) by replacing $P_n^r(r)$ with $Q_n^r(r)$ and $Q_n^\theta(r)$ with $-P_n^\theta(r)$.

Now since

$$\begin{Bmatrix} P_n^k(r') \\ \\ Q_n^k(r') \end{Bmatrix} = \frac{4(1-\nu)}{\pi}\int_0^{2\pi} \sigma_{zk}(r',\theta',0)\begin{Bmatrix} \cos n\theta' \\ \sin n\theta' \end{Bmatrix}d\theta',$$

equation (9.26) with expansion (4.5) gives, after summing:

$$\sigma_{rz}(r',\theta',0) = \frac{-2}{\pi^2(2-\nu)(r'^2-c^2)^{\frac{1}{2}}}\int_0^c\int_0^{2\pi} \frac{(c^2-s^2)^{\frac{1}{2}}}{P}\left\{\sigma_{rz}(s,\theta,0)D_1(r',s,\theta,\theta') - \right.$$

$$\left. - \sigma_{z\theta}(s,\theta,0)M_1(r',s,\theta,\theta')\right\}sd\theta ds,$$

$$\sigma_{z\theta}(r',\theta',0) = \frac{-2}{\pi^2(2-\nu)(r'^2-c^2)^{\frac{1}{2}}}\int_0^c\int_0^{2\pi}\frac{(c^2-s^2)^{\frac{1}{2}}}{P}\Big\{\sigma_{z\theta}(s,\theta,0)D_2(r',s,\theta,\theta') +$$

$$+\ \sigma_{zr}(s,\theta,0)M_2(r',s,\theta,\theta')\Big\}sd\theta ds \quad , \tag{9.27}$$

where

$$D_1(r',s,\theta,\theta') = \cos(\theta-\theta') - \frac{\nu s}{2r'} + \nu\frac{(s^2+r'^2)\cos(\theta-\theta') - 2sr'}{P}$$

$$D_2(r',s,\theta,\theta') = (1-\nu)\cos(\theta-\theta') - \frac{\nu s}{2r'} - \nu\frac{(s^2+r'^2)\cos(\theta-\theta') - 2sr'}{P}$$

$$M_1(r',s,\theta,\theta') = \left[1 + \nu\frac{r'^2-s^2}{P}\right]\sin(\theta-\theta')$$

$$M_2(r',s,\theta,\theta') = \left[1 - \nu - \nu\frac{r'^2-s^2}{P}\right]\sin(\theta-\theta')$$

$$P = r'^2 + s^2 - 2sr'\cos(\theta-\theta').$$

## 10. Some Applications of the Crack Result

A few applications of the results found in section 9 are presented here. The first consideration will be two models of penny-shaped cracks under axisymmetric loading.

### The Barenblatt Model

Consider a penny shaped crack whose surface is loaded by an axisymmetric normal traction, $\sigma_{zz}(r,0)$. As can be seen from equation (9.9) the stress becomes unbounded at the crack tip $r' = c$. Hence the material around the edge of the crack would not behave according to the linear theory of elasticity. In the neighbourhood of the crack tip there would be some zone where the material could be assumed to behave in a non-linear elastic manner. Barenblatt has proposed a model [17] (for a Griffith crack) in which there exists a zone of non-linear elastic behaviour extending a short distance from the crack tip. This zone is assumed to be coplanar with the crack itself. The Barenblatt model may easily be extended to the penny shaped crack. For this model the size of the non-linear region can be found as well as a non-linear integral equation for the displacement discontinuity in that region.

For a penny shaped crack of radius $c$ with a normal traction $\sigma_{zz}(r,0)$ given, suppose the non-linear region is a ring in the $z = 0$ plane extending beyond the crack tip to a distance $a$. Assume that $a - c$ is small compared to $c$. According to the model the stress is a non-linear function of the separation of the atomic layers which occurs in the edge region $c < r < a$. Denote $\Delta u_z(r)$ as this separation. Hence

$$\sigma_{zz}(r,0) = f[\Delta u_z(r)] \qquad c < r < a. \tag{10.1}$$

Further, according to the model the stress intensity factor at $r = a$ reduces to zero. From (9.10a) therefore

$$\int_0^c \frac{\sigma_{zz}(r,0)\,rdr}{(a^2-r^2)^{\frac{1}{2}}} + \int_c^a \frac{f[\Delta u_z(r)]\,rdr}{(a^2-r^2)^{\frac{1}{2}}} = 0. \tag{10.2}$$

Let $d = a - c$, and in the second integral let the variable $s = a - r$. Now both $d$ and $s$ are small compared with $c$ or $a$. Hence the second integral becomes approximately

$$\left(\frac{a}{2}\right)^{\frac{1}{2}} \int_0^d \frac{f[\Delta u_z(a-s)]\,ds}{s^{\frac{1}{2}}} .$$

If we denote the stress intensity factor

$$K_I = \frac{-2}{\pi(2c)^{\frac{1}{2}}} \int_0^c \frac{\sigma_{zz}(r,0)\,rdr}{(c^2-r^2)^{\frac{1}{2}}} ,$$

then equation (10.2) becomes approximatley:

$$K_I \simeq \frac{1}{\pi} \int_0^d \frac{f[\Delta u_z(a-s)]\,ds}{s^{\frac{1}{2}}} . \tag{10.2a}$$

Now from equation (9.5) for an axisymmetric load the displacement discontinuity in the non-linear region $c < r < a$ reduces to

$$\Delta u_z(r) = -\frac{4(1-\nu)}{\pi\mu} \int_r^a \frac{g(t)\,dt}{(t^2-r^2)^{\frac{1}{2}}} , \tag{10.3}$$

where

$$g(t) = \int_0^t \frac{s\sigma_{zz}(s,0)\,ds}{(t^2-s^2)^{\frac{1}{2}}} .$$

If $c < t < a$ then with (10.1) $g(t)$ is given by

$$g(t) = \int_0^c \frac{r\sigma_{zz}(r,0)\,dr}{(t^2-r^2)^{\frac{1}{2}}} + \int_c^t \frac{rf[\Delta u_z(r)]\,dr}{(t^2-r^2)^{\frac{1}{2}}} . \qquad (10.4)$$

Since $t \simeq c$ the first term of (10.4) is approximately equal to $-\frac{\pi}{2}(2c)^{\frac{1}{2}}K_I$. In the second term again let the variable $s = a - r$. Then the second term becomes approximately

$$\frac{(2c)^{\frac{1}{2}}}{2} \int_{a-t}^{d} \frac{f[\Delta u_z(a-s)]\,ds}{(t-a+s)^{\frac{1}{2}}} .$$

So

$$g(t) \simeq \frac{\pi}{2}(2c)^{\frac{1}{2}}\left\{\frac{1}{\pi}\int_{a-t}^{d} \frac{f[\Delta u_z(a-s)]\,ds}{(t-a+s)^{\frac{1}{2}}} - K_I\right\} . \qquad (10.5)$$

Now in (10.3) let the variables $p = a - r$ and $t' = a - t$. Then

$$\Delta u_z(a-p) \simeq -\frac{4(1-\nu)}{\pi\mu(2c)^{\frac{1}{2}}} \int_0^p \frac{g(a-t')\,dt'}{(p-t')^{\frac{1}{2}}} . \qquad (10.6)$$

From (10.5) we have that

$$g(a-t') \simeq \frac{\pi}{2}(2c)^{\frac{1}{2}}\left\{\frac{1}{\pi}\int_{t'}^{d} \frac{f[\Delta u_z(a-s)]\,ds}{(s-t')^{\frac{1}{2}}} - K_I\right\} .$$

Hence the equation (10.6) can be written

$$\Delta u_z(a-p) \simeq -\frac{2(1-\nu)}{\mu}\left\{\frac{1}{\pi}\int_0^p\int_{t'}^d \frac{f[\Delta u_z(a-s)]dsdt'}{(p-t')^{\frac{1}{2}}(s-t')^{\frac{1}{2}}} - K_I\int_0^p \frac{dt'}{(p-t')^{\frac{1}{2}}}\right\}.$$

After performing the $t'$ integration we have approximately

$$\Delta u_z(a-p) = \frac{4(1-\nu)}{\mu}\left\{p^{\frac{1}{2}}K_I - \frac{1}{2\pi}\int_0^d \ell n\left|\frac{s^{\frac{1}{2}}+p^{\frac{1}{2}}}{s^{\frac{1}{2}}-p^{\frac{1}{2}}}\right| f[\Delta u_z(a-s)]ds\right\}. \qquad (10.7)$$

Equation (10.7) thus provides a non-linear integral equation for the displacement discontinuity $\Delta u_z(p)$ in the non-linear edge region. For a given non-linear relation $\sigma_{zz}(r,0) = f[\Delta u_z(r)]$, $c < r < a$ and a given stress intensity factor $K_I$ equations (10.2a) and (10.7) determine the size $d$ of the non-linear region and the displacement in that region. Thus the behavior in the non-linear edge region depends on the external loading $\sigma_{zz}(r,0)$ only through $K_I$. Willis [18] found the same result when applying the Barenblatt model to a Griffith crack. So as long as the edge region is sufficiently small its size is the same for penny shaped and Griffith cracks.

## The BCS Model

Another approximation that deals with the unbounded stresses at the crack tip is the BCS model. In this model we assume the existence of a zone of plastic behavior near the crack tip. The BCS model can be readily applied to the penny shaped crack under a shear load. In this case the plastic zone is an annulus lying in the plane of the crack occupying the region $c < r < a$. In this application consider a penny shaped crack with the axisymmetric boundary conditions $\sigma_{\theta z}(r',0)$ given for $0 \leq r' \leq c$ and $\sigma_{\theta z}(r',0) = \sigma_1, c \leq r' < a$ ($\sigma_1$ is the yield stress of the material).

We may use the second of (9.17) to find an expression for $\sigma_{\theta z}(r',0)$. However some calculation can be spared by returning to equation (9.16). For the axisymmetric case all we need is the coefficient $P_0^\theta(r')$. Recall that $P_0^r(r')$ may be replaced by $P_0^\theta(r')$ in (9.16). With $n = 0$ the second term is zero since from (9.11) and (9.32) $A_n$ and $B_n$ are zero. Then from equations (9.16) and (9.5) we have

$$\sigma_{\theta z}(r',0) = \frac{-2}{\pi r'(r'^2-c^2)^{\frac{1}{2}}} \int_0^c \frac{\sigma_{\theta z}(s,0)s^2(c^2-s^2)^{\frac{1}{2}}ds}{(r'^2-s^2)} .$$

The stress intensity factor for a crack of radius $c$ becomes

$$K_{III} = \frac{-2}{\pi(1-\nu)c(2c)^{\frac{1}{2}}} \int_0^c \frac{s^2\sigma_{\theta z}(s,0)ds}{(c^2-s^2)^{\frac{1}{2}}} . \tag{10.8}$$

In applying (10.8) to the BCS model of the penny shaped crack, the stress intensity factor is to vanish at $r' = a$:

$$\int_0^c \frac{s^2\sigma_{\theta z}(s,0)\,ds}{(a^2-s^2)^{1/2}} + \int_c^a \frac{s^2\sigma_1\,ds}{(a^2-s^2)^{1/2}} = 0.$$

Thus

$$\int_0^c \frac{s^2\sigma_{\theta z}(s,0)\,ds}{(a^2-s^2)^{1/2}} = -\frac{\sigma_1}{2}[a^2\cos^{-1}(\frac{c}{a}) + c(a^2-c^2)^{1/2}]. \tag{10.9}$$

Equation (10.9) then enables the width $(a-c)$ of the plastic zone to be determined in terms of the applied load.

If the plastic width $(a-c)$ is small compared to $c$, then equation (10.9) together with (10.8) gives approximately

$$\frac{\pi(1-\nu)(2c)^{1/2}K_{III}}{2\sigma_1} \simeq (a^2-c^2)^{1/2}$$

or

$$\left\{\frac{\pi(1-\nu)K_{III}}{2\sigma_1}\right\}^2 \simeq (a-c).$$

This result for a small plastic zone agrees with the corresponding result obtained for a strip crack [12].

Unidirectional Shear Tractions

The final application of the results of ection 9 to be presented is the case of a unidirectional shear traction applied to the surface of a penny shaped crack. This application follows directly from the results by Lardner summarized in the second part of section 9. Again a summary of Lardner's work will be given here.

Let a penny shaped crack of radius $c$ be subjected to the traction $\sigma_{zx}(r,\theta,0) = k(r)$, $\sigma_{zy}(r,\theta,0) = 0$, $r < c$. In polars the boundary values become

$$\sigma_{zr}(r,\theta,0) = k(r)\cos\theta,\ \sigma_{z\theta}(r,\theta,0) = -k(r)\sin\theta,\ r < c.$$

The only non zero Fourier components of stress are $P_1^r$ and $Q_1^\theta$. So for $r < c$,

$$P_1^r(r) = \frac{4\pi(1-\nu)k(r)}{\mu}$$

$$Q_1^\theta(r) = \frac{-4\pi(1-\nu)k(r)}{\mu}.$$

For $r' > c$, from (9.16) and (9.5) we have

$$\sigma_{zr}(r',\theta',0) = \frac{-2\cos\theta'}{\pi r'^2(r'^2-c^2)^{\frac{1}{2}}}\int_0^c (c^2-p^2)^{\frac{1}{2}}k(p)\left[\frac{p^2}{r'^2-p^2} + \frac{2(1+\nu)}{2-\nu}\right]pdp$$

$$\sigma_{z\theta}(r',\theta',0) = \frac{2\sin\theta'}{\pi r'^2(r'^2-c^2)^{\frac{1}{2}}}\int_0^c (c^2-p^2)^{\frac{1}{2}}k(p)\left[\frac{p^2}{r'^2-p^2} + \frac{2(1-2\nu)}{2-\nu}\right]pdp,$$

and then

$$\sigma_{zx}(r',\theta',0) = -\frac{2}{\pi(r'^2-c^2)^{\frac{1}{2}}}\left\{\int_0^c \frac{(c^2-p^2)^{\frac{1}{2}}k(p)pdp}{r'^2-p^2} + \right.$$

$$\left. + \frac{3\nu}{1-\nu}\frac{\cos 2\theta'}{r'^2}\int_0^c (c^2-p^2)^{\frac{1}{2}}k(p)pdp\right\}.$$

## 11. Appendix I

Just after equation (5.14) an integral representation of the kernel $L_n(r,r')$ is used. This representation may be found as follows.

Recall from equation (4.23) that

$$I_n(r,r') = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\phi d\phi}{R}, \quad \text{where} \quad \phi = \theta - \theta'$$

$$= \frac{1}{\pi r'}\int_0^{2\pi} \frac{\cos n\phi d\phi}{\left[1 - \frac{2r}{r'}\cos + \frac{r^2}{r'^2}\right]^{\frac{1}{2}}}.$$

This expression is an integral representation of a hypergeometric function. In particular,

$$I_n(r,r') = \frac{2}{r'nB(\frac{1}{2},n)} \left(\frac{r}{r'}\right)^n {}_2F_1\left(\tfrac{1}{2},n+\tfrac{1}{2};n+1;\frac{r^2}{r'^2}\right), \tag{AI.1}$$

where $B(a,b)$ is the Beta function.

By the transformation formulae for ${}_2F_1$ we have [20]

$${}_2F_1\left(\tfrac{1}{2},n+\tfrac{1}{2};n+1;\frac{r^2}{r'^2}\right) = {}_2F_1\left(n+\tfrac{1}{2},\tfrac{1}{2};n+1;\frac{r^2}{r'^2}\right)$$

$$= \left[1 + \frac{r}{r'}\right]^{-2n-1} {}_2F_1\left(n+\tfrac{1}{2},n+\tfrac{1}{2};2n+1;\frac{4rr'}{(r+r')^2}\right) \tag{AI.2}$$

In turn the hypergeometric function in (AI.2) may be expressed as an integral involving the product of Bessel functions [19].

$$ {}_2F_1(n+\tfrac{1}{2},n+\tfrac{1}{2};2n+1;\ \frac{4rr'}{(r+r')^2}) = \frac{(r+r')^{2n+1}}{(rr')^n}\frac{\Gamma(n+1)\Gamma(\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2})}\int_0^\infty J_n(r\xi)J_n(r'\xi)d\xi \ . $$

Since

$$ B(\tfrac{1}{2},n) = \frac{\Gamma(n)\Gamma(\tfrac{1}{2})}{\Gamma(n+\tfrac{1}{2})} $$

we have finally that

$$ I_n(r,r') = 2\int_0^\infty J_n(r\xi)J_n(r'\xi)d\xi \quad . \tag{AI.3} $$

Now recall from equation (4.24) that

$$ L_n(r,r') = r^n \frac{\partial}{\partial r}\left[\frac{I_n(r,r')}{r^n}\right], $$

hence we have the important result that

$$ L_n(r,r') = -\,2\int_0^\infty J_{n+1}(r\xi)J_n(r'\xi)\xi d\xi \quad . \tag{AI.4} $$

---

In the derivation of equation (6.1) it was necessary to combine the two terms:

$$J_n(r\xi) + \frac{2}{\pi} r^n \int_a^\infty \int_a^p \frac{tJ_n(p\xi)dtdp}{(p^2-t^2)^{\frac{1}{2}}(t^2-r^2)^{\frac{3}{2}} p^{n-1}} \equiv T \,. \tag{AI.5}$$

The first term may be represented by the integral

$$J_n(r\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^n \xi^{\frac{1}{2}} \int_r^\infty \frac{J_{n+\frac{1}{2}}(\xi t)dt}{t^{n-\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} \,.$$

In the second term of equation (AI.5) the integral over p may be performed. Then (AI.5) can be written

$$T = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^n \left\{ \xi^{\frac{1}{2}} \int_r^\infty \frac{J_{n+\frac{1}{2}}(\xi t)dt}{t^{n-\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} + \frac{1}{\xi^{\frac{1}{2}} r} \frac{d}{dr} \int_a^\infty \frac{J_{n-\frac{1}{2}}(\xi t)dt}{t^{n-\frac{3}{2}}(t^2-r^2)^{\frac{1}{2}}} \right\} \,.$$

Now in the second term $\int_a^\infty = \int_r^\infty - \int_a^r$ . The integral $\int_r^\infty$ may be integrated by parts and then differentiated to give:

$$T = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^n \left\{ \xi^{\frac{1}{2}} \int_r^\infty \frac{J_{n+\frac{1}{2}}(\xi t)dt}{t^{n-\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} - \xi^{\frac{1}{2}} \int_r^\infty \frac{J_{n+\frac{1}{2}}(\xi t)dt}{t^{n-\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} - \right.$$
$$\left. - \frac{1}{\xi^{\frac{1}{2}} r} \frac{d}{dr} \int_a^r \frac{J_{n-\frac{1}{2}}(\xi t)dt}{t^{n-\frac{3}{2}}(t^2-r^2)^{\frac{1}{2}}} \right\} .$$

Combining terms we finally have

$$T = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{r^{n-1}}{\xi^{\frac{1}{2}}} \frac{d}{dr} \int_r^a \frac{J_{n-\frac{1}{2}}(\xi t)dt}{t^{n-\frac{3}{2}}(t^2-r^2)^{\frac{1}{2}}} \,. \tag{AI.6}$$

## 12. Appendix II

The following are some results previously referred to. They are placed in order of use.

$$\int_0^\infty J_n(r\xi)J_{n+\frac{1}{2}}(\xi t)\xi^{\frac{1}{2}}d\xi = \frac{2^{\frac{1}{2}}r^n H(t-r)}{\Gamma(\frac{1}{2})t^{n+\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} \tag{AII.1}$$

$$\int_0^\infty J_n(r\xi)J_{n+\frac{1}{2}}(\xi t)\xi^{-\frac{1}{2}}d\xi = \frac{\Gamma(\frac{2n+1}{2})r^n}{2^{\frac{1}{2}}t^{n+\frac{1}{2}}\Gamma(n+1)} \tag{AII.2}$$

$$\int_r^p \frac{tdt}{(p^2-t^2)^{\frac{1}{2}}(t^2-r^2)^{\frac{1}{2}}} = \frac{\pi}{2} \tag{AII.3}$$

$$\int_0^\infty J_{n-1}(r'\xi)J_{n-\frac{1}{2}}(\xi t)\xi^{\frac{1}{2}}d\xi = \frac{(\frac{2}{\pi})^{\frac{1}{2}}r'^{n-1}H(t-r')}{t^{n-\frac{1}{2}}(t^2-r'^2)^{\frac{1}{2}}} \tag{AII.4}$$

$$\int_0^\infty J_{n+1}(r\xi)J_{n+\frac{1}{2}}(\xi t)\xi^{\frac{1}{2}}d\xi = \frac{2^{\frac{1}{2}}t^{n+\frac{1}{2}}H(r-t)}{\Gamma(\frac{1}{2})r^{n+1}(r^2-t^2)^{\frac{1}{2}}} \tag{AII.5}$$

$$\int_0^t \frac{r^{n+1}J_n(r\xi)dr}{(t^2-r^2)^{\frac{1}{2}}} = (\frac{\pi}{2})^{\frac{1}{2}}\xi^{-\frac{1}{2}}t^{n+\frac{1}{2}}J_{n+\frac{1}{2}}(\xi t) \tag{AII.6}$$

$$\int_s^c \frac{tdt}{(r'^2-t^2)^{\frac{1}{2}}(t^2-s^2)^{\frac{1}{2}}} = \frac{(c^2-s^2)^{\frac{1}{2}}}{(r'^2-s^2)(r'^2-c^2)^{\frac{1}{2}}}, \quad r' > c \tag{AII.7}$$

Note that the integrals above involving the products of Bessel functions are just special cases of the Weber Schaftheitlin integral:

$$\int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda}\, dt =$$

$$\frac{\alpha^\nu \Gamma\left[\frac{\nu+\mu-\lambda+1}{2}\right]}{2^\lambda \beta^{\nu-\lambda+1} \Gamma\left[\frac{-\nu+\mu+\lambda+1}{2}\right] \Gamma(\nu+1)} \; F\left[\frac{\nu+\mu-\lambda+1}{2}, \; \frac{\nu-\mu-\lambda+1}{2}; \; \nu+1; \; \frac{\alpha^2}{\beta^2}\right]$$

$\mathrm{Re}(\nu+\mu-\lambda+1) > 0, \quad \mathrm{Re}\lambda > -1, \; 0<\alpha<\beta.$

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