A UNIVERSAL SUPERCOMPACTIFICATION

by

Douglas Bishop Super

B.A., Grinnell College, 1969

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

Name: Douglas Bishop Super

Degree: Master of Science

Title of Thesis: A Universal Supercompactification

Examining Committee:

    Chairman: Edgar Pechlaner

                                  S. K. Thomason
                                  Senior Supervisor

                                  D. Ryeburn

                                  A. R. Freedman

                                  H. Gerber
                                  External Examiner

                Date Approved: _April 6th 1977_

PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis or dissertation (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this thesis for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis/Dissertation:

A UNIVERSAL SUPERCOMPACTIFICATION

_____

_____

_____

Author:

(signature)'

Douglas B. Super
(name)

April 21, 1977
(date)

ABSTRACT

Supercompactness and supercompactifications are related to types of linked families in the same manner as compactness and compactifications are related to types of filters. The Stone-Čech compactification and Wallman compactification are known to be universal constructions in the sense of category theory. The main result is a supercompactification that likewise is universal. To obtain this result the superextension construction of deGroot is modified by the introduction of T_1-subbase spaces and the kinds of morphisms allowed are reduced from continuous functions to sure functions.

ACKNOWLEDGMENTS

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Prior to the formal presentation of our topic, which is deferred until the next chapter, we feel obliged to discuss briefly and informally the background and significance of our results. Although the succeeding chapters are self-contained, the present one presumes a familiarity with general topology and some category theory. An effort has been made throughout to use standard notation whenever possible.

Recall that a pair (Y,h) is called a compactification of a topological space X whenever Y is a compact space and X is homeomorphic with a dense subspace of Y via the function h. By a compactification for $\underline{C}$, where $\underline{C}$ is a category of topological spaces, we mean a collection $\{(\hat{C},r_C) : C \in \underline{C}\}$, where each $(\hat{C},r_C)$ is a compactification of C. Several methods for constructing compactifications are known which seem relatively "natural" and "uniform". More precisely, we say a compactification of $\underline{C}$ is a <u>universal</u> <u>construction</u> whenever each $\underline{C}$ function $f : C \to B$ with B a compact space has associated with it a unique $\underline{C}$ function $g : \hat{C} \to B$ such that $gr_C = f$. It is well known that the Stone-Čech compactification is a universal construction for the category of Tychonoff spaces [9, p. 137] whereas the one-point compactification [9, p. 136] is not one for the more restricted category of locally compact Hausdorff spaces.

Applications of the concept of universal construction extend beyond compactifications. In fact, MacLane [7] and others have defined and studied this concept within the realm of category theory and shown that instances of it occur throughout mathematics. So ubiquitous and

desirable is the universal property, that when a construction for a particular instance exists which is not universal, often steps are taken to study the situation further. Traditionally, this might involve a search for a different construction for the same instance which is more "natural" and "uniform" and thus possibly universal. Lately, several investigations, notably by Harris [4][5] and Bentley and Naimpally [1], have appeared which use a different approach. Since our main result also employs this new approach, which we call a fur strategy, an example of its use is mentioned next.

Let $\underline{\text{TOP}}_1$ denote the category of all T_1-spaces and continuous functions between these spaces, and $\underline{\text{COMP TOP}}_1$ be the full subcategory of $\underline{\text{TOP}}_1$ containing the compact T_1-spaces. A Wallman compactification $\{(X_w, w_X) : X \in \underline{\text{TOP}}_1\}$ for $\underline{\text{TOP}}_1$ exists which seems relatively "natural" and "uniform", but which is not universal. It was discovered by Harris [5] that, by considering a fur subcategory of $\underline{\text{TOP}}_1$ (one which has the same spaces but fewer functions), the problem could be "bypassed". Specifically, Harris found that the Wallman compactification for the category $\underline{\text{WOSEP}}$, a fur subcategory of $\underline{\text{TOP}}_1$, is a universal construction.

Supercompactness is a type of compactness which depends upon the subbase of a topological space. Suppose $\underline{\text{SUPERCOMP TOP}}_1$ is the full subcategory of $\underline{\text{TOP}}_1$ containing just supercompact T_1-spaces. A superextension construction, $\{(X_S, s_X) : X \in \underline{\text{TOP}}_1\}$, was discovered by deGroot [3] such that each X_S is a supercompact T_1-space and X is homeomorphic to a subspace of X_S via the map s_X . Several map extension situations have been investigated by Jensen [8, pp. 54 - 57].

Our main result is to define a reformulation of the superextension construction which is a universal construction for the category $\underline{SURE}$, a fur subcategory of a category related to $\underline{TOP}_1$.

Chapter II contains formal definitions of an epireflective subcategory (a notion stronger than that of universal construction) and of the fur strategy. In Chapter III is presented a restatement of supercompactness and of the superextension construction in a form which allows in Chapter IV our proof that a universal situation exists with respect to a suitable fur strategy. Chapter V reviews the results of Jensen mentioned above.

CHAPTER II

EPIREFLECTIVE SUBCATEGORIES

In this chapter, we introduce formally the category-theoretic notion *epireflective subcategory*, which is needed later in our discussion of the superextension construction of supercompact T_1-subbase spaces.

Definition 2.1 [6, p. 16] A *category* $\underline{C}$ is an ordered six-tuple $\underline{C} = <\text{Obj } \underline{C} \text{ , Mor } \underline{C} \text{ , dom, cod, } ^\circ \text{ , id}>$ where

(i) Obj $\underline{C}$ is a class whose members are called *objects*,

(ii) Mor $\underline{C}$ is a class whose members are called *morphisms*,

(iii) dom and cod are functions from Mor $\underline{C}$ to Obj $\underline{C}$,

(iv) $^\circ$ is a partial function from Mor $\underline{C}$ × Mor $\underline{C}$ to Mor $\underline{C}$, whose value at the pair $<f,g>$ (if this is defined) is denoted by fg and is called the composition of g and f , and

(v) id is a function from Obj $\underline{C}$ to Mor $\underline{C}$ whose value at an object X is denoted by id_X and is called the identity morphism of X

such that the following conditions are satisfied:

(1) fg is defined iff $cod(g) = dom(f)$;

(2) if fg is defined, then $dom(fg) = dom(g)$ and $cod(fg) = cod(f)$;

(3) if fg and hf are defined, then $h(fg) = (hf)g$;

(4) for each object X , $cod(id_X) = dom(id_X) = X$;

(5) if $cod(f) = X = dom(g)$, then $id_X f = f$ and $g \, id_X = g$;

(6) for any pair of objects (X,Y) , the class

$hom_{\underline{C}}(X,Y) = \{f \in Mor \ \underline{C} : dom(f) = X$ and

$cod(f) = Y\}$ is a set (rather than a proper class).

For example, SET denotes the category for which

(1) Obj SET is the class of all sets,

(2) Mor SET is the class of all functions between sets,

(3) for each function $f \in$ Mor SET , the sets dom(f)

and cod(f) are respectively the domain and

codomain of f ,

(4) for each set X , the function id_X is the identity

function for X , and

(5) the partial function ° corresponds with the usual

composition of functions.

It is customary to define a particular category, when little

chance for confusion is possible, by merely describing its class of

objects and class of morphisms.

The morphisms (objects) of any category $\underline{C}$ will be denoted

by lower (upper) case letters. Thus, by $f \in \underline{C}$ and $X \in \underline{C}$, we will

mean $f \in Mor \ \underline{C}$ and $X \in Obj \ \underline{C}$.   For f, X, Y $\in \underline{C}$, both $f : X \to Y$

and $X \xrightarrow{f} Y$ signify that $f \in hom_{\underline{C}}(X,Y)$.

Definition 2.2 [6, p. 23] We call $\underline{S}$ a subcategory of category $\underline{C}$

whenever $\underline{S}$ is a category such that

(1) Obj $\underline{S} \subseteq$ Obj $\underline{C}$,

(2) Mor $\underline{S} \subseteq$ Mor $\underline{C}$, and

(3) the functions dom, cod, id, and ° for $\underline{S}$ are restrictions

of the corresponding functions for $\underline{C}$.

Definition 2.3 [6, p. 24] A subcategory $\underline{S}$ of a category $\underline{C}$ is called a __full__ subcategory of $\underline{C}$ if and only if, for any objects $X, Y \in \underline{S}$, $\hom_{\underline{S}}(X,Y) = \hom_{\underline{C}}(X,Y)$.

Definition 2.4 A subcategory $\underline{S}$ of a category $\underline{C}$ is called a __fur__ subcategory of $\underline{C}$ if and only if $\mathrm{Obj}\,\underline{S} = \mathrm{Obj}\,\underline{C}$.

Definition 2.5 [6, p. 35] A morphism $f \in \underline{C}$ is a $\underline{C}$-isomorphism if and only if there exists a morphism $g \in \underline{C}$ such that $fg = \mathrm{id}_{\mathrm{cod}(f)}$ and $gf = \mathrm{id}_{\mathrm{dom}(f)}$.

Definition 2.6 [6, p. 40] A morphism $f \in \underline{C}$ is a $\underline{C}$-epimorphism if and only if, for all morphisms $g, h \in \underline{C}$, $gf = hf$ implies $g = h$.

Definition 2.7 [6, p. 275][5] A full subcategory $\underline{S}$ of a category $\underline{C}$ is an __epireflective subcategory of $\underline{C}$ with respect to a class__ $\underline{\{(r_X , X_R)\}_{X \in \underline{C}}}$ if and only if

(1) for each $X \in \underline{C}$, $X_R \in \underline{S}$ and $r_X : X \to X_R$ is a $\underline{C}$-epimorphism,

(2) if $X \in \underline{S}$, then r_X is an $\underline{S}$-isomorphism, and

(3) for any objects $Y, Z \in \underline{C}$ and morphism $f \in \hom_{\underline{C}}(Y,Z)$, there exists a morphism $f_R \in \hom_{\underline{S}}(Y_R, Z_R)$ such that $r_Z f = f_R r_Y$.

$$\begin{array}{ccc}
Y & \xrightarrow{\ r_Y\ } & Y_R \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_R} \\
Z & \xrightarrow{\ r_Z\ } & Z_R
\end{array}$$

A full subcategory $\underline{S}$ of a category $\underline{C}$ is called an epireflective subcategory of $\underline{C}$ whenever a class $\{(r_X, X_R)\}_{X \in \underline{C}}$ exists which satisfies the three previous conditions.

Defintion 2.8   Let $\underline{S}$ be a full subcategory of a category $\underline{C}$. Then a category $\underline{R}$ is called a successful fur strategy (with respect to the class $\{(r_X, X_R)\}_{X \in \underline{C}}$) if and only if

(1) $\underline{R}$ is a fur subcategory of $\underline{C}$ and

(2) $\underline{R} \cap \underline{S}$ is an epireflective subcategory of $\underline{R}$ (with respect to the class $\{(r_X, X_R)\}_{X \in \underline{C}}$) .

SUPEREXTENSIONS

The superextension construction and the property of super-
compactness, close relatives respectively of the Wallman-type compacti-
fication and the property of compactness, have been studied extensively
by deGroot [3], Császár [2], Verbeek [8], and others. In this chapter,
we introduce a formulation of these concepts which permits a successful
fur strategy.

Definition 3.1   [9, p. 24] A topology for a set X is a collection τ
of subsets of X such that:

   (1) any intersection of members of τ is a member of τ ,

   (2) any finite union of members of τ is a member of τ ,

   (3) the sets X and ϕ both belong to τ .

We call the members of a topology closed sets.   A topological space
is a pair (X,τ) where τ is a topology for X . We often will
abbreviate (X,τ) , when no confusion is likely, to X .

Definition 3.2   [9, pp. 38-39] Suppose (X,τ) is a topological space.
A base for τ is a collection $\beta \subseteq \tau$ such that $\tau = \{ \bigcap_{A \in \alpha} A : \alpha \subseteq \beta \}$.
A subbase for τ is a collection $\sigma \subseteq \tau$ such that $\beta = \{ \bigcup_{G \in \lambda} G : \lambda$
is a finite subcollection of $\tau \}$ is a base for τ .

Proposition 3.3   [9, p. 39] Any collection σ of subsets of set X
is a subbase for a topology for X .

Definition 3.4   Suppose α is a collection of subsets of X. The members of α are said to meet whenever $\bigcap_{A \in \alpha} A \neq \phi$. We call α an ip (intersection property) family if its members meet, a fip (finite intersection property) family if each finite subcollection of it consists of members which meet, and a linked family [8, p.1] if every two of its members meets.

Definition 3.5   [9, p. 118]   A topological space (X,τ) is compact if and only if each collection $\alpha \subsetneq \tau$ which is a fip family is also an ip family.

Definition 3.6   [8, p. 48]   A topological space (X,τ) is supercompact if and only if there exists a subbase σ for the topology τ such that each collection $\alpha \subseteq \sigma$ which is a linked family is also an ip family.

Proposition 3.7   [8, p. 48]   Each supercompact topological space is compact.

       This result is easily deduced from the Alexander subbase theorem [9, p. 129]: A topological space (X,τ) is compact if and only if there exists a subbase σ for the topology τ such that each collection $\alpha \subseteq \sigma$ which is a fip family is also an ip family.

Definition 3.8   [9, p. 86]   A topological space is a T_1-space if and only if each of its singletons is a closed set.

Remark 3.9   [8, p. 48]   It is known that every compact metrizable space is supercompact. Although compact T_1-spaces exist which are not super-compact, it is an open question whether there exists a compact T_2-space which is not supercompact.

-9-

In 1967, J. deGroot [3, p. 90] introduced the superextension construction by which a supercompact T_1-space can be constructed from any T_1-space. A formulation of the superextension construction is discussed in detail below.

<u>Defintion 3.10</u>   Suppose $f : X_1 \to X_2$ is a function and α_2 is a collection of subsets of X_2 . Then we let $f^-[\alpha_2] = \{f^-(A) : A \in \alpha_2\}$.

For topological spaces (X_1, τ_1) and (X_2, τ_2) , a function $f : X_1 \to X_2$ is a <u>continuous</u> <u>function</u> if and only if $f^-[\tau_2] \subseteq \tau_1$ [9, p. 44]. The function f is an <u>embedding</u> if and only if f is one-one and continuous and f^- is continuous.

<u>Remark 3.11</u>   Suppose $\underline{TOP}_1$ denotes the category whose class of spaces consists of all T_1-spaces and whose class of functions consists of all continuous functions between these spaces, and $\underline{SUPERCOMP\ TOP}_1$ is the full subcategory of $\underline{TOP}_1$ whose class of spaces consists of all supercompact T_1-spaces. For the categories $\underline{TOP}_1$ and $\underline{SUPERCOMP\ TOP}_1$ we have not found a satisfactory successful fur strategy with respect to the superextension construction mentioned above. We have, however, developed a closely related formulation of this problem for which a successful fur strategy does exist.

<u>Definition 3.12</u>   [8, p. 44] A collection R_X is a <u>T_1-subbase</u> for a topological space (X, τ) whenever

    (1) R_X is a subbase for τ which contains X and ϕ ,

    (2)   for each $x \in X$, $\{x\} = \cap \{R : x \in R$ and $R \in R_X\}$,

    (3) $x \in X$, $R \in R_X$, and $x \notin R$ imply there is a $T \in R_X$

         such that $x \in T$ and $R \cap T = \phi$.

Proposition 3.13 [8, p. 44] Each T_1-space has a T_1-subbase. If a topological space has a T_1-subbase, then it is a T_1-space.

The topology of each T_1-space is a T_1-subbase. Conversely, from (2) above it follows that $\{x\}$ is closed for each $x \in X$.

Definition 3.14 A T_1-subbase space is a pair (X, R_X) where X is a topological space and R_X is a T_1-subbase for X. A <u>supercompact</u> <u>T_1-subbase space</u> is a T_1-subbase space (X, R_X) such that X is supercompact with respect to the subbase R_X.

Let T_1-SUBBASE denote the category consisting of all T_1-subbase spaces and all continuous functions between these spaces and SUPERCOMPT T_1-SUBBASE denote the full subcategory of T_1-SUBBASE consisting of all supercompact T_1-subbase spaces.

Remark 3.15 T_1-SUBBASE and SUPERCOMPT T_1-SUBBASE are the categories to which we will eventually apply the fur strategy. The functions considered will be more directly related to the T_1-subbases than are continuous functions in general.

Definition 3.16 [2, p. 57] Let R_X be a T_1-subbase for X. An R_X-ultrasieve is a nonempty set σ with the properties

(1) $\sigma \subseteq R_X - \{\phi\}$

(2) if $B, D \in \sigma$, then $B \cap D \neq \phi$,

(3) if $B \in R_X$ and $B \cap D \neq \phi$ for each $D \in \sigma$, then $B \in \sigma$.

Definition 3.17 (cf. [1]) The R_X-bicontiguity relation, denoted by

c_{R_X} or just c , is defined by: for any sets $A_1, A_2 \subseteq X$,

$A_1 \, c_{R_X} \, A_2$ if and only if, for each pair $R_1, R_2 \in R_X$ with $A_1 \subseteq R_1$

and $A_2 \subseteq R_2$, $R_1 \cap R_2 \neq \phi$. If $A_1 \, c_{R_X} \, A_2$, we say A_1 and A_2 are

R_X-bicontiguous. We let $A_1 \, \not{c}_{R_X} \, A_2$ mean that $A_1 \, c_{R_X} \, A_2$ is not true.

Definition 3.18 (cf. [1]) An R_X-bicontiguity cluster is a non-empty

collection σ of subsets of X such that

(1) $\phi \not\in \sigma$,

(2) if $A, B \in \sigma$, then $A \, c_{R_X} \, B$,

(3) if $A \subseteq X$ and $A \, c_{R_X} \, B$ for each $B \in \sigma$, then $A \in \sigma$.

Defintion 3.19   Let $(X, R_X) \in T_1\text{-SUBBASE}$, $R \in R_X$, and $A \subseteq X$.

We define $R^+ = \{\sigma : \sigma$ is an R_X-ultrasieve containing $R\}$ and

$R_X^+ = \{R^+ : R \in R_X\}$ [8, p. 45]. In addition, we define

$^+A = \{\sigma : \sigma$ is an R_X-bicontiguity cluster containing $A\}$ and

$^+R_X = \{^+R : R \in R_X\}$ (cf. [1]).

Theorem 3.20   [8, pp. 46-48] If $(X, R_X) \in T_1\text{-SUBBASE}$ then $(X^+, R_X^+) \in$

SUPERCOMP $T_1\text{-SUBBASE}$ (where X^+ has the topology generated by the

subbase R_X^+).

   The function $\bar{\lambda}_{R_X} : (X, R_X) \to (X^+, R_X^+)$ defined so

$\bar{\lambda}_{R_X}(x) = \{R : x \in R$ and $R \in R_X\}$ is a T_1-subbase space embedding.

If $(X, R_X) \in$ SUPERCOMPT $T_1\text{-SUBBASE}$, then $\bar{\lambda}_{R_X}$ is a SUPERCOMP $T_1\text{-SUBBASE}$

- isomorphism.

__Theorem 3.21__ (cf. [1]) If $(X, R_X) \in$ __T_1-SUBBASE__, then $(^+X, ^+R_X) \in$

__SUPERCOMP T_1-SUBBASE__ (where ^+X has the topology generated by the

subbase ^+R_X).

The function $\lambda_{R_X} : (X, R_X) \to (^+X, ^+R_X)$ defined so

$\lambda_{R_X}(x) = \{A : \{x\} \, c_{R_X} \, A \text{ and } A \subseteq X\}$ is a T_1-subbase space embedding.

If $(X, R_X) \in$ __SUPERCOMP T_1-SUBBASE__, then λ_{R_X} is a __SUPERCOMP T_1-SUBBASE__

- isomorphism.

The function $\to : (^+X, ^+R_X) \to (X^+, R_X{}^+)$ defined by $\vec{\sigma} = \sigma \cap R_X$

for each $\sigma \in {}^+X$ is a __SUPERCOMP T_1-SUBBASE__ - isomorphism such that

$\to \lambda_{R_X} = \bar{\lambda}_{R_X}$.

Our proof of these theorems depends upon the following lemma:

__Lemma 3.22__   [8, pp. 45-46] Suppose $(X, R_X) \in$ __T_1-SUBBASE__ and $S, T \in R_X$.
Then,

(1) $S \subseteq T$ if and only if $S^+ \subseteq T^+$ and

(2) $S \cap T = \phi$ if and only if $S^+ \cap T^+ = \phi$.

(1)   Clearly, for any set $R \in R_X$ and R_X-ultrasieve σ ,

$\sigma \in R^+$ if and only if $R \in \sigma$. Suppose $S \subseteq T$ and $\sigma \in S^+$. Since

$S \in \sigma$ and σ is an R_X-ultrasieve, S meets each member of σ .

Thus, T meets each member of σ and by 3.16(3) $T \in \sigma$. Hence

$\sigma \in T^+$ and, since σ was arbitrary, $S^+ \subseteq T^+$.

Now suppose there is an $x \in X$ such that $x \in S$ and $x \notin T$.

We claim that $\bar{\lambda}_{R_X}(x) = \{R \in R_X : x \in R\}$ is an R_X-ultrasieve such that

$\bar{\lambda}_{R_X}(x) \in S^+$ and $\bar{\lambda}_{R_X}(x) \notin T^+$. Note that this would mean $S^+ \subseteq T^+$

implies $S \subseteq T$. Obviously, $\bar{\lambda}_{R_X}(x) \subseteq R_X - \{\phi\}$. If $R_1, R_2 \in \bar{\lambda}_{R_X}(x)$,

-13-

then $x \in R_1 \cap R_2$ and hence $R_1 \cap R_2 \neq \phi$. Since R_X is a T_1-subbase, if $Q \in R_X$ with $x \notin Q$, then there is an $R \in R_X$ with $x \in R$ such that $Q \cap R = \phi$. Hence, if Q meets each member of $\bar{\lambda}_{R_X}(x)$, then $x \in Q$ and thus $Q \in \bar{\lambda}_{R_X}(x)$. Consequently, $\bar{\lambda}_{R_X}(x)$ is an R_X-ultrasieve. Since we know that $S \in \bar{\lambda}_{R_X}(x)$ and $T \notin \bar{\lambda}_{R_X}(x)$, then $\bar{\lambda}_{R_X}(x) \in S^+$ and $\bar{\lambda}_{R_X}(x) \notin T^+$.

(2)   If $\sigma \in S^+ \cap T^+$, then $S \in \sigma$ and $T \in \sigma$. Since σ is an linked family, $S \cap T \neq \phi$. Thus, $S \cap T = \phi$ implies $S^+ \cap T^+ = \phi$.

Suppose $S \cap T \neq \phi$. Choose any $x \in S \cap T$. Since $\bar{\lambda}_{R_X}(x) = \{R \in R_X : x \in R\}$ is an R_X-ultrasieve with $\bar{\lambda}_{R_X}(x) \in S^+ \cap T^+$, clearly $S^+ \cap T^+ \neq \phi$.

Now, we resume our demonstration of Theorems 3.20 and 3.21.

Since R_X^+ is a collection of subsets of X, it is a subbase for a topology of X^+. To prove R_X^+ is a T_1-subbase, first note that $\phi^+ = \phi \in R_X^+$ and $X^+ \in R_X^+$ since $\phi \in R_X$ and $X \in R_X$. For any R_X-ultrasieve σ, surely $\sigma \in \cap \{R^+ \in R_X^+ : \sigma \in R^+\} = \cap \{R^+ : R \in \sigma\}$. If δ is an R_X-ultrasieve contained in $\cap \{R^+ : R \in \sigma\}$, then $\sigma \subseteq \delta$ since $R \in \sigma$ implies $R \in \delta$. But, if $S \in \delta$, then S meets each member of σ and hence belongs to σ. Therefore, $\{\sigma\} = \cap \{R^+ \in R_X^+ : \sigma \in R^+\}$. Finally, let us next assume $\sigma \in X^+$, $R^+ \in R_X^+$, and $\sigma \notin R^+$. Since $R \notin \sigma$, there exists an $S \in \sigma$ such that $R \cap S = \phi$. As a result, $R^+ \cap S^+ = \phi$ and $\sigma \in S^+$. We conclude that R_X^+ is a T_1-subbase.

To show that (X^+, R_X^+) is a supercompact T_1-subbase space, consider any linked family composed of members from R_X^+, say $\{R_i^+\}_{i \in I}$. Clearly, $\{R_i\}_{i \in I}$ is a linked family composed of members from R_X. But it is easy to show, using Zorn's lemma, that a class $\{\delta : \delta \subseteq R_X$ and δ is a linked set$\}$ has a maximal member σ (with respect to set inclusion). Further, σ is an R_X-ultrasieve by 3.16; for, if $R \in R_X$ meets each member of σ, then $R \in \sigma$ since otherwise $\{V : V \in \sigma$ or $V = R\}$ is a linked set properly containing σ. In particular then, $\{R_i\}_{i \in I}$ is contained in an R_X-ultrasieve, say σ. Since each $R_i \in \sigma$, then $\sigma \in \bigcap\limits_{i \in I} R_i^+$; that is $\{R_i^+\}_{i \in I}$ is an ip family.

For each $x \in X$, $\bar{\lambda}_{R_X}(x) = \{R \in R_X : x \in R\}$ is an R_X-ultrasieve; so $\bar{\lambda}_{R_X}$ is indeed a function from X to X^+. If $y, z \in X$ and $y \neq z$, then, since R_X is a T_1-subbase, there is an $R \in R_X$ with $y \in R$ and $z \notin R$; but then $\bar{\lambda}_{R_X}(y) \neq \bar{\lambda}_{R_X}(z)$. Thus $\bar{\lambda}_{R_X}$ is one-to-one.

From the fact that $x \in R$ iff $R \in \bar{\lambda}_{R_X}(x)$ iff $\bar{\lambda}_{R_X}(x) \in R^+$, we can deduce $\bar{\lambda}_{R_X}(R) \subseteq R^+$ and $\bar{\lambda}_{R_X}(R) = R^+ \cap \bar{\lambda}_{R_X}(X)$. Since $\bar{\lambda}_{R_X}$ is an injection, $\bar{\lambda}_{R_X}^-$ is a function. Note $\bar{\lambda}_{R_X}^-(R^+ \cap \bar{\lambda}_{R_X}(X)) = R$ and, incidentally, $\bar{\lambda}_{R_X}^-(R^+) = R$ and $\bar{\lambda}_{R_X}^-[R_X^+] = R_X$. Since $(X^+ \cap \bar{\lambda}_{R_X}(X), \{R^+ \cap \bar{\lambda}_{R_X}(X) : R \in R_X\})$ is surely a T_1-subbase space, we can conclude the function $\bar{\lambda}_{R_X} : (X, R_X) \to (X^+ \cap \bar{\lambda}_{R_X}(X), \{R^+ \cap \bar{\lambda}_{R_X}(X) : R \in R_X\})$ is a T_1-SUBBASE - isomorphism.

Assume (X, R_X) is a supercompact T_1-subbase space and $\sigma \in X^+$. Since σ is a linked family, it is an ip family. If $x \in X$ is contained in each member of σ, then $\sigma = \bar{\lambda}_{R_X}(x)$ by 3.16(3) since σ and $\bar{\lambda}_{R_X}(x)$ are R_X-ultrasieves with $\sigma \subseteq \bar{\lambda}_{R_X}(x)$. As a result $\bar{\lambda}_{R_X}(X) = X^+$, $\bar{\lambda}_{R_X}(R) = R^+$, $\bar{\lambda}_{R_X}[R_X] = R_X^+$, and $\bar{\lambda}_{R_X} : (X, R_X) \to (X^+, R_X^+)$ is a <u>SUPERCOMP T_1-SUBBASE</u> - isomorphism.

We claim $\lambda_{R_X}(x) = \{A \subset X : \{x\}\ c_{R_X}\ A\}$ is an R_X-bicontiguity cluster. Since $\{x\} \in \lambda_{R_X}(x)$, if B is R_X-bicontiguous with each member of $\lambda_{R_X}(x)$, then $B \in \lambda_{R_X}(x)$. Next, suppose $A, B \in \lambda_{R_X}(x)$ and $R_A, R_B \in R_X$ with $A \subseteq R_A$ and $B \subseteq R_B$. Since R_X is a T_1-subbase, $R_A\ c_{R_X}\ \{x\}$ implies $x \in R_A$. Clearly, $x \in R_A \cap R_B$ and hence $A\ c_{R_X}\ B$. Consequently, $\lambda_{R_X} : X \to {}^+X$ is a function.

Assume $\sigma \in {}^+X$ and $\vec{\sigma} = \sigma \cap R_X$. Obviously, $X \in \vec{\sigma}$. If $R_1, R_2 \in \vec{\sigma}$, then $R_1 \cap R_2 \neq \phi$ since $R_1\ c_{R_X}\ R_2$. Suppose $S \in R_X$ meets each $R \in \vec{\sigma}$. For any $A \in \sigma$, $S\ c_{R_X}\ A$ because each set $R \in R_X$ containing A is a member of σ. Thus $S \in \vec{\sigma}$ by 3.18(3) and hence $\vec{\sigma} \in X^+$ by definition 3.16. Thus $\to : {}^+X \to X^+$ is a function.

Next, we show there is a function $\leftarrow : X^+ \to {}^+X$ defined by $\overleftarrow{\sigma} = \{A \subseteq X : A\ c_{R_X}\ R$ for each $R \in \sigma\}$. Obviously, $X \in \overleftarrow{\sigma}$. Next suppose $A, B \in \overleftarrow{\sigma}$ and $R_a, R_b \in R_X$ with $A \subseteq R_a$ and $B \subseteq R_b$. Since A is bicontiguous with every $R \in \sigma$, R_a meets every $R \in \sigma$. Likewise R_b meets every $R \in \sigma$. Hence R_a and R_b each belong to σ and thus meet. Consequently, $A\ c_{R_X}\ B$; condition 3.18(2) is satisfied. Since $\sigma \subseteq \overleftarrow{\sigma}$, if a set D is bicontiguous with all members of $\overleftarrow{\sigma}$, then $D \in \overleftarrow{\sigma}$. Thus $\overleftarrow{\sigma} \in {}^+X$ and $\leftarrow : X^+ \to {}^+X$ is a function.

In fact, the function $\leftarrow$ is the inverse of $\rightarrow$. For if

$\sigma \in \,^+X$, then $\sigma = \overset{\leftarrow}{\overset{\rightarrow}{\sigma}}$ by 3.18 since $\sigma \subseteq \overset{\leftarrow}{\overset{\rightarrow}{\sigma}}$ and if $\sigma \in X^+$, then

$\sigma = \overset{\rightarrow}{\overset{\leftarrow}{\sigma}}$ by 3.16 since $\sigma \subseteq \overset{\rightarrow}{\overset{\leftarrow}{\sigma}}$. For any $R \in R_X$, consider $\rightarrow(\,^+R)$. If

$\sigma \in \,^+R$, then $R \in \sigma$, $R \in \overset{\rightarrow}{\sigma}$, and finally $\overset{\rightarrow}{\sigma} \in R^+$. Similarly, if

$\sigma \in R^+$, then $R \in \sigma$, $R \in \overset{\leftarrow}{\sigma}$, and so $\overset{\leftarrow}{\sigma} \in \,^+R$. Thus if $\sigma \in R^+$,

then $\sigma = \overset{\rightarrow}{\overset{\leftarrow}{\sigma}} \in \,^{\rightarrow}(\,^+R)$ and consequently $\,^{\rightarrow}(\,^+R) = R^+$. Likewise if

$\sigma \in \,^+R$, then $\sigma = \overset{\leftarrow}{\overset{\rightarrow}{\sigma}} \in \,^{\leftarrow}(R^+)$ and consequently $\,^{\leftarrow}(R^+) = \,^+R$. Now one

can easily deduce the remainder of Theorem 3.21.

Remark 3.23   From the previous results, we conclude that the construc-

tions $(\bar{\lambda}_{R_X}, (X^+, R_X{}^+))$ and $(\lambda_{R_X}, (\,^+X, \,^+R_X))$ are essentially the same for

our purposes. We call either construction the <u>superextension</u> of the

T_1-subbase space (X, R_X) . In addition, we say the morphisms $\bar{\lambda}_{R_X}$ and λ_{R_X}

are <u>superextension maps</u>. Our construction $(\lambda_{R_X}, (\,^+X, \,^+R_X))$, which is

similar to the Wallman-type compactification investigated by Bentley and

Naimpally [1], has been introduced, as in [1], to simplify certain aspects

of the presentation of a successful fur strategy.

Note that a superextension $(\lambda_{R_X}, (\,^+X, \,^+R_X))$ is not, in general,

a compactification of (X, R_X) since $\lambda_{R_X}(X)$ is seldom dense in $\,^+X$.

As an example, Verbeek [8, p. 47] has shown that, for any topological

space (X, τ) , which is a Hausdorff space containing at least three

elements, if we consider $(X, \tau) \in T_1\text{-SUBBASE}$, then the set $\lambda_\tau(X)$ is

not dense in $(\,^+X, \,^+\tau)$.

Example 3.24   Let the set $X = \{a,b\} \cup [0,1]$ be endowed with a subbase

R_X consisting of ϕ , X , and $\{a\} \cup [0,x]$, $\{a\} \cup [x,1]$, $\{b\} \cup [0,x]$,

and $\{b\} \cup [x,1]$ where $x \in (0,1)$.   It is easy to see R_X is a

T_1-subbase for   X .

Let $Y = [0,1] \times [0,1]$ be endowed with a T_1-subbase S_Y

consisting of ϕ , Y , and $[0,x] \times [0,1]$, $[x,1] \times [0,1]$, $[0,1] \times [0,x]$,

and $[0,1] \times [x,1]$ where $x \in (0,1)$.

Cszázár [2, p. 64] has shown there exists a   SUPERCOMP T_1-SUBBASE

- isomorphism $v : (X^+, R_X^+) \rightarrow (Y, S_Y)$ such that

$$v\lambda_{R_X}(x) = \begin{cases} (x,x) & \text{if} \quad x \quad [0,1] \\ (1,0) & \text{if} \quad x = a \\ (0,1) & \text{if} \quad x = b \end{cases}$$
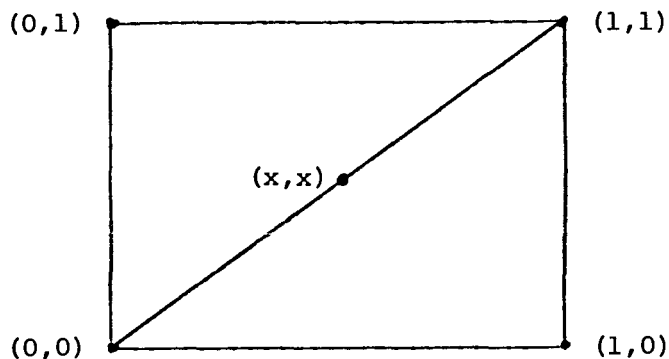
Note that S_Y is a subbase for the usual topology on   Y .

Clearly, there are an infinite number of continuous functions from

(Y, S_Y) to (Y, S_Y) that are invariant on the set $v\lambda_{R_X}(X)$.   We can

conclude $id_X : (X, R_X) \rightarrow (X, R_X)$ is a morphism in T_1-SUBBASE which does

not have a unique corresponding morphism $g : (X^+, R_X^+) \rightarrow (X^+, R_X^+)$ in

T_1-SUBBASE such that $\lambda_{R_X} id_X = g\lambda_{R_X}$.

Considerations of this example influenced the various fur

strategies which were tried. In the next chapter, we discuss the most

suitable successful fur strategy that was found.

CHAPTER IV

THE SURE CATEGORY

In this chapter, a fur subcategory of T_1-SUBBASE is defined and shown to be a successful fur strategy with respect to the superextension construction.

Definition 4.1   For T_1-subbase spaces (X,R_X) and (Y,S_Y) , let $f : (X,R_X) \to (Y,S_Y)$ be a function. We will call f a _sure_ map if and only if $f^-[S_Y] \subseteq R_X$ and, for any sets $A \subseteq X$ and $S \in S_Y$, $f(A) \ c_{S_Y} S$ implies $A \ c_{R_X} f^-(S)$.

Proposition 4.2   Every identity map in T_1-SUBBASE is a sure map. The composition of sure maps is a sure map. Each sure map is continous.

The proof of the first statement is trivial.

Let $f : (X,R_X) \to (Y,S_Y)$ and $g : (Y,S_Y) \to (Z,T_Z)$ be sure maps. Clearly, $(gf)^-[T_Z] \subseteq R_X$ since $g^-[T_Z] \subseteq S_Y$ and $f^-[S_Y] \subseteq R_X$. Suppose $gf(A) \ c_{T_Z} T$ for sets $A \subseteq X$ and $T \in T_Z$. Since g is a sure map, $f(A) \ c_{S_Y} g^-(T)$. Further, since f is a sure map and $g^-(T) \in S_Y$, $A \ c_{R_X} f^-(g^-(T))$. Since $f^-(g^-(T)) = (gf)^-(T)$, this shows that gf is a sure map.

Quite obviously, each sure map is continuous. For, suppose $f : (X,R_X) \to (Y,S_Y)$ is a sure map and S is a closed set in Y . Since S_Y is a subbase for Y , there exists a subcollection $\{S_{jn}\}_{j \in J \ n \in N_j}$ of S_Y , where each N_j is a finite index set dependent on j , such that $S = \bigcup_{j \in J} \bigcap_{n \in N_j} S_{jn}$. Thus, $f^-(S) = \bigcap_{j \in J} \bigcup_{n \in N_j} f^-(S_{jn})$ is a closed set in space X as each $f^-(S_{jn}) \in R_X$.

Definition 4.3 Let <u>SURE</u> be the fur subcategory of T_1-SUBBASE which has as its functions the sure maps and let <u>SUPERCOMP SURE</u> be the full subcategory of <u>SURE</u> which has as its spaces the supercompact T_1-subbase spaces.

Theorem 4.4 <u>SUPERCOMP SURE</u> is an epireflective subcategory of <u>SURE</u>.

Suppose $f : (X, R_X) \to (Y, S_Y)$ is a sure function and (T, T_Z) is a T_1-subbase space. Define ^+f so that, for each R_X-bicontiguity cluster σ , $^+f(\sigma) = \{C : \text{for each } D \in \sigma , C \ c_{S_Y} f(D)\}$. Then, our proof proceeds as follows: first, it is shown that the superextension map λ_{T_Z} is a sure map and that, if (T, T_Z) is a supercompact T_1-subbase space, then λ_{T_Z} is a <u>SURE</u> - isomorphism; next, it is proven that $^+f : {}^+X \to {}^+Y$ is a set map such that $^+f \ \lambda_{R_X} = \lambda_{S_Y} f$; then, it is demonstrated that $^+f : ({}^+X, {}^+R_X) \to ({}^+Y, {}^+S_Y)$ is a sure map; and finally, it is shown that λ_{T_Z} is a <u>SURE</u> - epimorphism. By the definition of epireflective subcategory, completing these steps finishes the proof.

For each T_1-subbase space (Z, T_Z) , recall $^+T_Z = \{^+T : T \in T_Z\}$, $\lambda_{T_Z}^- (^+T) = T$ for each $^+T \in {}^+T_Z$, and thus $\lambda_{T_Z}^- [^+T_Z] = T_Z$. For sets $A \subseteq Z$ and $^+T \in {}^+T_Z$, suppose $A \not\subset_{T_Z} \lambda_{T_Z}^- (^+T)$ or equivalently $A \not\subset_{T_Z} T$. Hence, there is a set $T_a \in T_Z$ such that $A \subseteq T_a$ and $T_a \cap T = \phi$. Recall this implies $^+T_a \cap {}^+T = \phi$. Since $\lambda_{T_Z}(A) \subseteq \lambda_{T_Z}(T_a) \subseteq {}^+T_a$, then $\lambda_{T_Z}(A) \not\subset_{+T_Z} {}^+T$ and so λ_{T_Z} is a sure map.

Now, assume (Z, T_Z) is a supercompact T_1-subbase space. It is already known λ_{T_Z} is a $\underline{T_1\text{-SUBBASE}}$ - isomorphism, $\lambda_{T_Z}(T) = {}^+T$ for

each $T \in T_Z$, and $\lambda_{T_Z}[T_Z] = {}^+T_Z$. For $B \subseteq {}^+Z$ and $T \in T_Z$, suppose

$B \not\subset_{+T_Z} (\lambda_{T_Z}{}^-)^-(T)$ or equivalently $B \not\subset_{+T_Z} {}^+T$. Thus, there exists a

set ${}^+T_b \in T_Z$ such that $B \subseteq {}^\bullet T_b$ and ${}^+T_b \cap {}^+T = \phi$. Since

${}^+T_b \cap {}^+T = \phi$ implies $T_b \cap T = \phi$, then $\lambda_{T_Z}{}^-(B) \subseteq \lambda_{T_Z}{}^-({}^+T_b) = T_b$

implies $\lambda_{T_Z}{}^-(B) \not\subset_{T_Z} T$. In other words, $\lambda_{T_Z}{}^-$ is sure map and so

λ_{T_Z} is a <u>SURE</u> - isomorphism.

Note that if $A, B \subseteq Y$, and $A \not\subset_{S_Y} B$, then $f^-(A) \not\subset_{R_X} f^-(B)$.

For suppose that $A \subseteq S_a \in S_Y$, $B \subseteq S_b \in S_Y$, and $S_a \cap S_b = \phi$.

Then $f^-(A) \subseteq f^-(S_a)$, $f^-(B) \subseteq f^-(S_b)$, and $f^-(S_a) \cap f^-(S_b) = \phi$. Since

$f^-[S_Y] \subseteq R_X$, then $f^-(S_a) \not\subset_{R_X} f^-(S_b)$ and hence $f^-(A) \not\subset_{R_X} f^-(B)$.

In particular, if $C, D \subseteq X$ and $f(C) \not\subset_{S_Y} f(D)$ then $C \not\subset_{R_X} D$;

for $C \subseteq f^- f(C)$, $D \subseteq f^- f(D)$, and $f^- f(C) \not\subset_{R_X} f^- f(D)$.

For each R_X-bicontiguity cluster σ , define $f\sigma = \{f(C) : C \in \sigma\}$.

We can conclude $f\sigma \subseteq {}^+f(\sigma)$ since, for any sets $C, D \in \sigma$, we know

$C \ c_{R_X} \ D$ and hence $f(C) \ c_{S_Y} \ f(D)$.

To show ${}^+f : {}^+X \to {}^+Y$ is a set map, it is necessary to prove

that, for any R_X-bicontiguity cluster σ , the set ${}^+f(\sigma)$ is an

S_Y-bicontiguity cluster. Suppose the set $A \subseteq Y$ is S_Y-bicontiguous with

each member of ${}^+f(\sigma)$. Then, since $f\sigma \subseteq {}^+f(\sigma)$, A is S_Y-bicontiguous

with each member of $f\sigma$ and, by the definition of ${}^+f(\sigma)$, $A \in {}^+f(\sigma)$.

Next, suppose $A, B \in {}^+f(\sigma)$. For any sets $S_a, S_b \in S_Y$ such that $A \subseteq S_a$

and $A \subseteq S_b$, clearly $S_a, S_b \in {}^+f(\sigma)$. If we assume $f^-(S_a), f^-(S_b) \in \sigma$,

then $f^-(S_a) \ c_{R_X} \ f^-(S_b)$ and hence $S_a \ c_{S_Y} \ S_b$. In other words,

$A, B \in {}^+f(\sigma)$ would imply $A \ c_{S_Y} \ B$. Consequently, ${}^+f(\sigma)$ would be an

an S_Y-bicontiguity cluster. Thus, we must demonstrate that, for any

set $S \in S_Y$, $S \in {}^+f(\sigma)$ implies $f^-(S) \in \sigma$. Given $S \in {}^+f(\sigma)$, then

$S \ c_{S_Y} \ f(C)$, for each $C \in \sigma$. Because f is a sure map, $f^-(S) \ c_{R_X} \ C$.

Thus, $f^-(S) \in \sigma$ since σ is an R_X-bicontiguity cluster.

For each $x \in X$, $\lambda_{R_X}(x) = \{A \subseteq X : \{x\} \ c_{R_X} A\}$ and

$\lambda_{S_Y}(f(x)) = \{B \subseteq Y : \{f(x)\}c_{S_Y} B\}$. Since $\{x\} \in \lambda_{R_X}(x)$, then

$\{f(x)\} \in f \lambda_{R_X}(x) \subseteq {}^+f(\lambda_{R_X}(x))$. Because the only S_Y-bicontiguity cluster

containing $\{f(x)\}$ is $\lambda_{S_Y}(f(x))$, clearly ${}^+f(\lambda_{R_X}(x)) = \lambda_{S_Y}(f(x))$.

Therefore, ${}^+f \ \lambda_{R_X} = \lambda_{S_Y} f$.

To prove that ${}^+f : ({}^+X, {}^+R_X) \to ({}^+Y, {}^+S_Y)$ is a sure map suppose,

for the moment, that we know ${}^+f({}^+B) \subseteq {}^+(f(B))$ for each $B \subseteq X$ and

${}^+(f^-(S)) = ({}^+f)^-({}^+S)$ for each $S \in S_Y$. Since $f^-(S) \in R_X$ and

${}^+(f^-(S)) = ({}^+f)^-({}^+S)$ for each $S \in S_Y$, obviously $({}^+f)^-[{}^+S_Y] \subseteq {}^+R_X$.

Now, assume $A \subseteq {}^+X$ and ${}^+S \in {}^+S_Y$ so that $A \not\!c_{{}^+R_X} ({}^+f)^-({}^+S)$ or

equivalently $A \not\!c_{{}^+R_X} {}^+(f^-(S))$. Thus, there exists a set $R \in R_X$ such

that $A \subseteq {}^+R$ and ${}^+R \cap {}^+(f^-(S)) = \phi$. Hence, $R \cap f^-(S) = \phi$ and so

$R \not\!c_{R_X} f^-(S)$. Since f is a sure map, then $f(R) \not\!c_{S_Y} S$. In other

words, there is a set $S_r \in S_Y$ such that $f(R) \subseteq S_r$ and $S_r \cap S = \phi$.

Since ${}^+S_r \cap {}^+S = \phi$, ${}^+(f(R)) \subseteq {}^+S_r$ would imply ${}^+(f(R)) \not\!c_{{}^+S_Y} {}^+S$. But,

for any S_Y-bicontiguity cluster σ , if $f(R) \in \sigma$ then $S_r \in \sigma$. Hence,

${}^+(f(R)) = \{\sigma \in {}^+Y : f(R) \in \sigma\} \subseteq \{\sigma \in {}^+Y : S_r \in \sigma\} = {}^+S_r$. By supposition

${}^+f({}^+R) \subseteq {}^+(f(R))$. We conclude ${}^+f(A) \not\!c_{S_Y} {}^+S$ since ${}^+f(A) \subseteq {}^+f({}^+R)$.

For any set $B \subseteq X$ and R_X-bicontiguity cluster σ, recall $\sigma \in {}^+B$ implies $B \in \sigma$. But if $B \in \sigma$, $f(B) \in f\sigma \subseteq {}^+f(\sigma)$. This means ${}^+f(\sigma) \in {}^+(f(B))$ and consequently ${}^+f({}^+B) \subseteq {}^+(f(B))$.

Consider $S \in S_Y$. Notice ${}^+(f^-(S)) \subseteq ({}^+f)^-({}^+S)$ if and only if ${}^+f({}^+(f^-(S)) \subseteq {}^+S$. But, since $f(f^-(S)) \subseteq S$, then ${}^+f({}^+(f^-(S))) \subseteq {}^+(f(f^-(S))) \subseteq {}^+S$. On the other hand, suppose for some S_Y-bicontiguity cluster $\sigma \in {}^+S$, there is an R_X-bicontiguity cluster δ such that ${}^+f(\delta) = \sigma$. Then, it has been demonstrated above that $f^-(S) \in \delta$ since $S \in \sigma \cap S_Y$ and f is a sure map. Since $\delta \in {}^+(f^-(S))$, we have shown ${}^+(f^-(S)) = ({}^+f)^-({}^+S)$. In conclusion, if f is a sure map, then ${}^+f$ is a sure map also.

The proof that each superextension map is a <u>SURE</u> - epimorphism requires two stages. In the first, we demonstrate, if $f : (X, R_X) \rightarrow (Y, S_Y)$ and $g : ({}^+X, {}^+R_X) \rightarrow ({}^+Y, {}^+S_Y)$ are sure maps so that $g \lambda_{R_X} = \lambda_{S_Y} f$, then $g = {}^+f$.

Assuming $S \in S_Y$, since $g^-({}^+S) \in {}^+R_X$ and $g \lambda_{R_X} = \lambda_{S_Y} f$, then $f^- \lambda_{S_Y}{}^-({}^+S) = \lambda_{R_X}{}^- g^-({}^+S) \in R_X$. Recalling that $\lambda_{S_Y}{}^-({}^+S) = S$ and ${}^+(\lambda_{R_X}{}^-({}^+R)) = {}^+R$ for each $R \in R_X$, clearly ${}^+(f^-(S)) = g^-({}^+S)$. Now, if σ is an R_X-bicontiguity cluster such that $S \in g(\sigma)$, then $g(\sigma) \in {}^+S$. This implies $\sigma \in g^-({}^+S) = {}^+(f^-(S))$ and so $f^-(S) \in \sigma$. Since $f^-(S) \in R_X$, $f^-(S) \in \sigma \cap R_X$. In summary, that $S \in g(\sigma) \cap S_Y$ implies $f^-(S) \in \sigma \cap R_X$ requires only the assumptions $g \lambda_{R_X} = \lambda_{S_Y} f$ and $g^-[{}^+S_Y] \subseteq {}^+R_X$.

Suppose $C \in \sigma$ and $S \in g(\sigma) \cap S_Y$. Since $f^-(S) \in \sigma \cap R_X$, clearly $C \ c_{R_X} \ f^-(S)$. Recalling $f\sigma \subseteq {}^+f(\sigma)$, obviously $f(C) \ c_{S_Y} \ f(f^-(S))$ and hence $f(C) \ c_{S_Y} \ S$. Since $f(C)$ is S_Y-bincontiguous with each set $S \in g(\sigma) \cap S_Y$, we can conclude $f(C) \in g(\sigma)$. In other words, $f\sigma \subseteq g(\sigma)$. But, this means $g(\sigma) \subseteq {}^+f(\sigma)$, since, if $A \in g(\sigma)$, then A is S_Y-bicontiguous with each set $D \in f\sigma$. Because $g(\sigma)$ and ${}^+f(\sigma)$ are S_Y-bicontiguity clusters, $g(\sigma) = {}^+f(\sigma)$. In summary, the sure map ${}^+f$ is the unique sure map g for which $g \lambda_{R_X} = \lambda_S f$.

The second part of the proof that each superextension map is a <u>SURE</u> - epimorphism follows by a strictly categorical argument from the result just proven. Let $\lambda_{R_X} : (X, R_X) \to ({}^+X, {}^+R_X)$ be a superextension map and $f, g : ({}^+X, {}^+R_X) \to (Y, S_Y)$ be sure maps so that $f \lambda_{R_X} = g \lambda_{R_X}$.
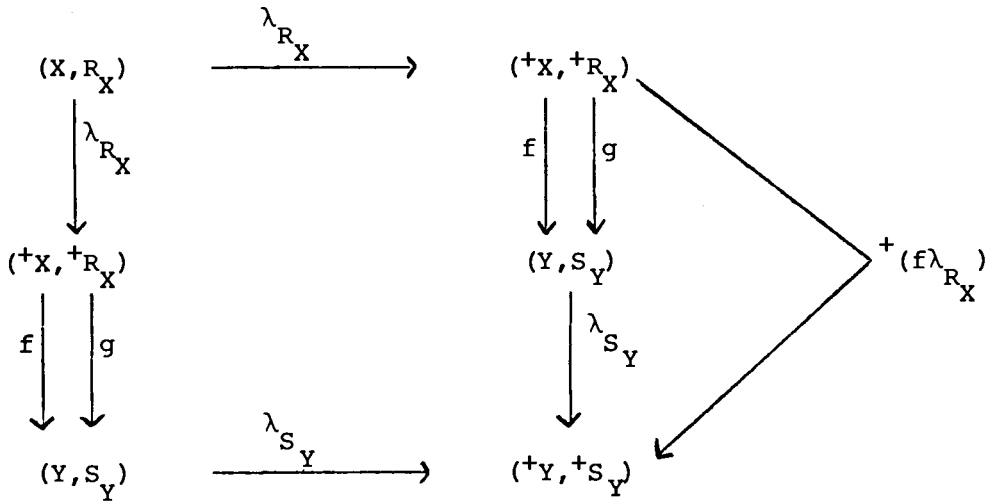
Since $f \lambda_{R_X} = g \lambda R_X$ is a sure map, we now know that ${}^+(f \lambda_{R_X})$ is the unique sure map such that $\lambda_{S_Y}(f \lambda_{R_X}) = {}^+(f \lambda_{R_X}) \lambda_{R_X} = \lambda_{S_Y}(g \lambda_{R_X})$. But obviously $\lambda_{S_Y} f$ and $\lambda_{S_Y} g$ are also sure maps such that

$$(\lambda_{S_Y} f) \lambda_{R_X} = \lambda_{S_Y}(f \lambda_{R_X}) \quad \text{and} \quad (\lambda_{S_Y} g) \lambda_{R_X} = \lambda_{S_Y}(g \lambda_{R_X}). \quad \text{Therefore,}$$

$\lambda_{S_Y} f = {}^+(f\lambda_{R_X}) = \lambda_{S_Y} g$. Since the superextension map λ_{S_Y} is one-to-one,

it is easy to see $f = g$. Thus, for each space (X, R_X) , the map λ_{R_X}

is a <u>SURE</u> - epimorphism.

It should be noted the previous categorical result is subsumed

by the following simple proposition [6,p.276]: If <u>S</u> is a full mono-

reflective subcategory of <u>C</u> , then <u>S</u> is an epireflective subcategory

of <u>C</u> .

<u>Remark 4.5</u>   The proof of 3.4 can, of course, be argued in terms of

ultrasieves rather than bicontiguity clusters. For example, we would

call $f : (X, R_X) \to (Y, S_Y)$ a "sure map" whenever $f^-[S_Y] \subseteq R_X$ and,

for sets $R \in R_X$ and $S \in S_Y$ such that $R \cap f^-(S) = \phi$, there exists

a set $S_r \in S_Y$ such that $R \subseteq f^-(S_r)$ and $S_r \cap S = \phi$. Also, for each

R_X-ultrasieve σ , we would define the map $f^+ : (X^+, R_X^+) \to (Y^+, S_Y^+)$ by

$f^+(\sigma) = \{S \in S_Y : f^-(S) \in \sigma\}$. Further details may be verified by the

interested reader. As justification of our approach, we note the

natural formulation of the sure maps and the function ${}^+f$ and cite the

methods used by Bentley and Naimpally in [1].

<u>Proposition 4.6</u>   If $f : (X, R_X) \to (Y, S_Y)$ is a set map, $g : ({}^+X, {}^+R_X) \to ({}^+Y, {}^+S_Y)$

is a sure map, and $g\lambda_{R_X} = \lambda_{S_Y} f$, then f is a sure map and so $g = {}^+f$.

In the previous proof, we have shown that $g^-[{}^+S_Y] \subseteq {}^+R_X$ and

$g\lambda_{R_X} = \lambda_{S_Y} f$ imply $f^-[S_Y] \subseteq R_X$ and, for each $S \in S_Y$, $g^-({}^+S) = {}^+(f^-(S))$.

Assume $A \subseteq X$, $S \in S_Y$, and $A \not\subset_{R_X} f^-(S)$. Then, $\lambda_{R_X}(A) \not\subset_{+R_X} {}^+(f^-(S))$

or equivalently $\lambda_{R_X}(A) \not\subset_{+R_X} g^-({}^+S)$. Since g is a sure map,

$g\lambda_{R_X}(A) \not\subset_{+S_Y} {}^+S$ and so $\lambda_{S_Y} f(A) \not\subset_{+S_Y} {}^+S$. Finally, we have $f(A) \not\subset_{S_Y} S$.

Thus, f is a sure map. Since f is a sure map, we know from 3.4 that $g = {}^+f$.

Proposition 4.7   Let $f : (X, R_X) \to (Y, S_Y)$ be a sure map. The map f is a SURE - epimorphism if and only if ${}^+f$ is a SURE - empimorphism. If f is a SURE - isomorphism, then ${}^+f$ is a SURE - isomorphism. Also, if f is a surjection, then ${}^+f$ is a surjection.

Suppose f is a SURE - epimorphism and $g, h : ({}^+Y, {}^+S_Y) \to (Z, T_Z)$ are sure maps such that $g\,{}^+f = h\,{}^+f$. Then, $g\lambda_{S_Y} f = g\,{}^+f\lambda_{R_X} =$ $= h\,{}^+f\lambda_{R_X} = h\lambda_{S_Y} f$. Since f and λ_{S_Y} are SURE - epimorphisms, $g = h$. Thus, ${}^+f$ is also a SURE - epimorphism.

Next, suppose ${}^+f$ is a SURE - epimorphism and $g, h : (Y, S_Y) \to (Z, T_Z)$ are sure maps such that $gf = hf$. Since f , g, and h are sure maps, then ${}^+f$, ${}^+g$, and ${}^+h$ are sure maps such that ${}^+f\lambda_{R_X} = \lambda_{S_Y} f$, ${}^+g\lambda_{S_Y} = \lambda_{T_Z} g$, and ${}^+h\lambda_{S_Y} = \lambda_{T_Z} h$. Thus, ${}^+g\,{}^+f$ and ${}^+h\,{}^+f$ are sure maps such that ${}^+g\,{}^+f\,\lambda_{R_X} = \lambda_{T_Z} gf = \lambda_{T_Z} hf = {}^+h\,{}^+f\,\lambda_{R_X}$. Since λ_{R_X} and ${}^+f$ are SURE - epimorphisms, then ${}^+g = {}^+h$. This means $\lambda_{T_Z} g = \lambda_{T_Z} h$ and so $g = h$. Thus, f is a SURE - epimorphism.

Notice that the preceding proof uses a strictly categorical argument. Likewise, via a trivial categorical proof, it can be shown that ${}^+f$ is a SURE - isomorphism if f is.

Finally, assume f is a surjection and $\sigma \in {}^+Y$. Consider $d = \{f^-(S) : S \in \sigma \cap S_X\}$. Since each pair of members from d has a nonempty intersection, clearly d is contained in some R_X-bicontiguity cluster, say δ. For each $D \in \delta$ and $S \in \sigma$, recall $D\, c_{R_X}\, f^-(S)$ implies

$f(D) \subset_{S_Y} S$ since $f(f^-(S)) \subseteq S$. Hence, $f(D) \in \sigma$ or equivalently

$f\delta \subseteq \sigma$. We can conclude $^+f(\delta) = \sigma$ since $^+f(\delta)$ and σ are

S_Y-bicontiguity clusters.

JENSEN'S RESULT

In this chapter, we present a mapping result due to G. A. Jensen [8, p. 56]. The situation Jensen describes was not motivated by nor is it particularly compatible with the notion of epireflective subcategories.

<u>Definition 5.1</u>   [8, p. 51]   A T_1-subbase R_X for space X is <u>normal</u> if, for any disjoint sets $R_1, R_2 \in R_X$, there exist sets $R_1^C, R_2^C \in R_X$ such that $R_1 \cap R_1^C = \phi$, $R_2 \cap R_2^C = \phi$, and $R_1^C \cup R_2^C = X$. A T_1-subbase space (X, R_X) is called <u>normal</u> if R_X is normal.

<u>Proposition 5.2</u>   [8, p. 52]   If R_X is a normal T_1-subbase for X , then R_X generates a Hausdorff topology.

Let x_1 and x_2 be different points in X . Since R_X is a T_1-subbase, we know there exist sets $R_1, R_2 \in R_X$ such that $x_1 \in R_1$, $x_2 \in R_2$, and $R_1 \cap R_2 = \phi$. Thus, there exist sets $R_1^C, R_2^C \in R_X$ such that $R_1^C \cup R_2^C = X$, $R_1 \cap R_1^C = \phi$, and $R_2 \cap R_2^C = \phi$. Clearly, $X - R^C$ and $X - R_2^C$ are open sets such that $(X - R_1^C) \cap (X - R_2^C) = \phi$, $x_1 \in X - R_1^C$, and $x_2 \in X - R_2^C$.

<u>Lemma 5.3</u>   [8, p. 13]   Suppose $(X, R_X) \in T_1\text{-SUBBASE}$ and $S, T \in R_X$. Then, $S \cup T = X$ if and only if $S^+ \cup T^+ = X^+$.

Assume $x \in X$ and $S^+ \cup T^+ = X^+$. Obviously, $\lambda_{R_X}(x) \in X^+$ and so, without loss of generality, $\lambda_{R_X}(x) \in S^+$. Hence, $S \in \lambda_{R_X}(x)$ and $x \in S$. Thus, $S \cup T = X$.

Next, suppose $S \cup T = X$ and $\sigma \in X^+$. If $\sigma \notin S^+ \cup T^+$, then $S, T \nleq \sigma$. Hence, there exist $R_S, R_T \in \sigma$ such that $R_S \cap S = R_T \cap T = \phi$. Since $R_S \cap R_T \subseteq$ complement $S \cap$ complement $T = \phi$, we have a contradiction. Thus, $S \cup T = X$ implies $S^+ \cup T^+ = X^+$.

<u>Theorem 5.4</u> [8, p. 56] Let (X, R_X) be a T_1-subbase space and (Y, S_Y) be a normal T_1-subbase space. If $f : (X, R_X) \to (Y, S_Y)$ is a map for which $f^-[S_Y] \subseteq R_X$, then there exists a continuous closed map $g : (X^+, R_X{}^+) \to (Y^+, S_Y{}^+)$ such that $g \lambda_{R_X} = \lambda_{S_Y} f$.

For each R_X-ultrasieve σ , consider the sets
$f^\sigma = \{S \in S_Y : f^-(S) \in \sigma\}$ and $\bar{f}(\sigma) = \{S \in S_Y :$ for each $S_\sigma \in f^\sigma$, $S \cap S_\sigma \neq \phi\}$. Clearly, f^σ is nonempty since at least $Y \in f^\sigma$. Also, if $S_a, S_b \in f^\sigma$, then $S_a \cap S_b \neq \phi$ since $f^-(S_a) \cap f^-(S_b) \neq \phi$. Thus, $f^\sigma \subseteq \bar{f}(\sigma)$. Consequently, if a set $S \in S_Y$ meets each member of $\bar{f}(\sigma)$, then $S \in \bar{f}(\sigma)$.

Now consider sets $S_1, S_2 \in \bar{f}(\sigma)$. Let us suppose $S_1 \cap S_2 = \phi$. Since S_Y is normal, we have sets $S_1^c, S_2^c \in S_Y$ such that $S_1^c \cap S_1 = \phi$, $S_2^c \cap S_2 = \phi$, and $S_1^c \cup S_2^c = Y$. Thus, $f^-(S_1^c) \cup f^-(S_2^c) = X$. Recall this and $f^-(S_1^c)$, $f^-(S_2^c) \in R_X$ imply that either $f^-(S_1^c) \in \sigma$ or $f^-(S_2^c) \in \sigma$. But, if $f^-(S_1^c) \in \sigma$, then $S_1^c \in f^\sigma$ and so $S_1 \cap S_1^c \neq \phi$. Similarly, if $f^-(S_2^c) \in \sigma$, then $S_2 \cap S_2^c \neq \phi$. The contradiction is resolved only if $S_1 \cap S_2 \neq \phi$. Thus $f(\sigma)$ is an S_Y-ultrasieve and $\bar{f} : X^+ \to Y^+$ is a set map.

Let $z \in X$. Recall $\lambda_{R_X}(z) = \{R : z \in R \in R_X\}$ and $\lambda_{S_Y}(f(z)) = \{S : f(z) \in S \in S_Y\}$. But, if $S \in \lambda_{S_Y}(f(z))$, then $f^-(S) \in \lambda_{R_X}(z)$. Thus, since $\lambda_{S_Y}(f(z)) \subseteq f^{\lambda_{R_X}(z)}$, clearly $\lambda_{S_Y}(f(z)) = \bar{f}(\lambda_{R_X}(z))$.

To prove $\bar{f}$ is a continuous map, it is sufficient to show $\bar{f}^-(S_1^+)$ is a closed set in space X for any $S_1 \in S_Y$. Suppose $\sigma \in X^+$ and $\sigma \notin \bar{f}^-(S_1^+)$. Then $\bar{f}(\sigma) \notin S_1^+$. This implies there is a set $S_2 \in \bar{f}(\sigma)$ such that $S_1 \cap S_2 = \phi$. Since (Y, S_Y) is normal, there exist sets $S_1^c, S_2^c \in S_Y$ such that $S_1^c \cap S = \phi$, $S_2^c \cap S_2 = \phi$, and $S_1^c \cup S_2^c = Y$. We claim the set $(f^-(S_2^c))^+$ has the properties that $\bar{f}^-(S_1^+) \subseteq (f^-(S_2^c))^+$ and $\sigma \notin (f^-(S_2^c))^+$. Since $f^-[S_Y] \subseteq R_X$, this would imply $f^-(S_1^+)$ is a closed set in X.

First, note since $S_2 \in \bar{f}(\sigma)$ and $S_2^c \cap S_2 = \phi$, then $S_2^c \notin \bar{f}(\sigma)$. In particular, $f^-(S_2^c) \notin \sigma$ and hence $\sigma \notin (f^-(S_2^c))^+$.

Next, $S_1^c \cap S_1 = \phi$ implies $(S_1^c)^+ \cap (S_1)^+ = \phi$ and $\bar{f}^-((S_1^c)^+) \cap \bar{f}^-(S_1^+) = \phi$. Then, $(f^-(S_1^c))^+ \cap \bar{f}^-(S_1^+) = \phi$ if we can show $(f^-(S_1^c))^+ \subseteq \bar{f}^-((S_1^c)^+)$. But, $\delta \in (f^-(S_1^c))^+$ iff $f^-(S_1^c) \in \delta$ only if $S_1^c \in \bar{f}(\delta)$ iff $\bar{f}(\delta) \in (S_1^c)^+$ iff $\delta \in \bar{f}^-(S_1^c)^+$. Moreover, since $S_1^c \cup S_2^c = Y$, then $f^-(S_1^c) \cup f^-(S_2^c) = X$ and hence $(f^-(S_1^c))^+ \cup (f^-(S_2^c))^+ = X^+$. Consequently, $\bar{f}^-(S_1^+) \subseteq (f^-(S_2^c))^+$.

Recall, for any $S_a, S_b \in S_Y$, that $S_a \subseteq S_b$ implies $S_a^+ \subseteq S_b^+$, $S_a \cap S_b = \phi$ implies $S_a^+ \cap S_b^+ = \phi$, and $S_a \cup S_b = Y$ implies $S_a^+ \cup S_b^+ = Y^+$. Hence, since (Y, S_Y) is normal, clearly (Y^+, S_Y^+) is normal. Thus, $\bar{f} : (X^+, R_X^+) \rightarrow (Y^+, S_Y^+)$, being a continuous map from a compact space into a Hausdorff space, is a closed map [9, p. 123].

Letting $g = \bar{f}$, the theorem is proven.

BIBLIOGRAPHY

[1]   Bentley, H.L. and Naimpally, S.A., "Wallman T_1-compactifications
        as epireflections", General Topology and Its Applications,
        vol. 4, 1974, pp. 29 - 41.

[2]   Császár, Á., "Wallman-type compactifications and superextensions",
        Periodico Mathematica Hungarica, vol. 1, 1971, pp. 55 - 80.

[3]   deGroot, J., "Supercompactness and superextensions", Contributions
        to extension theory of topological structures: Symposium
        Berlin 1967, Deutscher Verlag der Wissenschaften, Berlin,
        1967, pp. 89 - 90.

[4]   Harris, D., "The Wallman compactification as a functor", General
        Topology and Its Applications, vol. 1, 1971, pp. 273 - 281.

[5]   Harris, D., "The Wallman compactification as an epireflection",
        Proceedings of the American Mathematical Society, vol. 31,
        No. 1, 1972, pp. 263 - 267.

[6]   Herrlich, H. and Strecker, G.E., Category Theory, Allyn and Bacon,
        Inc., Boston, 1973.

[7]   MacLane, S., Categories for the Working Mathematician, Graduate
        Texts in Mathematics No. 5, Springer-Verlag, Berlin,
        Heidelberg, and New York, 1971.

[8]   Verbeek, A., Superextensions of Topological Spaces, Mathematical
        Centre Tracts, No. 41, Mathematisch Centru, Amsterdam, 1972.

[9]   Willard, S., General Topology, Addison Wesley Publishing Company,
        London and Don Mills, 1970.