FIRST ORDER THEORY OF THE ELASTIC DIELECTRIC

by

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ABSTRACT

The first part of this thesis presents a development of the theory of a continuous elastic dielectric. From the most general constitutive equations governing homogeneous, isotropic, elastic dielectric, the first order approximation is formulated.

In the second part, the first order theory so developed is applied to obtain solutions to four boundary value problems. In these problems, the finite deformation and the electric field are prescribed, and it is shown that the deformation can be supported without the mechanical body force and the charge distribution in every homogeneous, isotropic, incompressible elastic dielectric. In the end, it is shown that only homogeneous deformations and uniform electric fields are admissible when the dielectric is compressible.

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INTRODUCTION

There exists on the one hand the macroscopic theory of elasticity, and on the other what may be called the macroscopic theory of electrostatics. In recent years much interest has centered on the coupling of these two systems. Mostly, the literature in this area considered boundary-value problems in which the strains were considered infinitesimal at the outset, and the electric field was then superposed.

The first paper to treat simultaneous electrification and change of shape as physical phenomena worthy of exact analysis derived from fundamental laws of mechanics came from Toupin [1]. In this paper, Toupin considers a continuous deformable elastic dielectric solid subject to the simultaneous application of mechanical forces and an electric field. The existence of stored energy function is postulated, and with the formulation of a virtual work principle, the constitutive equations for a finite deformation theory of an elastic dielectric are derived.

The theory governing finite deformations of elastic dielectrics has also been derived by Singh and Pipkin [2]. In this paper, the authors have described various families of controllable states - the states in which a prescribed deformation and electric field can be supported without body forces or surface tractions in every homogeneous, isotropic, elastic dielectric. These states do not impose any restriction on the form of the stored energy function. The usefulness of such controllable states is that the comparison of theoretical results with measured data can determine the form of the strain energy function just what it should be. This feature of these special class of deformations has been utilized by Rivlin and Saunders [3] in finite elasticity theory. However, the experimentation of similar nature for elastic dielectrics has not been attempted so far.

Singh [4] formulated the theory of small but finite deformations for homogeneous and isotropic dielectrics. In such an analysis, the stored energy is expressed as a polynomial of its arguments which is then terminated after retaining the terms in the expansion that are appropriate to the approximation desired. In finite elasticity, the approximations of this nature are Mooney-Rivlin materials and Neo-Hookean materials. The approximate theories of constitutive equations not only provide mathematical simplifications enabling one to treat more difficult boundary-value problems but also reduce and simplify the response coefficients to make the experimental work for their determination possible. From the practical point of view such limitations are often a necessity rather than an exception, because the range of deformation to which materials can be subjected in the elastic range is often limited.

In Sections 2 to 4 is defined the macroscopic model of an elastic dielectric solid continuum.

The basic assumptions, field equations, and constitutive equations for a theory governing homogeneous, isotropic, elastic dielectrics are presented. In Sections 5 to 9, various approximate theories that can be derived are discussed. Their formulation is systematic and is obtained from the general constitutive equations. The classical electrostriction theory comes as a special case of the first order finite approximation. The first order approximation formulated in Section 9 is analogous to Mooney-Rivlin [5] approximation in finite elasticity theory.

The development so far, in Sections 2 to 9, is not new. However, the applications of the first order approximation considered in Sections 10 through 15 are original contributions of this presentation. The boundary-value problems solved, by using the constitutive equations of the first order approximation for incompressible, homogeneous, isotropic, elastic dielectric, are the rotation of a cylindrical tube about its axis in the presence of a radial electric field, expansion of a spherical shell in a radial field, flexural deformation of a block in a radial field. Within the application of the first order approximation, the deformations are controllable in the sense that they can be supported in every homogeneous, isotropic, incompressible, elastic dielectric without body forces or distributed charge. When the form of the stored-energy function is arbitrary, the flexural deformation of the block in a radial field is not controllable [2]. It is shown in Section 13 that such a state is controllable when first order theory is used.

It is well recognized that in some problems, it is more convenient and useful to take the dielectric displacement field as the independent variable instead of the electric field. The constitutive equations are formally equivalent to those when electric field is the independent variable [2]. In Section 14, we have solved the boundary-value problem of the flexural deformation of a block in presence of a uniform axial dielectric displacement field. This state, which is not possible with arbitrary form of stored-energy function is controllable with first order approximate theory.

Singh [6] has shown that with arbitrary form of stored-energy function, the <u>only</u> controllable states in homogeneous, isotropic, <u>compressible</u> elastic dielectrics are homogeneous deformations combined with uniform elastic fields.

The constitutive equations derived from the stored-energy function of the first order approximation are much less restrictive. One would, therefore, expect that certain non-homogeneous deformations may combine with non-uniform fields to become controllable. However, we show in Section 15 that even in first order approximate theory, the only controllable states are homogeneous deformations superposed with uniform fields.

2. Continuum Electrostatics

We consider a deformable elastic dielectric continuum that occupies volume V bounded by surface ∂V. The body is deformed and polarized by applied mechanical forces and an applied electric field. We use Cartesian coordinates to describe the deformation; the particle at coordinate position $X_A (A = 1,2,3)$ in the initial state is deformed to coordinate position $x_i (i = 1,2,3)$ referred to fixed rectangular Cartesian system. In the present paper, we deal only with the quasi-static case where the deformation takes place so slowly that at any instant of time the external forces are in equilibrium with the mechanical and electrical forces inside the dielectric; that is, inertial forces are negligible.

According to the Maxwell-Faraday theory of the electrostatic field, there exist in space two vector fields, the macroscopic electric field $E_i (i = 1,2,3)$ and the macroscopic dielectric displacement field $D_i (i = 1,2,3)$. In free space these fields are assumed to satisfy the integral equations:

$$\int_c E_i dx_i = 0 \quad , \tag{2.1}$$

$$\iint_S D_i n_i dS = Q \quad , \tag{2.2}$$

where c is an arbitrary closed curve, S an arbitrary closed surface, Q the total electric charge enclosed by S, and $n_i (i = 1,2,3)$ is the unit exterior normal to S.

We shall assume that (2.1) and (2.2) are also valid inside the dielectric.

We further assume that on any volume u of the dielectric, the resultant force F_i and resultant moment G_i, excluding gravitational or inertial moments and forces, are statically equivalent to a stress field t_i acting on the surface S of the volume. That is,

$$F_i = \iint t_i dS ,$$
$$G_i = \iint \varepsilon_{ijk} x_j t_k dS . \tag{2.3}$$

The stress vector t_i accounts for all electromechanical forces other than gravitational and inertial forces.

Applying Stokes Theorem to (2.1) and utilizing the fact that c is arbitrary, we obtain

$$E_{i,j} = E_{j,i} ,$$
$$\text{or} \qquad E_i = - \phi,_i , \tag{2.4}$$

where ϕ is the electrostatic potential. Here $,i$ denotes differentiation with respect to x_i coordinates.

Across the surface ∂V of the dielectric, it follows from (2.1) that the tangential component of E_i is continuous:

$$\varepsilon_{ijk}(E_j^+ - E_j^-)n_k = 0 , \tag{2.5}$$

where n_i is the unit outward normal to the surface ∂V, E_j^+ and E_j^- denote, respectively, the values of electric field outside and inside the dielectric surface.

We are restricting our considerations to the case in which the dielectric body and its surface are free of electrical charge. With this restriction and the fact that S is arbitrary, the application of divergence theorem to (2.2) yields

$$D_{i,i} = 0 \ . \tag{2.6}$$

Furthermore, applying (2.2) to a cylindrical "pill box" that contains the boundary of the dielectric, we can show that the normal component of D_i is continuous across the surface ∂V of the dielectric:

$$D_i^+ n_i = D_i^- n_i \ , \tag{2.7}$$

where, as before, D_i^+ and D_i^- are the values of the field D_i outside and inside the dielectric surface.

3. Equilibrium Equations

If V is an arbitrary volume of the dielectric having surface ∂V, then for equilibrium we must have

$$\begin{aligned} &\iint_{\partial V} t_i dS + \iiint_V \rho f_i dV = 0 \ , \\ \text{and} \quad &\iint_{\partial V} \varepsilon_{ijk} x_j t_k dS + \iiint_V \rho \varepsilon_{ijk} x_j f_k dV = 0 \ . \end{aligned} \tag{3.1}$$

Here, ρ is the mass density of the body and f_i represents the gravitational or inertial body force per unit mass. We will not consider surface couples or body couples.

If we apply the first of (3.1) to a tetrahedron we can show that the stress t_i on a surface with outward unit normal n_i is given by

$$t_i = \sigma_{ji} n_j \ , \tag{3.2}$$

where the quantities σ_{ij} represent the stress tensor.

It also follows from (3.1) that if mechanical surface tractions T_i per unit area of the deformed body are applied to the surface of the dielectric, then

$$T_i = (\sigma^-_{ij} - \sigma^+_{ij}) n_j \tag{3.3}$$

at the boundary. In the situation where electrical effects are absent, the stress σ^+_{ij} in the medium surrounding the dielectric is taken to be zero. In this paper, due to the electric field outside the dielectric, there is a stress σ^+_{ij}, called the Maxwell stress, present outside the dielectric.

Finally, if we substitute (3.2) into (3.1), apply the Divergence theorem, and utilize the arbitrariness of the region V, we obtain the equilibrium equations:

$$\sigma_{ij,j} + \rho f_i = 0 \ , \tag{3.4}$$

and $$\sigma_{ij} = \sigma_{ji} \ . \tag{3.5}$$

4. Constitutive Equations

The Maxwell equations (2.4) through (2.6) and the equilibrium equations (3.5) form an indeterminate system of equations; alone, they do not determine the behaviour of the medium. In order to obtain a determinate system, we need the constitutive equations which relate the material response with forces applied to the medium.

In the free space surrounding the dielectric body, we shall assume that the constitutive equations are simply those of Classical electrostatic theory:

$$D_i = \varepsilon E_i \, , \tag{4.1}$$

$$\sigma_{ij} = M_{ij} = \varepsilon[E_i E_j - \tfrac{1}{2} E_k E_k \delta_{ij}] \, . \tag{4.2}$$

Here, ε is the dielectric constant of free space and M_{ij} the Maxwell stress tensor. Clearly, M_{ij} satisfies equilibrium equations (3.4) and (3.5) identically, when body forces are considered zero as we shall in the applications to follow.

The deformation of the dielectric body is described by the relations:

$$x_i = x_i(X_A) \, . \tag{4.3}$$

Since the dielectric media that we shall be concerned with are homogeneous and perfectly elastic, we therefore assume that there exists in the dielectric a stored energy function W, defined as the energy per unit mass, which is a function of the electric field E_i and the deformation gradients

$x_{i,A} = \frac{\partial x_i}{\partial X_A}$:

$$W = W(x_{i,A};\ E_k) \ . \tag{4.4}$$

The constitutive equations for the homogeneous elastic dielectric that we use in this presentation are derived by Singh [4]:

$$\sigma_{ij} = \rho \frac{\partial W}{\partial x_{i,A}} \frac{\partial x_j}{\partial X_A} + \rho \frac{\partial W}{\partial E_i} E_j \ , \tag{4.5}$$

and $$D_i = \rho \frac{\partial W}{\partial E_i} \ ,$$

where ρ is the mass density measured in the deformed body.

Similar constitutive equations were derived by Toupin through a principle of virtual work. Our equations differ formally from his in that we have chosen the electric field as the independent variable instead of the polarization.

One of the restrictions on W is that it must satisfy the principle of material indifference. This principle states that if the dielectric body is subject to an arbitrary rigid rotation together with the electric field then the force system will undergo the same rigid rotation. Under this requirement, it can be shown that W must be expressible in the form:

$$W = W\left(\frac{\partial x_k}{\partial X_Q} \frac{\partial x_k}{\partial X_P} \ ; \ \frac{\partial x_p}{\partial X_Q} E_p\right) \ . \tag{4.7}$$

If we further restrict our considerations to the case of a dielectric which is isotropic in its undeformed, field-free state, then the stored energy W has to be a function of the six scalar invariants $I_k (k = 1,2,\ldots,6)$:

$$W = W(I_1, I_2, I_3, I_4, I_5, I_6) , \tag{4.8}$$

where

$$\begin{aligned} I_1 &= g_{ii} , \\ I_2 &= \tfrac{1}{2}(g_{ii}\, g_{jj} - g_{ij}\, g_{ij}) , \\ I_3 &= \det\,(g_{ij}) , \\ I_4 &= E_i\, E_i , \\ I_5 &= E_i\, g_{ij}\, E_j , \\ \text{and} \quad I_6 &= E_i\, g_{ij}\, g_{jk}\, E_k . \end{aligned} \tag{4.9}$$

Here g_{ij} is the Finger strain tensor defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_j}{\partial X_A} . \tag{4.10}$$

Substitution of (4.8) - (4.10) into (4.5) and (4.6), and the use of Cayley-Hamilton theorem, reduces the constitutive equations for an isotropic, homogeneous, elastic dielectric to the form

$$\begin{aligned} \sigma_{ij} = \frac{2\rho_0}{\sqrt{I_3}}\Bigg\{ & [\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}]\, g_{ij} - \frac{\partial W}{\partial I_2}\, g_{ij}^2 \\ & + I_3\frac{\partial W}{\partial I_3}\, g_{ij} + \frac{\partial W}{\partial I_4}\, E_i E_j \\ & + \frac{\partial W}{\partial I_5}\, [g_{ik}E_kE_j + g_{jk}E_kE_i] \end{aligned} \tag{4.11}$$

$$+ \frac{\partial W}{\partial I_6} [g^2_{ik}E_kE_j + g^2_{jk}E_kE_i + g_{ik}g_{jp}E_kE_p] \Bigg\},$$

and

$$D_i = \frac{2\rho_0}{\sqrt{I_3}} \left\{ \frac{\partial W}{\partial I_4} \delta_{ij} + \frac{\partial W}{\partial I_5} g_{ij} + \frac{\partial W}{\partial I_6} g^2_{ij} \right\} E_j \ , \qquad (4.12)$$

where $g^2_{ij} = g_{ik}g_{kj}$.

Here we have also used the relation

$$\rho = \frac{\rho_0}{\sqrt{I_3}} \ ,$$

where ρ_0 is the mass density of the undeformed dielectric.

5. Approximate Theories

Assuming that the stored energy function $W(I_1,I_2,I_3,I_4,I_5,I_6)$ can be expressed as a polynomial in the invariants I_k , we may write

$$W = \sum_{\alpha\beta\gamma\delta\lambda\mu} A_{\alpha\beta\gamma\delta\lambda\mu}(I_1-3)^{\alpha}(I_2-3)^{\beta}(I_3-1)^{\gamma}I_4{}^{\delta}I_5{}^{\lambda}I_6{}^{\mu} \ . \qquad (5.1)$$

Here $A_{\alpha\beta\gamma\delta\lambda\mu}$ represent material constants, and we have used the expression $(I_1-3)^{\alpha}(I_2-3)^{\beta}(I_3-1)^{\gamma}$, rather than simply $I_1^{\alpha}I_2^{\beta}I_3^{\gamma}$, so that in the field free, undeformed state, $W = 0$.

The approximate forms of W may now be obtained by terminating the series (5.1), retaining an appropriate number of terms. Using these approximate forms of W we obtain the approximate deformation theories for the homogeneous, isotropic, elastic dielectric.

At a point P in the dielectric, let $e_i (i = 1,2,3)$ denote the principal extensions and let $E_i (i = 1,2,3)$ denote the components of the electric field referred to the principal directions of strain at P . Then the invariants (4.9) can be written as

$$
\begin{aligned}
I_1 &= (1+e_1)^2 + (1+e_2)^2 + (1+e_3)^2 , \\
I_2 &= (1+e_1)^2(1+e_2)^2 + (1+e_2)^2(1+e_3)^2 + (1+e_3)^2(1+e_1)^2 , \\
I_3 &= (1+e_1)^2(1+e_2)^2(1+e_3)^2 , \\
I_4 &= E_1^2 + E_2^2 + E_3^2 , \\
I_5 &= (1+e_1)^2E_1^2 + (1+e_2)^2E_2^2 + (1+e_3)^2E_3^2 , \\
\text{and} \quad I_6 &= (1+e_1)^4E_1^2 + (1+e_2)^4E_1^2 + (1+e_3)^4E_3^2 .
\end{aligned} \tag{5.2}
$$

In order to rewrite the series (5.1) in the more purposeful manner, we define a new set of invariants J_k by the relations

$$
\begin{aligned}
J_1 &= I_1-3 , \\
J_2 &= (I_2-3) - 2(I_1-3) , \\
J_3 &= (I_3-1) - (I_2-3) + (I_1-3) ,
\end{aligned} \tag{5.3}
$$

$$J_4 = I_4 \quad , \; J_5 = I_5 - I_4 \; ,$$

and

$$J_6 = I_6 - 2(I_5 - I_4) - I_4 \; .$$

We note that these relations are invertible and since the I_k are a complete set of invariants, so are therefore J_k .

In terms of the new invariants J_k ,

$$W = \sum_{\alpha\beta\gamma\delta\lambda\mu} B_{\alpha\beta\gamma\delta\lambda\mu} \, J_1^{\alpha} J_2^{\beta} J_3^{\gamma} J_4^{\delta} J_5^{\lambda} J_6^{\mu} \; , \tag{5.4}$$

where $B_{\alpha\beta\gamma\delta\lambda\mu}$ once again represent material constants.

With relations (5.2) and (5.3) we observe that the J_k have the property:

$$\begin{aligned} J_1 &= O(e_i) \quad , \\ J_2 &= O(e_i^2) \quad , \\ J_3 &= O(e_i^3) \quad , \\ J_4 &= O(E_i^2) \quad , \\ J_5 &= O(e_k E_i^2) \; , \\ J_6 &= O(e_k^2 E_i^2) \; , \end{aligned} \tag{5.5}$$

where by $O(e_i^k)$ we mean that e_i^k is the lowest power term appearing in the expression for the invariants defined in (5.3).

Now we can approximate W to any desired order in the principal extensions and powers of the electric field by neglecting terms above an appropriate degree in the polynomial (5.4). Any such approximate form of W gives a complete theory in the sense that W will be invariant under all rigid rotations, even finite, of the dielectric and the electric field. These approximate theories are analogous to the approximate theories of finite elasticity such as the Mooney-Rivlin materials and the Neo-Hookean materials. It is apparent that the usefulness of any such approximate form of W depends on the magnitudes of the material constants $B_{\alpha\beta\gamma\delta\lambda\mu}$.

6. First Approximation

For small principle extensions and weak electric fields, we define the first approximation by retaining in W all terms involving principle extensions e_i up to second powers, terms involving the electric field to second powers in components E_i , and product terms of the type $e_i E_k^2$ only. Within this definition, (5.4) with (5.5) gives the following form of the energy function:

$$W = a_0 + a_1 J_1 + a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 , \tag{6.1}$$

where $a_0, a_1, \ldots, a_6$ are material constants.

We would expect that in the field free undeformed state, $W = 0$ and hence we set $a_0 = 0$. Furthermore, we notice that the expression (4.11) for σ_{ij} contains the term

$$\frac{\partial W}{\partial I_1} g_{ij} \quad \text{or} \quad (a_1+a_6J_4)\, g_{ij}$$

by virtue of (5.3) and (6.1). In the field free undeformed state, this expression reduces to $a_1\delta_{ij}$ and since we want vanishing stresses in this state, we set $a_1 = 0$. Hence, (6.1) becomes

$$W = a_2J_2 + a_3J_1^2 + a_4J_4 + a_5J_5 + a_6J_1J_4 \ . \tag{6.2}$$

We note that when the electric field vanishes, (6.2) reduces to

$$W = a_2J_2 + a_3J_1^2 \ ,$$

the form used as the first approximation of the stored energy function in finite elasticity theory [7] .

If we further neglect terms higher than second in the displacement gradients $\partial u_i/\partial X_j$ and field components E_k , and product terms of order higher than $E_k^2 \frac{\partial u_i}{\partial X_j}$, then (6.2) takes the form

$$\begin{aligned} W = {} & 2a_2(e_{ii}e_{jj}-e_{ij}e_{ij}) + 2a_3e_{ii}e_{jj} \\ & + a_4E_iE_i + 2a_5e_{ij}E_iE_j \\ & + 2a_6e_{ii}E_jE_j \ , \end{aligned} \tag{6.3}$$

where e_{ij} is the strain tensor of classical elasticity given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) .$$

Here $u_i = x_i - X_i$.

The W in (6.3) then is the stored energy function of the classical coupled theory of electrostriction.

The distinction between W given by (6.2) and W given by (6.3) is that the former allows arbitrary rigid rotations of the dielectric together with the electric field whereas the latter does not. In this sense, the classical theory of electrostinction is not a complete theory. In the present paper we will use the first approximate complete theory furnished by W in (6.2). Substitution of (6.2) in (4.11) and (4.12) gives the constitutive equations of the first order finite deformation theory of isotropic, homogeneous, elastic dielectrics:

$$\sigma_{ij} = \frac{2\rho_0}{\sqrt{I_3}}\Big\{[a_2 + (a_2+2a_3)J_1 + a_6J_4]g_{ij} - a_2g_{ij}^2 + (a_4-a_5)E_iE_j + a_5[g_{ik}E_kE_j + g_{jk}E_kE_i]\Big\} , \tag{6.4}$$

and

$$D_i = \frac{2\rho_0}{\sqrt{I_3}} [(a_4-a_5)\delta_{ij} + a_6J_1\delta_{ij} + a_5g_{ij}]E_j . \tag{6.5}$$

7. Classical Theory of Electrostriction

Although the classical theory of electrostriction is neither an essential nor an integral part of this paper, it is interesting to see how this theory comes out of the complete first approximate theory developed in Section 6. To obtain the constitutive equations of classical theory we neglect terms in (6.4) and (6.5) of order higher than first in components e_{ij} , quadratic in E_k , and product terms of order higher that $e_{ij}E_k^2$. Then

$$g_{ij} = \delta_{ij} + 2e_{ij} ,$$

$$g_{ij}^2 = \delta_{ij} + 4e_{ij} ,$$

$$g_{ik}E_kE_j = E_iE_j ,$$

$$J_1 = 2e \qquad (7.1)$$

$$J_1 g_{ij} = 2e\delta_{ij} ,$$

and

$$J_4 g_{ij} = E_kE_k\delta_{ij} ,$$

where

$$e = e_{ii} .$$

Substituting (7.1) in (6.4) and (6.5), we obtain

$$\sigma_{ij} = 4\rho_0(a_2+2a_3)e\delta_{ij} - 4\rho_0 a_2 e_{ij}$$

$$+ 2\rho_0 a_4 E_k E_k \delta_{ij} + 2\rho_0 (a_4 + a_5) E_i E_j \ , \tag{7.2}$$

and

$$D_i = 2\rho_0 [a_4 \delta_{ij} + 2a_4 e \delta_{ij} + 2a_5 e_{ij}] E_j \ . \tag{7.3}$$

These are the constitutive equations of the classical coupled theory of electrostriction. To derive the uncoupled theory, which is used more than often in literature, we have to neglect the coupling terms eE_i and $e_{ij}E_j$ in (7.3). For such a theory, the constitutive equations are:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + aE_k E_k \delta_{ij} + bE_i E_j \ ,$$

and

$$D_i = kE_i$$

where $\lambda, \mu, a, b,$ and k are material constants of the dielectric.

8. Second Approximation

We can define a second approximation to W by retaining terms in series (5.4) up to and including third powers in the principal extensions and field components, and product terms up to and including $e_k^2 E_i^2$. The corresponding expression for W is

$$W = a_2 J_2 + a_3 J_1^2 + a_4 J_4 + a_5 J_5 + a_6 J_1 J_4 + a_7 J_1^3$$

$$+ a_8 J_1 J_2 + a_9 J_3 + a_{10} J_2 J_4 + a_{11} J_1^2 J_4 \tag{8.1}$$

$$+ a_{12}J_6 + a_{13}J_1J_5 \ .$$

As before the a's are material constants. In writing (8.1) we have taken both W and σ_{ij} to vanish in the field free undeformed state.

We note that when the electric field is absent, (8.1) reduces to the Murnaghan [8] form for finite elasticity theory.

The expressions for stress σ_{ij} and dielectric displacement D_i may now be obtained by the substitution of (8.1) in (4.11) and (4.12).

9. Incompressible Dielectrics

So far we have been discussing elastic dielectrics which are homogeneous and isotropic. We now consider the dielectric which is also incompressible. Mathematically, this means $I_3 = 1$ for all deformations, and so W is a function of only five invariants:

$$W = W(I_1, I_2, I_4, I_5, I_6) \ . \tag{9.1}$$

In terms of principle extensions, the constraint $I_3 = 1$ implies

$$(1+e_1)^2(1+e_2)^2(1+e_3)^2 = 1 \ . \tag{9.2}$$

Because of (9.2), we have $J_1 = O(e_i^2)$. We thus introduce a new invariant J_2':

$$J_2' = J_2 - J_1 \ . \tag{9.3}$$

In view of (5.2) and (5.3), we see that

$$J_2' = O(e_i^3) \ . \tag{9.4}$$

Thus, for the incompressible dielectric, instead of (5.4), W has the form:

$$W = \sum_{\alpha\beta\gamma\delta\lambda} B_{\alpha\beta\gamma\delta\lambda} \, J_1^{\alpha} J_2'^{\beta} J_4^{\gamma} J_5^{\delta} J_6^{\lambda} \ . \tag{9.5}$$

For the first order complete theory outlined in Section 6,

$$W = b_1 J_1 + b_2 J_4 + b_3 J_5 \tag{9.6}$$

where, as before, the b's are constants of the material.

In a conservative system, a hydrostatic pressure p arises as a reaction to the constraint of no volume change. Keeping this in mind, the substitution of (9.6) into (4.11) and (4.12) yields the constitutive equations for the incompressible, homogeneous, isotropic, elastic dielectric:

$$\sigma_{ij} = -p\delta_{ij} + C_1 g_{ij} + C_2 E_i E_j + C_3 (g_{ik} E_k E_j + g_{jk} E_k E_i) \ , \tag{9.7}$$

and

$$D_i = C_2 E_i + C_3 g_{ij} E_j \ . \tag{9.8}$$

Here p represents arbitrary pressure and the constants c are material constants.

10. Applications

In this section of the paper we formulate the application of the first order theory of finite deformations for a homogeneous, isotropic, incompressible, elastic dielectric developed in Section 9.

The usual procedure would be to prescribe initially a set of applied mechanical surface tractions and an applied external electric field for the undeformed elastic dielectric. Then, using the constitutive equations, equilibrium equations, Maxwell equations, and boundary conditions, derive the displacement and electric field inside the dielectric. Unfortunately, the class of boundary value problems for which closed form solutions have been found by following this approach is confined to the classical linear uncoupled theory of electrostriction. To handle boundary value problems in this manner for the nonlinear elastic dielectric is, to say the least, rather complicated.

In view of this situation, in this presentation we use what is known as the inverse method. We prescribe the deformation and the electric field inside and outside the deformed dielectric. We then verify that Maxwell's equations and the equilibrium equations without body force are satisfied for a homogeneous, incompressible, isotropic, elastic dielectric within the formulation of the first order theory of finite deformations developed in previous Sections. The surface tractions which must then be applied to support such a prescribed deformation are calculated by the use of the boundary conditions.

We reproduce here the basic equations of the first order theory from the earlier Sections.

Constitutive equations:

Inside the dielectric:

$$\sigma_{ij} = -p\delta_{ij} + C_1 g_{ij} + C_2 E_i E_j + C_3 [g_{ik} E_k E_j + g_{jk} E_k E_i] \,, \tag{10.1}$$

$$D_i = C_2 E_i + C_3 g_{ik} E_k \,. \tag{10.2}$$

Outside the dielectric:

$$\sigma_{ij} = M_{ij} = \varepsilon [E_i^{(0)} E_j^{(0)} - \tfrac{1}{2} E_k^{(0)} E_k^{(0)} \delta_{ij}] \,, \tag{10.3}$$

$$D_i^{(0)} = \varepsilon E_i^{(0)} \,. \tag{10.4}$$

Balance equations:

Both outside and inside the dielectric:

$$\sigma_{ij,j} + \rho f_i = 0 \tag{10.5}$$

Maxwell equations:

$$D_{i,i} = 0 \,, \tag{10.6}$$

$$\varepsilon_{ijk} E_{k,j} = 0 \,. \tag{10.7}$$

Maxwell equations hold both inside and outside the dielectric.

Boundary conditions at the surface of the dielectric:

$$T_i = (\sigma_{ij}-M_{ij})n_j \ , \tag{10.8}$$

$$(D_i^{(0)}-D_i)n_i = 0 \ , \tag{10.9}$$

$$\varepsilon_{ijk}[E_j^{(0)}-E_j]n_k = 0 \ . \tag{10.10}$$

We have used symbols $E_i^{(0)}$ and $D_i^{(0)}$ to denote the fields outside the dielectric.

11. Rotation of a Right Circular Cylindrical Tube about its Axis in a Radial Electric Field

We consider an incompressible, homogeneous, isotropic elastic dielectric right circular cylindrical tube rotating with constant angular velocity ω about its axis of symmetry. Mechanically, it is equivalent to consider the tube stationary but subject to a body force $r\omega^2$ per unit mass acting in the radical direction.

The deformation we consider is a simultaneous extension and inflation of the tube which is described in cylindrical coordinate system by the mapping:

$$r = \lambda R \ , \ \theta = \Xi \ , \ z = \frac{Z}{\lambda^2} \ , \tag{11.1}$$

where λ is a constant. Here, (r,θ,z) denote the coordinates of the material particle in the deformed configuration whose initial coordinates are (R,Ξ,Z).

It can be verified that mapping (11.1) preserves volume.

From (4.10) and (11.1), the physical components of strain are

$$g_{rr} = \lambda^2 \ , \ g_{\theta\theta} = \lambda^2 \ , \ g_{zz} = \frac{1}{\lambda^4} \ ,$$
$$g_{\theta z} = g_{r\theta} = g_{rz} = 0 \ . \tag{11.2}$$

We consider the deformation to take place in a radial field, that is, $E_\theta^{(0)} = E_z^{(0)} = 0$. In view of (10.10) therefore, $E_\theta = E_z = 0$. Also, since $\mathrm{Curl}\,\vec{E} = 0$, both E_r and $E_r^{(0)}$ are functions of r alone.

By (10.4) and (10.2), the dielectric displacement fields are given by

$$D_r^{(0)} = \varepsilon E_r^{(0)} \ , \ D_\theta^{(0)} = D_z^{(0)} = 0 \ ,$$
$$\text{and} \qquad D_r = [C_2 + \lambda^2 C_3]E_r \ , \ D_\theta = D_z = 0 \ . \tag{11.3}$$

Equation (10.7) states that D_i must be solenoidal everywhere. With (11.3), this means that

$$E_r = \frac{K}{r} \ , \quad E_r^{(0)} = \frac{L}{r} \tag{11.4}$$

where K and L are constants. Substitution in (11.3) gives

$$D_r = (C_2 + \lambda^2 C_3)\frac{K}{r} \ ,$$

$$\text{and} \qquad D_r^{(0)} = \frac{\varepsilon L}{r} . \tag{11.5}$$

Since, according to (10.10), the normal component of the dielectric displacement is continuous across the boundary, (11.5) yields

$$K = \frac{\varepsilon L}{[C_2 + \lambda^2 C_3]} . \tag{11.6}$$

Let R_a and R_b denote the internal and external radii of the tube initially. In the deformed configuration, let these radii be r_a and r_b respectively.

The physical components of stress are furnished by substituting (11.2) and (11.4) into (10.1):

$$\begin{aligned}
\sigma_{rr} &= -p + C_1\lambda^2 + [C_2 + 2C_3\lambda^2]\frac{K^2}{r^2} , \\
\sigma_{\theta\theta} &= -p + C_1\lambda^2 , \\
\sigma zz &= -p + \frac{C_1}{\lambda^4} , \\
\sigma_{r\theta} &= \sigma_{rz} = \sigma_{\theta z} = 0 .
\end{aligned} \tag{11.7}$$

In cylindrical coordinates, the equilibrium equations are:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{1}{r}[\sigma_{rr} - \sigma_{\theta\theta}] + \rho f_r = 0 ,$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + \rho f_\theta = 0 \qquad (11.8)$$

$$\frac{\partial\sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + \rho f_\theta = 0 ,$$

which in this case reduce to

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}[\sigma_{rr} - \sigma_{\theta\theta}] + \rho f_r = 0 ,$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0 , \qquad (11.9)$$

$$\frac{\partial\sigma_{zz}}{\partial z} = 0 .$$

From the last two of these equations we see that the pressure is a function of r alone:

$$p = p(r) .$$

Substituting for σ_{rr} , $\sigma_{\theta\theta}$ from (11.7) and $f_r = r\omega^2$ in the first of the equilibrium equations (11.9), we obtain

$$\frac{\partial\sigma_{rr}}{\partial r} = - [C_2 + 2C_3\lambda^2]\frac{K^2}{r^3} - \rho r\omega^2 .$$

Integration yields:

$$\sigma_{rr} = \frac{C_2}{2}\frac{K^2}{r^2} + C_3\lambda^2 \frac{K^2}{r^2} - \rho\frac{r^2\omega^2}{2} - A ,$$

where A is an arbitrary constant. Comparing this with the expression for σ_{rr} in (11.7), we find

$$p = C_1\lambda^2 + \frac{1}{2}[C_2 + 2C_3\lambda^2] \frac{K^2}{r^2} + \rho\frac{r^2\omega^2}{2} - A . \tag{11.10}$$

The stress outside the dielectric body is given by (10.3):

$$M_{rr} = \frac{\varepsilon}{2} [E_r^{(0)}]^2 = \frac{\varepsilon}{2}\frac{L^2}{r^2} , \tag{11.11}$$

$$M_{\theta\theta} = M_{zz} = M_{r\theta} = M_{rz} = M_{\theta z} = 0 .$$

The surface tractions that must be applied on the exterior surface of the tube can now be calculated with use of (10.9):

$$T_\theta(r = r_b) = T_z(r = r_b) = 0 ,$$

$$T_r(r = r_b) = \sigma_{rr}(r = r_b) - M_{rr}(r = r_b) \tag{11.12}$$

$$= \frac{1}{2}[C_2 + 2C_3\lambda^2] \frac{K^2}{r_b^2} - \rho r_b^2 \frac{\omega^2}{2} + A - \frac{\varepsilon}{2}\frac{L^2}{r_b^2}$$

We may set $T_r(r = r_b) = 0$. This then yields an expression for the arbitrary constant A :

$$A = \rho r_b^2 \frac{\omega^2}{2} + \frac{1}{2r_b^2} [\varepsilon L^2 - [C_2 + 2C_3\lambda^2]K^2] .$$

or, substituting for K from (11.6) ,

$$A = \rho r_b^2 \frac{\omega^2}{2} + \frac{\varepsilon L^2}{2r_b^2} [1 - \varepsilon \frac{C_2 + 2C_3\lambda^2}{[C_2 + C_3\lambda^2]^2}] . \tag{11.13}$$

The surface tractions that must be applied on the inner surface of the tube are once again given by (10.8) as:

$$T_\theta(r = r_b) = T_z(r = r_a) = 0 ,$$

$$T_r(r = r_a) = M_{rr}(r = r_a) - \sigma_{rr}(r = r_a) \tag{11.14}$$

$$= \frac{\varepsilon}{2}\frac{L^2}{r_a^2} - \frac{1}{2}[C_2 + 2C_3\lambda^2] \frac{K^2}{r_a^2} + \rho r_a^2 \frac{\omega^2}{2} - A .$$

Substituting for A from (11.13), and K from (11.6),

$$T_r(r_a) = \frac{\omega^2}{2}[r_a^2 - r_b^2] + \frac{\varepsilon L^2}{2}[1 - \frac{C_2 + 2C_3\lambda^2}{[C_2 + C_3\lambda^2]^2}][\frac{1}{r_a^2} - \frac{1}{r_b^2}] .$$

12. <u>Expansion of a Spherical Shell in a Radial Field</u>

We consider a spherical shell of incompressible, isotropic, homogeneous elastic dielectric material. Initially the shell has internal radius R_a and external radius R_b . The particle initially at the point (R,Ξ,Φ) in a spherical coordinate system occupies the position (r,θ,ϕ) in the deformed state, given by

$$r(R) = [R^3 - R_a^3 + r_a^3]^{1/3} ,$$

$$\theta = \Xi \ , \ \phi = \Phi \ . \tag{12.1}$$

The constant r_a is the interior radius of the deformed shell and, in later analysis, we will let r_b be the exterior radius of the deformed shell. It is clear that the deformation (12.1) preserves volume. With (12.1), the physical components of strain are furnished by (4.10):

$$g_{rr} = (r')^2 \ , \ g_{\theta\theta} = g_{\phi\phi} = [\tfrac{r}{R}]^2 \ ,$$
$$g_{r\phi} = g_{r\theta} = g_{\theta\phi} = 0 \ , \tag{12.2}$$

whereas $r' = \frac{dr}{dR}$.

We consider the deformation to take place in a radial field. That is: $E_\theta^{(0)} = E_\phi^{(0)} = 0$. By virtue of boundary condition (10.11), we obtain $E_\theta = E_\phi = 0$. Also, since Curl $\vec{E} = 0$ both inside and outside the dielectric, $E_r^{(0)}$ and E_r will be functions of r alone.

The dielectric displacement fields are given by (10.4) and (10.2):

$$D_r^{(0)} = \varepsilon E_r^{(0)} \ , \ D_\theta^{(0)} = D_\phi^{(0)} = 0 \ ,$$

and $$D_r = [C_2 + C_3(r')^2]E_r \ , \ D_\theta = D_\phi = 0 \ . \tag{12.3}$$

In spherical coordinates, equation (10.6) becomes

$$\sin\theta \frac{\partial}{\partial r}(r^2 Dr) + \frac{\partial}{\partial\theta}(r \sin\theta \, D_\theta) + \frac{\partial}{\partial\phi}(rD_\phi) = 0 \ ,$$

which in our case yields

$$D_r^{(0)} = \frac{K}{r^2} ,$$

and (12.4)

$$D_r = \frac{L}{r^2} .$$

Since the normal component of the dielectric displacement has to be continuous across the boundary, $K = L$. In accordance with the usual conventions of electrostatics, we set

$$K = \frac{Q}{4\pi} ,$$

so that the dielectric displacement field both inside and outside the dielectric can be written as

$$D_r = \frac{Q}{4\pi} \frac{1}{r^2} ,$$

$$D_\theta = D_\phi = 0 . \qquad (12.5)$$

The electric field, outside and inside the dielectric, is therefore given by

$$E_r^{(0)} = \frac{Q}{4\pi\varepsilon} \frac{1}{r^2} , \quad E_\theta^{(0)} = E_\phi^{(0)} = 0 ,$$

$$E_r = \frac{1}{[C_2 + C_3(r')^2]} \frac{Q}{4\pi r^2} , \quad E_\theta = E_\phi = 0 , \qquad (12.6)$$

upon substitution of (12.5) in (12.3).

The physical components of stress are now obtained by substituting (12.2) and (12.6) into (10.1):

$$\sigma_{rr} = -p + C_1(r')^2 + C_2E_r^2 + 2C_3(r')^2E_r^2$$

$$= -p + S_{rr}(r) \ , \tag{12.7}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -p + C_1[\tfrac{r}{R}]^2 \ ,$$

$$\sigma_{r\theta} = \sigma_{\theta\phi} = \sigma_{\phi r} = 0 \ .$$

where

$$S_{rr}(r) = C_1(r')^2 + C_2E_r^2 + 2C_3(r')^2E_r^2$$

$$= C_1(r')^2 + \frac{C_2 + 2C_3(r')^2}{[C_2 + C_3(r')^2]^2}\,\frac{Q^2}{16\pi^2r^4} \ .$$

The equilibrium equations in spherical coordinates are:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}$$

$$+ \frac{1}{r}[2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + 2\sigma_{r\theta}\cot\theta] + \rho f_r = 0 \ ,$$

$$\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} \tag{12.8}$$

$$+ \frac{1}{r}[3\sigma_{r_\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta] + \rho f_\theta = 0 \ ,$$

$$\frac{\partial\sigma_{\phi r}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta}$$

$$+ \frac{1}{r} [3\sigma_{r\phi} + 2\sigma_{\phi\theta} \cot \phi] + \rho f_{\phi} = 0 ,$$

which in this case reduce to

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r} [2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}] = 0 ,$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0 , \qquad (12.9)$$

$$\frac{\partial\sigma_{\phi\phi}}{\partial\phi} = 0 .$$

From the last two of equations (12.9), we see that the pressure is a function of r alone:

$$p = p(r) .$$

Substituting (12.7) into the first of (12.9),

$$- \frac{\partial p}{\partial r} + \frac{\partial S_{rr}}{\partial r} + \frac{2}{r} [S_{rr} - C_1 \frac{r^2}{R^2}] = 0 ,$$

which on integration yields

$$p(r) = S_{rr} + 2\int_{r_a}^{r} \frac{1}{\eta} [S_{rr}(\eta) - C_1 \frac{\eta^2}{R^2(\eta)}] d\eta + p(r_a) - S_{rr}(r_a) .$$

or

$$p = S_{rr} + 2C_1 \left\{ \frac{1}{4}[\frac{R_a^4}{r_a^4} - \frac{R^4}{r^4}] + [\frac{R_a}{r_a} - \frac{R}{r}] \right\}$$

$$+ 2\int_{r_a}^{r} \frac{C_2 + 2C_3(r')^2}{[C_2 + C_3(r')^2]^2} \frac{Q^2}{16\pi^2\eta^5} d\eta - \sigma_{rr}(r_a) \ . \tag{12.10}$$

The stress outside the dielectric is given by (10.3):

$$M_{rr} = \frac{\varepsilon}{2} [E_r^{(0)}]^2 = \frac{1}{32\pi^2} \frac{Q^2}{r^4} , \tag{12.11}$$

with other $M_{ij} = 0$.

The surface tractions that must be applied to the interior surface of the shell can now be calculated from (10.6), (12.7), and (12.10):

$$T_\theta(r = r_a) = T_\phi(r = r_a) = 0$$

$$T_r(r = r_a) = M_{rr}(r = r_a) - \sigma_{rr}(r = r_a) \tag{12.12}$$

$$= \frac{1}{32\pi^2\varepsilon} \frac{Q^2}{r_a^2} - \sigma_{rr}(r = r_a) \ .$$

We can set $T_r(r = r_a) = 0$, and in doing so obtain

$$\sigma_{rr}(r_a) = \frac{1}{32\pi^2\varepsilon} \frac{Q^2}{r_a^2} \ . \tag{12.13}$$

The tractions to be applied to the exterior surface are once again furnished by (10.6), (12.7), and (12.10):

$$T_r(r = r_b) = \sigma_{rr}(r_b) - M_{rr}(r_b)$$

$$= -2C_1\left\{\frac{1}{4}\left[\frac{R_a^4}{r_a^4} - \frac{R_b^4}{r_b^4}\right] + \left[\frac{R_a}{r_a} - \frac{R_a}{r_b}\right]\right\}$$

$$- 2\int_{r_a}^{r_b} \frac{C_2 + 2C_3(r')^2}{[C_2 + C_3(r')^2]^2} \frac{Q^2}{16\pi^2\eta^5} \partial\eta \tag{12.14}$$

$$+ \frac{Q}{32\pi^2\varepsilon}\left[\frac{1}{r_a^4} - \frac{1}{r_b^4}\right] ,$$

$$T_\theta(r = r_b) \ , \ T_\phi(r = r_b) = 0 \ .$$

We note that we can set $T_r(r_b) = 0$ and hence obtain a relation between Q and r_a .

13. Flexural Deformations of a Block in a Radial Field

We consider a rectangular block of homogeneous, isotropic, incompressible, elastic dielectric material which has been deformed into a sector of a tube wall. Without the electrical effects, this deformation has been considered by Rivlin [9]. For the elastic dielectric with arbitrary form of stored energy function, Singh and Pipkin [2] have discussed this deformation combined with the Helical Electric Field. In this section, we show that within the formulation of first order complete theory, this deformation can also be supported with a radial field.

Under this deformation, the particle having Cartesian coordinates (X,Y,Z) initially moves to the position (r,θ,z) in a cylindrical system such that

$$r = AX^{\frac{1}{2}} \ , \ \theta = BY \ , \ z = \frac{2}{A^2B}Z + CY \ , \tag{13.1}$$

where A,B,C are any constants.

From (4.10), the physical components of strain are

$$\begin{aligned} g_{rr} &= \frac{A^4}{4r^2} \ , \ g_{r\theta} = g_{rz} = 0 \ , \\ g_{\theta\theta} &= B^2r^2 \ , \ g_{\theta z} = BCr \ , \ g_{zz} = C^2 + \left(\frac{2}{A^2B}\right)^2 \ . \end{aligned} \tag{13.2}$$

The electric field that we combine with (13.1) is radial:

$$E_r^{(0)} = E_r^{(0)}(r) \ , \ E_\theta^{(0)} = 0 \ , \ E_z^{(0)} = 0 \ . \tag{13.3}$$

In view of the condition (10.7),

$$E_r^{(0)} = \frac{L}{r} \ , \tag{13.4}$$

where L is any constant.

With (10.4) and (13.3),

$$D_r^{(0)} = \varepsilon\frac{L}{r} \ , \ D_\theta^{(0)} = D_z^{(0)} = 0 \ . \tag{13.5}$$

Because of the requirement of continuity of the tangential component of the electric field and normal component of the flux across the boundary,

$$E_\theta = E_z = 0 \ ; \ D_r = \varepsilon\frac{L}{r} \ . \tag{13.6}$$

Using the constitutive equation (10.2), we get

$$E_r = 4\varepsilon L \frac{r}{A^2 + 4C_2 r^2} \; ; \; D_\theta = D_z = 0 \tag{13.7}$$

It can be readily verified that Maxwell equations (10.6) and (10.7) are met by E_i and D_i both inside and outside the dielectric as given by (13.4) to (13.7).

From (10.1), the stresses are

$$\sigma_{rr} = -p + C_1 \frac{A^4}{4r^2} + (C_2 + C_3 \frac{A^4}{2r^2}) \left(\frac{4\varepsilon L r}{A^2 + 4C_2 r^2}\right)^2 ,$$

$$\sigma_{\theta\theta} = -p + C_1 B^2 r^2 ,$$

$$\sigma_{zz} = -p + C_1 [C^2 + (\frac{2}{A^2 B})^2] , \tag{13.8}$$

$$\sigma_{\theta z} = C_1 BCr ,$$

$$\sigma_{r\theta} = \sigma_{rz} = 0 .$$

Substituting the stress distribution (13.8) into the equilibrium equations (10.5) without the body forces, we obtain

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 ,$$

and

$$p = p(r) = C_1 \frac{A^4}{4r^2} + (C_2 + C_3 \frac{A^4}{2r^2}) \left(\frac{4\varepsilon L r}{A^2 + 4C_2 r^2}\right)^2$$

$$
\begin{aligned}
&+ C_1 \left(\frac{B^2}{2} r^2 + \frac{A^2}{8r^2}\right) \\
&- \frac{2L^2\varepsilon^2}{A(C_2C_3)^{\frac{1}{2}}} \left(\frac{2}{C_3A^2} + 1\right) \tan^{-1} \left(\frac{2C_2 r}{A(C_2C_3)^{\frac{1}{2}}}\right) \qquad (13.9) \\
&- \frac{2L^2\varepsilon^2}{A^2C_3} \; \frac{r}{\left[C_2 r^2 + \frac{C_3A^2}{4}\right]} + K \, ,
\end{aligned}
$$

where K is an arbitrary constant.

Substitution of (13.9) into (13.8) then gives the stress distribution σ_{ij}, whereas Maxwell stresses M_{ij} are furnished by substituting (13.3) into (10.3). The surface tractions that must be applied at the boundary can now be calculated by the relations

$$
T_i = (\sigma_{ij} - M_{ij}) n_j \; . \qquad (13.10)
$$

14. Incompressible Dielectrics. Flux as Independent Variable

In the problems we have considered so far, the electric field was taken as the independent variable. In some problems, it may be more convenient and practical to consider the dielectric displacement field as the independent variable. Following a development similar to one outlined in Sections 4 through 9, the field equations governing the complete first order theory for an incompressible, homogeneous, isotropic, and elastic dielectric are:

Inside the dielectric medium:

$$\sigma_{ij} = -p\ \delta_{ij} + K_1\ g_{ij} + K_2\ D_i\ D_j + K_3[g_{ik}D_kD_j + g_{jk}D_kD_i]\ , \qquad (14.1)$$

and $$E_i = K_2\ D_i + K_3\ g_{ik}\ D_k\ . \qquad (14.2)$$

Outside the medium:

$$\sigma_{ij} = M_{ij} = \frac{1}{\varepsilon} D_i^{(0)}\ D_j^{(0)} - \frac{1}{2\varepsilon} D_k^{(0)}\ D_k^{(0)}\ \delta_{ij}\ , \qquad (14.3)$$

and $$D_i^{(0)} = \varepsilon E_i^{(0)}\ . \qquad (14.4)$$

Inside and outside the medium:

$$\sigma_{ij,j} = 0\ , \qquad (14.5)$$

$$D_{i,i} = 0\ , \qquad (14.6)$$

and $$E_{i,j} - E_{j,i} = 0\ . \qquad (14.7)$$

Across the boundary of the dielectric, the tangential component of E_i and the normal component of D_i need to be continuous.

Flexural Deformations of a Block in a Uniform Axial Field of Flux

The flexural deformation of a block in a radial field of flux has been discussed by Singh and Pipkin. In fact, it is shown there that if the general constitutive equations, using arbitary form of the stored energy function, are used, then the only way to support the flexural deformation without body forces or charge distribution would be to superpose a radial field of flux. In this Section, we show that if the first order complete theory is used, then we can support flexural deformation with a uniform axial field of flux.

The deformation we consider is described by the mapping

$$r = AX^{\frac{1}{2}} \ , \ \theta = BY \ , \ z = \frac{2}{A^2 B} Z \ , \tag{14.8}$$

where A and B are any constants.

In this family of deformations, the particle initially at the point (X,Y,Z) in a Cartesian system is brought to the position (r,θ,z) in cylindrical coordinates.

Both Rivlin and Ericksen have discussed this deformation and its physical description in finite elasticity.

It can be easily verified that the deformation (15.8) preserves volume. The physical components of strain in the cylindrical system are given by (4.10):

$$\begin{aligned} g_{rr} &= \frac{A^4}{4r^2} \ , \ g_{r\theta} = g_{rz} = g_{\theta z} = 0 \ , \\ g_{\theta\theta} &= B^2 r^2 \ , \ g_{zz} = \left(\frac{2}{A^2 B}\right)^2 \ . \end{aligned} \tag{14.9}$$

On the deformation (14.8), we superpose the uniform axial field:

$$D_r^{(0)} = D_\theta^{(0)} = 0 \ , \ D_z^{(0)} = L \ , \tag{14.10}$$

where L is constant.

By virtue of (14.4),

$$E_r^{(0)} = E_\theta^{(0)} = 0 \ , \ E_z^{(0)} = \frac{1}{\varepsilon} L \ . \tag{14.11}$$

Since the tangential component of the electric field and normal components of flux have to be continuous across the boundary, we must have inside the dielectric,

$$E_\theta = 0 \ , \ E_z = \frac{1}{\varepsilon} L \ , \ D_r = 0 \ . \tag{14.12}$$

With (14.8) and (14.2),

$$\begin{aligned} E_r &= K_2 D_r + K_3 (A^2/4r^2) D_r \ , \\ E_\theta &= (K_2 + K_3 B^2 r^2) D_\theta \ , \\ E_z &= [K_2 + K_3 (\frac{2}{A^2 B})^2] D_z \ . \end{aligned} \tag{14.13}$$

Using (14.12), equations (14.13) give

$$E_r = E_\theta = 0 \ , \ E_z = \frac{1}{\varepsilon} L \ , \tag{14.14}$$

and

$$D_r = D_\theta = 0 \ , \ D_z = \frac{1}{\varepsilon} \frac{L}{\Delta} \ , \tag{14.15}$$

where $\Delta = K_2 + K_3 (\frac{2}{A^2 B})^2$.

It is easily observed that fields (14.10), (14.11), (14.14), and (14.15) satisfy the equations (14.6) and (14.7).

From (14.1),

$$\sigma_{rr} = -p + K_1 \frac{A^4}{4r^2}\,,\ \sigma_{\theta\theta} = -p + K_1 B^2 r^2,$$

$$\sigma_{zz} = -p + K_1 (\frac{2}{A^2 B})^2 + K_2 (\frac{L}{\varepsilon\Delta})^2 + 2K_3 (\frac{2}{A^2 B})^2 \ (\frac{L}{\varepsilon\Delta})^2\,, \tag{14.16}$$

$$\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{rz} = 0\ .$$

The last two of the equilibrium equations (14.5) require

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \tag{14.17}$$

for stress distribution (14.16) whereas the first of equilibrium reduces to

$$\frac{\partial}{\partial r}\ (\sigma_{rr}) + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0\ ,$$

or with (14.16),

$$\frac{\partial p}{\partial r} = \frac{3}{4} K_1 \ \frac{A^4}{r^3} \ - K_1 B^2 r\ .$$

Upon integration, we obtain

$$p(r) = -\frac{3}{8} K_1 \frac{A^4}{r^2} - \frac{1}{2} K_1 B^2 r^2\ . \tag{14.18}$$

Substitution of (14.18) into (14.16) yields the stresses σ_{ij} , and (14.4) inserted into (14.3) gives the Maxwell stress M_{ij} .

Finally, the surface tractions that must be applied at the boundary can now be calculated from the relations

$$T_i = (\sigma_{ij} - M_{ij})n_j \ . \tag{14.19}$$

15. Deformations Possible in Every Homogeneous, Isotropic, Compressible Elastic Dielectric Within the First Order Theory

Certain deformations in elastic dielectrics are called controllable. The deformation and the electric field are prescribed at the outset. It is then verified that such a state can be supported without body force or distributed charge for all arbitrary forms of the stored energy function of a homogeneous, isotropic elastic dielectric. Singh and Pipkin [2] described all such possible states for incompressible elastic dielectrics. However, Singh [6] proved that if the dielectric is incompressible, the only controllable states are the homogeneous deformations combined with uniform electric fields.

In this Section, we try to find out those states that are controllable in every compressible, homogeneous, isotropic elastic dielectrics which obey the constitutive equations of the first order complete theory as developed in Section 6.

Suppose we are given a symmetric and positive definite tensor field g_{ij} which is twice continuously differentiable. In order that g_{ij} be derived from a possible deformation $x_i(X_A)$ of the body, it is necessary and sufficient that g_{ij} meet the compatibility conditions:

$$R_{ijkl} = 2[g^{-1}_{il,kj} + g^{-1}_{jk,il} - g^{-1}_{ik,jl} - g^{-1}_{jl,ik}]$$

$$+ g_{mn}(A_{jkm} A_{iln} - A_{jlm} A_{ikn}) = 0 \ , \tag{15.1}$$

where g^{-1}_{ij} denotes the inverse of the matrix g_{ij} , and where

$$A_{ijk} = g^{-1}_{ik,j} + g^{-1}_{jk,i} - g^{-1}_{ij,k} \ . \tag{15.2}$$

Suppose a field satisfying (2.4) is prescribed inside the dielectric medium. If the given g_{ij} and E_i are to provide a controllable deformation, then D_i calculated from (6.5) and σ_{ij} calculated from (6.4) will satisfy (2.6) and (3.4) (with zero body forces) no matter what the constants of the material in (6.4) and (6.5) are.

To seek restrictions on possible g_{ij} and E_i , we substitute (6.4) into (3.4), and (6.5) into (2.6) to obtain

$$a_2\left(\frac{g_{ij} + J_1 g_{ij} - g^2_{ij}}{J}\right),j + 2a_3\left(\frac{J_1}{J} g_{ij}\right),j$$

$$+ a_4\left(\frac{E_i E_j}{J}\right),j + a_5\left(\frac{g_{ik}E_kE_j + g_{jk}E_kE_i - E_iE_j}{J}\right),j \tag{15.3}$$

$$+ a_6\left(\frac{J_4 g_{ij}}{}\right),j = 0 \ ,$$

and $$a_4\left(\frac{E_i}{J}\right),i + a_5\left(\frac{g_{ij}E_j - E_i}{J}\right), i + a_6\left(\frac{J_1 E_i}{J}\right),i = 0 \ , \tag{15.4}$$

Where the Jacobian $J = \det. \left(\frac{\partial x_i}{\partial X_j}\right) = \frac{\rho_0}{\rho} = (I_3)^{\frac{1}{2}}$ (15.5)

Necessary and sufficient that conditions (15.3) and (15.4) be satisfied for any choice of the material constants $a_1,...,a_6$, the coefficient of each of a's should separately vanish:

$$\left(\frac{g_{ij} - g_{ij}^2}{J}\right),_j = 0 , \quad (15.6)$$

$$\left(\frac{J_1}{J} g_{ij}\right),_j = 0 , \quad (15.7)$$

$$\left(\frac{E_i E_j}{J}\right),_j = 0 , \quad (15.8)$$

$$\left(\frac{g_{ik}E_k E_j + g_{jk}E_k E_i}{J}\right),_j = 0 , \quad (15.9)$$

$$\left(\frac{J_4\ g_{ij}}{J}\right),_j = 0 , \quad (15.10)$$

$$\left(\frac{E_i}{J}\right),_i = 0 , \quad (15.11)$$

$$\left(\frac{g_{ij}E_j}{J}\right),_i = 0 , \quad (15.12)$$

$$\left(\frac{J_1 E_i}{J}\right),_i = 0 . \quad (15.13)$$

Besides, the field E_i has to be conservative:

$$E_{i,j} = E_{j,i} . \quad (15.14)$$

Necessary and sufficent for a positive definite symmetric tensor g_{ij} and the field E_i to combine to form a controllable state is that the conditions

(15.6) - (15.14) as well as compatibility conditions (15.1) are all satisfied.

Equations (15.8) and (15.11) give

$$E_{i,j} E_j = 0 ,$$

which with (15.14) yields

$$(E_j E_j),i = 0 . \tag{15.15}$$

From (15.15) and (15.10), we obtain

$$(\frac{1}{J} g_{ij}),j = 0 . \tag{15.16}$$

When re-written, (15.16) becomes

$$0 = \frac{\partial}{\partial x_j} (\frac{1}{J} \frac{\partial x_i}{\partial X_A} \frac{\partial x_j}{\partial X_A})$$

$$= \frac{\partial}{\partial x_j} (\frac{1}{J} \frac{\partial x_j}{\partial X_A}) \frac{\partial x_i}{\partial X_A} + \frac{1}{J} \frac{\partial}{\partial x_j} (\frac{\partial x_i}{\partial X_A}) \frac{\partial x_j}{\partial X_A} \tag{15.17}$$

Since $\frac{\partial}{\partial x_j} (\frac{1}{J} \frac{\partial x_j}{\partial X_A})$ is identically zero, and $J \neq 0$, from (15.17) we get

$$x_{i,AA} = 0 \tag{15.18}$$

Also, with (15.16), equation (15.17) gives

$$\frac{1}{J} g_{ij} J_{1,j} = 0 . \tag{15.19}$$

Because the matrix g_{ij} is positive definite, (15.19) yields

$$J_{1,i} = 0 \,, \tag{15.20}$$

or that J_1 and hence I_1 is constant.
The Laplacian of I_1 is therefore zero. That is

$$\left(\frac{\partial x_i}{\partial x_B}\,\frac{\partial x_i}{\partial x_B}\right)_{,AA} = 0 \,, \tag{15.21}$$

or

$$x_{i,AAB}\, x_{i,B} + x_{i,AB}\, x_{i,AB} = 0 \,. \tag{15.22}$$

With (15.18), equation (15.22) gives

$$x_{i,AB}\, x_{i,AB} = 0 \,,$$

which, being the sum of squares, thus furnishes

$$x_{i,AB} = 0 \,. \tag{15.23}$$

The functions $x_i(X_A)$ are therefore linear in arguments X_A, thereby implying that g_{ij} has to be a constant tensor.

From (15.23), it also follows that $J = \det.\left|\frac{\partial x_i}{\partial X_A}\right|$ is a constant.
Using this with (15.11), we obtain

$$E_{i,i} = 0 \; . \tag{15.24}$$

From (15.14), (15.15), and (15.24) now

$$0 = (E_j\, E_j),ii = E_{j,ii}\, E_j + E_{j,i}\, E_{j,i} \quad ,$$

$$= E_{i,ij}\, E_j + E_{j,i}\, E_{j,i} \; ,$$

$$= E_{j,i}\, E_{j,i} \; ,$$

thus implying that E_i is uniform.

It is now readily seen that with g_{ij} and E_i both constant, the conditions (15.1), (15.6) to (15.14) are satisfied identically. Hence, the only controllable states for compressible elastic dielectrics, when first order complete approximation is used, are homogeneous deformations combined with uniform electric fields.

REFERENCES

[1] R.A. Toupin, "The Elastic Dielectric", J. Rational Mech. and Analysis, Vol. 5 (1956), pp. 849-915.

[2] M. Singh & A.C. Pipkin, "Controllable States of Elastic Dielectrics", Arch. Rational Mech. and Analysis, Vol. 21 (1966), pp. 169-210.

[3] R.S. Rivlin & D.W. Saunders, "Large Elastic Deformations of Isotropic Materials. VII. Experiments on the Deformation of Rubber", Phil. Trans. Roy.Soc., London, (A), Vol. 243 (1951), pp. 251-288.

[4] M. Singh, "Small Finite Deformations of Elastic Dielectrics", Quart. Appl. Math., Vol. 25 (1967), pp. 275-284.

[5] A.E. Green & W. Zerna, "Theoretical Elasticity", Oxford Univ. Press, New York, 1954.

[6] M. Singh, "Controllable States in Compressible Elastic Dielectric", ZAMP, Vol. 17 (1966), pp. 449-453.

[7] R.S. Rivlin, "Some topics in finite elasticity', Proceedings of the First Symposium on Naval Structural Mechanics, Pergamon Press, New York, 1960.

[8] F.D. Murnaghan, "Finite Deformation of an Elastic Solid", John Wiley & Sons, Inc., New York, 1951.

[9] R.S. Rivlin, "Large Elastic Deformations of Isotropic Materials, Part V. The Problem of Flexure", Proceedings of the Royal Society, London, Volume A 195 (1949), pp. 463-473.