SUMMABILITY-TOPOLOGICAL METHODS

by

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ABSTRACT

This thesis is a survey of applications of topological methods to summability. We also review and discuss some of the results obtained by A. Wilansky and K. Zeller.

Chapters 1 and 2 are of introductory nature. In Chapter 3 we discuss the classification of conservative matrices as co-null and co-regular matrices. In Chapter 4, we study the inclusion relations of c and l_A and give a detailed proof of a result due to Wilansky and Zeller. In Chapter 5, we study perfectness and type M for different classes of matrices.

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INTRODUCTION

This thesis is a survey of applications of topological methods to the theory of infinite matrix summability. We also review and discuss some of the results obtained by A. Wilansky and K. Zeller. Most of the materials are from [7], [8], [9] and [10].

Chapter 1 is of introductory nature; it consists of results of the theory of topological vector spaces, a sketch of the theory of FK spaces and some important results on infinite matrices (Theorem 1.22., Theorem 1.23., Theorem 1.25., and Proposition 1.27.). An example is given to show that multiplication of infinite matrices is, in general, not associative.

In Chapter 2, the general form of continuous linear functionals on the summability field c_A is given. This identification has numerous applications to the theory.

In Chapter 3 we define co-null and co-regular matrices in terms of the matrix entries. Also, we point out that co-nullity and co-regularity can be regarded as properties of the summability field rather than the matrix. In the second part of this chapter, we study the 'size' of summability fields of co-null and co-regular matrices. The construction of the matrix in Example 3.e. is based on the proof of Theorem 3 of [10].

In Chapter 4, we consider the inclusion relation between c and l_A. In the second part of this chapter, a detailed proof of part of Theorem 1 in [10] is given; the original proof in that paper is very precise. Theorem 4.3. and Proposition 1.12. assure that a co-null matrix must sum a bounded divergent sequence. This result was also obtained originally by K.Zeller.

In Chapter 5, perfectness and type M are studied for different classes of matrices in terms of different subsets of their summability fields. Concrete examples are given to show that these conditions are in general not equivalent.

A LIST OF SYMBOLS

$A, B, C, \cdots$	infinite matrices with complex entries
(a_{nk})	the infinite matrix whose element at the nth row and kth column is a_{nk}
x, y, z	sequences of complex numbers
$\{x^n\}$	sequence of sequences
$\dot{1}$	the sequence $(1, 1, 1, \cdots)$
δ^k	the sequence whose kth coordinate is 1 and others are zero
F	$\{\dot{1}\} \cup \{\delta^k \mid k = 1, 2, \cdots\}$
s	the space of all sequences
c	the space of all convergent sequences
c_o	the space of all sequences converging to zero
l_1	sequences such that $\sum_n \lvert x_n \rvert < \infty$
$\overline{X}$	the closure of the subset X in some topological space
c_A'	the set of all continuous linear functionals on c_A
$a, b, c \cdots$	complex numbers
v	vectors in some linear space
V	linear spaces over the complex numbers

CHAPTER I

SOME DEFINITIONS AND GENERAL RESULTS

1.1. Topological vector spaces

In what follows we will state some definitions and results from the theory of topological vector spaces. The details may be found in [3] and [7].

Definition : A seminorm on a vector space V is a map q from V to the non-negative real numbers satisfying

i) $q(av)=|a|q(v)$, for all complex numbers a and $v\in V$.

ii) $q(v_1+v_2)\leq q(v_1)+q(v_2)$.

It is known that given a family $(q_\iota)_{\iota\in I}$ of seminorms on V, a locally convex linear topology can be defined on V with the sets $\bigcap_{k=1}^{n}\varepsilon_k V_{\iota_k}$ as a fundamental system of neighborhoods of o, where $\varepsilon_k > o$ and $V_{\iota_k}=\{v|\, q_{\iota_k}(v)\leq 1\}$.

When the family of seminorms $(q_\iota)_{\iota\in I}$ is countable and total , that is, if $v\neq o$, there exists q_ι such that $q_\iota(v)\neq o$, we have the following result :

<u>Theorem 1.1</u>. If the locally convex topology on a vector space V is generated by a countable and total family of seminorms $(q_\iota)_{\iota\in N}$, then

$$q(v)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,\frac{r_n(v)}{1+r_n(v)}\quad , \qquad (1.1)$$

where $r_n(v)=\max_{1\leq i\leq n} q_i(v)$, satisfies

a) $|a| \leq 1$ implies $q(av) \leq q(v)$.

b) $a_k \to o$ implies $q(a_k v) \to o$.

c) $q(v)=o$ if and only if $v=o$.

d) $q(-v) = q(v)$.

e) $q(v_1+v_2) \leq q(v_1)+q(v_2)$.

Furthermore, the metric $d(v_1,v_2) \equiv q(v_1-v_2)$ defines the linear topology on V.

Proof : See Theorem 1 on p.111 and Proposition 2 on p.114 of [3].

Definition : A paranorm on a linear space V is a non-negative real function P satisfying

i) $P(o)=o$.

ii) $P(-v)=P(v)$.

iii) $P(v_1+v_2) \leq P(v_1)+P(v_2)$.

iv) If $\{a_n\}$ is a sequence of scalars with $a_n \to a$ and $\{v_n\}$ is a sequence of vectors such that $P(v_n-v) \to o$, then $P(a_n v_n-av) \to o$.

A paranorm is total if $P(v)=o$ implies that $v=o$. It can easily be seen that $q(v)$ in Theorem 1.1. is a total paranorm where iv) is justified by

$$q(a_n v_n-av) \leq q(a_n v_n-av_n)+q(av_n-av)=q((a_n-a)v_n)+q(a(v_n-v)).$$

The term $q((a_n-a)v)$ tends to zero as n increases by b). As for $q(a(v_n-v))$, we may assume $|a|>1$, otherwise a) assures that $q(a(v_n-v)) \to o$, as $n \to \infty$; now $|a|>1$ implies that $q(a(v_n-v))$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2} \frac{|a| r_n(v_n-v)}{1+r_n(v_n-v)} = |a|\, q(v_n-v),$$ hence $q(a(v_n-v))$ tends to zero as n increases.

Definition : A linear metric space is a linear topological space, the topology being generated by a metric d that arises from a total paranorm, that is, $d(x,y)=P(x-y)$ for some total paranorm P.

For linear metric spaces, we say that $\sum_n v_n$ converges to v if for any $\varepsilon>o$, there exists integer N such that $n_o>N$ implies $P(\sum_{n=1}^{n_o} v_n - v)<\varepsilon$.

Definition : A sequence of vectors $\{v_n\}$ is said to be a Schauder basis for a linear metric space V if, for every vector v in V, there is a unique sequence of scalars $\{a_n\}$ such that $v=\sum_n a_n v_n$.

If the locally convex topology on V is defined by a countable and total family of seminorms $(q_\iota)_{\iota\in N}$, then it clearly is a linear metric space.

<u>Theorem 1.2.</u> If the locally convex topology on V is generated by a countable and total family $(q_\iota)_{\iota\in N}$ of seminorms and $\{v_n\}$ is a sequence of vectors in V such that for every v in V there is a unique sequence of scalars $\{a_n\}$ such that $q_i(\sum_{k=1}^{n} a_k v_k - v)\underset{n}{\to} o$, for $i=1,2,3,\cdots$, then $\{v_n\}$ is a Schauder basis.

Proof : For any $\varepsilon>o$, choose integer N_1 so that $\sum_{j=N_1+1}^{\infty}\frac{1}{2^j} <\frac{\varepsilon}{2}$. Consider $q_1,\cdots,q_{N_1}$, Choose N_2 so that for $n>N_2$ we have $q_1(v-\sum_{k=1}^{n} a_k v_k)<\frac{\varepsilon}{2M}$, ...,

$q_{N_1}(v-\sum_{k=1}^{n}a_kv_k)<\frac{\varepsilon}{2M}$, where $M=\frac{1}{2}+\cdots+\frac{1}{2^{N_1}}$. For $n_1>N_2$,

$$d(v,\sum_{k=1}^{n_1}a_kv_k)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{r_n(v-\sum_{k=1}^{n_1}a_kv_k)}{1+r_n(v-\sum_{k=1}^{n_1}a_kv_k)}\leq$$

$$\sum_{n=1}^{N_1}\frac{1}{2}\frac{r_n(v-\sum_{k=1}^{n_1}a_kv_k)}{1+r_n(v-\sum_{k=1}^{n_1}a_kv_k)}+\frac{\varepsilon}{2}<(\sum_{n=1}^{N_1}\frac{1}{2^n}\frac{\varepsilon}{2M})+\frac{\varepsilon}{2}=M\frac{\varepsilon}{2M}+\frac{\varepsilon}{2}=\varepsilon$$

<u>Theorem 1.3.</u> If the locally convex linear topology on a vector space V is generated by a countable and total family of seminorms $(q_\iota)_{\iota\in N}$ and q is a seminorm not in $(q_\iota)_{\iota\in N}$, then the following conditions are equivalent.

i) q is discontinuous at the origin.

ii) q is discontinuous on V.

iii) The topology generated by $\{q\}\cup\{q_\iota \mid \iota\in N\}$ is strictly stronger than the topology generated by $\{q_\iota \mid \iota\in N\}$.

iv) For any positive real number M, any $(\iota_1,\cdots,\iota_n)$, there exists v in V such that $q(v)>M\max_{1\leq k\leq n}q_{\iota_k}(v)$.

v) For all $\varepsilon>0$, and integer N_1, there exists $v\in V$ such that $q(v)=1$ and $q_\iota(v)<\varepsilon$ for all $\iota\leq N$.

Proof: i) clearly implies ii). If q is continuous at the origin and v_1 is any vector in V, then

$$v_1+\{v \mid q(v)<\varepsilon\}=\{v_1+v \mid q(v)<\varepsilon\}\subseteq q^{-1}(N(q(v_1),\varepsilon))$$

implies that q is continuous at v_1. Hence i) and ii) are equivalent. If q is discontinuous at the origin, then for some $\varepsilon>0$, $\{v \mid q(v)<\varepsilon\}$ does not contain any open set of the topology generated by $(q_\iota)_{\iota\in N}$. Hence the topology generated by $\{q\}\cup\{q_\iota \mid \iota\in N\}$ is strictly stronger. iii) clearly implies i). The fact that iii) and iv) are equivalent is proved on p.98 of [3]. Now suppose iv) holds and for any $\varepsilon > o$ let $M = \frac{1}{\varepsilon}$., then there exists v_1 in V such that $q(v_1) > M_1 \max_{1 \leq k \leq n} q_{\iota_k}(v_1)$ for any $(\iota_1, \cdots, \iota_k)$. Let $v = \frac{v_1}{q(v_1)}$, then $q(v)=1$ and $q_{\iota_k}(v) < \varepsilon$. Conversely assume v). For any positive real number M and any $(\iota_1, \cdots \iota_k)$, choose integer N_1 so that $N_1 \geq \max(\iota_1, \cdots, \iota_n)$ and let $\varepsilon = \frac{1}{M}$, then there exists v_1 in V such that $q(v_1)=1$ and $q_{\iota_1}(v_1), \cdots, q_{\iota_n}(v_1)$ are all smaller than ε. Hence $q(v_1)=1 > \frac{1}{\varepsilon} \max_{1 \leq k \leq n} q_{\iota_k}(v_1) = M_1 \max_{1 \leq k \leq M} q_k(v_1)$.

<u>Theorem 1.4.</u> (Hahn-Banach) Let V_1 be a subspace of a linear space V,q be a seminorm defined on V and f a linear functional defined on V_1 such that $|f(v)| \leq q(v)$ for all v in V_1, then there is an extension F of f which is a linear functional on V and $|F(v)| \leq q(v)$ for all v in V.

Proof : See p.65 [7].

The following is a corollary of the Hahn-Banach extension theorem.

<u>Theorem 1.5.</u> Let V be a seminormed linear space, $V_1 \subseteq V$ be a linear subspace and v be a vector which does not belong to the closure of V_1, then there is a continuous linear functional f which vanishes on V, and $f(v) \neq o$.

Proof: See p.67 of [7].

It follows from the above theorem that if every continuous linear functional f that vanishes on V_1 is identically zero, then V_1 must be dense in V. This argument will be applied very frequently in the following chapters.

The following theorem contains two forms of the Banach-Steinhaus theorem.

Theorem 1.6. i) Let $(q_\iota)_{\iota\in I}$ be a pointwise bounded family of continuous seminorms on a complete seminormed space, then $\{\|q_\iota\| \mid \iota\in I\}$ is uniformly bounded.

ii) Let $\{f_n\}$ be a sequence of pointwise convergent continuous linear functions from a complete seminormed space to a normed space, then $f(x)=\lim_n f_n(x)$ defines a continuous linear function f.

Proof : See p.117 of [7].

1.2. Sequence Spaces and FK Spaces.

For every sequence $x=(x_1,\cdots x_n,\cdots)$ in c, define $\|x\| = \sup_n|x_n|$, and for every sequence $x=(x_1,\cdots x_n,\cdots)$ in l_1, define $\|x\| = \sum_n|x_n|$. It is well-known that c and l_1 become Banach spaces with these norms. Also c has F as Schauder basis where each $x\in c$ is represented by

$$x=(\lim x)\dot{1}+\sum_n(x_n-\lim x)\delta^n \tag{1.2}$$

If f is a continuous linear functional on c, then

$$f(x)=(\lim x)f(\dot{1})+\sum_n(x_n-\lim x)\ t_n \tag{1.3}$$

where $t_n=f(\delta^n)$, $n=1,2,\cdots\cdots$ (1.4)

and $\sum_n |t_n|<\infty$ (1.5)

For an arbitrary infinite matrix $A=(a_{ij})$ of complex numbers, denote a sequence $x=(x_1,\cdots x_n,\cdots)$ by a column vector

$$x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$$

By Ax we mean the column vector

$$\begin{pmatrix} \sum_k a_{1k}x_k \\ \sum_k a_{2k}x_k \\ \vdots \\ \sum_k a_{nk}x_k \\ \vdots \end{pmatrix}$$

if $\sum_k a_{nk}x_k$ exists for all n.

For an arbitrary matrix A, let $d_A=\{x \mid Ax$ exists, that is, $\sum_k a_{nk}x_k$ exists for all $n\}$. On d_A define

$$|P_n|(x)=|x_n| \quad \text{and} \quad h_n(x)=\sup\{|\sum_{k=1}^{r} a_{nk}x_k| \mid r=1,2,\cdots\}$$

for $n=1,2,3,\cdots$. From the triangular inequality $|P_n|(x+y)=|x_n+y_n|\leq|x_n|+|y_n|=|P_n|(x)+|P_n|(y)$. Also, $|P_n|(ax)=|a|\,|x_n|=|a|\,|P_n|(x)$. Hence $|P_n|$ is a seminorm for any n. For any n, $h_n(ax)=|a|\sup\{|\sum_{k=1}^{r} a_{nk}x_k| \mid r=1,2,\cdots\}$ and $\sup\{|\sum_{k=1}^{r} a_{nk}(x_k+y_k)| \mid r=1,2,\cdots\}\leq\sup\{|\sum_{k=1}^{r} a_{nk}x_k| \mid r=1,2,\cdots\}+\sup\{|\sum_{k=1}^{r} a_{nk}x_k| \mid r=1,2,\cdots\}$ implies that h_n is a seminorm.

Throughout this paper, the linear topology on d_A is defined to be the locally convex topology generated by $\{h_n \mid n=1,2,\cdots\}\cup\{|P_n| \mid n=1,2,\cdots\}$.

<u>Proposition 1.7.</u> For an arbitrary matrix A, $\{\delta^k \mid k=1,2,\cdots\}$ is a Schauder basis for d_A.

Proof: Let $x=(x_1,\cdots x_n,\cdots)\in d_A$. For any n, $|P_n|(x-\sum_{k=1}^{i} x_k\delta^k)=0$

if $i \geq n$. Hence $\lim_i |P_n|(x-\sum_{k=1}^{i} x_k\delta^k)$ is zero. For any n, $\sum_k a_{nk}x_k$ exists since $x \in d_A$. Given any $\varepsilon > 0$, choose integer K so that $|\sum_{k=k_1}^{k_2} a_{nk}x_k| < \varepsilon$ for any $k_2 \geq k_1 \geq k$. Let $i \geq K$, then $h_n(x-\sum_{k=1}^{i} x_k\delta^k) = \sup\{|\sum_{k=i+1}^{r} a_{nk}x_k| \mid r=i+1,i+2,\cdots\} < \varepsilon$. Hence $\lim_i h_n(x-\sum_{k=1}^{i} x_k{}^k)=0$. By theorem 1.2, $\{\delta^k \mid k=1,2,\cdots\}$ is a Schauder basis.

The particular type of topological space known as an FK space and introduced by Zeller has played an increasingly important role in summability. As examples of FK spaces we mention the spaces c_A and d_A. The general form of continuous linear functionals on c_A can be obtained from the general theory of FK spaces and this has numerous applications in summability theory. The details can be found in 11.3 and 12.4 of [7].

Definition: A complete linear metric space is called a Fréchet space.

Definition: Let H be a Hausdorff space and a linear space. An FH space is a Fréchet space such that

i) X is a linear subspace of H.

ii) The topology of X is stronger than that of H.

The special kind of FH spaces when H=s with the norm $\|x\|=\sum_n \frac{1}{2}\frac{|x_n|}{1+|x_n|}$ are called FK spaces.

Definition : Let X,Y be topological spaces and $f:X\to Y$ be a function, then f is said to be closed if the graph $\{(x,f(x)) \mid x \in X\}$ is closed in XxY with the product topology.

<u>Theorem 1.8.</u> Let X,Y be topological spaces, $f:X\to Y$ be

continuous and Y be Hausdorff, then f is closed.

Proof: See p.195 of [7].

It is clear that if f is closed and the topology on Y is replaced by a stronger topology then f remains closed.

Theorem 1.9. (The Closed-Graph Theorem) Let X,Y be Fréchet spaces and f:X→Y be a closed linear map, then f is continuous.

Proof : See p.200 of [7].

Theorem 1.10 : Let X be a Fréchet space, Y be an FH space with respect to some H and f:X→Y a linear function, then f is continuous if and only if it is continuous as a function from X to H.

Proof: If f:X→Y is continuous, then the topology of Y is stronger and $f(X)\subseteq Y$ imply that f:X→H is continuous. Conversely, if f:X→H is continuous then f is closed, by Theorem 1.8., hence f:X→Y is closed, by Theorem 1.9., it is continuous.

Corollary 1.11. Let X,Y be FH spaces with respect to the same H,$X\subseteq Y$,then the topology of X is stronger than that of Y, in particular a linear space of H has at most one topology that makes it an FH space.

Proof: Let i be the inclusion map i:X→H, then i is continuous since the topology on X is stronger than the subspace topology on it. Hence by Theorem 1.10., i:X→Y is continuous and the result follows.

Proposition 1.12. In Corollary 1.11.the topology on X is strictly stronger than the subspace topology if and only if X is not closed in Y.

Proof: If X is closed in Y then the subspace topology is complete, hence X is an FH space with the subspace topology. By Corollary 1.11. the topology on X is the same as the subspace topology. Conversely, suppose the two topologies on X are the same, then the subspace topology is complete and hence X is closed in Y.

Corollary 1.13. Let X be a Fréchet space, Y an FK space, $f:X\to Y$ a linear function, then f is continuous if and only if $f(x)=\{f_n(x)\}$ where each f_n is a continuous linear functional on X.

Proof: Recall that the norm on s is defined by $\|x\|= \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n|}{1+|x_n|}$ hence the coordinate projections $P_n(x)=x_n$ are continuous. Now if f is continuous as a mapping from X to Y then $f:X\to s$ is continuous hence if we let $f_n(x)=P_n of$, we have $f(x)=\{f_n(x)\}$ with each f_n continuous. Conversely let $d(u,v)$ be the metric of X, if $d(u_n,u)\to o$ then $f_i(u_n,u)\to o$ $(i=1,2,3,\cdots)$, hence $\sum_{i=1}^{\infty}\frac{1}{2}\frac{|f_i(u_n-u)|}{1+|f_i(u_n-u)|}\to o$, hence $f:X\to s$ is continuous.

Corollary 1.14. Let A be an infinite matrix and X,Y be FK spaces. If for every $x\in X$, Ax exists and belongs to Y, then A, considered as a mapping from X to Y, is continuous.

Proof: Consider $\sum_{k=1}^{\infty}a_{nk}x_k$, the nth coordinate of Ax, by Corollary 1.13. it suffices to show that $\sum_{k=1}^{\infty}a_{nk}x_k$ is a continuous linear functional on X. For this we define

$f_m(x)=\sum_{k=1}^{m} a_{nk}x_k$. Then for any $x\in X$, $\lim_m f_m(x)=\sum_{k=1}^{\infty} a_{nk}x_k$. Define $f:X\to c$ by $f(x)=(f_1(x),\cdots,f_m(x),\cdots)$. Now X is an FK space , hence convergence in X implies coordinatewise convergence, thus it also implies convergence in s, therefore $f:X\to s$ is continuous. By Theorem 1.10., $f:X\to c$ is continuous. Now in (1.3) let $t_n=o,(n=1,2,\cdots)$ and $f(i)=1$, then it follows that for every $(x_1,\cdots x_n,\cdots)$ in c, $\lim_n x_n$ is a continuous linear functional on c. Thus $\sum_{k=1}^{\infty} a_{nk}x_k$ is a continuous linear functional on X since it is the composite of f and $\lim_n x_n$.

<u>Theorem 1.15.</u> Let X,Y be FK spaces with their topologies generated by the families of seminorms $(q_\iota)_{\iota\in I}$ and $(r_\lambda)_{\lambda\in\Lambda}$ respectively. Let $f:X\to s$ be a continuous linear map. Then $f^{-1}(Y)$ with the linear topology generated by $(q_\iota)_{\iota\in I}$ and $(r_\lambda \circ f)_{\lambda\in\Lambda}$ is an FK space and $f:f^{-1}(Y)\to Y$ is continuous.

Proof: $f^{-1}(Y)$ is clearly a linear subspace of s and the topology generated by $(q_\iota)_{\iota\in I}\cup(r_\lambda \circ f)_{\lambda\in\Lambda}$ is stronger than the subspace topology relative to X, hence stronger than that relative to s. Now let $\{x^n\}$ be a Cauchy sequence in $f^{-1}(Y)$, then it is a $(q_\iota)_{\iota\in I}$ Cauchy sequence in X, hence $x^n\to x$ in X, on the other hand $\{f(x^n)\}$ is a Cauchy sequence in Y hence $f(x^n)\to y$ in Y, but f is continuous as a mapping from X to s, hence $f(x^n)\to f(x)$ in s, but the topology on Y is stronger than the subspace topology, hence $f(x^n)\to y$ in s, therefore $f(x)=y$. so $x\in f^{-1}(Y)$, hence the space $f^{-1}(Y)$ is complete.

Proposition 1.16. Under the assumption of Theorem 1.15. if f is one-one onto Y, then the linear topology generated by $(r_\lambda \circ f)_{\lambda\in\Lambda}$ alone is an FK space.

Proof: If f is one-one onto Y, then $f: f^{-1}(Y)\to Y$ is a congruence onto where $f^{-1}(Y)$ has the topology generated by $(r_\lambda \circ f)_{\lambda\in\Lambda}$. Now Y is an FK space, hence $f^{-1}(Y)$ is an FK space.

Lemma 1.17. d_A is an FK space for any matrix A.

Proof : For the mth row of A define $D_m=\{x \mid \sum_{k=1}^{\infty} a_{mk}x_k \text{ exists}\}$, then D_m with the seminorms $\{|P_n|\}$ and $h_m=\sup\{|\sum_{k=1}^{r} a_{mk}x_k| \mid r=1,2,\cdots\}$ is an FK space, for we can let X=s, Y=c in Theorem 1.15. and let f be defined by the matrix

$$A_m=\begin{pmatrix} a_{m1} & 0 & \cdots & & & \\ a_{m1} & a_{m2} & 0 & \cdots & & \\ a_{m1} & a_{m2} & a_{m3} & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & \cdots & \cdots & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

then f is continuous by Corollary 1.14 but clearly $c_{A_m}=D_m$ and the seminorm h_m is just the composite of the usual norm on c and f, hence D_m is an FK space by Theorem 1.15. Now $d_A=\bigcap_m D_m$, and $\{|P_n| \mid n=1,\cdots\}\cup\{h_n \mid n=1,2,3\cdots\}$ generate the topology on d_A hence the topology on d_A is clearly stronger than the subspace topology relative to s since it is stronger than that relative to D_m for any m. Let

$\{x^n\}$ be a Cauchy sequence in d_A then $\{x^n\}$ is a Cauchy sequence in D_m for each m, let it converge to y_m in D_m, also $\{x^n\}$ is a Cauchy sequence in s, hence it converges to x, but then $x=y_1=y_2=\cdots=y_m$, hence $x\in\bigcap_m D_m$, therefore d_A is complete, d_A is clearly a linear subspace of s, hence it is an FK space.

Theorem 1.18. Let A be a matrix, then c_A with the linear topology generated by the seminorms on d_A and the seminorm $P(x)=\sup\{|\sum_{k=1}^{\infty}a_{nk}x_k|\ n=1,\cdots\}$ is an FK space.

Proof: In Theorem 1.15., let $X=d_A$, $Y=c$, f be defined by A, then f is continuous by Corollary 1.14., now $c_A=f^{-1}(c)$ and $P(x)$ is the composite of f and the usual norm on c hence by Theorem 1.15., c_A is an FK space.

Definition : A matrix A is said to be reversible if it is a one-one onto mapping from c_A to c.

Proposition 1.19. Let A be reversible, then c_A is an FK space with the seminorm $P(x)=\sup\{|\sum_{k=1}^{\infty}a_{nk}x_k|\ n=1,2,\cdots\}$.

Proof: Follows from Theorem 1.18. and Proposition 1.16.

Lemma 1.20. Let q_1,q_2 be seminorms on a linear space V and f be a linear functional on V such that

$$|f(v)|\leqslant q_1(v)+q_2(v)$$

then there exist linear functionals f_1,f_2 on V such that

$$|f_1(v)|\leqslant q_1(v)\ ,\ |f_2(v)|\leqslant q_2(v)\quad \text{and } f(v)=f_1(v)+f_2(v).$$

Proof: Define $q:V\times V\to R^+$ (the positive reals) by $q(v_1,v_2)=q(v_1)+q(v_2)$, on the diagonal subspace $\{(v,v)|\ v\in V\}$ of $V\times V$, define $g(v,v)=f(v)$, then $g(v,v)$ is a linear functional and q is a seminorm on VxV, now $g(v,v) = f(v) \leq q_1(v)+q_2(v) = q(v,v)$, hence by Theorem 1.7., g can be extended to VxV with $|g(v_1,v_2)| \leq q_1(v_1)+q_2(v_2)$, let $f_1(v)=g(v,o)$, $f_2(v)=g(o,v)$, then $|g(v,o)| = |f_1(v)| \leq q_1(v)+o$, similarly $|g(o,v)| = |f_2(v)| \leq q_2(v)$, clearly $f(v)=g(v,o)+g(o,v)=f_1(v)+f_2(v)$.

<u>Theorem 1.21.</u> Let X,Y be FK spaces with their topologies generated by the families of seminorms $(q_\iota)_{\iota\in I}$ and $(r_\lambda)_{\lambda\in\Lambda}$ respectively. Let $f:X\to s$ be a continuous linear map and $f^{-1}(Y)$ has the linear topology generated by $(q_\iota)_{\iota\in I}$ and $(r_\lambda \text{ of})_{\lambda\in\Lambda}$. If g is a continuous linear functional on $f^{-1}(Y)$, then there exists $F\in X'$, $G\in Y'$ such that $g=F+G\text{of}$.

Proof: If g is a continuous linear functional, then $|g(x)|$ is a continuous seminorm, hence by Theorem 1.3.iv) there exists M and seminorms in $(q_\iota)_{\iota\in I}\cup(r_\lambda \text{of})_{\lambda\in\Lambda}$ such that

$$|g(x)| \leq M\max\{q_1(x),\cdots,q_n(x),r_1\text{of}(x),\cdots r_m\text{of}(x)\}$$
$$\leq M(q_1(x)+\cdots+q_n(x)+r_1\text{of}(x)+\cdots+r_m\text{of}(x)).$$

we may assume that $M(q_1+\cdots+q_n)\in(q_\iota)_{\iota\in I}$ and $M(r_1\text{of}+\cdots+r_m\text{of})\in(r_\lambda\text{of})_{\lambda\in\Lambda}$ since adding these seminorms to $(q_\iota)_{\iota\in I}$ and $(r_\lambda\text{of})_{\lambda\in\Lambda}$ does not change the topology on $f^{-1}(Y)$, hence $|g(x)| \leq q(x)+r\text{of}(x)$ where $q\in(q_\iota)_{\iota\in I}$ and $r\text{of}\in(r_\lambda\text{of})_{\lambda\in\Lambda}$.

By Lemma 1.20. there exist $F\in X'$ and $F_1\in X'$ such that $g=F+F_1$ and $|F|\leq q$, $|F_1|\leq r\text{of}$. Define G on $f(X)\cap Y$ by $G(y)=F_1(x)$ if $y=f(x)$, if $y=f(x_1)=f(x_2)$, then $|F_1(x_1)-F_1(x_2)| = |F_1(x_1-x_2)| \leq r\text{of}(x_1-x_2)=r(o)=o$, hence G is well-defined, by Theorem 1.4., G can be extended to Y, by construction of G we have $g=F+G\circ f$.

1.3. Infinite Matrices.

Definition : A matrix A is said to be conservative if $c_A \supseteq c$, that is, it transforms convergent sequences into convergent sequences.

Theorem 1.22. (Kojima-Schur) A matrix A is conservative if and only if

i) $\|A\|=\sup\{\sum_{k=1}^{\infty}|a_{nk}| \mid n=1,2,\dots\}<\infty$ and

ii) $c_A \supseteq \{1\}\cup\{\delta^k \mid k=1,2,\cdots\}$.

Proof : Suppose i), ii) hold , then $\lim_{n\to\infty} a_{nk}=\lim_{n\to\infty} A(\delta^k)$ exists for all k. Let $a_k=\lim_n a_{nk}$ and $\|A\|\leq M$ then $\sum_{k=1}^{m}|a_{nk}|\to \sum_{k=1}^{m}|a_k|$ for any finite m and $M\geq\sum_{k=1}^{m}|a_{nk}|$ imply that

$$\sum_{k=1}^{\infty}|a_k|\leq M \qquad (1.6).$$

Now if $x=(x_1,\dots x_k,\dots)$ and $\lim_k x_k=a$, write $x_k=a+\varepsilon_k$, hence for any $\varepsilon>0, \exists N(\varepsilon)$ such that $k>N(\varepsilon)$ implies $|\varepsilon_k|<\frac{\varepsilon}{3M}$ for $n\leq N(\varepsilon)$ choose N_1 great enough so that $|\sum_{k=1}^{N(\varepsilon)}(a_{nk}-a_k)\varepsilon_k|<\frac{\varepsilon}{3}$ for $n>N_1$, then

$$|\sum_{k=1}^{\infty}(a_{nk}-a_k)\varepsilon_k|\leq|\sum_{k=1}^{N_1(\varepsilon)}(a_{nk}-a_k)\varepsilon_k|+\sum_{k=N_1(\varepsilon)+1}^{\infty}(|a_{nk}|+|a_k|)|\varepsilon_k|$$

$$<\frac{\varepsilon}{3}+\left(\frac{2M}{3M}\right)\varepsilon=\varepsilon \quad , \text{ for } n>N_1$$

Therefore $\lim_n\sum_{k=1}^{\infty}a_{nk}\varepsilon_k=\sum_{k=1}^{\infty}a_k\varepsilon_k$. Now $(Ax)_n=\sum_{k=1}^{\infty}a_{nk}(a+\varepsilon_k)=a\sum_{k=1}^{\infty}a_{nk}+\sum_{k=1}^{\infty}a_{nk}\varepsilon_k$, but $1\in c_A$, hence $\lim_n\sum_{k=1}^{\infty}a_{nk}=b$ exists, Therefore $\lim_{n\to\infty}(Ax)_n=ab+\sum_k a_k\varepsilon_k$, (1.7).

Hence $c_A\supseteq c$. To prove the converse, we apply the Banach-Steinhaus Theorem twice. For any n define a sequence

$\{f_m\}$ of functional on c by

$$f_m(x)=\sum_{k=1}^{m} a_{nk}x_k \qquad ,(m=1,2,\cdots).$$

Then $\{f_m\}$ is a sequence of continuous linear functionals on c since convergence in c implies coordinate-wise convergence. Now by definition $\|f_m\|=\sup\{|\sum_{k=1}^{m} a_{nk}x_k| \ \|x\|\leq 1\}$, hence $\|f_m\|\leq\sum_{k=1}^{m}|a_{nk}|$, conversely we can let x be the sequence $(e^{-i\theta_1},\ldots,e^{-i\theta_m},o,o,o,\cdots)$ where $\theta_1,\cdots\theta_m$ are the arguments of $a_{n1},\cdots,a_{nm}$ respectively, then $\|x\|\leq 1$ and $|f_m(x)| = \sum_{k=1}^{m}|a_{nk}|\leq\|f_m\|$, thus $\|f_m\|=\sum_{k=1}^{m}|a_{nk}|$. The sequence $\{f_m\}$ is pointwise convergent hence pointwise bounded by Theorem 1.6. $\{\|f_m\| \mid m=1,\cdots\}$ is bounded, hence $\sum_{k=1}^{\infty}|a_{nk}|<\infty$ for any n. Now for any n, $g_n(x)=\sum_{k=1}^{\infty}a_{nk}x_k$ defines a continuous linear functional on c by Theorem 1.6., again $\|g_n\|=\sum_{k=1}^{\infty}|a_{nk}|$, now $\{|g_n|\}$ considered as seminorms on c is pointwise bounded, hence $\{\sum_{k=1}^{\infty}|a_{nk}| \mid n=1,2,\cdots\}$ is bounded.

Definition : A matrix A is said to be regular if for all $x\in c$, we have $\lim_n(Ax)_n=\lim_n x_n$.

<u>Theorem 1.23.</u> (Toeplitz-Silverman) A matrix A is regular if and only if

i) $\|A\|<\infty$,

ii) $\lim_n a_{nk}=o$, for each k

iii) $\lim_n\sum_{k=1}^{\infty}a_{nk}=1$.

Proof : Suppose A is regular then $c_A\supseteq c$ hence i) follows from Theorem 1.23., $\lim\delta^k=o$ for each k hence $\lim A\delta^k= \lim_n a_{nk}=o$, $\lim i=1$ hence $\lim Ai=\lim_n\sum_k a_{nk}=1$. The converse follows from (1.7).

Definition : Let $l_A=\{x\in s \mid Ax\in l_1\}$, then a matrix A is said to be an l_1-l_1 method if $l_A \supseteq l_1$.

In Theorem 1.15. let $X=d_A, Y=l_1$, f be defined by A then l_A becomes an FK space. Theorem 1.25. concerning l_1-l_1 methods is due to Mears, Knopp and Lorentz. (See Satz 1. of [5]).

<u>Lemma 1.24.</u> : The space l_1 has $\{\delta^k \mid k=1,2,3,\cdots\}$ as Schauder basis.

Proof : See p.86 of [7].

<u>Theorem 1.25.</u> : A matrix A is an l_1-l_1 method if and only if there exists M such that

$$\sum_n |a_{nk}| \leq M \qquad ,k=1,2,3,\cdots\cdots .$$

Proof : Suppose $l_A \supseteq l_1$, then considering A as a matrix transformation from l_1 to l_1, it is continuous by Corollary 1.14., hence there exists M such that $\|Ax\| \leq M\|x\|$, where the norm is the usual norm on l_1, hence $\|A\delta^k\| = \sum_n |a_{nk}| \leq M\|\delta^k\| = M$ for all k.

Conversely, let $x=(x_1,\cdots x_n,\cdots)\in l_1$, then $x=\sum_n x_n\delta^n$, by Lemma 1.24. Now if A is column bounded then $\sum_n x_n A(\delta^n)$ is convergent in l_1. For given $\varepsilon > o$, we may choose $N(\varepsilon)$ so that $\sum_{n=N(\varepsilon)}^{\infty} |x_n| < \frac{\varepsilon}{M}$ where M is the bound of the columns, then for $i,j \geq N(\varepsilon)$ $\|x_i A(\delta^i)+\cdots x_j A(\delta^j)\| \leq |x_i| M+\cdots+|x_j| M < \varepsilon$. Hence the partial sum of $\sum_n x_n A(\delta^n)$ form a Cauchy sequence, thus it is convergent in l_1. But the nth partial sum is just$(\sum_{k=1}^{n} a_{1k}x_k, \sum_{k=1}^{n} a_{2k}x_k, \cdots\cdots\cdots)$, therefore the limit of $\sum_n x_n A(\delta^n)$ must be $(\sum_{k=1}^{\infty} a_{1k}x_k, \sum_{k=1}^{\infty} a_{2k}x_k, \cdots\cdots\cdots)$ which is Ax, hence $Ax\in l_1$.

For any two infinite matrices $A=(a_{ij}),B=(b_{ij})$, the product AB is defined to be (c_{ij}) where $c_{ij}=\sum_k a_{ik}b_{kj}$, if each c_{ij} exists. With this definition multiplication is not associative in general, this can be seen as follows, let $\sum_n b_n$ be a convergent series which has a re-arrangement $\sum_n r_n$ that converges to a different limit, let $b_n=c_{f(n)}$, where $f(n)$ is the rearrangement. Now let B be the matrix b_{ij} where $b_{ij}=b_i$ if $f(i)=j$, $b_{ij}=o$ if $f(i)\neq j$, let A and C be the matrix whose elements are all equal to one, then the elements of $(AB)C$ are all equal to $\sum_n r_n$, whereas all elements of $A(BC)$ are $\sum_n b_n$.

Definition : A matrix A is called a lower semi-matrix if for $j>i$ $a_{ij}=o$.

<u>Proposition 1.26.</u> Lower semi-matrices are associative.

Proof: Let $(AB)C=(d_{ij})$, $A(BC)=(e_{ij})$, $A=(a_{ij})$, $B=(b_{ij})$ and $C=(c_{ij})$, then for $j>i$ clearly $d_{ij}=e_{ij}=o$ since both $(AB)C$ and $A(BC)$ are again lower semi-matrices. For $i\geq j$, we have $d_{ij}=(\sum_{k=j}^{i}a_{ik}b_{kj})c_{jj}+(\sum_{k=j+1}^{i}a_{ik}b_{kj})c_{j+1,j}+\cdots+a_{ii}b_{ii}c_{ij}$, this can be re-grouped to form $a_{i1}b_{jj}c_{jj}+a_{i2}(b_{j+1,j}c_{jj}+b_{j+1,j+1}c_{j+1,j})+\cdots+a_{ii}(b_{ij}c_{jj}+\cdots+b_{ii}c_{ij})=e_{ij}$, hence $(AB)C=A(BC)$.

Definition : A matrix is said to be row-bounded if there exists M such that $\sum_{k=1}^{\infty}|a_{nk}|\leq M$ for all n.

<u>Proposition 1.27.</u> : Row-bounded matrices are associative.

Proof : Let A,B,C be row-bounded matrices, $(AB)C=(d_{ij})$, $A(BC)=(e_{ij})$, without loss of generality consider d_{11} and

e_{11}, now $d_{11}=c_{11}(\sum_k a_{1k}b_{k1})+\cdots+c_{n1}(\sum_k a_{1k}b_{kn})+\cdots$, $e_{11}=$ $a_{11}(\sum_k b_{1k}c_{k1})+\cdots+a_{1m}(\sum_k b_{mk}c_{k1})=a_{11}(b_{11}c_{11}+\sum_{k=2}^{\infty}b_{1k}c_{k1})+$ $\cdots+a_{1m}(b_{m1}c_{11}+\sum_{k=2}^{\infty}b_{mk}c_{k1})+\cdots$, now $\{|b_{m1}c_{11}|\mid m=1,2,\ldots\}$ is bounded and $\sum_m |a_{1m}|<\infty$, therefore $e_{11}=c_{11}(\sum_k a_{1k}b_{k1})+$ $\left[a_{11}(\sum_{k=2}^{\infty}b_{1k}c_{k1})+\cdots+a_{1m}(\sum_{k=2}^{\infty}b_{mk}c_{k1})+\cdots\right]$, this step can be carried on for any n hence we have $e_{11}=c_{11}(\sum_k a_{1k}b_{k1})+$ $\cdots+c_{n1}(\sum_k a_{1k}b_{kn})+\left[a_{11}(\sum_{k=n+1}^{\infty}b_{1k}c_{k1})+\cdots+a_{1m}(\sum_{k=n+1}^{\infty}b_{mk}c_{k1})+\cdots\right]$. The last term tends to zero since all three matrices are row-bounded, to see this we can choose m great enough so that $|a_{1m}(\sum_{k=n}^{\infty}b_{mk}c_{k1})+\cdots|<\frac{\varepsilon}{2}$ for any n, then choose n great enough using row-boundedness of B so that $|a_{11}(\sum_{k=n+1}^{\infty}b_{1k}c_{k1})+\cdots+a_{1m-1}(\sum_{k=n+1}^{\infty}b_{m-1,k}c_{k1})|<\frac{\varepsilon}{2}$. Therefore $e_{11}=d_{11}$.

Definition : A matrix A is said to be normal if A is a lower semi-matrix with non-zero diagonal elements.

Proposition 1.28. If A is normal then the equation $Ax=y$ with x as unknown has a unique solution.

Proof: We have

$$a_{11}x_1=y_1$$
$$a_{21}x_1+a_{22}x_2=y_2$$
$$\cdots\cdots\cdots\cdots$$

hence $x_1=\dfrac{y_1}{a_{11}}$, $x_2=\dfrac{y_2-a_{21}x_1}{a_{22}}$, ...

Theorem 1.29. If the terms of a series $\sum_n r_n$ are defined by series, with $r_n=\sum_k a_{nk}$, and $\sum_n a_{nk}=s_k$ for each k, then $\sum_k |a_{nk}|=t_n$ and $\sum_n t_n$ is convergent imply that $\sum_n r_n=\sum_k s_k$.

Proof : See p.241 of [4].

CHAPTER II

CONTINUOUS LINEAR FUNCTIONALS ON c_A

Lemma 2.1. Let g be a continuous linear functional on d_A for an arbitrary matrix A, then $g((x_1,\dots x_n,\dots))=\sum_n x_n g(\delta^n)$ for all $(x_1,x_2\dots x_n,\dots)$ in d_A.

Proof : By Proposition 1.7. $\{\delta^n | n=1,2,\dots\}$ is a Schauder basis for d_A. Hence $\sum_{n=1}^{m} x_n \delta^n \to x$ as $m\to\infty$, therefore $\sum_{n=1}^{m} x_n g(\delta^n)=g(\sum_{n=1}^{m} x_n \delta^n)\to g(x)$ as $m\to\infty$, hence $g(x)=\sum_n x_n g(\delta^n)$.

Theorem 2.2. Let A be a conservative matrix, $f\in c_A'$. Then f may be expressed as

$$f(x)=\alpha \lim_A x+\sum_r t_r (Ax)_r+\sum_r \beta_r x_r \qquad (2.1)$$

where $\sum_r |t_r|<\infty$ and $\sum_r \beta_r x_r$ converges for all $x\in c_A$.

Proof : By Theorem 1.18. and Lemma 1.17., c_A and d_A are FK spaces. In Theorem 1.21., let $X=d_A$, $Y=c$, then by the same theorem every continuous linear functional f on c_A can be expressed as $f=G\circ A+F$ with $G\in c'$ and $F\in d_A'$. By (1.3) and Lemma 2.1. we may take $G(x)=\alpha \lim x+\sum_r x_r t_r$ and $F(x)=\sum_r x_r \beta_r$, where $\beta_n=F(\delta^n)$, hence $G\circ A=\alpha\lim_A x+\sum_r t_r (Ax)_r$ and the result follows.

In (2.1) let $x=\delta^k$, $(k=1,2,\dots)$, then $f(\delta^k)=\alpha a_k+\sum_r t_r a_{rk}+\beta_k$ where $a_k=\lim_n a_{nk}$. Hence $\beta_k=f(\delta^k)-\alpha a_k-\sum_r t_r a_{rk}$ and

$$f(x)=\alpha\lim_A x+\sum_r t_r (Ax)_r+\sum_k \left[f(\delta^k)-\alpha a_k-\sum_r t_r a_{rk}\right] x_k \qquad (2.2).$$

If A is conservative, by Theorem 1.22. $\sum_k |a_k|<\infty$, hence $\sum_k a_k$ is convergent. We define

$$\chi(A)\equiv \lim_A 1-\sum_k a_k=\lim_n \sum_k a_{nk}-\sum_k \lim_n a_{nk} \qquad (2.3).$$

In Chapter 3 we will classify the conservative matrices by means of this number.

In Theorem 2.2. if A is also reversible, then by Proposition 1.19., c_A and c are congruent. Let A^{-1} be the inverse map of A, then foA^{-1} is a continuous linear functional on c since $A^{-1}: c \to c_A$ is continuous. By (1.3), let $foA^{-1} = \alpha \lim x + \sum_n x_n t_n$, hence

$$foA^{-1}oA = f = \alpha \lim_A x + \sum_n (Ax)_n t_n \qquad (2.4).$$

A is a continuous linear transformation from c_A to c by Corollary 1.14. and the functional $f(x) = \lim x$ is a continuous linear functional on c, hence their composite $\lim_A x$ is a continuous linear functional on c_A. We also have the following result.

<u>Theorem 2.3.</u> If $c_B \supseteq c_A$, then $\lim_B x$ is a continuous linear functional on c_A.

Proof: If $c_B \supseteq c_A$ then we can consider B as a matrix transformation from c_B to c, it is linear and continuous by Corollary 1.14. Now lim x is a continuous linear functional on c, hence so is the composite $\lim_B x$. The topology of c_A is not weaker than the subspace topology relative to c_B, hence $\lim_B x \in c_A'$.

Definition : A conservative matrix A is said to be multiplicative m if for any $x \in c$, $\lim_A x = m \lim x$.

<u>Proposition 2.4.</u> A matrix A is multiplicative m if and only if $a_k = \lim_n a_{nk} = o$ for all k.

Proof : By Theorem 2.3. $\lim_A x$ is a continuous linear functional on c where c is considered as c_I, then by (1.3)

$\lim_A x = (\lim_A(\dot{1}) - \sum_k \lim_A(\delta^k))\lim x + \sum_k x_k \lim_A(\delta^k) = \chi(A)\lim x + \sum_k x_k a_k$,
but $\chi(A)=m$ if A is multiplicative m, hence $a_k=o$ for all k. Conversely if $a_k=o$ for all k then $\lim_A x = (\lim_A \dot{1})\lim x$ hence A is multiplicative.

For any continuous linear functional f on c_A where A is conservative, $\sum_k f(\delta^k)$ is convergent because we can consider f as a continuous linear functional on c then $\sum_k |f(\delta^k)| < \infty$ by (1.5), we define

$$\chi(f) \equiv f(\dot{1}) - \sum_k f(\delta^k) \qquad (2.5).$$

Proposition 2.5. If A is conservative, f is in c_A', and f is represented as in (2.1), then $\chi(f)=\alpha\chi(A)$.

Proof : $f(\dot{1}) = \alpha \lim_A \dot{1} + \sum_r t_r(\sum_k a_{rk}) + \sum_r \beta_r$, $\sum_n f(\delta^n) = \sum_n \alpha \lim_A \delta^n + \sum_k(\sum_r t_r a_{rk}) + \sum_r \beta_r$ hence $f(\dot{1}) - \sum_n f(\delta^n) = \alpha\chi(A) + \sum_r t_r(\sum_k a_{rk}) - \sum_k(\sum_r t_r a_{rk})$, now A is a conservative matrix hence row-bounded by Theorem 1.23., $\sum_r |t_r| < \infty$, hence by Theorem 1.29. $\sum_r t_r(\sum_k a_{rk}) = \sum_k(\sum_r t_r a_{rk})$, therefore $\chi(f)=\alpha\chi(A)$.

CHAPTER III

CO-NULL AND CO-REGULAR MATRICES

Definition : A conservative matrix A is said to be co-regular if $\chi(A)\neq o$, co-null if $\chi(A)=o$.

The above definition is due to Wilansky (p.61 of [9]).

Example 3a. The process of taking the arithmetic mean or Cesàro mean can be represented by the conservative matrix

$$A=\begin{pmatrix} 1 & o & o & \cdots & & & \\ \frac{1}{2} & \frac{1}{2} & o & o & \cdots & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & o & o & \cdots & \\ \cdots & & & & & & \\ \cdots & & & & & & \\ \frac{1}{n} & \cdots & \frac{1}{n} & o & \cdots & & \\ \cdots & & & & & & \end{pmatrix}$$

A is co-regular since $\chi(A)=1$. In fact, A is regular, and it follows from ii) and iii) of Theorem 1.23. that regular matrices are co-regular.

Example 3.b. The conservative matrix

$$\begin{pmatrix} 1 & o & o & \cdots & & & & \\ o & \frac{1}{2} & o & o & \cdots & & & \\ o & -1 & \frac{1}{3} & o & o & \cdots & & \\ o & 1 & -1 & \frac{1}{4} & o & o & \cdots & \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ o & o & \cdots & 1 & -1 & \frac{1}{n} & o & o \cdots \\ \cdots & & & & & & & \\ \cdots & & & & & & & \end{pmatrix}$$

is co-null since $\lim_n \sum_k a_{nk}=o$ and $\lim_n a_{nk}=o$ for each $k=1,2,\cdots$.

If A is multiplicative zero, then $\lim_A \dot{1}=o$ and $\lim_A \delta^k=o$ for all k. Hence A is co-null. Thus every multiplicative zero matrix is co-null.

For a conservative matrix A, we define

$$W_A \equiv \{x \in c_A \mid f(x)=\sum_n x_n f(\delta^n), \text{ for all } f \in c_A'\}$$

<u>Proposition 3.1.</u> A conservative matrix A is co-null if and only if $\dot{1} \in W_A$.

Proof: If $\dot{1} \in W_A$, consider $f(x)=\lim_A x$. $f(\dot{1})=\sum_n f(\delta^n)$ implies that A is co-null. Conversely, if A is co-null, then every $f \in c_A'$ we have $f(\dot{1})-{}_n f(\delta^n)=\alpha\chi(A)=o$, hence $f(\dot{1})=\sum_n f(\delta^n)$, thus $\dot{1} \in W_A$.

<u>Corollary 3.2.</u> A conservative matrix A is co-null if and only if for every $f \in c_A'$, $f(\sum_{n=1}^{k} \delta^n) \to f(\dot{1})$ as $k \to \infty$, that is, $\sum_{n=1}^{k} \delta^n$ converges weakly to $\dot{1}$ in c_A.

Proof : This follows immediately from Proposition 3.1.

From Corollary 3.2. it follows that we can regard coregularity as a property of c_A rather than the matrix A. This was done by Snyder, A.K. (Math.Z.90,1965,376-381)

<u>Proposition 3.3.</u> If c is closed in c_A, then A is co-regular.

Proof : If c is closed in c_A, by Proposition 1.12., the subspace topology and the usual topology on c are equivalent, if A is co-null then every continuous linear functional that vanishes on $\{\delta^n \mid n=1,2,\ldots\}$ must vanish at $\dot{1}$. In c consider the subspace V_1 generated by

$\{\delta^n | n=1,\ldots\}$ and the vector $\dot{1}$, clearly $d(\dot{1},V_1)=\inf\{\|\dot{1}-v\| \mid v\in V_1\}$ ≥ 1, hence $\dot{1}\notin\overline{V}_1$(in c_A). Thus by Theorem 1.5., there is a continuous linear functional $f\in c_A'$ satisfying the condition $f(V_1)$ $=0$ and $f(\dot{1})\neq 0$. This is a contradiction, hence A cannot be co-null.

The converse of the above Proposition is not true, to see this we consider the arithmetic mean in Example 3.a. This matrix is a reversible matrix by Proposition 1.28. Hence c_A and c are congruent under A. Now let $\{x^n\}=$ $\{(-1,0,0,\ldots),(-1,1,0,0,\ldots),(-1,1,-1,0,\ldots),\ldots\}$, then $\{x^n\}\subseteq c$, let $x=\{(-1)^n\}$, then $Ax^n\to Ax$ but $x\notin c$ hence $Ax\notin A(c)$. Therefore A(c) is not closed in c hence c is not closed in c_A.

Theorem 3.4. If A,B are conservative matrices and $c_A=c_B$, then both A and B are co-regular or both are co-null.

Proof : By Theorem 2.3., $\lim_A x\in c_B'$ and $\lim_B x\in c_A'$, by Proposition 2.5. $\chi(A)=\alpha_1\chi(B)$ and $\chi(B)=\alpha_2\chi(A)$ for some α_1, α_2, hence $\chi(A)$ and $\chi(B)$ are both non-zero or both zero. This completes the proof.

The above theorem shows that co-regularity is a property that depends on the summability field c_A alone and not the matrix A.

Proposition 3.5. If A,B are conservative matrices and $c_A\subseteq c_B$, then A is co-null implies that B is also co-null.

Proof : By Theorem 2.3., $\lim_B\in c_A'$, by Proposition 2.5. $\chi(B)=$ $\alpha\chi(A)$ hence the result follows.

We now turn to the study of the "size" of the summability field c_A. We will first assume that A is co-regular.

<u>Theorem 3.6.</u> (Steinhaus) If A is a regular matrix, then $c_A \nsupseteq m$.

Proof : By Theorem 1.23. we have i) $\sum_j |a_{ij}| < M$ for some M and for all i, ii) $\lim_i a_{ij} = o$ for all j and iii) $\sum_j a_{ij} = A_i \xrightarrow[i\to\infty]{} 1$. We will construct a sequence x that consists of o's and 1's such that Ax is not convergent. By iii) choose i_1 so that $|\sum_j a_{i_1,j}| > \frac{3}{4}$, by i) choose j_1 so that $\sum_{j=j_1+1}^{\infty} |a_{i_1,j}| < \frac{1}{12}$, for $1 \leq n \leq j_1$ let $x_n = 1$ then $(Ax)_{i_1} = \sum_{j=1}^{j_1} a_{i_1,j} + \sum_{j=j_1+1}^{\infty} a_{i_1,j} x_j = \sum_j a_{i_1,j} + \sum_{j=j_1+1}^{\infty} a_{i_1,j}(x_j - 1)$, hence $|(Ax)_{i_1}| \geq |\sum_j a_{i_1}| - \sum_{j=j_1+1}^{\infty} |a_{i_1,j}| \geq \frac{3}{4} - \frac{1}{12} = \frac{2}{3}$. Now choose $i_2 \geq i_1$ by ii) so that $\sum_{j=1}^{j_1} |a_{i_2,j}| < \frac{1}{6}$, choose $j_2 > j_1$ so that $\sum_{j=j_2+1}^{\infty} |a_{i_2,j}| < \frac{1}{6}$ by i) for $j_1 < n \leq j_2$ let $x_n = o$ then $|(Ax)_{i_2}| < \sum_{j=1}^{j_1} |a_{i_2,j}| + \sum_{j=j_2}^{\infty} |a_{i_2,j}| < \frac{1}{6} + \frac{1}{6} = \frac{1}{12}$. Next we choose $i_3 > i_2$ so that $|\sum_j a_{i_3,j}| > \frac{3}{4}$, and $\sum_{j=j_1+1}^{j_2} |a_{i_3,j}| < \frac{1}{24}$, choose $j_3 > j_2$ so that $\sum_{j=j_2+1}^{\infty} |a_{i_3,j}| < \frac{1}{24}$ then $|(Ax)_{i_3}| = |\sum_{j=1}^{j_1} a_{i_3,j} + \sum_{j=j_1+1}^{j_2} a_{i_3,j} + \sum_{j=j_3+1}^{\infty} a_{i_3,j} x_j| = |\sum_j a_{i_3,j} - \sum_{j=j_1+1}^{j_2} a_{i_3,j} + \sum_{j=j_3+1}^{\infty} a_{i_3,j}(x_j - 1)| \geq |\sum_j a_{i_3,j}| - |\sum_{j=j_1+1}^{j_2} a_{i_3,j}| - \sum_{j=j_3+1}^{\infty} |a_{i_3,j}| \geq \frac{3}{4} - \frac{1}{24} - \frac{1}{24} = \frac{2}{3}$. Continuing in this way we can construct $\{x_n\}$ so that $\{|(Ax)_n|\}$ is divergent, hence $\{(Ax)_n\}$ must be divergent.

<u>Theorem 3.7.</u> If A is co-regular then $c_A \nsupseteq m$.

Proof : Let $A = (a_{nk})$, consider $B = (a_{nk} - a_k)$ where $a_k = \lim_n a_{nk}$, then B is a multiplicative matrix, since $\sum_k |a_k| < \infty$ by (1.6), we have $c_A \cap m = c_B \cap m$, hence it suffices to show $c_B \nsupseteq m$. Now $\lim_B 1 = \lim_n \sum_k (a_{nk} - a_k) = \lim_n \sum_k a_{nk} - \sum_k a_k = \rho(A) \neq o$ hence B is multiplicative $\rho(A)$, thus $\frac{1}{\rho(A)}$ B is regular and $c_B=c_{\frac{1}{\rho(A)}B}\not\supseteq m$ by Theorem 3.6.

Example 3.c. Consider the arithmetic mean and the bounded sequence defined by the following rules

$$\begin{array}{ll} x_1=1 & \\ x_n=o, & 1<n\leq 3 \\ x_n=1, & 3<n\leq 3^2 \\ x_n=o, & 3^2<n\leq 3^3 \\ \cdots\cdots\cdots\cdots & \end{array}$$

The sequence is clearly bounded, but $(Ax)_1=1, (Ax)_3=\frac{1}{3}$, $(Ax)_{3^2}\geq\frac{2}{3}$, $(Ax)_{3^3}\leq\frac{1}{3}, \ldots (Ax)_{3^{2n}}\geq\frac{2}{3}$, $(Ax)_{3^{2n+1}}\leq\frac{1}{3}$, hence the sequence x is not in c_A.

For a co-regular matrix A, c_A may be a proper subset of m, for example if A=I, the identity matrix. However, the next main result (Theorem 3.10.) tells us that whenever a co-regular matrix sums a divergent bounded sequence, c_A is not a subset of m.

Lemma 3.8. If A is a co-regular matrix, then in $c_A, \bar{c}\supseteq c_A\cap m$.

Proof : Consider c as a linear subspace of $c_A\cap m$, by Theorem 1.5. it suffices to show that every continuous linear functional that vanishes on c must vanish on $c_A\cap m$. Let $f\in c_A'$ and $f(c)\equiv o$, then in the representation (2.1), $\alpha=o$, because $\chi(A)\neq o$, $\chi(f)=o$ and $\chi(f)=\alpha\chi(A)$, also $f(\delta^k)=o$ for all k, hence by (2.2).

$$f(x)=\sum_r t_r(Ax)_r-\sum_k\left(\sum_r t_r a_{rk}\right)x_k$$

But $\sum_r t_r(Ax)_r$ may be considered as $t(Ax)$ where t is the matrix whose first row is $(t_1, \ldots t_r, \ldots)$ and other rows are zero, x may be considered as the matrix whose first column is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \end{pmatrix}$$

and other columns are zero and $\sum_k(\sum_r t_r a_{rk})x_k$ may be considered as $(tA)x$ in the same way. Now all three matrices t, A and x are row bounded if $x \in c_A \cap m$, hence $t(Ax)=(tA)x$ by Proposition 1.27. hence $f(c_A \cap m) \equiv o$.

Lemma 3.9. If $c_A \subseteq m$, then c_A is closed in m.

Proof : Recall that $\sup_n |x_n|$ is the norm on m. Let $x \in \bar{c}_A$ in m, to show that $x \in c_A$ in m it suffices to show that Ax is a Cauchy sequence. For any $\varepsilon > o$, consider $N(x, \frac{\varepsilon}{4M})$ where $M=\|A\| \neq o$ (for if $\|A\|=o$, A is the zero matrix, then $s=c_A \nsubseteq m$), let $y \in c_A \cap N(x, \frac{\varepsilon}{4M})$ and $N(\varepsilon)$ be an integer such that for m, $n > N(\varepsilon)$ we have $|(Ay)_m-(Ay)_n| = |\sum_k(a_{mk}-a_{nk})y_k| < \frac{\varepsilon}{2}$. Let $x_k = y_k + c_k$ where $|c_k| < \frac{\varepsilon}{4M}$ by the choice of y. Thus we have

$$|(Ax)_m-(Ax)_n| = |\sum_k(a_{mk}-a_{nk})(y_k+c_k)| \leq |\sum_k(a_{mk}-a_{nk})y_k| + |\sum_k(a_{mk}-a_{nk})||c_k|$$
$$< \frac{\varepsilon}{2} + 2M.\frac{\varepsilon}{4M} = \varepsilon .$$

Hence Ax is a Cauchy sequence and $x \in c_A$.

Theorem 3.10. If a co-regular matrix sums a bounded divergent sequence, it must sum an unbounded sequence.

Proof : Suppose $c \subseteq c_A \subseteq m$, then by Lemma 3.9. and Proposition 1.12., the usual topology on c_A is the same as the subspace topology. But c is closed with respect to the usual topology of m, hence c is closed in c_A. By Lemma 3.8., $c=\bar{c}\supseteq c_A\cap m$, that is, c is all the bounded sequences in c_A, this contradicts the assumption that A sums a bounded divergent sequence, therefore $c_A\not\subseteq m$ and the result follows.

Example 3.d. The arithmetic mean sums the bounded divergent sequence $\{(-1)^n\}$, it also sums the unbounded sequence $(1,-1,\sqrt{2},-\sqrt{2},3,\sqrt{3},\ldots)$.

There exist matrices that sum unbounded sequences but do not sum any bounded divergent sequence. It will be seen in the next chapter that such matrices must be co-regular. We give now an example of such a matrix.

Example 3.e. We will define a matrix A whose diagonal elements are all equal to one. Construct a one-one correspondence k from the positive integers into themselves by the following rules

$$k(1)=2^2,\ k(2)=2^3,\ k(3)=2^5,\ldots,k(n+1)=2^n k(n),\ldots\ .$$

Let A be the matrix whose diagonal elements are one and $a_{n,k(n)}=-\frac{n}{k(n)}$, the other elements are zero, then A sums the sequence $(1,2,3,4,\ldots)$ since $(Ax)_n=n-n=o$. Now if x is a bounded divergent sequence then Ax is divergent, for otherwise $\lim_n (Ax)_n=\lim_n [x_n-(\frac{n}{k(n)})x_{k(n)}]$ exists but x is bounded and $\lim_n \frac{n}{k(n)}=o$ hence $\lim_n x_n$ exists, this is a contradiction.

As for co-null matrices, we will see that every co-null matrix must sum a bounded divergent sequence hence an unbounded one in the next chapter.

CHAPTER 4

c AS A SUBSET OF c_A

In the first part of this chapter we will study the conservative matrices that are also l_1-l_1 matrices and relationships between c and l_A. In the second part we will assume that c is closed in c_A and study the consequences.

For a conservative matrix A, the conditions $l_A \subseteq c$ and $c \subseteq l_A$ may or may not hold. For example if A=I, the identity matrix, then $l_A = l_1 \subseteq c = c_A$, but $c \nsubseteq l_A$. If

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & \\ 0 & \frac{1}{2} & 0 & \cdots & & & \\ 0 & 0 & \frac{1}{3} & \cdots & & & \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & 0 & \cdots & \frac{1}{n} & 0 & \cdots \\ \cdots & & & & & & \\ \cdots & & & & & & \end{pmatrix}$$

then $(1,2,3,\ldots) \in l_A$ but it is not in c, hence $l_A \nsubseteq c$. Also, it is easy to see that $c \subseteq l_A$.

Definition : A conservative matrix is said to be perfect if c is dense in c_A.

<u>Theorem 4.1.</u> If A is perfect, an l_1-l_1 method and $A(c_A)=c$, then $l_A \nsupseteq c$.

Proof : The matrix A considered as a mapping from c_A to c is continuous by Corollary 1.15., hence $\bar{c}=c_A$ implies $\overline{A(c)}=c$ since A is onto. Now if $l_A \supseteq c$, then $\overline{A(l_A)}=c$ with respect to the norm of c, but $A(l_A) \subseteq l_1$ since A is an

l_f-l_1 method, hence l_1 is dense in c. The last statement is not true because if we let $\varepsilon = \frac{1}{2}$, $x=(1,1,\ldots,1,\ldots)$, then $N(x,\varepsilon)$ contains no element of l.

It is obvious that, for an arbitrary matrix A, if $c\subseteq l_A$, then A is conservative and multiplicative zero. However for a conservative matrix A which is also an l_f-l_1 method, A multiplicative zero does not imply $c\subseteq l_A$. Consider

$$A=\begin{pmatrix} 1 & o & \cdots & \cdots & \cdots \\ \frac{1}{2^2} & \frac{1}{2^2} & o & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n^2} & \frac{1}{n^2} & \cdots & \frac{1}{n^2} & o \;\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

the matrix A is row-bounded, $\lim_A \delta^k = o$ for all k and $\lim_A \dot{1} = o$, hence by Theorem 1.22., A is conservative. Also, A is column-bounded, hence it is an l_f-l_1 method by Theorem 1.25. Now $\dot{1}\in c$ and $\sum_n |(A\dot{1})_n| = 1+\frac{1}{2}+\frac{1}{3}+\ldots=\infty$, hence $\dot{1}\notin l_A$.

Theorem 4.2. If A is an l-l method, then a necessary condition for $l_A\subseteq c$ is that for any subsequence $(r_1,r_2,\ldots,r_i\ldots)$ of the sequence $(1,2,3,\ldots)$ with $r_i+1<r_{i+1}$ for infinitely many r_i,

$$\sum_n \Big(\Big|\sum_{k=r_1,r_2\ldots} a_{nk}\Big| \Big) = \infty \qquad (4.1)$$

Proof : Suppose $l_A\subseteq c$, for any such sequence $(r_1,r_2,\ldots)$. Construct a sequence x whose r_ith term is 1 and others are zero, then x is a divergent sequence since $r_i+1<r_{i+1}$ for

infinitely many i. Hence $x \notin l_A$, that is, (4.1) holds.

The condition (4.1) is not sufficient, for example, let

$$A=\begin{pmatrix} -1 & o & o & \cdots & & & \\ -2 & 1 & o & \cdots & & & \\ o & -2 & 1 & o & \cdots & & \\ \cdots & & & & & & \\ \cdots & & & -2 & 1 & o & \cdot \\ \cdots & & & & & & \end{pmatrix}$$

then A is column-bounded hence an l_1-l_1 method, let x= (1,2,4,8,...), then Ax=(-1,o,o,o,...), thus $x \in l_A$, but $x \notin c$.

In what follows we will study the condition that c is closed in c_A, for this we arrange the seminorms that generate the linear topology on c_A in the following manner:

$$q_o(x) \equiv P_o(x) = \sup_n |\sum_k a_{nk}x_k|$$

$$q_{2n-1}(x) \equiv h_n(x) = \sup_m |\sum_{k=1}^{m} a_{nk}x_k|$$

$$q_{2n}(x) \equiv |P_n|(x) = |x_n| \; .$$

Recall that the locally convex linear topology on c_A is generated by

$$\sum_{n=o}^{\infty} \frac{1}{2^n} \frac{p_n(x)}{1+p_n(x)} \quad , \text{ where } p_n(x) = \max_{o \leq i \leq n} q_i(x) \; .$$

Also recall that c_A is complete; hence a series is convergent if the partial sums form a Cauchy sequence. The following interesting result is due to Wilansky and Zeller [10].

<u>Theorem 4.3.</u> For a conservative matrix A, c is closed in c_A if and only if A sums no bounded divergent sequence,

that is, $c_A \cap m \subseteq c$.

Proof : Suppose c is not closed in c_A and consider the subspaces

$$V_K = \{x \in c \mid x_k = o,\ k < K\} \quad , K = o, 1.2, \ldots.$$

These subspaces are not closed in c_A; for suppose V_{K_o} is closed for some K_o and let $\{x^m\} \subseteq c$ converge to x in c_A, furthermore, for each $x^m = (x_1^m, x_2^m, \ldots, x_{K_o-1}^m, x_{K_o}^m, \ldots)$, let $y^m = (x_1^m, \ldots, x_{K_o-1}^m, o, o, \ldots)$ and $z^m = (o, \ldots o, x_{K_o}^m, x_{K_o+1}^m, \ldots)$, then $x^m = y^m + z^m$ and $z^m \in V_{K_o}$. Now $q_i(x^m - x) \to o$ for all i by assumption, hence $q_{2n}(y^m - y) \to o$; but each y^m has zero coordinate after the K_o-1th coordinate, hence $q_o(y^m - y) \to o$ and $q_{2n-1}(y^m - y) \to o$. Now let $x = (x_1, \ldots x_{K_o-1}, x_{K_o}, \ldots) = (x_1, \ldots, x_{K_o-1}, o, \ldots) + (o, o, \ldots, x_{K_o}, x_{K_o+1}, \ldots)$ and let $y = (x_1, \ldots x_{K_o-1}, o, \ldots)$, $z = (o, o, \ldots x_{K_o}, x_{K_o+1}, \ldots)$, then $x = y + z$. For any i, $q_i(z^m - z) = q_i(x^m - x) + q_i(y^m - y)$; hence $q_i(z^m - z) \to o$. If V_{K_o} is closed in c_A, then $z \in V_{K_o}$, hence $x \in c$ and c is closed in c_A ——— contradiction.

By Proposition 1.12., the usual topology on $V_K \subseteq c_A$ is strictly stronger than the subspace topology relative to c_A , hence the seminorm $q(x) \equiv \|x\| = \sup_n |x_n|$ is discontinuous with respect to the subspace topology by definition. By v) of Theorem 1.3., for any $\varepsilon > o$, any intergers b,K, there exists $x \in V_K$ such that

$$q(x) = 1 \qquad (4.2)$$

$$p_k(x) < \varepsilon \qquad \text{for } k < b \qquad (4.3)$$

Case I. If A is a co-regular matrix. We may assume $\chi(A)=1$, for otherwise we may consider $\frac{1}{\chi(A)}A$; this matrix has the same summability field as A, also the identity map is a homeomorphism. Consider $\lim_A x$ as a continuous linear functional on c, and let $\lim_A x=\alpha \lim_I x+\sum_k a_k x_k$, where $a_k=\lim_n a_{nk}$; since $\chi(A)=1$, by Proposition 2.5., $\alpha=1$. By (1.6) $\sum_k |a_k|<\infty$. For any $\varepsilon>0$, choose K great enough so that $\sum_{k=K} |a_k|<\varepsilon$; by the preceding part, there exists $x\in V_K$ such that (4.2) and (4.3) hold; hence $|\lim_A x|<\varepsilon$, because $p_0(x)<\varepsilon$. Therefore $|\lim_I x|\leq|\lim_A x|+|\sum_{k=K}^{\infty} a_k x_k|\leq|\lim_A x|+\sup_n|x_n|\left(\sum_{k=K}^{\infty}|a_k|\right)$, but $q(x)=\sup_n|x_n|=1$, hence $|\lim_I x|\leq|\lim_A x|+\sum_{k=K}^{\infty}|a_k|<\varepsilon+\varepsilon=2\varepsilon$. If $\varepsilon<\frac{1}{2}$, then $|\lim x|<1$. Now $|x_n|<1$ for n sufficiently large, $x_n=0$ for $n=1,2,\ldots,K-1$ and $\sup_n|x_n|=1$, therefore there is a finite interval N(x) of natural numbers such that $|x_n|<1$ for $n\notin N(x)$ and $|x_n|=1$ for some $n\in N(x)$.

Let $\varepsilon_r=2^{-r-3}$, $r=1,2,3,\ldots$ and b=r; for each r choose $x^r\in V_{K_r}$ satisfying (4.2) and (4.3), furthermore, for each r, K_{r+1} is chosen in such a way so that $N(x^r)$ $r=1,2,\ldots.$ are pairwise disjoint and that infinitely many natural numbers are not in any $N(x^r)$. We claim that $\sum_r x^r$ is a convergent series in c_A. For any $\varepsilon>0$, choose r so that $\sum_{n=r}^{\infty}\frac{1}{2}<\frac{\varepsilon}{2}$, then for $j>i>r$,

$$\|x^i+\ldots+x^j\|=\sum_{n=0}^{i-1}\frac{1}{2}\,\frac{p_n(x^i+\ldots+x^j)}{1+p_n(x^i+\ldots+x^j)}\;+$$

$$\sum_{n=i}^{\infty} \frac{1}{2} \frac{p_n(x^i+..+x^j)}{1+p_n(x^i+..+x^j)} \leq \sum_{n=o}^{i-1} \frac{1}{2}[p_n(x^i)+...+p_n(x^j)]+$$

$$\frac{\varepsilon}{2} \leq \sum_{n=o}^{i-1} \frac{1}{2^n}[2^{-i-3}+...+2^{-j-3}]+\frac{\varepsilon}{2} \leq \sum_{n=o}^{i-1} \frac{1}{2^n}[2^{-i-3}(1+\frac{1}{2}+..+\frac{1}{2^{j-i}})]+$$

$$\frac{\varepsilon}{2} < (\sum_{n=o}^{i-1} \frac{1}{2^n})\, 2^{-i-1}+\frac{\varepsilon}{2} < 2^{-i+1}+\frac{\varepsilon}{2} \leq \varepsilon$$

therefore the partial sums form a Cauchy sequence, hence $\sum_r x^r$ is convergent to, say, x_o in c_A. The sequence $\sum_r x^r$ is bounded by construction; in fact, $|x_n| \leq 1+\sum_{r=1}^{\infty} 2^{-r-3}$ for all n, furthermore, it has a subsequence tending to 1 and a subsequence tending to zero, hence $\sum_r x^r$ is a divergent sequence; this completes the proof for co-regular matrices.

Case II. A is co-null. We first notice that c_o cannot be closed in c_A, for otherwise there exists $f \in c_A'$ such that $f(c_o) \equiv o$ and $f(1) \neq o$ by the Hahn-Banach Theorem. But A is co-null. Hence $f(c_o) \equiv o$ implies $f(1)=o$, therefore such f does not exist. Hence c_o cannot be closed in c_A. In the first part of this proof if we consider

$$V_K' = \{x \in c_o \mid x_k = o, k < K\}, \quad K=o,1,2,........$$

instead of V_K, then the V_K' (K=1,2,....) are not closed. The proof is exactly the same as the preceding one. Hence by v) of Theorem 1.3. for every $\varepsilon > o$, positive integers b,K, there exists $x \in V_K'$ such that (4.2) and (4.3) are satisfied. Now $\lim x = o$, hence the argument used in Case 1 can be applied to show that there is a bounded divergent sequence in c_A.

Corollary 4.4. A co-null matrix must sum a bounded divergent sequence.

Proof : By Proposition 3.3., c is not closed in c_A if A is co-null; by Theorem 4.3., A must sum a bounded divergent sequence.

Corollary 4.5. A co-null matrix must sum an unbounded sequence.

Proof : Suppose $c_A \subseteq m$, then c_A is closed in m by Lemma 3.9., hence by Proposition 1.12., the topology of c_A is the same as the subspace topology relative to m. c is complete with the usual topology, hence c is closed in c_A, thus A is co-regular —— contradiction. Therefore $c_A \nsubseteq m$, hence A sums an unbounded sequence.

Corollary 4.6. If A sums a bounded divergent sequence, then c is not closed in c_A.

CHAPTER V

PERFECTNESS AND MATRICES OF TYPE M

Definition : Let $A=(a_{nk})$ be an arbitrary matrix. Any sequence $\{\alpha_n\}$ in l_1 satisfying

$$\sum_n \alpha_n a_{nk}=0 \qquad \text{for } k=1,2,\dots. \qquad (5.1)$$

is said to be orthogonal to A. If the only sequence orthogonal to A is the zero sequence, A is said to be of type M.

All diagonal matrices with non-zero diagonal elements are of type M. For certain classes of matrices, perfectness and type M are closely related. In this chapter we will study these concepts for different classes of matrices. The concept of type M will be applied to consistency.

Definition : Let $\{\alpha_n\}$ be orthogonal to a conservative matrix A and let $f(x)=\sum_n \alpha_n (Ax)_n$. We call f(x) an orthogonal functional on c_A.

Proposition 5.1. If A is conservative, then every orthogonal functional vanishes on $c_A \cap m$.

Proof : Let $f=\sum_n \alpha_n (Ax)_n=\alpha(Ax)$ be an orthogonal functional, then by Proposition 1.27., $\alpha(Ax)=(\alpha A)x$ for every $x \in c_A \cap m$, hence $\alpha(Ax)=(\alpha A)x=0x=0$.

Proposition 5.2. Let $A=(a_{nk})$ be conservative and reversible, then $\overline{c_A \cap m}=c_A$ implies that A is of type M.

Proof : Suppose $\overline{c_A \cap m}=c_A$, then by Theorem 1.5., any continuous linear functional that vanishes on $c_A \cap m$ is identically zero on c_A. Let $\alpha=(\alpha_1,\alpha_2,\ldots)\in l_1$ and $\sum_n \alpha_n a_{nk}=o$ for all k. Suppose $\alpha_{n_o}\neq o$ for some n_o and let $Ay=\delta^{n_o}$. Such y exists because A is reversible, and clearly $y\in c_A$. Now the continuous linear functional $\sum_n \alpha_n (Ax)_n$ is identically zero on $c_A\cap m$ by Proposition 1.27., hence $\sum_n \alpha_n (Ax)_n$ is identically zero on c_A by assumption. But $\sum_n \alpha_n (Ay)_n = \alpha_{n_o}\neq o$ and this is a contradiction. Hence $\alpha=o$ and A is of type M.

Proposition 5.3. If A is co-regular, then $\bar{c}=\overline{c_A\cap m}$.

Proof : Clearly $\bar{c}\subseteq\overline{c_A\cap m}$. To show that $\bar{c}\supseteq\overline{c_A\cap m}$ it suffices to prove that $\bar{c}\subseteq c_A\cap m$, for then $\bar{c}=\bar{\bar{c}}\supseteq\overline{c_A\cap m}$. Let f be a continuous linear functional on c_A that vanishes on c, we show that $f(c_A\cap m)\equiv o$. By (2.2), $f(x)=\alpha \lim_A x+\sum_n t_n (Ax)_n+\sum_k [f(\delta^k)-\alpha a_k-\sum_n t_n a_{nk}]x_k$ and recall that $\chi(f)=\alpha\chi(A)$. Now $f(1)=o$ and $f(\delta^k)=o$ for all k, hence $\chi(f)=o$, but A is co-regular thus $\chi(A)\neq o$, hence $\alpha=o$. Also, the representation of f(x) is reduced to $f(x)=\sum_n t_n (Ax)_n-\sum_k(\sum t_n a_{nk})x_k$, hence $f(x)=t(Ax)-(tA)x$, where $t=(t_1,t_2,\ldots t_n,\ldots)$. By Proposition 1.27. f(x) vaniches on $c_A\cap m$.

Theorem 5.4. A reversible, co-regular matrix A is perfect if and only if it is of type M.

Proof : If A is perfect, then $\bar{c}=c_A$. Thus, by Proposition 5.3. $c_A=\overline{c_A\cap m}$. Hence by Proposition 5.2., A is of type M.

Conversely, suppose A is of type M. It suffices to show that every continuous linear functional that vanishes on c must vanish on c_A. By (2.4), $f(x)=\alpha \lim_A x+\sum_n t_n (Ax)_n$.

In exactly the same way as in the proof of Proposition 5.3., we obtain $f(x)=\sum_n t_n(Ax)_n$. Now f vanishes on c and $f(\delta^k)=\sum_n t_n a_{nk}=o$, hence by assumption $t=(t_1,t_2,\dots t_n,\dots)=o$, thus $f\equiv o$. This completes the proof.

In general, perfectness and type M are not equivalent conditions. For example the matrix

$$A=\begin{pmatrix} o & o & o & \dots & \\ o & 1 & o & \dots & \\ o & o & 1 & o & \dots \\ o & o & o & 1 & \dots \\ \dots & & & & \\ \dots & & & & \end{pmatrix}$$

is not of type M, since $(1,o,\dots)A=o$, but $c_A=c$, hence it is perfect. On the other hand, consider the matrix in Example 3.e., that is, the matrix A whose diagonal elements are 1, $a_{n,k(n)}=-\frac{n}{k(n)}$, where $k(1)=2^2$, $k(2)=2^3,\dots$, $k(n+1)=2^n k(n)$ and other elements are zero. This matrix does not sum any bounded divergent sequence, hence $\bar{c}=c$ in c_A, but $(1,2,3,\dots)\in c_A$, hence $c_A\neq c$, therefore the matrix is not perfect. The matrix is of type M. This can be seen as follows : Suppose $\alpha=(\alpha_1,\dots,\alpha_n,\dots)$ and $\alpha A=o$, then $\alpha_1=\alpha_2=\alpha_3=o$, also $\alpha_{k(1)}=\alpha_4=o$, because $\alpha_4.1+\alpha_1\cdot a_{1k(1)}=o$ but $\alpha_1=o$, hence $\alpha_{k(1)}=\alpha_4=o$. Similarly, we have $\alpha_1=\alpha_2=\dots=\alpha_{k(2)-1}=\alpha_7=o$ and $\alpha_8+\alpha_2\cdot a_{28}=o$ hence $\alpha_8=o$. Continuing in this way we have $\alpha=(o,o,\dots o,\dots)$, hence the matrix is of type M.

Now we will consider a different class of matrices, that is, the reversible and multiplicative matrices. It is a different class from the reversible co-regular matrices because the matrix

$$\begin{pmatrix} 2 & 0 & 0 & \cdots & & & & \\ 1 & 2 & 0 & 0 & \cdots & & & \\ 1 & 0 & 2 & 0 & 0 & \cdots & & \\ 1 & 0 & 0 & 2 & 0 & 0 & \cdots & \\ \cdots & & & & & & & \\ \cdots & & & & & & & \\ 1 & 0 & 0 & \cdots & 2 & 0 & 0 & \cdots \\ \cdots & & & & & & & \end{pmatrix}$$

is co-regular, reversible but not multiplicative since the first column does not tend to zero. On the other hand, the matrix in Example 3.b. is reversible, multiplicative and co-null.

Definition : A maximal subspace of a linear space is a subspace whose complementary subspace has dimension one.

Lemma 5.5. Let V_1 be a linear subspace of a linear space V. If there exist two independent linear functional f_1, f_2 such that $f_1(V_1) = f_2(V_1) \equiv 0$, then V_1 is not a maximal subspace.

Proof : Suppose V_1 is maximal in V and let v span the complementary subspace. Let $f_1(v) = \alpha_1$ and $f_2(v) = \alpha_2$, then $-\frac{\alpha_1}{\alpha_2} f_2(V) + f_1(V) \equiv 0$ and this contradicts the assumption, hence V_1 is not maximal.

Theorem 5.6. Let A be reversible and multiplicative. Then A is of type M if and only if c_0 is a maximal subspace of c_A.

Proof : Suppose A is of type M. It suffices to show that $\bar{c}_o$ is the kernel of some linear functional on c_A. Let $x_1 \notin \bar{c}_o$, by Theorem 1.5., there exists $f \in c_A'$ such that $f(c_o)=o$ and $f(x_1) \neq o$. By (2.4), we may let $f(x)=\alpha \lim_A x+\sum_n t_n(Ax)_n$. Now f is multiplicative and of type M, hence $f(\delta^k)=o+\sum_n t_n(a_{nk})=o$, therefore $t_n=o$ for all n, hence $f(x)=\alpha \lim_A x$. By assumption $f(x_1) \neq o$, hence $\lim_A x_1 \neq o$. Now consider the continuous linear functional $h(x)=\lim_A x$ on c_A. Since A is multiplicative, we have $\lim_A x=o$ for all $x \in c_o$, hence $\ker h(x) \supseteq \bar{c}_o$. By the preceding part of this proof, $h(x) \neq o$ for all $x \notin \bar{c}_o$, hence $\ker h(x)=c_o$, thus $\bar{c}_o$ is a maximal subspace.

Conversely, suppose A is not of type M. Let $t=(t_1,\ldots,t_n,\ldots) \in l_1$ be non-zero and $\alpha A=o$. Consider $f_1(x)=\alpha(Ax)$ and $f_2(x)=\lim_A x$; both $f_1(x)$ and $f_2(x)$ vanish on c_o. By Lemma 5.5., if $f_1(x)$ and $f_2(x)$ are independent, then $\bar{c}_o$ is not a maximal subspace. Let a_1,a_2 be two scalars such that $f(x)=a_1 \lim_A x+(a_2 \cdot t)(Ax) \equiv o$ on c_A and suppose $t_n \neq o$. Let $x_1 \in c_A$ be such that $Ax_1=\delta^n$, then $f(x_1)=o+a_2 t_n=o$, hence $a_2=o$. Let $x_2 \in c_A$ satisfy $Ax_2=\dot{1}$, then $f(x_2)=a_1 \cdot 1=o$, hence $a_1=o$, thus f_1 and f_2 are independent.

We have seen that in general the concepts of perfectness and type M are not equivalent. In what follows, we will look at some subsets of c_A and study some sufficient conditions on these subsets for A to be perfect or to be of type M.

For a conservative matrix A, we define

$B_A=\{x\in c_A \mid$ there exists $M>0$ depending on x such that

$|\sum_{k=1}^{m} a_{nk}x_k| < M$, for $m,n=1,2,\ldots\}$

$L_A=\{x\in c_A \mid (tA)x=\sum_k(\sum_n t_n a_{nk})x_k$ exists for all $t\in l_1\}$

$P_A=\{x\in c_A \mid (tA)x=t(Ax)$ for all $t\in l$ such that $(tA)y$

exists for all $y\in c_A.\}$

In general the subset B_A does not fill up c_A. For example, let A be the matrix in Example 3.b. and $x=(1,1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3},\ldots,1+\frac{1}{2}+\frac{1}{3}\ldots\frac{1}{n},\ \ldots)$, then $(Ax)_n=(1+\frac{1}{2}+\ldots+\frac{1}{n-2})-(1+\ldots+\frac{1}{n-1})+\frac{1}{n^2}(1+\frac{1}{2}+\ldots+\frac{1}{n})=-\frac{1}{n-1}+\frac{1}{n^2}(1+\ldots+\frac{1}{n})$, hence $\lim_n(Ax)_n=0$ and $x\in c_A$, but $x\notin B_A$ because $|\sum_{k=1}^{n-2}a_{nk}x_k|=|1+\ldots+\frac{1}{n-2}|$ which tends to infinity as n increases.

<u>Theorem 5.7.</u> If A is co-regular then $P_A=\bar{c}$.

Proof : Let f be a continuous linear functional vanishing on c. In the proof of Theorem 5.3. we proved that $f(x)$ is of the form $t(Ax)-(tA)x$, hence by the definition of P_A we have $f(x)$ vanishes on P, therefore $P_A\subseteq\bar{c}$.

Conversely, it is clear that $c\subseteq P_A$, hence it suffices to show that P_A is closed. Let $F=\{t\in l \mid (tA)x$ exists for all $x\in c_A\}$ and for every $t\in F$ define $f_t=(tA)x-t(Ax)$. Each f_t is a continuous linear transformation from c_A to s by Corollary 1.14., hence the kernel of f_t is closed. Now $P_A=\bigcap_{t\in F}\ker f_t$, hence P_A is closed. Therefore $P_A\supseteq\bar{c}$ and hence $P_A=\bar{c}$.

The above theorem characterizes c in case A is co-regular. Notice that $P\supseteq\bar{c}$ does not depend on the co-regularity of A. The following corollary follows trivially from Theorem 5.7.

<u>Corollary 5.8.</u> A co-regular matrix A is perfect if and only if $P_A = c_A$.

Corollary 5.8. is not true for co-null matrices, for example, consider the matrix

$$\begin{pmatrix} 1 & o & \cdots & & & & \\ o & \frac{1}{2} & o & \cdots & & & \\ o & o & \frac{1}{3} & o & \cdots & & \\ \cdots & & & & & & \\ \cdots & & & & & & \\ o & o & \cdots & \frac{1}{n} & o & \cdot & \\ \cdots & & & & & & \end{pmatrix}$$

Clearly $P_A = c_A$ and the sequence $x_o = (1,2,3,\cdots) \in c_A$. Recall that for normal matrices the topology is defined by the norm $\|x\| = \sup_n |(Ax)_n|$ for all $x \in c_A$. Let $\varepsilon = \frac{1}{2}$, then $N(x_o, \varepsilon)$ does not contain any element of c, thus A is not perfect.

<u>Proposition 5.9.</u> $B_A = L_A$

Proof : Let $x \in B_A$ and $t = (t_1, t_2, \cdots t_n, \cdots) \in l_1$. For any k, $(\sum_n t_n a_{nk}) x_k$ exists because $t \in l_1$ and $\{a_{nk} | n = 1,2,\cdots\}$ is bounded. Let

$S_1 = (\sum_n t_n a_{n1}) x_1$, $S_2 = (\sum_n t_n a_{n1}) x_1 + (\sum_n t_n a_{n2}) x_2 = \sum_n t_n (a_{n1} x_1 + a_{n2} x_2), \ldots$
$\ldots\ldots, S_k = (\sum_n t_n a_{n1}) x_1 + \cdots + (\sum_n t_n a_{nk}) x_k = \sum_n t_n (a_{n1} x_1 + \cdots + a_{nk} x_k), \ldots$

Let $S = t_1 (\sum_k a_{1k} x_k) + t_2 (\sum_k a_{2k} x_k) + \cdots$. We claim that S_k tends to S. For any $\varepsilon > o$, choose $N(\varepsilon)$ so that $\sum_{n=N(\varepsilon)+1} |t_n| < \frac{\varepsilon}{4M}$, and for $n = 1,2,\cdots,N(\varepsilon)$, choose K great enough so that $k_o > K$ implies

$|t_1||(a_{n1}x_1 + \cdots + a_{nk_o}x_{k_o}) - (_k a_{1k}x_k)| + \cdots + |t_{N(\varepsilon)-1}||(a_{N(\varepsilon)1}x_1 + \cdots + a_{N(\varepsilon)k_o}x_{k_o}) -$

$(\sum_k a_{N(\varepsilon)k}x_k)| < \frac{\varepsilon}{2}$.

Then for k>K, we have

$$|S_k - S| < \frac{\varepsilon}{2} + \sum_{n=N(\varepsilon)+1}^{\infty} |t_n||a_{n1}x_1 + \cdots + a_{nk}x_k| + \sum_{n=N(\varepsilon)+1}^{\infty} |t_n||\sum_k a_{nk}x_k|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

Hence $\sum_k(\sum_n t_n a_{nk})x_k = \sum_n(\sum_k a_{nk}x_k)t_n$ and thus $(tA)x$ exists.

Conversely, let $x=(x_1,x_2,\cdots)\in L$. Define a sequence of linear functional $\{f_m\}$ on l_1 by

$$f_m(t) = t_1(a_{11}x_1+\cdots+a_{1m}x_m) + t_2(a_{21}x_1+\cdots+a_{2m}x_m)+\cdots\cdots$$

Each f_m is well-defined since A is conservative. Recall that the norm on l_1 is defined by $\|t\| = \sum_n |t_n|$, hence it is easy to see that each f_m is a continuous linear functional on l_1. Now $\|f_m\| = \sup\{|\sum t_n(a_{n1}x_1+\cdots+a_{nm}x_m)|\,|\, \sum_n |t_n| \le 1\} \le \sup_n |a_{n1}x_1+\cdots+a_{nm}x_m|$. On the other hand, let $t=\delta^n$, then $\|f_m\| \ge \sup_n |a_{n1}x_1+\cdots+a_{nm}x_m|$, hence we have $\|f_m\| = \sup_n |a_{n1}x_1+\cdots+a_{nm}x_n|$. By definition of f_m we also have

$$f_m(t) = (t_1a_{11}+t_2a_{21}+\cdots)x_1+\cdots+(t_1a_{1m}+t_2a_{2m}+\cdots)x_m$$

Since $x\in L_A$, $\lim_m f_m(t) = \sum_k(\sum_n t_n a_{nk})x_k$ exists for each $t\in l$. By Theorem 1.6. $\{\|f_m\|\,|\, m=1,2,\cdots\}$ is uniformly bounded, hence there exists M such that $\sup_n |a_{n1}x_1+\cdots+a_{nm}x_m| \le M$ for all m, hence $x\in B_A$.

<u>Theorem 5.10.</u> If a conservative matrix A has a right inverse whose columns belong to B_A except for a finite number of them, then A is of type M.

Proof : Recall that in the proof of Proposition 5.10. we actually proved that $(tA)x = t(Ax)$ for all $x\in B_A$ and $t\in l$.

Suppose x is the nth column of A^{-1} belonging to B_A, $t\in l$ and t is orthogonal to A, then $(tA)x=o=t(Ax)=t(\delta^n)=t_n=o$, but all except a finite number of the columns of A^{-1} belong to B_A, hence $t_n=o$ except for a finite number of them. Let $t=(t_1,\cdots,t_n,o,o,o,\cdots)$ and let $u_1,u_2,\cdots u_n$ be the first n columns of A^{-1}, then $(tA)u_1=o=(t_1,o,o,\cdots)$ hence $t_1=o$, similarly $t_2=t_3=\cdots=t_n=o$.

Definition : A conservative matrix A is said to have the mean value property if $B_A=c_A$.

<u>Corollary 5.11.</u> A reversible matrix that has the mean value property is of type M.

Proof: Since A is reversible, there exists x^k such that $Ax^k=\delta^k$. Let D be the matrix whose kth column is x^k, then $D=A^{-1}$. If A has the mean value property, then $x^k\in c_A=B_A$. By Theorem 5.10., A is of type M.

<u>Proposition 5.12.</u> A co-regular matrix that has the mean value property is perfect.

Proof : In the proof of Proposition 5.9., we proved that for all $x\in B_A$ and $t\in l_1$, $(tA)x=t(Ax)$, hence $B_A\subseteq P_A$. By Theorem 5.7., when A is co-regular $P_A=\bar{c}$, hence if A has the mean value property $B_A=c_A\subseteq P_A=\bar{c}$, thus $c_A=\bar{c}$. Therefore A is perfect.

Definition : Two matrices A and B are said to be consistent if $\lim_A x=\lim_B x$ for all $x\in c_A\cap c_B$.

<u>Lemma 5.13.</u> Let A be a reversible conservative matrix,

then $f\in c_A'$ if and only if $f(x)=\lim_B x$ for some B such that $c_B\supseteq c_A$.

Proof : If $c_B\supseteq c_A$, then by Theorem 2.3., $\lim_B x\in c_A'$. Conversely, if $f\in c_A'$, let $f(x)=\alpha\lim_A x+\sum_n t_n(Ax)_n$ as in (2.4). Define a matrix $B=(b_{nk})$ where $b_{nk}=t_1a_{1k}+t_2a_{2k}+\cdots+t_{n-1}a_{n-1,k}+\alpha a_{n,k}$, then

$$(Bx)_m=\alpha\left(\sum_k a_{mk}x_k\right)+\sum_{n=1}^{m-1}t_n(Ax)_n,$$

hence $\lim_B x=f(x)$.

Theorem 5.14. Let A be reversible and co-regular, then a necessary and sufficient condition for A to be type M is that A is consistent with every matrix B such that

i) $c_B\supseteq c_A$

ii) $\lim_B x = \lim_A x$ for all $x\in\{\delta^k | k=1,2,\cdots\}\cup\{1\}=F$.

Proof : We will first prove that A is consistent with every B satisfying i) and ii) is equivalent to the condition that A is perfect. Then the theorem will follow from Theorem 5.4. Suppose A is consistent with every B satisfying i) and ii). Let f c_A' satisfy $f(F)=o$ and consider $f+\lim_A x=f_1$. Obviously $f_1\in c_A'$, by Lemma 5.13., we can let $f_1(x)=\lim_B x$, then i) and ii) are satisfied. Hence $\lim_A x=f_1(x)$ for all $x\in c_A$ and $f(A)\equiv o$. Now $F\subseteq c$, thus any continuous linear functional that vanishes on c must be identically zero on c_A. Hence by Theorem 1.5., c is dense in c_A, thus A is perfect. Conversely, let A be perfect and B be a matrix satisfying i) and ii), then f=

$\lim_A x - \lim_B x$ is in c_A' and f vanishes on F. Now F is a Schauder basis for c with the usual topology and this topology is stronger than the subspace topology relative to c_A, hence f also vanishes on c. But A is perfect, hence $c = c_A$ and thus $f(c_A) \equiv o$. Therefore $\lim_A x = \lim_B x$ for all $x \epsilon c_A$ and A,B are consistent.

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